Mat 1341 Cheat Sheet

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Whole Year Cheat Sheet

Introduction

Vector Operations [L1-L3]

If there are 2 vectors, they are equal if the first vector is the exact same as the second vector.

$$(2,3,1) = (2,3,1), \qquad (2,3,1) \neq (2,3,2)$$

When vectors are added, we can just add the individual components.

$$(2,3,1) + (0,0,1) = (2,3,2)$$

When we multiply a vector by a scalar, we just multiply all components by the scalar.

$$2 \cdot (2,3,1) = (4,6,2)$$

The dot product is a multiplication between 2 vectors. We just take the sum of the products of each individual component. It produces a scalar.

$$(2,3,1) \cdot (1,2,3) = 2 \cdot 1 + 3 \cdot 2 + 1 \cdot 3 = 2 + 6 + 3 = 11$$

Linear Systems [L4]

When we have something like:

$$x_1 + 2x_2 + 3x_3 = 4 \qquad x_2 + x_3 = 5 \qquad x_3 = 1$$

We can solve all three variables. We already know x_3, so we can sub it into the second equation to get x_2, and then we can sub both of those values into the first equation to get x_1.

This is known as a linear system of equations. Note that all of the variables have powers of 1.

Matrices

Matrices [L4]

When we have a system of linear equations, we can turn it into a matrix where the rows of the matrix correspond to the coefficients in front of the variables. If we want, we can add a second part to the matrix (separated by a vertical line) which is called the augmented matrix. We often put constant terms here.

Create a matrix using the following system of linear equations.

$$x_1 + 2x_2 + 3x_3 = 4 \qquad x_2 + x_3 = 5 \qquad x_3 = 1$$

We now can rewrite this as the following to give all variables coefficients:

$$x_1 + 2x_2 + 3x_3 = 4 \qquad 0x_1 + x_2 + x_3 = 5 \qquad 0x_1 + 0x_2 + x_3 = 1$$

Now, we can turn this into an augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 1 & 1 & | & 5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

We can see that the first column corresponds to the x_1 coefficients, the second column to x_2 coefficients, the third column to x_3 coefficients, and the fourth column to the constant terms.

Matrix Reduction [L5]

To get the solution, we often need to reduce a matrix to its row echelon form (REF).

For a matrix to be in REF, it needs to satisfy the following conditions:

- All zero rows are at the bottom
- The first nonzero entry in a row is a 1 (leading 1)
- Each leading 1 is to the right the leading 1s in the rows above.

If each leading 1 is the only nonzero entry in each column, then the matrix is said to be in reduced row echelon form (RREF).

Here are some examples:

Non-REF Matrix	REF Matrix	RREF Matrix
$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

To reduce a matrix to RREF, we follow the following steps:

1. If the matrix C is zero, stop.
2. Get the left-most non-zero column into the first row.
3. Divide/Multiply the first row, to get a leading 1.
4. Get rid of the rest of the column BELOW using this leading 1. That is, if a_i is the entry in this column of Row i, then add $-a_iR_1$ to R_i and put the result back in Row i.
5. Now we are done with row 1. Now, ignore the first row and go back to step 1 until all rows have been dealt with.

When all rows have been dealt with, the matrix will be in REF. Now time to put the matrix into RREF.

6. If the right most leading 1 is in row 1, go to step 8.
7. Start with the right most leading 1. Use it to get rid of all other coefficients above it in its column. That is, in row i, add $-a_i$ times this row to R_i and put result back into i.
8. Cover up the row you used and go to step 6.

These rules look very complicated, but in actuality they are very simple. An example can be found in the appendix.

Matrix Multiplication [L6]

We can multiply 2 matrices together. However, it is not the same thing as scalar multiplications; It has different rules.

If it not commutative so $AB \neq BA$ if A and B are matrices.

If AB is defined, this does not mean BA is defined.

Moreover, matrix multiplication only works if A is $m \times n$ and B is $n \times p$, which will create a product of $m \times p$.

When finding the value for row i and column j, we use the dot product of row i in matrix A, with the column j in matrix B.

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, find AB.

First, we notice that A is a 2×2 and B is 2×3. This means that YES we can do the multiplication (since the inner number are the same), and it will create a 2×3 matrix (using the outer values).

For the first row, first column, we will do the sum of the dot product of the elements of the first row in A, and the first column in B. For the first row, second column, it is the sum of the dot product of the elements in the first row of A, with the second column in B. And so on.

$$AB = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 4 & 1 \cdot 2 + 2 \cdot 5 & 1 \cdot 3 + 2 \cdot 6 \\ 3 \cdot 1 + 4 \cdot 4 & 3 \cdot 2 + 4 \cdot 5 & 3 \cdot 3 + 4 \cdot 6 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \end{bmatrix}$$

Matrix Rank [L6]

The rank of a matrix is simply the number of leading 1s in a matrix's REF.

$$rank\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix}\right) = 2$$

If the matrix is not in REF, then its rank will be the rank of its REF form.

