Owen Habeger

Math 4533/5533 Numerical Methods (Homework #4) Note that Due day is Thursday (3/17).

Submit through BB by 3PM.

1. (20points) Three different data (x1, y1), (x2, y2), (x3, y3) are given.

(a) (6points) Check the interpolation conditions for the interpolant p2.

1.) $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are given, where Lagrange interpolating poly $\rightarrow$ $p_2(x) = \sum_{i=1}^{3} y_i L_i(x)$ and

Lagrange interpolation basis $\Rightarrow$ $L_i(x) = \prod_{j \neq i, j=1}^{3} \frac{(x - x_j)}{(x_i - x_j)}$, $1 \le i \le 3$

a.) check interpolation conditions for interpolant p_2:

$$P_2(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$L_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$L_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$L_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Each $L_i(x)$ has degree $3 = P_3(x)$ has degree ≤ 3

$L_i(x_j) = 0$

$L_i(x_i) = 1$ where $j \neq i$ for $1 \le i, j \le 3$

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

So, $P_2(x)$ interpolates the data, $P_2(x) = y_i$ $i = 1, 2, 3$

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(b) (7points) Prove that there is a unique quadratic interpolant.

b.) Prove that there is a unique quadratic interpolant:

- Use contradiction to prove uniqueness
- Assume that there are two $P_{n-1}(x)$, $q_{n-1}(x)$

with degree $\deg(P_{n-1})$, $\deg(q_{n-1}) \leq n-1$.?. $p(x_i)=y_i$, $q(x_i)=y_i$ where $1 \leq i \leq n$

- Claim that $p(x) = q(x)$
- Define $r(x) = p(x) - q(x)$ with $\deg(r) \leq n-1$
- Since $p(x)$ and $q(x)$ interpolate all points

(x_1, y_1), (x_2, y_2), (x_3, y_3) then,

$$p(x_i) = q(x_i) \quad \text{for} \quad 1 \leq i \leq n$$

$$\Rightarrow r(x_i) = p(x_i) - q(x_i) = 0$$

$$\Rightarrow r(x) = (x - x_1)(x - x_2)(x - x_3) \cdot s(x)$$

- From this, we can prove that $\deg(r) \geq n$ unless $s(x) = 0$
- This is a contradiction from previous $\deg(r) \leq n-1$

If $s(x) = 0$, $r(x) = 0 \Rightarrow p(x) = q(x)$

(c) (7points) Find another interpolant.

c.) Find another interpolant

$$P_1(x) = y_1 L_1(x) + y_2 L_2(x)$$

$$L_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \qquad L_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$\boxed{P_1(x) = y_1 \left(\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \right) + y_2 \left(\frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} \right)}$$

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2. (20points)

(1) (5points) Estimate f(−1).

#2.)

1.) Estimate $f(-1)$. Here the function f is defined by

$$f(x) = \sum_{j=1}^{3}\left(\prod_{i=1}^{3} \delta_{ij}\, x^{2i+j}\right);$$

$$\left(\delta_{1j}\, x^{2+j} \cdot \delta_{2j}\, x^{4+j} \cdot \delta_{3j}\, x^{6+j}\right) = \delta_{11} x^{3}\, \delta_{21} x^{5}\, \delta_{31} x^{7} + (0) + (0)$$

$f(x) = 0$ for all $x \in \mathbb{R}$, since $i = j$

(2) (5points) Estimate f(1).

2.) Estimate $f(1)$. Here the function f is defined by

$$f(x) = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} \delta_{ij} + 1x^{i+2j}\right):$$

$$\left.\begin{aligned} i &= 2,\ 3 \\ j &= 1,\ 2 \end{aligned}\right\} \quad f(x) = x^{2+2\cdot 1} + x^{3+(2\cdot 2)}$$

$$f(x) = x^{4} + x^{7}$$

$$\boxed{f(1) = (1)^{4} + (1)^{7} = 2}$$

(3) (5points) We define a scalar function f:

#3.

