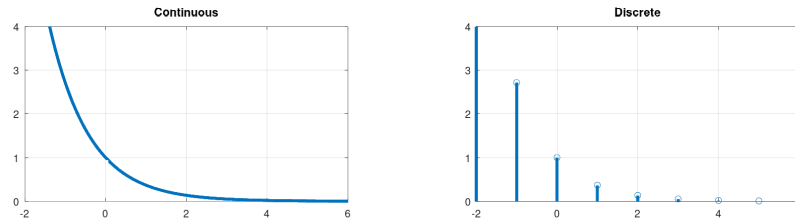# ELG 3125 Summary Sheet

1 Basics of Signals and Systems

We can have a signal that is either continuous or discrete. These signals are represented as math functions. Computers always will display and work with a discrete signal. However

often it is to model a continuous signal.

We call a signal a power signal if its average power is finite. Similarly, we call a signal an energy signal if its total energy is finite.

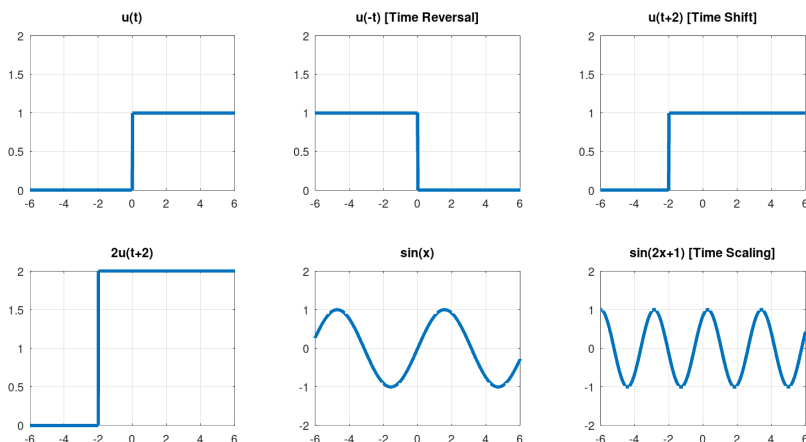**Ex.** A 120VAC wall outlet is a power signal.

This is because the average power is finite. If we have for example a 1500w heater connected, it will always draw 1500W.

However if we keep it running for a long time, it will use a very large amount of energy. So it is infinite energy.
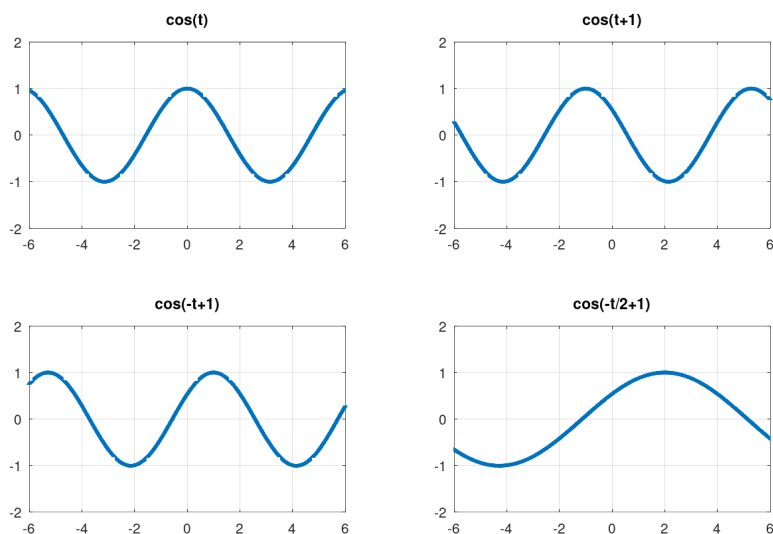

1.1 Transformations

There are many transformations.

When transforming a signal, we usually start with any time shifts. Then we apply other transformations such as a time reversal, or time scaling.

Ex. Plot $\cos(-t/2 + 1)$

We take it in 4 steps:

1.2 Periodicity

A signal is called periodic if:

$$x(t) = x(t + T) \;\forall\; t \tag{1}$$

We call the fundamental period T_0 the smallest *positive* value of T for which Equation 1 holds.

We have the same idea in discrete time except we change t for n, and T for N.

$$x[n] = x[n + N] \;\forall\; n \tag{2}$$

Note that any complex exponential in the form of $e^{j\omega_0 t}$ is periodic with period $T = \frac{2\pi}{\omega_0}$.

Ex. Find the period of $x[n]$.

$$x[n] = e^{j(\frac{2\pi}{3})n} + e^{j(\frac{3\pi}{4})n}$$

I need to find both periods, and find the greatest common multiple.

Note that since I am in discrete time, the period must be an integer.

$$T_1 = \frac{2\pi}{\frac{2\pi}{3}} = 3$$

$$T_2 = \frac{2\pi}{\frac{3\pi}{4}} = \frac{8}{3} = 8 \text{ since } \frac{8}{3} \notin \mathbb{Z}$$

$$T = LCM(3, 8) = 24$$

1.3 Even and Odd Functions

We call a signal Even if it satisfies Equation 3 or Odd if it satisfies Equation 4.
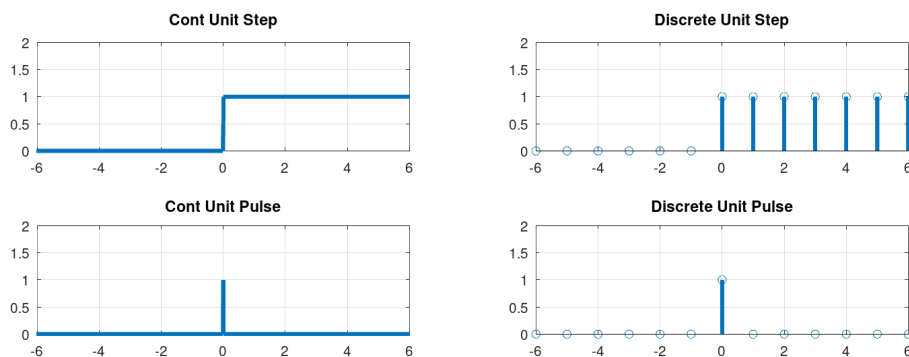
$$x(t) = x(-t) \tag{3}$$
$$x(t) = -x(-t) \tag{4}$$

We can construct any signal by using its even and odd portions.

$$x(t) = Ev\{x(t)\} + Od\{x(t)\} = \frac{1}{2}(x(t) + x(-t)) + \frac{1}{2}(x(t) - x(-t)) \tag{5}$$

1.4 Unit Impulse and Unit Step

These are two very useful functions as defined below. The unit step is 0 when x is negative,

and 1 when positive or 0. The impulse is 1 only when x is 0.

The unit step function is called $u(t)$ or $u[n]$. The unit pulse is called $\delta(t)$ or $\delta[n]$.

1.5 System Properties

1.5.1 Memory

A system is memoryless if the output is only dependant on the input at the same time. So basically we do not see any time shifts such as $t - 1$ or $t + 1$.

1.5.2 Invertibility

A system is invertible if distinct inputs lead to distinct outputs.

Ex. Example found in

1.5.3 Causality

A system is causal if the output at any time depends only on input values of present and in the past. It does not depend on any future values.

This is also referred to as being nonanticipative.

Note that all memoryless systems are causal.

Basically if we see something like $x(t+1)$, it is non causal. If we see something like $(t+1)x(t)$, it is likely causal.

1.5.4 Stability

A system is stable if when inputs are bounded, the outputs are always bounded. So:

$$|x(t)| < \infty \implies |y(t)| < \infty \tag{6}$$

Assuming $x(t)$ is input, and $y(t)$ is output.

1.5.5 Time Invariance

A system is time invariant if a time shift in the input results in an identical time shift in the output.

Ex. Is y(t) = x(6t) time invariant?

We can do:

$$
\begin{aligned}
y(t - t_0) &= x(6t - 6t_0) & &\text{Now we delay by } -t_0 \\
y(t) &= x(6t - 5t_0) & &\text{Now we delay by } +t_0 \\
&\neq x(6t) & &\text{So it is NOT time invariant.}
\end{aligned}
$$

1.5.6 Linearity

A system is linear if Equation 7 is satisfied.

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t) \tag{7}$$

Ex. Is $y(t) = x(t)^2$ linear?

We know intuitively that the answer is no here. But we can prove this using the relation:

$$
\begin{aligned}
y_1(t) &= x_1(t)^2 \\
y_2(t) &= x_2(t)^2 \\
x_3(t) &= ax_1(t) + bx_2(t) \\
y_3(t) &= ay_1(t) + by_2(t) \\
y_3(t) = ax_1(t)^2 + bx_2(t)^2 &\neq [ax_1(t) + bx_2(t)]^2 = x_3(t)^2
\end{aligned}
$$

Basically what we did here is find the left side of the equation by finding the output twice, vs finding the left side by doing the input twice.

Ex. Find all the properties of $y(t) = \cos(t)x(t+1)$.

We see that it is with memory since it relies on $t + 1$.

It is not causal since it depends on a future value $t + 1$.

If we have an input that is finite, so $x(t) < \infty$, then we also have $\cos(t) < \infty$ since $-1 \geq \cos(x) \geq 1$.

To determine if it is time variant, we need to do $t = t - t_0$ and then remove the t_0 from each side. We see if we are still the same.

$$y(t - t_0) = \cos(t - t_0)x(t + 1 - t_0)$$
$$\implies \quad y(t) = \cos(t - t_0)x(t + 1) \qquad\qquad X$$

So it is not time invariant. To see if it is linear we check the equation.

$$y_1(t) = \cos(t)x_1(t+1)$$
$$y_2(t) = \cos(t)x_2(t+1)$$
$$y_3(t) = y_1(t) + y_2(t) = \cos(t)x_1(t+1) + \cos(t)x_2(t+1)$$
$$= \cos(t)\left[x_1(t+1) + x_2(t+1)\right] = ay_1(t) + by_2(t)$$

---

2   LTI Systems

Many systems in the real world are linear and time invariant. An example is a signal amplifier.