The rank tells us the number of solutions in a matrix. Let A be a matrix, and $A|B$ be its augmented matrix.

- Inconsistent if $rank(A) < rank(A|B)$. This would cause something where $a = 0 \mid a \neq 0$ which is false.
- Unique solution if $rank(A) = rank(A|B)$. This means there are n equations, and n unknowns.
- Infinite solutions if $rank(A) = rank(A|B)$ and A is smaller than the number of columns in A.

Identity Matrix [L6]

The identity matrix is a matrix with ones on the diagonal (right to left), and zeroes everywhere else.

We say I_n is a $n \times n$ identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Vector Spaces

Vector Spaces and Subspaces [L7]

A vector space is for lack of a better word, a space where vectors can exist.

Many things can be classified as vectors such as traditional vectors, functions, and matrices.

For a vector space to be exist, three properties must exist:

- The sum of 2 vectors in the vectors space must be a vector in that space. (Closed under addition)
- Any scalar multiple of a vector must also be a vector in that space. (Closed under multiplication)
- The zero vector must exist.

There are also 7 other properties, but they are inherited from $\mathbb{R}^n$, so we don't really have to prove them.

If a vector space is inside another vector space, we call the inner one a subset of the outer one. For example, $\mathbb{R}^2$ is a subset of $\mathbb{R}^3$.

Prove that $v = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a + d = b + c\right\}$ is a subspace of $M_{22}(\mathbb{R})$.

This question is asking us to prove that v is a subspace of the 2×2 matrix space. So, we need to prove the three properties.

1. *Prove the existence of the zero vector:*

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *is in v since $a + d = 0 = b + c$*

2. *Prove that v is closed under addition:*

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in v$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in v$

$$(a_1 + a_2) + (d_1 + d_2) = (b_1 + b_2) + (c_1 + c_2)$$

$$(a_1 + d_1) + (a_2 + d_2) = (b_1 + b_2) + (c_1 + c_2)$$

Now, we can sub in the definition where $a + d = c + b$ to get:

$$(b_1 + c_1) + (b_2 + c_2) = (b_1 + b_2) + (c_1 + c_2)$$

$$(b_1 + b_2) + (c_1 + c_2) = (b_1 + b_2) + (c_1 + c_2)$$

3. *Prove that v is closed under scalar multiplication:*

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in v$, and $k \in \mathbb{R}$

$$kA = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}$$

$$ka_1 + kd_1 = kb_1 + kc_1$$

Now, I need to prove that the right side equals the left side.

$$k(a_1 + d_1) = k(b_1 + c_1)$$

I can sub in the definition to prove.

$$k(b_1 + c_1) = k(b_1 + c_1)$$

■

Span [L8]

If we have a set of vectors, $v = \overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$, then we say the span of v is all linear combinations of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$.

$$span\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$$

Linear Dependance [L9, L10, L11]

A set of vectors $\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}$ are linearly independent (LI) if there are a set of scalars $a_1, \ldots, a_n$ not all zero where $a_1\overrightarrow{v_1} + \cdots + a_n\overrightarrow{v_n} = \vec{0}$.

This is basically saying, if the solution to the above equation is $a_1 = \cdots = a_n = 0$, then it is linearly independent, otherwise, it is linearly dependent (LD).

We can say that a set of vectors in $\mathbb{R}^n$, can have at most n vectors to be LI.

Is the following set LI or LD? $\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$

We can just solve the equation to determine.

$$a_1\begin{pmatrix}0\\1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\0\\1\end{pmatrix} + a_3\begin{pmatrix}1\\1\\2\end{pmatrix} + a_4\begin{pmatrix}-1\\1\\0\end{pmatrix} = 0$$

We can come up with values of $1, 1, -1, 0$ *to satisfy this equation, so it is LD.*

$$1\begin{pmatrix}0\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\0\\1\end{pmatrix} - 1\begin{pmatrix}1\\1\\2\end{pmatrix} + 0\begin{pmatrix}-1\\1\\0\end{pmatrix} = 0$$

If we do not see an obvious solution, we can create a matrix to solve with the first column being $\begin{pmatrix}0\\1\\1\end{pmatrix}$*, second being* $\begin{pmatrix}1\\0\\1\end{pmatrix}$ *and so on. We will then come up with infinite solutions, which means that there it is LD.*

Basis [L11, L12]

A set in a vector space is a basis for that vector space if:

- The set is LI.
- The set spans the vector space.

Basically, a basis it the smallest possible spanning set for a vector space.

