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Using Kepler's laws and Rutherford scattering to chart the seven gravity assists in the epic sunward journey of the Parker Solar Probe

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On August 12, 2018, NASA launched the Parker Solar Probe (PSP) to explore regions very near the Sun. Losing enough energy and angular momentum to approach the Sun requires either an impractical amount of fuel or a maneuver called a gravity assist. A gravity assist is essentially an elastic collision with a massive, moving target—Rutherford scattering from a planet. Gravity assists are often used to gain energy in missions destined for the outer solar system, but they can also be used to lose energy. Reaching an orbit sufficiently close to the Sun requires that PSP undergoes not one but *seven* successive gravity assists off the planet Venus. This simple description poses several conceptual challenges to the curious physics student. Why is it so much more challenging to get to the Sun than to leave the Solar System? Why does it take more than one gravity assist to achieve this, and why does it require seven? Would it be more effective to use Mercury instead of Venus? These questions can be answered using the basic physics principles of Kepler's laws and Rutherford scattering. The reasoning can be presented in an illuminating graphical format to show that these and other seemingly arcane aspects of interplanetary exploration can be understood at the undergraduate level.
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I. INTRODUCTION

Interplanetary spacecraft feature prominently in the national and international news. They follow Keplerian orbits about the Sun, but reaching destinations across the solar system demands their orbits have significant eccentricities. This combination makes an interplanetary mission an ideal example with which to illustrate Kepler's laws of orbital motion in their fullest form[1]—more fully than possible using the nearly circular orbits of the planets themselves.

Missions to the outer planets almost always use gravity assists, also called slingshots or flybys, to give them extra energy and angular momentum. It is possible to treat these as elastic collisions between the spacecraft and the planet,[1–3] that is to say, as Rutherford scattering off a massive, moving target. Transforming first into, and then back out of, the center-of-mass frame shows how it is possible for the spacecraft to gain significant energy and angular momentum through such a scattering.[4,5] Recall that elastic scattering conserves energy of the system, but not of its individual components separately. It thereby provides another opportunity to illustrate fundamental principles of mechanics using these missions.

Kepler's laws can also be used to dispel the common misconception that it would be easy to "drop" something into the Sun. To send a spacecraft into such a plunging orbit would require its rocket engine to decrease its energy and to reduce its angular momentum about the Sun to nearly zero. It is shown below (Sec. III) that this is more than twice the velocity change required to reach escape speed from the solar system. The Sun is, therefore, the most difficult place in the Solar System to reach, in the sense that it requires the most fuel.

Rising to this challenge, NASA launched the Parker Solar Probe (PSP) on August 12, 2018, destined for points much closer to the Sun than ever reached by spacecraft before. The mission of the PSP is to sample the solar wind inside 0.05 astronomical units (AU), where some models predict it is subsonic and still undergoing significant acceleration. (1 AU $= 1.5 \times 10^{11}$ m is the mean distance between Earth and the Sun; it is the semi-major axis of Earth's orbit). In order to rid itself of energy and angular momentum without using excessive fuel, PSP will undergo unprecedented *seven* gravity assists, using the planet Venus.[6]

While gravity assists are often used to increase the energy of a spacecraft,[5] they are equally effective at reducing it as they do for PSP. In either case, the spacecraft's momentum is changed by the gravitational force from the planet, instead of a rocket motor's thrust. The result depends on the direction of that force, which depends in turn on the side of the planet to which spacecraft makes its closest approach. Passing a point just behind the planet (behind in the sense of the planet's motion), as shown in Fig. 1(a), results in a forward-directed gravitational force (i.e., less than 90° between $\mathbf{v}$ and $\mathbf{F}$) on the spacecraft which will increase its kinetic energy. Conversely, passing a point just ahead of the planet, as shown in Fig. 1(b), will decrease the spacecraft's kinetic energy. This is what PSP does to reduce its energy and angular momentum and thus approach the Sun.

It is reasonable to wonder if it would have been possible to achieve the needed change of energy and angular momentum in a single flyby of Venus. We show below that not only is this impossible, but at least six Venus flybys are needed to reach 0.05 AU. We also show that even with more Venus flybys, reaching closer than 0.05 AU would require significantly more fuel. We show further that flybys of Mercury are much less effective still and can produce the same proximity only through an order of magnitude more flybys.

The aim of this work is to use a few equations covered in a typical advanced undergraduate course on classical mechanics[1,7] to answer the questions posed above. In so doing, we hope to illustrate the power and the beauty of these principles. Accomplishing this requires a few simplifying assumptions, including approximating planetary orbits as circular and coplanar. The central assumption is that the spacecraft (PSP) is affected by gravitational force from only a single solar system body at a time, so it satisfies the equations of central

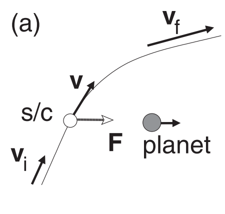

Fig. 1. Examples of gravity assists that (a) increase and (b) decrease the energy of the spacecraft. In each panel, the grey circle is the planet moving rightward, and the white circle is the spacecraft (s/c) following the orbit depicted by a solid curve. Each circle is rendered at the instant of closest approach although the motion of the planet makes it appear otherwise. The gravitational force from the planet, indicated by a grey arrow with white arrowhead labeled **F**, changes initial velocity $\mathbf{v}_i$ to final velocity $\mathbf{v}_f$.

force motion. Most of the time, the spacecraft is far enough from planets that it is affected by the Sun alone and follows a classic elliptic orbit with the Sun at its focus. When it approaches sufficiently close to Venus,[8] however, we consider only that gravitational influence and ignore the Sun. This brief interval of planetary influence is the gravity assist, before and after which the spacecraft orbits the Sun. This approximation essentially treats the gravity assist as an elastic collision, i.e., Rutherford scattering, which changes the spacecraft's velocity, placing it into a different Keplerian orbit. A similar approach has been proposed before[2,3,9] and succeeds in making computations easy to perform and understand. It also sheds new light on a second topic usually covered in the same course: Rutherford scattering.

As reasonable as these the approximations are, they do compromise the results by a percent or so, and could not be used when designing the actual mission. The accuracy that is achieved (around a percent) shows the student how well-chosen approximations can yield reasonable, albeit not perfect, results while also providing enormous intuitive insight. We perform the computational graphically, thereby showing rather clearly what is, and is not, possible using gravity assists.

This work presents its analysis in several steps. Section II introduces elements from Kepler's theory and assembles them into a velocity-space map to graphically analyze orbital dynamics. Section III uses the velocity-space map to compare orbits of a few interplanetary missions, including the PSP. Doing so shows the difficulty of reaching the Sun, and thus the need for gravity assists. Section IV uses the velocity-space map to show how gravity assists from Venus are used to approach the Sun. This reveals the need for at least 6 flybys for PSP to reach its target. Finally, Sec. V shows how the flybys must be selected to assure each flyby is followed by another.