2.1   Convolution in Discrete Time

The convolution sum contains 3 parts.

- $h[n]$ is the response of the LTI system

- $x[n]$ is the input to the LTI system

- $y[n]$ is the output of the LTI system

Ex. Consider a signal amplifier such as a microphone amplifier, or a speaker amplifier.

If we have a microphone or speaker, the input $y[n]$ is the input such as the voice or the demodulated radio signal.

The response $h[n]$ is the algorithms that do all the amplification, and maybe some noise reduction.

The output $y[n]$ is the final audio signal such as what we hear from a speaker.

The convolution sum is how we calculate the output given an input and response. We can think of this as we have the response $h[n]$, and then we send through the input $x[n]$ throughout the entire response.

The convolution is represented in Equation 8.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] \tag{8}$$

Note that when finding the convolution, we now consider k as the x axis variable, and n as a constant.

To calculate the convolution sum, we draw $h[n-k]$ and $x[k]$. Then we analyze all overlap regions and find $x[k] \cdot h[n-k]$ for that region.

2.2   Convolution in Continuous Time

This is very similar to the convolution in discrete time except now we have $h(t), y(t), x(t)$. We also use an integral to calculate the convolution instead of a sum as shown in Equation 9.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = x(t) * h(t) \tag{9}$$

Again to solve, we break it up into all the overlap regions and find the product $(x(t) \cdot h(t))$ for each region.

Ex. We have a system response $h(t)$. The input $x(t) = \delta(t)$. What is the output $y(t)$?

This is very simple. We could go about calculating the convolution sum using equation 9, but we can also think this through logically.

Recall that the response $h(t)$ is just the stuff that is applied to the input signal $x(t)$. Since the input signal is just a single pulse, it will travel through the system visiting each point only **once**.

This means that the output will just be the response.

$$y(t) = x(t) * h(t) = \delta(t) * h(t) = h(t)$$
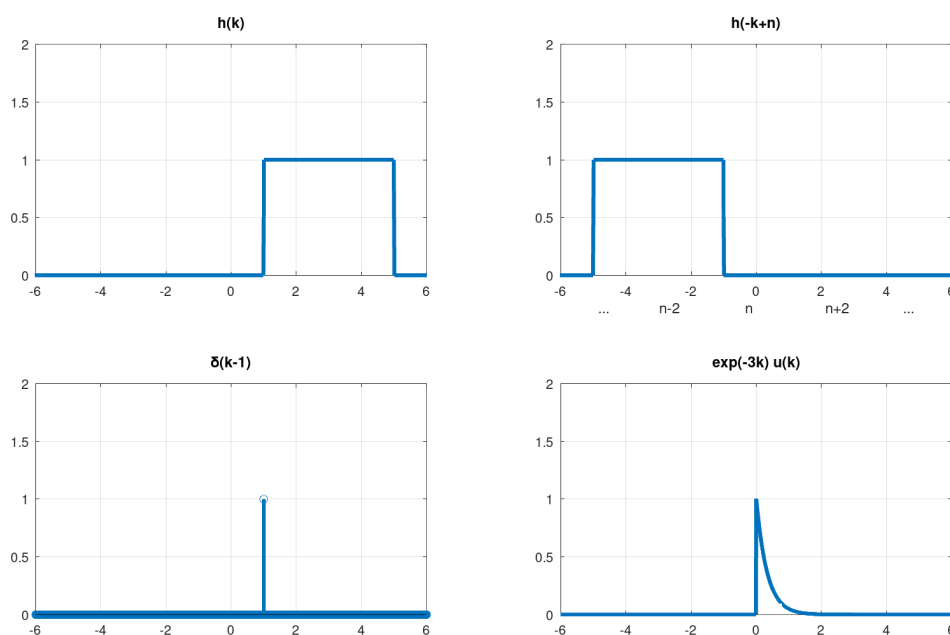
6

This is actually a general equation where:

$$x(t) * \delta(t - t_0) = x(t - t_0) \tag{10}$$

Ex. We have an input signal and a convolution. Find the output.

$$x(t) = \delta(t - 1) + e^{-3t}u(t) \qquad h(t) = u(t - 1) - u(t - 5)$$

To find the output, I need to take the convolution of these two signals. I will need to plot $h(-k + n)$, and it will be helpful to have a plot of the input as well.

I see that $x(t)$ looks complicated to plot. So I will do it in two plots.

Now we need to run the response $h(n - k)$ through both other functions.

Note that k is the variable that we change. So we change

Using the first delta function, I have 3 areas:

$$\begin{aligned} n - 1 < 1 &\implies n < 2 & &= 0 \\ 1 < n - 1 < 5 &\implies 2 < n < 6 & &= 1 \\ n - 1 > 5 &\implies n > 6 & &= 0 \end{aligned}$$

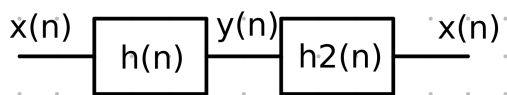Then for the second function, I also have 3 regions.

$$k - 1 < 0 \implies k < 1 \qquad\qquad\qquad = 0$$

$$0 < k - 1 < 4 \implies 1 < k < n \qquad \int_1^n e^{-3(t-k)}\mathrm{d}k = e^{-3t}\int_1^n e^{3k}\mathrm{d}k$$

$$k - 1 > 4 \implies k > 5 \qquad\qquad\qquad = 0$$

Now we can add up the 2 functions to get 5 regions.

2.3 Properties

An LTI system is memoryless if $h[n] = K\delta[n]$ where $k \in \mathbb{R}$. This means $y[n] = Kx[n]$.

An LTI system is invertable if we can get back the input after applying a convolution to the output.

An LTI system is causal if $h[n] = 0$ for $n < 0$.

An LTI system is stable if for all finite inputs to the response, the output os finite: $\int_{-\infty}^{\infty}|h(t)|\mathrm{d}\tau < \infty$.

2.4 Differential and Difference Equations

These are equations in the form of:

$$y[n] = y[n-1] + x[n] \qquad \text{Difference Equation}$$

$$\frac{dy(t)}{dt} + y(t) = x(t) \qquad \text{Differential Equation}$$

We can solve difference equations by using the initial rest condition which is a time when the output is 0. Then we can solve for consecutive values of n starting at initial rest.

For differential equations we need to find the homogeneous solution (input is 0) and the particular solution. Then we sum them together.

Ex. We have a causal LTI system described by $y[n] - \frac{1}{3}y[n-1] = x[n]$

The impulse response $h[n]$ can be found by remembering that if we let $x[n] = \delta[n]$ then $y[n] = h[n]$.

$$h[n] = \delta[n] + \frac{1}{3}h[n-1]$$

Then since it is causal, we know that when $n < 0$, $h[n] = 0$. So I can try a few values to get a pattern.

$$h[0] = \delta[0] + \frac{1}{3}h[-1] = 1 + \frac{1}{3}(0) = 1$$

$$h[1] = \delta[1] + \frac{1}{3}h[0] = \frac{1}{3}(1) = \frac{1}{3}$$

$$h[2] = 0 + \frac{1}{3}\frac{1}{3} = \left[\frac{1}{3}\right]^2$$

$$h[3] = \left[\frac{1}{3}\right]^3$$

$$h[n] = \left[\frac{1}{3}\right]^n$$

Now we know $h[n]$. We can do many things.

We know the system is stable since if we increase $n \to \infty$ then $h[n]$ is finite.

We also can find the inverse impulse response $h'[n]$ by just swapping x for y in the original equation and then solving using the same method.

If we want to find the output given a certain input we can. We just use the convolution sum to do so. We would end up with three regions, the first of which $y[n] = 0$.

Ex.

3   Fourier Series

Any periodic signal can be represented by sinusoids (or complex exponentials).

We have the signal of $e^{j\omega_0 t}$ which is periodic. It contains both sin and cos components.

If we change it to $e^{jk\omega_0 t}$, for $k = 0, \pm 1, \pm 2, ...$ then we say that these are harmonically related. The first harmonic has $k = \pm 1$, the second $k = \pm 2$, and so on. We have:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right] \tag{11}$$

We see that the e terms are constant, only the coefficients change a meaningful amount. We can calculate the coefficients using Equation 12.

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \tag{12}$$

$$a_0 = \frac{\text{AREA}}{T} \qquad \text{Specific Case for DC offset } (a_0) \tag{13}$$

Ex. Find a_k and a_0

We need to use Equation 12.

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} \mathrm{d}t = \frac{1}{T} \int_{T_1}^{-T_1} 1 \cdot e^{-jk\omega_0 t} \mathrm{d}t = \frac{1}{Tjk\omega_0} e^{-jk\omega_0 t} \big|_{-T_1}^{T_1} = \frac{2}{k\omega_0 T} \left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j} \right]$$

$$= \frac{2}{k\omega_0 T} \cdot \sin(k\omega_0 T_1) \qquad \text{Using an identity}$$

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad \text{Using } T = \frac{2\pi}{\omega_0}$$

$$a_0 = \frac{\text{AREA}}{T} = \frac{2T_1}{T} \qquad \text{We could also use } a_k \text{ to get 0/0, and use l'Hopital}$$

3.1   Properties of Continuous Fourier Series

If we have coefficients a_k already for a signal $x(t)$, or maybe more than one signal $x(t)$ and $y(t)$, then after applying some changes to the signals we come up with new coefficients by these properties.

