A Notation and Preliminaries

We establish some notation and review some elements of representation theory. For a comprehensive review of representation theory, please see [53, 37]. The identity element of any group G will be denoted as e. A subgroup H of G will be denoted as $H \subseteq G$. We will always work over the field $\mathbb{R}$ unless otherwise specified.

A.0.1 Group Actions

Let Ω be a set. A group action Φ of G on Ω is a map $\Phi : G \times \Omega \to \Omega$ which satisfies

$$\begin{aligned} &\text{Identity: } \forall \omega \in \Omega, \quad \Phi(e, \omega) = \omega \\ &\text{Compositionality: } \forall g_1, g_2 \in G, \ \ \forall \omega \in \Omega, \quad \Phi(g_1 g_2, \omega) = \Phi(g_1, \Phi(g_2, \omega)) \end{aligned} \tag{3}$$

We will often suppress the Φ function and write $\Phi(g, \omega) = g \cdot \omega$.

$$\begin{tikzcd} \Omega \arrow[r, "\Psi"] \arrow[d, "\Phi(g,\cdot)"] & \Omega' \arrow[d, "\Phi'(g,\cdot)"] \\ \Omega \arrow[r, "\Psi"] & \Omega' \end{tikzcd}$$

Figure 8: Commutative Diagram For G-equivariant function: Let $\Phi(g, \cdot) : G \times \Omega \to \Omega$ denote the action of G on Ω. Let $\Phi'(g, \cdot) : G \times \Omega' \to \Omega'$ denote the action of G on Ω' The map $\Psi : \Omega \to \Omega'$ is G-equivariant if and only if the following diagram is commutative for all $g \in G$.

Let G have group action Φ on Ω and group action Φ' on Ω'. A mapping $\Psi : \Omega \to \Omega'$ is said to be G-equivariant if and only if

$$\forall g \in G, \forall \omega \in \Omega, \quad \Psi(\Phi(g, \omega)) = \Phi'(g, \Psi(\omega)) \tag{4}$$

Diagrammatically, Ψ is G-equivariant if and only if the diagram A.0.1 is commutative.

A.0.2 Induced and Restriction Representations

Let V be a vector space over $\mathbb{C}$. A *representation* (ρ, V) of G is a map $\rho : G \to \mathrm{Hom}[V, V]$ such that

$$\forall g, g' \in G, \ \ \forall v \in V \quad \rho(g \cdot g')v = \rho(g) \cdot \rho(g')v$$

Restriction Representation Let $H \subseteq G$. Let (ρ, V) be a representation of G. The restriction representation of (ρ, V) from G to H is denoted as $\mathrm{Res}^G_H[(\rho, V)]$. Intuitively, $\mathrm{Res}^G_H[(\rho, V)]$ can be viewed as (ρ, V) evaluated on the subgroup H. Specifically,

$$\forall v \in V, \quad \mathrm{Res}^G_H[\rho](h)v = \rho(h)v \tag{5}$$

Note that the restricted representation and the original representation both live on the same vector space V.

Induction Representation The induction representation is a way to construct representations of a larger group G out of representations of a subgroup $H \subseteq G$. Let (ρ, V) be a representation of H. The induced representation of (ρ, V) from H to G is denoted as $\mathrm{Ind}^G_H[(\rho, V)]$. Define the space of functions

$$\mathcal{F} = \{ \ f \ | \ f : G \to V, \ \ \forall h \in H, \ \ f(gh) = \rho(h^{-1})f(g)\}$$

Then the induced representation is defined as $(\pi, \mathcal{F}) = \mathrm{Ind}^G_H[(\rho, V)]$ where the induced action π acts on the function space $\mathcal{F}$ via

$$\forall g, g' \in G, \ \ \forall f \in \mathcal{F} \quad (\pi(g) \cdot f)(g') = f(g^{-1}g')$$

Induced Representation for Finite Groups There is also an equivalent definition of the induced representation for finite groups that is slightly more intuitive [54]. Let G be a group and let $H \subset G$. The set of left cosets of G/H form a partition of G so that

$$G = \bigcup_{i=1}^{|G/H|} g_i H$$

where $\{g_i\}_{i=1}^{|G/H|}$ are a set of representatives of each unique left coset. Now, left multiplication by the element $g \in G$ is an isomorphism of G. Left multiplication by g must thus permute left cosets of G/H so that

$$\forall g \in G, \quad g \cdot g_i = g_{j_g(i)} h_i(g)$$

where $j_g : \{1, 2, ..., m\} \to \{1, 2, ..., m\} \in S_m$ is a permutation of left coset representatives. The $h_i(g) \in H$ is an element of subgroup H. The map $j_g(i)$ and group element $h_i(g) \in H$ satisfy a compositionality property. Specifically, we have that

$$\forall g, g' \in G, \quad j_{g'} \circ j_g = j_{g'g}, \quad h_i(g'g) = h_{j_g(i)}(g') \cdot h_i(g)$$

which can be seen by acting on the left cosets with g followed by g' versus acting on the left cosets with $g'g$. Note that

$$e \cdot g_i = g_i \cdot e = g_{j_e(i)} h_i(e)$$

holds so $j_e = e$ and $h_i(e) = e$ holds. Now, let (ρ, V) be a representation of the group H. Let us define the vector space W as

$$W = \bigoplus_{i=1}^{|G/H|} g_i V_{(i)}$$

where the (somewhat confusing) notation $g_i V_{(i)}$ denotes an independent copy of the vector space V. This notation is simply a labeling and all copies of $g_i V_{(i)}^H$ are isomorphic to V^H,

$$V \cong g_1 V_1 \cong g_2 V_2 \cong ... \cong g_{|G/H|} V_{|G/H|}$$

so that the space $W \cong \bigoplus_{i=1}^{|G/H|} V$ is just $|G/H|$ independent copies of V. The induced representation lives on this vector space, $(\pi, W) = \mathrm{Ind}_H^G[(\rho, V)]$. The induced action $\pi = \mathrm{Ind}_H^G \rho$ acts on the vector space W via

$$\forall g \in G, \ \forall w = \sum_{i=1}^{|G/H|} g_i v_i \in W, \quad \pi(g) \cdot w = \sum_{i=1}^{|G/H|} \sigma(h_i(g)) v_{j_g(i)} \in W$$

where $v_i \in V_{(i)}$ is in the i-th independent copy of the vector space V. Using the compositionality property of j_g and $h_i(g)$, it is easy to see that this is a valid group action so that $(\pi, W) = \mathrm{Ind}_H^G[(\rho, V)]$ is a valid representation. Note that the induced action π acts on the vector space W by permuting and left action by the H-representation $\rho(h)$.

A.0.3 G-Intertwiners

Let (ρ, V) and (σ, W) be two G-representations. The set of all G-equivariant linear maps between (ρ, V) and (σ, W) will be denoted as

$$\mathrm{Hom}_G[(\rho, V), (\sigma, W)] = \{\Phi \mid \Phi : V \to W, \ \forall g \in G, \ \Phi(\rho(g)v) = \sigma(g)\Phi(v)\}$$

Hom_G is a vector space over $\mathbb{C}$. A linear map $\Phi \in \mathrm{Hom}_G[(\rho, V), (\sigma, W)]$ is said to *intertwine* the representations (ρ, V) and (σ, W). An intertwiner Φ is a map that makes the A.0.3 diagram commutative.

$$\begin{tikzcd}
(\rho, V) \arrow[r, "\Phi"] \arrow[d, "\rho(g)"] & (\sigma, W) \arrow[d, "\sigma(g)"] \\
(\rho, V) \arrow[r, "\Phi"] & (\sigma, W)
\end{tikzcd}$$

Figure 9: Commutative Diagram For G-intertwiner. The map $\Psi :\in \mathrm{Hom}_G[(\rho, V), (\sigma, W)]$ if and only if the following diagram is commutative for all $g \in G$.

Computing a basis for the vector space $\mathrm{Hom}_G[(\rho, V), (\sigma, W)]$ is one of the triumphs of classical group theory [37].

A.0.4 $(H \subseteq G)$-Intertwiners

We will also consider another definition of intertwiners between different groups. Let $H \subseteq G$. Let (ρ, V) be a H-representation. Let (σ, W) be a G-representation. We define the vector space of intertwiners of (ρ, V) and (σ, W) as

$$\mathrm{Hom}_H[(\rho, V), \mathrm{Res}_H^G[(\sigma, W)]] = \{\Phi \mid \Phi : V \to W, \ \forall h \in H, \ \Phi(\rho(h)v) = \sigma(h)\Phi(v)\}$$

We say that a linear map $\Phi : V \to W$ is an $(H \subseteq G)$-intertwiner of the H-representation (ρ, V) and the G-representation (σ, W) if $\Phi \in \mathrm{Hom}_H[(\rho, V), \mathrm{Res}_H^G[(\sigma, W)]]$. The induced and restriction representations are adjoint functors [38]. By the Frobinous reciprocity theorem [38],

$$\mathrm{Hom}_H[(\rho, V), \mathrm{Res}_H^G[(\sigma, W)]] \cong \mathrm{Hom}_G[\mathrm{Ind}_H^G[(\rho, V)], (\sigma, W)]$$

and so for every $\Phi : V \to W$ which intertwines (ρ, V) and $\mathrm{Res}_H^G[(\sigma, W)]$ over H there is a unique $\Phi^{\uparrow} : \mathrm{Ind}_H^G[V] \to W$ that intertwines $\mathrm{Ind}_H^G[(\rho, V)]$ and (σ, W) over G. Not every H-representation can be realized as the restriction of a G-representation. Thus, the universe of $(H \subseteq G)$-intertwiners is a proper subset of the universe of H-intertwiners. $(SO(2) \subseteq SO(3))$-intertwiners arise naturally when trying to design $SO(3)$-equivarient neural networks for image data.

$$\begin{tikzcd}
(\rho, V) \arrow[r, "\Phi"] \arrow[d, "\rho(h)"'] & (\sigma, W) \arrow[d, "\sigma(h)"', "\sigma(g)"] \\
(\rho, V) \arrow[r, "\Phi"] & (\sigma, W)
\end{tikzcd}$$

Figure 10: Commutative Diagram For $(H \subseteq G)$-intertwiner. $\Phi : V \to W$. The map $\Phi \in \mathrm{Hom}_G[(\rho, V), \mathrm{Res}_H^G[(\sigma, W)]]$ if and only if the following diagram is commutative for all $h \in H$. Note that $\sigma(g)$ also has G action on the vector space W.

A map $\Phi : V \to W$ is a $(H \subseteq G)$-intertwiner if and only if the diagram in A.0.4 is commutative.

B Additional Experiments

ModelNet10-SO(3) Results The first dataset, ModelNet10-SO(3) [33], is composed of rendered images of synthetic, untextured objects from ModelNet10 [55]. The dataset includes 4,899 object instances over 10 categories, with novel camera viewpoints in the test set. Each image is labelled with a single 3D rotation matrix, even though some categories, such as desks and bathtubs, can have an ambiguous pose due to symmetry. For this reason, the dataset presents a challenge to methods that cannot reason about uncertainty over orientation.

ModelNet10-SO(3) Results

Table 4: Rotation prediction on ModelNetSO(3). First column is the average over all categories.

	Median rotation error in degrees (↓)										
	avg	*bathtub*	*bed*	*chair*	*desk*	*dresser*	*monitor*	*stand*	*sofa*	*table*	*toilet*
Mohlin et al. [48]	17.1	89.1	4.4	5.2	13.0	6.3	5.8	13.5	4.0	25.8	4.0
Prokudin et al. [35]	49.3	122.8	3.6	9.6	117.2	29.9	6.7	73.0	10.4	115.5	4.1
Deng et al. [34]	32.6	147.8	9.2	8.3	25.0	11.9	9.8	36.9	10.0	58.6	8.5
Liao et al. [33]	36.5	113.3	13.3	13.7	39.2	26.9	16.4	44.2	12.0	74.8	10.9
Murphy et al. [14]	21.5	161.0	4.4	5.5	7.1	5.5	5.7	7.5	4.1	9.0	4.8
Klee et al. [12]	**16.3**	124.7	**3.1**	**4.4**	**4.7**	**3.4**	**4.4**	**4.1**	**3.0**	**7.7**	**3.6**
Ours	17.8	123.7	4.6	5.5	6.9	5.2	6.1	6.5	4.5	12.1	4.9

The performance on the ModelNet dataset is reported in Table 4. Our induction layer outputs signals on S^2, and naturally allows for capturing uncertainty as a distribution over $\mathrm{SO}(3)$. Both our method and [12] use equivariant layers to improve generalization but our method slightly under-performs [12] on the ModelNet dataset. ModelNet-10 is a synthetic dataset consisting of totally opaque objects and it seems that the image formation model used in [12] is a good approximation to the true image formation model.

C Image to $\mathbb{R}^3 \times S^2$ for 6DOF-Pose Estimation

The goal in 6DOF-pose estimation is to estimate the location of an object in three-dimensional space and the orientation of said object. Orientation estimation is a sub-problem of pose estimation where the goal is to estimate just the orientation of an object and disregard the objects position in three-dimensional space.