c.) We define a scalar function $f: \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x_1, x_2, x_3) = \cos(x_1 x_2) - x_3 e^{x_1} + \ln(x_1^2 + x_3^2 + 1)$$

then find,

$$\delta_{ij} \frac{\delta^2 f}{\delta x_i \delta x_j} \quad \text{for} \quad 1 \le i, j \le 3$$

Note that $\delta^2 f / \delta x_i \delta x_j$ is a Hessian matrix:

$$\delta_{ij} \frac{d^2 f}{dx_i \delta x_j} = \begin{bmatrix} \frac{d^2 f}{dx_1^2} & 0 & 0 \\ 0 & \frac{d^2 f}{dx_2^2} & 0 \\ 0 & 0 & \frac{d^2 f}{dx_3^2} \end{bmatrix}$$

$$\frac{d^2 f}{dx_1^2} = \frac{df}{dx_1}\left(-x_2 \sin(x_1 x_2) - x_3 e^{x_1} + \frac{2x_1}{x_1^2 + x_3^2 + 1}\right)$$

$$\frac{d^2 f}{dx_2^2} = \frac{df}{dx_2}\left(-x_1 \sin(x_1 x_2)\right) = \boxed{-x_1^2 \cos(x_1 x_2)} = B$$

$$\frac{d^2 f}{dx_3^2} = \frac{df}{dx_3}\left(-e^{x_1} + \frac{2x_3}{x_1^2 + x_3^2 + 1}\right) = \boxed{\frac{2x_1^2 - 2x_3^2 + 2}{(x_1^2 + x_3^2 + 1)^2}} = C$$

$$\Rightarrow \boxed{-x_2^2 \cos(x_1 x_2) - x_3 e^{x_1} + \frac{-2x_1^2 + 2x_3^2 + 2}{(x_1^3 + x_3^2 + 1)^2}} = A$$

$$= \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

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(4) (5points) We define a vector-valued function f :

#3.)

4.) We define a vector-valued function $f: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$f = \begin{cases} x_1 \sin x_2 - e^{x_2} \\ \sqrt{x_2^2 + x_3^2 + 1} \\ \dfrac{\cos x_3}{x_1^2 + 1} \end{cases}, \quad \text{find } \delta_{ij} \frac{\partial f_i}{\partial x_j} \text{ for } 1 \leq i, j \leq 3$$

<u>Jacobian Matrix:</u>

$$\delta_{ij} \frac{\partial f_i}{\partial x_j} = \begin{bmatrix} \sin x_2 & 0 & 0 \\ 0 & \dfrac{x_2}{\sqrt{x_2^2 + x_3^2 + 1}} & 0 \\ 0 & 0 & -\dfrac{\sin x_3}{x_1^2 + 1} \end{bmatrix}$$

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3. (30points) We want to get the Lagrange basis functions over the local interval [xn, xn+1].

(1) (10points) Find the three basis functions L0, L1 , and L2 on the local interval [xn, xn+1].

3.) We want to get the Lagrange basis functions over the local interval $[x_n, x_{n+1}]$. They are quadratic and written by,

$$L_n^\ell(x) = \prod_{m=0,\, m\neq\ell}^{2} \frac{(x - x_n + m/2)}{(x_{n+\ell/2} - x_{n+m/2})} \quad \text{for } 0 \le \ell \le 2$$

a.) Find the three basis functions L_n^0, L_n^1, L_n^2 on the local interval $[x_n, x_{n+1}]$: $L^0 = \frac{x(x-1)}{2}$, $L^1 = \frac{(x+1)(x-1)}{-1}$, $L^2 = \frac{x(x+1)}{2}$

$$P_2(x) = y_0 L_n^0(x) + y_1 L_n^1(x) + y_2 L_n^2(x), \quad \begin{cases} y_i = P_2(x_i) \\ L_n^i(x_j) = \delta_{ij} \end{cases}$$

$$\ell = 0,\ L_n^0 = \frac{(x - x_n + 1/2)(x - x_{n+1})}{(x_n - x_n + 1/2)(x_n - x_n + 1)} \Rightarrow \frac{(x)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2}$$

$$\ell = 1,\ L_n^1 = \frac{(x - x_n)(x - x_n + 1)}{(x_n + 1/2 - x_n)(x_n + 1/2 - x_n + 1)} \Rightarrow \frac{(x+1)(x-1)}{(0+1)(0+1)} = \frac{(x+1)\cdot(x-1)}{-1}$$

$$\ell = 2,\ L_n^2 = \frac{(x - x_n)(x - x_n + 1/2)}{(x_n + 1 - x_n)(x_n + 1 - x_n + 1/2)} \Rightarrow \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{x(x+1)}{2}$$

(2) (10points) Use the Scilab to sketch the three functions on the reference interval [−1, 1].