Property	Periodic Signal	Coefficients
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0} a_k$
Frequency Shifting	$e^{jM\omega_0 t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a_{-k}^*
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \ \alpha > 0$	a_k
Periodic Convolution	$\int_T x(\tau) y(t - \tau) \mathrm{d}\tau$	$Ta_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$jk\omega_0 a_k = jk\frac{w\pi}{T} a_k$
Parseval's Relation	$\frac{1}{T} \int_T |x(t)|^2 \mathrm{d}t = \sum_{k=-\infty}^{\infty} |a_k|^2$	

3.2   Properties of Discrete Fourier Series

It is the same idea with discrete signals. We have the equations for the coefficients, and the properties.

$$x[n] = \sum_{k=N} a_k e^{jk\omega_0 n} \tag{14}$$

$$a_k = \frac{1}{N} \sum_{n=N} x[n] e^{-jk\omega_0} n \tag{15}$$

Property	Periodic Signal	Coefficients
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$e^{-jk\omega_0 n_0} a_k$
Frequency Shifting	$e^{jM\omega_0 n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_m[n]$	a_k
Periodic Convolution	$\sum_{r=N} x[r]y[n - r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=N} a_l b_{k-l}$
Differentiation	$x[n] - x[n - 1]$	$(1 - e^{-jk\omega_0})a_k$
Parseval's Relation	$\frac{1}{T} \sum_{n=N} |x[n]|^2 = \sum_{n=N} |a_k|^2$	

4   Fourier Transformations

Fourier Transforms is a very important concept in this class. It is used as the basis for the Laplace transform, and Z transform.

We can also represent non periodic (aperiodic) signals by sinusoids using Fourier Transformations.

We do this by taking a periodic signal, and increasing the period to be very large. This makes it so each pulse of the periodic signal is infinitesimally far apart, which means it is basically a non periodic signal.

We can represent a signal in either the time domain (what we usually do) or the frequency domain. The frequency domain is similar to the fourier series representation of a periodic function.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\jmath\omega)e^{\jmath\omega t}\mathrm{d}\omega \qquad \text{Time Domain (Inverse Fourier Transform)} \tag{16}$$

$$x(\jmath\omega) = \int_{-\infty}^{\infty} x(t)e^{-\jmath\omega t}\mathrm{d}t \qquad \text{Frequency Domain (Fourier Transform)} \tag{17}$$

We can also fourier transform periodic signals into the frequency domain using:

$$X(\jmath\omega) = \sum_{-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \tag{18}$$

These are some common fourier transform pairs:

$e^{-at}u(t) \iff \frac{1}{a+j\omega}, \quad a > 0$	$e^{-a\lvert t\rvert} \iff \frac{2a}{a^2+\omega^2}, \quad a > 0$
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \iff \frac{2\sin\omega T_1}{\omega} \\ 0, & \lvert t\rvert > T_1 \iff \frac{2\sin\omega T_1}{\omega} \end{cases}$	$\frac{\sin Wt}{\pi t} \iff X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$
$\delta(t) \iff 1$	$\delta(t - t_0) \iff e^{-j\omega t_0}$
$1 \iff 2\pi\delta(\omega)$	$e^{j\omega_0 t} \iff 2\pi\delta(\omega - \omega_0)$
$\sin\omega_0 t \iff \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$	$\cos\omega_0 t \iff \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \iff \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega - k\omega_0)$	$\sum_{k=-\infty}^{\infty} \delta(t - kT) \iff \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T_0}\right)$

4.1 Properties

We also have a lot of properties of these transforms such as the derivative which is very useful.

Below is a table with some of the ones:

Time Shifting	$x(t - t_0) \iff e^{-j\omega t_0}X(j\omega)$
Time Reversal	$x(-t) \iff X(-j\omega)$
Time and Frequency Scaling	$x(\alpha t) \iff \frac{1}{\lvert\alpha\rvert}X\left(\frac{j\omega}{\alpha}\right)$
Conjugate	$x^*(t) \iff X^*(-j\omega)$
Differentiation	$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \iff j\omega X(j\omega)$
Parseval's Relation	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\mathrm{d}t \iff \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\mathrm{d}\omega$
Convolution	$y(t) = h(t) * x(t) \iff Y(j\omega) = H(j\omega)X(j\omega)$
Multiplication	$r(t) = s(t)p(t) \iff R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega - \theta))\mathrm{d}\theta$

Ex.

Ex.

Ex.

5 Discrete Time Fourier Transformation

This is very similar to the fourier transforms in continuous time, but there are slight differences.

The idea is exactly the same, it is just a few equations that we use are a bit different.

We get the two equations of:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega \qquad \text{Time Domain (Inverse Fourier Transform)} \qquad (19)$$

$$x[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{Frequency Domain (Fourier Transform)} \qquad (20)$$

$\delta[n] \Longleftrightarrow 1$	$a^n u[n], |a| < 1 \Longleftrightarrow \frac{1}{1-ae^{-j\omega}}$
$a^{|n|}, |a| < 1 \Longleftrightarrow \frac{1-a^2}{1-2a\cos\omega+a^2}$	$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases} \Longleftrightarrow \frac{\sin\omega(N_1+1/2)}{\sin(\omega/2)}$
$e^{j\omega_0 n} \Longleftrightarrow 2\pi\delta(\omega-\omega_0), -\pi \le \omega \le \pi$	$\sum_{k=-\infty}^{+\infty} \delta[n-kN] \Longleftrightarrow \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$
$\cos\omega_0 n \Longleftrightarrow \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$	$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \Longleftrightarrow \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$

5.1 Properties

We also have a lot of properties of these transforms.

Below is a table with some of the ones:

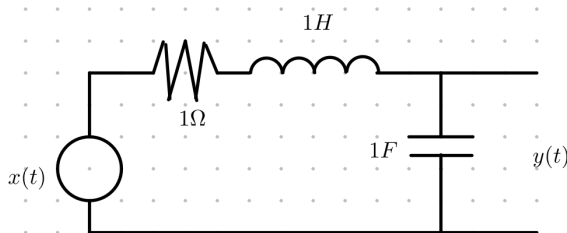| Time Shifting	$x[n-n_0] \Longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$
Time Reversal	$x[-n] \Longleftrightarrow X(e^{-j\omega})$
Conjugate	$x^*[n] \Longleftrightarrow X^*(e^{-j\omega})$
Time Difference	$x[n] - x[n-1] \Longleftrightarrow (1 - e^{-j\omega})X(e^{j\omega})$
Parseval's Relation	$\int_{-\infty}^{\infty} |x[n]|^2 dt \Longleftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega$
Convolution	$y[n] = h[n] * x[n] \Longleftrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
Multiplication	$r[n] = s[n]p[n] \Longleftrightarrow R(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(e^{j\omega})P(e^{j(\omega-\theta)}) d\theta$

> **Ex.**

6 Filters

Filters are used to pass/stop certain frequencies. Low pass filters pass low frequencies and block high frequencies, high pass filters block low frequencies and pass high frequencies, and band pass filters pass a certain band and block low and high frequencies.

These filters are made up of a combination of Resistors, Capacitors, and Inductors. A common design is an RL (Resistor and Inductor) or RC (Resistor and Capacitor) filter.

When analyzing these circuits, we usually need to create a differential equation by using fourier transforms. It is useful to know information about inductors and capacitors:

$$\text{Capacitors} \qquad I_C = C\frac{dV}{dt}$$

$$\text{Inductors} \qquad V_L = L\frac{dI}{dt}$$

Ex. Using the following circuit, find the differential equation relating x and y.

I start with a KVL:

$$x(t) = 1^{\Omega} I + 1^{H} \frac{\mathrm{d}I}{\mathrm{d}t} + y(t)$$

Then I need to find what I is in relation to y or x. I can get this relation using the capacitor equation noting that the voltage over the cap is just y.

$$I_C = C\frac{\mathrm{d}V}{\mathrm{d}t} = 1 \cdot \frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{\mathrm{d}y(t)}{\mathrm{d}t}$$

To do this easily I can go into the frequency domain. I can then just sub this I_C as I into the main KVL equation in the frequency domain.

$$I_C = \frac{\mathrm{d}Y(\jmath\omega)}{\mathrm{d}t} = \jmath\omega Y(\jmath\omega)$$

$$X(\jmath\omega) = \jmath\omega Y(\jmath\omega) + \frac{\mathrm{d}\jmath\omega Y(\jmath\omega)}{\mathrm{d}t} + Y(\jmath\omega) = \jmath\omega Y(\jmath\omega) + (\jmath\omega)^2 Y(\jmath\omega) + Y(\jmath\omega)$$

I can then easily find the frequency response of this, and calculate the output $Y(\jmath\omega)$ and then get $y(t)$ given some $x(t)$.

7   Bode Plot

This is a way to graphically represent fourier transforms. If we want to plot $x(t)$, we can actually plot $20\log_{10}|X(\jmath\omega)|$ and we will get a straight line equation. This is for the magnitude

Here we have the x axis being 10^t so $1, 10, 100, 1000, ...$

The y axis would be in decibals (dB) linearly so $..., -20, -10, 0, 10, 20, ...$

Here knowing the log properties are very useful such as:

$\log_a(xy) = \log_a x + \log_a y$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
$\log 1 = 0$	$\log_a(xn) = n\log_a(x)$
$\log_a\left(\frac{1}{x}\right) = -\log_a(x)$	

When finding magnitude bode plots, we want to get the equation into the form of:

$$\frac{\jmath\omega}{n} + 1 \qquad\qquad \text{OR} \qquad\qquad \frac{1}{\frac{\jmath\omega}{n} + 1}$$

Then on the x axis, we have the values of n. This is where it changes. If we have the first form above, it is a positive increasing form, and the second form is a negative decreasing form. This can more easily be seen in the example below. There is math that shows this, but it is complicated.

