Final Exam

Class scores distribution Show

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Instructions

- You have three hours to complete the exam. There are nine questions worth 10 marks each.
- You must justify all your work, with knowledge and notation from this course, unless specified otherwise.
- A table of potentially useful generating functions is available on the back of the cover page. No other documentation is allowed. Calculators and smarter devices are forbidden.
- You can use the back pages as extra space for answering each question. No extra booklet will be allowed.
- There are two extra pages at the end of the booklet if you need more space. If you want them to be graded for a specific question, you must clearly tell the grader, on the appropriate question's page, to have a look. No extra booklet will be allowed.
- The exam will be scanned and graded on a computer (with Crowdmark). So please do not write anything over the QR code in the corner of any page, and write dark enough and not too close to the edge of the paper.
- Good luck!

$\frac{1-x^{N+1}}{1-x} = 1+x+x^2+\ldots+x^N$	$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$	$\frac{1}{(1-x)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} x^n$
$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n+1}}{4^n(2n-1)} x^n$	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cosh(x) = \frac{e^x+e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$\sinh(x) = \frac{e^x-e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

Q1 10

Question 1 (10 marks)
Find a combinatorial proof of the following identity.

$$\sum_{k=2}^{n} \binom{n}{k} \cdot \binom{k}{2} = \binom{n}{2} \cdot 2^{n-2}$$

The LHS is the number of possible ways to choose k-sized subsets of a set of size n, where $2 \leq k \leq n$, and choose 2 elements from each of those subsets. This is the same as the RHS, because the RHS is the # of ways to choose 2 elements from [n], times the # of subsets we can make with the remaining n-2 elements. If we add the 2 elements we chose from [n] to each of the 2^{n-2} subsets we can make with n-2 elements, we can see that this is also the number of ways to choose every possible subset of [n] with size ≥ 2 and choose 2 elements from that subset. □

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*Extra space for Question **1**.*

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Q2 10

Question 2 (10 marks)
Use the Generating Functions Method to solve the recurrence $a_n = 3a_{n-1} + 4^n$ with initial condition $a_0 = 1$.

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} (3a_{n-1} + 4^n) x^n$$

$$= 1 + \sum_{n=1}^{\infty} 3a_{n-1} x^n + \sum_{n=1}^{\infty} 4^n x^n$$

$$= \cancel{1} + 3x \sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} 4^k x^k - \cancel{1}$$

$a_0 = 1$
$a_1 = 7$
$a_2 = 37$
$a_3 = 175$
$\vdots$ ✓

$$A(x) = 3x A(x) + \frac{1}{1-4x}$$

$$A(x)(1-3x) = \frac{1}{1-4x}$$

$$A(x) = \frac{1}{(1-3x)(1-4x)} = \frac{A}{1-3x} + \frac{B}{1-4x}$$

$A(1-4x) + B(1-3x) = 1$

$A + B = 1 \Rightarrow A = 1 - B$

$-4A - 3B = 0$

$-4(1-B) - 3B = 0$

$-4 + 4B - 3B = 0$

$-4 + B = 0$

$B = 4$

$A = -3$

$$A(x) = -\frac{3}{1-3x} + \frac{4}{1-4x}$$

$$= -3\sum_{n=0}^{\infty} 3^n x^n + 4\sum_{n=0}^{\infty} 4^n x^n$$

$$A(x) = \sum_{n=0}^{\infty} (4^{n+1} - 3^{n+1}) x^n$$

$$\boxed{a_n = 4^{n+1} - 3^{n+1}}$$

Extra space for Question 2.

Q3 10

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Question 3 (10 marks)
Let $G = (V, E)$ be a simple graph. Prove that there are distinct vertices $u, v \in V$ such that $\deg(u) = \deg(v)$.
Hint: *Use the Pigeonhole Principle.*

$* \; n = |V|$

$\forall x \in V$, $0 \leq \deg(x) \leq n-1$, which gives n options (bins) for each vertex's (balls) degree. However, if there exists a vertex in G s.t. it's degree is zero, then $\Delta(G) \leq n-2$. So $\forall x \in V$, we actually have that $0 \leq \deg(x) \leq n-2$ or $1 \leq \deg(x) \leq n-1$, which gives $n-1$ options for the different degrees of the n vertices in G. Therefore, by the Pigeonhole Principle, we have that two vertices in G must have the same degree, because $\lceil \frac{n}{n-1} \rceil = 2$. $\square$

Extra space for Question 3.

Q4 10

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Question 4 (10 marks)
True or false? Justify with a proof or a counter-example. In each case, G denotes a simple graph.

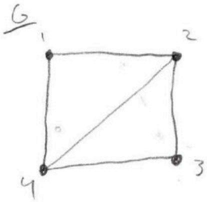
(a) If G has a perfect matching, then G has an even number of vertices.

