Mat 1320 Cheat Sheet

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University of Ottawa

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Integrals represent the area under a curve.

Whole Year Cheat Sheet

Introduction

Limits [L4]

If we have a function f on an interval I, with the number a except possibly at a, then the limit of f as f approaches a is the value that f tends to as it approaches a.

Limits only exist if the right- and left-hand limits are equal. These are the values that f approaches from the left and right sides.

When we are calculating limits as $x \to \infty$, or $x \to -\infty$, then we need to get x on its own on both the numerator and the demonator. Then, we can cancel out all terms that are $\frac{n}{x}$ since anything over infinity is zero. Finally, we can evaluate.

Evaluate $\lim\limits_{x\to\infty} \frac{x-1}{\sqrt{2x^2+4x}}$

We need to get x in all the terms we can.

$$= \lim_{x\to\infty} \frac{x\left(1-\frac{1}{x}\right)}{\sqrt{x^2\left(2+\frac{4}{x}\right)}} = \lim_{x\to\infty} \frac{x\left(1-\frac{1}{x}\right)}{|x|\sqrt{2+\frac{4}{x}}}$$

We use $|x|$ in the demonator because originally it is x^2, so this means that regardless of the sign of x, it will give a positive value. So, removing that square still means that it only takes the magnitude of x.

This trick will allow us to cancel out all terms with the x in the demonator since these will become 0.

$$= \lim_{x\to\infty} \frac{x(1)}{|x|\sqrt{2}}$$

Now, if we sub in positive infinity, the infinities will cancel each other out, and it will be $\frac{1}{\sqrt{2}}$.

Likewise, if we sub in negative infinity, again they will cancel each other out and it will be $-\frac{1}{\sqrt{2}}$.

Continuity and Differentiability [L5]

A differentiable function must be continuous. The opposite is not necessarily true.

Differentiable on a domain I means that the function's left and right-side limits exist and are the same for all points in I. All points must also exist in I.

This function f is continuous on the interval $[-2,1]$ but is not continuous over $[-2,\infty]$.

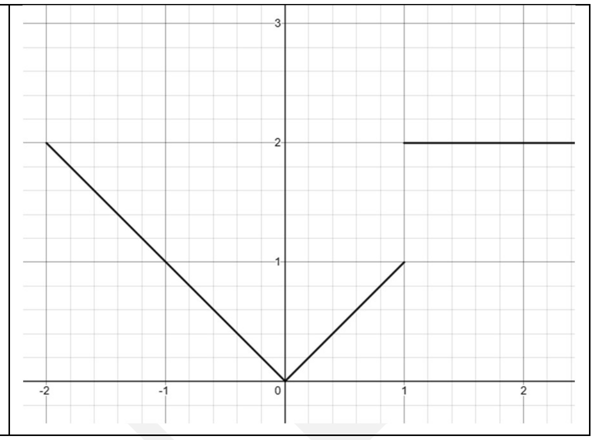
f is not not differentiable over $[-2,1]$ since it is not differentiable at x=0. It is also not differentiable at x=0.

At x=1, right and left limits are different.

$$\lim_{x\to 1^-} f(x) = 1, \qquad \lim_{x\to 1^+} f(x) = 2$$

At x=0, the rate of change to the left and right are different.

$$f'(-0.1) = -1, \qquad f'(0.1) = 1$$



Derivatives

Differentiation Rules [L5, L6]

The derivative is the rate of change of a function at a certain point. The general formula is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is complicated though, so there are lots of simple rules to differentiate functions. Let f and g be differentiable functions:

Constants	$f(x) = c$	$f'(x) = 0$
Powers	$f(x) = x^n$	$f'(x) = nx^{n-1}$
Euler's Number	$f(x) = e^x$	$f'(x) = e^x$
Products	$(f \cdot g)'$	$fg' + gf'$
Quotients	$\left(\frac{f}{g}\right)'$	$\frac{gf' - fg'}{g^2}$

Trig Derivatives [L7]

These are some derivatives for common trig functions. They just need to be memorized.

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arccos x) = \frac{1}{-\sqrt{1-x^2}}$
$\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$	

Chain Rule [L7]

When we are applying any of the differentiation rules and there is a composition function, we need to apply the chain rule.

$$\big(f(g(x))\big)' = f'(g(x)) \cdot g'(x)$$

Find $f'(x)$ if $f(x) = (x^3 - 1)^{1000}$

We can use both the power rule, and the chain rule where $g(x) = x^3 - 1$

$$f'(x) = \frac{d(g(x))^{1000}}{dx} \cdot \frac{d(x^3-1)}{dx} = 1000 \cdot g(x)^{999} \cdot 3x^2 = 3000(x^3-1)^{999} \cdot x^2$$

Exponential Differentiation [L7]

This is very simple. If $b > 0$ and b is constant, then:

$$f(x) = b^x, \qquad f'(x) = b^x \cdot \ln b$$

Note that if the exponent is anything other than x, we need to use the chain rule.

Implicit Differentiation [L8]

We use implicit differentiation if we need to find the derivative of a second variable such as y.

When we differentiate this other variable, we are not sure what it represents. So, we need to use the chain rule for all instances of this second variable.

$$\frac{d\big(f(y)\big)}{dx} = f'\big(y(x)\big) \cdot \frac{dy}{dx}$$

Find $\frac{dy}{dx}$ of $x^2y^3 + xe^y = 6$.

