Vector Analysis

Owen Cox

May 31, 2023

Contents

1 Sequences

1.1 SEQUENCE

Sequence ordered collection of numbers defined by function f. Usually denoted a_n.

$a_n = f_n$ known as **terms**

n is the **sequence**

1.2 CONVERGE AND DIVERGENCE

if $\lim_{x\to\infty} f(x)$ exists, then $a_n = f(n)$ converges to the same limit

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) \tag{1}$$

generally testing this is straightforward.

1.3 FUNCTION RAISED TO A FUNCTION- INCOMPLETE

$$\lim_{x\to\infty} (1 + \frac{a}{n})^n \tag{2}$$

take natural log, move exponent to coefficient, and find limit

$$n \ln(1 + \frac{a}{n}) \tag{3}$$

can be formatted as a fraction for L'Hopital's rule

$$\frac{\ln(1 + \frac{a}{n})}{\frac{1}{n}} \tag{4}$$

$$e^n \tag{5}$$

1.4 GEOMETRIC SEQUENCE AND SERIES

for $a_n = cr^n$

1. if $|r| < 1$,
2. if $r > 1$ then $\lim_{n\to\infty} a_n = \infty$
3. if $r = 0$, $\lim_{n\to\infty} = 0$

Geometric Series

1. if $|r| > 1$, diverges
2. if $|r| < 1$, converges to $a\frac{1}{1-r}$

1.5 TRIGONOMETRIC FUNCTIONS, DIVERGENCE BY OSCILLATION

$$\sin(x), \cos(x) \tag{6}$$

diverge by **oscillation**

1.6 SQUEEZE THEOREM

$$a_n = e^{-2n}\cos(n) \tag{7}$$

squeezing $\cos(n)$;

$$-1 \leq \cos(n) \leq 1 \tag{8}$$

$$-e^{-2n} \leq -e^{2n}\cos(n) \leq -3^{-2n} \tag{9}$$

because $-e^{2n}$ approaches 0 when $\lim_{n\to\infty}$, the whole function converges

1.7 BOUNDED, MONOTONIC

Bounded, has a maximum or minimum value

Monotonic, either increasing or decreasing.

1. If increasing and bounded above, converges.
2. If decreasing and bounded below, converges.
3. If converges, bounded

To determine if a series is monotone, take derivative. This sometimes can indicate monotonicity.

2 Series

2.1 SERIES

Series, adding every term in a sequence.

$$\sum_{n=1}^{\infty} a_n$$

S_n corresponds to a_n, but rather than that place in the sequence, it is the sum of every previous term and that term in the series.

Some have infinite sums were the answer approaches a value.

$$S_n = \sum_{n=1}^{\infty} 0.1^n \to 0.11111\ldots \to \frac{1}{9} \tag{10}$$

Series can sometimes be written as sequences.

$$S_n = \sum_{n=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{n}{2n+1} \tag{11}$$

If the sequence of sums, S_n diverges, then the series diverges.

Intuition can be deceiving

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{12}$$

Diverges

2.2 SUM DOES NOT START AT N=0

$$\sum_{k=2}^{\infty} ar^k \tag{13}$$

equivalent to

$$\sum_{k=0}^{\infty} ar^{k+2} \tag{14}$$

2.3 TELESCOPING SERIES

A **telescoping series** is one in which the terms cancel. One is often left with an initial term and a final term that has not yet canceled. This latter will have n in it. not so if n goes to infinity.

Some things don't look like they're telescoping but require partial fractions

2.4 DIVERGE AND INTEGRAL TEST

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_k = 0$. Equivalently if $\lim_{n\to\infty} a_n \neq 0$, series diverges.

If $\lim_{n\to\infty} \to 0$, inconclusive. If $\lim_{n\to\infty} = a \in \mathbb{R}$, converges

Integral Test let $a_n = f_n$, where f is positive, decreasing, and continuous. If $\int f(n)$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

Ex.

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{15}$$

make sure positive, decreasing, continuous

$$\int_1^{\infty} \frac{1}{n} \tag{16}$$

$$\ln(n)|_1^{\infty} \tag{17}$$

$$\ln(\infty) - \ln(1) = \infty - 0 \tag{18}$$

diverges.

2.5 CONVERGENCE OF THE P-SERIES

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{k^p} \tag{19}$$

converges when $p > 1$ and diverges when $p \leq 1$

2.6 COMPARISON TESTS

Assume there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq M$. Past a certain value, M, b is forever greater than a.