An ordered basis is a basis where the order of the vectors in the basis matter.

The following are equivalent if they are regular bases, but unique if they are ordered bases.

$$\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}$$

Dimension [L11]

The dimension of a vector space is the number of vectors in its basis.

$$\dim(\mathbb{R}^2) = 2 \qquad \dim(\mathbb{R}^n) = n$$
$$\dim(\mathbb{P}^2) = 3 \qquad \dim(\mathbb{P}^n) = n+1$$
$$\dim\big(M_{22}(\mathbb{R})\big) = 4 \qquad \dim\big(M_{mn}(\mathbb{R})\big) = m \times n$$

Column Space [L13, L14]

If A is a matrix, then $col(A) = span\{\overrightarrow{c_1}, \overrightarrow{c_2}, \dots, \overrightarrow{c_n}\}$ where $\overrightarrow{c_1}, \overrightarrow{c_2}, \dots, \overrightarrow{c_n}$ are the column vectors of A.

We can use the column space to find a basis for a set of vectors. The created basis will be made up of vectors of the given set.

Find a basis for $V = span\left\{\begin{pmatrix}2\\3\\4\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}9\\11\\15\end{pmatrix},\begin{pmatrix}3\\1\\1\end{pmatrix},\begin{pmatrix}7\\2\\4\end{pmatrix}\right\}$ which is a subset of V.

We can create a matrix where the vectors of V are the columns of the matrix and then row reduce it.

$$\begin{bmatrix}2 & 1 & 9 & 3 & 7\\ 3 & -1 & 11 & 1 & 2\\ 4 & 1 & 15 & 1 & 4\end{bmatrix} \sim \dots \sim \begin{bmatrix}1 & 0 & 4 & 0 & 1\\ 0 & 1 & 1 & 0 & 2\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}$$

We can see that the leading ones are in columns 1, 2, and 4.

So, the basis will be the corresponding vectors $\left\{\begin{pmatrix}2\\3\\4\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}3\\1\\1\end{pmatrix}\right\}$.

Note how there are 3 vectors in the basis, and each vector has three coordinates ($\mathbb{R}^3$)

Row Space [L13]

If A is a matrix, then $Row(A) = span\{\overrightarrow{r_1}, \overrightarrow{r_2}, \dots, \overrightarrow{r_m}\}$ where $\overrightarrow{r_1}, \overrightarrow{r_2}, \dots, \overrightarrow{r_m}$ are the row vectors of A.

We know that if $A \sim B$, then $Row(A) = Row(B)$.

We can use the row space to find a basis for a set of vectors. This basis will not necessarily be made up of the vectors of the given set.

Find a basis of $\mathbb{R}^4$ containing $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ and $\begin{pmatrix}2\\4\\7\\8\end{pmatrix}$.

We know that a basis for $\mathbb{R}^4$ will need 4 vectors.

So, using the row space, we will end up needing a leading one in each column.

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 7 & 8\end{bmatrix} \sim \dots \sim \begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 0 & 1 & 0\end{bmatrix}$$

We will add two vectors, each with a leading one in columns 2 and 4.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, we can say that $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ *is a basis for* $\mathbb{R}^4$.

Null Space [L13]

This is a vector space of an $n \times m$ matrix A where $A\vec{x} = 0$.

We can say that $\dim(Null(A)) + rank(A) = n$ where n is the number of columns.

Find a basis for $Null(A)$ if $A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

To find the null space, we need to solve $Ax = 0$.

$$= \begin{bmatrix} 1 & 2 & 3 & -3 & | & 0 \\ 0 & 1 & 1 & 2 & | & 0 \\ 1 & 0 & 1 & 1 & | & 0 \end{bmatrix}$$

Now we can row reduce the matrix and get:

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & | & 0 \\ 0 & 1 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since this matrix is $Ax = 0$, *we can see that:*

$$x_1 + x_3 + x_4 = 0$$

$$x_2 + x_3 - 2x_4 = 0$$

Since x_3 *and* x_4 *are free (they are not leading ones), we can solve in terms of them.*

$$\begin{aligned} x_1 &= -x_3 - x_4 \\ x_2 &= 2x_4 - x_3 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

Now, we can turn this into a basis.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$Null(A) = span \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This is a basis for $Null(A)$.

Matix Inverse [L14, L15]

If $\mathrm{AB} = \mathrm{BA} = \mathrm{I_n}$, and A and B are $n \times n$, then A is the inverse of B, and B is the inverse of A.

$$B = A^{-1}, \qquad A = B^{-1}$$

To find the inverse of an $n \times n$ matrix A, we solve the following system:

$$(A|I_n) \sim \ldots \sim (I_n|A^{-1})$$

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Since A is an $n \times n$ matrix, we can do this. Solve the system $(A|I_3) \sim \ldots \sim (I_3|A^{-1})$

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & | & 1 & 0 & -3 \\ 0 & 1 & 0 & | & 0 & 1 & -1 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 1 & -2 & -3 \\ 0 & 1 & 0 & | & 0 & 1 & -1 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

So, we can see that the inverse of A is:

$$A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonality [L15, L16]

Two vectors are orthogonal if the dot product is zero.

