Assignment 2

Class scores distribution Show

My score

100% (70/70)

Q1

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Let G be a simple graph with n vertices, and assume that every vertex has a degree $\geq \frac{n}{2}$. Show that G is connected.

$\boxed{\text{Asgn 2}}$

1) The connected component with the least number of vertices such that each vertex has degree $\frac{n}{2}$ is $K_{(\frac{n}{2}+1)}$ and has $\frac{n}{2}+1$ vertices. This is because G is a simple graph and thus cannot have reflexive edges (self-loops), and each vertex must have an edge to another vertex. Suppose that it is possible for G to be disconnected and is thus made up of connected components of size k, s.t. $\frac{n}{2}+1 \leq k < n$. However, if we have a connected component of size k in G, then there are only at most $\frac{n}{2}-1$ vertices in G which are not a part of this connected component. Thus, we cannot create anymore connected components with each vertex having degree $\frac{n}{2}$, and have reached a conflict with the fact that each vertex in G has degree $\frac{n}{2}$. Therefore, it is impossible for G to be disconnected and must, in fact, be connected.

Q2

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The *chromatic polynomial* of a graph G is the function $P_G(k)$ that gives the number of ways to color the vertices of G using k colors, such that no two adjacent vertices have the same color.

(a) If L is a path (linear graph) with n vertices, show that $P_L(k) = k(k-1)^{n-1}$.

(b) Let C_n be the cyclic graph with n vertices. Find $P_{C_3}(k)$.

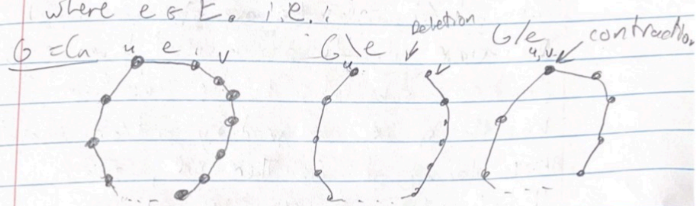
(c) Show that for $n > 3$, $P_{C_n}(k)$ satisfies the recurrence:

$$P_{C_n}(k) = k(k-1)^{n-1} - P_{C_{n-1}}(k) .$$

2a) Suppose you start coloring L at one end and work your way to the other end coloring each vertex one at a time. When you color the first vertex, there are no constraints. You can choose to color it any of the k colors. For the remaining $n-1$ vertices, you cannot color the vertex the same as the one before it. Thus, there are $k-1$ possible colors for each of these vertices, and $P_L(k) = k \cdot \underbrace{(k-1)(k-1)\cdots(k-1)}_{n-1} = k(k-1)^{n-1}$.

2b) Using a similar argument, we start at an arbitrary vertex in C_3 and color it one of k colors. We then move in an arbitrary direction to another vertex, for which there are $k-1$ coloring options. We then move to the final vertex which has edges to the other two vertices, which have already been colored two different colors, leaving us with $k-2$ coloring options. Therefore, $P_{C_3}(k) = k(k-1)(k-2)$

2c) Deletion-contraction recurrence: For any simple graph $G=(V,E)$, $P_G(k) = P_{G\setminus e}(k) - P_{G/e}(k)$ where $e \in E$. i.e.:



Proof: If $e=uv$, then a k-coloring of G is the same as a k-coloring of $G \setminus e$ minus the k-colorings where $c(u)=c(v)$, which is the same as the k-colorings of G/e.

It is easily observed that $C_n \setminus e = L_n$ and $C_n/e = C_{n-1}$ where $n \geq 3$, and is shown in the drawing above. Therefore, by the Deletion-contraction recurrence:

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$P_{C_n}(k) = P_{L_n}(k) - P_{C_{n-1}}(k)$

$P_{C_n}(k) = k(k-1)^n - P_{C_{n-1}}(k)$ * from (2a)

This holds for $n>3$, because we need at least 3 vertices to define a cyclic graph and when $n>3$, $n-1 \geq 3$. This allows us to define C_{n-1} and maintains the validity of the recurrence.