Let us see how induced and restriction representations arise naturally in the design of neural architectures for 6DOF-pose estimation. Let V and $V^\uparrow$ be vector spaces. Let $\mathcal{F}$ be the vector space of all V-valued signals defined on the plane

$$\mathcal{F} = \{f| \ f : \mathbb{R}^2 \to V\}$$

The group $SE(2) = \mathbb{R}^2 \rtimes SO(2)$ acts on the vector space $\mathcal{F}$ via in some representation π,

$$\forall f \in \mathcal{F}, \ \forall h = \bar{h}h_c \in SE(2), \quad \pi(h) \cdot f(r) = \rho(h_c) f(h^{-1}r)$$

where (ρ, V) is an $SO(2)$-representation describing the transformation law of the fibers of f and $(\pi, \mathcal{F}) = \mathrm{Ind}_{SO(2)}^{SE(2)}[(\rho, V)]$ so that $(\pi, \mathcal{F})$ furnishes a representation of the group $SE(2)$. The space $\mathcal{F}$ will serve as the space on which input signals are defined on. In pose reconstruction tasks, the output of our neural network will be functions from $\mathbb{R}^3 \times S^2$ into the vector space $V^\uparrow$. Let $\mathcal{F}^\uparrow$ be the vector space of all such outputs defined as

$$\mathcal{F}^\uparrow = \{f| \ f : \mathbb{R}^3 \times S^2 \to V^\uparrow\}$$

The group $SE(3) = \mathbb{R}^3 \rtimes SO(3)$ acts on the vector space $\mathcal{F}^\uparrow$ via

$$\forall f^\uparrow \in \mathcal{F}^\uparrow, \ \forall (p, \hat{n}) \in \mathbb{R}^3 \times S^2, \quad \forall g = \bar{g}g_c \in SE(3), \quad \pi^\uparrow(g) \cdot f^\uparrow(p, \hat{n}) = \rho^\uparrow(g_c) f^\uparrow(g^{-1}p, g_c^{-1}\hat{n})$$

where $\rho^\uparrow(g_c)$ is a representation of $SO(3)$.

Analogous to the argument presented in the main text. We would like to characterize all maps from $\mathcal{F}$ to $\mathbf{F}^\uparrow$ that preserve $SE(2)$-equivarience. Consider the space of linear maps $\Phi : \mathcal{F} \to \mathcal{F}^\uparrow$ that intertwine $(\pi, \mathcal{F})$ and $(\pi^\uparrow, \mathcal{F}^\uparrow)$. The map $\Phi : \mathcal{F} \to \mathcal{F}^\uparrow$ must satisfy the relation

$$\forall h \in SE(2), \ \forall f \in \mathcal{F}, \quad \Phi(\pi(h) \cdot f) = \mathrm{Res}_{SE(2)}^{SE(3)}[\pi^\uparrow](h) \cdot \Phi(f)$$

where $\mathrm{Res}_{SE(2)}^{SE(3)}[\pi^\uparrow]$ is the restriction of the $SE(3)$-representation $(\pi^\uparrow, \mathcal{F}^\uparrow)$ to a $SE(2)$ subgroup.

C.0.1 Deriving the Kernel Constraint

The most general linear map $\Phi : \mathcal{F} \to \mathcal{F}^\uparrow$ between $(\pi, \mathcal{F})$ and $(\pi^\uparrow, \mathcal{F}^\uparrow)$ can be written as

$$\forall (p, \hat{n}) \in \mathbb{R}^3 \times S^2, \quad [\Phi(f)](p, \hat{n}) = \int_{r \in \mathbb{R}^2} dr \ \kappa(p, \hat{n} : r) f(r)$$

where $\kappa : (\mathbb{R}^3 \times S^2) \times \mathbb{R}^2 \to \mathrm{Hom}[V, V^\uparrow]$. Let us enforce the $(H \subseteq G)$-equivarience condition

$$\forall h \in SE(2), \quad \pi^\uparrow(h) \cdot \Phi(f) = \Phi(\pi(h) \cdot f)$$

This constraint places a restriction on the allowed space of kernels. We have that

$$\forall h \in SE(2), \quad \Phi[\pi(h) \cdot f] = \int_{r \in \mathbb{R}^2} dr \ \kappa(p, r)[\pi(h) \cdot f(r)] = \int_{r \in \mathbb{R}^2} dr \ \kappa(p, \hat{n} : r) \rho(h_c) f(h^{-1}r)$$

Now, making the change of variables $r \to hr$ gives

$$\forall h \in SE(2), \quad \Phi[\pi(h) \cdot f] = \int_{r \in \mathbb{R}^2} dr \ \kappa(p, \hat{n} : h \cdot r) \rho(h_c) f(r)$$

Now, by assumption $\Phi(f) \in (\pi^\uparrow, \mathcal{F}^\uparrow)$ so

$$\forall h \in SE(2), \quad \pi^\uparrow(h) \cdot \Phi(f) = \int_{r \in \mathbb{R}^2} dr \ \rho^\uparrow(h_c) \kappa(h^{-1}p, h^{-1}\hat{n} : r) f(r)$$

Thus, the kernel κ satisfies the constraint

$$\forall h \in SE(2), \quad \rho^{\uparrow}(h_c)\kappa(h^{-1} \cdot p, h^{-1}\hat{n} : r) = \kappa(p, \hat{n} : h \cdot r)\rho(h_c)$$

We can write this in the more compact form as

$$\forall h \in SO(2), \quad \kappa(h \cdot p, h \cdot \hat{n} : h \cdot r) = \rho^{\uparrow}(h_c)\kappa(p, \hat{n} : r)\rho(h_c^{-1})$$

This constraint is linear and solutions κ form a vector space over $\mathbb{R}$. We reduce this constraint to the steerable kernel constraint considered in [7, 21, 9, 8].

First, note that the $SO(2)$ action does not mix the z-component of $[\Phi(f)](\hat{n}, x, y, z)$. Thus, the most general linear map can be written as

$$[\Phi(f)](\hat{n}, x, y, z) = \int_{(r_x, r_y) \in \mathbb{R}^2} dr_x dr_y \; \kappa(\hat{n}, x - r_x, y - r_y, z) f(r_x, r_y)$$

where for each fixed z, the kernel κ is an intertwiner of $\mathrm{Res}^{SO(3)}_{SO(2)}[(\rho^{\uparrow}, V^{\uparrow})]$ and (ρ, V) and satisfies

$$\forall h \in SO(2), \quad \kappa(h \cdot \hat{n}, h \cdot r : z) = \rho^{\uparrow}(h)\kappa(\hat{n}, r : z)\rho(h^{-1})$$

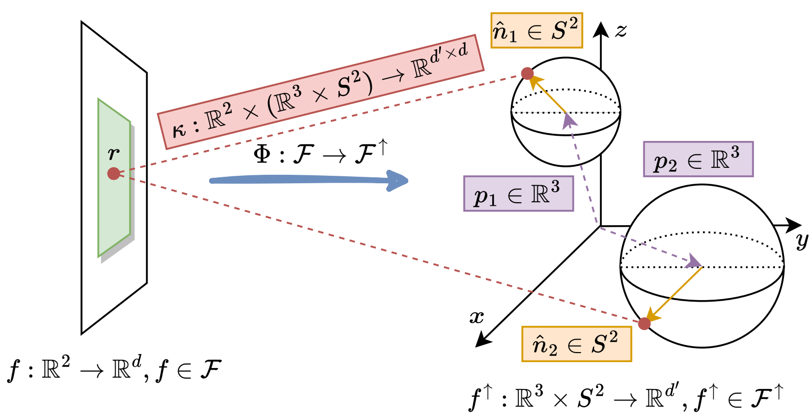

Figure 11: Right: Diagram of an Equivariant Image to Sphere Convolution. At each point $p = (x, y, z) \in \mathbb{R}^3$ and each unit vector $\hat{n} \in S^2$ the kernel $\kappa(\hat{n}, p : p')$ is dependent on the image point $p' = (x', y') \in \mathbb{R}^2$. Equivarience constraints put restrictions on the allowed form of $\kappa(\hat{n}, p : p')$ C.0.1. Similar to a standard convolution, the kernel κ has a user defined receptive field.

Let simplify this constraint further. The set of spherical harmonics form an orthonormal basis for functions on S^2. We can expand the kernel κ as

$$\kappa(\hat{n}, r : z) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} F_{\ell}^{k}(r, z) Y_{\ell}^{k}(\hat{n})$$

where $F_{\ell}^{k}(r, z) : \mathbb{R}^2 \times \mathbb{R} \to \mathrm{Hom}[V, V^{\uparrow}]$. The kernel constraint places additional restrictions on the set of allowed $F_{\ell}^{k}(r, z)$. We have that,

$$\forall h \in SO(2), \quad \kappa(h \cdot \hat{n}, h \cdot r : z) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} F_{\ell}^{k}(h \cdot r, z) Y_{\ell}^{k}(h \cdot \hat{n}) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} F_{\ell}^{k}(h \cdot r, z) D_{kk'}^{\ell}(h) Y_{\ell}^{k'}(\hat{n})$$

and,

$$\forall h \in SO(2), \quad \rho^{\uparrow}(h)\kappa(\hat{n}, z : r)\rho(h^{-1}) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} \rho^{\uparrow}(h) F_{\ell}^{k}(r, z)\rho(h^{-1}) Y_{\ell}^{k}(\hat{n})$$

Thus, the functions $F_\ell^k(r,z): \mathbb{R}^2 \times \mathbb{R} \to \mathrm{Hom}[V, V^\uparrow]$ must satisfy,

$$\forall h \in SO(2), \quad \rho^\uparrow(h) F_\ell^k(r,z)\rho(h^{-1}) = \sum_{k'=-\ell}^{\ell} F_\ell^{k'}(h\cdot r, z) D^\ell_{k'k}(h)$$

Now, the Wigner D-matrices are unitary and the above constraint is equivalent to

$$\forall h \in SO(2), \quad F_\ell^k(h\cdot r, z) = \rho^\uparrow(h)\sum_{k'=-\ell}^{+\ell} F_\ell^{k'}(r,z)\rho(h^{-1})D^\ell_{k'k}(h^{-1}) = \rho^\uparrow(h)\sum_{k'=-\ell}^{+\ell} F_\ell^{k'}(r,z)[D^\ell_{k'k}(h)\rho(h)]^{-1}$$

Now, let us vectorize the matrix valued functions $F_\ell^k(r,z)$ as

$$F_\ell(r,z) = \begin{bmatrix} F_\ell^\ell(r,z), & F_\ell^{\ell-1}(r,z), & \ldots & F_\ell^{-\ell+1}(r,z), & F_\ell^{-\ell}(r,z)\end{bmatrix} \in \mathrm{Hom}[V\otimes W^\ell, V^\uparrow]$$

Let us define the tensor product representation of (ρ, V) and $\mathrm{Res}^{SO(3)}_{SO(2)}[(D^\ell, W^\ell)]$ as

$$(\rho^\ell, V^\ell) = (\rho, V) \otimes \mathrm{Res}^{SO(3)}_{SO(2)}[(D^\ell, W^\ell)]$$

which is a $SO(2)$-representation. Then the functions $F_\ell(r): \mathbb{R}^2 \to \mathrm{Hom}[V\otimes W^\ell, V^\uparrow]$ satisfy the constraint

$$\forall h \in SO(2), \quad F_\ell(h\cdot r, z) = \rho^\uparrow(h) F_\ell(r,z)\rho^\ell(h^{-1})$$

For fixed z, this is exactly the constraint on an $SO(2)$-steerable kernel with input representation $(\rho^\ell, V^\ell) = (\rho, V)\otimes \mathrm{Res}^{SO(3)}_{SO(2)}[(D^\ell, W^\ell)]$ and output representation $\mathrm{Res}^{SO(3)}_{SO(2)}[\rho^\uparrow, V^\uparrow)]$. [20, 8] give a complete classification of kernel spaces that satisfy this constraint. Note that by demanding that $SE(3)$ has action on the space $(\pi^\uparrow, \mathcal{F}^\uparrow)$ we have added additional constraints to the set of allowed kernels. Specifically, instead of mapping arbitrary $SO(2)$-input representation to arbitrary $SO(2)$-output representation, the allowed input and output representations must satisfy additional constraints. Specifically, not every representation can be realized as the restriction of an $SE(3)$ to $SE(2)$ representation. The Induction/Restriction mappings of $SO(2)\subset SO(3)$ are shown in 2.

In practice, once the multiplicities of the input $SO(2)$-representation and the output $SO(3)$-representation are specified, the $SO(2)$-steerable kernels can be explicitly constructed using numerical programs defined in [20]. To summarize, all equivariant linear maps between a function $f:\mathbb{R}^2\to V$ and a function $f^\uparrow: \mathbb{R}^3\times S^2 \to V^\uparrow$ can be written as

$$f^\uparrow(\hat{n}, x, y, z) = \sum_{\ell=0}^{\infty}(F_{\ell,z}\star f)(x,y)\cdot Y_\ell(\hat{n}) = \sum_{\ell=0}^{\infty}\int_{(x',y')\in\mathbb{R}^2} dA\ f(x',y')F_{\ell,z}(x-x', y-y')\cdot Y_\ell(\hat{n})$$

where for each fixed z, $F_{\ell,z}(x,y)$ is a $SO(2)$-steerable kernel that takes input representation $(\rho^\ell, V^\ell) = (\rho, V)\otimes \mathrm{Res}^{SO(3)}_{SO(2)}[(D^\ell, W^\ell)]$ to output representation $\mathrm{Res}^{SO(3)}_{SO(2)}[(\rho^\uparrow, V^\uparrow)]$. Once the coefficients of the spherical harmonics

$$C_\ell(x,y,z) = (F_{\ell,z}\star f)(x,y) = \int_{(x',y')\in\mathbb{R}^2} dA\ f(x',y')F_{\ell,z}(x-x', y-y')$$

are computed, the resultant function $f^\uparrow(\hat{n}, x, y, z) = \sum_{\ell=0}^{\infty} C_\ell^T(x,y,z)Y^\ell(\hat{n})$ is defined on a homogeneous space of $SE(3)$ and we can utilize $SE(3)$-steerable CNNs to make predictions about 6DoF poses [21, 56, 57].

