APPM 4600 Homework #4

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1)

a) $$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$T(x,t) = T_s + (T_i - T_s)\,\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

we want $T(x, 60 \text{ days}) = 0$

$$0 = T_s + (T_i - T_s)\,\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$T_i = 20,\ T_s = -15$

$$0 = -15 + 35\,\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$\boxed{\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \frac{15}{35} = 0 = f(x)}$$

solve $f(x) = 0$ for x using

$\alpha = 0.138 \times 10^{-6}$, $t = 60$ days (convert to s)

to compute $f'(x)$ consider def of erf

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-s^2)\, ds$$

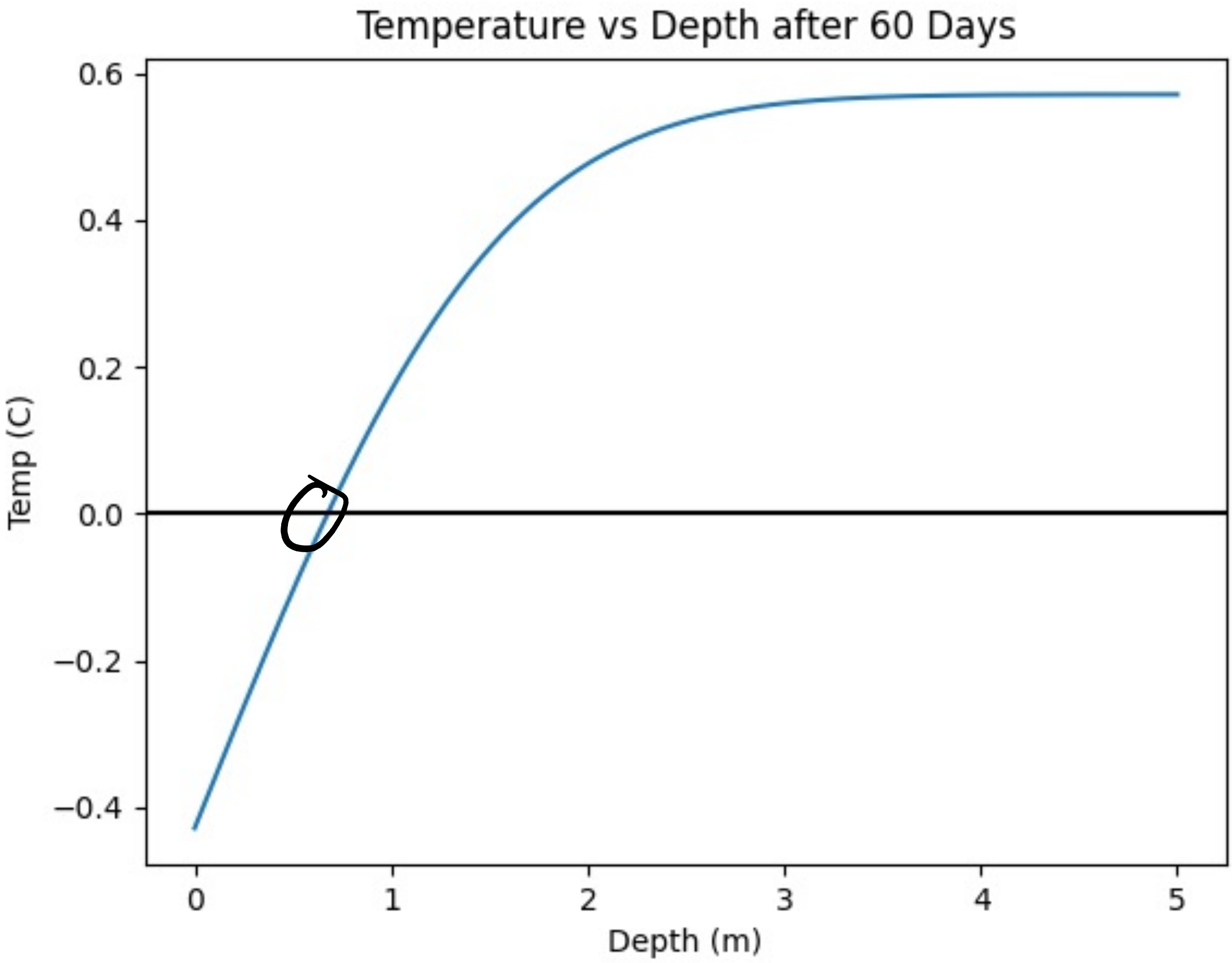
taking derivative leaves only $\exp(-s^2)$

$$f'(x) = \frac{2}{\sqrt{\pi}} \exp\left[-\left(\frac{x}{2\sqrt{\alpha t}}\right)^2\right]\left(\frac{1}{2\sqrt{\alpha t}}\right)$$

$$f'(x) = \frac{1}{\sqrt{\alpha t \pi}} \exp\left[-\left(\frac{x}{2\sqrt{\alpha t}}\right)^2\right]$$

Choose $\bar{x} = 5$ meters

see code on Github for plot:



b) See code on Github:

approx depth = 0.6769618544819167 m.

c) See code on Github:

for $x_0 = 0.01$,

approx depth = 0.6769618544819366

for $x_0 = 5$, we get nan as initial guess is too far from root so we end up dividing by zero somewhere.

