Dual Groups of Ribbon Fusion Categories

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These notes are about Fusion Categories. We review and state some open conjectures about fusion categories. Ribbon fusion categories (RFCs) are rigid fusion categories equipped with extra duality, braid and twist structure. Unitary Modular Tensor Categories (UMTC) are RFCs with an additional non-degeneracy condition. We study the properties of UMTC generated by irreducible representations of finite groups, which are important for defining Levin-Wen models. It is conjectured that Levin-Wen models are error correcting codes.

I. INTRODUCTION

Ribbon Fusion Categories are an important object in topological quantum computing. Fusion categories are generalizations of finite groups. Fusion categories are conjectured to always have unitary and piviotal structures. Ribbon Fusion Categories are fusion categories equipped with additional duality, twist and braiding structures. A complete classification of finite groups has been completed [2], but a complete classification of Ribbon Fusion Categories (or fusion categories for that matter) does not yet exist [14]. Modular tensor categories are unitary ribbon fusion categories with an additional non-degeneracy requirement on the S-matrix. In this work, we try and resolve some open questions about ribbon fusion categories. To the best of the author's knowledge, there are a few unproven conjectures in the theory of Fusion Categories:

- Every fusion category admits a unitary structure.
- Every fusion category admits a pivotal structure.
- Every fusion category with pivotal structure and positive quantum dimensions admits a unitary structure.
- State the topological charge conjecture of UTMCs
- Let G be a finite group. The ground state of the Levin-Wen model [12] of the fusion category generated by G is an error correcting code (see XI for formal definition of error correcting code).

The connection between fusion categories and braid groups is well understood. Specifically, the Kaufman bracket [11] maps a braid group element into the Temperly-Lieb-Jones algebras, which are the prototypical example of unitary modular ribbon categories. Similarly, there exist an interesting set of open conjectures about braid group representations:

- Is the Jones representation faithful?
- Give a complete characterization of unitary and anti-unitary linear representations of the braid group.
- Compute the group cohomology of the braid group.

A better understanding and classification of fusion categories would have applications to a wide variety of fields in pure and applied mathematics. These notes are based on [13, 14], which are both excellent comprehensive expositions of the subject. Furthermore, there has been recent interest trying to use machine learning to characterize representations of braid groups and Temperly-Lieb-Jones (TLJ) algebras [6, 9]. There has been intense research interest in using deep learning to find knot invariants [6].

a. Summary of Notes: The goal of these notes is threefold:

- We try to state open conjectures related to fusion categories and braid group representations.
- We propose a numerical bootstrap method to try and give a exhaustive classification of UTMCs of given rank.

$$\text{General Category} \xrightarrow[\otimes]{\text{Bifunctor}} \text{Tensor/Monodial Category} \xrightarrow{\text{Strict}} \text{Fusion Category}$$

FIG. 1: Fusion Categories are General Categories equipped with additional structures. Specifically, a fusion category is a strict tensor/monodial category.

II. FUSION CATEGORIES

Fusion categories are some of the most interesting objects in mathematics. Fusion categories are natural generalizations of finite groups. Additionally, the objects of interest in topological quantum computing are Unitary Modular Tensor Categories (UMTCs) which are fusion categories equipped with additional structure.

A. Algebroids and Tensor Categories

Let $\mathbb{F}$ be a field of characteristic zero. A $\mathbb{F}$-linear category, also called an $\mathbb{F}$-algebroid, is a category such that the morpism spaces $\mathrm{Hom}(x, y)$ are vector spaces over $\mathbb{F}$ and the composition operation

$$\mathrm{Hom}(y, z) \circ \mathrm{Hom}(x, y) = \mathrm{Hom}(x, z)$$

is bilinear over $\mathbb{F}$. A monoidal category (also called a tensor category) is a category equipped with a bifunctor $\otimes$ such that

$$\text{Identity: } I \otimes A \cong A \cong A \otimes I$$
$$\text{Associativity: } (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

where $\cong$ is the isomorphism relation. When the isomorpism condition $\cong$ is additional required to be equality $=$, the category is called strict. I.e. a monoidal category is called strict if the associativity and unit constraints are identities. A Fusion Category is a finite $\mathbb{F}$-algebroid that is also strict monodial category. Specifically, a fusion category $\mathcal{R} = (L, 1, \hat{\ }, \otimes)$ consists of a finite label set L with a distinguished 'identity' element 1, an involution operation $\hat{\ } : L \to L$ and an associative operation $\otimes : L \times L \to L$ called the fusion rule. The distinguished element $1 \in L$ is always required to satisfy $\hat{1} = 1$. The fusion rule $\otimes : L \times L \to L$ is an associative operation that satisfies the decomposition