D Plane to Space for Object Reconstruction

Another problem of interest in single view geometric construction is monocular density reconstruction (also sometimes called monocular depth estimation). The goal in monocular density reconstruction problems is to build a three-dimensional model of the world given a single two-dimensional images [58, 59]. Monocular depth reconstruction tasks are of specific interest in endoscopy [60] and autonomous driving [61, 62].

In monocular reconstruction tasks, the output of our neural network will be a density map which is a function from $\mathbb{R}^3$ into a vector space $V^\uparrow$. Let $\mathcal{F}^\uparrow$ be the vector space of all such outputs

$$\mathcal{F}^\uparrow = \{f| \ f : \mathbb{R}^3 \to V^\uparrow\}$$

The group $\mathbb{R}^3 \rtimes SO(3)$ acts on the vector space $\mathcal{F}^\uparrow$ via

$$\forall f^\uparrow \in \mathcal{F}^\uparrow, \ \ \forall g \in SE(3), \quad \pi^\uparrow(g) \cdot f^\uparrow(r) = \rho^\uparrow(g_c) f^\uparrow(g^{-1} r)$$

where $\rho^\uparrow(g_c)$ is a representation of $SO(3)$. Now, consider the space of linear maps $\Phi : \mathcal{F} \to \mathcal{F}^\uparrow$ that intertwine $(\pi, \mathcal{F})$ and $(\pi^\uparrow, \mathcal{F}^\uparrow)$. The map $\Phi : \mathcal{F} \to \mathcal{F}^\uparrow$ must satisfy the relation

$$\forall h \in SE(2), \ \ \forall f \in \mathcal{F}, \quad \Phi(\pi(h) f) = \pi^\uparrow(h)\Phi(f)$$

by definition of the restricted representation this is equivalent to

$$\forall h \in SE(2), \ \ \forall f \in \mathcal{F}, \quad \Phi(\pi(h) f) = \mathrm{Res}^G_H[\pi^\uparrow](h)\Phi(f)$$

where $\mathrm{Res}^{SO(3)}_{SO(2)}[(\pi^\uparrow, \mathcal{F}^\uparrow)]$ is the restriction of the $SE(3)$-representation $(\pi^\uparrow, \mathcal{F}^\uparrow)$ to a $SE(2)$ subgroup. Similar to C, the most general linear map between $(\pi, \mathcal{F})$ and $(\pi^\uparrow, \mathcal{F}^\uparrow)$ can be written as

$$\forall p \in \mathbb{R}^3, \quad (k \cdot f)(p) = \int_{r \in \mathbb{R}^2} dr \ \kappa(p, r) f(r)$$

where $\kappa : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathrm{Hom}[V, V^\uparrow]$ satisfies the constraint

$$\forall h \in SE(2), \quad \rho^\uparrow(h_c)\kappa(h^{-1} \cdot p, r) = \kappa(p, h \cdot r)\rho(h_c)$$

We can write this in the more compact form

$$\forall h \in SO(2), \quad \kappa(h \cdot p, h \cdot r) = \rho^\uparrow(h_c)\kappa(p, r)\rho(h_c)$$

Note that the $SO(2)$ action does not mix the z-component of $[\Phi(f)](x, y, z)$. Thus, the most general linear map can be written as

$$[\Phi(f)](x, y, z) = \int_{r \in \mathbb{R}^2} dr_x dr_y \ \kappa(x - r_x, y - r_y, z) f(r_x, r_y) = (\kappa_z \star f)(x, y)$$

where for each fixed z, the kernel κ is an intertwiner of $\mathrm{Res}^{SO(3)}_{SO(2)}[(\rho^\uparrow, V^\uparrow)]$ and (ρ, V) and satisfies

$$\forall h \in SO(2), \quad \kappa(g \cdot r, z) = \rho^\uparrow(h)\kappa(r, z)\rho(h^{-1})$$

To summarize, a function $f : \mathbb{R}^2 \to V$ can be mapped into a function

$$f^\uparrow(x, y, z) = \Phi(f)(x, y, z) = \int_{r \in \mathbb{R}^2} dr \ k(x - x', y - y', z) f(x', y') = [\kappa_z \star f](x, y)$$

where for fixed z, κ_z is an $SO(2)$-steerable kernel with input representation (ρ, V) and output representation $\mathrm{Res}^{SO(3)}_{SO(2)}[(\rho^\uparrow, V^\uparrow)]$.

E Solving the Kernel Constraint

Let us solve the kernel constraint derived in the previous section. The most general linear map $\Phi : \mathcal{F} \to \mathcal{F}^\uparrow$ between $(\pi, \mathcal{F})$ and $(\pi^\uparrow, \mathcal{F}^\uparrow)$ can be written as

$$\forall \hat{n} \in S^2, \quad [\Phi(f)](\hat{n}) = \int_{r \in \mathbb{R}^2} dr \ \kappa(\hat{n}, r) f(r)$$

where $\kappa : S^2 \times \mathbb{R}^2 \to \mathrm{Hom}[V, V^\uparrow]$. Let us enforce the $SO(2)$-equivarience condition derived in 1. We have that,

$$\forall h \in SE(2), \quad \pi^\uparrow(h_c) \cdot \Phi(f) = \Phi(\pi(h) \cdot f)$$

This constraint places a restriction on the allowed space of kernels. We have that, $\forall h = \bar{h}h_c \in SE(2)$,

$$\Phi[\pi(h) \cdot f] = \int_{r\in\mathbb{R}^2} dr\ \kappa(p,r)[\pi(h) \cdot f(r)] = \int_{r\in\mathbb{R}^2} dr\ \kappa(p,\hat{n} : r)\rho(h_c)f(h^{-1}r)$$

Now, making the change of variables $r \to hr$ gives

$$\forall h \in SE(2), \quad \Phi[\pi(h) \cdot f] = \int_{r\in\mathbb{R}^2} dr\ \kappa(p,\hat{n} : h \cdot r)\rho(h_c)f(r)$$

Now, by assumption $\Phi(f) \in (\pi^{\uparrow}, \mathcal{F}^{\uparrow})$ so

$$\forall h_c \in SO(2), \quad \pi^{\uparrow}(h_c) \cdot \Phi(f) = \int_{r\in\mathbb{R}^2} dr\ \rho^{\uparrow}(h_c)\kappa(h_c^{-1}\hat{n} : r)f(r)$$

Thus, the kernel κ satisfies the linear constraint

$$\forall h \in SE(2), \quad \rho^{\uparrow}(h_c)\kappa(h_c^{-1}\hat{n} : r) = \kappa(p,\hat{n} : h \cdot r)\rho(h_c)$$

Fiber representations are unitary and left multiplying, we can the kernel constraint in the more compact form

$$\forall h \in SO(2), \quad \kappa(h_c \cdot \hat{n} : h \cdot r) = \rho^{\uparrow}(h_c)\kappa(\hat{n} : r)\rho(h_c^{-1})$$

We can further reduce this to a standard steerable kernel constraint studied in [7, 21, 9]. The set of spherical harmonics Y_ℓ^k form an orthonormal basis for functions on S^2. We can expand the kernel κ as

$$\kappa(\hat{n}, r) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} F_\ell^k(r)Y_\ell^k(\hat{n})$$

where $F_\ell^k(r) : \mathbb{R}^2 \to \mathrm{Hom}[V, V^{\uparrow}]$. The kernel constraint places additional restrictions on the set of allowed $F_\ell^k(r)$. We have that,

$$\forall h = \bar{h}h_c \in SO(2), \quad \kappa(h_c \cdot \hat{n}, h \cdot r) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} F_\ell^k(h \cdot r)Y_\ell^k(h_c \cdot \hat{n}) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} F_\ell^k(h \cdot r)D_{kk'}^\ell(h_c)Y_\ell^{k'}(\hat{n})$$

and,

$$\forall h = \bar{h}h_c \in SO(2), \quad \rho^{\uparrow}(h)\kappa(\hat{n} : r)\rho(h^{-1}) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} \rho^{\uparrow}(h)F_\ell^k(r,z)\rho(h^{-1})Y_\ell^k(\hat{n})$$

Thus, the functions $F_\ell^k(r) : \mathbb{R}^2 \to \mathrm{Hom}[V, V^{\uparrow}]$ must satisfy,

$$\forall h \in SO(2), \quad \rho^{\uparrow}(h)F_\ell^k(r)\rho(h^{-1}) = \sum_{k'=-\ell}^{\ell} F_\ell^{k'}(h \cdot r)D_{k'k}^\ell(h)$$

Now, the Wigner D-matrices are unitary and the above constraint is equivalent to

$$\forall h \in SO(2), \quad F_\ell^k(h \cdot r) = \rho^{\uparrow}(h)\sum_{k'=-\ell}^{+\ell} F_\ell^{k'}(r)\rho(h^{-1})D_{k'k}^\ell(h^{-1}) = \rho^{\uparrow}(h)\sum_{k'=-\ell}^{+\ell} F_\ell^{k'}(r)[D_{k'k}^\ell(h)\rho(h)]^{-1}$$

Now, let us vectorize the matrix valued functions $F_\ell^k(r)$ as

$$F_\ell(r) = \left[F_\ell^\ell(r), \quad F_\ell^{\ell-1}(r), \quad \ldots \quad F_\ell^{-\ell+1}(r), \quad F_\ell^{-\ell}(r)\right] \in \mathrm{Hom}[V \otimes W^\ell, V^{\uparrow}]$$

We define the tensor product representation of (ρ, V) and $\mathrm{Res}_{SO(2)}^{SO(3)}[(D^\ell, W^\ell)]$ as

$$(\rho^\ell, V^\ell) = (\rho, V) \otimes \mathrm{Res}_{SO(2)}^{SO(3)}[(D^\ell, W^\ell)]$$

which is a $SO(2)$-representation. Then the functions $F_\ell(r) : \mathbb{R}^2 \to \mathrm{Hom}[V \otimes W^\ell, V^{\uparrow}]$ satisfy the constraint

$$\forall h \in SO(2), \quad F_\ell(h \cdot r) = \rho^{\uparrow}(h)F_\ell(r)\rho^\ell(h^{-1})$$

This is exactly the constraint on an $SO(2)$-steerable kernel with input representation $(\rho^\ell, V^\ell) = (\rho, V) \otimes \mathrm{Res}_{SO(2)}^{SO(3)}[(D^\ell, W^\ell)]$ and output representation $\mathrm{Res}_{SO(2)}^{SO(3)}[(\rho^{\uparrow}, V^{\uparrow})]$. [20, 8] give a complete classification of kernel spaces that satisfy this constraint. Note that by enforcing that the output transforms in an $SO(3)$-representation, we have added additional constraints to the set of allowed kernels.

F Image to $SO(3)$ for Rotation Estimation

Instead of inducing from signals on the plane to signals on the S^2 as in 4, we can induce directly from image to $SO(3)$. Let $\mathcal{F}^\uparrow$ be the vector space of all such outputs

$$\mathcal{F}^\uparrow = \{f| \ f : SO(3) \to V^\uparrow\}$$

The group $SO(3)$ acts on the vector space $\mathcal{F}^\uparrow$ via

$$\forall f^\uparrow \in \mathcal{F}^\uparrow, \ \forall g, g' \in SO(3), \quad \pi^\uparrow(g) \cdot f^\uparrow(g') = \rho^\uparrow(g) f^\uparrow(g^{-1}g')$$

where $\rho^\uparrow(g)$ is a representation of $SO(3)$. Now, consider the space of linear maps $\Phi : \mathcal{F} \to \mathcal{F}^\uparrow$ that intertwine $(\pi, \mathcal{F})$ and $(\pi^\uparrow, \mathcal{F}^\uparrow)$. The map $\Phi : \mathcal{F} \to \mathcal{F}^\uparrow$ must satisfy the relation

$$\forall h \in SO(2), \ \forall f \in \mathcal{F}, \quad \Phi(\pi(h)f) = \mathrm{Res}^{SO(3)}_{SO(2)}[\pi^\uparrow](h)\Phi(f) = \pi^\uparrow(h)\Phi(f)$$

where $\mathrm{Res}^{SO(3)}_{SO(2)}[\pi^\uparrow]$ is the restriction of the $SO(3)$-representation $(\pi^\uparrow, \mathcal{F}^\uparrow)$ to a $SO(2)$ subgroup. Using an argument similar to C, the most general linear equivariant map from functions on $\mathbb{R}^2$ to functions on the $SO(3)$ is

$$\forall g \in SO(3), \quad [\Phi(f)](g) = \int_{(x,y)\in\mathbb{R}^2} dA \, \kappa(g, x, y) f(x, y)$$

where the map $\kappa : SO(3) \times \mathbb{R}^2 \to \mathrm{Hom}[V, V^\uparrow]$. The kernel κ satisfies

$$\forall h \in SO(2), \quad \kappa(h^{-1}g, h^{-1}r) = \rho^\uparrow(h)\kappa(g, r)\rho(h^{-1})$$

The set of Wigner D-matrices form an orthonormal basis for functions on $SO(3)$ and we can uniquely expand κ as

$$\kappa(g, x, y) = \sum_{\ell=0}^{+\infty} \sum_{k,k'=-\ell}^{\ell} F_\ell^{kk'}(x, y) D^\ell_{kk'}(g)$$

where $F_\ell^{kk'}(x, y) : \mathbb{R}^2 \to \mathrm{Hom}[V, V^\uparrow]$ are matrix valued coefficients. The kernel constraint places restrictions on the allowed form of $F_\ell^{kk'}(x, y)$. Let us define the $SO(2)$-representations

$$(\rho_\ell, V_\ell) = (\rho, V) \otimes \mathrm{Res}^{SO(3)}_{SO(2)}[(D^\ell, W^\ell)], \quad (\rho_\ell^\uparrow, V_\ell^\uparrow) = \mathrm{Res}^{SO(3)}_{SO(2)}[(\rho^\uparrow, V^\uparrow) \otimes (D^\ell, W^\ell)]$$

Then, the kernel constraint holds only if

$$\forall h \in SO(2), \ \forall r \in \mathbb{R}^2, \quad F^\ell_{kk'}(h \cdot r) = \rho^\uparrow(h)[\sum_{nn'=-\ell}^{\ell} D^\ell_{kn}(h) F^\ell_{nn'}(r) D^\ell_{n'k'}(h^{-1})]\rho(h^{-1})$$

We can reduce this constraint to a standard $SO(2)$-kernel constraint by considering the $F_\ell(r)_{kk'} = F^\ell_{kk'}$ as a larger matrix. Then, the matrixed $F_\ell(x, y) : \mathbb{R}^2 \to \mathrm{Hom}[V \otimes W^\ell, V^\uparrow \otimes W^\ell]$ are constrained to satisfy

$$\forall h \in SO(2), \quad F_\ell(h \cdot r) = \rho_\ell^\uparrow(h) F_\ell(r) \rho_\ell(h^{-1})$$

so that each $F_\ell(x, y)$ is an $SO(2)$-steerable kernel with input representation $(\rho_\ell, V_\ell) = (\rho, V) \otimes \mathrm{Res}^{SO(3)}_{SO(2)}[(D^\ell, W^\ell)]$ and output representation $(\rho_\ell^\uparrow, V_\ell^\uparrow) = \mathrm{Res}^{SO(3)}_{SO(2)}[(\rho^\uparrow, V^\uparrow) \otimes (D^\ell, W^\ell)]$. The type of F_ℓ is determined by the Clebsch-Gordon coefficients and the branching/induction rules of $SO(2)$ and $SO(3)$.