II. A VELOCITY-SPACE MAP OF KEPLERIAN ORBITS

An interplanetary spacecraft uses rocket engines infrequently and only for very short intervals. It experiences significant forces from planets also at brief, infrequent intervals. Except for these events, it moves as an inert body orbiting the Sun according to Kepler's laws of orbital motion. According to Kepler's first law, its orbit is an ellipse, which we assume to lie in the same plane as all planets (the ecliptic plane), with the Sun at one focus (taken as the origin); Fig. 2 shows an example of such an orbit.

An elliptical orbit is characterized by the extreme distances called perihelion, r_p (smallest distance), and aphelion, r_a

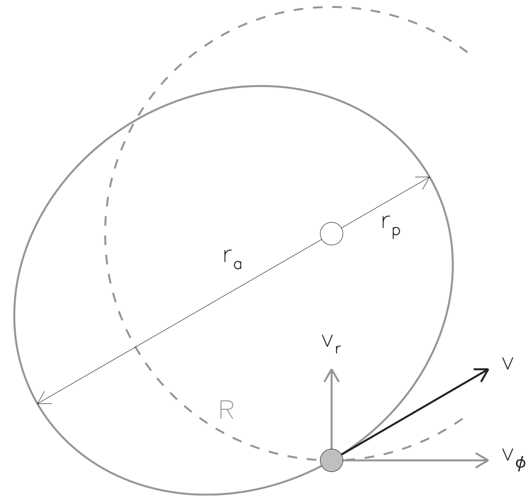

Fig. 2. An elliptic orbit (solid grey curve) crossing the reference radius R (dashed grey curve). The point of crossing is marked by a grey circle. The instantaneous velocity is shown as a dark arrow tangent to the orbit, and its radial and azimuthal components by grey arrows. The perihelion, r_p, and aphelion, r_a, are indicated by arrows originating the Sun (white circle).

(greatest distance), indicated by thin arrows in Fig. 2. The values of these extrema are the most important aspects of an orbit for the purposes of exploration. For example, the success of the PSP depends on its achieving an orbit with $r_p < 0.05$ AU. A spacecraft can control its orbit by changing its instantaneous velocity through either rocket propulsion or gravity assist. The relation between the velocity and the key orbital characteristics is concisely presented in the velocity-space map, Fig. 3, showing values of r_p and r_a as contours in a space of the velocity at a fixed reference radius R. In this velocity-space map, the azimuthal and radial velocities, v_ϕ and v_r, are scaled to $v_o = \sqrt{GM/R}$, characteristic of the

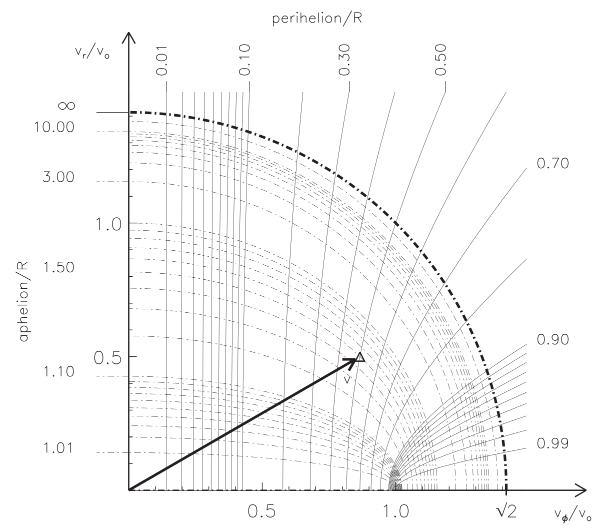

Fig. 3. The velocity-space plot used to determine perihelion and aphelion from velocity **v** at a fixed reference radius. The black arrow shows the same velocity vector from Fig. 2, scaled to $v_o = \sqrt{GM/R}$. Thin solid contours mark the values of r_p labeled along the top and right; thin broken (dash-dot) contours mark the values of r_a, labeled along the left. A thick broken contour occurs at escape speed, $|\mathbf{v}| = \sqrt{2}\, v_o$, where $r_a \to \infty$. The triangle at the end of the velocity arrow is the entire orbit plotted as a grey ellipse in Fig. 2. The orbital parameters, $r_p = 0.5$ and $r_a = 1.5R$, are found from the contours crossing at that point.

reference radius, and each contour is labeled by perihelion and aphelion scaled to the reference radius, r_p/R and r_a/R.

The normalized velocity-space map in Fig. 3 is a powerful tool for understanding orbital dynamics. (While the plot is a fairly simple to construct, the author is not aware of other publications where it was previously presented in detail, highlighted, or used for this purpose.) It shows the perihelion and aphelion of an orbit given the spacecraft's instantaneous velocity at a specified reference radius. Every orbit crossing the reference radius R is a single point on the plot. Since perihelion and aphelion are constant for a Keplerian orbit, the point remains fixed in time. While the velocity of the orbiting body does change in time, the coordinates here represent the velocity only at the instant the orbit crosses the fixed reference radius R. All later analysis will use a map from a single reference radius.

The orbit crosses the reference radius twice, once inbound ($v_r < 0$) and once outbound ($v_r > 0$), both times with the same v_ϕ. To avoid double counting orbits on our plot, we use only the $v_r > 0$ quadrant and interpret the ordinate as $|v_r|$. All the calculations presented below can be performed graphically using this one plot.

To better appreciate how to work with the velocity-space map, consider the orbit shown in Fig. 2, whose velocity is $\mathbf{v}/v_o = 0.866\,\hat{\phi} - 0.5\,\hat{\mathbf{r}}$, shown by a black arrow, at the instant it crosses radius R (the dashed circle). That same arrow is plotted on Fig. 3, after being scaled to the reference velocity v_o appropriate to the reference radius R. (We have reversed the sign of v_r, as discussed above.) The point at the arrow's vertex, indicated by a triangle, corresponds to an orbit with $r_p = 0.5R$ and $r_a = 1.5R$, matching the ellipse in Fig. 2. In this way, the velocity-space map translates the velocity vector at the reference radius to a description of the entire orbit. (It is noteworthy that our illustration happens to use an orbit whose speed at the reference radius is $|\mathbf{v}| = v_0 = \sqrt{GM/R}$.)

The velocity-space map expresses r_p and r_a in terms of instantaneous velocity $\mathbf{v}$. This mapping can be derived using orbital relations typically covered in undergraduate mechanics.[1,7] The first relation is that angular momentum (per unit mass) $\ell = r\,v_\phi$ is a constant of the motion. The second is that the total energy (per unit mass)

$$E = \frac{1}{2}|\mathbf{v}|^2 - \frac{GM}{r} \tag{1}$$

is also constant, where M is the mass of the Sun and G is Newton's gravitational constant. The total energy combines the kinetic energy and gravitational potential from the central body (the Sun). (Since angular momentum and kinetic and gravitational potential energy are all proportional to the mass of the orbiting body, all equations, such as Eq. (1), are hereafter divided by that mass.)