1. if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ also diverges

2.7 LIMIT COMPARISON

Let a_n, b_n be positive sequences. Assume the following limit exists:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} \tag{20}$$

1. if $L > 0$ then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.
2. if $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.
3. if $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

2.8 ALTERNATING SERIES AND ABSOLUTE CONVERGENCE

Alternating Series, a series whose terms switch between positive and negative

$$\sum_{n=1}^{\infty}(-1)^n b_n, \text{ or } \sum_{n=1}^{\infty}(-1)^{n+1} b_n \tag{21}$$

for $b_n \geq 0$

Absolute Convergence The series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

If a_n has absolute convergence, a_n converges

alternating series test if b_n is a positive sequence that is decreasing and converges to 0, then

$$S = \sum_{n=1}^{\infty}(-1)^{n-1} b_n \tag{22}$$

converges.

Furthermore,

$$0 < S < b_1, \text{ and } S_{2N} < S < S_{2N}, N \geq 1 \tag{23}$$

2.9 CONDITIONAL CONVERGENCE

An infinite series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

2.10 RATIO AND ROOT TESTS

Ratio Test * good for factorials

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \tag{24}$$

1. if $\rho < 1, \sum a_n$ converges absolutely
2. if $\rho > 1, \sum a_n$ diverges
3. if $\rho = 1$, the test is inconclusive

Root Test

$$L = \lim_{n\to\infty} \sqrt[k]{|a_n|} \tag{25}$$

1. if $L < 1, \sum a_n$ converges absolutely
2. if $L > 1, \sum a_n$ diverges
3. if $L = 1$, the test is inconclusive

2.11 CONVERGENCE AND FUNCTION APPROXIMATION. POWER SERIES

Power series can sometimes be approximated by the less intensive solution to geometric series with $|r| < 1$

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^k \tag{26}$$

For

$$\sum_{n=0}^{\infty} (ax - b)^n \tag{27}$$

This approximation converges only when $|ax - b| < 1$. Solving this and finding mid point can find the radius of convergence

Not all power series have the form of a geometric series. Coefficients c_k might not be the same for all terms. The **Ratio Test** is often useful in these situations.

$$a_k = \sum_{k=1}^{\infty} \frac{(x-7)^k}{k} \tag{28}$$

evaluate:

$$L = \lim_{n\to\infty} |\frac{a_{k+1}}{a_k}| \tag{29}$$

1. if $L < 1$, convergence
2. if $L > 1$, divergence
3. if $L = 1$, inconclusive

The values that make $L = 1$ must be evaluated to find specific character

2.12 TAYLOR SERIES

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{30}$$

A Maclaurin series is a Taylor series centered at $x = 0$

3 Polar, Cylindrical and Spherical Coordinates

In a rectangular coordinate system: a point at (x,y) has length r and is above the horizontal axis at θ

x and y can be represented

$$x = r\cos\theta, y = r\sin\theta \tag{31}$$

and

$$r^2 = x^2 + y^2 \tag{32}$$

$$\tan\theta = \frac{y}{x} \tag{33}$$

3.1 GENERAL POLAR EQUATION FORMS

cardioid

$$r = a(1 \pm \cos\theta) \tag{34}$$

$$r = a(1 \pm \sin\theta) \tag{35}$$

rose

$$r = a\cos(b\theta), r = a\sin(b\theta) \tag{36}$$

3.2 CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

Cylindrical Coordinate Systems, $P(x, y, z)$ is represented by $P(r, \theta, z)$. if

$$z = r \tag{37}$$

A cone has been formed.

Spherical Coordinates $P(\rho, \theta, \phi)$

1. ρ is the distance between P and origin.
2. θ is the angle used in cylindrical or polar coordinates
3. ϕ is the angle between the z axis and the line segment OP, where is the origin and $0 \leq \phi \leq \pi$

3.3 BETWEEN SPHERICAL, CYLINDRICAL, AND RECTANGULAR COORDINATES

$$x = \rho\sin\phi\cos\theta \tag{38}$$

$$y = \rho\sin\phi\sin\theta \tag{39}$$

$$z = \rho\cos\phi \tag{40}$$

$$\sqrt{x^2 + y^2} = \rho\sin\phi \tag{41}$$

and

$$\rho^2 = x^2 + y^2 + z^2 \tag{42}$$

$$\tan\theta = \frac{y}{x} \tag{43}$$

$$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \tag{44}$$

Relationship between cylindrical and spherical coordinates

$$r = \rho\sin\phi \tag{45}$$

$$\theta = \theta \tag{46}$$

$$z = \rho\cos\phi \tag{47}$$

and

$$\rho = \sqrt{r^2+z^2} \tag{48}$$

$$\phi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right) \tag{49}$$

4 Vectors

$$r(t) = f(t)i + g(t)j = \langle f(t), g(t)\rangle \tag{50}$$

This makes a function. Following the vector across t makes its own curve.

Initial point: (x_0, y_0).

