Question 1: "Structural Induction"

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For this question, you must consider the following definition and functions for the NTree data type (below) and prove (using structural induction) that for every binary tree Z:

count Z ≤ 2^(height Z) - 1

Remember that, if you are trying to prove that b ≤ d, and you know that a ≤ b ≤ c, you can replace b with c (to get c ≤ d) but you cannot replace b with a (to get a ≤ d). To receive full marks for this question you must show every step in your proof and you must use the source code line labels (shown below, in red) whenever you use equational reasoning.

data NTree = NilT | Node Int NTree NTree
[ctb] count NilT = 0
[ctr] count (Node n x y) = (count x) + (count y) + 1

[htb] height NilT = 0
[htr] height (Node n x y) = (max (height x) (height y)) + 1 max a b

[mc1] | a >= b = a
[mc2] | otherwise = b

Solution:

Step 1: Proof base case
Base case z = NilT

count Z ≤ 2^(height Z) – 1

count NilT ≤ 2^(height NilT) – 1   by: by ctb and htb

0 ≤ 2^0 − 1   by: by ctb and htb

0 ≤ 0 = True   by: Mathz

Step 2: prove inductive assumption

count(Node n x y) $\leq$ ^2(height Node n x y) – 1−1 by: ctr and htr

When X=NiLt Y=NiLt

$0 + 0 + 1 \leq 2^{0+1} - 1$ by: by ctb and htb

$1 \leq 2 - 1$ by: Mathz

$1 \leq 1$ = True By: Mathz

Case When Z is a Complete: height(x) = height(y)

count Z $\leq 2^{(\text{height Z})} - 1$

(count x) + (count y) + 1 $\leq 2^{(\max\ (\text{height x})\ (\text{height y}))+1} - 1$ by: ctr and htr

$(2^{(\text{height x})} - 1 + 2^{(\text{height y})} - 1) + 1 \leq 2^{(\max\ (\text{height x})\ (\text{height y}))+1} - 1$ By: Inductive assumption

$(2^{(\text{heightx})}-1+2^{(\text{heighty})}-1)+1 \leq 2^{(\text{heightx})+1}-1$ by: mc1

$(2*2^{(\text{heightx})}-2)+1 \leq 2^{(\text{heightx})+1}-1$ by: Inductive Assumption

$(2^{(1+\text{height x})} - 2) + 1 \leq 2^{(\text{height x})+1} - 1$ by: Inductive Assumption

$2^{(\text{height x})} - 1 \leq 2^{(\text{height x})} - 1$ = true by: mathz

Case Where Z is an incomplete: X = NiLt XOR Y = NiLt

count Z $\leq 2^{(\text{height Z})} - 1$

$(\text{count } x) + (\text{count } y) + 1 \leq 2^{(\max (\text{height } x)\ (\text{height } y))+1} - 1$ by: ctr and htr

$(\text{count N iLt}) + (\text{count } y) + 1 \leq 2^{(\max (\text{height N iLt})\ (\text{height } y))+1} - 1$ by: ctr and htr

$(0) + (\text{count } y) + 1 \leq 2^{(\max (0)\ (\text{height } y))+1} - 1$ by: ctr and htr

$(2^{(\text{height } y)} - 1) + 1 \leq 2^{(\text{height } y)+1} - 1$ by: Inductive Assumption and htb

$2^{(\text{height } y)} \leq 2^{(\text{height } y)+1} - 1 = \text{true}$ by: Inductive Assumption and htb and mathz