Central Limit Theorem Proof

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1 Central Limit Theorem

Let $W_1, W_2, ...$ be an infinite sequence of indpendent random variables, each with the same distribution. Suppose the mean μ and variance σ^2 of $f_W(w)$ are both finite. Then, for any numbers a, b:

$$\lim_{n\to\infty} P\left(a \leq \frac{W_1 + ... + W_n - n\mu}{\sqrt{n}\sigma} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{\frac{-z^2}{2}} dz$$

1.0.1 Lemma:

Let $W_1, W_2, ...$ be a set of random variables such that $\lim_{n\to\infty} M_{W_n}(t) = M_W(t)$ for all t in some interval about 0. Then $\lim_{n\to\infty} F_{W_n}(w) = F_W(w)$ for all $-\infty < w < \infty$.

1.1 Proof:

Let $W_1, W_2, ...$ be an infinite sequence of independent random variables, each with the same distribution $f_W(w)$. Suppose $E(W_i) = \mu$, $Var(W_i) = \sigma^2$ for all $i \in \mathbb{N}$, and that both of these values are finite.

For each of notation, let:

$$\frac{W_1 + ... + W_n - n\mu}{\sqrt{n}\sigma} = \frac{S_1 + ... + S_n}{\sqrt{n}}$$

Where $S_i = \frac{W_i - \mu}{\sigma}$. Therefore, $E(W_i) = \mu$ implies $E(S_i) = 0$ and $Var(W_i) = \sigma^2$ implies $Var(S_i) = 1$.

By Theorem 3.12.3,

$$M_{\frac{S_1+...+S_n}{\sqrt{n}}}(t) = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]\left[M\left(\frac{t}{\sqrt{n}}\right)\right]... = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

Using the expected values and variance for S_i's, the moment generating function (MGF) for S_i's is gives:

$$M(0) = 1$$

$$M^{(1)}(0) = E(S_i) = 0$$
$$M^{(2)}(0) = E(S_i^2) = Var(S_i) + [E(S_i)]^2 = Var(S_i) = 1$$

By Taylor's Theorem, we can expand $M(t)$:

$$M(t) = \frac{M(0)t^0}{0!} + \frac{M^{(1)}(0)t^1}{1!} + \frac{M^{(2)}(t)t^2}{2!} = 1 + \frac{M^{(2)}(r)t^2}{2}$$

for some r such that $|r| < |t|$. Therefore,

$$\lim_{n\to\infty} \left[M\Big(\frac{t}{\sqrt{n}}\Big) \right]^n = \lim_{n\to\infty} \left[1 + \frac{M^{(2)}(r)t^2}{2} \right]^n$$

where $|s| < \frac{|t|}{\sqrt{n}}$

$$= exp\left[\lim_{n\to\infty} nln \left[1 + \frac{M^{(2)}(r)t^2}{2} \right] \right]$$

$$= exp\left[\lim_{n\to\infty} \frac{t^2}{2} M^{(2)}(s) \frac{ln\left[1 + \frac{t^2}{2n} M^{(2)}(s)\right] - ln(1)}{\frac{t^2}{2n} M^{(2)}(s)} \right]$$

We know $M^{(3)}(t)$ exists, so $M^{(2)}(t)$ is continuous. Therefore, $\lim_{t\to 0} M^{(2)}(t) = M^{(2)}(0) = 1$. Since $|s| < \frac{|t|}{\sqrt{n}}$, $s \to 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} M^{(2)}(s) = M^{(2)}(0) = 1$.

If we let $h = \frac{t^2}{2n} M^{(2)}(s)$, then:

$$\lim_{n\to\infty} \left[M\Big(\frac{t}{\sqrt{n}}\Big) \right]^n = exp\left[\lim_{n\to\infty} \frac{t^2}{2} M^{(2)}(s) \frac{ln[1+h] - ln(1)}{h} \right]$$

As $n \to 0$, $\frac{t^2}{2n} M^{(2)}(s) \to 0$, so $h \to 0$.

$$\lim_{h\to 0} \frac{ln[1+h] - ln(1)}{h} = ln^{(1)}(1)$$

Thus,

$$lim_{n\to\infty} \left[M\Big(\frac{t}{\sqrt{n}}\Big) \right]^n = exp\Big[\frac{t^2}{2} ln^{(1)}(1)\Big] = e^{\frac{t^2}{2}}$$

This is the MGF for the standard normal random variable. Thus,

$$M\left(\frac{W_1 + ... + W_n - n\mu}{\sqrt{n}\sigma} \right) = M(\mathcal{N}(0,1))$$

And it follows from this that:

$$\lim_{n\to\infty} P\left(a \leq \frac{W_1 + ... + W_n - n\mu}{\sqrt{n}\sigma} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{\frac{-z^2}{2}} dz$$

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