G Including Non-linearities

In section 4.2, we considered the most general linear maps that satisfied the generalized equivariance constraint. After applying the linear layer described in C, we apply an additional RELU activation to the signal on S^2. It is also possible to use tensor-product based non-linearities analogous to the results of [18, 6]. In this section, we will consider how to include non-linearities for the general $H \subseteq G$

case where G is a compact group. Let (ρ, V) and (σ, W) be two irreducible H-representations. The tensor product representation of (ρ, V) and (σ, W) will in general not be irreducible and will break down into irreducibles as

$$(\rho, V) \otimes (\sigma, W) = \bigoplus_{\tau \in \hat{H}} c^{\tau}_{\rho\sigma} (\tau, V_\tau)$$

where $c^{\tau}_{\rho\sigma}$ counts the number of copies of the H-irreducible (ρ, V_τ) in the tensor product representation. Analogous to the Clebsch-Gordon coefficients [8], we can define $C^{\tau}_{\rho_1\rho_2}$ to be the coefficients of the representation (τ, V_τ) in the tensor product basis. Specifically, let

$$|\tau i_\tau\rangle = \sum_{j_1=1}^{d_1} \sum_{j_2=1}^{d_2} \underbrace{\langle \rho_1 j_1, \rho_2 j_2 | \tau i_\tau \rangle}_{(C^{\tau}_{\rho_1\rho_2})_{i_\tau, j_1 j_2}} |\rho_1 j_1, \rho_2 j_2\rangle$$

with $C^{\tau}_{\rho_1\rho_2}$ we can use the results of [18] to project the tensor product unto a desired output representation. By choosing the output representation (τ, V_τ) to be the restriction of an G representation, we can use tensor products as non-linearites in the induction layer. One difficulty with this procedure is that it is too computationally expensive for practical use. It may be possible to simplify the complexity of implementation using the results of [63]. Tensor product based non-linearities for the construction in 1 is a promising future direction that we leave for future work.

H Generalization to Arbitrary Homogeneous Spaces

The results of C.0.1 can be generalized to any $H \subseteq G$. Let G be a compact group and let $H \subseteq G$. Let $H_c \subseteq H$ and let $X_H = H/H_c$ be a homogeneous space of H. Let $\mathcal{F}(X_H)$ be the set of functions on X_H that transform in representation (ρ_H, V_H) of H,

$$\mathcal{F}(X_H) = \{f | \ f : X_H \to V_H, \quad [h \cdot f](x) = f(h^{-1} \cdot x) = \rho_H(h) f(x)\}$$

Similarly, let $G_c \subseteq G$ and let $X_G = G/G_c$ be a homogeneous space of G. Let $\mathcal{F}(X_G)$ be the set of functions on X_G that transform in the representation (ρ_G, V_G) of G,

$$\mathcal{F}(X_G) = \{f | \ f : X_G \to V_G, \quad [g \cdot f](x) = f(g^{-1} \cdot x) = \rho_G(g) f(x)\}$$

We are interested in characterizing all equivariant maps $\Phi : \mathcal{F}(X_H) \to \mathcal{F}(X_G)$ from $\mathcal{F}(X_H)$ to $\mathcal{F}(X_G)$. Now, generalizing the consistency condition derived in 1 to any $H \subseteq G$, the condition we seek to enforce is that

$$\forall h \in H, \quad \Phi(\rho_H(h) \cdot f) = \rho_G(h) \cdot \Phi(f) \tag{6}$$

By definition of the restriction representation, 3, this is equivalent to the condition,

$$\forall h \in H, \quad \Phi(\rho_H(h) \cdot f) = \mathrm{Res}^G_H[\rho_G(h)] \cdot \Phi(f) \tag{7}$$

Now, the most general linear map $\Phi : \mathcal{F}(X_H) \to \mathcal{F}(X_G)$ between the function spaces $\mathcal{F}(X_H)$ and $\mathcal{F}(X_G)$ can be written as

$$\Phi(f)(x_g) = \int_{x_h \in X_H} dx_h \, \kappa(x_g, x_h) f(x_h)$$

where the kernel $\kappa(x_g, x_h) : X_G \times X_H \to \mathrm{Hom}[V_H, V_G]$ must satisfy the relation

$$\forall h \in H, \quad k(h \cdot x_g, h \cdot x_h) = \rho_G(h) k(x_g, x_h) \rho_H(h)$$

This is a generalization of the steerable kernel constraint first derived in [9] and solved completely in [8]. Let us simplify this constraint to a more tractable form. Using a result stated in [8], the functions on any homogeneous space of a compact group can always be decomposed into a sum of harmonic functions. Let G be a compact group, and X a homogeneous space of G, then for every $(\rho, V_\rho) \in \hat{G}$, there exist multiplicities $0 \leq m_\rho \leq d_\rho$ such that there exist a orthonormal basis $\{Y^{\rho}_{ij}\}$ where the indices range over $\rho \in \hat{G}$ and $i \in \{1, 2, ..., d_\rho\}, j \in \{1, 2, ..., m_\rho\}$ such that

$$\forall j \in 1, 2, ..., m_\rho, \quad \forall g \in G, \ \forall x \in X, \quad Y^{\rho}_{ij}(g^{-1}x) = \sum_{i=1}^{d_j} \rho_{ii'}(g) Y^{\rho}_{i'j}(x)$$

Let us denote the harmonic basis functions on the homogeneous space X_G as Y_{ij}^{σ}. Using the orthogonality of harmonic functions, we can expand the κ uniquely in terms of harmonics as

$$k(x_g, x_h) = \sum_{\sigma \in \hat{G}} \sum_{i=1}^{d_\sigma} \sum_{j=1}^{m_\sigma} F_{ij}^{\sigma}(x_h) Y_{ij}^{\sigma}(x_g)$$

where $F_{ij}^{\sigma} : X_H \to \mathrm{Hom}[V_H, V_G]$ are the matrix valued expansion coefficients of κ. We can simplify this expression for κ by vectorizing,

$$k(x_g, x_h) = \sum_{\sigma \in \hat{G}} [Y^{\sigma}(x_g)]^T F^{\sigma}(x_h)$$

where

$$F^{\sigma}(x_h) : X_H \to \mathrm{Hom}[V_H, V_G \otimes (\underbrace{V_\sigma \oplus V_\sigma \oplus ... \oplus V_\sigma}_{m_\sigma \text{ copies}})]$$

Let us denote $(m_\sigma \sigma, m_\sigma V_\sigma)$ as m_σ copies of the G-irreducible (σ, V_σ),

$$(m_\sigma \sigma, m_\sigma V_\sigma) = \underbrace{(\sigma, V_\sigma) \oplus (\sigma, V_\sigma) \oplus ... \oplus (\sigma, V_\sigma)}_{m_\sigma \text{ copies}}$$

The kernel constraint places a restriction on the allowed form of the $F^{\sigma}(x_h)$. We have that

$$\forall h \in H, \quad k(h \cdot x_g, h \cdot x_h) = \sum_{\sigma \in \hat{G}} [Y^{\sigma}(h \cdot x_g)]^T F^{\sigma}(h \cdot x_h) = \sum_{\sigma \in \hat{G}} [m_\sigma \sigma(h^{-1}) \cdot Y^{\sigma}(x_g)]^T F^{\sigma}(h \cdot x_h)$$

Using the identity $\sigma(h^{-1})^T = \sigma(h)$, we have that,

$$\forall h \in H, \quad k(h \cdot x_g, h \cdot x_h) = \sum_{\sigma \in \hat{G}} [Y^{\sigma}(x_g)]^T [m_\sigma \sigma(h) \cdot F^{\sigma}(h \cdot x_h)]$$

Now, using 6, $k(h \cdot x_g, h \cdot x_h)$ must be equal to $\rho_G(h) k(x_g, x_h) \rho_H(h)$. This is only satisfied if and only if

$$\forall h \in H, \quad F^{\sigma}(h \cdot x_h) = (\rho_G \otimes m_\sigma \sigma)(h) \cdot F^{\sigma}(x_h) \cdot \rho_H(h)$$

Thus, F^{σ} is a H-steerable kernel with input representation ρ_H and output representation $\mathrm{Res}_H^G[(\rho_G \otimes m_\sigma \sigma)]$. Note that the Clebsch-Gordon coefficients, the multiplicities m_σ and the induction/restriction coefficients completely determine the output representation type of the H-steerable kernels F^{σ}.

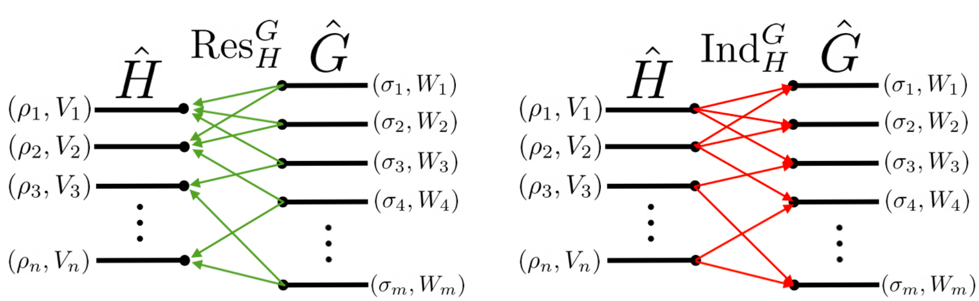

Figure 12: Left: Restriction representation Res_H^G from G to H of G-irreducibles (σ_i, W_i) to H-irreducibles (ρ_j, V_j). Not every H-representation can be realized as the restriction of a G-representation. Right: Induction representation Ind_H^G from H to G of H-irreducibles (ρ_j, V_j) to G-irreducibles (σ_i, W_i). Not every H-representation can be realized as the induction of a H-representation. The Restriction and Induction operations are adjoint functors. In general, the restriction and induction operations are generically *sparse*. This sparsity places restrictions on what irreducibles can appear in $(H \subseteq G)$-equivariant maps.

I A Completeness Property For Induced Representations

Much of the early work on machine learning focused on proving that sufficiently wide and deep neural networks can approximate any function within some accuracy [64]. A network that can approximate any function is said to be expressive. The induced representation satisfies a completeness property.

I.1 Group Valued Functions and Completeness

Can every function $f : G \to \mathbb{R}^c$ be realized as the induced mapping of functions in $\mathbb{R}^H$? We show that this is the case. We have the following compositional property of induced representations [54]: Let $K \subseteq H \subseteq G$. Let (ρ, V) be any representation of K. Then,

$$\text{Ind}_K^G[(\rho, V)] = \text{Ind}_H^G[\text{Ind}_H^K[(\rho, V)]] \tag{8}$$

which states that the induced representation of (ρ, V) from K to G can be constructed by first inducing (ρ, V) from K to H and then inducing from H to G.

