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May 31, 2023

Contents

1 Determinants

1.1 ROW OPERATIONS AND DETERMINANTS

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B (replacement operation) then $\det(B) =$

$\det(A)$.

2. If two rows of A are interchanged to produce B then $\det(B) = -\det(A)$
3. If a row of A is multiplied by a scalar to produce B, then $\det(B) = k\det(A)$

Suppose a matrix A has been reduced to Echelon form U by row replacements and row interchanges. If there are r total interchanges performed

$$\det(A) = (-1^r)\det(U) \tag{1}$$

Since U is in echelon form (triangular) $\det(U)$ is the product of the diagonal entries.

- If A is invertible the entries on the diagonal are all pivots $(A \sim I_n)$
- If A is non-invertible, at least one of these entries $u_1, u_2, \ldots, u_{nn}$ must be zero, so the product

$$u_{11}, u_{22}, \ldots, u_{nn} = 0 \tag{2}$$

So, for a matrix A to be invertible,

$$\det(A) \neq 0 \tag{3}$$

1.2 DETERMINANT OF TRANSPOSE

if A $m \times n$

$$\det(A^T) = \det(A) \tag{4}$$

1.3 DETERMINANT OF PRODUCTS

If A and B are $n \times n$ matrices:

$$\det(AB) = \det(A)\det(B) \tag{5}$$

1.4 CRAMMER'S RULE

Useful for solving systems of equations.
For any $n \times n$ matrix A and any $b \in \mathbb{R}$, let $A_i(b)$ be the matrix obtained from A by replacing column i by vector b

Let A be an invertible $n \times n$ matrix, for any $b \in \mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det(A_i b)}{\det(A)} \tag{6}$$

Solving $Ax = b$

1. Row reduction of augmented form
2. $A^{-1}b = x$
3. Crammer's Rule

1.5 AREAS AND VOLUMES BY DETERMINANT

1. If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det(A)|$
2. if A is a 3×3 matrix, the volume f the parallelepiped determined by the columns of a is $|\det(A)|$

1.6 THEOREM 10

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation determined by a 2×2 matrix A, if S is a parallelogram in $\mathbb{R}^2$, then

$$A_{T(S)} = |\det(A)| \times A_S \tag{7}$$

Theorem 10 holds whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

2 Day 9 -

2.1 OBJECTIVES

1. Know hat conditions for a set to be a subspace of a larger vector space
2. Find the column space and null space of a matrix
3. Show that a transformation between vector spaces in linear
4. Find the kernel/ Null space and range of a linear transformation
5. know how to find the basis for a vector space especially for the null space and column space

2.2 VECTOR SPACE

A vector space is a non-empty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by sacalars subject to the 10 axioms listed below. The axioms must be held for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and for all scalars c and d

1. The sum of $\vec{u}$ and $\vec{v}$, directed by $\vec{u} + \vec{v}$, is in V
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $(\vec{u} + \vec{v}) + w = \vec{u} + (\vec{v} + w)$
4. There is a zero vector 0 in V such that $\vec{u} + 0 = \vec{u}$
5. For each $\vec{u}$ in V, there is a vector $-\vec{u} \in V$ such that $\vec{u} + (-\vec{u}) = 0$
6. The scalar multiple of $\vec{u}$ by c , denoted by $c\vec{u}$, is in V
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$

Can show:

1. $0\vec{u} = 0$
2. $c0 = 0$
3. $-\vec{u} = (-1)\vec{u}$

P_n for $n \geq 0$, polynomials with degree of at most n

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$
$$q(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$$

Addition

$$(p + q)(t) = p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n$$

2.3 SUBSPACES

Subspace - A subspace of a vector space V is a subset H of V that has these properties

1. the zero vector of V is in H
2. H is closed under vector addition

   For any u and $v \in H$, $u + v \in H$
3. H is closed under multiplication by scalars

   For any u in H, and scalar c, $cu \in H$

$$\left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : \text{s and t are real} \right\} \tag{8}$$

Vectors in H can be written in the form $u = s_1 v_1 + s_2 v_2$ for some scalars s_1, s_2.

