APPM4600 Homework #10

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1) $f(x) = \sin(x)$

a) $$P_3^3 = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3}$$

Compare w/ Taylor exp: $f(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120}$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = (1 + b_1 x + b_2 x^2 + b_3 x^3)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

Comparing coeffs:

x^0: $a_0 = 0$	x^3: $a_3 = -\frac{1}{6} + b_2$
x^1: $a_1 = 1$	x^4: $0 = \frac{-b_1}{6} + b_3$
x^2: $a_2 = b_1$	x^5: $0 = \frac{1}{120} - b_2/6$
	x^6: $0 = \frac{b_1}{120} - b_3/6$

$b_1 = b_3 = 0, \quad b_2 = \frac{6}{120}$

$a_0 = 0, \quad a_1 = 1, \quad a_2 = b_1 = 0$

$a_3 = -\frac{1}{6} + b_2 = \frac{6}{120} - \frac{1}{6}$ so

$$P_3^3 = \frac{x + \left(\frac{6}{120} - \frac{1}{6}\right)x^3}{1 + 6/120\, x^2}$$

b) for $P_4^7 \Rightarrow$

$$a_0 + a_1 x + a_2 x^2 = (1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4)\left(x - x^3/3 + \frac{x^5}{120}\right)$$

Comparing coeffs:

$x^0: \; a_0 = 0$

$x^1: \; a_1 = 1$

$x^2: \; a_2 = b_1$

$x^3: \; 0 = -\frac{1}{6} + b_2$

$x^4: \; 0 = -\frac{b_1}{3} + b_3$

$x^5: \; 0 = \frac{1}{120} - \frac{b_2}{6} + b_4$

$x^6: \; 0 = \frac{b_1}{120} - \frac{b_3}{6}$

$b_1 = b_3 = 0, \quad b_2 = \frac{1}{6}, \quad b_4 = \left(\frac{-1}{120} + \frac{1}{36}\right)$

$a_0 = 0, \; a_1 = 1, \; a_2 = b_1 = 0$ so

$$P_4^2 = \frac{x}{1 + \frac{1}{6}x + \left(\frac{1}{36} - \frac{1}{120}\right)x^4}$$

c) $\underline{\underline{P_2^4}}$

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) = (1 + b_1 x + b_2 x^2)\left(x - \frac{x^3}{3} + \frac{x^5}{120}\right)$$

Comparing Coeffs:

