HW 3 MATH423 - Probability

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1 Question 1

Theorem: Suppose $A_1, A_2, ..., A_n$ are n events in the sample space S. Then, it follows from:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Then we have:

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) + (-1)^1 \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} P(A_i \cap A_j) +$$

$$(-1)^2 \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n-2} P(A_i \cap A_j \cap A_k) + ... + (-1)^{n-1} P(A_1 \cap A_2 \cap ... \cap A_n)$$

1.1 Proof:

Suppose $A_1, A_2, ..., A_n$ are n events in the sample space S. We know that $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. Thus, we will compute our base case with 3 events:

1.1.1 Base Case:

In this case, let $n = 3$:

$$P((A_1 \cup A_2) \cup A_3) = P(A_1 \cup A_2) + P(A_3) - P((A_1 \cup A_2) \cap A_3)$$

We know: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$
Need to simplify:

$$P((A_1 \cup A_2) \cap A_3) = P((A_1 \cap A_3) \cup (A_2 \cap A_3)) = P(A_1 \cap A_3) + P(A_2 \cap A_3) - P((A_1 \cap A_3) \cap (A_2 \cap A_3))$$

Since: $(A_1 \cap A_3) \cap (A_2 \cap A_3) = (A_1 \cap A_2 \cap A_3)$,

$$P((A_1 \cap A_3) \cup (A_2 \cap A_3)) = P(A_1 \cap A_3) + P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3)$$

Thus,

$$P(A_1 \cup A_2 \cup A_3) = (P(A_1) + P(A_2) - P(A_1 \cap A_2)) + P(A_3) - \Big[P(A_1 \cap A_3) + P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3)\Big]$$

With some rearranging, this is equal to:

$$P(\bigcup_{i=1}^{3} A_i) = \sum_{i=1}^{3} P(A_i) + (-1)^1 \sum_{i=1}^{3} \sum_{j=i+1}^{2} P(A_i \cap A_j) + (-1)^2 P(A_1 \cap A_2 \cap A_3)$$

So, the theorem holds for the case of n = 3.

1.1.2 Induction Step:

Suppose the formula holds for some $m \in \mathbb{Z}$. Then we have, by basic probability theorem:

$$P(\bigcup_{i=1}^{m+1} A_i) = P(\bigcup_{i=1}^{m} A_i \cup A_{m+1}) = P(\bigcup_{i=1}^{m} A_i) + P(A_{m+1}) - P(\bigcup_{i=1}^{m} A_i \cap A_{m+1})$$

By the inductive hypothesis, we have that: $P(\bigcup_{i=1}^{m} A_i) = \sum_{i=1}^{m} P(A_i) + (-1)^1 \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P(A_i \cap A_j) + ... + (-1)^{m-1} P(A_1 \cap A_2 \cap ... \cap A_m)$. Also note that since intersections distribute across unions:

$$P(\bigcup_{i=1}^{m} A_i \cap A_{m+1}) = P(\bigcup_{i=1}^{m} (A_i \cap A_{m+1}))$$

Thus, our formula thus far simplifies to:

$$P(\bigcup_{i=1}^{m+1} A_i) = \left[\sum_{i=1}^{m} P(A_i) + (-1)^1 \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P(A_i \cap A_j) + ... + (-1)^{m-1} P(A_1 \cap A_2 \cap ... \cap A_n)\right]$$

$$+P(A_{m+1}) - P(\bigcup_{i=1}^{m} (A_i \cap A_{m+1}))$$

Incorporating the $P(A_{m+1})$ term into our sum, we obtain:

$$P(\bigcup_{i=1}^{m+1} A_i) = \left[\sum_{i=1}^{m+1} P(A_i) + (-1)^1 \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P(A_i \cap A_j) + ... + (-1)^{m-1} P(A_1 \cap A_2 \cap ... \cap A_m)\right]$$

$$-P(\bigcup_{i=1}^{m} (A_i \cap A_{m+1}))$$

Let us examine the $P(\bigcup_{i=1}^{m}(A_i \cap A_{m+1}))$ term. Let $B_i = A_i \cap A_{m+1}$. Then, by the induction hypothesis from the formula for $P(\bigcup_{i=1}^{m} A_i)$:

$$P(\bigcup_{i=1}^{m} B_i) = \sum_{i=1}^{m} P(B_i) + (-1)^1 \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P(B_i \cap B_j) + ... + (-1)^{m-1} P(B_1 \cap B_2 \cap ... \cap B_m)$$