Process for determining if H is a subspace

1. consider $s_1 = s_2 = 0$

$$0v_1 + 0v_2 = 0 \in H$$

2. consider $u, w \in H$

$$u = s_1 v_1 + s_2 v_2$$

$$w = t_1 v_1 + t_2 v_2$$

$$u + w = (s_1 + t_1)v_1 + (s_2 + t_2)v_2 \in H$$

3. consider

$$cu = c(s_1 v_1 + s_2 v_2) = (cs_1)v_1 + (cs_2)v_2$$

3 Space

3.1 NULL SPACE

Let

$$x_1 - 3x_2 - 2x_3 = 0$$

$$-5x_1 + 9x_2 + x_3 = 0$$

Solution set:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We define the set of x that satisfy $Ax = 0$ the null space of A.

Null Space definition: The null space of an $m \times n$ matrix A, denoted by Nul A is the set of all solutions that satisfy $Ax = 0$

3.1.1 Null space as a subspace theorem

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$

Proof:

$$\mathbf{A}0 = 0$$

$$\vec{u}, \vec{v} \in \mathrm{Nul}\mathbf{A} \to \mathbf{A}\vec{u} = 0, \mathbf{A}\vec{v} = 0 \to \vec{u} + \vec{v}\mathbf{A}(\vec{u} + \vec{v}) = \mathbf{A}\vec{u} + \mathbf{A}\vec{v} =$$

$$\vec{u} \in \mathrm{Nul}\mathbf{A} \to \mathbf{A}\vec{u}\mathbf{A}(c\vec{u}) = c\mathbf{A}\vec{u} = c0 = 0$$

3.2 COLUMN SPACE DEFINITION

The column space of an $m \times n$ matrix $\mathbf{A}$ (Col $\mathbf{A}$) is the set of all linear combinations of the columns of $\mathbf{A}$, $\mathbf{A} = [\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots \boldsymbol{a_n}]$

Col$\mathbf{A}$ =span $\{a_1, \ldots, a_n\}$

3.3

The column space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^n$

if Col$\mathbf{A}$ is a subspace of $\mathbb{R}^k$, what is k?

3.4 FINDING A VECTOR IN COL A

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \in \mathrm{Col}\mathbf{A}$$

Nul A

$$\begin{bmatrix} \mathbf{A} & 0 \end{bmatrix} \sim \begin{bmatrix} a_1 & a_2 & \cdots & a_n & 0 \end{bmatrix}$$

A linear transformation T from a vector space V into a vector space W is a rule that assigns each vector $x \in V$ to a unique vector $T(x)$ in W such that

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u,v} \in V$
2. $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u} \in V$ and scalars c
3. The kernel (or null space) is the set of all $\vec{u}$ in V such that $T(\vec{u}) = 0$
4. The range of T is the set of all vectors in W of the form $T(\vec{x}) = 0$

3.5 LINEAR INDEPENDENCE AND DEPENDENCE

The definition of Linear Independence in $\mathbb{R}^n$

Consider: $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_p\vec{v}_p = \mathbf{0}$

1. If this system has a non-trivial solution, $v_1, \ldots, v_p$, are **linearly dependent**
2. If this system has **only** the trivial solution, $v_1, \ldots, v_p$ are **linearly independent**.

3.6 LINEAR INDEPENDENCE (THM 4.4)

An indexed set (put in a certain order) of 2 or more vectors with $v_1 \neq 0$, is linearly dependent iff some $\vec{v}_j, (j > 1)$ is a linear combination of the preceding vectors $v_{11}, \ldots, v_{j-1}$

4 Basis Definition

Let H be a subspace of a vector space V_1.