Now, choose $K = \{e\}$ to be the identity element of G. Let (ρ, V) be the trivial one dimensional representation of $K = \{e\}$ with

$$\dim V = 1, \quad \rho(e)v = v$$

Consider the set of left cosets of H in $K = \{e\}$. We have that

$$H/K = H/\{e\} = \{he | h \in G\} = H$$

so the set of coset representatives of H/K is just elements of H. Using a from [54], the induced representation of (ρ, V) from $K = \{e\}$ to H is the left regular representation of H. By the same argument, the induced representation of (ρ, V) from $K = \{e\}$ to G is the left regular representation of G. Thus,

$$\text{Ind}_K^H[(\rho, V)] = (L, \mathbb{C}^H), \quad \text{Ind}_K^G[(\rho, V)] = (L, \mathbb{C}^G)$$

Using the compositionality property of the induced representation (8), we thus have that

$$(L, \mathbb{C}^G) = \text{Ind}_H^G[(L, \mathbb{C}^H)]$$

Thus, the induced representation from H to G of the left regular representation of H is the left regular representation of G.

$$\begin{tikzcd}
(L, \mathbb{C}^H) \arrow[r, "{\text{Ind}_H^G[(L,\mathbb{C}^H)]}"] \arrow[d, "L(h)"'] & (L, \mathbb{C}^G) \arrow[d, "L(h)"', "L(g)"] \\
(L, \mathbb{C}^H) \arrow[r, "{\text{Ind}_H^G[(L,\mathbb{C}^H)]}"] & (L, \mathbb{C}^G)
\end{tikzcd}$$

Figure 13: Commutative Diagram for Completeness Property of Induced Representations. L_h denotes the left regular action of H on $\mathbb{C}^H$. L_g denotes the left regular action of G on $\mathbb{C}^G$. The induced representation of the left regular representation of H is the left regular representation of G, $(L, \mathbb{C}^G) = \text{Ind}_H^G[(L, \mathbb{C}^H)]$. The induced representation makes the diagram commutative. This should be contrasted with the definition of G-equivarience defined in A.0.1.

Thus, the induced representation maps the space of all group valued functions on H into the space of all group valued functions on G.

J Irridicibility and Induced and Restriction Representations

Let H be a subgroup of compact group G. We can use the induced representation to map representations of H to representations of G and the restriction representation to map representations of G to representations of H. All representations of H break down into direct sums of irreducible representations of H. Similarly, all representations of G break down into direct sums of irreducible representations of G. Let use denote $\hat{H}$ as a set of representatives of all irreducible representations of H and $\hat{G}$ as a set of representatives of all irreducible representations of G

$$\hat{H} = \{(\rho, V_\rho) |\text{ Representative irreducibles of } H\} \qquad \hat{G} = \{(\sigma, W_\sigma) |\text{ Representative irreducibles of } G\}$$

We want to understand how the restriction and induction representations transform H-irreducibles to G-irreducibles and vice versa. We can completely characterize how irreducibles change under restriction and induction using *branching rules* and *induction rules*, respectively.

J.1 Restriction Representation and Branching Rules

Let (σ, W) be any G-representation. The restricted representation from G to H of (σ, W) is denoted as $\mathrm{Res}_H^G[(\sigma, W)]$ and defined as

$$\forall h \in H,\ \ \forall w \in W, \quad \mathrm{Res}_H^G[\sigma](h)w = \sigma(h)w$$

The restriction operation is linear and

$$\mathrm{Res}_H^G[(\sigma \oplus \sigma', W \oplus W')] = \mathrm{Res}_H^G[(\sigma, W)] \oplus \mathrm{Res}_H^G[(\sigma', W')]$$

We can study the restriction operation by looking at restrictions of the set of G-irreducibles $\hat{G}$. The restriction of an G-irreducible is not necessarily irreducible in H and will decompose as a direct sum of H-irreducibles. Let $(\sigma, W_\sigma) \in \hat{G}$. We can define a set of integers $B_{\sigma,\rho} : \hat{G} \times \hat{H} \to \mathbb{Z}^{\geq 0}$,

$$\mathrm{Res}_H^G[(\sigma, W_\sigma)] = \bigoplus_{\rho \in \hat{H}} B_{\sigma,\rho}(\rho, W_\rho)$$

so that $B_{\sigma,\rho}$ counts the multiplicities of the H-irreducible (ρ, W_ρ) in the restriction representation of the G-irreducible (σ, W_σ). The $B_{\sigma,\rho}$ are called *branching rules* and they have been well studied in the context of particle physics [53]. Let (σ', W') be any G-representation. (σ', W') will decompose into G-irreducibles as

$$(\sigma', W') = \bigoplus_{\sigma \in \hat{G}} m_\sigma(\sigma, W_\sigma)$$

where m_σ counts the number of copies of the G-irreducible (σ, W_σ) in (σ', W'). Then, the restriction representation decomposes into H-irreducibles as

$$\mathrm{Res}_H^G[(\sigma', W')] = \bigoplus_{\sigma \in \hat{G}} m_\sigma \,\mathrm{Res}_H^G[(\sigma, W_\sigma)] = \bigoplus_{\rho \in \hat{G}} \sum_{\sigma \in \hat{G}} [m_\sigma B_{\sigma,\rho}](\rho, W_\rho)$$

So that the multiplicity of the (ρ, W_ρ) irreducible in the restriction of (σ', W') is $\sum_{\sigma \in \hat{G}} m_\sigma B_{\sigma,\rho}$. Thus, the branching rules $B_{\sigma,\rho}$ completely determine how an arbitrary G-representation restricts to an H-representation.

J.2 Induced Representation and Induction Rules

The induced representation acts linearly on representations composed of direct sums of representations. Specifically, if (ρ_1, V_1) and (ρ_2, V_2) are representations of H, then

$$\mathrm{Ind}_H^G[(\rho_1, V_1) \oplus (\rho_2, V_2)] = \mathrm{Ind}_H^G[(\rho_1, V_1)] \oplus \mathrm{Ind}_H^G[(\rho_2, V_2)]$$

The induced representation Ind_H^G maps every irreducible representation $(\rho, V_\rho) \in \hat{H}$ to a G-representation. The induced representation of an irreducible representation of H is not necessarily irreducible in G and will break into irreducibles in $\hat{G}$ as

$$\mathrm{Ind}_H^G(\rho, V_\rho) = \bigoplus_{\sigma \in \hat{G}} I_{\rho,\sigma}(\sigma, W_\sigma)$$

where the integers $I_{\rho,\sigma} : \hat{H} \times \hat{G} \to \in \mathbb{Z}^{\geq 0}$ denotes the number of copies of the irreducible $(\sigma, W_\sigma) \in \hat{G}$ in the induced representation $\mathrm{Ind}_H^G(\rho, V_\rho)$ of the irreducible (ρ, V_ρ). The $I_{\rho,\sigma}$ are called *Induction*

Rules and completely determine the multiplicities of G-irreducibles in the induced representation of any H-representation. Specifically, let (ρ', V') be any representation of H. Then, (ρ', V') breaks into H-irreducibles as

$$(\rho', V') = \bigoplus_{\rho \in \hat{H}} n_\rho (\rho, V_\rho)$$

The induced representation is linear and maps (ρ', V') into a representation of G which will break into G-irreducibles as

$$\mathrm{Ind}_H^G[(\rho', V')] = \bigoplus_{\rho \in \hat{H}} n_\rho \, \mathrm{Ind}_H^G(\rho, V_\rho) = \bigoplus_{\sigma \in \hat{G}} (\sum_{\rho \in \hat{H}} n_\rho I_{\rho,\sigma})(\sigma, W_\sigma)$$

so that the multiplicity of $(\sigma, W_\sigma) \in \hat{G}$ in the induced representation of $(\rho, V_\rho) \in \hat{H}$ is given by $\sum_{\rho \in \hat{H}} m_\sigma I_{\rho,\sigma}$. Thus, the induction rules $I_{\rho,\sigma}$ completely determine the multiplicities of G-representations in the induced representation of any H-representation.

J.3 Irriducibility and Frobinous Reciprocity

The induction rules $I_{\rho\sigma} : \hat{H} \times \hat{G} \to \mathbb{Z}^{\geq 0}$ and the branching rules $B_{\sigma\rho} : \hat{G} \times \hat{H} \to \mathbb{Z}^{\geq 0}$ are related by the Frobinous reciprocity theorem. Let (ρ', V') be any H-representation and let (σ', W') be any G-representation. Then,

$$\mathrm{Hom}_H[(\rho', V'), \mathrm{Res}_H^G[(\sigma', W')]] \cong \mathrm{Hom}_G[\mathrm{Ind}_H^G(\rho', V'), (\sigma', W')]$$

Choosing $(\rho', V') = (\rho, V_\rho) \in \hat{H}$ and $(\sigma', W') = (\sigma, W_\sigma) \in \hat{G}$ gives $I_{\rho,\sigma} = B_{\sigma,\rho}$. So that when viewed as matrices, $B = I^T$. All information about how H-representations are induced to G-representations and G-representations are restricted to H-representations is encoded in both $B_{\sigma,\rho}$ and $I_{\rho,\sigma}$. It should be noted for many cases of interest, $B_{\sigma,\rho}$ and $I_{\rho,\sigma}$ are sparse, and have non-zero entries for only a small number of ρ and σ pairs. In the next section, we discuss how the structure of $B_{\sigma,\rho}$ and $I_{\rho,\sigma}$ constraint the design of equivariant neural architectures.

J.4 Induced and Restriction Representation Based Architectures

Heuristically, convolutional neural networks are compositions of linear functions, interleaved with non-linearities. At each layer of the network, we have a set of functions from a homogeneous space of a group into some vector space [6]. Let X_i^H be a set of homogeneous spaces of the group H and let X_j^G be a set homogeneous spaces of the group G. Let V_i^H and W_j^G be a set of vector spaces .Then, consider the function spaces

$$\mathcal{F}_i^H = \{f | \ f : X_i^H \to V_i^H\} \quad \mathcal{F}_j^G = \{f' | \ f' : X_j^G \to W_j^G\}$$

The group H acts on the homogeneous spaces X_i^H and the group G acts on the homogeneous spaces X_j^G so that the function spaces $\mathcal{F}_i^H$ and $\mathcal{F}_j^G$ form representations of H and G, respectively

Suppose we wish to design a downstream G-equivariant neural network that accepts as signals functions that live in the vector space $\mathcal{F}_0^H$ and transform in the ρ_0 representation of H. Thus, $(\rho_0, \mathcal{F}_0^H)$ is a H-representation, but not necessarily a G-representation. At some point, in the architecture, a layer $\mathcal{F}_i^H$ must be H equivariant on the left and both H and G-equivariant on the right. Let us call the layer that is both H and G-equivariant $\mathcal{F}_1^G$.

$$\begin{array}{ccccc} \cdots \xrightarrow{\Phi_{i-1}} & (\rho_i, \mathcal{F}_i^H) & \xrightarrow{\Psi} & (\sigma_1, \mathcal{F}_1^G) & \xrightarrow{\Psi_1} \cdots \\ & \Big\downarrow{\scriptstyle \rho_i(h)} & & \Big\downarrow{\scriptstyle \sigma_1(g)} & \\ \cdots \xrightarrow[\Phi_{i-1}]{} & (\rho_i, \mathcal{F}_i^H) & \xrightarrow[\Psi]{} & (\sigma_1, \mathcal{F}_1^G) & \xrightarrow[\Psi_1]{} \cdots \end{array} \quad \cong \quad \begin{array}{ccccccc} \cdots \xrightarrow{\Phi_{i-1}} & (\rho_i, \mathcal{F}_i^H) & \xrightarrow{\Phi_{\rho_i}} & \mathrm{Ind}_H^G[(\rho_i, \mathcal{F}_i^H)] & \xrightarrow{\Psi^\uparrow} & (\sigma_1, \mathcal{F}_1^G) & \xrightarrow{\Psi_1} \cdots \\ & \Big\downarrow{\scriptstyle \rho_i(h)} & & \Big\downarrow{\scriptstyle \mathrm{Ind}_H^G[\rho_i]} & & \Big\downarrow{\scriptstyle \sigma_1(g)} & \\ \cdots \xrightarrow[\Phi_{i-1}]{} & (\rho_i, \mathcal{F}_i^H) & \xrightarrow[\Phi_{\rho_i}]{} & \mathrm{Ind}_H^G[(\rho_i, \mathcal{F}_i^H)] & \xrightarrow[\Psi^\uparrow]{} & (\sigma_1, \mathcal{F}_1^G) & \xrightarrow[\Psi_1]{} \cdots \end{array}$$

Figure 14: Factorization of Generic Architecture Using Universal Property of Induced Representation 5.1 $\Psi = \Psi^\uparrow \circ \Phi_{\sigma_i}$

Suppose that Ψ is an intertwiner between $(\rho_i, \mathcal{F}_i^H)$ and $(\sigma_1, \mathcal{F}_1^G)$. Using 5.1, there is a canonical basis of the space $\mathrm{Hom}_H[(\rho_i, \mathcal{F}_i^H), \mathrm{Res}_H^G[(\sigma_1, \mathcal{F}_1^G)]] \cong \mathrm{Hom}_G[\mathrm{Ind}_H^G[(\rho_i, \mathcal{F}_i^H)], (\sigma_1, \mathcal{F}_1^G)]$ and we may write Ψ uniquely as $\Psi = \Psi^\uparrow \circ \Phi_\rho$ where Φ_ρ is an H-equivariant map and $\Psi^\uparrow$ is a G-equivariant map.

$$
\begin{tikzcd}
(\rho_0, \mathcal{F}_0^H) \arrow[r, "\Phi_0"] \arrow[d, "\rho_0(h)"] & (\rho_1, \mathcal{F}_1^H) \arrow[r, "\Phi_1"] \arrow[d, "\rho_1(h)"] & \dots \arrow[r, "\Phi_{i-1}"] \arrow[d] & (\rho_i, \mathcal{F}_i^H) \arrow[r, "\mathrm{Ind}_H^G"] \arrow[d, "\rho_i(h)"] & (\sigma_1, \mathcal{F}_1^G) \arrow[r, "\Psi_1"] \arrow[d, "\sigma_1(g)"] & (\sigma_2, \mathcal{F}_2^G) \arrow[r, "\Psi_2"] \arrow[d, "\sigma_2(g)"] & \dots \arrow[r, "\Psi_{j-1}"] \arrow[d] & (\sigma_j, \mathcal{F}_j^G) \arrow[d, "\sigma_j(g)"] \\
(\rho_0, \mathcal{F}_0^H) \arrow[r, "\Phi_0"'] & (\rho_1, \mathcal{F}_1^H) \arrow[r, "\Phi_1"'] & \dots \arrow[r, "\Phi_{i-1}"'] & (\rho_0, \mathcal{F}_0^H) \arrow[r, "\mathrm{Ind}_H^G"'] & (\sigma_1, \mathcal{F}_1^G) \arrow[r, "\Psi_1"'] & (\sigma_2, \mathcal{F}_2^G) \arrow[r, "\Psi_2"'] & \dots \arrow[r, "\Psi_{j-1}"'] & (\sigma_j, \mathcal{F}_j^G)
\end{tikzcd}
$$

Figure 15: Most general downstream G-equivariant architecture that accepts signals of capsule type ρ_0 that live in vector space $\mathcal{F}_0^H$. Using the universal property of the induction layer, all downstream G-equivariant architectures can be written in this form.