It is traditional to express the elliptical orbit using its semi-major axis a and eccentricity e which are together related to the angular momentum (per unit mass) ℓ, as $a(1-e^2) = \ell^2/GM$. In polar coordinates, the elliptical orbit is described by[1,7]

$$r(\phi) = \frac{a(1-e^2)}{1 + e\,\cos(\phi - \phi_p)}, \tag{2}$$

where perihelion occurs at the azimuthal angle ϕ_p. The perihelion and aphelion are the minimum and maximum values of Eq. (2),

$$r_p = a(1-e), \quad r_a = a(1+e), \tag{3}$$

found by taking $\phi = \phi_p$ and $\phi = \phi_p + \pi$, respectively.

The final property of a Keplerian orbit is that energy (per unit mass) is inversely proportional to the semi-major axis

$$E = -\frac{GM}{2a}. \tag{4}$$

Using this relation in Eq. (1), it is easy to obtain an expression for the semi-major axis

$$\frac{a}{R} = \frac{1}{2 - |\mathbf{v}|^2/v_o^2}, \tag{5}$$

where $v_o = \sqrt{GM/R}$ is the speed of a circular orbit of radius R. A second relation follows from squaring the angular momentum (per unit mass)

$$R^2 v_\phi^2 = \ell^2 = GM\,a(1-e^2) = v_o^2\,aR\,(1-e^2). \tag{6}$$

Solving for e^2 yields the explicit relation

$$e^2 = 1 - \frac{R}{a}\frac{v_\phi^2}{v_o^2} = 1 - \frac{v_\phi^2}{v_o^2}\left(2 - \frac{|\mathbf{v}|^2}{v_o^2}\right), \tag{7}$$

after using Eq. (5) to replace R/a.

Introducing Eqs. (5) and (7) into Eq. (3) yields expressions for r_p/R (upper sign) and r_a/R (lower sign) explicitly in terms of normalized velocity components

$$\frac{r_{p,a}}{R} = \frac{v_\phi^2}{v_o^2}\left[1 \pm \sqrt{1 - \frac{v_\phi^2}{v_o^2}\left(2 - \frac{v_r^2 + v_\phi^2}{v_o^2}\right)}\right]^{-1}. \tag{8}$$

Figure 3 plots contours of these expressions. These relations have several notable properties, evident in Fig. 3. Aphelion contours (broken curves) form elliptic-like curves focused on $\mathbf{v} = (v_o, 0)$. The line from the origin to that focus ($0 \le v_\phi \le v_o$, plotted as a broken horizontal segment) is a degenerate ellipse corresponding to orbits grazing the reference radius at their aphelion: $R_a = R$. This degenerate contour has the minimum possible value $r_a/R = 1$ since any orbit with $r_a < R$ will not cross the reference radius. The outermost contour is at the escape speed from the reference radius, $|\mathbf{v}| = \sqrt{2GM/R} = \sqrt{2}v_o$, where Eq. (5) diverges. Points along this contour are parabolic orbits, and any point outside this contour corresponds to a hyperbolic orbit whose perihelion is given by the thin solid contour out there.

Perihelion contours (thin solid curves) are quasi-hyperbolic and are also focused at $\mathbf{v} = (v_o, 0)$. A line extending along the v_ϕ axis from this point, indicated by a black arrow along the axis, is a degenerate hyperbola corresponding to orbits, either elliptic or hyperbolic, which graze the reference radius at their perihelions, $r_p = R$. This is the maximum permissible value of r_p on the plot. The intersection between this degenerate curve and the degenerate ellipse, $0 \le v_\phi \le v_o$, is the single point, $\mathbf{v} = (v_o, 0)$, corresponding to a circular orbit, $r_p = r_a = R$. In all that follows, we will assume planets to have perfectly circular orbits, and therefore to lie at the focal point $\mathbf{v} = (v_o, 0)$.

III. ORBITS FROM EARTH

All spacecraft launched from Earth will, at least initially, follow orbits crossing the orbit of Earth. For simplicity, we take Earth's orbit to be perfectly circular with radius $R_e = 1$ AU and orbital speed $v_{o,e} = 29.8$ km/s. We further assume the spacecraft's orbit lies in the same plane, the ecliptic, to keep our problem two-dimensional. After those two assumptions we can place any Earth-crossing spacecraft orbit on a velocity-space plot with reference radius $R = R_e$, as done in Fig. 4. The point on the plot, $\mathbf{v} = v_r\hat{\mathbf{r}}+v_\phi\hat{\phi}$, is the velocity, represented in the inertial Solar-system frame, with which the spacecraft left the region dominated by the gravitational influence of Earth. (The Sun's gravitational force is dominant at distances exceeding 2.5×10^5 km $= 1.7 \times 10^{-3}$ AU, from Earth.[8]) Once the spacecraft is outside the region of influence, and is still at approximately $R = R_e = 1$ AU from the Sun, it has velocity relative to Earth,

$$\Delta\mathbf{v}_e = \mathbf{v} - v_{o,e}\,\hat{\phi}, \tag{9}$$

exemplified by the black arrow on Fig. 4.

Squares and triangles on Fig. 4 show orbits of several spacecraft,[10] launched to explore the inner heliosphere (points toward the left) and the outer solar system and beyond (points toward the right); the points refer to the orbits once the craft have left the gravitational influence of Earth. The square shows the initial orbit of PSP, and the triangle labeled H shows Helios 2. The latter was launched in 1976 to study the solar wind and long held the record for the closest approach to the Sun with $r_p = 0.29$ AU. It is clear from the $r_p = 0.21$ AU contour that the record was broken by PSP on its first orbit. Triangles labeled P10 and NH show the

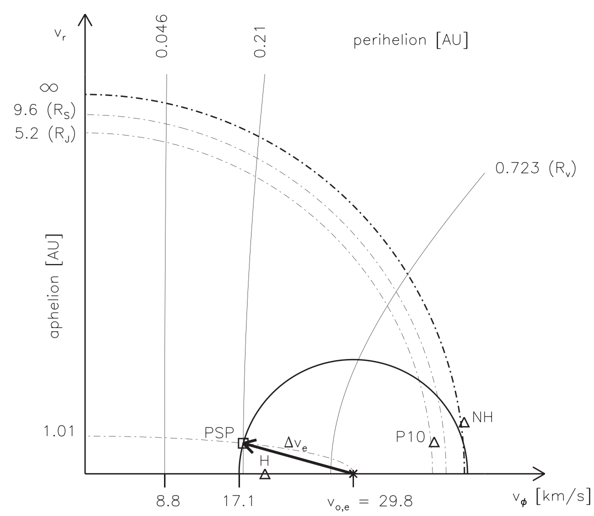

Fig. 4. Velocity-space map of orbits crossing $R = 1$ AU— the orbit of Earth. The format is identical to Fig. 3, but values have been given dimensions using $R = 1$ AU and $v_o = 29.8$ km/s. Values of v_ϕ corresponding to two of the contour levels, 8.8 km/s, and 17.1 km/s, are noted along with 29.8 km/s. For maximum clarity only a small number of perihelion (thin solid) and aphelion (thin broken) contours are plotted; these are labeled in units of AU. For further clarity, none of the velocities, v_r, are labeled, so all labels along the vertical axis designate r_a contour levels in units of AU. The square labeled PSP shows the initial orbit of PSP ($r_p = 0.21$ AU and $r_a = 1.01$ AU), and triangles show several other spacecraft launched to explore the inner heliosphere (H for Helios 2) and the outer solar system and beyond (P10 for Pioneer 10 and NH for New Horizons).