Firstly, note that we are finding the derivative of this with respect to x.
Next, note that there are two products on the left side, and a constant on the right side.

$$\frac{d}{dx}(x^2y^3) + \frac{d}{dx}(xe^y) = 0$$
$$\left(\frac{d(x^2)}{dx} \cdot y^3 + \frac{d(y^3)}{dx} \cdot x^2\right) + \left(\frac{d(x)}{dx} \cdot e^y + \frac{d(e^y)}{dx} \cdot x\right) = 0$$
$$2xy^3 + 3y^2y'x^2 + e^y + e^y y'x = 0$$

Note that $y' = \frac{dy}{dx}$

This looks good. Next, we need to isolate for y' to get:

$$\frac{dy}{dx} = y' = \frac{-2xy^3 - e^y}{3x^2y^2 + xe^y}$$

Logarithmic Differentiation [L8]

The concept of logarithmic differentiation is very simple.

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \ln a}$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

The great part about logarithmic differentiation is that we can use it in otherwise very hard to compute functions to simplify it.

Calculate $f'(x)$ if $f(x) = x^x$

This looks *very hard, so we can use logarithmic differentiation. First, we need to get it into a form with two variables.*

Let $y = f(x)$

$$y = x^x$$

$$ln(y) = ln(x^x)$$

$$ln(y) = x\, ln(x)$$

Now we can differentiate.

$$\frac{d}{dx} ln(y) = \frac{d}{dx}(x\, ln\, x)$$

$$\frac{1}{y} y' = ln\, x + 1$$

$$y = y(1 + ln\, x)$$

We can turn this back to $f'(x)$ by remembering that $y = x^x$.

$$f'(x) = x^x(1 + ln\, x)$$

L'Hospitals Rule [L19]

If f and g are differentiable, and the both approach 0, ∞, or $-\infty$, l'Hospitals Rule states the following:

$$\lim_{x \to n} \frac{f(x)}{g(x)} = \lim_{x \to n} \frac{f'(x)}{g'(x)}$$

This is useful if the functions f and g produce an indeterminate form such as $\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}$. Even with other indeterminate forms, we can often turn them into one of the above forms and then use l'Hospitals rule to solve.

- With indeterminate products, such as $0 \times \infty$, we can use the following information to turn it into a correct form:

$$fg = \frac{f}{\frac{1}{g}}, \qquad fg = \frac{g}{\frac{1}{f}}$$

- With indeterminate differences $\infty - \infty$, we can get a common demonator, and then we will have a correct form.
- With indeterminate powers such as $0^0, \infty^0, 1^\infty$, we use log laws to turn it into the correct form.

$$f(x)^{g(x)} = g(x) \ln f(x)$$

Compute $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

First, we see that if we plug in 1, we get an indeterminate form.

We see that this is an indeterminate difference. So, we will get a common demonator.

$$= \lim_{x \to 1^+} \frac{(x-1) - ln\, x}{ln\, x \cdot (x-1)}$$

Now, we can find the derivatives of the numerator, and the demonator.

NOTE: We are not finding the derivative of the whole rational function, we will do the demonator and numerator separately.

$$= \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + ln(x)} = \lim_{x \to 1^+} \frac{x - 1}{x - 1 + x\,ln\,x}$$

Now, we still have an indeterminate form, but it is simpler. So, we can apply l'Hospital's rule again.

$$\lim_{x \to 1^+} \frac{1}{1 + ln\,x + 1} = \lim_{x \to 1^+} \frac{1}{2 + ln\,x} = \frac{1}{2}$$

An example of indeterminate powers is found in the appendix.

Integration

Riemann Sums [L10, L11]

If we want to calculate the area under a curve, we can divide that curve into n different rectangles that sit under the curve. As n gets progressively larger, the calculated area will approach the true area.

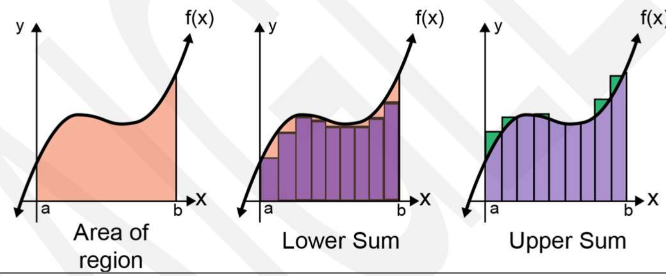
This is the equation to calculate the Riemann sum:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}[f(x_{i1}^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

Where x_i^* is the start or end of each subinterval (depending on if using upper or lower endpoints), and Δx is the interval width. Keep in mind that in practice, we cannot truly evaluate $n \to \infty$ so we set n to some realistic value.

We use either the left (lower) or right (upper) endpoints to create these rectangles.

- The upper will overestimate the area.
- The lower will underestimate the area.