Using this decomposition, we may write any G-equivariant neural architecture that accepts signals in the function space $\mathcal{F}_0^H$ as J.4. Each layer $\mathcal{F}_i^H$ transforms in the ρ_i representation of the group H. Each layer $\mathcal{F}_j^G$ transforms in the σ_j representation of the group G. Each map $\Phi_i \in \mathrm{Hom}_H[(\rho_i, \mathcal{F}_i^H), (\rho_{i+1}, \mathcal{F}_{i+1}^H)]$ is an intertwiner of H representations. Each map $\Psi_i \in \mathrm{Hom}_G[(\sigma_i, \mathcal{F}_i^G), (\sigma_{i+1}, \mathcal{F}_{i+1}^G)]$ is an intertwiner of G representations. All layers preceding the induced mapping are H-equivariant. All layers succeeding the induced mapping are G-equivariant.

Uniformly G-equivariant networks are the topic of a significant amount of research. End to end G-equivariant networks can be essentially fully categorized [8]. Each layer is labeled by the number of multiplicity of irreducibles that it falls into and the non-linear activation function. Thus, an architectures of the form J.4 can be completely specified by decomposition of each layer into irreducibles

$$
\begin{aligned}
&(\rho_0, \mathcal{F}_0^H) = \bigoplus_{\rho \in \hat{H}} m_{0\rho}(\rho, V_\rho) \\
&(\rho_1, \mathcal{F}_1^H) = \bigoplus_{\rho \in \hat{H}} m_{1\rho}(\rho, V_\rho), \quad (\rho_2, \mathcal{F}_2^H) = \bigoplus_{\rho \in \hat{H}} m_{2\rho}(\rho, V_\rho), \quad \dots, \quad (\rho_i, \mathcal{F}_i^H) = \bigoplus_{\rho \in \hat{H}} m_{i\rho}(\rho, V_\rho) \\
&(\sigma_1, \mathcal{F}_1^G) = \bigoplus_{\sigma \in \hat{G}} n_{1\tau}(\sigma, W_\sigma), \quad (\sigma_2, \mathcal{F}_2^G) = \bigoplus_{\sigma \in \hat{G}} n_{2\sigma}(\sigma, W_\sigma), \quad \dots, \quad (\sigma_j, \mathcal{F}_j^G) = \bigoplus_{\sigma \in \hat{G}} n_{j\sigma}(\sigma, W_\sigma)
\end{aligned}
$$

where $m_{i,\rho}$ are the multiplicities of the H-irreducible (ρ, V_ρ) in the i-th H-equivariant layer and $n_{j,\sigma}$ are the multiplicities of the G-irreducible (σ, W_σ) in the j-th G-equivariant layer. [6] introduced the concept of *fragments*, which label how a layer breaks into irreducibles. For networks that are initially H-equivariant but downstream G-equivariant, we need to specify the group as well as the fragment type.

A induced representation based network is characterized by the non-linearities and $(i{+}1)$ H-fragments and j G-fragments,

H-Equivariant Input Space: $(m_{0,1}, m_{0,2}, \dots m_{0,|\hat{H}|})$

H-Equivariant Layers: $(m_{1,1}, m_{1,2}, \dots m_{1,|\hat{H}|})$ $(m_{1,1}, m_{1,2}, \dots m_{1,|\hat{H}|})$ $\dots$ $(m_{i,1}, m_{i,2}, \dots m_{i,|\hat{H}|})$

G-Equivariant Layers: $(n_{1,1}, n_{1,2}, \dots n_{1,|\hat{G}|})$, $(n_{1,1}, n_{1,2}, \dots n_{1,|\hat{G}|})$ $\dots$ $(n_{i,1}, n_{i,2}, \dots n_{i,|\hat{G}|})$

where each of the i H-equivariant layers is specified by a fragment $(m_{x,1}, m_{x,2}, \dots m_{x,|\hat{H}|})$ which specifies the decomposition of the x-th layer into H-irreducibles. Similarly, each of the j G-equivariant layers is specified by a fragment $(n_{y,1}, n_{y,2}, \dots n_{y,|\hat{G}|})$ which specifies the decomposition of the y-th layer into G-irreducibles. The fragments $(m_{i,1}, m_{i,2}, \dots m_{i,|\hat{H}|})$ and $(n_{1,1}, n_{1,2}, \dots n_{1,|\hat{G}|})$ can not be arbitrarily chosen and are related by induced and restriction representations. Specifically, the representations at the boundary must be related by

$$
\mathrm{Res}_H^G[(\sigma_1, \mathcal{F}_1^G)] = (\rho_i, \mathcal{F}_i^H)
$$

Specifically, if $(\rho_i, \mathcal{F}_i^H)$ and $(\sigma_1, \mathcal{F}_1^G)$ decompose into irreducibles as

$$
(\rho_i, \mathcal{F}_i^H) = \bigoplus_{\sigma \in \hat{H}} m_{i\rho}(\rho, V_\rho), \qquad (\sigma_i, \mathcal{F}_1^G) = \bigoplus_{\sigma \in \hat{G}} n_{1\sigma}(\sigma, W_\sigma)
$$

These equations gives a set of linear equations that must be satisfied at each boundary in terms of the branching rules $B_{\sigma\rho}$, we have that

$$
\forall \rho \in \hat{H}, \ \sum_{\sigma \in \hat{G}} n_{1,\sigma} B_{\sigma,\rho} = m_{i,\rho}
$$

However, in practice the true latent G-representation is unknown. The universal property of the induction layer can be used to guarantee that features are expressive. With an induction layer, the the representations at the boundary must be related by

$$\mathrm{Ind}_H^G[(\rho_i, \mathcal{F}_i^H)] = (\sigma_1, \mathcal{F}_1^G)$$

Specifically, if $(\rho_i, \mathcal{F}_i^H)$ and $(\sigma_1, \mathcal{F}_1^G)$ decompose into irreducibles as

$$(\rho_i, \mathcal{F}_i^H) = \bigoplus_{\sigma \in \hat{H}} m_{i\rho}(\rho, V_\rho), \qquad (\sigma_i, \mathcal{F}_1^G) = \bigoplus_{\sigma \in \hat{G}} n_{1\sigma}(\sigma, W_\sigma)$$

These equations gives a set of linear equations that must be satisfied at each boundary in terms of the induction rules $I_{\sigma\rho}$, we have that

$$\forall \sigma \in \hat{G}, \ \sum_{\rho \in \hat{H}} m_{i,\rho} I_{\rho,\sigma} = n_{1\sigma}$$

J.4.1 Generalization to Multiple Groups

We have chosen to consider the case where we induce directly from $H \subset G$ to G. It should be noted that this induction procedure can also be performed incrementally for any sequence of nested ascending subgroups $H = G_1 \subset G_2 ... \subset G_{N-1} \subset G = G_N$. A network architecture is then completely specified by a set of layers that decompose into G_i-irreducibles,

$$(\rho_0^{G_1}, \mathcal{F}_0^{G_1}) = \bigoplus_{\sigma \in \hat{G}_1} n_{0\sigma}^{G_1}(\sigma, V_\sigma), \quad (\rho_1^{G_1}, \mathcal{F}_1^{G_1}) = \bigoplus_{\sigma \in \hat{G}_1} n_{1\sigma}^{G_1}(\sigma, V_\sigma), \quad \ldots \quad (\rho_{i_1}^{G_1}, \mathcal{F}_{i_1}^{G_1}) = \bigoplus_{\sigma \in \hat{G}_1} n_{i_1\sigma}^{G_1}(\sigma, V_\sigma)$$
$$(\rho_1^{G_2}, \mathcal{F}_1^{G_2}) = \bigoplus_{\sigma \in \hat{G}_2} n_{1\sigma}^{G_2}(\sigma, V_\sigma), \quad (\rho_2^{G_2}, \mathcal{F}_2^{G_2}) = \bigoplus_{\sigma \in \hat{G}_2} n_{2\sigma}^{G_2}(\sigma, V_\sigma), \quad \ldots \quad (\rho_{i_2}^{G_2}, \mathcal{F}_{i_2}^{G_2}) = \bigoplus_{\sigma \in \hat{G}_2} n_{i_2\sigma}^{G_2}(\sigma, V_\sigma),$$
$$\ldots$$
$$(\rho_1^{G_N}, \mathcal{F}_1^{G_N}) = \bigoplus_{\sigma \in \hat{G}_N} n_{1\sigma}^{G_N}(\sigma, V_\sigma), \quad (\rho_2^{G_N}, \mathcal{F}_2^{G_N}) = \bigoplus_{\sigma \in \hat{G}_N} n_{2\sigma}^{G_N}(\sigma, V_\sigma), \quad \ldots \quad (\rho_{i_N}^{G_N}, \mathcal{F}_{i_N}^{G_N}) = \bigoplus_{\sigma \in \hat{G}_N} n_{i_N\sigma}^{G_N}(\sigma, V_\sigma)$$

If the true representations at each layer are known, the equivarience conditions require that

$$\begin{aligned}
&\mathrm{Res}_{G_1}^{G_2}[(\rho_1^{G_2}, \mathcal{F}_1^{G_2})] = (\rho_{i_1}^{G_1}, \mathcal{F}_{i_1}^{G_1}) \\
&\mathrm{Res}_{G_2}^{G_3}[(\rho_1^{G_3}, \mathcal{F}_1^{G_3})] = (\rho_{i_2}^{G_2}, \mathcal{F}_{i_2}^{G_2}) \\
&\ldots \\
&\mathrm{Res}_{G_{N-1}}^{G_N}[(\rho_1^{G_N}, \mathcal{F}_1^{G_N})] = (\rho_{i_{N-1}}^{G_{N-1}}, \mathcal{F}_{i_{N-1}}^{G_{N-1}})
\end{aligned}$$

Equivarience constraints give a set of linear equations for the allowed irreducibles of the representations at each boundary,

$$\begin{aligned}
&\forall \rho \in \hat{G}_1, \quad \sum_{\sigma \in \hat{G}_2} n_{1\sigma}^{G_2} B_{\sigma,\rho}^{G_1 G_2} = n_{0\rho}^{G_1} \\
&\forall \rho \in \hat{G}_2, \quad \sum_{\sigma \in \hat{G}_3} n_{1\sigma}^{G_3} B_{\sigma,\rho}^{G_2 G_3} = n_{0\rho}^{G_2} \\
&\ldots \\
&\forall \rho \in \hat{G}_{N-1}, \quad \sum_{\sigma \in \hat{G}_N} n_{1\sigma}^{G_N} B_{\sigma,\rho}^{G_{N-1} G_N} = n_{0\rho}^{G_{N-1}}
\end{aligned}$$

If the latent representation layers are unknown, then expressively can be guaranteed by using induction layers are each boundary,

$$\begin{aligned}
&\mathrm{Ind}_{G_1}^{G_2}[(\rho_{i_1}^{G_1}, \mathcal{F}_{i_1}^{G_1})] = (\rho_1^{G_2}, \mathcal{F}_1^{G_2}) \\
&\mathrm{Ind}_{G_2}^{G_3}[(\rho_{i_2}^{G_2}, \mathcal{F}_{i_2}^{G_2})] = (\rho_1^{G_3}, \mathcal{F}_1^{G_3}) \\
&\ldots \\
&\mathrm{Ind}_{G_{N-1}}^{G_N}[(\rho_{i_{N-1}}^{G_{N-1}}, \mathcal{F}_{i_{N-1}}^{G_{N-1}})] = (\rho_1^{G_N}, \mathcal{F}_1^{G_N})
\end{aligned}$$

Let $I^{G_iG_{i+1}} : \hat{G}_i \times \hat{G}_{i+1} \to \mathbb{Z}^{\geq 0}$ and $B^{G_iG_{i+1}} : \hat{G}_{i+1} \times \hat{G}_i \to \mathbb{Z}^{\geq 0}$ be the induction rules and the branching rules for the groups $G_i \subset G_{i+1}$, respectively. Equivarience constraints give a set of linear equations for the allowed irreducibles of the representations at each boundary,

$$
\begin{aligned}
&\forall \sigma \in \hat{G}_2, \quad \sum_{\rho \in \hat{G}_1} n_{0\rho}^{G_1} I_{\rho,\sigma}^{G_1G_2} = n_{1\sigma}^{G_2} \\
&\forall \sigma \in \hat{G}_3, \quad \sum_{\rho \in \hat{G}_2} n_{0\rho}^{G_2} I_{\rho,\sigma}^{G_2G_3} = n_{1\sigma}^{G_3} \\
&\ldots \\
&\forall \sigma \in \hat{G}_N, \quad \sum_{\rho \in \hat{G}_{N-1}} n_{0\rho}^{G_{N-1}} I_{\rho,\sigma}^{G_{N-1}G_N} = n_{1\sigma}^{G_N}
\end{aligned}
$$

Thus, the induced representation allows for the design of networks that are equivariant with respect a sequence of ascending nested larger groups. It should be noted that it is also possible to move in the 'other direction'. The restriction representation can be used for *coset pooling* [20] to design networks that are equivariant with respect to a descending sequence of nested subgroups $G'_1 \supset G'_2 \supset \ldots \supset G'_N$. Thus, the induced representation, combined with coset pooling allow for the design of neural networks that are at different stages equivariant with respect to an arbitrary sequence of groups $G_1, G_2, \ldots, G_N$, so long as each group in the sequence either contains or is contained by the previous group.

