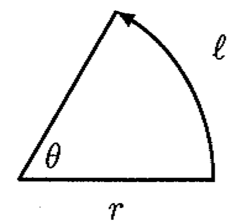
LINEAR (or TRANSLATION) VS ANGULAR (or ROTATION)

By linear and translation, I mean the center of mass of the object is moving in some sort of a line, but not necessarily a straight line, it could curve, or it could not.

Circular Motion of an object's Center of Mass:

All objects move in some sort of a curvy line, but sometimes that curve is circular. When it is, we have a nice geometry to work with, where we can make headway on the problem. In general, an object sweeps out some arclength as it moves through some angle:



Here, l is the arclength, relating to the other quantities by:

$l = r\theta$ (1)

This actually defines the SI unit of angle, θ: the radian, which is a weird unit…because it is dimensionless.

Notice, the left-hand side of the equation has dimensions of length (meters), the right-hand side has dimensions of meters*radians. This means radians has no dimensions. It is a "dimensionless unit".

This means, you have to use radians for this equation, where 2π is the number of radians in a full circle. If you converted to degrees or # or revolutions, you'd get some crazy answers.

Finding the change in both sides of the arclength equation with time, and noting that the radius stays constant, you get the following relating the speed of an object (tangent to the circle) along an arc, with the angular speed, ω in radians/second:

$v_{tangent} = r\omega$ (2)

Again, you need to use radians for this to make sense. If you do the same thing again, you relate the acceleration of an object, tangent to the circle, to its angular acceleration, α, in radians/second^2 is:

$a_{tangent} = r\alpha$ (tangent to circle) (3)

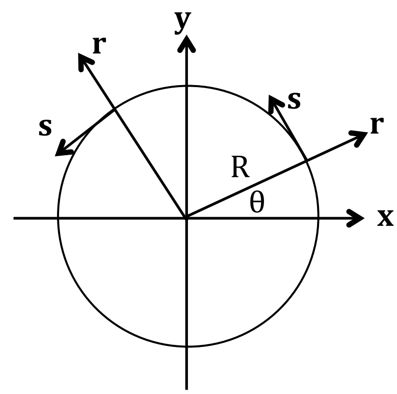
This isn't the only acceleration when an object moves in a circle. If the object's tangential speed isn't changing, $a_{tangent} = 0$ because its angular velocity, ω, is constant so its $\alpha = 0$. But the object is still accelerating because its direction of motion is changing, so the velocity *vector* is changing.

From the pure geometry of the problem, it turns out the acceleration of an object moving at a constant speed in a circle points radially inward (think of the force of gravity on the moon pointing radially inward toward Moon's circular orbit – toward the Earth). For the case of an object moving at a constant speed in a circle, the acceleration pointing radially inward, called "centripetal" acceleration (meaning center-seeking acceleration) is:

$a_{radial} = v^2_{tangent}/r$ (radially inward) (4)

If an object is moving in a circle and changing speed, the total acceleration is the vector sum of the two components.

Proof that $a_{radial} = v^2_{tangent}/r$:



Let **r** and **s** stand for unit vectors that point radially and tangentially (respectively) to the circle and change direction as θ changes (as shown). Let $\mathbf{v} = v_{tan}\,\mathbf{s}$, meaning that the velocity points entirely tangent to the circle but its magnitude is constant. Take the derivative to find the acceleration:

$$\mathbf{a} = d\mathbf{v}/dt = d(v_{tan}\,\mathbf{s})/dt = (dv_{tan}/dt)\mathbf{s} + v_{tan}(d\mathbf{s}/dt) = v_{tan}(d\mathbf{s}/dt)$$

The first term in the sum is zero because the speed is constant (if not, this would become $a_{tan} = r\alpha$). Since the unit vector depends on θ and θ depends on time, the unit vector depends on time. It is best to represent the unit vector **s** in terms of constant unit vectors **x** and **y** (from analyzing the picture):

$$\mathbf{r} = (\mathbf{r}\bullet\mathbf{x})\mathbf{x} + (\mathbf{r}\bullet\mathbf{y})\mathbf{y} = (\cos\theta)\mathbf{x} + (\sin\theta)\mathbf{y}$$
$$\mathbf{s} = (\mathbf{s}\bullet\mathbf{x})\mathbf{x} + (\mathbf{s}\bullet\mathbf{y})\mathbf{y} = (-\sin\theta)\mathbf{x} + (\cos\theta)\mathbf{y}$$

