APPM4600 Homework #11

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1a) See code on GitHub

b) Error formula for Composite trap:

$$|E_T| = \frac{-(b-a)^3}{12n_T^2} \max |f''|_{[a,b]}$$

$$f(s) = \frac{1}{1+s^2}$$

$$f'(s) = \frac{-2s}{(1+s^2)^2}$$

$$f''(s) = \frac{-2}{(1+s^2)^2} + \frac{8s^2}{(1+s^2)^3}$$

$$= \frac{-2(1+s^2) + 8s^2}{(1+s^2)^3}$$

$$f''(s) = \frac{6s^2 - 2}{(1+s^2)^3}$$

$$\max |f''(s)|_{[-5,5]} = 2 \text{ when } s = 0$$

$$\Rightarrow |E_T| = \frac{(5+5)^3}{12 n_T^2}(2) < 10^{-4}$$

$$n_T^2 > \frac{2(10)^3}{12(10^{-4})}$$

$$n_T > 1290.9$$

$$\underline{\underline{n_T = 1291}}$$

for composite Simpson's:

$$|E_s| = \frac{-(b-a)^5}{180 n^4} \max |f^{(4)}|_{[a,b]}$$

$$f''(s) = \frac{6s^2-2}{(1+s^2)^3}$$

$$f'''(s) = \frac{8s}{(1+s^2)^3} - \frac{6s(s^2-2)}{(1+s^2)^4}$$

$$f'''(s) = \frac{8s(1+s^2) - 6s(s^2-2)}{(1+s^2)^4}$$

$$f'''(s) = \frac{2s^3-4s}{(1+s^2)^4}$$

$$f^{(4)}(s) = \frac{6s^2-4}{(1+s^2)^4} - \frac{8s(2s^3-4s)}{(1+s^2)^5}$$

$$f^{(4)}(s) = \frac{(6s^2-4)(1+s^2) - 16s^4 - 32s^2}{(1+s^2)^5}$$

$$= \frac{6s^2 - 6s^4 - 4 - 4s^2 - 16s^4 - 32s^2}{(1+s^2)^5}$$

$$f^{(4)}(s) = \frac{-20s^4 - 30s^2 - 4}{(1+s^2)^5}$$

max = 4 at $s = 0$:

$$|E_s| = \frac{10^5}{180 n_s^4}(4) < 10^{-4}$$

$$n_s^4 > \frac{4(10)^5}{180(10^{-4})}$$

$$n_s > 68.65$$

$\underline{\underline{n_s = 70}}$ since n must be even.

Using n_T and n_s in code, I get the following results:

$I_{trap} = 2.744673731677375$

$$I_{simps} = 2.7130049798793787$$

$$I_{quad} = 2.7468015338900327$$

$$I_{quad}\ (tol = 10^{-4}) = 2.746801533909586$$

2) $\displaystyle\int_{1}^{\infty} \frac{\cos(x)}{x^{3}}\,dx$

$t = \frac{1}{x} \Rightarrow t^{3} = \frac{1}{x^{3}}$

$x = \frac{1}{t}$

$dx = -\frac{1}{t^{2}}\,dt$

when $x = 1$, $t = 1$

when $x = \infty$, $t \to 0$

$\displaystyle\Rightarrow \int_{1}^{0} t^{3}\cos\left(\frac{1}{t}\right)\left(-\frac{1}{t^{2}}\,dt\right)$

$\displaystyle= \int_{0}^{1} t\cos\left(\frac{1}{t}\right)dt$

$t=0$ will be a problem so change lower limit to

1×10^{-8}:

$$= \int_{1\times10^{-8}}^{1} t\cos\left(\frac{1}{t}\right)dt$$

evaluating using code from Q1:

$$= \underline{\underline{0.011748}}$$

$\Downarrow$

3)

Assume $I - I_n = \frac{c_1}{n\sqrt{n}} + \frac{c_2}{n^2} + \frac{c_3}{n^2\sqrt{n}} + \frac{c_4}{n^3} + \cdots$

$$I_n = I - \frac{c_1}{n\sqrt{n}} - \frac{c_2}{n^2} - \frac{c_3}{n^2\sqrt{n}} - \cdots$$

$$I_{n/2} = I - \frac{c_1}{\frac{n}{2}\sqrt{\frac{n}{2}}} - \frac{c_2}{\left(\frac{n}{2}\right)^2} - \frac{c_3}{\left(\frac{n}{2}\right)^2\sqrt{\frac{n}{2}}} - \cdots$$

$$I_{n/4} = I - \frac{c_1}{\frac{n}{4}\sqrt{\frac{n}{4}}} - \frac{c_2}{\left(\frac{n}{4}\right)^2} - \frac{c_3}{\left(\frac{n}{4}\right)^2\sqrt{\frac{n}{4}}}$$

$$I' = aI_n + bI_{n/2} + cI_{n/4}$$

$$= a\left(I - \frac{c_1}{n\sqrt{n}} - \frac{c_2}{n^2} - \cdots\right) + b\left(I - \frac{c_1}{\frac{n}{2}\sqrt{\frac{n}{2}}} - \cdots\right)$$

$$+ c\left(I - \frac{c_1}{\frac{n}{4}\sqrt{\frac{n}{4}}} - \cdots\right)$$

$$= I(a+b+c) - c_1\left(\frac{a}{n\sqrt{n}} + \frac{b}{\frac{n}{2}\sqrt{\frac{n}{2}}} + \frac{c}{\frac{n}{4}\sqrt{\frac{n}{4}}}\right)$$

$$- c_2\left(\frac{a}{n^2} + \frac{b}{\left(\frac{n}{2}\right)^2} + \frac{c}{\left(\frac{n}{4}\right)^2}\right) + \cdots$$

we want $c_1 = c_2 = 0$

$$\frac{a}{n\sqrt{n}} + \frac{b}{\frac{n}{2}\sqrt{\frac{n}{2}}} + \frac{c}{\frac{n}{4}\sqrt{\frac{n}{4}}} = 0$$

$$a + 2\sqrt{2}b + 8c = 0 \quad ①$$

$$\frac{a}{n^2} + \frac{b}{(\frac{n}{2})^2} + \frac{c}{(\frac{n}{4})^2} = 0$$

$$a + 4b + 16c = 0 \quad ②$$

$$a + b + c = 1 \quad ③$$

$$\begin{bmatrix} 1 & 2\sqrt{2} & 8 \\ 1 & 4 & 16 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving with python:

$a = 2.06255755$, $b = -1.2448636$, $c = 0.18230605$

$$\underline{\underline{I' = 2.06 I_n - 1.24 I_{n/2} + 0.182 I_{n/4}}}$$