An indexed set of vectors $B = \{b_1, \ldots, b_p\}$ in V is a basis for H if

- B is a linearly independent set
- H is the span of the vectors of B

4.1 SPAN IN THE CONTEXT OF REMOVING LINEAR COMBINATIONS

Let $S = \{v1, \ldots, vp\}$ be a set in V and let $H = \text{Span}\{v_1, \ldots, v_p\}$

1. If one of the vectors in S, denoted v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k will still span H

4.2 BASIS OF N VECTORS

4.2.1 Theorem 9

If a vector space V has a basis $B = \{b_1, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

4.2.2 Theorem 10

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Proof

let B_1 be a basis of n vectors and B_2 be another basis of V

1. Since B_1 is a basis and B_2 is linearly independent,
2. B_2 has no more than n vectors by theorem Theorem 9

4.3 DIMENSIONS

If V is spanned by a finite set, then V is said to be finite dimensional and **the dimension of V, written as**

$$\mathbf{dim}V \tag{9}$$

is the number of vectors in a basis for V.

if V is not spanned by a finite set, then V is infinite dimensional.

The standard basis of $\mathbb{R}^n$ contains n vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix} \cdots e_n = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \end{bmatrix} \tag{10}$$

The standard polynomial basis $\{1, t, t^2\}$ spans $\mathbb{P}_2$. $\text{dim}\mathbb{P}_2 = 3$.

In general $\text{dim}\mathbb{P}_n = n + 1$

Let $H = \text{Span}\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

then H is a plane, $\{v_1, v_2\}$ is a basis for H $\text{dim}H = 2$

The subspaces of $\mathbb{R}^3$ can be classified by dimension

1. 0-dimensional subspace: only the zero subspace $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
2. 1-dimensional subspace: Any subspace spanned by a non-zero vector: lines through origin
3. 2 dimensional subspaces: spanned by two linearly independent vectors: plane through origin
4. 3 dimensional subspaces: only $\mathbb{R}^3$ itself: any 3 vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$

4.4 THEOREM 11

Let H be a subspace of a finite-dimensional vector space V

Any LI set in H can be expanded, if necessary, to form a basis for H. Also, H is finite dimensional and

$$\text{dim}H \leq \text{dimV} \tag{11}$$

$$S = \{v_1, \ldots, v_p\} \in H \tag{12}$$

4.5 THEOREM 12 - THE BASIS THEOREM

1. Let V be a p-dimensional vector space ,$p \geq 1$
2. Any linearly independent set of exactly p elements in V is automatically a basis for V.
3. Any set of exactly p elements that spans V is automatically a basis of V

The pivot columns of a matrix A form the basis of Col A

$$A = [a_1, \ldots, a_n] \tag{13}$$

Then we know the dimension of Cl A as soon as we know the dimensions of the pivot columns

For the dimensions of Nul A

Assume A is $m \times n$ and $Ax = 0$ has k free variables

The method for producing a basis for Nul A will produce exactly k linearly independent vectors

dim Nul A is the number of free variables in the equation $Ax = 0$

dim Col A is the number of pivot columns of A

4.6 ROW SPACE

The row space is the set of all linear combinations of the **row vectors**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \tag{14}$$

$$r_1 = [1, 2, 3, 4], r_2 = [5, 6, 7, 8], etc \tag{15}$$

The rows of A correspond to the columns of A^T

if we know the linear dependence relation among the rows of A, we could use the spanning theorem to shrink the set to a basis

4.7 THEOREM 13

If 2 matrices A and B are row equivalent then their row spaces are the same

If B is in echelon form, the non-zero rows of B form a basis for the row space of A as well as that of B

4.8 RANK A

the rank of A is the dimension of the column space of A

rank A=dim Col A

Since Row A = Col A^T, dim Row A = rank A^T

4.9 THE RANK THEOREM

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal

This common dimension, the rank of A, also equals the number of pivot positions in A,and satisfies:

$$\text{Rank A} + \text{dim Nul A} = n(\text{number of columns}) \tag{16}$$

addition to imt Let A $n \times n$, A is invertible (all this applies to A^T btw

The columns of A fom a basis of $\mathbb{R}^n$, (it spans)