Approximate the area under the curve of the function $x^2 - 2x + 3$ on $[1,3]$ using 4 rectangles using the right endpoints. *First, we need to find the interval width (Δx). This will be* $\Delta x = \frac{3-1}{4} = 0.5$ *Now, we need to find the right endpoints. I can create a set of all values in* $[1,3]$. $x_i^* = [1, 1.5, 2, 2.5, 3]$ *Since we are using the right endpoints, we will ignore leftmost value* 1. *Then we can sub into the equation.* $\sum f(x_i^*)\Delta x = 0.5\big(f(1.5) + f(2) + f(2.5) + f(3)\big)$ *Then we can simply compute the answer.*

Definite and Indefinite integrals [L10, L11]

Integrals are just antiderivatives. If we have a function f, then f' is the derivative of f, and F is the antiderivative of f.

Take a velocity vs time example. The rate of change of velocity (derivative) of the graph would be acceleration, then the antiderivative would be the position vs time graph. This is because the derivative of position is velocity.

Therefore, we can say that the antiderivative of a function f is $F(x) + C$ where C is the integration constant which can be any real number.

Because integrals are just antiderivatives:

$$F(x) = \int f(x)dx + C$$

Solve $\int x^3 dx$

Note that this is just finding the antiderivative of x^3.

So, we need to find an equation that's derivative is x^3.

This would be $\frac{x^4}{4}$, and with the integration constant, $\frac{x^4}{4} + C$

So, the integral (antiderivative) of x^3 is $\frac{x^4}{4} + C$

This example shows that the integral of a power is $\int x^n = \frac{x^{n+1}}{n+1}$

Fundamental Theorem of Calculus [L11]

The fundamental theorem of calculus (FTC) is the theorem that establishes a relationship between derivatives and integrals.

If f is continuous on $[a, b]$:

1) If $g(x) = \int_a^x f(t)dt$ then $g'(x) = f(x)$
2) $\int_a^b f(x)dx = F(b) - F(a)$

Part 1 says that the derivative of an integral $\int_a^x f(t)$ is just $f(x)$

Part 2 says that the integral of $f(x)$ is just its antiderivative if it is an indefinite integral, and with a definite integral, it is the difference between the antiderivative of the boundaries.

u Substitution [L12]

We know of the chain rule for evaluating derivatives, and for integrals, this is known as u substitution.

If $u = g(x)$, then $\int f(g(x))g'(x)dx = \int f(u)du$

For the definite integral: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

This basically means that when we are evaluating an integral, we need to take one function as u, and then differentiate that function. Then, we can use this to simplify the integral and solve using the power rule.

Find $\int x^3 \cos(x^4 + 2)\, dx$

First, notice that there are 2 terms in the equation. x^3, and $cos(x^4 + 2)$.

Since the derivative of $x^4 + 2$ is $4x^3$, then that will cancel out the x^3.

Let $u = x^4 + 2, du = 4x^3dx$

Therefore, we can say $dx = \frac{du}{4x^3}$

Subbing all this into the equation (replace all x instances), we get:

$$= \int \frac{x^3\, cos(u)\, du}{4x^3}$$

Cancelling out the x^3 terms, and moving the constant out of the integral, we get:

$$= \frac{1}{4}\int cos(u)\,du$$

This is simple to solve. We know the derivative of sin is cos, so the antiderivative of cos is sin.

$$= \frac{1}{4} sin(u) + C$$

However, we want this in terms of x, not u. So, we can sub x back into the equation since $u = x^4 + 2$.

$$= \frac{1}{4} sin(x^4 + 2) + C$$

Integration by Parts [L13]

If we cannot solve an integral by u substitution, and there is a multiplication of terms, we can use integration by parts (IBP). This is akin to the product rule for derivatives.

Let $u = f(x), \quad v = g(x), \quad dv = g'(x)dx, \quad du = f'(x)dx$

$$\int udv = uv - \int vdu$$

We can see that we will need to find one part of the integral to be u, and one part to be dv. The part assigned to u, should be relatively easy to derive, likewise, dv should be relatively easy to integrate.

Note: We may need to do multiple IBPs to solve. The complexity of the integral should be decreasing with each round though.

Compute $\int x \sin(x)\, dx$

Here, we can see that there is a product of functions.

We see that the derivative of x is just 1. So, we can take that as u, and therefore take $sin\,(x)$ as dv.

$$u = x \qquad dv = sin(x)$$

$$du = 1 \qquad v = -cos\,(x)$$

Subbing these values into the IBP formula, we have:

$$\int x\,sin(x)\,dx = x(-cos(x)) - \int -cos(x)\,(1)$$

Then, this next integral looks very easy.

$$= -x\,cos(x) - (-\,sin(x) + C) = -x\,cos(x) + sin(x) + C$$

Integration of Sin / Cos [L13]

Integrating $\sin x$ or $\cos x$ is very simple. But what if we have something like $\int \sin^5 x \cos^2 x$? We could use IBP, but this would be long and annoying. Thankfully, there are some tricks we can do using the trig identities.

Evaluating $\int \sin^m x \cos^n x\, dx$

- If n (cos) is odd, replace *all but one* cos factor with $\cos^2 x \ = 1 - \sin^2 x$
- If m (sin) is odd, replace *all but one* sin factor with $\sin^2 x = 1 - \cos^2 x$

- If both powers are even, use the half angle identies.
  - $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
  - $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
  - $\sin x \cos x = \frac{1}{2}\sin 2x$

It is similar if we want to evaluate $\int \tan^m x \sec^n x$ function, except we use the identity $\tan^2 x = \sec^2 x - 1$.