Ex. Find the magnitude bode plot of the following:

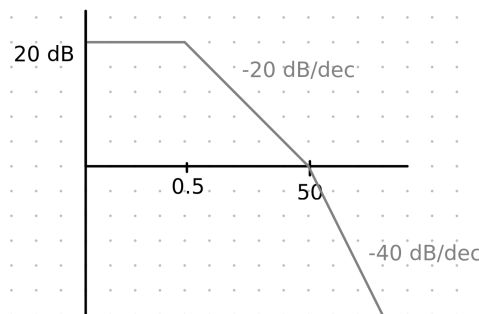$$\frac{250}{(\jmath\omega)^2 + 50.5\jmath\omega + 25}$$

I start by modifying the equation to be in the correct form. I can factor the denominator and then break up into two parts. Then I can apply log rules.

$$\implies \frac{250}{1} \cdot \frac{1}{\frac{\jmath\omega}{1} + 50} \cdot \frac{1}{\frac{\jmath\omega}{1} + 0.5} = \frac{250}{50 \cdot 0.5} \cdot \frac{1}{\frac{\jmath\omega}{50} + 1} \cdot \frac{1}{\frac{\jmath\omega}{0.5} + 1}$$

$$= 10 \cdot \frac{1}{\frac{\jmath\omega}{50} + 1} \cdot \frac{1}{\frac{\jmath\omega}{0.5} + 1}$$

$$\text{Applying Log} \implies 20\log(10) + 20\log(\frac{1}{\frac{\jmath\omega}{50} + 1}) + 20\log(\frac{1}{\frac{\jmath\omega}{0.5} + 1})$$

$$= 20 + 20\log(\frac{1}{\frac{\jmath\omega}{50} + 1}) + 20\log(\frac{1}{\frac{\jmath\omega}{0.5} + 1})$$

We see that both of these slopes will be decreasing due to it being a fraction.

We see that before 0.5, we will have a constant of 20 dB. After 0.5, but before 50, we have slope of -20dB. After 50, we add on a slope of -20dB to get -40dB. We get a graph of the similar form as below. This is not to scale.

To get the phase bode plots, we simply use the fact that a term in the numerator gives a positive 45 degree slope, and on in the denomonator gives a -45 degree slope. We approximate by extending this slope one decade before and after the actual given time.

If we have a second order term, we simply change the slope to 90 degrees or -90 degrees.

We do this for each term and then add the terms up.

Ex.

<div align="center">wrong</div>

7.1 Damping

This is a value that describes how systems respond to being disturbed. Lower values mean it oscillates for a long time and returns to equilibrium slowly, 1 means it returns to equilibrium quickly, and higher values mean it does not oscillate and returns to equilibrium slowly.

- $\zeta < 1$: Underdamped

- $\zeta = 1$: Critically Damped

- $\zeta > 1$: Overdamped

These damping values can be found from the equation of the impulse response of a system in the frequency domain. Specifically the coefficients for the $Y(\jmath\omega)$, $Y(\jmath\omega)\jmath\omega$, and $Y(\jmath\omega)(\jmath\omega)^2$ terms.

$$H(\jmath\omega) = \frac{\omega_n^2}{(\jmath\omega_n)^2 + 2w_n\zeta(\jmath\omega) + \omega_n^2} \tag{21}$$

ω_n and ζ are just resistance, inductance, and capacitance:

$$\zeta = \frac{R}{2\omega_n L} = \frac{R}{2\sqrt{\frac{L}{C}}} \qquad\qquad \omega_n = \frac{1}{\sqrt{LC}}$$

Ex. Find the damping of this system:

$$5\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 5y(t) = 7x(t)$$

I will use equation 21 to solve.

$$5(\jmath\omega)^2 Y(\jmath\omega) + 4\jmath\omega Y(\jmath\omega) + 5Y(\jmath\omega) = 7X(\jmath\omega)$$
$$\implies Y(\jmath\omega)\left[5(\jmath\omega)^2 + 4(\jmath\omega) + 5\right] = 7X(\jmath\omega)$$
$$\implies H(\jmath\omega) = \frac{Y(\jmath\omega)}{X(\jmath\omega)} = \frac{7}{5(\jmath\omega)^2 + 4(\jmath\omega) + 5}$$

Then to get it in the correct form, I will make it so the numerator ω_n^2 is the same as the last term on the denominator ω_n^2.

$$\implies \frac{7}{5}\frac{1}{(\jmath\omega)^2 + \frac{4}{5}(\jmath\omega) + 1}$$

And now it is in the right form, so I can solve for ω_n and then ζ.
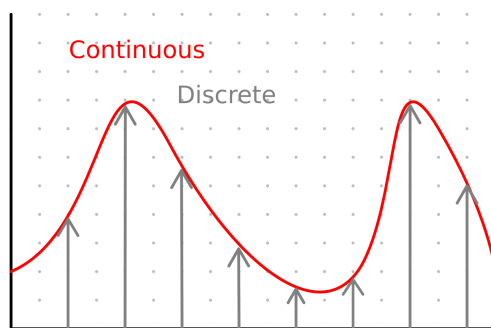
$$\omega_n^2 = 1 \implies w_n = 1$$

$$2\omega_n\zeta = \frac{4}{5} \implies \zeta = \frac{\frac{4}{5}}{2\omega_n} = \frac{\frac{4}{5}}{2} = \frac{4}{10} = \frac{2}{5}$$
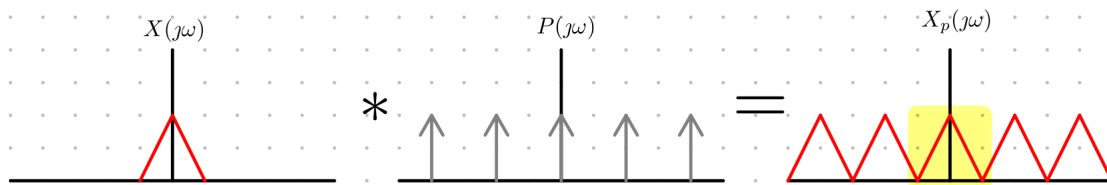
Therefore since $\zeta < 1$ it is underdamped.

8   Sampling

Sampling is how we convert a continuous signal to a discrete signal. We sample using a large impulse train.
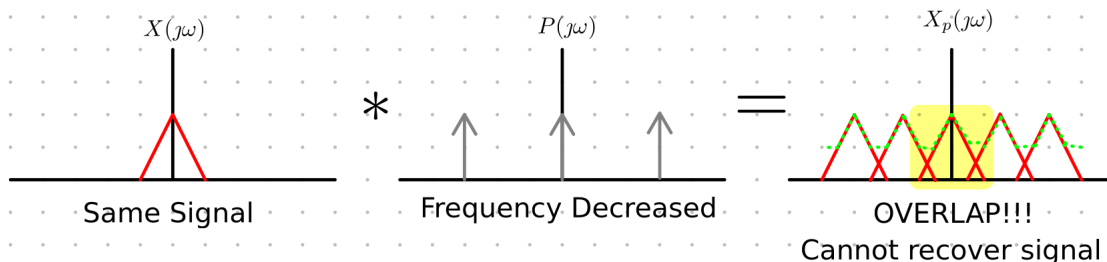

When we sample a signal, we will multiply the signal $x(t)$ by the impulse train $p(t)$. If we go into the frequency domain, we will get the convolution $X_p(\jmath\omega) = X(\jmath\omega) * P(\jmath\omega)$.

If we want to get the original signal $X(\jmath\omega)$ back from the sampled one $X_p(\jmath\omega)$, then we just need to use a bandpass filter over the middle repetition of the sampled signal.

A problem can occur if the sampling frequency is too low which would cause the period in the time domain to increase. Because of math, the period in the frequency domain would then decrease which would cause overlap preventing us from recovering the original signal.
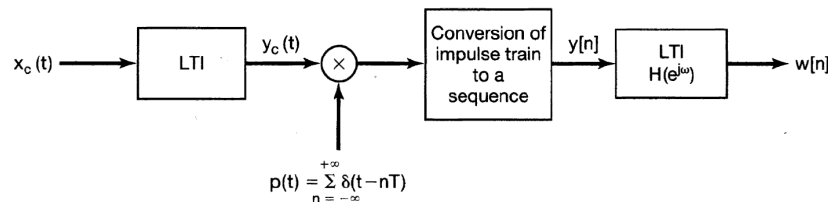
We say that the angular frequency (period in frequency domain) needs to be greater than the highest frequency we are using.

$$\omega_s > 2\omega_M \text{ where } \omega_s = \frac{2\pi}{T}$$

We call $2\omega_M$ the Nyquist rate

A useful fact to know that helps solve these problems is that when dealing with the convolution, specifically in the frequency domain $Z(\jmath\omega) = X(\jmath\omega) * Y(\jmath\omega)$, if the two functions have maximum frequencies of ω_X and ω_Y respectively, then the max frequency of Z is $\omega_Z = \omega_X + \omega_Y$.

