HW4, MATH 423

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1 Example 3.8.2

Suppose that X and Y are two independent binomial random variables, each with the same success probability but defined on m and n trials, respectfully. i.e.:

$$p_X(k) = \binom{m}{k} p^k (1-p)^{m-k}, k = 0, 1, ..., m$$

and

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, ..., n$$

We are to find $p_W(w)$ where $W = X + Y$.

By Theorem 3.8.3, we know that $p_W(w) = \sum_x p_X(x) p_Y(w-x)$. The binomial probability distribution is defined to be 0 when $x < 0$. Therefore, we only want to sum values where $p_X(x)$ and $p_Y(w-x)$ are nonzero, i.e. where $(w-x) \geq 0$. This occurs when we sum over the values $x \in \{0, 1, ..., w\}$. Thus:

$$p_W(w) = \sum_{x=0}^{w} p_X(x) p_Y(w-x) = \sum_{x=0}^{w} \binom{m}{x} p^x (1-p)^{m-x} \binom{n}{w-x} p^{w-x} (1-p)^{n-(w-x)}$$

$$= \sum_{x=0}^{w} \binom{m}{x} \binom{n}{w-x} p^w (1-p)^{n+m-w}$$

Consider an urn having m red chips and n white chips. If w chips are drawn without replacement, then the probability that exactly x red chips are drawn is given by the hypergeometric distribution:

$$P(x) = \frac{\binom{m}{x}\binom{n}{w-x}}{\binom{m+n}{w}}$$

Since this is a probability density function (pdf), we know:

$$\sum_{x=0}^{w} \frac{\binom{m}{x}\binom{n}{w-x}}{\binom{m+n}{w}} = 1$$

Thus,

$$\sum_{x=0}^{w} \binom{m}{x} \binom{n}{w-x} = \binom{m+n}{w}$$

Plugging this into our $p_W(w)$ equation, we get:

$$p_W(w) = \binom{m+n}{w} p^w (1-p)^{n+m-w}$$

This is a binomial distribution with parameters $m+n$ and p.

2 Theorem 3.8.4

Thm. 3.8.4: Let X and Y be independent continuous random variables, with pdf's $f_X(x)$ and $f_Y(y)$, respectively. Assume that X is zero for at most a set of isolated points. Let $W = X/Y$. Then:

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx) dx$$

Proof:

$$F_W(w) = P(Y/X \leq w)$$

$$= P(Y/X \leq w \cap X \geq 0) + P(Y/X \geq w \cap X < 0)$$

$$= P(Y \leq wX \cap X \geq 0) + P(Y \geq wX \cap X < 0)$$

because

$$(Y \geq wX \cap X < 0)^c = (Y \leq wX \cap X < 0)$$

$$P((Y \geq wX \cap X < 0)^c) = 1 - P(Y \leq wX \cap X < 0)$$

Thus,

$$F_W(w) = P(Y \leq wX \cap X \geq 0) + 1 - P(Y \leq wX \cap X < 0)$$

By definition of cumulative distribution function:

$$F_W(w) = \int_0^{\infty} \int_{-\infty}^{wx} f_X(x) f_Y(y) dy dx + 1 - \int_{-\infty}^{0} \int_{-\infty}^{wx} f_X(x) f_Y(y) dy dx$$

Differentiate: $\frac{d}{dw} F_W(w) = f_W(w)$

$$f_W(w) = \frac{d}{dw} \int_0^{\infty} \int_{-\infty}^{wx} f_X(x) f_Y(y) dy dx - \frac{d}{dw} \int_{-\infty}^{0} \int_{-\infty}^{wx} f_X(x) f_Y(y) dy dx$$

$$= \int_0^\infty f_X(x)\left[\frac{d}{dw}\int_{-\infty}^{wx} f_Y(y)dy\right]dx - \int_{-\infty}^0 f_X(x)\left[\frac{d}{dw}\int_{-\infty}^{wx} f_Y(y)dy\right]dx$$

By the Fundamental Theorem of Calculus, we have that:

$$\frac{d}{dw}F_W(w) = \frac{d}{dw}\int_{-\infty}^{wx} f_Y(y)dy = f_Y(wx)\frac{d}{dw}wx = xf_Y(wx)$$

$$f_W(w) = \int_0^\infty xf_X(x)f_Y(wx)dx + \int_{-\infty}^0 (-x)f_X(x)f_Y(wx)dx$$

$$= \int_0^\infty |x|f_X(x)f_Y(wx)dx + \int_{-\infty}^0 |x|f_X(x)f_Y(wx)dx$$

Thus,

$$f_W(w) = \int_{-\infty}^\infty |x|f_X(x)f_Y(wx)dx \blacksquare$$