initial orbits of Pioneer 10 (launched in 1972) and New Horizons (launched in 2006). Both used gravity assists from Jupiter to send their orbits outside the solar system. The position of the NH point shows that its initial orbit (i.e., before assist) would escape the Sun. The assist allowed it to reach its destination, Pluto, sooner.[11]

The initial orbit of PSP, shown as a square labeled PSP, separates from Earth with relative speed $|\Delta\mathbf{v}_e| = 29.8 - 17.1 = 12.7$ km/s: the length of the black arrow. The set of all orbits with the same relative speed is shown by a black semi-circle. It is noteworthy that the other spacecraft on the plot were launched at comparable relative speeds, so their points fall inside or just outside this semi-circle. New Horizon's speed was slightly higher (the highest value achieved to date[6]), and those launched in the 1970s were somewhat lower.

To send a spacecraft away from Earth with relative speed $|\Delta\mathbf{v}_e|$ requires an impulse (i.e., change of momentum) beyond that required to simply leave the region dominated by Earth's gravity. The required impulse comes from fuel mass m_f carried out of Earth's gravitational well, along with the payload mass m_p, (i.e., the mass after all fuel has been consumed). The relative velocity is given by the solution to the so-called *rocket* equation[1]

$$|\Delta\mathbf{v}_e| \simeq u \ln\left(1 + \frac{m_f}{m_p}\right), \tag{10}$$

where u, called the *specific impulse*, is the differential impulse per mass of fuel burned, which is essentially the speed the exhaust gas leaving the engine, and is related to the type of fuel. For a hydrogen-oxygen mixture it can be about $u \simeq 4$ km/s. Comparing the velocity-space points of the various missions shows how their fuel-to-payload mass ratios compare.

Sending PSP directly to its target perihelion, $r_p = 0.046$ AU, would require a greater velocity change, $|\Delta\mathbf{v}_e| = 29.8 - 8.8 = 21.0$ km/s, and therefore more fuel, than it was actually given. Equation (10) shows that the relative velocity depends logarithmically on fuel-to-payload mass ratio, so producing an additional $17.1 - 8.8 = 8.3$ km/s of relative velocity can require m_f/m_p larger by a factor

$$\frac{\exp\left(\dfrac{21.0\ \text{km/s}}{4\ \text{km/s}}\right) - 1}{\exp\left(\dfrac{12.7\ \text{km/s}}{4\ \text{km/s}}\right) - 1} = 8.3, \tag{11}$$

after assuming specific impulse $u = 4$ km/s. This means either eight times more fuel, for the same payload, or a payload only one-eighth the mass for the same fuel. Neither option is attractive, so it became necessary to use gravity assists to send the PSP to its final destination. The rationale behind the ultimate choice of orbit will become clearer as we explore the details of gravity assists in Secs. IV and V.

The foregoing exercise has shown that it is far more difficult to reach the Sun than to reach interstellar space. Hyperbolic orbits leaving the solar system require $|\Delta\mathbf{v}_e| = \left(\sqrt{2} - 1\right)v_{o,e} = 12.3$ km/s, while an orbit plunging into the sun ($r_p = 0$) requires $|\Delta\mathbf{v}_e| = v_{o,e} = 29.8$ km/s. Since the latter is greater by 17.5 km/s, it requires exponentially more fuel acccording to Eq. (10).

IV. GRAVITY ASSISTS AS RUTHERFORD SCATTERINGS

The initial orbit of PSP, with $r_p = 0.21$ AU, would take it past the orbit of Venus, $R_v = 0.723$ AU. To obtain a gravity assist, the spacecraft must pass close enough to Venus itself to be significantly accelerated by its gravitational force. As mentioned in the Introduction, we simplify the analysis by assuming this interaction is brief, intense, and localized: Venus has no effect before or after the interaction, and the Sun has no effect during the interaction. This means that before and after the interaction the spacecraft follows simple Keplerian orbits about the Sun representable as points on a velocity-space map with reference radius $R = R_v$; see Fig. 5. The square labeled a in that plot corresponds to the initial PSP orbit $r_a = 1.01$ AU, $r_p = 0.21$ AU, previously depicted by a square in Fig. 4. Those two plots use different reference radii, $R = R_e$ *vs.* $R = R_v$, so the same orbit lies at a different point in each one.

Orbit a has semi-major axis $a = (r_p + r_a)/2 = 0.61$ AU and eccentricity $e = r_a/a - 1 = 0.66$. Its position on Fig. 5 is found using $a/R = 0.61/0.723 = 0.84$ in Eq. (5) to solve for $v/v_o = 0.90$, and then using $e^2 = 0.44$ in (7) to solve for $v_\phi/v_o = 0.69$. Multiplying these by $v_o = 35.0$ km/s gives $v = 31.6$ km/s, $v_\phi = 24.1$ km/s, and therefore $v_r = \sqrt{v^2 - v_\phi^2} = 20.4$ km/s. This is where the square appears and is naturally very near the intersections of the contours for $r_p = 0.2$ AU and $r_a = 1$ AU.

To further simplify the analysis, we assume the interval of Venus's influence to be sufficiently brief and intense that the gravitational force from the Sun may be ignored during it. In other words, we treat the interaction as *an elastic collision* between Venus and the spacecraft. Collisional interaction

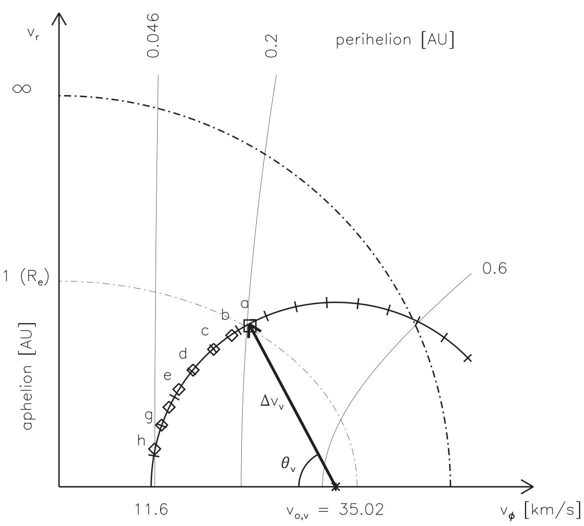

Fig. 5. Velocity-space map of orbits crossing $R = R_v = 0.723$ AU—the orbit of Venus. The format is identical to Fig. 3, but values have been given dimensions using $R = 0.723$ AU and $v_o = 35.02$ km/s. Only a small number of perihelion (thin solid) and aphelion (thin broken) contours are plotted; these are labeled in units of AU (rather than R_v) along the top and right (for perihelion) and left (for aphelion). For further clarity, none of the velocities, v_r, are labeled, so all labels along the vertical axis designate r_a contour levels in units of AU. The initial orbit of PSP is shown as a square, labeled a, with a black arrow to it labeled $\Delta\mathbf{v}_v$. The set of orbits with the same relative speed, $|\Delta\mathbf{v}_v|$, is shown by a black semi-circle with tick marks spaced by $\max(\Delta\theta_v) = 9.7°$. The seven subsequent orbits of PSP are shown by diamonds with labels on select orbits.