Ex. We have a differential equation modeling an LTI system shown below. Find $y_c(t)$, and then find the impulse response $h[n]$ if $w[n] = \delta[n]$.

$$\frac{\mathrm{d}y_c(t)}{\mathrm{d}t} + y_c(t) = \delta(t)$$

We start by finding what $y_c(t)$ is. We can do this by putting the DE into the frequency domain, which allows us to get rid of the derivative. Then we can simply isolate for y, and the put back into time domain.

$$\implies \jmath\omega Y(\jmath\omega) + Y(\jmath\omega) = 1$$
$$\implies Y(\jmath\omega)\left[\jmath\omega + 1\right] = 1$$
$$\implies Y(\jmath\omega) = \frac{1}{\jmath\omega + 1}$$
$$\implies y(t) = e^{-t}u(t)$$
$$\implies y[n] = e^{-nT}u[n]$$

Now we have it in discrete time. We now need to find out the response function. We can do this by converting to frequency domain. We consider the e^{-T} as a coefficient which allows us to use this transform.

$$Y(e^{\jmath\omega}) = \frac{1}{1 - e^{-T}e^{-\jmath\omega}}$$
$$H(e^{\jmath\omega}) = \frac{W(e^{\jmath\omega})}{Y(e^{\jmath\omega})} = \frac{1}{\frac{1}{1-e^{-T}e^{-\jmath\omega}}} = 1 - e^{-T}e^{-\jmath\omega}$$

$$h[n] = \delta[n] - e^{-T}\delta[n-1]$$

The reason we get the $\delta[n-1]$ is because of the property of $x[n-n_0] \Longleftrightarrow e^{\jmath\omega n_0}X(e^{\jmath\omega})$ and we consider $X(e^{\jmath\omega})$ to be the invisible 1 coefficient.
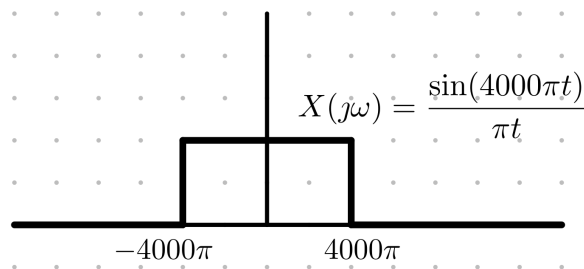

Ex. Find the Nyquist rate of $x(t) = \left(\frac{\sin(4000\pi t)}{\pi t}\right)^2$.

To get the nyquist rate, I need to know the maximum frequency. So I need to put it into the frequency domain. I do not know a fourier transform for $x(t)$, but if I break it up into a product, I can transform each part, and multiplication becomes a convolution in frequency domain.
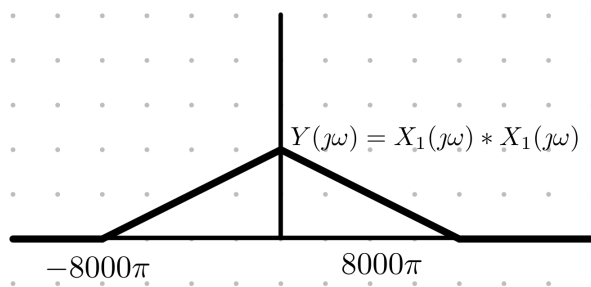
$$x(t) = \left(\frac{\sin(4000\pi t)}{\pi t}\right)\left(\frac{\sin(4000\pi t)}{\pi t}\right) = x_1(t) \cdot x_1(t)$$
$$X(\jmath\omega) = X_1(\jmath\omega) * X_1(\jmath\omega)$$

Now I can fourier transform this into:

$$X(\jmath\omega) = \frac{\sin(4000\pi t)}{\pi t}$$

-4000π 4000π

If I then convolve that with itself, I can get:

$$Y(\jmath\omega) = X_1(\jmath\omega) * X_1(\jmath\omega)$$

-8000π 8000π

Here we see that the maximum frequency is doubled, so the nyquist rate is just the maximum frequency doubled.

$$\omega_n = 2\omega_m = 2 \cdot 8000\pi = 16000\pi$$

---

9   Laplace Transformations

Laplace transformations are similar to fourier transformations in how they work. But rather than converting to from $x(t)$ to $X(\jmath\omega)$, we go to $X(s)$ where $s = \sigma + \jmath\omega$. So it has both a real **and** complex component. This allows it to work for some cases where the standard
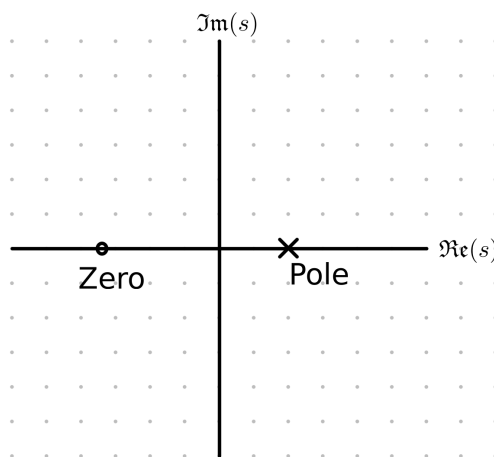
fourier component would not work. But we do also need to define the bounds for each transformation.

We have the following equation for the Laplace transform:

$$X(s) = X(\jmath\omega + \sigma) = \int_{-\infty}^{\infty} x(t)e^{-st}\mathrm{d}t = \int_{-\infty}^{\infty} x(t)e^{-(\jmath\omega+\sigma)t}\mathrm{d}t$$

$$x(t) = \frac{1}{2\pi\jmath} \int_{\sigma-\jmath\infty}^{\sigma+\jmath\infty} X(s)e^{st}\mathrm{d}t$$

We often display these transformations on the Cartesian plane with axis for $\Re(s)$ and $\Im(s)$ (Real and Imaginary parts of s). Here we have zeroes (points where numerator goes to 0), and poles (points where denominator goes to 0).

Using this graph, we can tell if the function is causal and/or stable. If it is causal, then the ROC will be right sided and unbounded on that side (it will be $\Re\{s\} > a, a \in \mathbb{R}$). If it is stable, then the y axis ($\Im$) axis will be contained in the ROC.

We have a table of Laplace transforms as shown below:

$e^{-at}u(t) \longleftrightarrow \frac{1}{a+s}, \mathfrak{Re}\{s\} > -a$	$u(t) \longleftrightarrow \frac{1}{s}, \mathfrak{Re}\{s\} > 0$
$-e^{-at}u(-t) \longleftrightarrow \frac{1}{a+s}, \mathfrak{Re}\{s\} < -a$	$\delta(t) \longleftrightarrow 1$
$\cos(\omega_0 t)u(t) \longleftrightarrow \frac{s}{s^2+\omega_0^2}, \mathfrak{Re}\{s\} > 0$	$\sin(\omega_0 t)u(t) \longleftrightarrow \frac{\omega_0}{s^2+\omega_0^2}, \mathfrak{Re}\{s\} > 0$

We have a table of laplace properties shown below:

Time Shifting	$x(t - t_0) \longleftrightarrow e^{-st_0}X(s)$
Flipping	$x(-t) \longleftrightarrow X(-s)$, Inverses ROC
S Domain Shifting	$e^{s_0 t}x(t) \longleftrightarrow X(s - s_0)$, ROC is shifted
Time Scaling	$x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right)$, ROC is scaled
Conjugation	$x^*(t) \longleftrightarrow X^*(s^*)$
Convolution	$x_1(t) * x_2(t) \longleftrightarrow X_1(s)X_2(s)$, ROC is union
Differentiation	$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \longleftrightarrow sX(s)$
S Domain Differentiation	$-tx(t) \longleftrightarrow \frac{\mathrm{d}X(s)}{\mathrm{d}s}$
Integration	$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \longleftrightarrow \frac{1}{s}X(s)$, ROC is $R_{X(s)} \cap \{\mathfrak{Re}\{s\} > 0\}$

Ex. Find $h(t)$ for this differential equation for all cases of the ROC.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = x(t)$$

To do this, I need to get the ROC. This will be found by getting the poles and then finding possible ROCs.

I start with laplace transform.

$$X(s) = s^2 Y(s) - sY(s) - 2Y(s)$$
$$\implies \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)}$$
$$\implies \frac{Y(s)}{X(s)} = \frac{1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \qquad \text{Do partial fractions}$$
$$\implies 1 = A(s+1) + B(s-2) \implies 0 = A + B, 1 = A - 2B \implies A = \frac{1}{3}, B = \frac{-1}{3}$$
$$\implies \frac{Y(s)}{X(s)} = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

So now I have two poles of 2 and -1 and therefore 3 possible ROCs.

We need to find the equation for each ROC. We use the property that
$e^{-at}u(t) \longleftrightarrow \frac{1}{a+s}, \Re\{s\} > -a$ and $-e^{-at}u(-t) \longleftrightarrow \frac{1}{a+s}, \Re\{s\} < -a$ are the same,
except their ROC is slightly different.

$$
\begin{aligned}
&(1) \quad a < -1 \\
&(2) \quad -1 < a < 2 \\
&(3) \quad a > 2
\end{aligned}
$$

Now I can actually get the equations for these.

$$
\begin{aligned}
&(1) \quad h(t) = \frac{1}{3}\left(-e^{2t}u(-t) - (-)e^{-t}u(-t)\right) \\
&(2) \quad h(t) = \frac{1}{3}\left(-e^{2t}u(-t) - e^{-t}u(t)\right) \\
&(3) \quad h(t) = \frac{1}{3}\left(e^{2t}u(t) - e^{-t}u(t)\right)
\end{aligned}
$$

We can also say which ROCs are causal, and stable through the properties, or through logic of $h(t)$.

$$
\begin{aligned}
&(1) \quad \text{NOT causal, NOT stable} \\
&(2) \quad \text{NOT causal, stable} \\
&(3) \quad \text{causal, NOT stable}
\end{aligned}
$$