Col A = $\mathbb{R}^n$

dim Col A = n

Nul A = 0

dim Nul A = 0

5 4.4 Coordinate Systems and 4.6 Change of Basis

5.1 OBJECTIVE

1. Find the coordinate vector relative to a given basis
2. Understand how this coordinate mapping as an isomorphism that allows us to answer questions abut abstract vector spaces in Euclidian Space
3. Find the change of basis matrix and use it t convert from one basis to another

5.2 COORDINATE SYSTEMS

a basis B creates a coordinate system for vector space

can map abstract vector maps to $\mathbb{R}^n$

different coordinate systems in $\mathbb{R}^n$ offer different "views" of vector spaces

5.3 THEOREM 2: THE UNIQUE REPRESENTATION THEOREM

let $B = \{b_1, \dots, b_n\}$ be a basis for vector space V

Then for each $x \in V$ there exists a unique set of scalars so that we can write **x** as a linear combination of these vectors

Definition: Basis Coordinates

Suppose $B = \{b_1, \dots, b_n\}$ is a basis fr V and $x \in V$,

The coordinates of x relative t B (or the B coordinates of x) are the weights $c_1, \dots, c_n$ such that $x = c_1 b_1 + \cdots + c_n b_n$

$$[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix} \tag{17}$$

The coordinate vector of x relative to B, or the B-coordinate vector of x

standard basis: $\varepsilon = \{e_1, e_2\}$

The coordinate mapping determined by $B = \{b_1, b_2\}$ for $\mathbb{R}^n$, $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Suppose $[x]_b = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, find x

$$x = (-2)b_1 + 3b_2 \tag{18}$$

The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ changes the b coordinates of a vector (c_1, c_2) into the standard coordinates for x.

in general, for a basis $B = \{b_1, \dots, b_n\}$, Let $P_B = [b_1, b_2, \dots, b_n]$

Then the vector equation $x = c_1 b_1 + \cdots + c_n b_n$ is equivalent to $P_B[x]_b = x$

P_B is called the change of coordinate matrix from B to the standard basis in $\mathbb{R}^n$. Furthermore, the columns of P_B form a basis for $\mathbb{R}^n$, so P_B is invertible by IMT.

$$P_B^{-1} x = [x]_B \tag{19}$$

By creating a basis in a vector space V, can create a coordinate system to relate to $\mathbb{R}^n$

5.4 THEOREM 8

Let $B = \{b_1, \ldots, b_n\}$ be a basis for a vector space V

Then the coordinates mapping $x \to [x]_B$ is a one to one linear transformation from V onto $\mathbb{R}^n$. This transformation is both one-to-one and onto.

Define: Isomorphism

Isomorphisms are transformations that are both one to one and onto.

For a set of polynomials, for example

$$1 + 2t^2, 4 + t + 5t^2, 3 + 2t \tag{20}$$

We can set an augmented matrix and solve Ax=0 to show if the set is linearly dependent or independent, with the row a_1 being the coefficients of the non t value and so on.

It can be worthwhile to look at vectors in different coordinate systems

Consider vector **x** and bases B and C. Relate $[x]_B$ to $[x]_C$.

$B = \{b_1, b_2\}, C = \{c_1, c_2\}$ for a vector space V such that $b_1 = 4c_1 + c_2 and b_2 = -6c_1 + c_2$. Suppose $x = 3b_1 + b_2$.

$$[x]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{21}$$

$$[x]_c = [3b_1 + b_2]_c = 3[b_1]_c + [b_2]_c. \tag{22}$$

$$[x]_c = \begin{bmatrix} [b_1]_c [b_2]_c \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \tag{24}$$

$$x = 6c_1 + 4c_2 \tag{25}$$

5.5 THEOREM 15 - CHANGE OF BASIS MATRIX P

Let $B = \{b_1, \dots, b_n\}, C = \{c_1, \dots, c_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $P_{C \Leftarrow B}$ such that

$$[x]_c = P_{C \Leftarrow B}[x]_b \tag{26}$$

The columns of $P_{C \Leftarrow B}$ are the C-coordinate vectors of the vectors in the change of coordinate matrix from B to C

To find $P_{C \Leftarrow B}$

with b_1, b_2, c_2, c_2. Row reduce $[c_1, c_2, b_1, b_2]$, the left should reduce to identity matrix and the right half reduces to P. Vice Versa for C to B.

