Weak Law of Large Numbers Proof

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1 Weak Law of Large Numbers Theorem

Let $X_1, X_2, ..., X_n$ be a random sample from the same distribution $\widetilde{X}$ with $E(\widetilde{X}) = \mu$ and $Var(\widetilde{X}) = \sigma^2$. Let $\bar{X_n} = \frac{X_1 + X_2 ... + X_n}{n}$. Then for any $\epsilon > 0$:

$$\lim_{n \to \infty} P\big(|\bar{X_n} - \mu| \geq \epsilon\big) = 0$$

1.1 Chebyshev's Inequality

If X has finite mean μ and variance σ^2, and $k > 0$, then:

$$P\big(|X - \mu| \geq k\big) \leq \frac{\sigma^2}{k^2}$$

1.2 Proof of Weak Law of Large Numbers

Let $X_1, X_2, ..., X_n$ be a random sample from the same distribution $\widetilde{X}$. Assume that $E(\widetilde{X}) = \mu$ and $Var(\widetilde{X}) = \sigma^2$ are both finite.

Let $S_n = X_1 + ... + X_n$ and let $X_n = \frac{S_n}{n}$. By the Central Limit Theorem, we know that as $n \to \infty$, S_n has a Normal distribution with Var$(S_n) = n\sigma^2$. By theorem, Var$(\frac{S_n}{n}) = \left(\frac{1}{n}\right)^2Var(S_n) = \frac{\sigma^2}{n}$ =Var$(\bar{X_n})$. In addition, we know from CLT that E$(S_n) = n\mu$. This implies that E$\left(\frac{S_n}{n}\right) = \mu$ by theorem.

By Chebyshev's Inequality, for any $\epsilon > 0$, we have that:

$$P\Big(|\bar{X_n} - \mu| \geq \epsilon\Big) = \frac{Var(\bar{X_n})}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

Thus, for a fixed ϵ,

$$\lim_{n \to \infty} P\big(|\bar{X_n} - \mu| \geq \epsilon\big) = 0$$

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