We will replace our B_i terms with $A_i \cap A_{m+1}$ one-by-one:

$$\sum_{i=1}^{m} P(B_i) = \sum_{i=1}^{m} P(A_i \cap A_{m+1})$$

$$\sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P(B_i \cap B_j) = \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P((A_i \cap A_{m+1}) \cap (A_j \cap A_{m+1}))$$

Since $(A_i \cap A_{m+1}) \cap (A_j \cap A_{m+1}) = (A_i \cap A_j \cap A_{m+1})$,

$$\sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P(B_i \cap B_j) = \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P(A_i \cap A_j \cap A_{m+1})$$

$$\sum_{i=1}^{m} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m-2} P(B_i \cap B_j \cap B_k) = \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m-2} P((A_i \cap A_{m+1}) \cap (A_j \cap A_{m+1}) \cap (A_k \cap A_{m+1}))$$

Since $(A_i \cap A_{m+1}) \cap (A_j \cap A_{m+1}) \cap (A_k \cap A_{m+1}) = (A_i \cap A_j \cap A_k \cap A_{m+1})$,

$$\sum_{i=1}^{m} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m-2} P(B_i \cap B_j \cap B_k) = \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m-2} P((A_i \cap A_j \cap A_k \cap A_{m+1}))$$

This pattern will continue for every other term, with the repetitive A_{m+1} terms condensing down to one intersection for each term within the sums. Let us consider the final term for completion:

$$P(B_1 \cap B_2 \cap ... \cap B_m) = P((A_1 \cap A_{m+1}) \cap (A_2 \cap A_{m+1}) \cap ... \cap (A_m \cap A_{m+1})$$

$$= P(A_1 \cap A_2 \cap ... \cap A_m \cap A_{m+1})$$

Therefore, we can go back to our original $P(\bigcup_{i=1}^{m}(A_i \cap A_{m+1}))$ formula:

$$P(\bigcup_{i=1}^{m}(A_i \cap A_{m+1})) = \sum_{i=1}^{m} P(A_i \cap A_{m+1}) + (-1)^1 \sum_{i=1}^{m} \sum_{j=i+1}^{m-1} P(A_i \cap A_j \cap A_{m+1})$$

$$+(-1)^2\sum_{i=1}^{m}\sum_{j=i+1}^{m-1}\sum_{k=j+1}^{m-2}P((A_i\cap A_j\cap A_k\cap A_{m+1}))+...+(-1)^{m-1}P(A_1\cap A_2\cap...\cap A_m\cap A_{m+1})$$

Therefore, if we multiply this by -1 before adding back to the original equation:

$$-P(\bigcup_{i=1}^{m}(A_i\cap A_{m+1}))=(-1)^1\sum_{i=1}^{m}P(A_i\cap A_{m+1})+(-1)^2\sum_{i=1}^{m}\sum_{j=i+1}^{m-1}P(A_i\cap A_j\cap A_{m+1})$$

$$+(-1)^3\sum_{i=1}^{m}\sum_{j=i+1}^{m-1}\sum_{k=j+1}^{m-2}P((A_i\cap A_j\cap A_k\cap A_{m+1}))+...+(-1)^{m}P(A_1\cap A_2\cap...\cap A_m\cap A_{m+1})$$

Now we can add this back into our original equation. Recall:

$$P(\bigcup_{i=1}^{m+1}A_i)=P(\bigcup_{i=1}^{m}A_i)+P(A_{m+1})-P(\bigcup_{i=1}^{m}A_i\cap A_{m+1})$$

Then,

$$P(\bigcup_{i=1}^{m+1}A_i)=\left[\sum_{i=1}^{m}P(A_i)+(-1)^1\sum_{i=1}^{m}\sum_{j=i+1}^{m-1}P(A_i\cap A_j)+...+(-1)^{m-1}P(A_1\cap A_2\cap...\cap A_m)\right]$$

$$+\left[(-1)^1\sum_{i=1}^{m}P(A_i\cap A_{m+1})+(-1)^2\sum_{i=1}^{m}\sum_{j=i+1}^{m-1}P(A_i\cap A_j\cap A_{m+1})\right.$$

$$\left.+(-1)^3\sum_{i=1}^{m}\sum_{j=i+1}^{m-1}\sum_{k=j+1}^{m-2}P((A_i\cap A_j\cap A_k\cap A_{m+1}))+...+(-1)^{m}P(A_1\cap A_2\cap...\cap A_m\cap A_{m+1})\right]$$