via an inverse-square-law force is the scenario considered in Rutherford scattering, another standard topic in undergraduate physics.[1,7] As in any collision, we transform to the frame co-moving with the center of mass, which in this case is the velocity of Venus. The craft's velocity in that frame, $\Delta\mathbf{v}_v = \mathbf{v} - v_{o,v}\hat{\phi}$, is shown by the black arrow in Fig. 5.

Because the collision is perfectly elastic, the magnitudes of the velocity in the center-of-mass frame must be the same before and after scattering

$$|\Delta\mathbf{v}_{v,f}| = |\Delta\mathbf{v}_{v,i}| = \sqrt{(24.1 - 35.0)^2 + (20.4)^2} = 23.4 \text{ km/s}. \tag{12}$$

The scattering event thus produces a new orbit lying somewhere along the black semi-circle in Fig. 5, which we call the *elastic scattering arc*. While the magnitude of the velocity vector is unchanged, its direction, θ_v (see Fig. 5), is changed by some amount $\Delta\theta_v$ depending on the impact parameter of the scattering event.

The intersection of the semi-circle with the ordinate, $v_\phi = 35.0 - 23.4 = 11.6$ km/s, is the orbit with minimum perihelion reachable by elastic scattering, $r_p \simeq 0.045$ AU. Plunging into the Sun (i.e., obtaining $r_p = 0$) is so difficult that it is not even possible with help of a planet—at least not Venus. The diamond labeled h in Fig. 5 has a perihelion just above the minimum, $r_p \simeq 0.046$ AU, and is the final orbit of the PSP mission. Reaching that orbit directly from the initial orbit, a, would require a single gravity assist with scattering angle $\Delta\theta_v = 50.2°$.

Since the scattering events are elastic, the Venus/PSP system as a whole neither gains nor loses energy. In the reference frame of Venus, the spacecraft's own energy is conserved, meaning that $|\Delta\mathbf{v}_v|^2$ remains constant. But the scattering moves the orbital point toward the origin in Fig. 5, meaning $|\mathbf{v}|$ decreases in the solar system frame. This means the spacecraft loses energy to Venus. Since v_ϕ decreases as well (the point moves left in the plot), the spacecraft also loses angular momentum. For these changes, the PSP must pass just in front of Venus as in Fig. 1(b). Conservation demands that energy and angular momentum lost by PSP must be gained by Venus, but these small contributions will be utterly negligible to the much more massive planet. This exchange between spacecraft and planet is the secret to a gravity assist.

A. The scattering angle

A particle (or spacecraft) undergoing Rutherford scattering follows a hyperbolic trajectory with the center of mass (i.e., Venus) at its focus, as shown in Fig. 6. This may be treated as a Keplerian orbit with eccentricity $\tilde{e} > 1$ and radius $\tilde{r}$ measured from Venus rather than the Sun. (We henceforth use a tilde to distinguish quantities defined for the PSP/Venus system from those in the PSP/Sun system; it remains the case that $e < 1$ for the PSP/Sun system.) Before and after the scattering, the particle follows the hyperbola's asymptotes whose angles are ϕ_i and ϕ_f. The equation for a hyperbola resembles Eq. (2) and has the same denominator,[1,7] but with eccentricity greater than one. The angles of the asymptotes are those that make the denominator vanish

$$\cos(\phi_{i,f} - \phi_p) = -\frac{1}{\tilde{e}}, \tag{13}$$

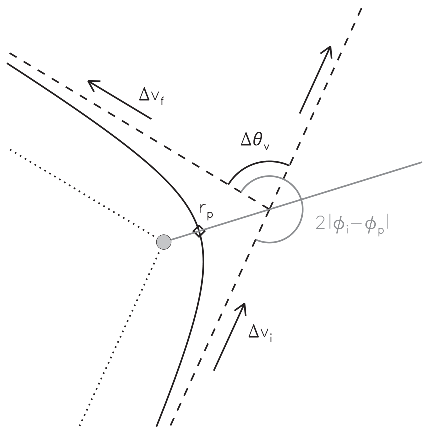

Fig. 6. A hyperbolic orbit (solid black) representing scattering from the stationary Venus (grey circle). The asymptotes of the hyperbola are dashed lines, and arrows show the initial and final velocity. Directions ϕ_i and ϕ_f are shown as dotted lines from Venus, parallel to the asymptotes. A grey line runs from Venus through the pericenter (diamond) and out along ϕ_p. Grey and black arcs show the angles $2|\phi_i - \phi_p|$ and $\Delta\theta_v$, respectively.

where $\tilde{e} > 1$ is the eccentricity in Venus-centered coordinates. Inverting the cosine gives $\phi_i - \phi_p$ and $\phi_f - \phi_p$ of equal magnitudes but opposite signs. This means the angle of the pericenter falls midway between the asymptotes, as is clear from Fig. 6. The right hand side of Eq. (13) is negative so both angles are obtuse: $|\phi_i - \phi_p| = |\phi_f - \phi_p| > \pi/2$. Figure 6 shows the scattering angle to be

$$|\Delta\theta_v| = 2|\phi_i - \phi_p| - \pi, \tag{14}$$

where we use absolute values since the scattering can increase or decrease θ_v depending on the side of the planet approached and thus which sense the hyperbola is traversed, as illustrated in Fig. 1. (The sign of v_r is likewise irrelevant, so either sense of scattering is possible whether the spacecraft is moving toward or away from the Sun.) Using this in Eq. (13) yields an expression relating the scattering angle to the eccentricity of the hyperbolic orbit

$$\sin\left(\frac{1}{2}|\Delta\theta_v|\right) = \frac{1}{\tilde{e}}. \tag{15}$$

Expressions (3) and (4) may be combined to eliminate a and solve for eccentricity

$$\tilde{e} = 1 + \frac{2\tilde{E}}{GM_v}\tilde{r}_p, \tag{16}$$

where M_v is the mass of Venus and $\tilde{r}_p$ is the distance from pericenter. The energy (per unit mass) of a scattering orbit is positive and equal to the kinetic energy (per unit mass) at large distance: $\tilde{E} = |\Delta\mathbf{v}_{v,i}|^2/2$. Using this yields eccentricity

$$\tilde{e} = 1 + \frac{|\Delta\mathbf{v}_{v,i}|^2}{GM_v/\tilde{r}_p} = 1 + \frac{2|\Delta\mathbf{v}_{v,i}|^2}{v_{\mathrm{esc},p}^2}, \tag{17}$$

where we have introduced $v_{\mathrm{esc},p} = \sqrt{2GM_v/\tilde{r}_p}$, the escape speed from pericenter. This provides us an explicit relation for the scattering angle

$$\sin\left(\frac{1}{2}|\Delta\theta_v|\right) = \frac{1}{1 + 2|\Delta\mathbf{v}_{v,i}|^2/v_{\mathrm{esc},p}^2}. \tag{18}$$

