APPM4600 Homework #9

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1) $$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

apply A^T to both sides:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$u = 1, \quad v = \frac{1}{2}$

2) minimize $E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$

given:

$$\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

multiply by weight matrix W:

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$ to give weights:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - W \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = W\underline{b}$$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix} + \begin{bmatrix} b_1 \\ 2b_2 \\ 5b_3 \\ 3b_4 \end{bmatrix}$$

now we can apply LS by applying A^T:

$$\begin{bmatrix} 1 & 12 & 20 & 6 \\ 3 & -2 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 & 12 & 20 & 6 \\ 3 & -2 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 581 & 105 \\ 105 & 454 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 421 \\ 247 \end{bmatrix}$$

$$581x_1 + 105x_2 = 421$$

$$105x_1 + 454x_2 = 247$$

$x_1 = 0.654$ solved using python

$x_2 = 0.393$

3)

a) $p(x) = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n = 0$

$p'(x) = C_1 + 2C_2 x + \dots + n C_n x^{n-1} = 0$

$p^n(x) = n! C_n = 0$

$n! \neq 0$ so $\underline{\underline{C_n = 0}}$

if $C_n = 0$ then

$p(x) = C_0 + C_1 x + \dots + C_{n-1} x^{n-1} = 0$

$p^n(x) = (n-1)! C_{n-1} = 0$

$(n-1)! \neq 0$ so $C_{n-1} = 0$

applying this to all coefficients means that $c_k = 0$ for $k \in 0, \dots, n$

so $\{1, x, x^2, \dots, x^n\}$ is linearly independent

since if $\sum_k c_k f_k(x) = 0$ then if linearly ind.

$c_k = 0$ for $k \in 0, \dots, n$ which we showed.

b) $p(x) = c_0 + c_1 \cos x + c_2 \cos 2x + \dots, c_n \cos nx$
$+ d_1 \sin x + d_2 \sin 2x + \dots, + d_n \sin nx$
$= 0$

if we integrate in $[0, 2\pi]$ integral, all cos, sin terms go to 0:

$$\int_0^{2\pi} c_0 \, dx = 0$$

$$2\pi c_0 = 0$$

$$c_0 = 0$$

now integrate $\cos(mx)\,p(x)$ over $[0, 2\pi]$:

$$\int_0^{2\pi} \left[c_1 \cos x \cos mx + c_2 \cos 2x \cos mx + \dots + c_n \cos mx + \dots + d_n \sin nx \cos mx \right] dx = 0$$

for $m = 1, 2, \dots, n$

all terms $= 0$ due to sin/cos orthogonality except when indices match for cosine:

$$\int_0^{2\pi} c_m \cos^2(mx)\, dx = 0$$

$$c_m \underbrace{\int_0^{2\pi} \cos^2(mx)\, dx}_{} = 0$$

↳ this in general $\neq 0$ so

$c_m = 0$ for all m to satisfy equation

if you apply same process by integrating over $p(x)\sin(mx)$, you get result for d coeffs:

$d_m = 0$ for all m.

since $c_0 = c_m = d_m$ for $m = 1, \ldots, n$,

$\{1, \cos x, \ldots \cos nx, \sin x, \ldots, \sin nx\}$ is

<u>linearly independent</u>

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4) Assume

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k \phi_{k-2}(x)$$

$$- \{a_{k-3}\phi_{k-3}(x) + a_{k-4}\phi_{k-4} + \ldots + a_0\phi_0\}$$

take inner product of ϕ_k, ϕ_j:

$$\langle \phi_k, \phi_j \rangle = \langle (x - b_k)\phi_{k-1} - c_k\phi_{k-2}$$

$$- \{a_{k-3}\phi_{k-3} + a_{k-4}\phi_{k-4} + \ldots + a_0\phi_0\}$$

$$, \phi_j \rangle$$

$$\begin{array}{c} \langle \phi_k, \phi_j \rangle \\ \| \\ 0 \end{array} = \left.\begin{array}{l} \langle (x-b_k)\phi_{k-1}, \phi_j \rangle \\ \\ - c_k \langle \phi_{k-1}, \phi_j \rangle \end{array}\right\} = 0$$

$$- \sum_{i=0}^{k-3} a_i \langle \phi_i, \phi_j \rangle$$

first two terms are $= 0$ since polynomials are orthogonal by definition

only non-zero term in sum is when $i=j$:

$$0 = a_j \langle \phi_j, \phi_j \rangle$$

Since $\langle \phi_j, \phi_j \rangle \neq 0$ in general, $\underline{\underline{a_j = 0}}$

so we have original assumption w/o a terms:

$$\phi_k(x) = (x-b_k)\phi_{k-1} - c_k \phi_{k-2}$$

take inner product with ϕ_{k-1}:

$$\underset{\substack{\| \\ 0}}{\langle \phi_k, \phi_{k-1} \rangle} = \langle (x-b_k)\phi_{k-1} - c_k \phi_{k-2}, \phi_{k-1} \rangle$$

$$0 = \langle (x-b_k)\phi_{k-1}, \phi_{k-1} \rangle - c_k \overbrace{\langle \phi_{k-2}, \phi_{k-1} \rangle}^{0}$$

$$0 = \langle x\phi_{k-1}, \phi_{k-1} \rangle - b_k \langle \phi_{k-1}, \phi_{k-1} \rangle$$

$$b_k = \frac{\langle x\phi_{k-1}, \phi_{k-1} \rangle}{\langle \phi_{k-1}, \phi_{k-1} \rangle} \quad \text{as required}$$

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take inner product of $\langle \phi_k, \phi_{k-2} \rangle$:

$$0 = \langle (x-b_k)\phi_{k-1}, \phi_{k-2} \rangle - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle$$

$$0 = \langle x\phi_{k-1}, \phi_{k-2} \rangle - b_k \cancel{\langle \phi_{k-1}, \phi_{k-2} \rangle} \rightarrow 0 - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle$$

$$0 = \langle x\phi_{k-1}, \phi_{k-2} \rangle - c_k \langle \phi_{k-2}, \phi_{k-2} \rangle$$

$$c_k = \frac{\langle x\phi_{k-1}, \phi_{k-2} \rangle}{\langle \phi_{k-2}, \phi_{k-2} \rangle}$$ as required

5) Standard def:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$$

where $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$

using above, try $n=0$, $n=1$:

$$T_0(x) = \frac{1}{2}\left(z^0 + \frac{1}{z^0}\right) = 1 \quad ✓✓$$

$$T_1(x) = \frac{1}{2}\left(z + \frac{1}{z}\right) = x \quad ✓✓$$

try $n = n+1$ and $n = n-1$:

$$T_{n+1}(x) = \frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right)$$

$$T_{n-1}(x) = \frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right)$$

and $2x\,T_n(x) = 2\left[\underbrace{\frac{1}{2}\left(z+\frac{1}{z}\right)}_{x}\underbrace{\left(\frac{1}{2}\left(z^n+\frac{1}{z^n}\right)\right)}_{T_n(x)}\right]$

$$2x\,T_n(x) = \frac{1}{2}\left(z+\frac{1}{z}\right)\left(z^n + \frac{1}{z^n}\right)$$

$$2x\,T_n(x) = \frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right) + \frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right)$$

using standard def:

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

Substituting expressions in z:

$$\cancel{\frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right)} = \cancel{\frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right)} + \cancel{\frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right)} - \cancel{\frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right)}$$

$0 = 0$ ✓✓

so both formulations are identical