Let us consider two similar terms from each of the formulas in brackets. Let $G=(-1)^1\sum_{i=1}^{m}\sum_{j=i+1}^{m-1}P(A_i\cap A_j)$ and $H=(-1)^1\sum_{i=1}^{m}P(A_i\cap A_{m+1})$:

$$G=(-1)\left[\Big[P(A_1\cap A_2)+...+P(A_1\cap A_m)\Big]+\Big[P(A_2\cap A_3)+...+P(A_2\cap A_m)\Big]+...+P(A_{m-1}\cap A_m)\right]$$

$$H=(-1)\Big[P(A_1\cap A_{m+1})+...+P(A_m\cap A_{m+1})\Big]$$

Thus, if we add G and H, we can combine these terms:

$$G+H=(-1)\Bigg[\Big[P(A_1\cap A_2)+...+P(A_1\cap A_m)+P(A_1\cap A_{m+1}\Big]+$$

$$\Big[P(A_2\cap A_3)+...+P(A_2\cap A_m)+P(A_2\cap A_{m+1})\Big]+...+\Big[P(A_{m-1}\cap A_m)+P(A_{m-1}+A_{m+1})\Big]+P(A_m\cap A_{m+1})\Bigg]$$

Thus:

$$G+H=(-1)^1\sum_{i=1}^{m}\sum_{j=i+1}^{m-1}P(A_i\cap A_j)+(-1)^1\sum_{i=1}^{m}P(A_i\cap A_{m+1})=(-1)^1\sum_{i=1}^{m+1}\sum_{j=i+1}^{m}P(A_i\cap A_j)$$

You can repeat this process for every other similar term between the two bracketed sums. Therefore, our result is:

$$P(\bigcup_{i=1}^{m+1}A_i)=\sum_{i=1}^{m+1}P(A_i)+(-1)^1\sum_{i=1}^{m+1}\sum_{j=i+1}^{m}P(A_i\cap A_j)+$$

$$(-1)^2\sum_{i=1}^{m+1}\sum_{j=i+1}^{m}\sum_{k=j+1}^{m-1}P(A_i\cap A_j\cap A_k)+...+(-1)^m P(A_1\cap A_2\cap...\cap A_{m+1})$$

Thus, by induction, our original formula holds for all $m\in\mathbb{Z}$. ■

2 Question 2

Theorem 3.2.2: Suppose an urn contains r red chips and w white chips, where $r+w=N$. If n chips are drawn out at random, without replacement, and if k denotes the number of red chips selected, then:

$$\text{P}(k\text{ red chips are chosen})=\frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}}$$

2.1 Proof:

Suppose an urn contains r red chips and w white chips, where $r+w=N$. We need to find how many elements are in the event of drawing exactly k red chips and $n-k$ white chips. The ordered possibilities of drawing k red chips is $_rP_k$, similarly it is $_wP_{n-k}$ for drawing $n-k$ white chips. These numbers represent the number of ordered possibilities for drawing the respective chips. The number of combinations for drawing k chips from our n chips drawn is $\binom{n}{k}$.

Thus, the number of elements (possibilities) in our event of interest is:

$$\binom{n}{k}(_rP_k)(_wP_{n-k})$$

We also know that the number of ordered permutations of drawing n chips from our N total is: ${}_N P_n$. Since every element in our event of interest (drawing exactly k red chips) is equal, the probability of drawing exactly k red chips is:

$$P(k) = \frac{\binom{n}{k}({}_r P_k)({}_w P_{n-k})}{{}_N P_n}$$

We can simplify this:

$$P(k) = \frac{(\frac{n!}{k!(n-k)!})(\frac{r!}{(r-k)!})(\frac{w!}{(w-n+k)!})}{\frac{N!}{(N-n)!}}$$

$$P(k) = \frac{(\frac{r!}{k!(r-k)!})(\frac{w!}{(n-k)!(w-(n-k))!})}{\frac{N!}{n!(N-n)!}}$$

$$P(k) = \frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}}$$

This the form of our original formula. ■