A set of vectors is an orthogonal set if the dot product between any and all vectors in that set is zero. None of the vectors can be $\vec{0}$.

We know that any orthogonal set is LI so an orthogonal set in $\mathbb{R}^n$ has up to n vectors, and is a basis of $\mathbb{R}^n$.

If we have an orthogonal basis for a vector space W, then any vector in W can be written as:

$$\vec{w} = \frac{\vec{w} \cdot \overrightarrow{w_1}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}} \overrightarrow{w_1} + \cdots + \frac{\vec{w} \cdot \overrightarrow{w_m}}{\overrightarrow{w_m} \cdot \overrightarrow{w_m}} \overrightarrow{w_m}$$

Consider the orthogonal set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$. Write $\begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix}$ as a linear combination of the set.

We can verify that it is an orthogonal set by finding the dot product between all the vectors, and it will be zero.

Then, we can use the above formula.

$$\begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{20}{6} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{6}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{10}{3}\begin{pmatrix}1\\2\\1\end{pmatrix} - 3\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Vector Projections [L16]

When we are projecting a vector v onto a space U, we are finding the closest approximation of v that is in U.

To find the projection of $\vec{v}$ onto w where $\{\overrightarrow{w_1}, \dots, \overrightarrow{w_m}\}$ is an orthogonal basis for w:

$$proj_w \vec{v} = \frac{\vec{v} \cdot \overrightarrow{w_1}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}}\overrightarrow{w_1} + \cdots + \frac{\vec{v} \cdot \overrightarrow{w_m}}{\overrightarrow{w_m} \cdot \overrightarrow{w_m}}\overrightarrow{w_m}$$

Gram Schmidt Algorithm [L16]

If we do not have an orthogonal basis, then we can use the Gram Schmidt algorithm to find one.

If $\{\overrightarrow{u_1}, \dots, \overrightarrow{u_m}\}$ is a basis for a vector space U, then the orthogonal basis $\{\overrightarrow{w_1}, \dots, \overrightarrow{w_m}\}$ is:

$$\begin{aligned}
&\overrightarrow{w_1} = \overrightarrow{u_1} \\
&\overrightarrow{w_2} = \overrightarrow{u_2} - proj_{\overrightarrow{w_1}}\overrightarrow{u_2} \\
&\overrightarrow{w_3} = \overrightarrow{u_3} - proj_{\overrightarrow{w_1}}\overrightarrow{u_3} - proj_{\overrightarrow{w_2}}\overrightarrow{u_3} \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
&\overrightarrow{w_m} = \overrightarrow{u_m} - proj_{\overrightarrow{w_1}}\overrightarrow{u_m} - proj_{\overrightarrow{w_2}}\overrightarrow{u_m} - \cdots - proj_{\overrightarrow{w_{m-1}}}\overrightarrow{u_m}
\end{aligned}$$

Find an orthogonal basis for $w = span\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\-1\\1\end{pmatrix}\right\}$.

First, we need to find a basis for the vectors. We can use the column space method.

$$\begin{bmatrix}1&1&2\\0&1&1\\0&1&-1\\1&0&1\end{bmatrix} \sim \begin{bmatrix}1&1&2\\0&1&1\\0&1&-1\\0&-1&-1\end{bmatrix} \sim \begin{bmatrix}1&1&2\\0&1&1\\0&0&1\\0&0&0\end{bmatrix}$$

We see a leading one in every column, so the basis is just the set itself. This is because all vectors are LI.

Next, we can use the Gram Schmidt algorithm to turn this basis into an orthogonal basis.

$$\overrightarrow{w_1} = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\overrightarrow{w_2} = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}}{\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}} \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\1\\1\\-\frac{1}{2}\end{pmatrix}$$

$$\overrightarrow{w_3} = \begin{pmatrix}2\\1\\-1\\1\end{pmatrix} - \frac{\begin{pmatrix}2\\1\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}}{\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}} \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}2\\1\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}\frac{1}{2}\\1\\1\\-\frac{1}{2}\end{pmatrix}}{\begin{pmatrix}\frac{1}{2}\\1\\1\\-\frac{1}{2}\end{pmatrix} \cdot \begin{pmatrix}\frac{1}{2}\\1\\1\\-\frac{1}{2}\end{pmatrix}} \begin{pmatrix}\frac{1}{2}\\1\\1\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}2\\1\\-1\\1\end{pmatrix} + \frac{3}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{0.5}{\frac{10}{4}}\begin{pmatrix}\frac{1}{2}\\1\\1\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{2}{5}\\\frac{4}{5}\\-\frac{6}{5}\\-\frac{2}{5}\end{pmatrix}$$

