APPM4600 Homework #2

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1a) Taylor expand $(1+x)^n$ around $x=0$:

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n-1)x^2 + \ldots$$

as $x \to 0$, terms after nx are negligible thus:

$$(1+x)^n = 1 + nx + o(x^2)$$

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b) Taylor expand $\sin x$ around $x=0$:

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots$$

$$\text{so } \sin\sqrt{x} \approx \sqrt{x} + \frac{x^{3/2}}{6} + \ldots$$

$$x\sin\sqrt{x} \approx x\sqrt{x} + \frac{x^{5/2}}{6} + \ldots$$

$$x\sin\sqrt{x} \approx x^{3/2} + \frac{x^{5/2}}{6} + \ldots$$

so as $x \to 0$, $x^{3/2}$ will be dominant thus:

$$x \sin\sqrt{x} = O(x^{3/2})$$

c) Take limit of $e^{-t}/(\frac{1}{t^2})$ as $t \to \infty$

$$\Rightarrow \lim_{t\to\infty} t^2 e^{-t} = \lim_{t\to\infty} \frac{t^2}{e^t}$$

apply L-Hopital twice:

$$\lim_{t\to\infty} \frac{t^2}{e^t} = \lim_{t\to\infty} \frac{2}{e^t} = \lim_{t\to\infty} 2e^{-t}$$

$$\underline{\underline{= 0}}$$

Since $\lim_{t\to\infty} \frac{g(t)}{f(t)} = 0$ for $g(t) = e^{-t}$, $f(t) = 1/t^2$

$\frac{1}{t^2}$ falls off faster than e^{-t}

$$\underline{\underline{e^{-t} = o\left(\frac{1}{t^2}\right)}}$$

d) Taylor expand e^{-x^2} around $x=0$:

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

$$e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \ldots$$

as $x \to 0$, $e^{-x^2} \approx 1 + O(x^2)$

$$\int_0^\varepsilon e^{-x^2} dx = \int_0^\varepsilon (1 + O(x^2))\, dx$$

$$= \left[x + O(x^2)x\right]_0^\varepsilon$$

$$= \varepsilon(1 + O(x^2))$$

as $\varepsilon \to 0$

$$\underline{\underline{= O(\varepsilon)}}$$

2) $\underline{A} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{A}^{-1} = \begin{pmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{pmatrix}$

a) $\underline{A}\,\underline{x} = \underline{b}$ new $\underline{b}$, $\underline{b}' = \underline{b} + \Delta\underline{b}$

new sol $\underline{x}' = \underline{x} + \Delta\underline{x}$

$\underline{A}\,\underline{x}' = \underline{b}'$

$\underbrace{\underline{A}\,\underline{x}}_{= \underline{b}} + \underline{A}\,\Delta x = \underline{b} + \Delta\underline{b}$

$\Rightarrow \underline{A}\,\Delta\underline{x} = \Delta\underline{b} = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}$

$$\Delta\underline{x} = \underline{A}^{-1}\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}$$

b) $K(A) = \frac{|\Delta \underline{x}|}{|\underline{x}|}\left(\frac{|b|}{|\Delta b|}\right) = \frac{|A^{-1}\cancel{\Delta}\underline{b}|}{|x|}\frac{|b|}{\cancel{|\Delta b|}}$

$$k(A) = \frac{|\underline{A}^{-1}||\underline{b}|}{|x|} = \frac{|\underline{A}^{-1}||\underline{A}\cancel{\underline{x}}|}{\cancel{|x|}}$$

$$k(A) = |\underline{A}^{-1}||\underline{A}|$$

using python to eval (see Github repo)

$$k(A) = 1.999999425 \times 10^{9}$$

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b) $err_{rel} = \frac{|\Delta \underline{x}|}{|x|} = k(A)\frac{|\Delta \underline{b}|}{|\underline{b}|}$

using $\Delta \underline{b} \approx \begin{pmatrix} 10^{-5} \\ 10^{-5} \end{pmatrix}$

$$err_{rel} = k(A)\left(\frac{\sqrt{2(10^{-5})^2}}{\sqrt{2}}\right) = k(A) \times 10^{-10}$$

$$err_{rel} = 0.1999999425$$

3) $f(x) = e^x - 1$

a) $\kappa(f(x)) = \dfrac{f'(x)x}{f(x)}$

$$\kappa(f(x)) = \frac{xe^x}{e^x - 1}$$

this blows up as $x \to 0$
as denominator $\to 0$
so this is ill conditioned at $x = 0$.

b) $y = $ math.e^x
return $y - 1$

algorithm is conditionally stable since at small x values, output varies a lot for small changes in input.
But for other values of x, algorithm is perfectly stable.

c) Using python, algorithm provides

$$f(x) = \underbrace{9.9999}_{\text{5 correct digits}}8606957 66 \times 10^{-10}$$

so we lost 11 digits of precision due to input close to 0.

d) Taylor expand e^x around $x = 0$:

$$e^x \approx 1 + x + \frac{x^2}{2} + \dots$$

so $e^x - 1 \approx x + \frac{x^2}{2} + \dots$

terms after x^2 we can ignore:

$$e^x - 1 = x + \frac{x^2}{2} = f(x)$$

e) See code on Github repo under folder

Homework/Homework 2

4)

Pushed code to Github repo
under folder Homework/Homework 2