Q3

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In this problem, we investigate the edge-chromatic number $\chi^E(K_n)$ of complete graphs K_n.

(a) Show that coloring the edge ij with color $i + j \pmod n$ gives a valid edge coloring of K_n.

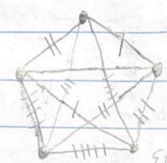
(b) Show that if n is odd, then $\chi^E(K_n) = n$.

(c) Show that if n is even, then $\chi^E(K_n) = n - 1$.

Hint for (c): K_n is K_{n-1} with one extra vertex, and $n - 1$ is odd|...

3a) For each edge ij if we examine either endpoint i or j, we see that the colors of the other edges incident i or j are $(i+x_i) \bmod n$ and $(j+x_j) \bmod n$ where x_i and x_j represent the variables neighboring i and j, respectively. Without loss of generality, let's consider just the vertex i. If the proposed edge coloring were invalid, then there would need to be vertices x_i, x_i' neighboring i s.t. $x_i \neq x_i'$ and $i + x_i \equiv i + x_i' \bmod n$. It follows that $x_i \equiv x_i' \bmod n$. However, no two vertices in K_n can be elements of the same equivalence class mod n because there are only n vertices, which are $\{1, 2, \ldots, n\}$. Therefore, this method of coloring is valid.

3b) ~~It follows from (3a) that K_n can be colored with n colors. Thus, it's sufficient to show that K_n can't be colored with $n-1$ colors when n is odd~~



Each color class can contain at most $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ edges, because this is the maximum size of a matching in K_n.

There is a total of $\binom{n}{2}$ edges in K_n.

$\binom{n}{2} = \frac{n(n-1)}{2}$, $\frac{n(n-1)}{2} \cdot \frac{2}{n-1} = n$.

Therefore, we need at least n color classes to edge color K_n.

This comes from the fact that the size of a matching times 2 is the number of vertices involved in the matching and $\frac{n+1}{2} \cdot 2 > n$, while $\frac{n-1}{2} \cdot 2 < n$.

3c) A similar argument for (3b) applies. Since n is even, $\frac{n}{2}$ is a natural number and $\frac{n}{2} \cdot 2 = n$, the number of vertices in K_n. Therefore, we can make color classes of size at most $n/2$ when we edge color our graph. There are $\frac{n(n-1)}{2}$ edges in K_n. Thus, $\chi^E(K_n) = \frac{n(n-1)}{2} \cdot \frac{2}{n} = n-1$.

Q4

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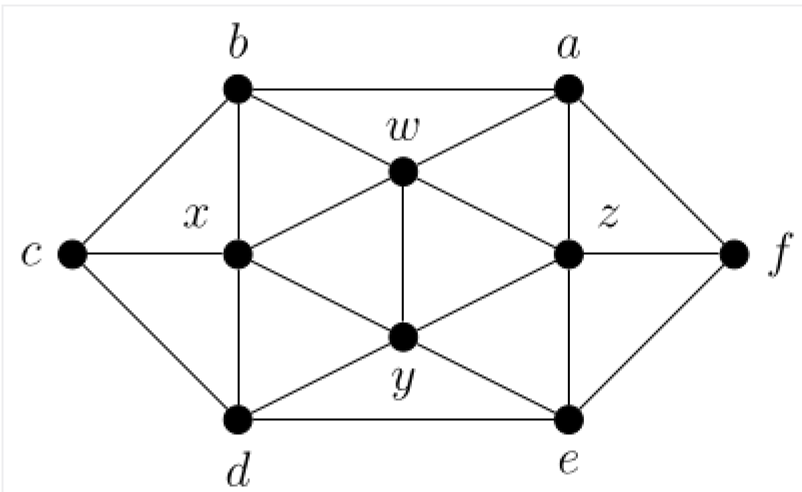
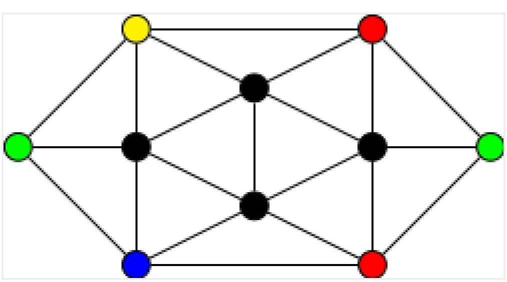
The goal of this problem is to prove *some* of the many cases of the 4-colors theorem. Let our four colors be: red, blue, green and yellow.