Evaluate $\int \sin^5 x \cos^2 x \, dx$

We immediately notice that sin has an odd power, so we can convert all but one power of sin to cos.

$$\int sin^4 x \, sin\, x \, cos^2 x \, dx = \int (1 - cos^2 x)^2 \, cos^2 x \, sin\, x \, dx$$

Now we can let $u = cos\, x$, and therefore $du = -sin\, x \, dx$ and $dx = \frac{du}{-sin\, x}$:

$$= \int \frac{(1-u^2)^2 u^2 \, sin\, x \, du}{-sin\, x} = -\int (1 - 2u^2 + u^4) u^2 du = -\int (u^2 - 2u^4 + u^6) du$$

Now this form looks solvable. We can just use the regular power rule to get:

$$= -\left(\frac{u^3}{3} - \frac{2}{5}u^5 + \frac{u^7}{7}\right) + C$$

Remember, this is not solved yet. We need it in terms of x.

$$-\frac{cos^3 x}{3} + \frac{2}{5} cos^5 x - \frac{cos^7 x}{7} + C$$

Also ensure to add the integration constant C.

Trig Substitution [L14]

Just like IBP, and u substitution, trig substitution is another way to take a complex integral and simplify it.

This table shows the three common trig substitutions. If we see something that looks like one of the expressions, we can apply the substitution shown, and then use its corresponding identity to solve.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \cdot \sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \cdot \tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \cdot \sec\theta, 0 \le \theta < \frac{\pi}{2} \quad OR \quad \pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

These can be a bit more complicated than other substitutions. This is because we end up getting an answer in terms of θ, and then we need to use the trigonometrical relationship of x with θ to get the answer in terms of x.

It becomes much easier if we draw a triangle IMMIDIATELY after we make the substitution with θ labelled, as well as the sides of the triangle. This is best shown by an example.

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2} dx$

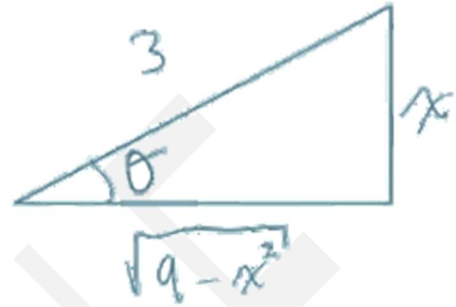
We see right away that in the numerator, $\sqrt{9-x^2}$ is in the form of $\sqrt{a^2-x^2}$ where $a=3$, $x=x$.

We can substitute $x = 3 \cdot \sin\theta$, and $dx = 3 \cdot \cos\theta$.

Now we can make a triangle. We can isolate for theta relative to x (we do not want theta relative to dx) to get $\theta = \arcsin\left(\frac{x}{3}\right)$.

This means that the opposite side to θ is x, the hypotenuse is 3, and using Pythagorean theorem, we get:

$$o^2 + a^2 = h^2 \quad \rightarrow \quad x^2 + a^2 = 3^2 \quad \rightarrow \quad a = \sqrt{9-x^2}$$



Now that we have the triangle for later, we can solve the integral by making the substitution:

$$\int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2} \cdot 3\cos\theta\, d\theta = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta = \int \frac{\sqrt{9(1-\sin^2\theta)}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

Now, we see in this final form we can substitute the trig identity $(1-\sin^2\theta) = \cos^2\theta$.

$$= \int \frac{\sqrt{9\cos^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta = \int \frac{3\cos\theta}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

Now, knowing the trig identity $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and therefore $\cot(x) = \frac{\cos(x)}{\sin(x)}$, we can simplify this further:

$$= \int \cot^2\theta\, d\theta$$

Then using another trig identity (I know, this is a lot, but it is really only 3 identities) $\cot^2\theta = \csc^2\theta - 1$:

$$= \int (\csc^2\theta - 1) d\theta = -\cot\theta - \theta + C$$

Almost done. Now we just need to solve in terms of x. This is where the triangle is useful. We see $\cot(\theta) = \frac{adj}{opp}$ so the opposite of θ is x, and the adjacent is $\sqrt{9-x^2}$, and then for θ, we can use any relation we want. I will use $\theta = \arcsin\left(\frac{x}{3}\right)$. So we get:

$$= -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C$$

This method looks very complicated and long. So, there are more examples in the appendix.

Method of Partial Fractions [L15]

When we have a rational function in the form of $f(x) = \frac{P(x)}{Q(x)}$, we can use the method of partial fractions to solve. This is where we break up the function into multiple simpler parts.

To do this, we need to ensure that the highest power is on the demonator. If it is not, we need to use long division to divide the numerator in the denominator.

Complete the following long division:

$$x-1|\overline{x^3+x}$$

This is confusing to do by text, so it is better to write it out by hand.

$$\begin{array}{r l}
 & x^2 + x + 2 \\
x-1 \big) & \overline{x^3 + x} \\
 & \underline{x^3 - x^2} \\
 & 0 \quad x^2 + x \\
 & \quad \underline{x^2 - x} \\
 & \quad 0 \quad +2x \\
 & \qquad \underline{2x - 2} \\
 & \qquad\quad -2
\end{array}$$

In this line, we need to find what $(x-1)$ multiplies by to get (x^3+x). We get x^2 which goes at the top. Then we still need to multiply by -1 which gives the $-x^2$.
On the next line, the differences are taken. $x^3 - x^3 = 0$, $0 - -x^2 = x^2$, and finally, $x - 0 = x$.
Note that we do not subtract x^2 from x.

$$\therefore \frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

In the above example, it is then very easy to solve. However, we will often still have something more complex.