10 Z Transform

The Z transform is basically the Laplace transform but in discrete time. Rather than using $X(s)$ with $s = \sigma + \jmath\omega$, we have $X(z)$ where $z = re^{\jmath\omega}$ (similar to the discrete fourier transform of $e^{\jmath\omega}$).
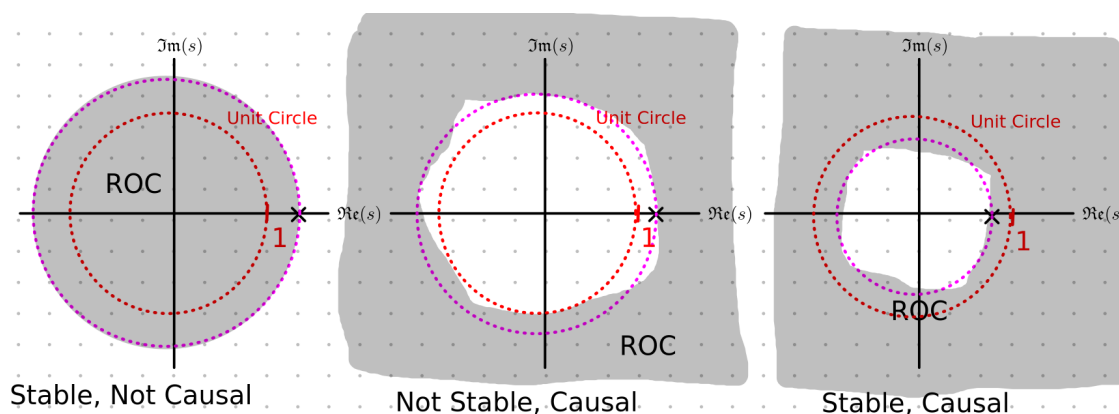
By definition, we have it as:

$$
X(z) = \sum_{k=-\infty}^{\infty} x[n]z^{-n} \qquad\qquad \text{Z Transform}
$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} \mathrm{d}z \qquad\qquad \text{Inverse Z Transform}$$

We can create the Zero Pole plot as well, and this allows us to see if it is stable and or causal. We however use circles rather than lines (like laplace). If the outside of the circle to the bounds is the ROC, then it is causal. If the unit circle (radius of 1) is in the ROC, then it is stable.

In the below figure, the first image has the ROC inside the pole (so it does not extend to the bounds), and the unit circle is in the ROC. For the second the ROC extends to the bounds but does not include the unit circle, and the third also extends to the bounds and includes the ROC.

We have a stock list of z transforms, just like the fourier transforms and laplace transforms.

$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$	$-a^n u[-n-1] \longleftrightarrow \frac{1}{1-az^{-1}}, \quad |z| < |a|$	
$u[n] \longleftrightarrow \frac{1}{1-z^{-1}}, \quad |z| > 1$		

We also have a list of properties of z transforms just like fourier transforms and laplace transforms.

Time Shifting	$x[n-n_0] \longleftrightarrow z^{-n_0} X(z)$
Scaling	$z_0^n x[n] \longleftrightarrow X\left(\frac{z}{z_0}\right)$, new ROC is $|z_0|R$
Time Reversal	$z[-n] \longleftrightarrow X\left(\frac{1}{z}\right)$, new ROC is $\frac{1}{R}$
Conjugation	$x^*[n] \longleftrightarrow X^*(z^*)$
Convolution	$x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$, new ROC is union
Differentiation	$nx[n] \longleftrightarrow -z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$

Ex.

11   Appendix

A few formulas

Euler

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Summations, geometric series

$$\sum_{k=n_1}^{\infty} a^k = \frac{a^{n_1}}{1-a} \quad |a|<1 \qquad \sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1} - a^{n_2+1}}{1-a} \quad \begin{array}{l} a \neq 1 \\ n_2 \geq n_1 \end{array}$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad |a|<1 \qquad \sum_{k=0}^{n_1} a^k = \frac{1 - a^{n_1+1}}{1-a} \quad \begin{array}{l} a \neq 1 \\ n_1 \geq 0 \end{array}$$

Even and odd parts

$$x_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t) \quad x_o(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$

$$x_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n] \quad x_o[n] = \frac{1}{2}x[n] - \frac{1}{2}x[-n]$$

Convolutions

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$h(t)$ response for differential equations describing LTI systems (single order roots)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$h(t) = \sum_{k=0}^{N-1} A_k e^{s_k t} u(t) + \sum_{k=0}^{M-N} B_k \frac{d^k \delta(t)}{dt^k}$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = x(t) \quad h'(t) = \sum_{k=0}^{N-1} A_k e^{s_k t} u(t) \text{ (simpl. sys.)}$$

$h[n]$ response for difference equations describing LTI systems (single order roots)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$h[n] = \sum_{k=0}^{N-1} A_k \alpha_k^n u[n] + \sum_{k=0}^{M-N} B_k \delta[n-k]$$

$$\sum_{k=0}^{N} a_k y[n-k] = x[n] \quad h'[n] = \sum_{k=0}^{N-1} A_k \alpha_k^n u[n] \text{ (simpl. sys.)}$$

LTI systems and eigenfunctions

$$e^{st} \xrightarrow{LTI(cont.)} H(s)e^{st}$$

$$z^n \xrightarrow{LTI(discr.)} H(z)z^n$$

$$e^{j\omega t} \xrightarrow{LTI(cont.)} H(j\omega)e^{j\omega t}$$

$$e^{j\omega n} \xrightarrow{LTI(discr.)} H(e^{j\omega})e^{j\omega n}$$

$$\cos(\omega t) \xrightarrow{LTI(cont.)} |H(j\omega)|\cos(\omega t + \angle H(j\omega))$$

$$\cos(\omega n) \xrightarrow{LTI(discr.)} |H(e^{j\omega})|\cos(\omega n + \angle H(e^{j\omega}))$$

Standard first and second order low-pass systems, continuous time

$$H(j\omega) = \frac{1}{1 + j\omega\tau} \qquad H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

Standard first and second order recursive systems, discrete time

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad |a| < 1$$

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}} \quad 0 \leq r < 1, \, 0 \leq \theta \leq \pi$$

Continuous time sampling

$$x_p(t) = x(t) \times p(t) \qquad x_d[n] = x(nT)$$

$$X_p(j\omega) = f_S \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \qquad \omega_s = \frac{2\pi}{T} = 2\pi f_s$$

$$X_d(e^{j\omega}) = X_p(j\omega f_s)$$

$$H_0(j\omega) = e^{-j\pi\omega/\omega_s} 2\sin(\pi\omega/\omega_s)/\omega \text{ (sample \& hold)}$$

Other formulas

$$A\cos(\phi + \theta) = A\sin(\phi + \theta + \pi/2) = B\cos\phi - C\sin\phi$$

$$(B = A\cos\theta \quad C = A\sin\theta \quad A^2 = \sqrt{B^2 + C^2} \quad \theta = \tan^{-1}\left(\frac{C}{B}\right))$$

$$ae^{j\phi} + a^* e^{-j\phi} = 2\operatorname{Re}\{ae^{j\phi}\} = 2|a|\cos(\phi + \angle a)$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1) + c$$

$$\frac{d\,\text{atan}(x)}{dx} = \frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}$$

Properties – Continuous time Fourier series (C.T.F.S.)

Definitions:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad\qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_0 = \frac{1}{T}\int_T x(t)dt$$

$x(t)$ periodic with period T sec.,

Fundam. angular frequency $\omega_0 = 2\pi f_0 = 2\pi/T$ rad./sec.

$x(t)\xleftrightarrow{\ C.T.F.S.\ } a_k \qquad y(t)\xleftrightarrow{\ C.T.F.S.\ } b_k$

If $x(t)\xrightarrow{\ LTI\ } y(t)$ then $b_k = a_k\, H(j\omega)\big|_{\omega=k\omega_0}$

Linearity: $Ax(t) + By(t)\xleftrightarrow{\ C.T.F.S.\ } A\,a_k + B\,b_k$

Shifting: $x(t-t_0)\xleftrightarrow{\ C.T.F.S.\ } e^{-jk\omega_0 t_0}\,a_k$

Scaling: $x(\alpha t)\xleftrightarrow{\ C.T.F.S.\ } a_k$
$(\alpha > 0$, period $T/\alpha)$

Flipping: $\quad x(-t)\xleftrightarrow{\ C.T.F.S.\ } a_{-k}$

Conjugate: $x^*(t)\xleftrightarrow{\ C.T.F.S.\ } a_{-k}^*$

$\qquad\qquad x^*(-t)\xleftrightarrow{\ C.T.F.S.\ } a_k^*$

Symmetries:

if $x(t)$ is real: $a_k = a_{-k}^*$, $|a_k| = |a_{-k}|$, $\angle a_k = -\angle a_{-k}$

$x(t)$ real and even : a_k real and even $a_k = a_{-k}$

$x(t)$ real and odd: a_k imaginary and odd $a_k = -a_{-k}$

Periodic convolution:

$$\int_T x(\tau)\,y(t-\tau)d\tau \xleftrightarrow{\ C.T.F.S.\ } T\,a_k\,b_k$$

Modulation: $x(t)\,y(t)\xleftrightarrow{\ C.T.F.S.\ } a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$

$$e^{jm\omega_0 t}\,x(t)\xleftrightarrow{\ C.T.F.S.\ } a_{k-m}$$

Differentiation: $\dfrac{dx(t)}{dt}\xleftrightarrow{\ C.T.F.S.\ } jk\omega_0 a_k$

Integration: $\displaystyle\int_{\tau=-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\ C.T.F.S.\ } \dfrac{a_k}{jk\omega_0}$ (if $a_0 = 0$)

Parseval: $\dfrac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

Table of continuous time Fourier series (C.T.F.S.)