P is square and has LI columns, so it is invertible by the IMT

$B = \{b_1, \dots, b_n\}$ and $\epsilon = \{e_1, \dots, e_n\}$

$$[b_1]_\epsilon = b_1 \text{and} P_{\epsilon \Leftarrow B} = P_B \tag{27}$$

6 5.1 Eigenvectors and Eigenvalues, 5.2 The character

6.1 EIGENVECTORS

Eigenvector: An eigenvector of an $n \times n$ matrix is a nonzero vector such that $Ax = \lambda x$ for some scalar λ

λ is an eigenvalue of A if there is a nontrivial solution **x** of $Ax = \lambda x$.

To test if a vector is an eigenvector of a matrix, multiply the vector and matrix, then see if the result is a scalar multiple of the original vector.

To show that a value is an eigenvalue of a matrix and finding the corresponding eigenvalue:

given

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \tag{28}$$

$$Ax = 7x \to Ax - 7x = 0 \to (A - 7I)x = 0 \to \tag{29}$$

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \tag{30}$$

$$Bx = 0 \rightarrow \begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{31}$$

$x_1 = x_2$ and x_2 is free.

Eigenspace = Span$\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$

1. row reduction can be used to find eigenvectors, but does not help find eigenvalues
2. In general λ is an eigenvalue of A iff
$$(A - \lambda I)x = 0 \tag{32}$$
has a nontrivial solution
3. The set of all solutions is the null space of the matrix $A - \lambda I$
4. This space is a subspace of $\mathbb{R}^n$ called the Eigenspace of A corresponding to λ

6.2 EIGENVALUE

To find the eigenvalues of A

Find all λ such that
$$(A - \lambda I)x = 0 \tag{33}$$
has nontrivial solution.
$$\det(A - \lambda I) = 0 \tag{34}$$

$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \tag{35}$$

$$\det(A - \lambda I) = \det(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}) = 0. \tag{36}$$

A scalar λ is an eigenvalue of an $n \times n$ matrix A iff λ satisfies the scalar equation
$$\det(A - \lambda I) = 0 \tag{37}$$

which we call the characteristic equation

if A is an $n \times n$ matrix, then the characteristic equation is a polynomial with degree n called the characteristic polynomial. The algebraic multiplication of an eigenvalue i its multiplicity as a root of the characteristic equation

6.3 THEOREM 1

The eigenvalues of a triangular matrix are the diagonal entries.

6.4 THEOREM 2

If $v_1, \ldots, v_p$ that correspond t distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ of an $n \times n$ matrix A, then the set of these vectors is linearly independent.

6.5 INVERTIBLE MATRIX THEOREM CONT.

Let A be an $n \times n$ matrix then A is invertible iff

1. The number zero is not an eigenvalue of A

6.6 SIMILAR MATRICES

Similarity: A matrix A is similar to a matrix B iff there is an invertible matrix P such that

$$P^{-1}AP = B, \text{or}, A = PBP^{-1} \tag{38}$$

so B is similar to A, A and B are similar. Changing A into $P^{-1}AP$ is a similarity transformation.