This is true, because for a perfect matching M, the number of vertices in G must be $2|M|$, which is even.

* because all vertices are part of one edge in the matching, and each edge has 2 vertices.

(b) If G has a maximum matching M and a minimum vertex cover S of the same size ($|M| = |S|$), then G is bipartite.

False; G



$M = \{\{1,4\}, \{2,3\}\}$ is maximal

$S = \{2, 4\}$ is minimum

$|M| = |S|$ but G is not bipartite because it contains an odd cycle.

Extra space for Question 4.

Q5 8

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Question 5 (10 marks)
Apply the Girl Proposal Algorithm to find a stable matching given the preference lists below.

$\boldsymbol{G_1}$: $B_5 > \boxed{B_2} > B_1 > B_3 > B_4$
$\boldsymbol{G_2}$: $B_3 > \boxed{B_4} > B_5 > B_1 > B_2$
$\boldsymbol{G_3}$: $B_4 > B_5 > B_3 > \boxed{B_1} > B_2$
$\boldsymbol{G_4}$: $\boxed{B_5} > B_1 > B_3 > B_4 > B_2$
$\boldsymbol{G_5}$: $B_4 > \boxed{B_3} > B_1 > B_5 > B_2$

$\boldsymbol{B_1}$: $G_1 > \boxed{G_3} > G_4 > G_5 > G_2$
$\boldsymbol{B_2}$: $G_3 > G_4 > G_2 > G_5 > \boxed{G_1}$
$\boldsymbol{B_3}$: $G_1 > G_4 > \boxed{G_5} > G_2 > G_3$
$\boldsymbol{B_4}$: $G_4 > \boxed{G_2} > G_3 > G_1 > G_5$
$\boldsymbol{B_5}$: $\boxed{G_4} > G_5 > G_1 > G_3 > G_2$

B_1 : $\emptyset$
B_2 : $\emptyset$
B_3 : $\boxed{G_2}$
B_4 : $\boxed{G_3}, G_5$
B_5 : $G_1, \boxed{G_4}$

B_1 : $\emptyset$
B_2 : G_1
B_3 : $\cancel{G_2}\ \boxed{G_5}$
B_4 : $\boxed{G_3}$
B_5 : $\boxed{G_4}$

B_1 : $\emptyset$
B_2 : G_1
B_3 : G_5
B_4 : $\cancel{G_3}, G_2$
B_5 : G_4

nothing changes x2

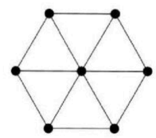
Please explain... -2

B_1 : G_3
B_2 : G_1
B_3 : G_5
B_4 : G_2
B_5 : G_4

*Extra space for Question **5**.*

Q6 10

Question 6 (10 marks)
The *wheel graph* W_n is like a cyclic graph C_n, with an extra vertex in the middle that connects to all vertices of the cycle. For instance, a picture of W_6 is presented on the right. Show that for all $n \geq 3$,



$$\chi(W_n) = \begin{cases} 4 \text{ if } n \text{ is odd;} \\ 3 \text{ if } n \text{ is even.} \end{cases}$$

We can say that C_n is 2-colorable when n is even and 3-colorable when n is odd. Thus, if we remove the vertex in the middle, the remaining C_n subgraph is either at least 2-colorable or at least 3-colorable, depending on the parity of n. Therefore, when we add the vertex in the middle, it is either adjacent to vertices of 2 colors and we need $2+1=3$ colors to color W_n (when n is even), or it is adjacent to vertices of 3 different colors (n is odd) and we need $3+1=4$ colors to color W_n. □

* We showed in class the fact that $\chi(C_n) = 2$ when n is even and $\chi(C_n) = 3$ when n is odd.

Extra space for Question ***6***.

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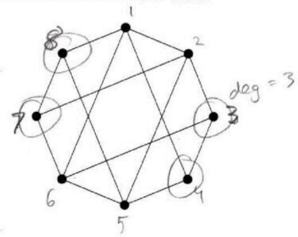
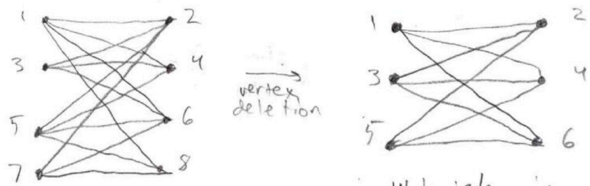
Q7 10

Question 7 (10 marks) Is the following graph planar? If you think it is, then draw it accordingly. If not, then prove it with results from class.





Which is isomorphic to $K_{3,3}$ and a minor of the above graph. By Kuratowski's theorem, the above graph is not planar.

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Extra space for Question 7.

Q8 10

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Question 8 (10 marks)
You have two dice. One is fair but the other is loaded, and their probabilities of throwing a six are $\frac{1}{6}$ and 1, respectively. But you do not know which is fair and which is loaded. So you choose one of the dice at random and throw it.