$x(t)$ periodic, period $T = \dfrac{2\pi}{\omega_0} = \dfrac{1}{f_0}$ sec.	Fourier series coefficients a_k
$e^{j\omega_0 t}$	$a_1 = 1$ $a_k = 0$ elsewhere
$\cos(\omega_0 t)$	$a_1, a_{-1} = 1/2$ $a_k = 0$ elsewhere
$\sin(\omega_0 t)$	$a_1, a_{-1} = 1/(2j)$ $a_k = 0$ elsewhere
$\begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$ (periodic T)	$a_k = \dfrac{\sin(k\omega_0 T_1)}{k\pi}$ $k \neq 0$ $a_k = \dfrac{2T_1}{T} = \dfrac{T_1\omega_0}{\pi}$ $k = 0$
1	$a_0 = 1$ $a_k = 0$ elsewhere
$\displaystyle\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$a_k = \dfrac{1}{T}$

Properties – Discrete time Fourier series (D.T.F.S.)

Definitions:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j(k\frac{2\pi}{N})n} \qquad x[n] = \sum_{k=<N>} a_k e^{j(k\frac{2\pi}{N})n}$$

$$a_0 = \frac{1}{N}\sum_{n=<N>} x[n]$$

$x[n]$ periodic with period N samples (fundamental angular frequency $\omega_0 = \dfrac{2\pi}{N}$ rad./sample)

$x[n]\xleftrightarrow{\ D.T.F.S.\ } a_k \quad y[n]\xleftrightarrow{\ D.T.F.S.\ } b_k$

If $x[n]\xrightarrow{\ LTI\ } y[n]$ then $b_k = a_k\,H(e^{j\omega})\big|_{\omega=k\frac{2\pi}{N}}$

Periodicity: $x[n]\xleftrightarrow{\ D.T.F.S.\ } a_k = a_{k+N}$

Linearity: $Ax[n] + By[n]\xleftrightarrow{\ D.T.F.S.\ } Aa_k + Bb_k$

Shifting: $x[n - n_0]\xleftrightarrow{\ D.T.F.S.\ } e^{-jk\frac{2\pi}{N}n_0}a_k$

Flipping: $x[-n]\xleftrightarrow{\ D.T.F.S.\ } a_{-k}$

Conjugate: $x^*[n]\xleftrightarrow{\ D.T.F.S.\ } a^*_{-k}$

$\qquad\qquad x^*[-n]\xleftrightarrow{\ D.T.F.S.\ } a^*_k$

Symmetries:

if $x[n]$ is real : $a_k = a^*_{-k}$, $|a_k| = |a_{-k}|$, $\angle a_k = -\angle a_{-k}$

$x[n]$ real and even : a_k real and even $a_k = a_{-k}$

$x[n]$ real and odd: a_k imaginary and odd $a_k = -a_{-k}$

Periodic convolution:

$$\sum_{m=<N>} x[m]y[n-m]\xleftrightarrow{\ D.T.F.S.\ } N\,a_k\,b_k$$

Modulation: $x[n]y[n]\xleftrightarrow{\ D.T.F.S.\ } \sum_{l=<N>} a_l b_{k-l}$

$$e^{jm\frac{2\pi}{N}n}x[n]\xleftrightarrow{\ D.T.F.S.\ } a_{k-m}$$

Accumulation : $\displaystyle\sum_{m=-\infty}^{n} x[m]\xleftrightarrow{\ D.T.F.S.\ } \frac{1}{\left(1-e^{-jk\frac{2\pi}{N}}\right)}a_k$

$\qquad\qquad\qquad\qquad\qquad$ (if $a_0 = 0$)

Parseval: $\dfrac{1}{N}\displaystyle\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$

Duality : if $x[n]\xleftrightarrow{\ DTFS\ } a_k$ then $a[n]\xleftrightarrow{\ DTFS\ } \dfrac{1}{N}x_{-k}$

Table of discrete time Fourier series (D.T.F.S.)

$x[n]$ periodic, period N samples	Fourier series coefficients a_k (periodic with period N)
$e^{j\omega_0 n}$	If $x[n]$ periodic with $\omega_0 = \dfrac{2\pi m}{N}$: $a_k = 1 \quad k = m, m\pm N, m\pm 2N,\ldots$ $a_k = 0$ elsewhere
$\cos(\omega_0 n)$	If $x[n]$ periodic with $\omega_0 = \dfrac{2\pi m}{N}$: $a_k = 1/2$ $\qquad k = \pm m, \pm m \pm N, \pm m \pm 2N,\ldots$ $a_k = 0$ elsewhere
$\sin(\omega_0 n)$	If $x[n]$ periodic with $\omega_0 = \dfrac{2\pi m}{N}$: $a_k = 1/(2j)$ $\qquad k = \pm m, \pm m \pm N, \pm m \pm 2N,\ldots$ $a_k = 0$ elsewhere
$\begin{cases}1 & \|n\| \le N_1 \\ 0 & N_1 < \|n\| \le N/2\end{cases}$ (periodic N, N even)	$a_k = \dfrac{\sin\left(\dfrac{2\pi}{N}k(N_1 + 1/2)\right)}{N\sin(\dfrac{\pi}{N}k)}$ $\qquad\qquad k \ne 0, \pm N, \pm 2N,\ldots$ $a_k = (2N_1 + 1)/N \ \ k = 0, \pm N, \pm 2N,\ldots$
1	$a_k = 1 \quad k = 0, \pm N, \pm 2N,\ldots$ $a_k = 0$ elsewhere
$\displaystyle\sum_{m=-\infty}^{\infty} \delta[n - mN]$	$a_k = \dfrac{1}{N}$

Properties – Continuous time Fourier transform (C.T.F.T.)

Definitions:

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

ω in rad./sec.

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi\, a_k \delta(\omega - k\omega_0) \text{ if } x(t) \text{ periodic}$$

$$x(t) \xleftrightarrow{\;CTFT\;} X(j\omega) \quad y(t) \xleftrightarrow{\;CTFT\;} Y(j\omega)$$

Linearity: $ax(t) + by(t) \xleftrightarrow{\;CTFT\;} aX(j\omega) + bY(j\omega)$

Shifting: $x(t - t_0) \xleftrightarrow{\;CTFT\;} e^{-j\omega t_0} X(j\omega)$

Scaling: $x(at) \xleftrightarrow{\;CTFT\;} \dfrac{1}{|a|} X(\dfrac{j\omega}{a})$

Flipping: $x(-t) \xleftrightarrow{\;CTFT\;} X(-j\omega)$

Conjugate: $x^*(t) \xleftrightarrow{\;CTFT\;} X^*(-j\omega)$

$\qquad\qquad x^*(-t) \xleftrightarrow{\;CTFT\;} X^*(j\omega)$

Symmetries:

if $x(t)$ is real : $X(j\omega) = X^*(-j\omega)$,

$\qquad |X(j\omega)| = |X(-j\omega)|, \quad \angle X(j\omega) = -\angle X(-j\omega)$

$x(t)$ real and even : $X(j\omega)$ real and even $X(j\omega) = X(-j\omega)$

$x(t)$ real and odd: $X(j\omega)$ imag., odd $X(j\omega) = -X(-j\omega)$

Convolution:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \xleftrightarrow{\;CTFT\;} X(j\omega)Y(j\omega)$$

Modulation:

$$x(t)y(t) \xleftrightarrow{\;CTFT\;} \frac{1}{2\pi}X(j\omega)*Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\omega-\theta))d\theta$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\;CTFT\;} X(j(\omega - \omega_0))$$

$$\cos(\omega_0 t)x(t) \xleftrightarrow{\;CTFT\;} \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0))$$

Differentiation: $\dfrac{dx(t)}{dt} \xleftrightarrow{\;CTFT\;} j\omega\, X(j\omega)$

Integration: $\displaystyle\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\;CTFT\;} \dfrac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$

Differentiation in freq.: $tx(t) \xleftrightarrow{\;CTFT\;} j\dfrac{dX(j\omega)}{d\omega}$

Integration in freq.:

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{\;CTFT\;} \int_{-\infty}^{\omega} X(j\eta)d\eta$$

Parseval: $\displaystyle\int_{-\infty}^{+\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2 d\omega$

Duality : if $x(t) \xleftrightarrow{\;CTFT\;} X(j\omega)$ then

$$X(t) \xleftrightarrow{\;CTFT\;} 2\pi\, x(-j\omega)$$

Table of continuous time Fourier transforms (C.T.F.T.)

signal $x(t)$ typ. aperiodic	$X(j\omega)$ (ω in rad./sec.)
if $x(t)$ is periodic, with period $T = \dfrac{2\pi}{\omega_0} = \dfrac{1}{f_0}$ sec.	$2\pi \sum\limits_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 t)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin(\omega_0 t)$	$\dfrac{\pi}{j}\delta(\omega - \omega_0) - \dfrac{\pi}{j}\delta(\omega + \omega_0)$
$\begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$ (periodic T)	$2\sum\limits_{k=-\infty}^{+\infty} \dfrac{\sin(k\omega_0 T_1)}{k}\delta(\omega - k\omega_0)$ $\qquad\qquad k \neq 0$ $\dfrac{4\pi T_1}{T}\delta(\omega) = 2T_1\omega_0\delta(\omega)$ $\qquad\qquad k = 0$
1	$2\pi\delta(\omega)$
$\sum\limits_{n=-\infty}^{\infty} \delta(t - nT)$	$\omega_s \sum\limits_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)\;\; \omega_s = \dfrac{2\pi}{T}$
$\begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$	$\dfrac{2\sin(\omega T_1)}{\omega}$
$\dfrac{\sin(Wt)}{\pi t} \quad W > 0$	$\begin{cases} 1 & |\omega| \leq W \\ 0 & |\omega| > W \end{cases}$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$
$e^{-at}u(t) \qquad \mathrm{Re}\{a\} > 0$	$\dfrac{1}{a + j\omega}$
$-e^{-at}u(-t) \qquad \mathrm{Re}\{a\} < 0$	$\dfrac{1}{a + j\omega}$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t) \quad \mathrm{Re}\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$
$-\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(-t) \;\mathrm{Re}\{a\} < 0$	$\dfrac{1}{(a + j\omega)^n}$
$e^{-at}\sin(\omega_0 t)u(t)$ $a > 0 \quad \omega_0 \geq 0 \quad a, \omega_0\, \text{real}$	$\dfrac{\omega_0}{(j\omega + a)^2 + \omega_0^2}$
$e^{-at}\cos(\omega_0 t)u(t)$ $a > 0 \quad \omega_0 \geq 0 \quad a, \omega_0\, \text{real}$	$\dfrac{j\omega + a}{(j\omega + a)^2 + \omega_0^2}$
$-e^{-at}\sin(\omega_0 t)u(-t)$ $a < 0 \quad \omega_0 \geq 0 \quad a, \omega_0\, \text{real}$	$\dfrac{\omega_0}{(j\omega + a)^2 + \omega_0^2}$
$-e^{-at}\cos(\omega_0 t)u(-t)$ $a < 0 \quad \omega_0 \geq 0 \quad a, \omega_0\, \text{real}$	$\dfrac{j\omega + a}{(j\omega + a)^2 + \omega_0^2}$

Properties – Discrete time Fourier transform (D.T.F.T.)