In this case, we will need to break up the demonator into its factors. If we can, break it up into linear factors, if not, quadratic is fine. To do this, we can ideally use simple factoring, but if there are no apparent factors, then we will need to guess a root, and then long divide that root out.

Once we have a rational function, with the highest degree on the demonator, and all linear/quadratic factors, we then need to apply the following formulas:

Case	Demonator	Formula
1	Unique Linear Factors	$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_r}{a_rx+b_r}$
2	Repeat Linear Factors	$\frac{R(x)}{Q(x)} = \frac{B_1}{a_1x+b_1} + \frac{B_2}{(a_1x+b_1)^2} + \cdots + \frac{B_s}{(a_1x+b_1)^s}$
3	Unique Quad Factors	$\frac{R(x)}{Q(x)} = \cdots + \frac{Ax+B}{ax^2+bx+c} + \cdots$
4	Repeat Quad Factors	$\frac{R(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$

We will apply these formulas for EACH and EVERY factor in the demonator.

Then, we can set all those formulas equal to the original function, and solve by creating a system of equations.

Because of the complexity of this method, it is often used as a last resort.

Write out the partial fraction decomposition (no need to solve) of $f(x) = \frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$

This looks very complicated. But, we can just take it one step at a time.

I can see that there are 2 unique linear factors, x, and $x-1$. There is a unique quadratic factor x^2+x+1, and a repeated quadratic factor $(x^2+1)^3$.

We can now use the above formulas to set this up.

For the first linear factor, it will be $\frac{A}{x}$

For the second linear factor it will be $\frac{B}{x-1}$

For the unique quadratic factor, it will be $\frac{Cx+D}{x^2+x+1}$

Finally, for the repeated quadratic factor $\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$

Putting this all together, we get:

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$$

Now if we wanted to solve, we could get a common denominator, create a system of equations, and solve.

Now for an actual example.

Evaluate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$

We see that the highest degree is on the demonator so we do not need to long divide.

Then we see that on the demonator, it is already factored. We have a unique linear term and a repeated quadratic term.

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Now, I can find a common denominator by multiplying each term by what is missing.

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2}=\frac{A(x^2+1)^2}{x(x^2+1)^2}+\frac{(Bx+C)(x^2+1)(x)}{x(x^2+1)^2}+\frac{(Dx+E)x}{x(x^2+1)^2}$$

Now, I can cancel out the demonator, and gather all like terms.

$$1-x+2x^2-x^3=A(x^4+2x^2+1)+(Bx+C)(x^3+x)+Dx^2+Ex$$

$$1-x+2x^2-x^3=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A$$

Now, we have the system of:

$$\begin{aligned}&A+B=0\\&C=-1\\&2A+B+D=2\\&C+E=-1\\&A=1\end{aligned}$$

And we get the solution $A=D=1,\quad C=B=-1,\quad E=0$

Now, we can go back to the equation with A, B, C, D, and E, and sub in and solve the integrals.

$$\int\frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx=\int\left(\frac{1}{x}-\frac{x+1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx=ln|x|-\frac{1}{2\,ln(x^2+1)}-arctan\,x-\frac{1}{2(x^2+1)}+C$$

Approximate Integration [L16]

Sometimes we do not have a function, we just have experimental data. So we cannot find the integral of that data.

In these situations, we can use approximation methods such as Riemann Sums. However, Riemann Sums are not the most reliable since they tend to either over or underestimate the data.

Each method has an interval width Δx which is calculated by taking the difference between the upper and lower bound, and dividing by the number of rectangles we want: $\Delta x = \frac{b-a}{n}$

- One method is the trapezoidal rule.
$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$
- Another method is Simpson's rule.
$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$
    - Simpson's rule only works where n is even.
- There is also the midpoint rule which will take the midpoint of 2 values.
$$\int_a^b f(x)dx \approx M_n = \Delta x[f(\overline{x_1}) + \cdots + f(\overline{x_n})]$$
    - Where $\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$.

$f(x) = \sqrt{x^2 + x}$ for $1 \le x \le 7$. Estimate $\int_1^7 f(x)dx$ using the trapezoidal rule. Use 4 sub intervals.

First, we need to find the interval width.

$$\Delta x = \frac{7-1}{4} = \frac{3}{2}$$

So, we have the intervals of:

$$[1, 2.5, 4, 5.5, 7]$$

Now we can sub this into the trapezoidal rule formula.

$$T_4 = \frac{\frac{3}{2}}{2}\Big(f(1) + 2\big(f(2.5)\big) + 2\big(f(4)\big) + 2\big(f(5.5)\big) + f(7)\Big) = 26.79$$

If we wanted, we could also use the midpoint rule, or Simpson's rule with this (since n is even).

Extra Stuff

Related Rates [L9]

This concept is not that hard. It delves into word problems that we can solve combining regular variables, and their derivatives.

We often need to find an equation that relates 2 values, differentiate it, and then solve for the unknown value.

It is best shown with examples.

Air is being pumped into a spherical balloon, so its volume increases at a rate of $100\frac{cm^3}{s}$. How fast is its radius increasing when the diameter is $50cm$?.