K Toy Example: Tetrahedral Signals

We work out one toy example to help build intuition for induced representations.

Let $\bar{T}$ denote a tetrahedron in three dimensional space. $\bar{T}$ is composed of four vertices and four equilateral triangular faces. Let T be the projection of $\bar{T}$ in a direction normal to a face of $\bar{T}$. As show in **??**, the image of a projection in a direction normal to a face is a equilateral triangle which we will call T.

The group of orientation preserving symmetries of the equilateral triangle T is $\mathbb{Z}_3$ which corresponds to rotations through the origin an angle of 0, $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. The group of orientation preserving symmetries of $\bar{T}$ is A_4.

Let $f : T \to \mathbb{R}^c$ be a signal defined on T. Take $\{\Phi_k\}_{k=1}^4$ to be four independent filters with $\Phi_k : T \to \mathbb{R}^{K \times c}$. We can then convolve each Φ_k with f,

$$
\forall g \in \mathbb{Z}_3, \quad \Psi_k(g) = (\Phi_k \star f)(g) = \int_{x \in T} \Phi_k(x) f(g^{-1}x)
$$

so that each $\Psi_k : \mathbb{Z}_3 \to \mathbb{R}^K \in (\mathbb{R}^K)^{\mathbb{Z}_3}$. The group $\mathbb{Z}_3$ has action on each Ψ_k. Now, let us vectorize the Ψ_k group valued functions into one variable Ψ with $\Psi : \mathbb{Z}^3 \to \mathbb{R}^{4K}$,

$$
g \in \mathbb{Z}_3, \quad \Psi(g) = \begin{bmatrix} \Psi_1(g) \\ \Psi_2(g) \\ \Psi_3(g) \\ \Psi_4(g) \end{bmatrix}
$$

We can now compute the induced action. The computations involved with this map are straightforward but somewhat tedious and are described in L. We just state the results in this section. Let $\Phi^{\uparrow}$ be the function defined on A_4, which has A_4 induced action. First, consider $\Psi^{\uparrow}$ on elements of $\mathbb{Z}_3 = \{e, (1,2,3), (1,3,2)\}$,

$$
\Psi^{\uparrow}[e] = \begin{bmatrix} \Psi_1[e] \\ \Psi_2[e] \\ \Psi_3[e] \\ \Psi_4[e] \end{bmatrix}, \quad \Psi^{\uparrow}[(1,2,3)] = \begin{bmatrix} \Psi_1[(1,2,3)] \\ \Psi_4[(1,2,3)] \\ \Psi_2[(1,2,3)] \\ \Psi_3[(1,2,3)] \end{bmatrix} \quad \Psi^{\uparrow}[(1,3,2)] = \begin{bmatrix} \Psi_1[(1,3,2)] \\ \Psi_3[(1,3,2)] \\ \Psi_4[(1,3,2)] \\ \Psi_2[(1,3,2)] \end{bmatrix}
$$

Note that on $\mathbb{Z}_3$ coset $\Psi^{\uparrow}$ acts only via permutations.

Now, consider the $(1,2,4)H$ coset, we have that

$$\Psi^{\uparrow}[(1,2,4)] = \begin{bmatrix} \Psi_2[e] \\ \Psi_4[(1,3,2)] \\ \Psi_3[(1,3,2)] \\ \Psi_1[(1,2,4)] \end{bmatrix}, \quad \Psi^{\uparrow}[(1,3)(2,4)] = \begin{bmatrix} \Psi_2[(1,2,3)] \\ \Psi_1[(1,3,2)] \\ \Psi_4[e] \\ \Psi_3[e] \end{bmatrix} \quad \Psi^{\uparrow}[(2,4,3)] = \begin{bmatrix} \Psi_2[(1,3,2)] \\ \Psi_3[(1,2,3)] \\ \Psi_1[e] \\ \Psi_4[(1,2,3)] \end{bmatrix}$$

Similarly, for the $(2,3,4)H$ coset, we have that,

$$\Psi^{\uparrow}[(2,3,4)] = \begin{bmatrix} \Psi_3[e] \\ \Psi_1[(1,2,3)] \\ \Psi_2[(1,3,2)] \\ \Psi_4[(1,3,2)] \end{bmatrix}, \quad \Psi^{\uparrow}[(1,2)(3,4)] = \begin{bmatrix} \Psi_3[(1,2,3)] \\ \Psi_4[e] \\ \Psi_1[(1,3,2)] \\ \Psi_2[e] \end{bmatrix} \quad \Psi^{\uparrow}[(3,4,1)] = \begin{bmatrix} \Psi_3[(1,3,2)] \\ \Psi_2[(1,2,3)] \\ \Psi_4[(1,2,3)] \\ \Psi_1[e] \end{bmatrix}$$

Lastly for the $(3,1,4)H$ coset, we have that

$$\Psi^{\uparrow}[(3,1,4)] = \begin{bmatrix} \Psi_4[e] \\ \Psi_2[(1,3,2)] \\ \Psi_1[(1,2,3)] \\ \Psi_3[(1,3,2)] \end{bmatrix}, \quad \Psi^{\uparrow}[(2,3)(1,4)] = \begin{bmatrix} \Psi_4[(1,2,3)] \\ \Psi_3[e] \\ \Psi_2[e] \\ \Psi_1[(1,3,2)] \end{bmatrix} \quad \Psi^{\uparrow}[(1,4,2)] = \begin{bmatrix} \Psi_4[(1,3,2)] \\ \Psi_1[e] \\ \Psi_3[(1,2,3)] \\ \Psi_2[(1,2,3)] \end{bmatrix}$$

Thus, we have constructed a function $\Psi^{\uparrow} : A_4 \to \mathbb{R}^{4K}$ from a set of four filters $\Phi_k : T \to \mathbb{R}^{K \times c}$ defined on the triangle T. It should be noted that unlike the projection trick used in [10], this construction requires no padding or projections. Furthermore, it is not even required that the signal f be lifted from T into $\bar{T}$.

K.0.1 Comparison With Orthographic Projection

In analogy with [12, 10, 11], another way to create a signal on $\bar{T}$ would be to first lift the signal from T to $\bar{T}$ via orthographic projection and then use an A_4-equivariant neural network to extract features. Note that this approach is a specific instance of our construction in K and corresponds to setting

$$\Phi_1 = \Phi(x) \quad \Phi_2 = \Phi_3 = \Phi_4 = 0$$

where $\Phi(x) : T \to T$ is a feature map defined on the equilateral triangle. With this choice of Φ_k, occluded faces of the tetrahedron have no signal defined on them.

L Group Calculations for Induced Representation of $\mathbb{Z}_3$ to A_4

This section details the calculations in computing induced representations of $\mathbb{Z}_3$ on A_4. Computations were done with symbolic computer program, which is available upon request. Let us take $\mathbb{Z}_3 \subset A_4$ to be the group

$$\mathbb{Z}_3 = \langle (1,2,3) \rangle = \{e, (1,2,3), (1,3,2)\}$$

Let us calculate the representatives of the four left cosets of $A_4/\mathbb{Z}_3$. We have that

$$\begin{aligned} e \cdot \mathbb{Z}_3 &= \{e, (1,2,3), (1,3,2)\} \\ (1,2,4) \cdot \mathbb{Z}_3 &= \{(1,2,4), (1,3)(2,4), (2,4,3)\} \\ (2,3,4) \cdot \mathbb{Z}_3 &= \{(2,3,4), (1,2)(3,4), (3,4,1)\} \\ (3,1,4) \cdot \mathbb{Z}_3 &= \{(1,4,3), (2,3)(1,4), (1,4,2)\} \end{aligned}$$

Thus, the elements $g_1 = e$, $g_2 = (1,2,4)$, $g_3 = (2,3,4)$, $g_4 = (3,1,4)$ are representatives of $A_4/\mathbb{Z}_3$. Now, we know that,

$$\forall g \in A_4, \quad \forall g_i \in \{g_1, g_2, g_3, g_4\}, \quad \exists h_i(g) \in \mathbb{Z}_3 \text{ s.t. } g \cdot g_i = g_{j_g(i)} h_i(g)$$

where j_g is a permutation and $h_i(g) \in H$. We thus need to compute the permutations $j_g \in S_4 : \{1,2,3,4\} \to \{1,2,3,4\}$ and $h_i(g) \in H$. The identity element coset has

$$
\begin{aligned}
j_e &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad j_{(1,2,3)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{bmatrix}, \quad j_{(1,3,2)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}, \\
h(e) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ e & e & e & e \end{bmatrix}, \\
h(1,2,3) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ (1,2,3) & (1,2,3) & (1,2,3) & (1,2,3) \end{bmatrix}, \\
h(1,3,2) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ (1,3,2) & (1,3,2) & (1,3,2) & (1,3,2) \end{bmatrix}
\end{aligned}
$$

Now, for the $g_2 = (1,2,4)$ coset,

$$
\begin{aligned}
j_{(1,2,4)} &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix}, \quad j_{(1,3)(2,4)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}, \quad j_{(2,4,3)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}, \\
h(1,2,4) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ e & (1,3,2) & (1,3,2) & (1,2,3) \end{bmatrix}, \\
h((1,3)(2,4)) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ (1,2,3) & (1,3,2) & e & e \end{bmatrix}, \\
h(2,4,3) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ (1,3,2) & (1,2,3) & e & (1,2,3) \end{bmatrix}
\end{aligned}
$$

Similarly, for the $(2,3,4)$ coset,

$$
\begin{aligned}
j_{(2,3,4)} &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{bmatrix}, \quad j_{(1,2)(3,4)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \quad j_{(3,4,1)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}, \\
h(2,3,4) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ e & (1,2,3) & (1,3,2) & (1,3,2) \end{bmatrix}, \\
h((1,2)(3,4)) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ (1,2,3) & e & (1,3,2) & e \end{bmatrix}, \\
h(3,4,1) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ (1,3,2) & (1,2,3) & (1,2,3) & e \end{bmatrix}
\end{aligned}
$$

And lastly for the $(1,4,3)$ coset,

$$
\begin{aligned}
j_{(1,4,3)} &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix}, \quad j_{(2,3)(1,4)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \quad j_{(1,4,2)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}, \\
h(1,4,3) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ e & (1,3,2) & (1,2,3) & (1,3,2) \end{bmatrix}, \\
h((2,3)(1,4)) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ (1,2,3) & e & e & (1,3,2) \end{bmatrix}, \\
h(1,4,2) &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ (1,3,2) & e & (1,2,3) & (1,2,3) \end{bmatrix}
\end{aligned}
$$

Now that we have explicit formulae for j_g and $h(g)$ we can construct the induction of a function from domain $\mathbb{Z}_3$ to A_4.

L.1 Counting Degrees of Freedom

$\mathbb{Z}_3$ has three one dimensional irreducible representations $(\rho_1, V_1), (\rho_+, V_+)$ and (ρ_-, V_-). The actions are given by

$$
\begin{aligned}
v \in V_1, \quad & \rho_1(g)v = v \\
v \in V_\pm, \quad & \rho_\pm(g)v = \exp(\pm\frac{2\pi i}{3})v
\end{aligned}
$$

where (ρ_1, V_1) is the trivial representation and (ρ_+, V_+) and (ρ_-, V_-) are conjugate representations.

We can now find the induced representation of (ρ_k, V_k) on A_4. The index is given by $|A_4 : \mathbb{Z}_3| = 4$. Let g_1, g_2, g_3, g_4 be representatives of the four left cosets in $A_4/\mathbb{Z}_3$. So that

$$A_4/\mathbb{Z}_3 = \{g_1\mathbb{Z}_3, g_2\mathbb{Z}_3, g_3\mathbb{Z}_3, g_4\mathbb{Z}_3\} \tag{9}$$

Note that $\mathbb{Z}_3$ is not normal in A_4 so $A_4/\mathbb{Z}_3$ is not a group. Despite this, the decomposition in (9) holds, via the fact that the set of representatives of cosets partitions G. The induced representation of the irreducible (ρ_k, V_k) representation of $\mathbb{Z}_3$ on A_4 acts on the vector space

$$k \in \{1, +, -\}, \quad W_k = \mathrm{Ind}_{\mathbb{Z}_3}^{A_4}(V_k) = \bigoplus_{i=1}^{4} g_i V_k^{(i)}$$

were the notation $g_i V_k^{(i)}$ is a label denoting the i-th independent copy of the vector space V_k. Let $R_k = \mathrm{Ind}_{\mathbb{Z}_3}^{A_4}(\rho_k)$ denote the action of A_4 on W_k. We have that,

$$\forall g \in A_4, \quad R_k(g) \cdot \sum_{i=1}^{4} g_i v_i = \sum_{i=1}^{4} g_{j_g(i)} \rho_k(h_i(g)) v_i \in W_k$$

where $\forall g \in A_4$, $j_g(i) \in S_4 : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ is a permutation of the coset representatives and $h_i(g) \in \mathbb{Z}_3$.

