APPM 4600 — HOMEWORK # 5

1. Suppose we want to find a solution located near $(x, y) = (1, 1)$ to the nonlinear set of equations

$$\begin{aligned} f(x, y) &= 3x^2 - y^2 = 0, \\ g(x, y) &= 3xy^2 - x^3 - 1 = 0 \end{aligned} \tag{0.1}$$

   (a) Iterate on this system numerically (in Python), using the iteration scheme

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \quad n = 0, 1, 2, \ldots$$

   starting with $x_0 = y_0 = 1$, and check how well it converges.

   (b) Provide some motivation for the particular choice of the numerical 2×2 matrix in the equation above.

   (c) Iterate on (0.1) using Newton's method, using the same starting approximation $x_0 = y_0 = 1$, and check how well this converges.

   (d) Spot from your numerical result what the exact solution is, and then verify that analytically.

2. Consider the nonlinear system of equations

$$\begin{cases} x = \dfrac{1}{\sqrt{2}}\sqrt{1 + (x + y)^2} - \dfrac{2}{3} \\ y = \dfrac{1}{\sqrt{2}}\sqrt{1 + (x - y)^2} - \dfrac{2}{3} \end{cases}$$

Theorem 10.6 in the textbook (on page 633 in the 9th edition reads as follows:

Let $D = \{(x_1, x_2, \ldots, x_n)^t : a_i \leq x_i \leq b_i,\}$ some collection of constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. Supposed that $\mathbf{G}$ is a continuous function from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with the property that $\mathbf{G}(\mathbf{x}) \in D$ whenever $\mathbf{x} \in D$. Then $\mathbf{G}$ has a fixed point in D

Moreover, supposed that all the component functions of $\mathbf{G}$ have continuous partial derivative and a constant $K \leq 1$ exists with

$$\left| \frac{\partial g_i(\mathbf{x})}{\partial x_j} \right| \leq \frac{K}{n}$$

whenever $\mathbf{x} \in D$, for each $j = 1, \ldots, n$ and each component function g_i. Then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrary selected $\mathbf{x}^{(0)}$ in D and generated by

$$\mathbf{x}^{(k)} = \mathbf{G}(\mathbf{x}^{(k-1)}), \quad \text{for each } k \geq 1$$

converges to the unique fixed point $\mathbf{p} \in D$ and

$$\|\mathbf{x}^{(k)} - \mathbf{p}\|_\infty \leq \frac{K^k}{1 - K} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty.$$

Based on this theorem, find a region D in the $x, y-$plane for which the fixed point iteration

$$\begin{cases} x_{n+1} = \dfrac{1}{\sqrt{2}}\sqrt{1+(x_n+y_n)^2} - \dfrac{2}{3} \\ y_{n+1} = \dfrac{1}{\sqrt{2}}\sqrt{1+(x_n-y_n)^2} - \dfrac{2}{3} \end{cases}$$

is guaranteed to converge to a unique solution for any starting point $(x_0, y_0) \in D$.

3. Let $f(x, y)$ be a smooth function such that $f(x, y) = 0$ defines a smooth curve in the $x, y-$plane. We want to find some point on this curve that lies in the neighborhood of a start guess (x_0, y_0) that is off the curve. i.e. your goal is to move from an initial guess to the curve $f(x, y) = 0$.

   (a) Derive the iteration scheme

   $$\begin{cases} x_{n+1} = x_n - d f_x \\ y_{n+1} = y_n - d f_y \end{cases}$$

   for solving the task outlined above. Here $d = f/(f_x^2 + f_y^2)$, and it is understood that f, f_x, f_y are all evaluated at the location (x_n, y_n).

   Hint: One way to proceed is to look for a new iterate (x_{n+1}, y_{n+1}) that (i) lies on the gradient (i.e. normal) line through (x_n, y_n) and (ii) also obeys $f(x, y) = 0$. Apply Newton to this 2×2 system.

   (b) The iteration scheme above generalizes in an obvious way to moving from a start location (x_0, y_0, z_0) onto a surface $f(x, y, z) = 0$. With this iteration, find a point the ellipsoid $x^2 + 4y^2 + 4z^2 = 16$ when starting from $x_0 = y_0 = z_0 = 1$. Give numerical evidence showing that the iteration indeed is quadratically convergent.