Assume that the 4-colors theorem is proven for all planar graphs with less than n vertices and that G is a graph with n vertices that contains the following subgraph.

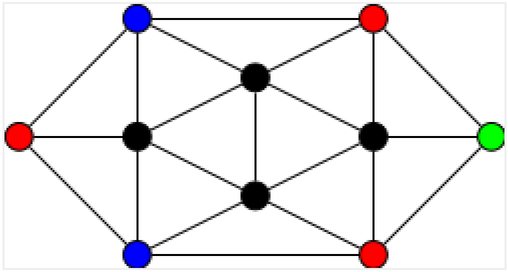


We remove the four vertices w, x, y, z in the middle, resulting in a smaller planar graph, and let the induction hypothesis give us a coloring of the six outer vertices a, b, c, d, e, f. We call such a coloring a *configuration*.

(a) Suppose that the induction hypothesis gave us the following configuration. Conclude that G is 4-colorable.

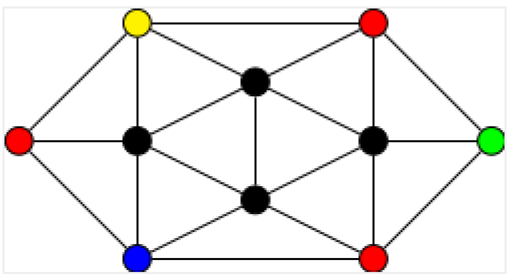


(b) Suppose instead that we have the following configuration. Use the Kempe-Chain argument to *modify* the coloring on G so that you can complete it and conclude that G is 4-colorable.



Hint for (b): Consider the possible connected components of Red-Yellow subgraph G_{RY} or the Green-Blue subgraph G_{GB}|...

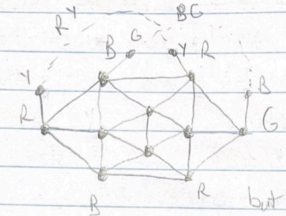
(c) Now, say that the induction hypothesis gave us the following configuration. Use another Kempe-Chain argument to modify the coloring in order to end up with the coloring in part **(a)** or **(b)**. Deduce that G is 4-colorable again in that case.



Note: Now you have a fairly good idea of how the 4-colors theorem was originally proved. The idea is to build a tree (actually, a forest) of configurations that are reducible to one another by a Kempe-Chain argument, so that the leaves of this forest can be completed directly like in (a). The issue is that the forest built by Appel and Haken's program has 1936 nodes rather than just the 3 nodes you just checked, which is why the help of a computer was required.

4a) Let G' be the graph with vertices w, x, y, z removed. By our assumption, G' is 4-colorable, because it has fewer than n vertices and is planar. In the given configuration, we can define the coloring $c : \{w, x, y, z\} \to \{red, blue, green, yellow\}$ as $c(w) = blue$, $c(x) = red$, $c(y) = green$, $c(z) = yellow$. This completes a valid 4-coloring of G, thus proving that G is 4-colorable.

4b)



It is possible that either of the Kempe chains shown on the left exist in G, but not both, per the Kempe chain argument made in lecture. If the shown RY chain exists, then we can flip the colors of the part of the BG subgraph connected to vertex b, making it now green instead of blue. We then color x yellow, w blue, z yellow, and y green, providing a 4-coloring for G.

Now suppose that the BG chain exists

We can use a similar argument as before to flip the coloring of the part of the RY subgraph connected to vertex a, making it yellow rather than red. We then color w red, x yellow, y green, and z blue, providing a 4-coloring for G.

