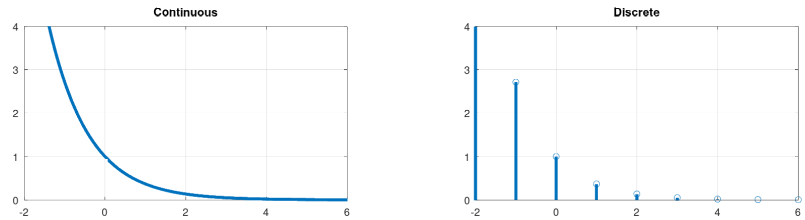
# ELG 3125 Summary Sheet

## 1 Basics of Signals and Systems

We can have a signal that is either continuous or discrete. These signals are represented as math functions. Computers always will display and work with a discrete signal. However




often it is to model a continuous signal.

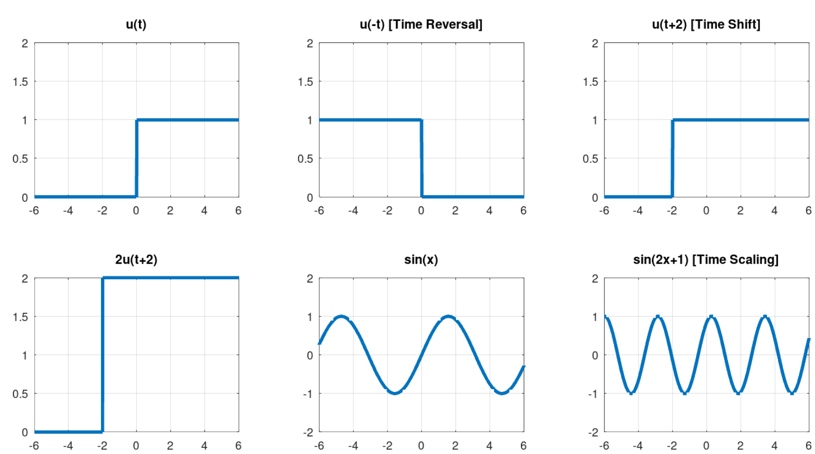
We call a signal a power signal if its average power is finite. Similarly, we call a signal an energy signal if its total energy is finite.

> **Ex.** A 120VAC wall outlet is a power signal.
>
> This is because the average power is finite. If we have for example a 1500w heater connected, it will always draw 1500W.
>
> However if we keep it running for a long time, it will use a very large amount of energy. So it is infinite energy.

### 1.1 Transformations

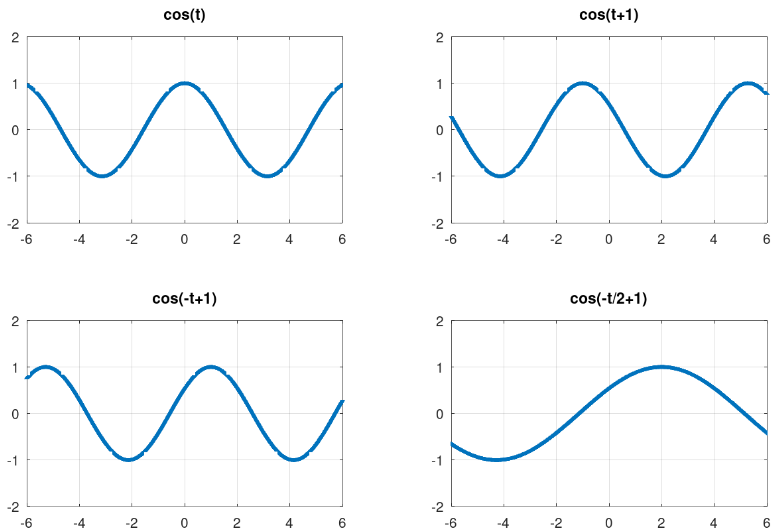
There are many transformations.

When transforming a signal, we usually start with any time shifts. Then we apply other transformations such as a time reversal, or time scaling.



**Ex.** Plot $\cos(-t/2+1)$

We take it in 4 steps:




## 1.2 Periodicity

A signal is called periodic if:

$$x(t) = x(t+T)\ \forall\ t \tag{1}$$

We call the fundamental period $T_0$ the smallest *positive* value of $T$ for which Equation 1 holds.