$$a \otimes b = \bigoplus_{c \in L} N_{ab}^c c$$

where N_{ab}^c are integers. Strictness requires that the fusion rule $\otimes$ is additionally required to satisfy $1 \otimes a = a \otimes 1 = a$. Furthermore, the identity element only appears in the tensor product of an element and its inverse so that

$$N_{ab}^1 = N_{ba}^1 = \delta_{a\hat{a}}$$

The integers N_{ab}^c are required to satisfy a set of equations stemming from the strictness requirement. Specifically, the integers N_{ab}^c which satisfy,

$$\text{Associativity: } \sum_{e=1}^{L} N_{ab}^e N_{ec}^d = \sum_{e=1}^{L} N_{ae}^d N_{bc}^e \tag{1}$$

$$\text{Unit: } N_{a1}^c = \delta_{ac} = N_{1a}^c \tag{2}$$

$$\text{Identity: } N_{ab}^1 = \delta_{a\hat{a}} = N_{ba}^1 \tag{3}$$

Furthermore, a fusion category is called semi-simple if the endomorpism spaces of each object are direct sums of $\mathbb{C}$. A fusion category (FC) that has either zero or one fusion rules so that $N_{ab}^c \in \{0, 1\}$ always is called a multiplicity free FC. A FC that has commutative tensor product $a \otimes b = b \otimes a$ is called a commutative FC. The rank of an FC is the number of elements in the set L. We would like to give a complete classification of fusion categories, analogous to the complete classification of simple groups.

There are currently a set of known algebraic objects that generate fusion categories. These include representations of finite groups, Temperley-Lieb-Jones algebras, central extensions of Lie algebras and quantum groups [10]. What is so surprising (and interesting!) is that all known fusion categories seem to come with inherent additional structures. Specifically, it is conjectured that all fusion categories admit both pivotal and unitary structures [14].

B. Unitary Fusion Categories

A conjugation σ on a fusion category $\mathcal{C}$ is a operation on morpisms of $\mathcal{C}$

$$\sigma : \mathrm{Hom}(x, y) \to \mathrm{Hom}(y, x)$$

satisfying the relations

$$\sigma(\sigma(f)) = f, \quad \sigma(f \otimes g) = \sigma(f) \otimes \sigma(g) \quad \sigma(f \circ g) = \sigma(g) \circ \sigma(f)$$

A fusion category $\mathcal{C}$ is said to have a unitary structure if there exists a conjugation σ on $\mathcal{C}$ such that $\sigma(f) \circ f = 0$ if and only if $f = 0$ holds. The tuple $(\mathcal{C}, \sigma)$ is called a unitary fusion category. It is conjectured [14], but not proven, that every fusion category admits a unitary structure. This is conjecture is the category theoretic generalization of the fact that all representations of finite groups are equivalent to a unitary representation.

- This should be easy to prove: Analogous to the finite group case, define the inner product $\langle x, y\rangle_U = \int_{g\in G} dg\ \langle gx, gy\rangle$. This generates a unitary representation of G.
- Let $f \in \mathrm{Hom}(x, y)$ what objects can we define that are conjugation operations. One natural object is $\Pi(x) = \frac{1}{|L|}\sum_{a\in L} a \otimes x \otimes \hat{a}$, similar to the G-averaging Reynolds operator on finite groups.

III. RIGID FUSION CATEGORIES

Rigid fusion categories were originally defined by Grothendieck (cite original paper). Rigid fusion categories, also called fusion categories with dual, are generalizations of dual vector spaces. Let V be a vector space over the field $\mathbb{F}$. We can define the dual $V^\star$ as the space of functions

$$V^\star = \{\ \alpha \ |\ \alpha : V \to \mathbb{F},\ \ \alpha \text{ is } \mathbb{F}\text{-linear}\ \}$$