So, $\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{2}\\1\\1\\-\frac{1}{2}\end{pmatrix}, \begin{pmatrix}\frac{2}{5}\\\frac{4}{5}\\-\frac{6}{5}\\-\frac{2}{5}\end{pmatrix}\right\}$ *is an orthogonal basis for the above set.*

Determinants and Eigen

Matrix Determinant [L18]

This is only for $n \times n$ matrices.

If we have a 2×2 matrix, then $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

If $n \geq 3$, then we have to use the definition. This has a very complicated and confusing equation, so I will provide an example.

Find $\det\left(\begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right)$.

$$A = \begin{pmatrix} 2^{+} & 3^{-} & 4^{+} & 5^{-} \\ 1^{-} & 0^{+} & 0^{-} & 1^{+} \\ 0^{+} & 1^{-} & 0^{+} & 1^{-} \\ 0^{-} & 0^{+} & 1^{-} & 0^{+} \end{pmatrix}$$

The first step here, is to create a checkerboard of plus and minus signs beside each of the matrix coefficients starting with a plus at the top left value.

Next, we can pick any row or column which will be used to compute the determinant. It works best to pick one with lots of zeros.

I will pick row 4.

Now we will go across row 4, write the sign corresponding to that cell in the matrix (starting from column 1, then 2...), then the coefficient on row 4, and then the determinant of the whole matrix MINUS row 4, and minus column i.

$$\det(A) = -0 \cdot \det\left(\begin{bmatrix} 3 & 4 & 5 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix} 2 & 4 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right) - 1 \cdot \det\left(\begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)$$

Since the first 2, and the 4th term are 0, we can remove them. Now we are left with:

$$\det(A) = -1 \cdot \det\left(\begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)$$

We can go and calculate the whole thing again using this matrix. We will use the first column this time.

$$\det(A) = -1 \cdot \det\left(\begin{bmatrix} 2^{+} & 3^{-} & 5^{+} \\ 1^{-} & 0^{+} & 1^{-} \\ 0^{+} & 1^{-} & 1^{+} \end{bmatrix}\right) = -1\left(2 \cdot \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) - 1 \cdot \det\left(\begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}\right)\right)$$

Now this looks much better. We can use the non-recursive 2×2 formula to solve.

$$= -1\big(2(0 \cdot 1 - 1 \cdot 1) - 1(3 \cdot 1 - 5 \cdot 1) + 0(3 \cdot 1 - 5 \cdot 0)\big) = -1(-2 + 2) = 0$$

We can see that having lots of zeroes in the matrix really helps. We are able to row reduce a matrix, but it may change the determinant.

Let A be a matrix, and B be its row reduced form:

- If the row operation interchanges 2 rows, then $\det(B) = -\det(A)$
- If the row operation multiplies a row by r, then $\det(B) = r \cdot \det(A)$
- If the row operation adds a multiple of another row, then $\det(A) = \det(B)$

If the matrix is triangular (only zeroes below the triangle), then we can say that the determinant is just the product of those values.

Find $\det\left(\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 5 & 6 & 7\\ 0 & 0 & 8 & 9\\ 0 & 0 & 0 & 10\end{bmatrix}\right)$

This looks long, but since there are only zeroes below the diagonal, we can use the shortcut for triangular matrices.

$$det\left(\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 5 & 6 & 7\\ 0 & 0 & 8 & 9\\ 0 & 0 & 0 & 10\end{bmatrix}\right) = 1\cdot 5\cdot 8\cdot 10 = 400$$

Eigenvalues and Eigenvectors [L19, L20]

If A is an $n \times n$ matrix, λ is a scalar, and $\vec{x}$ is a non zero vector, and $A\vec{x} = \lambda\vec{x}$, then $\vec{x}$ is an eigenvector of A and λ is an eigenvalue of A.

To find all eigenvalues for a matrix, we solve the system $\det(A - \lambda I) = 0$

To find the set of eigenvectors for one eigenvalue, we solve the system $(A - \lambda I)\,\vec{x} = 0$

Find all eigenvectors, and all eigenspaces of $A = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$

To find the eigenvectors, we first need to find the eigenvalues.