Equation (18) is a version of the classic formula for the Rutherford scattering angle although it would appear more familiar if expressed using the impact parameter rather than $v_{\mathrm{esc},p}$. We choose here to use the escape speed from pericenter because it relates more clearly to the depth into the gravitational well the particle actually drops. That depth is compared to the particle's energy (per unit mass) in the planet's reference frame. A relatively deep plunge, $v_{\mathrm{esc},p} \gg |\Delta\mathbf{v}_{v,i}|$, will result in a large scattering angle, while a shallow plunge, $v_{\mathrm{esc},p} \ll |\Delta\mathbf{v}_{v,i}|$, will produce a small scattering angle: the particle (spacecraft) is only weakly deflected.

A single scattering to the final perihelion, $r_p = 0.046$ AU, requires $\Delta\theta_v = 50.2^\circ$ as observed above. Using this in Eq. (18) shows that the spacecraft must drop deeply into Venus's gravitational well: $v_{\mathrm{esc},p} = 1.2\,|\Delta\mathbf{v}_{v,i}| = 28$ km/s. This turns out to be impossibly deep, below the surface in fact, since the escape speed from Venus's surface is a mere 10.4 km/s. We cannot find an escape speed larger than this and must not descend even that deep. We adopt, as a practical maximum, the escape speed from 400 km above the Venusian surface: $v_{\mathrm{esc,max}} = 10.0$ km/s. This maximum escape speed defines a maximum scattering angle

$$\max(\Delta\theta_v) = 2\,\sin^{-1}\left(\frac{1}{1 + 2|\Delta\mathbf{v}_{v,i}|^2/v_{\mathrm{esc,max}}^2}\right). \tag{19}$$

The PSP orbit being considered, with $|\Delta\mathbf{v}_{v,i}| = 23.4$ km/s, has a maximum scattering angle $\max(\Delta\theta_v) = 9.7^\circ$. This is less than one fifth of the value needed to achieve the target orbit ($r_p = 0.046$ AU) in a single assist. Thus, the target orbit can only be reached by a sequence of at least six gravity assists, each scattering by some angle less than 9.7°. The actual mission uses seven assists[6] to progressively move into orbits b – h indicated by diamonds along the elastic scattering arc in Fig. 5.

Properties of all eight orbits are listed in Table I. Values of r_p and r_a are taken from the mission design,[10] and all other quantities are computed using relations provided above. It can be seen in Fig. 5 that all orbits fall along the elastic scattering arc and, from the value of θ_v in Table I, that no two orbits are separated by more than 9.7°. Of course there are many other ways to arrange eight points to satisfy these conditions, i.e., consistent with Rutherford scattering. We show in Sec. V that only these particular orbits will allow each

Table I. The eight orbits achieved by PSP through seven gravity assists. These correspond to the selectively labeled squares and diamonds in Fig. 5. The period of each orbit is listed both in days and in units of Venusian years, $P_v = 225$ days. The last column gives the maximum speed that is attained at perihelion, in km/s.

Orbit	rp (AU)	ra (AU)	θv	ℓ/ℓ_e	e	P (days)	P (P_v)	vp (km/s)
a	0.207	1.013	62.0°	0.59	0.66	174.2	0.774	84.2
b	0.166	0.938	55.5°	0.53	0.70	149.8	2/3	95.3
c	0.130	0.874	48.4°	0.48	0.74	129.9	0.577	109.2
d	0.095	0.817	39.3°	0.41	0.79	112.4	1/2	129.6
e	0.074	0.783	31.9°	0.37	0.83	102.4	0.455	147.7
f	0.062	0.761	25.5°	0.34	0.85	96.3	3/7	162.9
g	0.053	0.745	19.5°	0.32	0.87	92.1	0.409	176.4
h	0.046	0.731	11.8°	0.29	0.88	88.4	0.393	190.7

scattering event to follow the previous; they are the only choice that works.

V. REPEATED ASSISTS VIA RESONANT AND NON-RESONANT ORBITS

The foregoing has shown that it is only possible to reach the final perihelion, $r_p = 0.046$ AU, through repeated gravity assists from Venus. After one such event, both PSP and Venus will depart along their respective orbits. The two orbits, PSP's ellipse and Venus's circle, generically intersect at two points: one where PSP is inbound and the other where it is outbound (see Fig. 2). In order to undergo the next gravity assist, PSP and Venus must reach one of these intersections simultaneously, after leaving together from that or the other intersection. In order to meet again at the same one, the periods of their orbits must be commensurable, $P/P_v = m/n$, where m and n are whole numbers, and P and P_v are the orbital periods of PSP and Venus, respectively. In that case, called an m:n resonance, n complete orbits of PSP and m orbits of Venus will bring the two back together at the same point for the next gravity assist.

Scattering onto a resonant orbit with $P/P_v = m/n$ is one way to assure a subsequent flyby. In order to keep the mission time reasonably short, we prefer numerators m as small as possible. According to Kepler's third law, the period of an orbit scales only with its semi-major axis as $P \propto a^{3/2}$. Similarly the period of Venus's orbit, P_v, scales with its semi-major axis, R_v, as $P_v \propto R_v^{3/2}$, with the same constant of proportionality. The constants cancel out of the ratio

$$\frac{P}{P_v} = \left(\frac{a}{R_v}\right)^{3/2}. \tag{20}$$

Using Eq. (5) gives a relation

$$\frac{|\mathbf{v}|}{v_o} = \sqrt{2 - \left(\frac{P}{P_v}\right)^{-2/3}}, \tag{21}$$

describing concentric circles on the velocity-space plot, shown in Fig. 7. The particular values 2/3, 1/2, and 3/7 are shown in Fig. 7. Orbits b, d, and f fall at the intersections of these resonances with the elastic scattering curve. These orbits, listed in Table I with their rational periods, are selected based on this intersection.

There are no suitable rationals (i.e., with numerators less than 3) between resonant orbits b and d or between d and f. Nor is it possible to jump between these directly since they are separated by $\Delta\theta_v > 9.7°$. The orbit with $P/P_v = 2/5$ (not shown) is also too far from f to reach in a single jump ($\Delta\theta_v = 10.2°$). Non-resonant orbits must therefore be found to fill the gaps. Rather than returning to the same intersection, the non-resonant orbit we seek follows an inbound flyby with a subsequent rendezvous on an outbound leg, or *vice versa*. Identifying such partial-orbit meetings requires computations more complicated than finding resonances (i.e., more complicated than Eq. (21)) which will not be described here.[12]

One version of this computation finds the unique orbit along the elastic scattering curve that follows an inbound flyby with an outbound rendezvous during the incomplete mth orbit of Venus and incomplete nth orbit of PSP. Choosing m:n to be 1:2 and 2:5 yields orbits c and g, respectively (see Table I).