The dual $V^\star$ forms a vector space over the field $\mathbb{F}$. The evaluation pairing i is a canonical mapping $i : V \otimes V^\star \to \mathbb{F}$ defined as

$$\forall v \in V,\ \ \forall \alpha \in V^\star,\ \ i(v, \alpha) = \alpha(v)$$

which is $\mathbb{F}$-bilinear. We can also define the co-evaluation mapping $e : \mathbb{F} \to V \otimes V^\star$. Let v_i be an orthonormal basis of V. Let v^i be the corresponding dual basis of $V^\star$ with $v^i(v_j) = \delta^i_j$. The co-evaluation map is then defined as

$$\forall f \in \mathbb{F}, \quad e(f) = \sum_i f v_i \otimes v^i$$

Note that the co-evaluation map is a basis independent quantity. Specifically, under the change of basis $v_i \to U^j_i v_j$, the dual basis transforms as $v^i \to U^i_j v^j$ so the co-evaluation map

$$e(f) \to \sum_i f U^j_i v_j \otimes U^i_k v^k = \sum_i f \underbrace{(U^j_i U^i_k)}_{\delta^j_k} v_j \otimes v^k = e(f)$$

is invariant. The composition of evaluation and co-evaluation maps gives an isomorphism $e \circ i : V \otimes V^\star \to V \otimes V^\star$ of the tensor product of V and its dual. The evaluation and co-evaluation maps are used to define rigid categories.

Rigid categories are category theory generalizations of dual vector spaces. In a (right) rigid category each object x has a dual object $x^\star$ such that there exist special morpisms $b_x \in \mathrm{Hom}(1, x \otimes x^\star)$ and $d_x \in \mathrm{Hom}(x^\star \otimes x, 1)$ satisfying

$$\begin{aligned} &\text{Left Straightening: } (\mathbb{I}_x \otimes d_x) \circ (b_x \otimes \mathbb{I}_x) = \mathbb{I}_x \\ &\text{Right Straightening: } (d_{x^\star} \otimes \mathbb{I}_{x^\star}) \circ (b_{x^\star} \otimes \mathbb{I}_x) = \mathbb{I}_x \end{aligned}$$

The map b_x is called the co-evaluation (birth) morpism and the map d_x is called the evaluation (death) morpism. Rigid Fusion categories also allow for definition of dual morpisms. Specifically, for any morpism $f \in \mathrm{Hom}(x, y)$ we can define a dual morpism $f^\star \in \mathrm{Hom}(y^\star, x^\star)$ given by

$$f^\star = (d_x \otimes \mathbb{I}_x) \circ (\mathbb{I}_{y^\star} \otimes f \otimes \mathbb{I}_{x^\star}) \circ (\mathbb{I}_{x^\star} \otimes b_x)$$

$$\begin{tikzcd}
 & \text{Fusion Category} \arrow[dl, "\text{Rigidity}"'] \arrow[dr, "\text{Braiding}"] & \\
\text{Rigid Fusion Category} \arrow[dr, "\text{Compatible Braiding}"'] & & \text{Braided Tensor Category} \arrow[dl, "\text{Compatible Dual}"] \\
 & \text{Ribbon Fusion Category} \arrow[d, "\text{Unitary}"] & \\
 & \text{Unitary Ribbon Fusion Category} \arrow[d, "\text{Non-Degenerate } S\text{-Matrix}"] & \\
 & \text{Unitary Modular Tensor Category (UMTC)} &
\end{tikzcd}$$

FIG. 2: Fusion Categories are general Categories equipped with additional structures. Ribbon Fusion categories are fusion categories with additional braid and dual structures. Unitary Ribbon Fusion categories are Ribbon Fusion categories with unitary structure. Unitary Modular Tensor Categories (UMTCs) are Unitary Ribbon Fusion categories with non-degenerate S-matrix

A. Pivotal Fusion Categories

Let V be a vector space over the field $\mathbb{F}$. The double dual of a vector space $(V^\star)^\star$ has a canonical identification with V. General fusion categories are conjectured to have this property, but this has never been proven. Fusion categories equipped with a canonical isomorphism between the object x and $(x^\star)^\star$ are called piviotal fusion categories. In a pivotal category, every object is isomorphic to its double dual, justifying the name dual. Dual vector spaces behave well under tensor product. Specifically,

$$(V \otimes W)^\star = V^\star \otimes W^\star$$

so that the dual of the tensor product of is the tensor product of the duals. Thus the dual vector spaces

$$\begin{aligned} &\text{Double Dual: } (V^\star)^\star = V \\ &\text{Tensor Product of Dual: } (V \otimes W)^\star = V^\star \otimes W^\star \end{aligned} \tag{4}$$

A pivotal category abstracts these properties 4.