4(c) In this case, we consider the possible connected components of the Blue-Yellow subgraph and the Red-Green subgraph. There can be either a Blue-Yellow Kempe chain from vertex b to d or a Red-Green Kempe chain from c to f, but not both.
Thus, it may be possible to flip the colors of the Blue-Yellow connected component containing vertex b, coloring b Blue without recoloring any other vertices in the subgraph.
If this is not possible, then it must be possible to recolor the Red-Green connected component containing vertex c, recoloring c to be Green, without recoloring any other vertices in the subgraph.
In the first case, we have made the subgraph

equal to the one in (4b), while in the second part, we have constructed the subgraph equal to the one given in (4a). We have already proven that given either of those subgraphs by the induction hypothesis, we can modify and/or complete the 4-coloring of G. Therefore, G is 4-colorable with the given subgraph.

Q5

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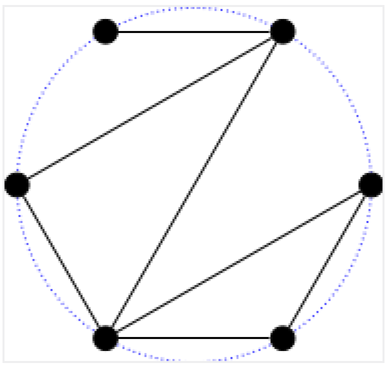
An *outerplanar graph* is a simple graph that has a **planar** drawing for which all vertices touch the outer face of the drawing (equivalently: such that all vertices are on the boundary of some imaginary circle that contains the whole drawing). For example, the following graph is outerplanar.



(a) Show that the complete graph K_4 and the complete bipartite graph $K_{2,3}$ are *not* outerplanar.

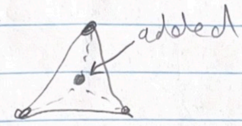
Hint: Use Kuratowski's Theorem

(b) Show that every outerplanar graph is 3-colorable.

Hint: Use the 4-Colors Theorem

5a) It is easily observable that we can add a vertex to a face in a planar graph and add edges connecting it to all vertices incident to that face without inducing a crossing, maintaining planarity of the graph.

example:



In an outerplanar graph G, all vertices are incident with the exterior face. If we add a vertex to the exterior face of G, and connect it to all vertices in G, we are left with a planar graph. This means that G could not have been K_4 or $K_{2,3}$, or else we would have induced K_5 or $K_{3,3}$ as subgraphs by adding the vertex and edges to all other vertices. Therefore, K_4 & $K_{2,3}$ are not outerplanar.

5b) As stated in (5a), if we take an outerplanar graph G and add a vertex v along with edges from v to all vertices of G, we form a planar graph G' which is 4-colorable. Since G' is 4-colorable, v must be adjacent to vertices with at most 3 different colors. Because v is adjacent to all vertices of G, we can conclude that G is 3-colorable.

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Q6

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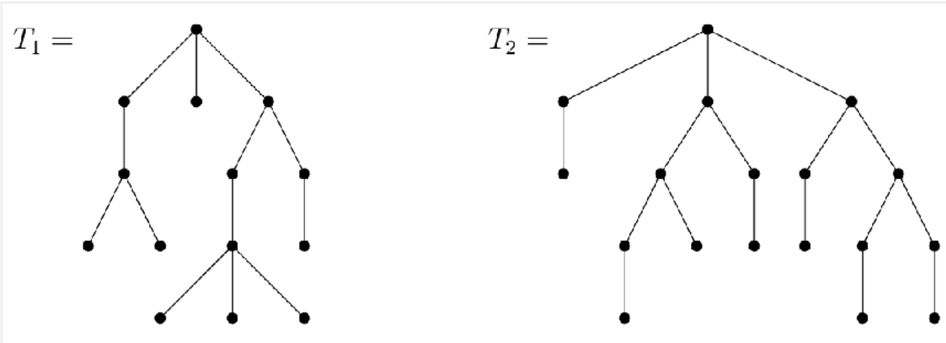
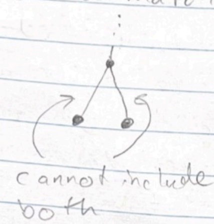
(a) How many perfect matchings do each of the following trees T_1 and T_2 contain? Justify your answer.