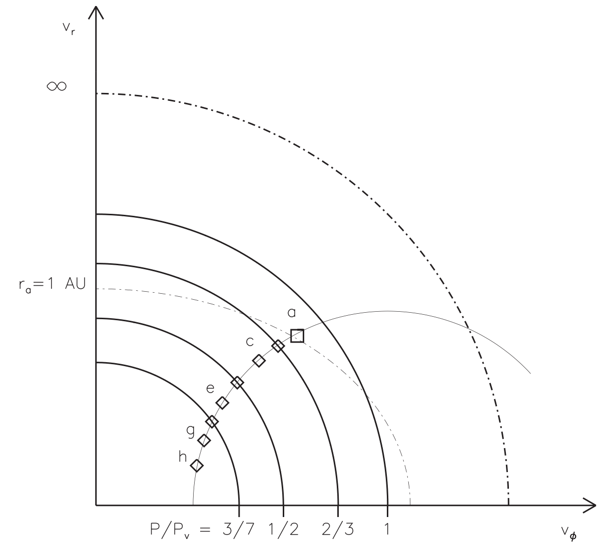

Fig. 7. Velocity-space plots of orbits crossing $R = R_v = 0.723$ AU—the orbit of Venus. This is a copy of Fig. 5, but with only the broken contours, $r_a = 1$ AU and $r_a = \infty$, and labeled on the left. The same velocity axes are used, but ticks and values are omitted for clarity. The orbits of PSP are shown by a square (orbit a) and diamonds (orbits b–h) with labels on select orbits. The set of orbits with the same relative speed, $|\Delta \mathbf{v}_v| = 23.4$ km/s, known as the elastic scattering arc, is shown by a thin black semi-circle. Solid black quarter circles, centered at the origin, show curves of resonant orbits given by Eq. (21). The rational values of P/P_v are listed along the bottom of each quarter circle.

Since the omitted segment of the last PSP orbit would take longer than the omitted segment of Venus's orbit, such orbits have $P/P_v > m/n$. Similarly, following an outbound flyby, there is a single orbit in which an inbound rendezvous occurs beyond the mth full orbit of Venus and the nth full orbit of PSP. For m:n = 1:2, this orbit is e, and it has $P/P_v < 1/2$.

The logic above dictates the choice of all the orbits and the sequence of Venus flybys between them. Unsurprisingly, these correspond to the ones actually selected for the mission, whose details are given in Table II. The need for six intermediate orbits linked by seven flybys, as well as their uneven spacing, is all explained by the unique requirements they had to satisfy. From these, it is possible to chart the entire journey of PSP, depicted in physical space in Fig. 8. The closest pass, #3, involves a scattering angle $\Delta\theta_v \simeq 9.1°$, which demands, according to Eq. (18), a pericenter $\tilde{r}_p = 1.15\,\tilde{r}_v$—900 km above Venus's surface. This is comfortably above the atmosphere, and all other passes are even more comfortable.

PSP was launched from Earth into orbit a which brought it very quickly to Venus flyby #1. (Table II omits details of this brief segment along the top row, since it depends on the precise launch date.) Flyby #1 scatters the PSP by $\Delta\theta_v = 6.4°$ into orbit b. That orbit is in 2:3 resonance with Venus, so after 2 Venusian years (450 days) PSP meets Venus at the same inbound point for flyby #2. There it is scattered into orbit c which, as described above, brings them back together after less than two orbits and less than one Venusian year, as PSP crosses R_v outbound (see Table II for specifics).

The remainder of the journey, involving a total of seven flybys, is illustrated in Fig. 8 and detailed in Table II. The table shows how much time is required in each orbit before the next flyby. Orbit f, which is in 3:7 resonance with Venus,

Table II. Properties of the 7 Venus flybys used by PSP to achieve orbit with $r_p = 0.046$ AU (orbit h). The flybys, numbered 1–7, scatter PSP by an angle $\Delta\theta_v$ (column 5) taking it from initial orbit i to final orbit f. These are labeled a–h in reference to Table I. Column 6 uses $\Delta\theta_v$ in Eq. (18) to compute the pericenter distance of the flyby, $\tilde{r}_p$, in units of the surface radius $\tilde{r}_v$. Columns 7–9 give the total time spent in the initial orbit, prior to a flyby, in units of Venusian years (P_v) spacecraft orbits (P_i) and days. The rightmost columns are a running total beginning at flyby #1.

	Orbits					Δt_i			$\sum_i \Delta t_i$	
Flyby	i	f	In/out	$\Delta\theta_v$	$\tilde{r}_p$ ($\tilde{r}_v$)	(P_v)	(P_i)	(days)	(orbits)	(days)
1	a	b	In	6.4°	1.67	—	—	—	0.0	0
2	b	c	In	7.1°	1.49	2	3	450	3.0	450
3	c	d	Out	9.1°	1.15	0.87	1.51	197	4.5	647
4	d	e	Out	7.4°	1.43	1	2	225	6.5	871
5	e	f	In	6.3°	1.70	1.06	2.33	239	8.8	1110
6	f	g	In	6.0°	1.79	3	7	675	15.9	1785
7	g	h	Out	7.7°	1.37	1.96	4.79	441	20.6	2226

consumes 3 full Venusian years (675 days) between flyby 5 and 6. Such intervals combine to a total of 2,226 days (more than 6 Earth years) between the first and last flybys.

VI. DISCUSSION

The foregoing has shown how Kepler's laws and Rutherford scattering provide constraints on the use of gravity assists by interplanetary spacecraft. We have used these tools to understand the constraints dictating the seemingly complex design of the PSP mission.[6] Taking orbits from the actual mission plan (Table II), our tools reveal how physical constraints dictated their selection. The orbits needed to lie along the elastic scattering arc in Fig. 5, and some had to fall on resonances in Fig. 7. The actual orbits satisfy both conditions at least to the accuracy of the plots. We then used our tools to map out the entire journey, as reported in Table II. Our values there differ from the actual design by a percent or so. For example, we calculate 239 days between flybys number 4 and 5, while the actual plan calls for 237 days. This 0.8% discrepancy arises from our approximations, including replacing the slightly elliptical orbit of Venus ($e = 0.007$) by