We have the same idea in discrete time except we change $t$ for $n$, and $T$ for $N$.

$$x[n] = x[n+N]\ \forall\ n \tag{2}$$

Note that any complex exponential in the form of $e^{j\omega_0 t}$ is periodic with period $T = \frac{2\pi}{\omega_0}$.

**Ex.** Find the period of $x[n]$.

$$x[n] = e^{j(\frac{2\pi}{3})n} + e^{j(\frac{3\pi}{4})n}$$

I need to find both periods, and find the greatest common multiple.

Note that since I am in discrete time, the period must be an integer.

$$T_1 = \frac{2\pi}{\frac{2\pi}{3}} = 3$$

$$T_2 = \frac{2\pi}{\frac{3\pi}{4}} = \frac{8}{3} = 8 \text{ since } \frac{8}{3} \notin \mathbb{Z}$$

$$T = LCM(3, 8) = 24$$

## 1.3 Even and Odd Functions

We call a signal Even if it satisfies Equation 3 or Odd if it satisfies Equation 4.

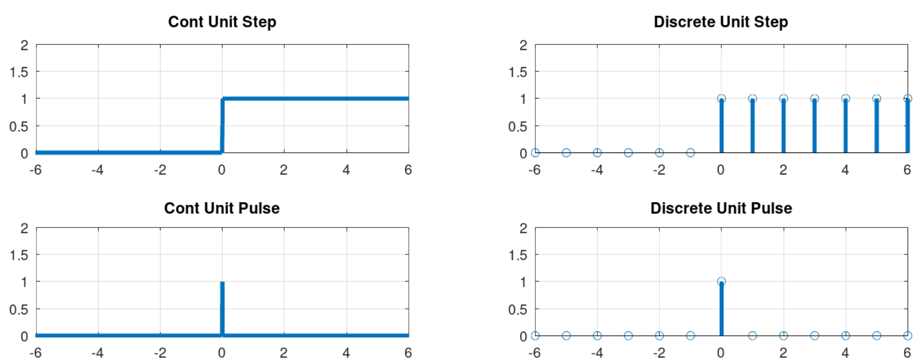
$$x(t) = x(-t) \tag{3}$$
$$x(t) = -x(-t) \tag{4}$$

We can construct any signal by using its even and odd portions.

$$x(t) = Ev\{x(t)\} + Od\{x(t)\} = \frac{1}{2}(x(t) + x(-t)) + \frac{1}{2}(x(t) - x(-t)) \tag{5}$$

## 1.4 Unit Impulse and Unit Step

These are two very useful functions as defined below. The unit step is 0 when x is negative,




and 1 when positive or 0. The impulse is 1 only when x is 0.

The unit step function is called $u(t)$ or $u[n]$. The unit pulse is called $\delta(t)$ or $\delta[n]$.

## 1.5 System Properties

### 1.5.1 Memory

A system is memoryless if the output is only dependant on the input at the same time.

So basically we do not see any time shifts such as $t - 1$ or $t + 1$.

### 1.5.2 Invertibility

A system is invertible if distinct inputs lead to distinct outputs.

### 1.5.3 Causality

A system is causal if the output at any time depends only on input values of present and in the past. It does not depend on any future values.

This is also referred to as being nonanticipative.

Note that all memoryless systems are causal.

| **Ex.** |
|---|

### 1.5.4 Stability

A system is stable if when inputs are bounded, the outputs are always bounded. So:

$$|x(t)| < \infty \implies |y(t)| < \infty \tag{6}$$

Assuming $x(t)$ is input, and $y(t)$ is output.

### 1.5.5 Time Invariance

A system is time invariant if a time shift in the input results in an identical time shift in the output.

| **Ex.** |
|---|

### 1.5.6 Linearity

A system is linear if Equation 7 is satisfied.

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t) \tag{7}$$

| **Ex.** |
|---|

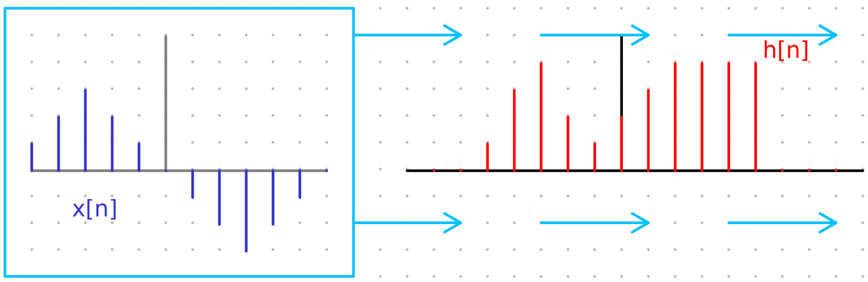
# 2 LTI Systems

Many systems in the real world are linear and time invariant. An example is a signal amplifier.