6.7 THEOREM 4

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and the same eigenvalues.

if $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P \tag{39}$$

$$\det(B - \lambda I) = \det[P^{-1}(A - \lambda I)P] = \det(P^{-1}) \tag{40}$$

7 5.3 Diagonalization and 5.4 Eigenvectors and Linear Transformations

7.1 DIFFERENCE EQUATIONS AND DIAGONALIZATION

Difference Equation: A difference equation - a reurcive relation of the form $x_{k+1} = Ax_k$ for some matrix A and x_0, x_1, x_2

Assume we start with initial condition x_0. $x_1 = Ax_0$. $x_2 = Ax_1 = AAx_0 = A^2x_0$. Generalized to:

$$A^k = PD^kP^{-1} \tag{41}$$

Diagonalizable Definition: A square matrix A is said to be diagonalizable if A is similar t a diagonal matrix, that is $A = PDP^{-1}$ for some invertible P and some diagonal D.

7.2 THEOREM 5: THE DIANOGALIZATION THEOREM

1. An $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors
2. $A = PDP^{-1}$ with a diagonal matrix D, iff the columns of P are n linearly independent eigenvectors of A
3. The diagonal entries of D are eigenvalues of A that correspond to

i

7.3 DETERMINING DIAGONAL

1. Find eigenvalues $det(A - \lambda I)$
2. Find 3 linearly independent eigenvectors of A
3. Construct matrix P from vectors from step 2- Line up eigenvectors in matrix.
4. Construct matrix D as a diagonal matrix of the corresponding eigenvalues

7.4 THEOREM 6 AND 7

6 A $n \times n$ matrix with n distinct eigenvalues is diagonalizable

7 Let a $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$

for $1 \leq x \leq$, the dimensions of the eigenspace for λ_k is less than or equal to the multiplicity of λ_k

The matrix A is diagonalizable iff the sum of the dimensions of the eigenspaces equals n

1. The characteristic polynomial factors completely
2. The dimension of the eigenspace of each λ_k=multiplicity of λ_k
3. if A is diagonalizable and B_k is a basis for the eigenspace for each k, then the total collection of basis vectors is a basis for $\mathbb{R}^n$

Can use basis coordinates to relate abstrac transformations to matrix multiplication on vectors in real spaces

V is an n-dimensional vector space and W is a m-dimensional vector space

1. B is a basis for V, the B-coordinate vector $[x]_B$ is in $\mathbb{R}^n$
2. C is a basis for W, the C-cordinate vector $[u]_c$ is in $\mathbb{R}^n$.

3. $\{b_1, \ldots, b_n\}$ is the basis for B for V.
4. if $x \in r_1 b_1 + \cdots + r_n b_n$, $[x]_B = \begin{bmatrix} r_1 \\ r_2 \\ \cdots \\ r_n \end{bmatrix}$

$T : v \to W$ is a linear transformation

$T(x) = T(r_1 b_1 + \cdots + r_n b_n = r_1 T(b_1) + \cdots + r_n T(b_1) \in W$

Since the C coordinate mapping from W to $\mathbb{R}^n$ is linear:

$$[T(x)]_c = [r_1 T(b_1) + \cdots r_n T(b_1)]_c = r_1[T(b_1)] + \cdots + r_n[T(b_n)]_c \in \mathbb{R}^n \tag{42}$$

$$[T(x)]_c = M[x]_B, M = \left[[T(b_1]_c \ldots [T(b_n)]_c\right] \tag{43}$$

Linear transformations in $\mathbb{R}^n$

1. A linear transformation T represented by marix multiplication $x \to Ax$
2. if A is diagonalizable then there is a basis B fr $\mathbb{R}^n$ consisting of eigenvectors of A.
3. it turns out that the B matrix for T is diagonal

7.5 THEOREM 8 DIAGONAL MATRIX REPRESENTATION

Suppose $A = PDP^{-1}$. D is diagonal, P is invertible

if B is the basis for $\mathbb{R}^n$ formed from the columns of P, call them $b_1, b_2, \ldots, b_n$ are eigenvalues of A.

D is the B matrix for the transformations $x \to Ax$.

8 6.1 Inner Product, Length, Orthogonality

8.1 INNER PRODUCT

If **u** and **v** vectors in $\mathbb{R}^n$ consider them as $n \times 1$ matrices.