x^0: $a_0 = 0$

x^1: $a_1 = 1$

x^2: $a_2 = b_1$

x^3: $a_3 = \frac{-1}{6} + b_2$

x^4: $a_4 = -b_1/6$

x^5: $0 = \frac{1}{120} - \frac{b_2}{6}$

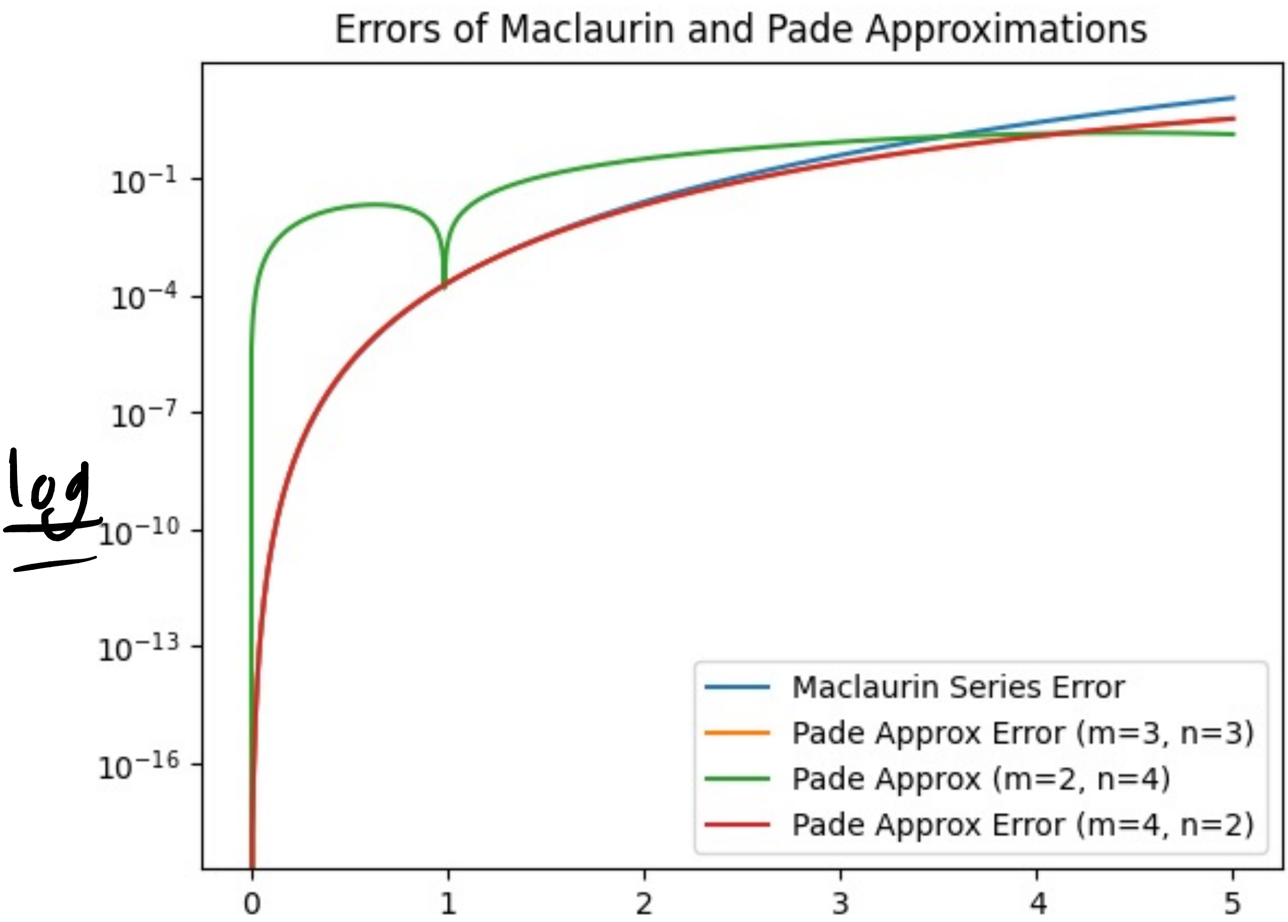
x^6: $0 = \frac{b_1}{120}$

$b_1 = 0,\ b_2 = \frac{6}{120}$

$a_0 = 0,\ a_1 = 1,\ a_2 = b_1 = 0,\ a_3 = \left(\frac{6}{120} - \frac{1}{6}\right)$