## 2.1 Convolution in Discrete Time

The convolution sum contains 3 parts.

- $h[n]$ is the response of the LTI system
- $x[n]$ is the input to the LTI system
- $y[n]$ is the output of the LTI system




**Ex.** Consider a signal amplifier such as a microphone amplifier, or a speaker amplifier.

If we have a microphone or speaker, the input $y[n]$ is the input such as the voice or the demodulated radio signal.

The response $h[n]$ is the algorithms that do all the amplification, and maybe some noise reduction.

The output $y[n]$ is the final audio signal such as what we hear from a speaker.

The convolution sum is how we calculate the output given an input and response. We can think of this as we have the response $h[n]$, and then we send through the input $x[n]$ throughout the entire response.

The convolution is represented in Equation 8.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] \tag{8}$$

Note that when finding the convolution, we now consider $k$ as the x axis variable, and $n$ as a constant.

To calculate the convolution sum, we draw $h[n-k]$ and $x[k]$. Then we analyze all overlap regions and find $x[k] \cdot h[n-k]$ for that region.

## 2.2 Convolution in Continuous Time

This is very similar to the convolution in discrete time except now we have $h(t), y(t), x(t)$. We also use an integral to calculate the convolution instead of a sum as shown in Equation 9.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = x(t) * h(t) \tag{9}$$

Again to solve, we break it up into all the overlap regions and find the product $(x(t) \cdot h(t))$ for each region.

**Ex.** We have a system response $h(t)$. The input $x(t) = \delta(t)$. What is the output $y(t)$?

This is very simple. We could go about calculating the convolution sum using equation 9, but we can also think this through logically.

Recall that the response $h(t)$ is just the stuff that is applied to the input signal $x(t)$. Since the input signal is just a single pulse, it will travel through the system visiting each point only **once**.

This means that the output will just be the response.

$$y(t) = x(t) * h(t) = \delta(t) * h(t) = h(t)$$

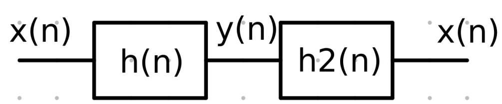
This is actually a general equation where:

$$x(t) * \delta(t - t_0) = x(t - t_0) \tag{10}$$

## 2.3 Properties

An LTI system is memoryless if $h[n] = K\delta[n]$ where $k \in \mathbb{R}$. This means $y[n] = Kx[n]$.

An LTI system is invertable if we can get back the input after applying a convolution to the output.




An LTI system is causal if $h[n] = 0$ for $n < 0$.

An LTI system is stable if for all finite inputs to the response, the output os finite: $\int_{-\infty}^{\infty} |h(t)| \mathrm{d}\tau < \infty$.

## 2.4 Differential and Difference Equations

These are equations in the form of:

$$y[n] = y[n-1] + x[n]$$

We can solve by using the initial rest condition which is a time when the output is 0. Then we can solve for consecutive values of $n$ starting at initial rest.

**Ex.**

# 3 Fourier Series

Any periodic signal can be represented by sinusoids (or complex exponentials).

We have the signal of $e^{j\omega_0 t}$ which is periodic. It contains both sin and cos components.

If we change it to $e^{jk\omega_0 t}$, for $k = 0, \pm1, \pm2, ...$ then we say that these are harmonically related. The first harmonic has $k = \pm1$, the second $k = \pm2$, and so on. We have:

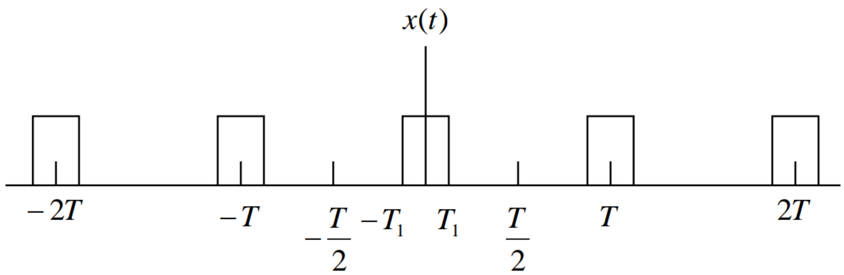
$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right] \tag{11}$$

We see that the $e$ terms are constant, only the coefficients change a meaningful amount. We can calculate the coefficients using Equation 12.