HW4.sce (C:\Users\owenh\Desktop\Spring 2022\SciLab\HW4.sce) - SciNotes

BisectionMethod.sce | Question1.sce | NewtonMethod.sce | *FPI.sce | SecantMethod.sce | HW4.sce

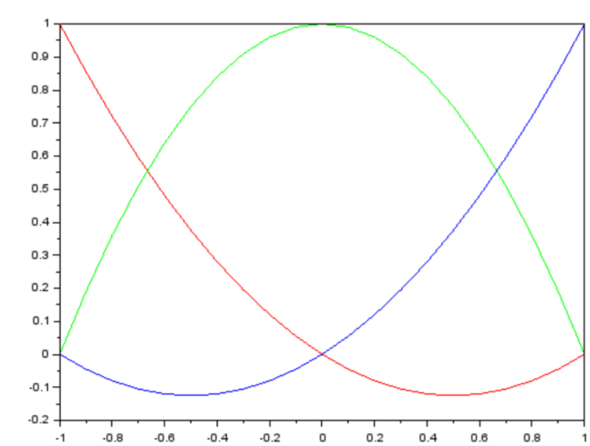
```
///////////////////////////TEST FUNCTIONS///////////////////////
//
function f = Ln0(x)
    f = x.*(x -1) / 2;
endfunction

function f = Ln1(x)
    f = (x + 1).*(x - 1) / -1;
endfunction

function f = Ln2(x)
    f = (x.*(x + 1)) / 2;
endfunction

x=-1:0.1:1
y=Ln0(x);
plot(x,y, 'r')
y=Ln1(x);
plot(x,y, 'g')
y=Ln2(x);
plot(x,y, 'b')
```

Graphic window number 0



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(3) (10points) Check the interpolation conditions at each point x = −1, 0, 1.

#3.)

c.) Check the interpolation conditions at each point $X = -1, 0, 1$

$$L^0_n = \frac{X(X-1)}{2} = f(-1) = \frac{-1(-1-1)}{2} = 1$$

$$L^1_n = \frac{(X-1)(X+1)}{-1} = f(-1) = \frac{(-1-1)(-1+1)}{-1} = 0$$

$$L^2_n = \frac{X(X+1)}{2} = f(-1) = \frac{-1(-1+1)}{2} = 0$$

Conditions met

$$L^0_n = f(0) = \frac{0(0-1)}{2} = 0$$

$$L^1_n = f(0) = \frac{(0-1)(0+1)}{-1} = 1$$

$$L^2_n = f(0) = \frac{0(0+1)}{2} = 0$$

Conditions met

$$L^0_n = f(1) = \frac{1(1-1)}{2} = 0$$

$$L^1_n = f(1) = \frac{(1-1)(1+1)}{-1} = 0$$

$$L^2_n = f(1) = \frac{1(1+1)}{2} = 1$$

Conditions met

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4. (30points) We want to interpolate the four data (0, −1), (2, 1), (3, 3), (1, 2). Use Scilab.

(1) (10points) Use Lagrange interpolation.