$a_4 = {-b_1}/6 = 0$ so

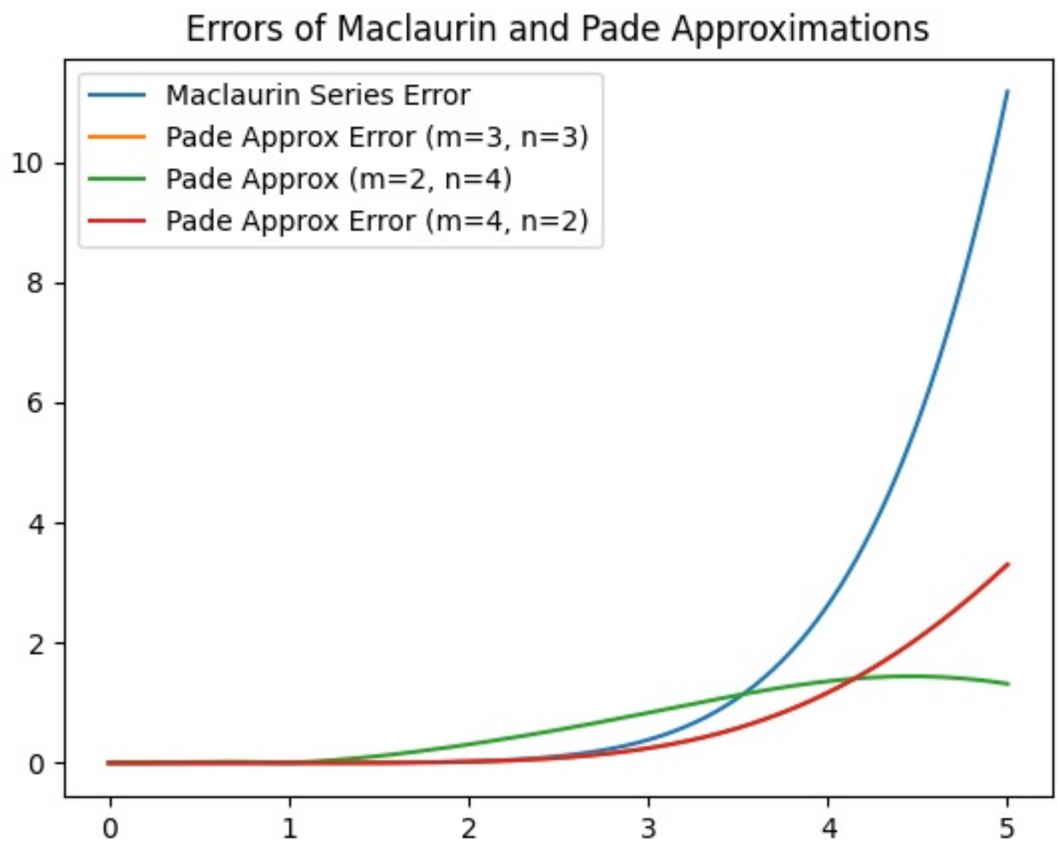
$$P_2^4 = \frac{x + \left(\frac{6}{120} - \frac{1}{6}\right)x^3}{1 + (6/120)x^2} = P_3^3 \text{ !!}$$



See GitHub for full code of plot

$P_3^3 = P_2^4$ and the error is reduced compared to Maclaurin expansion

in fact this is a log scale plot which makes Maclaurin error look better..



as seen in this plot, Maclaurin error blows up near end of interval

2) $\int_0^1 f(x)\,dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$

insert $f(x) = 1$:

$$\int_0^1 1\,dx = 1 = \frac{1}{2} + c_1$$

so $\boxed{c_1 = \frac{1}{2}}$

insert $f(x) = x$:

$$\int_0^1 x\,dx = \frac{1}{2} = \frac{1}{2} x_0 + c_1 x_1 = \frac{1}{2}[x_0 + x_1]$$

$$x_0 + x_1 = 1 \quad \Rightarrow x_1 = 1 - x_0 \quad (1)$$

insert $f(x) = x^2$:

$$\int_0^1 x^2\,dx = \frac{1}{3} = \frac{1}{2} x_0^2 + \frac{1}{2} x_1^2 \quad (2)$$

sub in (1) into (2)

$$\frac{2}{3} = x_0^2 + (1 - x_0)^2$$

$$\frac{2}{3} = x_0^2 + 1 - 2x_0 + x_0^2$$

$$2x_0^2 - 2x_0 - \frac{1}{3} = 0$$

$$x_0 = \frac{2 \pm \sqrt{4 - 4(2(-\frac{1}{3}))}}{2(2)}$$

$$x_0 = \frac{2 \pm \sqrt{\frac{12}{3} + 8/3}}{4}$$

$$\boxed{x_0 = \frac{2 \pm \sqrt{\frac{20}{3}}}{4}}$$

$$x_1 = \frac{2 \pm \sqrt{\frac{20}{3}}}{4} - 1$$

$$\boxed{x_1 = \frac{-2 \pm \sqrt{\frac{20}{3}}}{4}}$$