Definitions:

$x[n] = x(nT) = x(t)\big|_{t=nT}$, where $T = 1/f_s = 2\pi/\omega_s$ is the sampling period in sec., and n is an integer, results in:

$$X_p(j\omega) = f_s \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)), \qquad X(e^{j\omega}) = X_p(j\omega f_s)$$

where $X(j\omega)$ is the original CTFT of $x(t)$, and $X(e^{j\omega})$ is the DTFT of $x[n]$ defined as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Periodicity: $x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$

Linearity: $ax[n] + by[n] \xleftrightarrow{DTFT} aX(e^{j\omega}) + bY(e^{j\omega})$

Shifting: $x[n - n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$ n_0 integer

Expansion, insertion of zeros:

$x_{(k)}[n] \xleftrightarrow{DTFT} X(e^{jk\omega})$ where k is a positive integer

$\qquad x_{(k)}[n] = x[n/k]$ if n is a multiple of k

$\qquad x_{(k)}[n] = 0$ elsewhere

Flipping: $x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$

Conjugate: $x^*[n] \xleftrightarrow{DTFT} X^*(e^{-j\omega})$

$\qquad x^*[-n] \xleftrightarrow{DTFT} X^*(e^{j\omega})$

Symmetries:

if $x[n]$ is real : $\quad X(e^{j\omega}) = X^*(e^{-j\omega})$,

$\qquad\qquad \left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|$, $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$

$x[n]$ real and even : $X(e^{j\omega})$ real, even $X(e^{j\omega}) = X(e^{-j\omega})$

$x[n]$ real, odd $X(e^{j\omega})$ imag., odd $X(e^{j\omega}) = -X(e^{-j\omega})$

Convolution:

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k] \xleftrightarrow{DTFT} X(e^{j\omega})Y(e^{j\omega})$$

Modulation: $x[n]y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

$\qquad e^{j\omega_0 n}x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$

Accumulation:

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{DTFT} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega - m2\pi)$$

Differentiation in freq.: $nx[n] \xleftrightarrow{DTFT} j\dfrac{dX(e^{j\omega})}{d\omega}$

Parseval: $\displaystyle\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega$

Duality : If $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ then $X(t) \xleftrightarrow{CTFS} x_{-k}$

Table of discrete time Fourier transforms (D.T.F.T.)

signal $x[n]$ typ. aperiodic	$X(e^{j\omega})$ (periodic 2π, ω in rad./sample)
if $x[n]$ is periodic, with period N samples	$2\pi \displaystyle\sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\frac{2\pi}{N})$
$e^{j\omega_0 n}$	$2\pi \displaystyle\sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi)$
$\cos(\omega_0 n)$	$\pi \displaystyle\sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi)$ $+\pi \displaystyle\sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - l2\pi)$
$\sin(\omega_0 n)$	$\dfrac{\pi}{j} \displaystyle\sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi)$ $-\dfrac{\pi}{j} \displaystyle\sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - l2\pi)$
1	$2\pi \displaystyle\sum_{l=-\infty}^{\infty} \delta(\omega - l2\pi)$
$\displaystyle\sum_{m=-\infty}^{\infty} \delta[n - mN]$	$\dfrac{2\pi}{N} \displaystyle\sum_{m=-\infty}^{+\infty} \delta(\omega - m\frac{2\pi}{N})$
$\begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$	$\sin(\omega(N_1 + \tfrac{1}{2}))/\sin(\omega/2)$
$\dfrac{\sin(Wn)}{\pi n}$ $0 < W < \pi$	$\begin{cases} 1 & 0 \le |\omega| \le W \\ 0 & W < |\omega| \le \pi \end{cases}$ period. 2π
$\delta[n]$	1
$u[n]$	$\dfrac{1}{1-e^{-j\omega}} + \pi \displaystyle\sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$
$a^n u[n]$ $|a| < 1$	$1/(1 - ae^{-j\omega})$
$-a^n u[-n-1]$ $|a| > 1$	$1/(1 - ae^{-j\omega})$
$\dfrac{(n+r-1)!}{n!(r-1)!} a^n u[n]$ $|a| < 1$	$\dfrac{1}{\left(1 - ae^{-j\omega}\right)^r}$
$\dfrac{-(n+r-1)!}{n!(r-1)!} a^n u[-n-1]$ $|a| > 1$	$\dfrac{1}{\left(1 - ae^{-j\omega}\right)^r}$
$r^n \sin(\omega_0 n)u[n]$ $0 \le r < 1$ $0 \le \omega_0 \le \pi$	$\dfrac{r\sin(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$r^n \cos(\omega_0 n)u[n]$ $0 \le r < 1$ $0 \le \omega_0 \le \pi$	$\dfrac{1 - r\cos(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$-r^n \sin(\omega_0 n)u[-n-1]$ $r > 1$ $0 \le \omega_0 \le \pi$	$\dfrac{r\sin(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$-r^n \cos(\omega_0 n)u[-n-1]$ $r > 1$ $0 \le \omega_0 \le \pi$	$\dfrac{1 - r\cos(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$

Properties – bilateral (two-sided) Laplace transform

Definitions:

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \qquad x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

$$x(t)\xleftrightarrow{\ LT\ } X(s) \quad ROC_x \qquad y(t)\xleftrightarrow{\ LT\ } Y(s) \quad ROC_y$$

Linearity: $ax(t)+by(t)\xleftrightarrow{\ LT\ } aX(s)+bY(s)$

$$ROC_x \cap ROC_y$$

Shifting: $x(t-t_0)\xleftrightarrow{\ LT\ } e^{-st_0}X(s) \quad ROC_x$ unchanged

Scaling: $x(at)\xleftrightarrow{\ LT\ } \dfrac{1}{|a|}\,X\!\left(\dfrac{s}{a}\right)$

ROC_x dilated factor $|a|$ or compressed factor $\dfrac{1}{|a|}$,

and ROC_x inversed if $a<0$

Flipping: $x(-t)\xleftrightarrow{\ LT\ } X(-s) \quad ROC_x$ inversed

Conjugate: $x^*(t)\xleftrightarrow{\ LT\ } X^*(s^*) \quad ROC_x$ unchanged

Symmetry: if $x(t)$ real : $X(s)=X^*(s^*)$,

$$\left|X(s)\right| = \left|X(s^*)\right|$$

Convolution:

$$x(t)*y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau \xleftrightarrow{\ LT\ } X(s)Y(s)$$

$$ROC_x \cap ROC_y$$

Modulation: $e^{s_0 t}x(t)\xleftrightarrow{\ LT\ } X(s-s_0)$

$$ROC_x \text{ shifted to right by } \mathrm{Re}\{s_0\}$$

Differentiation: $\dfrac{dx(t)}{dt}\xleftrightarrow{\ LT\ } s\,X(s) \quad ROC_x$ unchanged

Integration: $\displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\ LT\ } \dfrac{1}{s}\,X(s)$

$$ROC_x \cap \left(\mathrm{Re}\{s\}>0\right)$$

Differentiation in freq.: $-tx(t)\xleftrightarrow{\ LT\ } \dfrac{dX(s)}{ds}$

$$ROC_x \text{ unchanged}$$

Table of bilateral (two-sided) Laplace transforms

Signal $x(t)$	Laplace transform $X(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\dfrac{1}{s}$	$\mathrm{Re}\{s\}>0$
$-u(-t)$	$\dfrac{1}{s}$	$\mathrm{Re}\{s\}<0$
$e^{-at}u(t)$	$\dfrac{1}{s+a}$	$\mathrm{Re}\{s\}>-a$
$-e^{-at}u(-t)$	$\dfrac{1}{s+a}$	$\mathrm{Re}\{s\}<-a$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\dfrac{1}{(s+a)^n}$	$\mathrm{Re}\{s\}>-a$
$-\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\dfrac{1}{(s+a)^n}$	$\mathrm{Re}\{s\}<-a$
$e^{-at}\sin(\omega_0 t)u(t)$ $\omega_0 \geq 0 \quad a,\omega_0$ real	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-a$
$e^{-at}\cos(\omega_0 t)u(t)$ $\omega_0 \geq 0 \quad a,\omega_0$ real	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-a$
$-e^{-at}\sin(\omega_0 t)u(-t)$ $\omega_0 \geq 0 \quad a,\omega_0$ real	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}<-a$
$-e^{-at}\cos(\omega_0 t)u(-t)$ $\omega_0 \geq 0 \quad a,\omega_0$ real	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}<-a$