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\mathrm{d}t \tag{12}$$

$$a_0 = \frac{\text{AREA}}{T} \qquad \text{Specific Case for DC offset } (a_0) \tag{13}$$

**Ex.** Find $a_k$ and $a_0$




We need to use Equation 12.

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\mathrm{d}t = \frac{1}{T}\int_{T_1}^{-T_1} 1\cdot e^{-jk\omega_0 t}\mathrm{d}t = \frac{1}{Tjk\omega_0}e^{-jk\omega_0 t}|_{-T_1}^{T_1} = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1 - e^{-jk\omega_0 T_1}}}{2j}\right]$$

$$= \frac{2}{k\omega_0 T}\cdot \sin(k\omega_0 T_1) \qquad \text{Using an identity}$$

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad \text{Using } T = \frac{2\pi}{\omega_0}$$

$$a_0 = \frac{\text{AREA}}{T} = \frac{2T_1}{T} \qquad \text{We could also use } a_k \text{ to get } 0/0\text{, and use l'Hopital}$$

## 3.1 Properties of Continuous Fourier Series

If we have coefficients $a_k$ already for a signal $x(t)$, or maybe more than one signal $x(t)$ and $y(t)$, then after applying some changes to the signals we come up with new coefficients by these properties.

| Property | Periodic Signal | Coefficients |
|---|---|---|
| Linearity | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| Time Shifting | $x(t - t_0)$ | $e^{-jk\omega_0 t_0} a_k$ |
| Frequency Shifting | $e^{jM\omega_0 t} x(t)$ | $a_{k-M}$ |
| Conjugation | $x^*(t)$ | $a^*_{-k}$ |
| Time Reversal | $x(-t)$ | $a_{-k}$ |
| Time Scaling | $x(\alpha t),\ \alpha > 0$ | $a_k$ |
| Periodic Convolution | $\int_T x(\tau) y(t-\tau) \mathrm{d}\tau$ | $Ta_k b_k$ |
| Multiplication | $x(t)y(t)$ | $\sum_{l=-\infty}^{\infty} a_l b_{k-l}$ |
| Differentiation | $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ | $jk\omega_0 a_k = jk\frac{w\pi}{T} a_k$ |
| Parseval's Relation | $\frac{1}{T}\int_T \|x(t)\|^2 \mathrm{d}t = \sum_{k=-\infty}^{\infty} \|a_k\|^2$ | |

## 3.2 Properties of Discrete Fourier Series

It is the same idea with discrete signals. We have the equations for the coefficients, and the properties.

$$x[n] = \sum_{k=N} a_k e^{jk\omega_0 n} \tag{14}$$

$$a_k = \frac{1}{N} \sum_{n=N} x[n] e^{-jk\omega_0} n \tag{15}$$

| Property | Periodic Signal | Coefficients |
|---|---|---|
| Linearity | $Ax[n] + By[n]$ | $Aa_k + Bb_k$ |
| Time Shifting | $x[n - n_0]$ | $e^{-jk\omega_0 n_0} a_k$ |
| Frequency Shifting | $e^{jM\omega_0 n} x[n]$ | $a_{k-M}$ |
| Conjugation | $x^*[n]$ | $a^*_{-k}$ |
| Time Reversal | $x[-n]$ | $a_{-k}$ |
| Time Scaling | $x_m[n]$ | $a_k$ |
| Periodic Convolution | $\sum_{r=N} x[r] y[n-r]$ | $Na_k b_k$ |
| Multiplication | $x[n]y[n]$ | $\sum_{l=N} a_l b_{k-l}$ |
| Differentiation | $x[n] - x[n-1]$ | $(1 - e^{-jk\omega_0}) a_k$ |
| Parseval's Relation | $\frac{1}{T}\sum_{n=N} \|x[n]\|^2 = \sum_{n=N} \|a_k\|^2$ | |

# 4 Fourier Transformations

We can also represent non periodic (aperiodic) signals by sinusoids using Fourier Transformations.

# 5 Discrete Time Fourier Transformation

# 6 Filters

# 7 Bode Plot

# 8 Sampling

# 9 Laplace Transformations