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{bmatrix}\right) = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$(3-\lambda)(-1-\lambda) = 0$$

$$\lambda = 3, \qquad \lambda = -1$$

Now that we have the eigenvalues, we can first find the eigenvectors for $\lambda = 3$

$$(A - 3I)\vec{x} = 0$$

Solve the matrix:

$$\begin{bmatrix}-2 & 2 & | & 0\\ 2 & -2 & | & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1 & | & 0\\ 0 & 0 & | & 0\end{bmatrix} \Rightarrow \begin{matrix}x = y\\ y = y\end{matrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix} = y\begin{pmatrix}1\\ 1\end{pmatrix}$$

We can say that a basis for the eigenspace of $\lambda = 1$ *is* $E_3 = \left\{\begin{pmatrix}1\\ 1\end{pmatrix}\right\}$.

Now, we can find the eigenvectors of $\lambda = -1$

$$(A - (-1)I)\vec{x} = 0$$

$$\begin{bmatrix} 2 & 2 & | & 0 \\ 2 & 2 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \quad => \quad \begin{matrix} x = -y \\ y = y \end{matrix} \quad => \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So, the basis for the eigenspace $E_{-1} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

Diagonalizable

A matrix A is diagonalizable if there is a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors of A.

Basically, a matrix in $\mathbb{R}^n$ is diagonalizable if it has n eigenvectors.

Is $\mathrm{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ diagonalizable?

We will need to find the number of eigenvectors for A. If it is 2, then Yes, if not, then No.

First, find the eigenvalues.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \right) = (3-\lambda)^2 = 0$$

$$\lambda = 3$$

Now, we can find the eigenvectors associated with that eigenvalue.

$$(A - 3I)\vec{x} = 0$$

$$\begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \quad => \quad \begin{matrix} x = x \\ y = 0 \end{matrix} \quad => \quad \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$E_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

Since there is only one eigenvector for the matrix A which is in $\mathbb{R}^2$, then A is not diagonalizable.

If we find the eigen values of a matrix A, and A ends up being diagonalizable, then we can use this information to calculate high powers of the matrix A.

We can use the fact that if P is a matrix made up of eigenvectors, and P^{-1} is its inverse, and D is the matrix with eigenvalues on the diagonal, then:

$$A^n = PD^nP^{-1}$$

Find A^{2000} if $\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

This will be very hard to do using regular methods.

So, we can find the eigenvectors of this matrix to calculate A^{2000}.

Find the eigenvalues:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \quad => \quad \lambda = 1, \qquad \lambda = -1, \qquad \lambda = 2$$

Now we can find the eigenvectors (I will skip lots of steps since it is a long problem):

$$A-1I=\begin{bmatrix}0&1&1\\0&-2&0\\0&0&1\end{bmatrix}\sim\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}\quad=>\quad\begin{pmatrix}x\\y\\z\end{pmatrix}=x\begin{pmatrix}1\\0\\0\end{pmatrix}\quad=>\quad E_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$A-(-1)I=\begin{bmatrix}2&1&1\\0&0&0\\0&0&3\end{bmatrix}\quad=>\quad\begin{pmatrix}x\\y\\z\end{pmatrix}=x\begin{pmatrix}1\\-2\\0\end{pmatrix}\quad=>\quad E_{-1}=\begin{pmatrix}1\\-2\\0\end{pmatrix}$$

$$A-2I=\begin{bmatrix}-1&1&1\\0&-3&0\\0&0&0\end{bmatrix}\quad=>\quad\begin{pmatrix}x\\y\\z\end{pmatrix}=z\begin{pmatrix}1\\0\\1\end{pmatrix}\quad=>\quad E_2=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

Now we can write a matrix using the eigenvectors and a matrix using the eigenvalues on the diagonal.

$$P=\begin{bmatrix}1&1&1\\0&-2&0\\0&0&1\end{bmatrix}$$

$$D=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&2\end{bmatrix}$$

Now we can find the inverse of P

$$P^{-1}=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-2&0&0&1&0\\0&0&1&0&0&1\end{array}\right]\sim\ldots\sim\left[\begin{array}{ccc|ccc}1&0&0&1&1/2&-1\\0&1&0&0&-1/2&0\\0&0&1&0&0&1\end{array}\right]=\begin{bmatrix}1&1/2&-1\\0&-1/2&0\\0&0&1\end{bmatrix}$$

Now, we can finally calculate A^{2000}

$$A^{2000}=PD^{2000}P^{-1}=\begin{bmatrix}1&1&1\\0&-2&0\\0&0&1\end{bmatrix}\begin{bmatrix}1^{2000}&0&0\\0&(-1)^{2000}&0\\0&0&2^{2000}\end{bmatrix}\begin{bmatrix}1&1/2&-1\\0&-1/2&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1&1&2^{2000}\\0&-2&0\\0&0&2^{2000}\end{bmatrix}\begin{bmatrix}1&1/2&-1\\0&-1/2&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&2^{2000}\\0&1&0\\0&0&2^{2000}\end{bmatrix}$$

Multiplicity

The algebraic multiplicity of λ is the number of times its factor appears in $\det(A-\lambda I)$.

