HW5 MATH 423

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1 Example 3.10.6

Let Y_1, Y_2, Y_3 be a random sample of size $n = 3$ from a uniform pdf defined over the unit interval, $f_Y(y) = 1$, $0 \le y \le 1$. By definition, the range R is:

R = range = $Y_{max} - Y_{min} = Y_3' - Y_1'$

We are to find $f_R(r)$, the pdf for the range:

First, find the joint pdf of Y_1' and Y_3'. Then integrate $f_{Y_1',Y_3'}(u, v)$ is integrated over the region $Y_3' - Y_1' \le r$ to find the cdf, $F_R(r) = P(R \le r)$. Then we differentiate this cdf to obtain the pdf (by theorem of pdf and cdf).

$f_Y(y) = 1, 0 \le y \le 1$ implies:

$$F_Y(y) = \begin{cases} 0, y < 0 \\ y, 0 \le y \le 1 \\ 1, y > 1 \end{cases}$$

Applying Equation 3.10.5:

We have $n = 3$, $i = 1$, $j = 3$.This gives joint pdf of Y_1' and Y_3'.

$$f_{Y_1',Y_3'}(u, v) = \frac{3!}{0!1!0!} u^0 (v - u)^1 (1 - v)^0 \cdot 1 \cdot 1 = 6(v - u), 0 \le u < v \le 1$$

We can write the cdf for R in terms of Y_1' and Y_3':

$$F_R(r) = P(R \le r) = P(Y_3' - Y_1' \le r) = P(Y_3' \le Y_1' + r)$$

Integrate the joint pdf of Y_1' and Y_3' over this region:

$$F_R(r) = P(R \le r) = \int_0^{1-r} \int_u^{u+r} 6(u - v)dvdu + \int_{1-r}^1 \int_u^1 6(v - u)dvdu$$

$$\int_0^{1-r} \int_u^{u+r} 6(u - v)dvdu = 3r^2 - 3r^3$$

$$\int_{1-r}^1 \int_u^1 6(v - u)dvdu = r^3$$

This implies:

$$F_R(r) = 3r^2 - 3r^3 + r^3 = 3r^2 - 2r^3$$

Therefore,

$$\frac{d}{dr}F_R(r) = f_R(r) = 6r - 6r^2, 0 \le r \le 1$$

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2 Example 3.12.4

The pdf of the normal distribution is:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} exp\Bigg[-\frac{1}{2}\Big(\frac{y-\mu}{\sigma}\Big)^2\Bigg], -\infty < y < \infty$$

where $\mu = E(Y)$ and $\sigma^2 = Var(Y)$. We are to derive the moment-generating function for this distribution:

Since Y is a continuous random variable,

$$M_Y(t) = E(e^{tY}) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{ty} exp\Bigg[-\frac{1}{2}\Big(\frac{y-\mu}{\sigma}\Big)^2\Bigg] dy$$

The computation of this integral can be quite complicated, and it requires completing the square of the numerator.

$$e^{ty} exp\Bigg[-\frac{1}{2}\Big(\frac{y-\mu}{\sigma}\Big)^2\Bigg] = exp\Bigg[-\frac{y^2 - 2\mu y - 2\sigma^2 ty + \mu^2}{2\sigma^2}\Bigg]$$

$$y^2 - 2\mu y - 2\sigma^2 ty + \mu^2 = \big[y - (\mu + \sigma^2 t)\big]^2 - \sigma^4 t^2 + 2\mu t\sigma^2$$

$$\frac{\big[y - (\mu + \sigma^2 t)\big]^2 - \sigma^4 t^2 + 2\mu t\sigma^2}{2\sigma^2} = \mu t + \frac{\sigma^2 t^2}{2} - \frac{1}{2}\Big[\frac{y - (\mu + t\sigma^2)}{\sigma}\Big]^2$$

Thus,

$$M_Y(t) = exp\Bigg(\mu t + \frac{\sigma^2 t^2}{2}\Bigg)\Bigg(\frac{1}{\sqrt{2\pi}\sigma}\Bigg)\int_{-\infty}^{\infty} exp\Bigg[-\frac{1}{2}\Big[\frac{y - (\mu + t\sigma^2)}{\sigma}\Big]^2\Bigg] dy$$

Let us define defining a new normal distribution Y' with $Var(Y') = \sigma^2$ and $E(Y') = \mu + t\sigma^2$. Therefore, our last two factors in the above equation constitute the pdf for this new distribution, so by definition of pdf, we have:

$$\Bigg(\frac{1}{\sqrt{2\pi}\sigma}\Bigg)\int_{-\infty}^{\infty} exp\Bigg[-\frac{1}{2}\Big[\frac{y - (\mu + t\sigma^2)}{\sigma}\Big]^2\Bigg] dy = 1$$

Therefore, we have:

$$M_Y(t) = exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

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