4.) We want to interpolate the four data (0,-1),(2,1), (3,3), (1,2). Use Scilab.

a.) Using Lagrange interpolation:

$$P_3(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) + y_4 L_4(x)$$

$$P_3(x) \Rightarrow -1 L_1(x) + L_2(x) + 3 L_3(x) + 2 L_4(x)$$

$$L_1(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} = \frac{(x-2)(x-3)(x-1)}{(0-2)(0-3)(0-1)}$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} = \frac{(x-0)(x-3)(x-1)}{(2-0)(2-3)(2-1)}$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} = \frac{(x-0)(x-2)(x-1)}{(3-0)(3-2)(3-1)}$$

$$L_4(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$

$$L_1(x) = -\frac{(x-2)(x-3)(x-1)}{6}$$

$$L_2(x) = -\frac{x(x-3)(x-1)}{2}$$

$$L_3(x) = \frac{x(x-2)(x-1)}{6}$$

$$L_4(x) = \frac{x(x-2)(x-3)}{2}$$

$$\boxed{P_2(x) \Rightarrow \frac{(x-2)(x-3)(x-1)}{6} - \frac{x(x-3)(x-1)}{2} + \frac{x(x-2)(x-1)}{2} + x(x-2)(x-3)}$$

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(2) (10points) Use the Newton's divided differences.

#4.)

6.) Use the Newtons divided differences:

	x_1	x_2	x_3	x_4
x	0	2	3	1
$f(x)$	-1	1	3	2
	y_1	y_2	y_3	y_4

x	$f(x)$	$f[\,,\,]$	$f[\,,,\,]$	$f[\,,,,\,]$
0	-1	$\frac{2}{2}$	$\frac{1}{3}$	$\boxed{\frac{7}{6}}$
2	1	$\frac{2}{1}$	$\frac{3}{2}$	
3	3	$\frac{-1}{-2}$		
1	2			

(3) (5points) Use the linear system. (Use Scilab).

title

c.) $P(x) = ax^3 + bx^2 + cx + d$ → Solve using linear System

$(0,-1) \Rightarrow p(0) = d = -1$

$(2,1) \Rightarrow p(2) = 8a + 4b + 2c + d = 1$

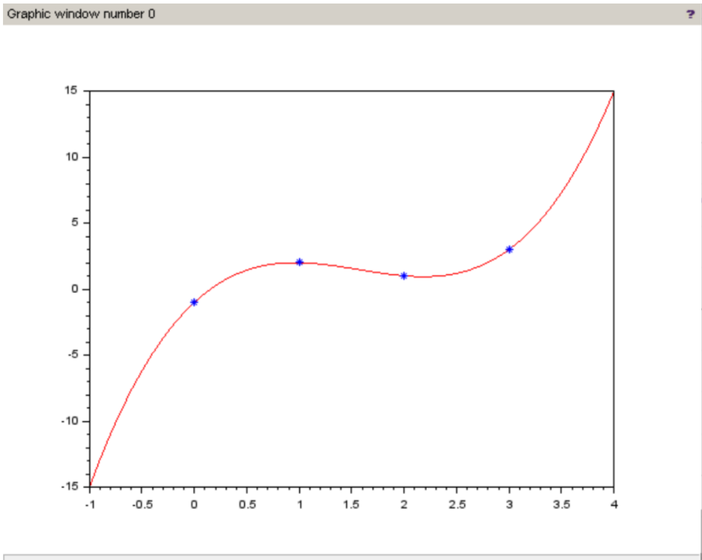
$(3,3) \Rightarrow p(3) = 27a + 9b + 3c + d = 3$

$(1,2) \Rightarrow p(1) = a + b + c + d = 2$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 2 \end{bmatrix} \Rightarrow \text{Scilab}$$

$$\boxed{P_3(x) = 1.166x^3 - \frac{11}{2}x^2 + \frac{22}{3}x - 1}$$

(4) (5points) Graph the cubic interpolant.



BisectionMethod.sce | Question1.sce | NewtonMethod.sce | FPI.sce | SecantMethod.sce | HW4.sce

```
endfunction

function f = Ln1(x)
    f = (x + 1).*(x - 1) / -1;
endfunction

function f = Ln2(x)
    f = (x.*(x + 1)) / 2;
endfunction

function f = Q4(x)
    f = (1.1666666)*x.^3 - (11/2)*x.^2 + (22/3)*x - 1
endfunction

x=-1:0.1:4
y=Q4(x);
plot(x,y, 'r')
plot(0, -1, 'b*')
plot(2, 1, 'b*')
plot(3, 3, 'b*')
plot(1, 2, 'b*')
```