To summarize, irreducible representations of $\mathbb{Z}_3 = \langle g \rangle$ are given by (ρ_k, V_k) with

$$\begin{aligned} &v \in V_1, \quad \rho_1(g)v = v \\ &v \in V_\pm, \quad \rho_\pm(g)v = \exp(\frac{\pm 2\pi i}{3})v \end{aligned}$$

The induced representations of $\mathbb{Z}_3$ on A_4 are given by (R_k, W_k) with

$$\begin{aligned} &k \in \{1, +, -\}, \quad W_k = \bigoplus_{i=1}^{4} g_i V_k^{(i)} \\ &R_k(g) \cdot \sum_{i=1}^{4} g_i v_i = \sum_{i=1}^{4} g_{j_g(i)} \rho_k(h_i(g)) v_i \\ &\text{with } g \cdot g_i = g_{j_g(i)} \cdot h_i(g) \end{aligned}$$

Let us construct the induced representation of each irreducible of $\mathbb{Z}_3$ explicitly.

L.1.1 Trivial Representation (ρ_1, V_1)

Consider first the trivial representation (ρ_1, V_1) of $\mathbb{Z}_3$. The induced action $R_1 = \mathrm{Ind}_{\mathbb{Z}_3}^{A_4}(\rho_1)$ is then given by

$$R_1[e] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \quad R_1[(1,2,3)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_4 \\ v_2 \\ v_3 \end{bmatrix} \quad R_1[(1,3,2)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_3 \\ v_4 \\ v_2 \end{bmatrix}$$

$$R_1[(1,2,4)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_4 \\ v_3 \\ v_1 \end{bmatrix} \quad R_1[(1,3)(2,4)] \cdot \begin{bmatrix} v_2 \\ v_1 \\ v_4 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \quad R_1[(2,4,3)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_1 \\ v_4 \end{bmatrix}$$

$$R_1[(2,3,4)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_3 \\ v_1 \\ v_2 \\ v_4 \end{bmatrix} \quad R_1[(1,2)(3,4)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_3 \\ v_4 \\ v_1 \\ v_2 \end{bmatrix} \quad R_1[(3,4,1)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_3 \\ v_2 \\ v_4 \\ v_1 \end{bmatrix}$$

$$R_1[(1,4,3)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_4 \\ v_2 \\ v_1 \\ v_3 \end{bmatrix} \quad R_1[(2,3)(1,4)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_4 \\ v_3 \\ v_2 \\ v_1 \end{bmatrix} \quad R_1[(2,4,3)] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_4 \\ v_1 \\ v_3 \\ v_2 \end{bmatrix}$$

Working in the standard Euclidean basis, we may write this as

$$R_1[e] = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \quad R_1[(1,2,3)] = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix} \quad R_1[(1,3,2)] = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}$$
$$R_1[(1,2,4)] = \begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{bmatrix} \quad R_1[(1,3)(2,4)] = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \quad R_1[(2,4,3)] = \begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$$
$$R_1[(2,3,4)] = \begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix} \quad R_1[(1,2)(3,4)] = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix} \quad R_1[(3,4,1)] \begin{bmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$$
$$R_1[(1,4,3)] = \begin{bmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix} \quad R_1[(2,3)(1,4)] = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} \quad R_1[(2,4,3)] = \begin{bmatrix}0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0\end{bmatrix}$$

Note that the induced action of a trivial representation acts only via permutation for all groups.

L.1.2 (ρ_+, V_+) and (ρ_-, V_-) Representations

Now, consider the two complex representations (ρ_+, V_+) and (ρ_-, V_-). These representations are conjugate representations,

$$\overline{(\rho_+, V_+)} = (\rho_-, V_-) \quad \overline{(\rho_-, V_-)} = (\rho_+, V_+)$$

The induced representation of the conjugate is the conjugate of the induced representation,

$$\mathrm{Ind}_H^G[\overline{(\rho, V)}] = \overline{\mathrm{Ind}_H^G[(\rho, V)]}$$

Thus, we have that

$$R_\pm[e] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} \quad R_\pm[(1,2,3)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \omega_\pm \begin{bmatrix}v_1\\v_4\\v_2\\v_3\end{bmatrix} \quad R_\pm[(1,3,2)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \omega_\mp \begin{bmatrix}v_1\\v_3\\v_4\\v_2\end{bmatrix}$$
$$R_\pm[(1,2,4)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}v_2\\\omega_\pm v_4\\\omega_\mp v_3\\\omega_\mp v_1\end{bmatrix} \quad R_\pm[(1,3)(2,4)] \cdot \begin{bmatrix}v_2\\v_1\\v_4\\v_3\end{bmatrix} = \begin{bmatrix}\omega_\pm v_1\\\omega_\mp v_2\\v_3\\v_4\end{bmatrix} \quad R_\pm[(2,4,3)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}\omega_\mp v_2\\\omega_\pm v_3\\v_1\\\omega_\pm v_4\end{bmatrix}$$
$$R_1[(2,3,4)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}v_3\\\omega_\pm v_1\\\omega_\mp v_2\\\omega_\mp v_4\end{bmatrix} \quad R_\pm[(1,2)(3,4)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}\omega_\pm v_3\\v_4\\\omega_\mp v_1\\v_2\end{bmatrix} \quad R_\pm[(3,4,1)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}\omega_\mp v_3\\\omega_\pm v_2\\\omega_\pm v_4\\v_1\end{bmatrix}$$
$$R_\pm[(1,4,3)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}v_4\\\omega_\mp v_2\\\omega_\pm v_1\\\omega_\mp v_3\end{bmatrix} \quad R_\pm[(2,3)(1,4)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}\omega_\pm v_4\\v_3\\v_2\\\omega_\mp v_1\end{bmatrix} \quad R_\pm[(2,4,3)] \cdot \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}\omega_\mp v_4\\v_1\\\omega_\pm v_3\\\omega_\pm v_2\end{bmatrix}$$

Working in the standard Euclidean basis, we may write this as

$$R_\pm[e] = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \quad R_\pm[(1,2,3)] = \omega_\pm \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix} \quad R_\pm[(1,3,2)] = \omega_\mp \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}$$

$$R_\pm[(1,2,4)] = \begin{bmatrix}0&1&0&0\\0&0&0&\omega_\pm\\0&0&\omega_\mp&0\\\omega_\mp&0&0&0\end{bmatrix} \quad R_\pm[(1,3)(2,4)] = \begin{bmatrix}0&\omega_\pm&0&0\\\omega_\mp&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \quad R_\pm[(2,4,3)] = \begin{bmatrix}0&\omega_\mp&0&0\\0&0&\omega_\pm&0\\1&0&0&0\\0&0&0&\omega_\pm\end{bmatrix}$$

$$R_\pm[(2,3,4)] = \begin{bmatrix}0&0&1&0\\\omega_\pm&0&0&0\\0&\omega_\mp&0&0\\0&0&0&\omega_\mp\end{bmatrix} \quad R_\pm[(1,2)(3,4)] = \begin{bmatrix}0&0&\omega_\pm&0\\0&0&0&1\\\omega_\mp&0&0&0\\0&1&0&0\end{bmatrix} \quad R_\pm[(3,4,1)] = \begin{bmatrix}0&0&\omega_\mp&0\\0&\omega_\pm&0&0\\0&0&0&\omega_\pm\\1&0&0&0\end{bmatrix}$$

$$R_\pm[(1,4,3)] = \begin{bmatrix}0&0&0&1\\0&\omega_\mp&0&0\\\omega_\pm&0&0&0\\0&0&\omega_\mp&0\end{bmatrix} \quad R_\pm[(2,3)(1,4)] = \begin{bmatrix}0&0&0&\omega_\pm\\0&0&1&0\\0&1&0&0\\\omega_\mp&0&0&0\end{bmatrix} \quad R_\pm[(2,4,3)] = \begin{bmatrix}0&0&0&\omega_\mp\\1&0&0&0\\0&0&\omega_\pm&0\\0&\omega_\pm&0&0\end{bmatrix}$$

	e	$(1,2,3)$	$(1,3,2)$	$(12)(34)$
χ_{R_1}	4	1	1	0
χ_{R_+}	4	ω_+	ω_-	0
χ_{R_-}	4	ω_-	ω_+	0

Table 5: Character Table for induced representations of the irreducibles (ρ_1, V_1), (ρ_+, V_+) and (ρ_-, V_-) of $\mathbb{Z}_3$ on A_4, $R_+ = \text{Ind}_{\mathbb{Z}_3}^{A_4}(\rho_+)$ and $R_- = \text{Ind}_{\mathbb{Z}_3}^{A_4}(\rho_-)$. $\omega_+ = \exp(\frac{2\pi i}{3}) = \bar{\omega}_-$.

The group A_4 has four conjugacy classes: e, $(1,2,3)$, $(1,2)(3,4)$ and $(1,3,2)$. The four irreducible representations of A_4 are: The trivial (σ_1, W_1) representation, two conjugate one-dimensional representations $(\sigma_{1,+}, W_{1,+}), (\sigma_{1,-}, W_{1,-})$ and one three dimensional representation (σ_3, W_3).

	e	$(1,2,3)$	$(1,3,2)$	$(12)(34)$
χ_1	1	1	1	1
$\chi_{1,-}$	1	ω_+	ω_-	1
$\chi_{1,+}$	1	ω_-	ω_+	1
χ_3	3	0	0	-1

Table 6: Character Table for A_4. $\omega_+ = \exp(\frac{2\pi i}{3}) = \bar{\omega}_-$. $(\sigma_{1,+}, W_{1,+})$ and $(\sigma_{2,-}, W_{2,-})$ are conjugate representations.

We can thus compute the structure coefficients of the induced representation of $\mathbb{Z}_3$ on A_4. We have that

$$\begin{aligned}\text{Ind}_{\mathbb{Z}_3}^{A_4}[(\rho_1, V_1)] &= (\sigma_3, W_3) \oplus (\sigma_1, W_1)\\ \text{Ind}_{\mathbb{Z}_3}^{A_4}[(\rho_+, V_+)] &= (\sigma_3, W_3) \oplus (\sigma_{1,+}, W_{1,+})\\ \text{Ind}_{\mathbb{Z}_3}^{A_4}[(\rho_-, V_-)] &= (\sigma_3, W_3) \oplus (\sigma_{1,-}, W_{1,-})\end{aligned}$$

We are only interested in real representations. The most general real representation of $\mathbb{Z}_3$ is given by

$$(\rho, V) = m_1(\rho_1, V_1) \oplus m_c[(\rho_+, V_+) \oplus (\rho_-, V_-)]$$

where m_1 and m_c are integers. The dimension of the vector space V is $\dim V = m_1 + m_c$. The induced representation of (ρ, V) is

$$(R, W) = \text{Ind}_{\mathbb{Z}_3}^{A_4}[(\rho, V)] = [m_1 + 2m_c](\sigma_3, W_3) \oplus m_c[(\sigma_{1,+}, W_{1,+}) \oplus (\sigma_{1,-}, W_{1,-})] \oplus m_1(\sigma_1, W_1)$$

where the vector space W of the induced representation has dimension $\dim W = 3(m_1 + 2m_c) + 2m_c + m_1 = 4m_1 + 8m_c = 4(m_1 + 2m_c) = 4\dim V$ as expected. This result, although simple is extremely satisfying as it shows that any function on A_4 can be lifted from a function on $\mathbb{Z}_3$. To see this, note the following: By the Peter-Weyl theorem, the left regular representation $(L, \mathbb{R}^{\mathbb{Z}_3})$ decomposes as

$$(L, \mathbb{R}^{\mathbb{Z}_3}) = (\rho_1, V_1) \oplus [(\rho_+, V_+) \oplus (\rho_-, V_-)]$$

Thus, the induced representation of $(L, \mathbb{R}^{\mathbb{Z}_3})$ is from $\mathbb{Z}_3$ to A_4 is thus

$$(R, W) = \text{Ind}_{Z_3}^{A_4}[\mathbb{R}^{\mathbb{Z}_3}] = 3(\sigma_3, W_3) \oplus [(\sigma_{1,+}, W_{1,+}) \oplus (\sigma_{1,-}, W_{1,-})] \oplus (\sigma_1, W_1)$$

Now, again by the Peter-Weyl theorem, the left regular representation $(L, \mathbb{R}^{A_4})$ of A_4 decomposes as

$$(L, \mathbb{R}^{A_4}) = 3(\sigma_3, W_3) \oplus [(\sigma_{1,+}, W_{1,+}) \oplus (\sigma_{1,-}, W_{1,-})] \oplus (\sigma_1, W_1)$$

So the induced representation of the left regular representation of $\mathbb{Z}_3$ has the same decomposition into irreducibles as the left regular representation of A_4. Representations are completely determined by their decomposition into irreducibles and

$$(L, \mathbb{R}^{A_4}) = \text{Ind}_{Z_3}^{A_4}[(L, \mathbb{R}^{\mathbb{Z}_3})] \tag{10}$$

Ergo, the space of functions from A_4 into $\mathbb{R}$ is identical to the induced representation from $\mathbb{Z}_3$ to A_4 of the space of functions of $\mathbb{Z}_3$ into $\mathbb{R}$. Using the linearity of the induced representation and taking the c-fold direct sum of both sides of (10), we have that

$$(L, (\mathbb{R}^c)^{A_4}) = \text{Ind}_{Z_3}^{A_4}[(L, (\mathbb{R}^c)^{\mathbb{Z}_3})]$$

Thus, as expected, the induced representation bijectively maps group valued functions from $\mathbb{Z}_3 \to \mathbb{R}^c$ into group valued functions from $A_4 \to \mathbb{R}^{4c}$.