Terminal point: (x_1, y_1)

A vector is in **standard position** if the initial point is at the origin When graphing we usually graph vectors in the domain of the function in standard position

A **plane curve** is created by a function of $\hat{i}, \hat{j}$

A **space curve** is created by a function of $\hat{i}, \hat{j}, \hat{k}$

4.1 PARAMETRIZATION PARAMETRIC

Line segment

$$< x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1) > \tag{51}$$

Circle

$$< r\cos t, r\sin t > \tag{52}$$

4.2 DIFFERENTIATING AND OPERATING ON VECTOR VALUED FUNCTIONS

Differentiating a vector value function at a point gives a tangent vector at that point

$$r(t) = f(t)i + g(t)j \tag{53}$$

$$r'(t) = f'(t)i + g'(t)j \tag{54}$$

dot product: given (x_1, y_1) and (x_2, y_2)

$$u\dot{v} = x_1x_2 + y_1y_2 \tag{55}$$

cross product: say $u_1 = (a_1, a_2, a_3)$, and $v_1 = (b_1, b_2, b_3)$. Magnitude of cross product is area bounded by vectors.

$$\begin{bmatrix} i & j & k \\ a1 & a2 & a3 \\ b1 & b2 & b3 \end{bmatrix} \tag{56}$$

$$u \times v = (a2b3 - b2a3)i - (a1b3 - b1a3)j + (a1b2 - b1a2)k \tag{57}$$

4.3 LENGTH OF A VECTOR VALUED FUNCTION

fr $r(t) = f(t)i + g(t)j \cdots + z(t)z$

$$\int \sqrt{f'(t)^2 + g'(t)^2 \cdots + z'(t)^2}dt = \int ||r'(t)||dt \tag{58}$$

Arclength Function

$$s(t) = \int ||r'(u)||du \tag{59}$$

$$\frac{ds}{dt} \tag{60}$$

4.4 UNIT TANGENT VECTOR

$$T(t) = \frac{r'(t)}{||r'(t)||} \tag{61}$$

4.5 CURVATURE

often measured in relation to **radius of curvature.** If a circle where overlaid at that point, what would the radius be to match the curve.

$$\kappa = \frac{||T'(t)||}{||r'(t)||} = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}} = ||\frac{dT}{ds}|| = ||T'(s)|| \tag{62}$$

$$\kappa = \frac{||r'(x) \times r''(x)||}{||r'(x)||} \tag{63}$$

4.6 PRINCIPAL UNIT NORMAL VECTOR AND BINORMAL VECTOR

Principal Unit Normal Vector: vector of length one perpendicular to curve at a point.

$$N(t) = \frac{T'(t)}{||T'(t)||} \tag{64}$$

Binormal Vector is orthogonal to T and N

$$T(t) \times N(t) \tag{65}$$

$$||B|| = ||T \times N|| = ||T||||N|| \sin\theta = 1 \tag{66}$$

4.7 ACCELERATION

$$a(t) = v'(t) \cdot T(t) + [v(t)]^2 \cdot \kappa \cdot N(t) \tag{67}$$

5 Integrals

for f continuous on a rectangular region $a \leq x \leq b$ and $c \leq y \leq d$. Either order of standard double integration will work

$$\iint f(x,y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x,y) \tag{68}$$

$$\text{Volume} = \int A(x)dx \tag{69}$$

with $A(x)$ being a function for area, $A(x) = \int f(x)dy$

5.1 NON-RECTANGULAR

Some shapes are bounded by two functions. If they are functions of x, initially integrate with respect to y, with the functions as the bounds of the integral.

Let R be a regin bounded below and above by the graphs of their continuous functions $y = g(x)$ and $y = h(x)$, and by the lines a=x and x=b. If f is continuous of R, then

$$\iint f(x,y)dA = \iint f(x,y)dxdy \tag{70}$$

5.2 2 FUNCTIONS

$$\iint g(x,y) - f(x,y)A \tag{71}$$

use the intersection of these forms projected on xy axis as the bounds of integrals.

5.3 INTEGRATION WITH POLAR COORDINATES

$R = (r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta$

$$\iint f(r,\theta)dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r,\theta) \cdot rdrd\theta \tag{72}$$

where $f(r,\theta)$ is z

This can be used for overlapping circles and a lot of other things. The main difference being the bounds of the integrals. One may have to ad further integrated integrals

5.4 TRIPLE INTEGRALS

$$\iiint f(x,y,z)dV \tag{73}$$

5.5 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

$$\Delta V = \rho^2 \sin\phi \cdot d\rho \cdot d\phi \cdot d\theta \tag{74}$$

$$\iiint = f(\rho,\theta,\phi)\rho^2 \sin\phi \tag{75}$$

6 Vector Fields

insane. Can be used to model all sorts of fields.