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2) $f(x)$ has root α, multiplicity m.

a) $f(\alpha) = 0$

$f'(\alpha) = 0$

$f''(\alpha) = 0$

...

$f^{m-1}(\alpha) = 0$

all derivatives at α disappear except m^{th} derivative.

$$f^{i}(\alpha) = \begin{cases} \neq 0 & i = m \\ 0 & i \neq m \end{cases}$$

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b) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

taylor expand $f(x)$ around α:

$$f(x) \approx f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2}(x-\alpha)^2 + \dots$$

but all derivatives are 0 except m^{th} and $f(\alpha)=0$

so $$f(x) \approx \frac{f^m(\alpha)}{m!}(x-\alpha)^m$$

$$\Rightarrow f'(x) \approx \frac{m f^m(\alpha)}{m!}(x-\alpha)^{m-1}$$

$$\Rightarrow x_{n+1} = x_n - \frac{\cancel{f^m(\alpha)}(x_n-\alpha)^m \cancel{m!}}{\cancel{m!}\, m \cancel{f^m(\alpha)}(x_n-\alpha)^{m-1}}$$

$$x_{n+1} = x_n - \frac{(x_n-\alpha)}{m}$$

$$x_{n+1} - \alpha = x_n - \alpha - \frac{(x_n-\alpha)}{m}$$

$$e_{n+1} = e_n - \frac{e_n}{m}$$

$$e_{n+1} = e_n\left(1 - \frac{1}{m}\right)$$

so error reduces linearly with m

c) $g(x) = x - m\dfrac{f(x)}{f'(x)}$

$$\Rightarrow x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

using equation from part b,

$$x_{n+1} = x_n - m\left(\frac{(x_n - \alpha)}{m}\right)$$

$$\underline{\underline{x_{n+1} = \alpha}}$$

so for root with m multiplicity, this modified method will find root in 1 iteration.

3) order of convergence of seq $\{x_k\}_{k=1}^{\infty}$ that converges to α:

$$|x_{k+1} - \alpha| = C|x_k - \alpha|^p$$

$$\log(|x_{k+1} - \alpha|) = \log(C|x_k - \alpha|^p)$$

$$\log(|x_{k+1} - \alpha|) = p\log|x_k - \alpha| + \log C$$

linear relationship where p is the slope

4) $f(x) = e^{3x} - 27x^6 + 27x^4e^x - 9x^2e^{2x}$

in interval $[3,5]$.

using code, applied each method then passed array of values at each iter into function calculating order of convg. (See Github)

c) Newton's method:

Order of Convergence $= 0.926 \approx 1$

linear as expected

ii) Modified Newton

had trouble with code for this function,
unable to complete Q4.

I prefer Newton method because I got it
to work...