$$\mathbf{a} = v_{tan}d\mathbf{s}/dt = v_{tan}\{d(-\sin\theta)/dt\,\mathbf{x} + d(\cos\theta)/dt\,\mathbf{y}\} = v_{tan}\{-(d\theta/dt)\cos\theta\mathbf{x} - (d\theta/dt)\sin\theta\mathbf{y}\} = v_{tan}\{-\omega\cos\theta\mathbf{x} - \omega\sin\theta\mathbf{y}\} = -\omega v_{tan}\{\cos\theta\mathbf{x} + \sin\theta\mathbf{y}\} = -\omega v_{tan}\mathbf{r}$$

The unit vector in {}-brackets is equal to the radius unit vector, **r**. We can now substitute $v_{tan} = R\omega$ to use what form we find most convenient:

$$\mathbf{a} = -(v^2_{tan}/R)\mathbf{r} = -(R\omega^2)\mathbf{r}$$

Rotation of an object about a point:

An object's center of mass can translate in a circle, but the extended body of an object can rotate around some point. For example, the Earth moves around the Sun but it also spins on its axis.

We have freedom to define our system how we want. Some choices make a problem easier to solve than others.

*We can always treat any motion of a rigid object as a combination of 2 types of motion:

1. Translation of the object's center of mass
   a. Newton's 2nd law gives us the acceleration of the object's center of mass
2. Rotation of the object around its center of mass
   a. We have a variation of Newton's 2nd law just for this
- Sometimes it is easier to use a different rotation point (like how a door rotates about its hinge)

In general, you have the same 2 models at your disposal, but adapted for rotation (see the next page for details):

1. Energy model – we have a new type of motion, rotation; so we have a new type of kinetic energy Ek,rotation or Ek,rot. This extra energy type gets incorporated just as any other type.

2. Force model – the rotational equivalent of force is torque, which depends on where you apply the force: this requires a new diagram: the extended force diagram (you can't just put all forces on the center of mass for this)

   Using the force model, you need to solve BOTH PROBLEMS (sometimes) to figure out the full motion – make a force diagram for the center of mass AND an extended force diagram to show its rotational motion and torques.

	LINEAR/TRANSLATION		ANGULAR/ROTATION	
	Symbol/descrip.	Units	Symbol/descrip.	Units
Position	**x**, **y**, **r** vector points from origin to location	meters, m	**θ** vector is _|_ to plane of rotation: RHR1	radians, rad (dimensionless units)
Displacement	Δ**x**, Δ**y**, Δ**r** Vector points in direction of motion, initial → final position	m	Δ**θ** Vector is _|_ to plane of rotation, direction of motion $\theta i \rightarrow \theta f$: RHR1	rad (dimensionless units)
Velocity	**v** = d**r**/dt vector points same direction as displacement (instantaneous vs average)	m/s	**ω** = d**θ**/dt Vector in same direction as displacement: RHR1	rad/s
Acceleration	**a** = d**v**/dt vector points along change in velocity	m/s^2	**α** = d**ω**/dt vector points same as change of ang. velocity	rad/s^2
Momentum	**p** = m**v**	kg m/s	**L** = I**ω** = $\|r\|\|p\| \sin\Theta$ dir: same as ω (usually), or dir for 2nd eq: RHR2	$kg\ m^2\ rad/s$
Inertia-ish thing	mass, m	kg	Rotational Inertia, usu. $I = CmL^2$, C = const. dep. on mass distrib. L = length, dep on mass distrib. Look up C, L for your shape (there are ways to calculate this)	$kg\ m^2$
Kinetic Energy	$E_{kin,trans} = \frac{1}{2} mv^2$	Joules, $J = kg\ m^2/s^2$	$E_{kin,rot} = \frac{1}{2} I\omega^2$	Joules, $J = kg\ m^2/s^2$
Force-ish thing	Force, **F**	$N = kg\ m/s^2$	Torque, **τ** $\|\tau\| = \|r\|\|F\| \sin\Theta$ r = moment arm: points from rotation point to where force acts Θ is angle between moment arm and F, Direction: RHR2	Nm (NOT a "Joule")