A vector field F in $\mathbb{R}^n$ is an assignment of an n dimensional vector $F(x,y,etc)$ t each point of a subset D in $\mathbb{R}$

A vector field is **continuous** if both components are continuous.

two kinds of vector fields. In a **radial field** all vectors either point toward or away from the origin

Rotational field is tangent t a circle with radius $r = \sqrt{x^2 + y^2}$. **Dot Product** is zero.

a **Unit vector field** is a field in which every vector has magnitude 1.

6.1 NORMALIZING A VECTOR FIELD

$$F = < P, Q, R > \tag{76}$$

unit field:

$$\frac{F}{||F||} \tag{77}$$

6.2 GRADIENT

$$\text{grad} f = \nabla f =< f'x, f'y > \tag{78}$$

A field is a **gradient field** or **conservative vector field** if there is a single scalar function f such that $\nabla f = F$ f must be a function where if differentiated for each component $(x, y, etc,)$, it yields the components of F in accordance to variable differentiated for.

f is called a **potential function**

Cross Partial Property of Conservative Vector fields

If $F(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

This can be used to show a field is conservative, not vice versa.

6.3 VECTOR LINE INTEGRAL

the **Vector Line Integral** of a vector field F along an oriented smooth curve C is

$$\int_C F \cdot Tds = F(r(t)) \cdot \frac{r'(t)}{||r'(t)||} \cdot ||r'(t)||dt \tag{79}$$

$$\int_C = F \cdot Tds = \int_a^b F(r(t)) \cdot r'(t)dr \tag{80}$$

$$\int_C F \cdot Tds = \int_C F \cdot dr \tag{81}$$

Piecewise Smooth Function a function made of a finite number of smooth curves

$$\sum_{m=1}^{n} \int_{C_m} F \cdot ds \tag{82}$$

6.4 FLUX

$$\int_C F \cdot Nds \tag{83}$$

$$\int_C F(r(t)) \cdot n(t)dt \tag{84}$$

All these variables are vectors

$$n =< y', -x' > \tag{85}$$

6.5 CIRCULATION

Circulation of F along C: line integral of F along oriented **closed** curve

$$\oint_C F \cdot T ds \tag{86}$$

$$\int F(r(t)) \cdot r'(t) \tag{87}$$

Simple Curves do not cross themselves.

A region D is a **connected region** for any two points if there is a path where the trace is entirely within D. A region is **simply connected** if you can shrink it to a straight line. If there is a hole/excepted area within the region it is not simply connected.

6.6 FUNDAMENTAL THEOREM FOR LINE INTEGRALS

C must be a piecewise smooth curve

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a)) \tag{88}$$

Gradient fields are **path independent**

6.7 GREEN'S THEOREM

Let D be an open, simply connected region with a boundary curve C that is piecewise smooth, simple closed curve oriented counterclockwise. Only for 2 dimensional vector fields.

$$\oint_C F \cdot dr = \oint_c Pdx + Qdx = \iint_D (Q_x - P_y) dA = \int_C = F \cdot Tds \tag{89}$$

$$\iint_D (Q_X - P_y) dA \tag{90}$$

Given an equation that satisfies, identify P, which is with dx, and Q, which is with dy, and then put them in the form. If it is going clockwise, make it negative.

If $Q_x - P_y$=1, dA is integrated and is equal to the initial integral.

6.8 PARAMETRIZE AN ELLIPSE

$$< a \cos t, b \sin t > \tag{91}$$

with a as top radius in x, b top radius in y

6.9 FLUX FORM OF GREEN'S THEOREM

Let D be an open, simply connected region with a boundary curve C that is piecewise smooth, simple closed curve oriented counterclockwise. Only for 2 dimensional vector fields.

$$\oint_C F \cdot N ds = \iint_D (P_x + Q_y) dA \tag{92}$$

6.10 HARMONIC FUNCTIONS

A **source free vector field** is a conservative field but with flux instead.

Conservative and source free vector fields on simply connected domain: any potential function satisfied Laplace's Equation: $f_{xx} + f_{yy} = 0$. f is a harmonic function.

6.11 NON SIMPLY CONNECTED REGIONS

Split the integrals up until they are simply connected

6.12 DIVERGENCE

Divergence measures the 'genesis' of a certain point, $divF(x, y)$. Divergence is negative if it 'flows in', and positive if its generative

6.13 GRADIENT OPERATOR

$\nabla =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$ (93)

$$\text{div}F = \nabla \cdot F \tag{94}$$

Let $F =< P, Q >$ be a simply connected vector field.

$$\text{div}F = 0 \tag{95}$$

iff F is source-free

6.14 CURL

for $F =< P, Q, R >$, a vector field whose component derivatives all exist,

$$\text{curl}F = \nabla \times F \tag{96}$$

$$\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \tag{97}$$

$$\text{div (curl (F))} = 0 \tag{98}$$

for a conservative vector field curl$F = 0$

divergence of a gradient is 0