Honors by Contract HW - Owen Queen

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February 13, 2021

1 Question 1

Let X be a uniform random variable with distribution function $p_X(x) = \frac{1}{b-a}$ for $a \leq x \leq b$. Let $\{X_1, ..., X_n\}$ be a random sample of X and let $\{x_1, ..., x_n\}$ be the observed values from the sample. Using the method of moments, prove that estimators a_e and b_e where

$$a_e = \bar{x} - \sqrt{\frac{3\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

$$b_e = \bar{x} + \sqrt{\frac{3\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

are estimators for a and b.

1.1 Proof

Let X be a uniform random variable with distribution function $p_X(x) = \frac{1}{b-a}$ for $a \leq x \leq b$. Let $\{X_1, ..., X_n\}$ be a random sample of X and let $\{x_1, ..., x_n\}$ be the observed values from the sample. Let us compute the first and second moments for X. Using the method of moments, we will set $E(X) = \frac{\sum\limits_{i=1}^{n} x_i}{n}$ and $E(X^2) = \frac{\sum\limits_{i=1}^{n} x_i^2}{n}$:

$$E(X) = \frac{a+b}{2} = \frac{\sum\limits_{i=1}^{n} x_i}{n} \tag{1}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

Therefore,

$$E(X^2) = Var(X) + [E(X)]^2$$

$$E(X^2) = \frac{(b-a)^2}{12} + \left(\frac{a+b}{2}\right)^2 = \frac{\sum\limits_{i=1}^{n} x_i^2}{n} \tag{2}$$

Let $\frac{\sum\limits_{i=1}^{n} x_i}{n} = \bar{x}$ and $\frac{\sum\limits_{i=1}^{n} x_i^2}{n} = \bar{\psi}$.

We will now use the method of moments to solve for a and b. We know that since $\bar{x} = \frac{a+b}{2}$, then $b = 2x - a$. Thus,

$$\bar{\psi} = \frac{[(2\bar{x} - a) - a]^2}{12} + \left(\frac{[a + (2\bar{x} - a)]}{2}\right)^2$$

$$\bar{\psi} = \frac{(\bar{x} - a)^2}{3} + \bar{x}^2$$

$$3\left(\bar{\psi} - \bar{x}^2\right) = (\bar{x} - a)^2$$

$$(\bar{x} - a) = \sqrt{3\left(\bar{\psi} - \bar{x}^2\right)}$$

Therefore, we have the estimate for a:

$$a_e = \bar{x} - \sqrt{3\left(\bar{\psi} - \bar{x}^2\right)} \tag{3}$$

A similar process follows for b. We know that $a = 2x - b$, and if we substitute it in 2:

$$\bar{\psi} = \frac{[b - (2\bar{x} - b)]^2}{12} + \left(\frac{[(2\bar{x} - b) + b]}{2}\right)^2$$

$$\bar{\psi} = \frac{(b - \bar{x})^2}{3} + \bar{x}^2$$

$$3\left(\bar{\psi} - \bar{x}^2\right) = (b - \bar{x})^2$$

$$(b - \bar{x}) = \sqrt{3\left(\bar{\psi} - \bar{x}^2\right)}$$

Therefore,

$$b_e = \bar{x} + \sqrt{3\left(\bar{\psi} - \bar{x}^2\right)} \tag{4}$$

Substituting for $\bar{\psi}$ into **??**eq:3 and 4, we obtain:

$$a_e = \bar{x} - \sqrt{3\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2\right)}$$

$$b_e = \bar{x} + \sqrt{3\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2\right)}$$

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2 Question 2

Let X be a uniform random variable on the interval $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$. Given a sample $X_1, ..., X_n$ with values $x_1, ..., x_n$, find maximum likelihood estimates and corresponding estimators, for θ.

2.1 Proof

Let X be a uniform random variable on the interval $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ with a sample $X_1, ..., X_n$. Values from the sample are $x_1, ..., x_n$. We know that X has a distribution function:

$$f_X(x;\theta) = \frac{1}{(\theta + \frac{1}{2}) - \theta + \frac{1}{2}} = 1, x \in \left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$$

and

$$f_X(x;\theta) = 0, x \notin \left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$$

The likelihood function of θ with respect to the sample is:

$$L(\theta) = \prod_{i=1}^{n} f_X(x;\theta)$$

for every $x_i \in \left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$. Thus, $L(\theta) = 1$ if every value from the sample x_i is in interval $\left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$, but : $L(\theta) = 0$ if there is any x_i not in this interval. Therefore, $L(\theta)$ is maximized whenever, for every x_i, $x_i \in \left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$. Maximizing this function is equivalent to maximizing the probability that every x_i falls into this range, i.e. maximizing $P(\theta - \frac{1}{2} \leq x_i \leq \theta + \frac{1}{2})$.

This problem is equivalent to the formulation of finding the probability that $x_i \leq \theta + \frac{1}{2}$ and $\theta - \frac{1}{2} \leq x_i$. Therefore, we can also say that θ must lie in the bounds:

$$max(x_i) \leq \theta + \frac{1}{2}$$

$$max(x_i) - \frac{1}{2} \leq \theta$$

and

$$\theta - \frac{1}{2} \leq min(x_i)$$

$$\theta \leq min(x_i) + \frac{1}{2}$$

Thus, we know that for all θ_e's (the maximum likelihood estimations), they must be real numbers in the range of:

$$max(x_i) - \frac{1}{2} \leq \theta_e \leq min(x_i) + \frac{1}{2}$$

We can generalize this for the maximum likelihood estimators:

$$max(X_1, ..., X_n) - \frac{1}{2} \leq \hat{\theta_e} \leq min(X_1, ..., X_n) + \frac{1}{2}$$

for any sample of size n.

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