The transpose u^T is a $1 \times n$

The matrix product: $u^T * v = 1 \times 1$ matrix, a scalar, the inner product.

$$u = \begin{bmatrix} u \\ \ldots u_n \end{bmatrix} \begin{bmatrix} v = v \\ \ldots v_n \end{bmatrix} \tag{44}$$

The inner product is

$$\begin{bmatrix} u & \ldots & u_n \end{bmatrix} \times \begin{bmatrix} v \\ \ldots \\ v_n \end{bmatrix} \tag{45}$$

8.2 THEOREM 6.1

u,v,w are vectors in $\mathbb{R}^n$. $c \in \mathbb{R}$.

1. $u \cdot v = v \cdot u$
2. $(u+v) \cdot w = u \cdot w + v \cdot w$
3. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
4. $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$

Definition: Length The length or form of **v** is the non-negative scalar $||v||$

$$||v|| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{46}$$

$$||v|| = \sqrt{v^T v} \tag{47}$$

For any scalar c cv is $|c|$ times the length of v

Definition: Unit Vector: A vector whose length is one

$$v, u = \frac{1}{||v||} v. \tag{48}$$

u is a unit vector in the direction of v, "normalized v"

Definition: distance For **u,v**$\in \mathbb{R}^n$, the distance between u and v, written as dist(u,v) is the length of the vector **u-v**.

$$\text{dist}(u,v) - ||u - v|| \tag{49}$$

Definition: Orthogonal: $u, v \in \mathbb{R}^n$ are orthogonal if $u \cdot v = 0$. They are perpendicular.

8.3 THEOREM 2: PYTHAGOREAN THEOREM

if we have two vectors u,v are orthogonal iff $||u+v||^2 = ||u||^2 + ||v||^2$.

Orthogonal Set Definition A set of vectors $\{u_1, \ldots, u_n\}$ in $\mathbb{R}^n$ is an orthogonal set if each pair of distinct vectors from the set is orthogonal

$$u_i \cdot u_j = 0 \tag{50}$$

for any i,j

$W \in \mathbb{R}^n$ if a vector z is orthogonal to every vector in W (plane) z is orthogonal to W.

8.4 THEOREM 4

if $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and is a basis fr the subspace spanned by S.

Definition: Orthogonal Basis: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set

Theorem 5: Let $[u]_1, \ldots, [u]_p$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each y$\in$W the weights of the linear combination $y = c_1[u]_1 + \cdots, c_p[u]_p$ are given by $c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$

$$[y]_B = \begin{bmatrix} \frac{y \cdot u_1}{u_1 \cdot u_1} \\ \ldots \\ \frac{y \cdot u_p}{u_p \cdot u_p} \end{bmatrix} \tag{51}$$

$$\hat{y} = (\frac{y \cdot u}{u \cdot u}) = \text{proj}_u y \tag{52}$$

which is the orthogonal projection of y onto u.

8.5 THEOREM 8: ORTHOGONAL DECOMPOSITION THEOREM

Let W be a subspace of $\mathbb{R}^n$. Then every y $\in$ $\mathbb{R}^n$ can be written uniquely in the form $y = \hat{y} + z$, where $\hat{y} \in W$ and $z \in W^T$, and if $[u]_1, \ldots, [u]_n$ is an orthogonal basis of W

8.6 THEOREM 9: BEST APPROXIMATION THEOREM

W is a subspace f $\mathbb{R}^n$, let y be any vector in $\mathbb{R}^n$, $\hat{y}$ is the orthogonal projection of y into W

then $\hat{y}$ is the closest point in W to y

$$||y - \hat{y}|| < ||y - v|| \tag{53}$$

for all v in W distinct from $\hat{y}$

9 6.4 The Gram-Schmidt Process and 6.5 Least Squares Problem

9.1 GRAM SCHMIDT

If we have $[v]_1, \ldots, [v]_n$

Let $W = [x]_1, \ldots, [x]_n$, $x1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W

$$v_1 = x_1 \tag{54}$$

$$v_2 = x_2 - p \tag{55}$$
$$p = \text{proj}_{x_1} x_2 = \frac{x_2 \cdot x_1}{x_1 \cdot x_1} x_1 \tag{56}$$