	LINEAR/TRANSLATION		ANGULAR/ROTATION	
	Symbol/descrip.	Units	Symbol/descrip.	Units
Newton's 2nd law	$\mathbf{a} = \mathbf{F}net/m$ object's center of mass accelerates (linearly) if there is a net force on them		$\boldsymbol{\alpha} = \boldsymbol{\tau}net/I$ objects accelerate rotationally if there is a net torque on them	
Newton's 3rd Law	$\mathbf{F}_{1\to2} = -\ \mathbf{F}_{2\to1}$		$\boldsymbol{\tau}_{1\to2} = -\boldsymbol{\tau}_{2\to1}$ $(r_1\sin\theta_1)F_{1\to2} = (r_2\sin\theta_2)F_{2\to1}$ (**_|_ comp. of moment arms '=' for collinear, eq/opp forces & same origin)	
Force diagram	All forces are placed on the object's center of mass		All forces are placed where they act on the object ("extended")	
Work	$W_{trans} = \|F\|\|\Delta x\|\cos\Theta$ (Constant Force) Θ angle between F and Δx vectors	Joules	$W_{rot} = \|\tau\|\|\Delta\theta\|\cos\Theta$ (Constant torque) Θ angle between τ and $\Delta\theta$ vectors	Joules
Work-Energy Thm	$W_{total,trans} = \Delta E_{k,trans}$ (center of mass displacement)		$W_{total,rot} = \Delta E_{k,rot}$	
Momentum Consv. Eq	$\mathbf{p}_{total}(init) + (\mathbf{F}_{total}*\Delta t) = \mathbf{p}_{total}(fin)$ (Constant Force)		$\mathbf{L}_{total}(init) + (\boldsymbol{\tau}_{total}*\Delta t) = \mathbf{L}_{total}(fin)$ (Constant Torque)	

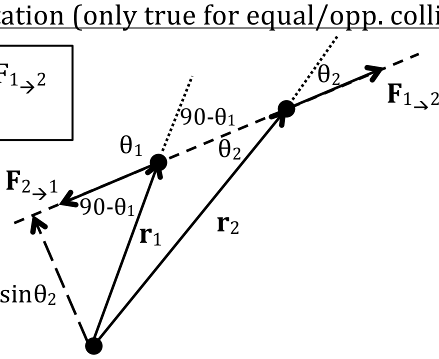
- For rotation, you often need to individually define the magnitude and direction for vectors, if you aren't familiar with vector operations such as dot products and cross products (beyond the scope of this course).
- The directions for vectors need some convention if they point perpendicular to a plane of rotation. Why? There are 2 vectors perpendicular to any plane (e.g. up and down are perpendicular to the surface of the Earth). We use the right-hand rules (RHR1 and RHR2) for this.

RHRs (Right-Hand Rules) – Direction Symbols: (Like a feathered-arrow) Into Page = $\otimes$, Out of Page = $\odot$

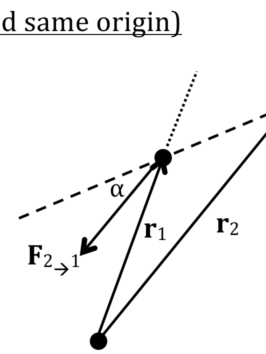
- RHR1: Point your RIGHT hand's fingers in the direction of rotation (they point within the plane of rotation), the thumb gives the direction for the vector quantity you want (e.g. $\boldsymbol{\theta}$, $\Delta\boldsymbol{\theta}$, $\boldsymbol{\omega}$)
- RHR2: 2 vectors on the right side of the equation, as they're written now: curl RIGHT hand's fingers from 1st to 2nd vector: thumb gives resultant vector (e.g. $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, or $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Here the x is "cross product" of the vectors – a tool for finding both magnitude and direction)

**Newton's 3rd law for rotation (only true for equal/opp. collinear forces and same origin)

$r_1\sin\theta_1 F_{2\to1} = r_2\sin\theta_2 F_{1\to2}$
$\boldsymbol{\tau}_{2\to1} = -\boldsymbol{\tau}_{1\to2}$



$r_1\sin(90-\theta_1)=r_1\sin\theta_1=r_2\sin\theta_2$



$\theta_1 \neq 0$
$\tau_{2\to1} \neq 0$
$\theta_2 = 0$
$\tau_{1\to2} = 0$