(a) What is the probability of throwing a six? Prob of picking either die is $\frac{1}{2}$

$$\frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot 1 = \frac{1}{12} + \frac{1}{2} = \boxed{\frac{7}{12}}$$

(b) Suppose you have thrown a six. What is the probability that the die you chose was the fair one? *Use Bayes' theorem.*

Let F be the Bernoulli r.v. for picking the fair die and L the Bernoulli r.v. for picking the loaded die. Let D be the r.v. for the number rolled.

$$P(F=1 \mid D=6) = \frac{P(D=6 \mid F=1)\, P(F=1)}{P(D=6)}$$

$$= \frac{\frac{1}{6} \cdot \frac{1}{2}}{\frac{7}{12}} = \frac{1}{12} \cdot \frac{12}{7} = \boxed{\frac{1}{7}}$$

*Extra space for Question **8**.*

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Q9 10

Question 9 (10 marks)
A plane company sells 10% more tickets than the plane capacity, knowing that 10% of their clients cancel their flight at the last minute (we assume that they choose to do so independently of each other client). Let n be the number of seats that the plane actually contains.

(a) Assuming that all tickets are sold (including the extra 10%), find the expected number of people who actually take the plane. *Hint: The answer is not n: it is less than that!*

Tickets sold $= 1.1n$

expected # on plane $= 1.1n - \sum_{k=0}^{1.1n} .1 = 1.1n - .1(1.1n) = 1.1n - .11n$

$= \boxed{0.99n}$

(b) Use a suitable Chernoff bound to estimate the probability that the plane is over its capacity, and show that this probability converges to 0 as $n \to \infty$.

Let X be the r.v. for the number of people who keep their tickets.

~~$P(X \geq n+1) \Rightarrow n+1 = (1+\delta)0.99n$~~

~~$n+1 = 0.99n + 0.99n\delta$~~

~~$0.01n + 1 = 0.99n\delta$~~

~~$\delta = \frac{0.01n + 1}{0.99n} = \frac{1 \cdot 10^{-2} n}{99 \cdot 10^{-2} n} + \frac{100}{99n}$~~

~~$= \frac{1}{99} + \frac{100}{99n}$~~

~~$0 \leq \delta \leq 1$~~

more on Page 21

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Extra space for Question 9.

$$P(X \geq n+1) \leq e^{-\frac{\delta^2}{3}} = e^{-\frac{\left(\frac{n+100}{99n}\right)^2}{3}} = e^{-\frac{(n+100)^2}{3(99n)^2}} = \frac{1}{}$$

$$\delta = \frac{1}{99} + \frac{100}{99n} = \boxed{\frac{n+100}{99n}}$$

$$n = (1+\delta)\,0.99n$$

$$0.01n = 1+\delta$$

$$0.01n - 1 = \delta$$

$$\delta > 1$$

$$P(X \geq n) \leq e^{-\frac{\delta}{3}0.} = e^{-\frac{(0.01n-1)}{3}} \leq \frac{1}{e^{\frac{0.01n-1}{3}}} \to 0 \text{ as } n \to \infty$$

$$P(X > n) = 1 - P(X \leq n)$$

$$n = (1-\delta)\,0.99n$$

$$n = 0.99n - 0.99n\,\delta$$

$$0.01\cancel{n} = -0.99\cancel{n}\,\delta$$

$$-\frac{0.01}{0.99} = \delta$$

$$-\frac{1}{99} = \delta$$

$$-\frac{\delta^2}{2} = -\frac{\frac{1}{99^2}}{2}$$

$$P(X \leq n) \leq e^{-\frac{\delta^2}{2}0.99n} \leq e^{-\frac{1}{2(99)^2}\cancel{99}n \cdot 10^{-2}} = e^{-\frac{n \cdot 10^{-2}}{198}} = e^{-\frac{n}{19800}}$$

$$\therefore P(X > n) = 1$$

Extra

*Extra space for **any question**. Do not forget to refer the graders to this page from the relevant question if you want this page to be graded.*

Q9) $n = (1 + \delta)\, 0.99n$

$n = 0.99n + 0.99n\delta$

$0.01\cancel{n} = 0.99\cancel{n}\,\delta$

$\frac{1}{99} = \delta$

$0 \le \delta \le 1$

$P(X \ge n) \le e^{-\frac{\left(\frac{1}{99}\right)^2}{3} 0.99n}$

$= e^{-\frac{1}{3 \cdot 99^{\cancel{2}}} \cancel{99} \cdot 10^{-2} n}$

$= e^{-\frac{n \cdot 10^{-2}}{3 \cdot 99}} \longrightarrow 0$ as $n \to \infty$

$P(X > n) \le P(X \ge n)$

$\therefore P(X > n) \to 0$ as $n \to \infty$

$n > E(X) = 0.99n$

Course: MATH 340

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