9.2 THEOREM 11, THE GRAM SCHMIDT PROCESS

Given a basis $[x]_1, \ldots, [x]_n$ fr a nonzero space $W \in \mathbb{R}^n$, define Step 1:

$$v_1 = x_1 \tag{57}$$

Step 2:

$$v_2 = x_2 - proj_{w_1} x_2 \tag{58}$$

Step 3:

$$W_2 = \text{Span}\{x_1, x_2\} \tag{59}$$
$$v_3 = x_3 - \text{proj}_{w_2} x_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \right) \tag{60}$$
$$\cdots \tag{61}$$
$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1} \tag{62}$$

9.3 LEAST SQUARES

1. in applications $Ax = b$ won't often be consistent
2. **Definition: Least Squares** if A is $m \times n$ and $b \in \mathbb{R}^n$, a least squares solution of $Ax = b$ is an x in $\mathbb{R}^n$ such that
$$||b - Ax|| \leq ||b - Ax|| \tag{63}$$
for all $x \in \mathbb{R}^n$
3. no matter what x e pick Ax will always be in the column space of A
4. Find x that makes Ax the closest point in Col A to b
5. Apply the approximation thm
$$\hat{b} = \text{proj}_{ColA} b \tag{64}$$
6. because b in in Col A $Ax = \hat{b}$ is consistent

Suppose x satisfies $Ax = b$, by the orthogonal decomposition theorem $b - \hat{b}$ is rthogonal to Col A, meaning $b - \hat{b}$ is orthogonal to each column of A.

$$a_j \cdot (b - \hat{b}) = 0 \tag{65}$$
$$a_j^T (b - \hat{b}) = 0 \tag{66}$$
$$A^T (b - \hat{b}) = A^T (b - Ax) = 0 \tag{67}$$
$$A^T b - A^T Ax = 0 \tag{68}$$
$$A^T Ax = A^T b \tag{69}$$

9.4 THEOREM 13

The set of least squares solutions of Ax=b coincides with the non-empty set of solution to the novel equations.

9.5 THEOREM 14

Let A be an $m \times n$ matrix, the following statements are logically equivalent

1. The equation Ax=b has a unique least squares solution for each b in $\mathbb{R}^n$
2. The columns of A are linearly independent
3. The matrix $A^T A$ is invertible

When these statements are true, the least squares slution $\hat{x}$ is given by

$$\hat{x} = (A^T A)^{-1} A^T b \tag{70}$$

10 Final Additions

Prove Linear Transformation

$$T(u+v) = T(u) + T(v) \tag{71}$$

and

$$T(cu) = cT(u) \tag{72}$$

Linear Combination Matrix T(x) represented as Ax=b

$$A = [T(e_1), T(e_2), \ldots, T(e_n)] \tag{73}$$

with $e_1, \ldots, e_n$ vectors/columns of the identity matrix

Change of Basis Transformation Matrix For matrix transformations between coordinate systems

$$M = \big[[T(b_1)]_C \quad \ldots \quad [T(b_n)]_C \big] \tag{74}$$

$T(b_1)$ is the transformation of the original first basis vector. Possibly identity, depending on question.

For putting in it coordinates of C, say $[v]_C$, row reduce the matrix made from the basis vectors in C to v, or in the above case $T(b_n)$

Theorem one to one matrix transformations logically equivalent:

1. t is one to one

2. for every b in Rm the equation T(x)=b has at mst one solution
3. for every b in rm AX=b is either unique or is inconsistent
4. Ax-0 has only trivial
5. Columns linearly independent
6. A has a pivot in every column
7. The range of T has dimensions n

Theorem onto

1. T is onto
2. T(x)=b has at least one solution fr ever b in R
3. Ax=b is consistent for every b in R
4. The columns of A span R
5. A has a pivot in every row
6. The range of T has dimensions m