We have the following information:

$$\frac{dV}{dt} = 100\frac{cm^3}{s}$$
$$r = \frac{d}{2} = 25cm$$

We do not know the rate of change of the radius, but we do know when we need that rate of change.
$\frac{dr}{dt} = ?$ when $r = 25$

An equation that relates the volume to radius is $v = \frac{4}{3}\pi r^3$

Differentiating and solving for $\frac{dr}{dt}$ gives us $\frac{dr}{dt} = \frac{\frac{dv}{dt}}{4\pi r^2}$

Subbing in all the values gives:

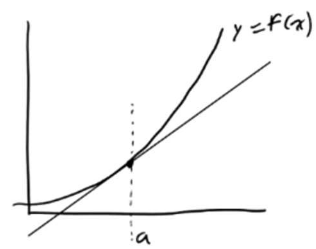
$$\frac{dr}{dt} = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi}\frac{cm}{s}$$

More examples can be found in the appendix.

Linear Approximation [L9]

Linear approximation is the process of approximating certain values on a function f by using the tangent line at a point a.

It is used when it is very hard to calculate exact values of a function, so the tangent line of a close point can be calculated which will approximate the values of nearby points.



The equation is:

$$L(x) = f(a) + f'(a)(x - a)$$

Find the linear approximation of $f(x) = \frac{1}{x}$ at $a = 2$, and use it to approximate 2.1, and 10.

First, we need to calculate the first derivative.

$$f'(x) = -x^{-2}$$

Now, we can use the formula:

$$L(x) = f(2) + f'(2)(x - 2)$$

$$L(x) = 1 - \frac{x}{4}$$

We can use this to approximate 2.1 *fairly accurately, but approximating* 10 *will not give a good result since* 10 *is far from* 2*.*

$$L(2.1) = 1 - \frac{2.1}{4} = 0.475$$

$$L(10) = 1 - \frac{10}{4} = -1.5$$

Curve Sketching [L17, L18, L19]

To sketch a curve using a given equation, we need to gather lots of information using the first and second derivatives.

Once we have all this information, we can easily sketch the function.

- Domain of the function
- x and y intercepts of the function
- Symmetry of the function (if it exists)
- Vertical and Horizontal asymptotes of the function
- Intervals of increase and decrease
- Maximums and minimums of the function
- Concavity of the function

Once we have all this information, we should have enough to draw out the graph of the function.

Sketch the graph of $y = \frac{2x^2}{x^2-1}$

First, we can find the domain of the function. This will be $x \in \mathbb{R}$ *where* $x \neq 1, -1$ *since these points will cause the demonator to be zero.*

Next, we can find the intercepts. Finding $f(0)$*, we get* 0*, and* $f(x) = 0$ *gives* 0 *as well.*

Subsequently, we can see that since it is an even function, it will be symmetric about the y axis.

Next, we can test the behavior as x *goes to infinity to find the horizontal asymptotes.*

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2x^2}{x^2\left(1-\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{2}{1-\frac{1}{x^2}} = 2$$

So, $y = 2$ *is a horizontal asymptote.*

We can test the behaviors of y *as* x *approaches the points where the function is undefined.*

$\lim_{x\to1^+} \frac{2x^2}{x^2-1} = \infty$	$\lim_{x\to1^-} \frac{2x^2}{x^2-1} = -\infty$
$\lim_{x\to-1^+} \frac{2x^2}{x^2-1} = -\infty$	$\lim_{x\to-1^-} \frac{2x^2}{x^2-1} = \infty$

Now, we can find the intervals of increase, and decrease.

Optimization [L20]

Often, we will have a problem and we will need to find the optimal solution.

For example, when a company produces a product, they need to find the price that will give them the maximum profit, or the minimum number of employees required to meet demand.

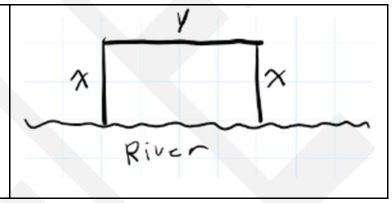
The methods to solve these problems have to do with finding a function to model the situation, finding its derivative, and then using that to find the maximum/minimum. Then, this value needs to be correctly interpreted.

A farmer has 2400ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

We need to maximize the area of the field using a constant amount of fencing (2400ft).

It is always a good idea to start with a diagram.

We can see that the area is length times the width.



$$A = x \cdot y$$

This equation has 2 variables, so we can use the relation between y, x, and the total amount of fencing to get y in terms of x.

$$y = 2400 - 2x$$

Now we have:

$$A = x \cdot (2400 - 2x) = -2x^2 + 2400x$$

$$A' = -4x + 2400$$

We need the maximum of A. First, we can find the critical points of A'.

$$0 = -4x + 2400$$

$$x = \frac{2400}{4} = 600$$

To find the maximum, we will need to test all critical points, and the bounds of the domain. x must be a positive integer that is no larger than 1200 (or there will not be enough fence).

$A(600) = 720000$ $\qquad$ $A(0) = 0$ $\qquad$ $A(2400) = 0$

So, this means that the x dimension is 600, and the y would be $2400 - 2(600) = 1200$.

Therefore, the optimal dimensions are 600ft by 1200ft.