5) $f(x) = x^6 - x - 1$

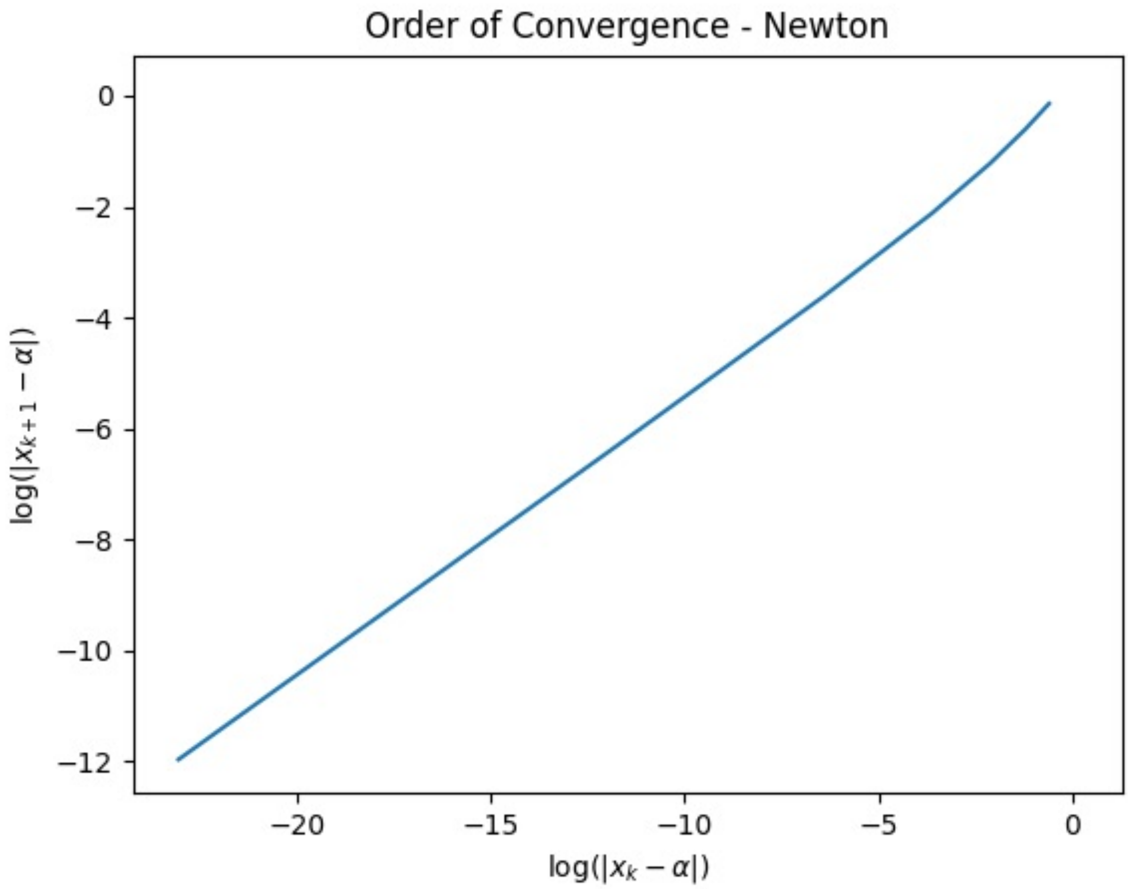
a) See code on Github for tables

error decreases to 0 in both cases

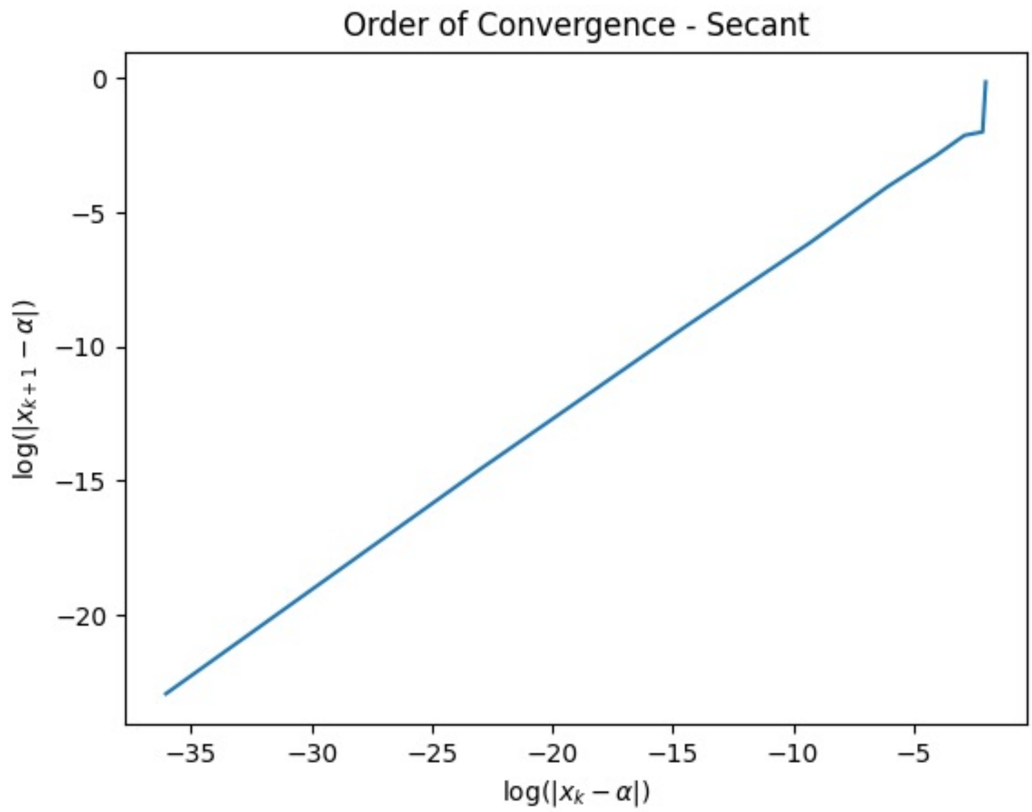
$\text{err Newt} = [0.86, 0.54, \ldots, 9.8 \times 10^{-11}]$

$\text{err sec} = [0.86, 0.13, \ldots, 2.2 \times 10^{-16}]$

but the secant method errors decrease faster and reach a lower absolute value compared to the Newton method b)



slope = order = 1.916



slope = order = 1.569