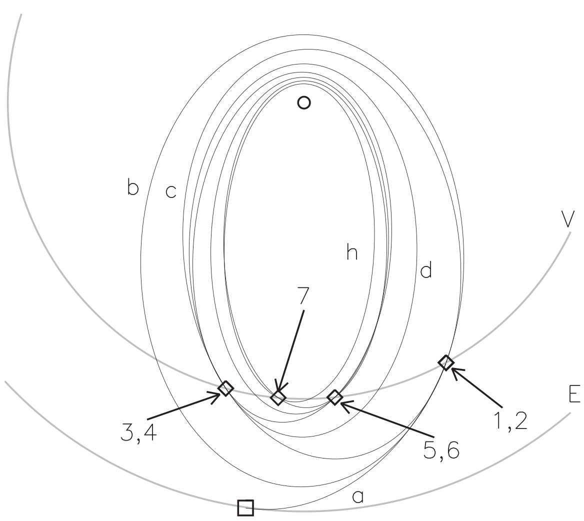

Fig. 8. The orbits of PSP viewed from above the ecliptic in a non-rotating frame. Each orbit is represented by a thin black ellipse, with several labeled. The orbits of Earth (E) and Venus (V) are shown as grey circles. Venus flybys are indicated by diamonds labeled 1–7. The ellipses are constructed using values of r_p and r_a taken from Table II. Consecutive ellipses intersect at inbound or outbound flybys according to the fourth column of Table II.

a perfect circle ($e = 0$). Accounting for an elliptical planetary orbit in Fig. 5 would be difficult, cumbersome, and would interfere with the physical intuition we gain from our simplified graphical analysis.

Among the design choices, we find that Keplerian and elastic scattering considerations, along with a practical limit on the relative velocity from Earth, $|\Delta \mathbf{v}_e| < 13$ km/s, ultimately prevent PSP from reaching closer than $r_p = 0.046$. Reaching a smaller final perihelion would require the first orbit, a, to fall on a larger semi-circle on Fig. 5—one that intersects the v_ϕ axis closer to the origin. The initial orbit, found where the larger semi-circle intersects the $r_a = 1$ AU contour (i.e., Earth), would have a perihelion smaller than $r_p = 0.2$. In order to have a smaller initial perihelion, $r_p < 0.2$, this proposed initial orbit would lie further from Earth than the square in Fig. 4. That would require a greater initial velocity change and thus more fuel or a lighter payload.

Even if it had been possible to launch PSP with greater $|\Delta \mathbf{v}_e|$, the redesigned mission would have ended up taking longer than the current 6 years. The new orbits, lying on a larger semi-circle in Fig. 5, would have a larger speed approaching Venus, $|\Delta \mathbf{v}_v| > 24$ km/s. This would, in turn, lead to a smaller maximum scattering angle according to Eq. (19). Reaching the final orbit would then require even more than seven gravity assists, requiring even more than the 6 years PSP is scheduled for. In the end, limits on both fuel and time combined with basic physical principles to dictate the ultimate target of the PSP mission.[6]

It is worth asking whether another planet, perhaps Mercury, could be used instead of Venus to produce a closer approach. According to Eq. (19), the maximum scattering angle a planet can produce depends on the speed it is approached, $|\Delta \mathbf{v}|$, and the escape speed at its surface, or above its atmosphere, $v_{\rm esc,max}$. The first will be comparable to the orbital speed of the planet, v_o, since that is used to scale the axes of the velocity-space plot. The planet's typical effectiveness in gravity assists is thus characterized by the ratio of its surface-escape-speed to its orbital speed: $\xi = v_{{\rm esc},s}/v_o$. This can be used in an expression analogous to Eq. (19),

$$\Delta\theta_{\rm char} = 2\,\sin^{-1}\left(\frac{1}{1 + 2\xi^{-2}}\right), \tag{22}$$

to define a *characteristic* scattering angle for each planet.

Using the value for Venus, $\xi = 0.30$, gives a characteristic scattering angle $\Delta\theta_{\rm char} = 5°$. We found the limit for the particular orbit of PSP to be twice that, showing the value is merely an estimate. Mercury's higher orbital speed, $v_o = 48$ km/s, and lower surface-escape-speed, $v_{\mathrm{esc},s} = 4.2$ km/s, reduce its effectiveness ratio to $\xi = 0.09$. This results in a characteristic scattering angle an order of magnitude smaller than Venus's: $\Delta\theta_{\mathrm{char}} = 0.4°$. It seems it would take an order of magnitude more Mercury flybys than Venus flybys to achieve the same result.

Jupiter, on the other hand, has a surface-escape-speed, $v_{\mathrm{esc},s} = 60$ km/s, larger than its orbital velocity, $v_o = 13$ km/s, by a factor $\xi = 4.6$. This means it is capable of scattering a spacecraft to virtually any angle. This explains why gravity assists from Jupiter are used so often for outward exploration, for example, by Pioneer 10 and New Horizons. But Jupiter is also useful for inward exploration. The Ulysses mission used a gravity assist from Jupiter to send it out of the ecliptic plane, and back toward the Sun. An earlier version of the PSP mission had planned to use Jupiter to send the spacecraft to $r_p \simeq 0.03$ AU in a single assist. Other considerations led this to be dropped in favor of the present seven-Venus-flybys plan.[6]

The calculation presented here combines Kepler's laws and Rutherford scattering into a powerful graphical tool for visualizing gravity assists. The rescaled velocity-space map in Fig. 3 can be used, with a compass, to perform similar analyses for other spacecraft using one or more gravity assists off any planet. By taking the analysis a bit further, it is possible to see how specific launch times are required to send a spacecraft to a particular flyby. These times recur with a period governed by the orbital periods of Earth and the target planet. With slightly more analysis, one can deduce the allowable range of launch times known as the launch window. The upshot is that many seemingly arcane aspects of interplanetary exploration can be understood by a physics student using concepts from an undergraduate mechanics course.

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[8]The region over which a planet's gravitational force dominates the Sun's is called its *sphere of influence*. A simple estimate is found by equating the two forces. At a distance $\tilde{r}$, a planet of mass m_p will produce an acceleration of magnitude $Gm_p/\tilde{r}^2$. This matches the acceleration at distance r from the Sun (mass M) when $\tilde{r} = r\sqrt{m_p/M}$. This turns out to be sufficiently small that we can replace $r = R_p$, the semi-major radius of the planet's orbit. More detailed analysis yields $\tilde{r} = R_p(m_p/M)^{2/5}$ (see G. R. Hintz, *Orbital Mechanics and Astrodynamics* (Springer, New York, 2015)), but this is close enough that we will quote the simpler versions: $\tilde{r} = 1.6 \times 10^5\ \mathrm{km} = 1.1 \times 10^{-3}$ AU for Venus and $\tilde{r} = 2.5 \times 10^5\ \mathrm{km} = 1.7 \times 10^{-3}$ AU for Earth. At distances greater than these, the gravitational force from that planet may be neglected. At distances far smaller, the gravitational force from the Sun may be ignored

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[12]Computing the time for a partial orbit requires, by Kepler's second law, computing the area within a wedge of a full ellipse. This is given by an integral that cannot be performed in closed form and must be calculated numerically. We perform these numerical integrals to obtain the values of Δt_i in Table II. An alternative approach would be to transform the integral to a form that can be done and then work with the resulting transcendental equation known as Kepler's equation. This obviates the need for numerical integration but requires multiple steps. (See H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA 1980), for details.)