Most optimization problems are similar to this. Another example can be found in the appendix.

Newton's Method [L21]

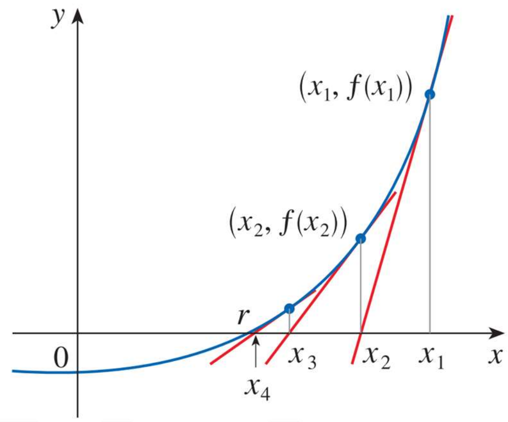
This is the method that is used by calculators to find the roots to complex functions such as $(x-1)^{60}+(x+2)^{45}$.

This is done by taking a linear approximation of a random point on a function, and then doing it again, except this time, using the x intercept of the previous point's linear approximation.

It is found by using the formula:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

For the first iteration, we pick a random point on the function.



Starting with $x=2$, find the third approximation x_3 to the solution of $x^3-2x-5=0$.

Let $f(x)=x^3-2x-5$

Then, $f'(x)=3x^2-2$

Now, we can use the first iteration where $n=1$*, and* $x_1=2$.

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=2-\frac{f(2)}{f'(2)}=2.1$$

Now, to find the third approximation, I can use $n=2$*, and* $x_2=2.1$.

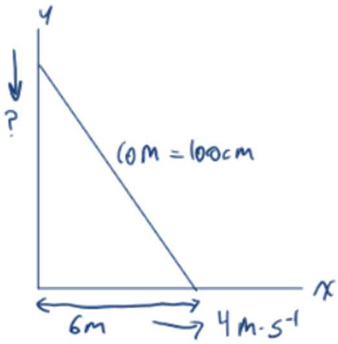
$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}=2.1-\frac{f(2.1)}{f'(2.1)}\approx 2.095$$

Appendix

More Related Rates examples

The key with this example is the fact that the length of the ladder does not change, therefore it is a constant term and its derivative is 0.

Ex. A ladder 10m long rests against a vertical wall. If the bottom slides away from wall at a rate of $4ms^{-1}$ how fast is the top of the ladder sliding down the wall when the bottom is 6m away from the wall?



$$\frac{dx}{dt} = 4m\cdot s^{-1}$$

$$x = 6m$$

$$\frac{dy}{dt} = ?$$

The relation between x and y is:

$$x^2 + y^2 = 100$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

We do not know y but we can use pythag to solve.

$$y = \sqrt{100 - x^2}$$

$$y = \sqrt{100 - 36}$$

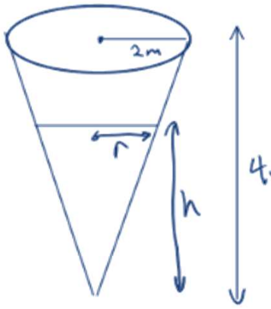
$$y = \sqrt{64} = 8$$

$$\frac{dy}{dt} = -\frac{6}{8} \cdot 4$$

$$\frac{dy}{dt} = -3\,m\cdot s^{-1}$$

The key with this example is getting the radius in terms of height. Originally there are 2 variables, and we cannot really work with 2 variables.

Ex. A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2m^3 \cdot min^{-1}$ find the rate of increase of the water level when the water is 3m deep.



$$\frac{dV}{dt} = 2m^3 \cdot min^{-1}$$

$$\frac{dh}{dt} = ? \text{ when } h=3$$

$$V = \frac{1}{3}\pi r^2 h$$

↳ We have $\frac{dV}{dt}$, h, but not anything about r?

Use a ratio to relate pink r to blue r, and pink h to blue h.

$$\frac{h}{4} = \frac{r}{2} \implies r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{3}\pi \frac{h^2}{4} h$$

$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = 3 \cdot \frac{1}{12} \cdot \pi \cdot h^2 \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi (3)^2} \cdot 2 = \frac{8}{9\pi}$$

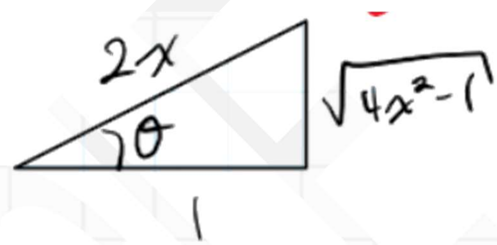
More trig substitution examples

Solve $\int \frac{dx}{x^2\sqrt{4x^2-1}}$

Notice that there is a spare x^2 term in the demonator. This might make you think IBP but no. Trig substitutions are higher in priority since they do a better job of simplifying. We see that it is a $\sqrt{x^2-a^2}$

We can say that:

$$2x = sec(\theta), \qquad x = \frac{1}{2}sec(\theta), \qquad dx = \frac{1}{2}sec(\theta)\,tan(\theta)\,d\theta$$



The reason we use $2x$ rather than x is because $(2x)^2 = 4x^2$.