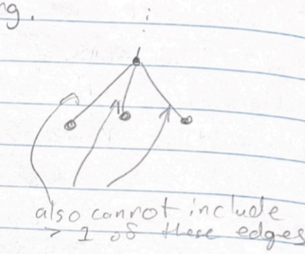


(b) Prove that a tree always contains **at most one** perfect matching.

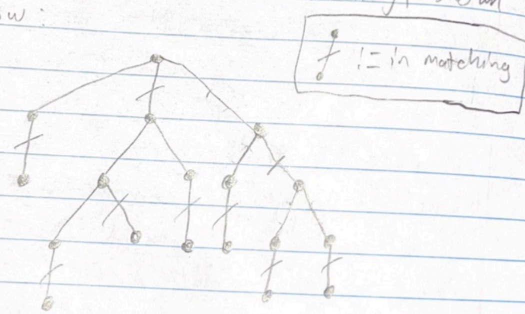
Hint: Proceed by induction on the number of vertices...

6a) T_1 has 0 perfect matchings. If we examine the leaf nodes, we see that there are multiple leaf nodes that share a parent. In order to construct a perfect matching, these leaf nodes must have their edges to their parent included in the matching. However, this violates the conditions of a matching.

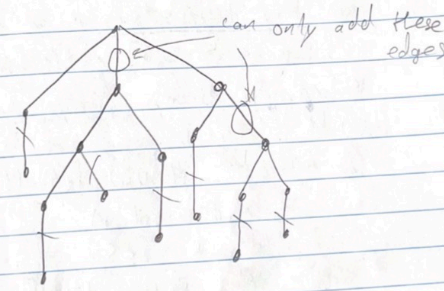




T_2 has 1 perfect matching, shown below:



The matching shown is the only perfect matching possible in T_1. This is because we must include all of the edges to the leaves, as leaves only belong to one edge each. We observe that there is no more than one way to complete the perfect matching once we add all of the leaf-edges:



6b) Proof by induction:

Base case: $n=0$, the empty tree has one perfect matching: $\emptyset$. $n=1$, the tree containing 1 vertex does not contain any perfect matchings.

Induction Hypothesis: assume that every tree with less than or equal to n vertices has at most one perfect matching.

Inductive step: Suppose we have a tree made up of $n+1$ vertices. If we can add all of the edges which contain a leaf to our matching, we do so. Otherwise we cannot form a perfect matching. Suppose we can add these edges to our matching. We proceed to remove all of the vertices in these edges from our tree. This leaves us with a new tree which has fewer than $n+1$ vertices (all trees w/ ≥ 2 vertices have leaves which belong to an edge). This tree has at most 1 perfect matching by the inductive hypothesis. If this tree has no perfect matchings, then neither does our original tree. However, if it does have a perfect matching, there won't be any conflicts when we add our previously deleted edges to this matching. There was only one way to pick these edges, thus we have constructed the only perfect matching in our original tree.

Q7

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Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \cup Y$, such that $|X| = |Y| = 42$ and $\deg(v) \geq 21$ for every vertex $v \in V$. Show that G has a perfect matching.

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7) Suppose G does not satisfy Hall's Condition and $\exists\, S \subseteq X$ s.t. $|S| > |N(S)|$. This is only possible if $|S| > 21$, because every vertex in G has degree ≥ 21. It follows that $|X \setminus S| < 42 - 21 = 21$. However, this is a contradiction, because the elements of $Y \setminus N(S)$ need to be able to map to at least 21 vertices in $X \setminus S$, as each element of $Y \setminus N(S)$ has degree ≥ 21. Therefore, G satisfies [illegible] Condition and by Hall's Theorem, there is a matching on G that saturates X or Y. However, $|X| = |Y|$, so any matching that saturates X or Y is perfect. $\square$

good 10