The geometric multiplicity of λ is $\dim(E_\lambda)$.

Find the algebraic and geometric multiplicities of $A=\begin{bmatrix}2&4&-3\\0&3&5\\0&0&3\end{bmatrix}$.

First, we will need to find the eigenvalues.

$$\det(A-\lambda I)=\begin{bmatrix}2-\lambda&4&-3\\0&3-\lambda&5\\0&0&3-\lambda\end{bmatrix}$$

Since this is a triangular matrix, the determinant is just:

$$(2-\lambda)(3-\lambda)(3-\lambda) = (2-\lambda)(3-\lambda)^2$$

$$\lambda = 3, \qquad \lambda = 2$$

Since the $(3-\lambda)^2$ factor is order 2, its algebraic multiplicity is 2. The algebraic multiplicity for $\lambda = 2$ is just 1.

Now, we can find the eigenvectors for these eigenvalues starting with $\lambda = 2$

$$A - 2I = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 4 & -3 & | & 0 \\ 0 & 1 & 5 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = x \\ y = 0 \\ z = 0 \end{matrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$E_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Onto $\lambda = 3$

$$A - 3I = \begin{bmatrix} -1 & 4 & -3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 4 & -3 & | & 0 \\ 0 & 0 & 5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = 4y \\ y = y \\ z = 0 \end{matrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$E_3 = \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Since the dimension of both E_2 and E_3 is 1, then the geometric multiplicities of both eigenvalues is 1.

Linear Transformations [L21]

If U and V are 2 vector spaces, then a linear transformation T is a function from U to V where:

$$T(\overrightarrow{u_1} + \overrightarrow{u_2}) = T(u_1) + T(\overrightarrow{u_2})$$

$$T(c\overrightarrow{u_1}) = cT(\overrightarrow{u_1})$$

Basically, there must be addition, and multiplication.

The standard matrix A is found where $T(\vec{u}) = A\vec{u}$.

The columns of the standard matrix A are $T(e_1), T(e_2), \dots, T(e_n)$ where e_1 is $(1,0,0,\dots,0)$, e_2 is $(0,1,0,\dots,0)$, and e_n is $(0,0,0,\dots,1)$.

The kernel of a linear transformation $\ker(A)$ are the roots of T. The image of a linear transformation $im(T)$ represents the range.

$$\ker(T) = Null(A), \qquad Im(T) = Col(A), \qquad \dim(\ker(T)) + \dim\big(Im(T)\big) = n$$

Verify that $T: \mathbb{R}^2 \to \mathbb{R}^3$ where $T(x,y) = (x, y, x+y)$ is a linear transformation.

First prove addition.

$$T(v_1 + v_2) = T(x_1 + x_2, y_1 + y_2) = (x_1 + x_2, y_1 + y_2, x_1 + x_2 + y_1 + y_2)$$

$$T(v_1) + T(v_2) = T(x_1, y_1) + T(x_2, y_2) = (x_1, y_1, x_1 + y_1) + (x_2, y_2, x_1 + y_1)$$
$$= (x_1 + x_2, y_1 + y_2, x_1 + x_2 + y_1 + y_2)$$

Since the two statements are equal, then the addition part is proven.

Now prove multiplication.

$$T(kv_1) = T(kx_1, ky_1) = (kx_1, ky_1, kx_1 + ky_1)$$

$$kT(v_2) = k(x_2, y_2, x_2 + y_2) = (kx_1, ky_1, kx_1 + ky_1)$$

Since those two statements are also equal, then the multiplication part is proven as well.

Therefore, $T(x, y)$ *is a linear transformation.*

More examples are found in the appendix.

Extras

Complex Numbers [L17]

The complex numbers are in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. So, $i^2 = \left(\sqrt{-1}\right)^2 = 1$.

So, for a number to be $\in \mathbb{C}$, it must be in the correct form.

Turn the following into a complex number: $\frac{3+2i}{-2+4i}$ $$\frac{3+2i}{-2+4i} \cdot \frac{-2+4i}{-2+4i} = \frac{2-16i}{20} = -\frac{1}{10} - \frac{4}{5}i \text{ which is } \in \mathbb{C}$$

List of Equivalent Statements

If A is an $n \times m$ matrix, the following statements are equivalent (all are true, or all are false):

- Each variable is a leading variable.
- There is a leading one in every column of the RREF of A.
- $A\vec{x} = 0$ has a unique solution.
- The columns of A are LI.
- $Null(A) = \{\vec{0}\}$
- $\dim\left(Null(A)\right) = 0$
- $rank(A) = n$

If A is an $n \times n$ matrix:

- $rank(A) = n$
- $A\vec{x} = \vec{b}$ has a unique solution if $\vec{b} \in \mathbb{R}^n$
- The RREF of A is I_n
- $Null(A) = \{\vec{0}\}$
- $col(A) = \mathbb{R}^n$
- $row(A) = \mathbb{R}^n$
- Columns of A are LI.
- Rows of A are LI.
- The columns of A is a basis for $\mathbb{R}^n$.
- The rows of A is a basis for $\mathbb{R}^n$.
- A is invertible.
- $\det(A) \neq 0$
- $\det(A^T) \neq 0$ This means, that all the properties are also true about A^T.