Now we can make a triangle using the above ratios noting that $sec\,\theta = \frac{h}{a}$

Now we can solve with respect to θ.

$$= \int \frac{\frac{1}{2}sec\,\theta\,tan\,\theta\,d\theta}{\frac{1}{4}sec^2\,\theta\,\sqrt{sec^2\,\theta - 1}} = 2\int \frac{sec\,\theta\,tan\,\theta\,d\theta}{sec^2\,\theta\,tan\,\theta} = 2\int \frac{1}{sec\,\theta} = 2\int cos\,\theta = 2\,sin\,\theta + C$$

Now we can give the answer with respect to x.

Note that $sin(\theta) = \frac{opp}{hy} = \frac{\sqrt{4x^2-1}}{2x}$

$$= 2\frac{\sqrt{4x^2-1}}{2x} + C = \frac{\sqrt{4x^2-1}}{x} + C$$

Solve $\int \frac{x + \arcsin x}{\sqrt{1-x^2}}dx$

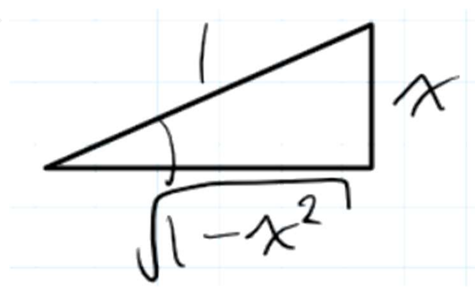
We see that there is a trig substitution in the demonator of the form $\sqrt{a^2-x^2}$ so we know to sub $x = a \cdot sin\,(\theta)$.

$$x = sin\,\theta, \qquad dx = cos\,\theta\,d\theta$$

Since we have the $arcsin\,x$ on the numerator, we can also solve what $arcsin\,x$ would be. We get this by applying $arcsin$ to $x = sin\,(\theta)$.

$$arcsin(x) = \theta$$

Now we can draw the triangle and solve.



$$= \int \frac{sin(\theta) + \theta}{\sqrt{1 - sin^2\,\theta}} cos\,\theta\,d\theta = \int \frac{(sin(\theta)+\theta)\,cos(\theta)}{cos(\theta)}d\theta = \int (sin(\theta)+\theta)d\theta = -\,cos(\theta) + \frac{\theta^2}{2} + C$$

Now we can solve in terms of x. Note that for the $\frac{\theta^2}{2}$ term, we can use whatever relation we want. I will use $arcsin$ since it is already calculated.

$$= -\frac{x}{\sqrt{1-x^2}} + \frac{arcsin^2(x)}{2} + C$$

L'Hospitals Rule

This is an indeterminate powers example:

Calculate $\lim_{x \to 0^+} (1 + \sin(4x))^{\cot(x)}$

We see that this is undefined in the form of 1^∞ *since the base gives* $1 + sin\big(4(0)\big) = 1$ *and the exponent gives* $cot(0) = \frac{1}{tan(0)}$ *which is undefined, BUT if we take a value close to 0 such as* 0.001*, we get a value close to 0.*

So, we need to use logarithms. I will apply log to both sides of the equation.

Let $y = (1 + sin(4x))^{cot(x)}$

$$ln(y) = cot(x) \cdot ln(1 + sin(4x))$$

Now, it is still not a good form, we need it to be a rational function.

$$ln(y) = \frac{ln(1 + sin(4x))}{tan(x)}$$

Using l'Hospitals rule, and reintroducing the limits, we can get:

$$\lim_{x \to 0^+} (ln(y)) = \lim_{x \to 0^+} \frac{\left(\frac{4\,cos(4x)}{1 + sin(4x)}\right)}{sec^2 x} = \frac{\frac{4}{1}}{1} = 4$$

This is not the answer to the question though. This is the limit of $ln\,(y)$*.*

We can rearrange though to get y.

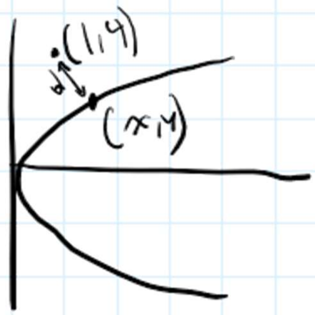
$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{ln(y)} = e^4$$

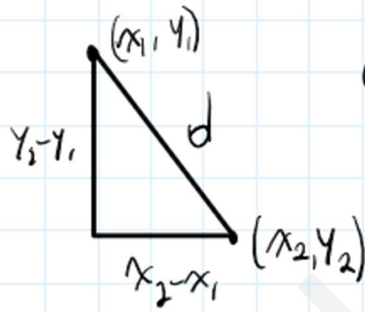
Optimization Problems

This example is a bit confusing. The main trick is that the maximum of d is the same as the maximum of d^2.

To set up the problem, the definition of the slope is used. This will relate the minimum distance to the two points.

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.





$$d = (y_2 - y_1)^2 + (x_2 - x_1)^2$$

Note: $y^2 = 2x$, $x = \frac{y^2}{2}$

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$$

We know that the minimum of d = minimum of d^2

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)2y + 2(y-4)$$

$$= y^3 - 8$$

$f'(y) = 0$ when $y = 2$

$f'(y) < 2$: $-$ So min

$f'(y) > 2$: $+$

$y^2 = 2x$

$x = \frac{y^2}{2}$ $\quad\therefore (2,2)$

$x = \frac{4}{2} = 2$