Appendix

Row Reducing a Matrix Example

■ **Example 12.2.1** Let's run this on $C = \begin{bmatrix} 0 & 0 & -2 & 2 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$.

Step 1 The matrix is not zero, so we proceed.

Step 2 The left-most nonzero column is column 1, but we need to get a non-zero entry in row 1, so let's interchange rows 1 and 2:

$$\begin{bmatrix} 0 & 0 & -2 & 2 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & -2 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

Step 3 There's no need to rescale: we already have a leading 1 in row 1.

Step 4 We need to clear the column below our leading 1: we subtract the first row from rows 3 and 4:

$$\begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & -2 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{matrix} -R_1 + R_3 \to R_3 \\ \sim \\ -R_1 + R_4 \to R_4 \end{matrix} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

Step 5 We ignore the first row and go to Step 1.

Step 1 Even ignoring the original first row, the matrix is not zero.

Step 2 The left-most non zero column now (remember: we ignore row 1) is column 3, and there is a non-zero entry (2) in the second row (which is the *first* row of the matrix left when we ignore the first row of the whole matrix), so there's nothing to do now.

Step 3 Let's divide row 2 by -2 to get a leading one in the second row.

$$\begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \underset{\sim}{-\tfrac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

Step 4 Now we need to clear column 3 below our new leading one:

$$\begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{matrix} R_2 + R_3 \to R_3 \\ 3R_2 + R_4 \to R_4 \\ \sim \end{matrix} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5 Now we ignore the first two rows, and go to Step 1

Step 1 Well if we we ignore the first two rows, the matrix we see is zero, so we stop this part and proceed to Step 6

Step 6 The *right most* leading 1 is in row 2, so we proceed to Step 7

Step 7 We use the leading one in row two to clear its column above it:

$$\begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{-3R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 8 We cover up row 2 and go back to Step 6

Step 6 Ignoring row 2, now the right-most leading 1 *is* in row 1! So we stop. The matrix is now in RREF.

We're not machines, of course, and we could see at the end of Step 7 that the matrix was in RREF, so there was no need to have done Steps 8 and 6, but we did it above to illustrate the algorithm.

Linear Transformation Examples

This example shows that we can work with linear transformations for non-standard functions. I skipped some steps here, but the important part is the concept, which is demonstrated.

1. Determine if the following functions are linear transformation. If they are, write the associated standard matrix. If they are not, justify your answer.

c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (1,-1,2) \times (x,y,z)$ ($\times$ is the cross product).

Do cross Product.

$$\begin{matrix} -1 & 2 & 1 & -1 \\ -z-2y & 2x-z & y+x & \\ y & z & x & y \end{matrix}$$

$$(-z-2y,\ 2x-z,\ y+x)$$

Verify Addition

$$T(V_1+V_2) = (-(z_1+z_2)-2(y_1+y_2),\ 2(x_1+x_2)-(z_1+z_2),\ y_1+y_2+x_1+x_2)$$

$$= (-z_1-z_2-2y_1-2y_2,\ 2x_1+2x_2-z_1-z_2,\ x_1+x_2+y_1+y_2)$$

$$T(V_1)+T(V_2) = (-z_1-2y_1,\ 2x_1-z_1,\ y_1+x_1)+(-z_2-2y_2,\ 2x_2-z_2,\ y_2+x_2)$$

$$= \ldots$$

we see they are equal

Test multiplication

$$T(kV_1) = (-kz_1-2ky_1,\ \ldots) = k(T(V)) \checkmark$$

Now standard matrix.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + z\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

This next one piggybacks on the previous.

2. For each of the linear transformations T in question 1, find a basis for $Im(T)$ and for $Ker(T)$.

c) $A = \begin{bmatrix} 6 & -2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$Col(A) = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} = Im(T)$

$Null(A) = Ker(T) = Ax = 0$

$\Rightarrow \begin{aligned} x + y &= 0 \\ -2y + z &= 0 \end{aligned}$

$\begin{aligned} x &= -y \\ y &= y \\ z &= 2y \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$Null(A) = Ker(T) = \left\{ \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$