Let $\mathcal{R} = (L, 1, \hat{\ }, \otimes)$ be a fusion category. The fusion category $\mathcal{R}$ is said to have a pivotal structure if and only if for each $x \in L$ there exists a isomorphism from x to the double dual of x defined as $\Phi_x : x \to (x^\star)^\star$ satisfying the tensor property,

$$\Phi_{x \otimes y} = \Phi_x \otimes \Phi_y$$

This gives a isomorphism between tensor products and double dual spaces

$$\text{Hom}[x \otimes y, (x^\star)^\star \otimes (x^\star)^\star] \cong \text{Hom}[x \otimes y, ((x \otimes y)^\star)^\star]$$

Note that we do not require the stronger strict equality $x = (x^\star)^\star$, which is a stronger condition than the isomorphism condition. Furthermore, the double dual of a morpism must be identical to that morpism

$$(f^\star)^\star = f$$

Conseqence or assumption? Pivotal structures are important as they allow for the definition of a trace operator. Specifically, in a rigid category with a pivotal structure, we can define a consistent trace operator.

Conjecture: We state a famous unproved conjecture due to [7]: Every fusion category has a pivotal structure. It is shown in [7] that there is an isomorphism between x and the quadruple dual $(((x^\star)^\star)^\star)^\star$.

B. Spherical Tensor Categories

A pivotal category is a rigid category equipped with an additional isomorphism between an object x and the double dual $(x^\star)^\star$. Let $\Phi_x : x \to (x^\star)^\star$ be said isomorphism. For a pivotal category we can define a left and right trace,

$$\text{Left trace:}\ \mathrm{Tr}_L[f] = d_x \circ (\Phi_{x^\star} \otimes f) \circ (\mathbb{I}_x \otimes \Phi_x^{-1}) \circ b_{x^\star}$$
$$\text{Right trace:}\ \mathrm{Tr}_R[f] = d_{x^\star} \circ (\Phi_x \otimes \mathbb{I}_{x^\star}) \circ (f \otimes \mathbb{I}_{x^\star}) \circ b_x$$

The left trace and right trace do not have to be equal. In a spherical category, the trace behaves like a standard trace under tensor product,

$$\mathrm{Tr}_L[f \otimes g] = \mathrm{Tr}_L[f]\mathrm{Tr}_L[g]\mathrm{Tr}_R[f \otimes g] = \mathrm{Tr}_R[f]\mathrm{Tr}_R[g]$$

A category that has equal left and right trace is called a spherical category. The dimension of an object x in a spherical category is defined as

$$\text{Dimension:}\ d_x = \mathrm{Tr}[1_x]$$

where $1_x \in \mathrm{Hom}(x, x)$ is the trivial endomorpism.

IV. BRAIDING AND TWIST

- Need to write a little section to motivate the braid and twist structures here. What is the best way to motivate this?

A fusion category can be equipped with Braiding and Twist structures. Let $\mathcal{C}$ be a fusion category.

A. Braiding

The elements $x \otimes y$ and $y \otimes x$ are not required to be equal. A braiding on the fusion category $\mathcal{C}$ is a set of isomorphisms $\hat{B}_{x,y} \in \mathrm{Hom}(x \otimes y, y \otimes x)$ for all objects $x, y \in L$. The braiding operator is required to transform in the natural way under tensor product,

$$\text{Left Tensor:}\ \hat{B}_{x\otimes y,z} = \hat{B}_{x,z} \otimes \hat{B}_{y,z}$$
$$\text{Right Tensor:}\ \hat{B}_{x,y\otimes z} = \hat{B}_{x,y} \otimes \hat{B}_{x,z}$$

So that the braid of a tensor product is the tensor product of the braid. The composition of two braids $\hat{B}_{x,y}$ and $\hat{B}_{y,x}$ is again a braid.

B. Twist

A twist on the fusion category $\mathcal{C}$ is a set of isomorphisms $\hat{T}_x \in \mathrm{Hom}(x, x)$ for each object x. The set of twists for a given $x \in L$ forms a group under composition. Specifically, if $\hat{T}_x$ and $\hat{T}_{x'}$ are invertable endomorpisms of $\mathrm{Hom}(x, x)$, then the composition $\hat{T}_x \circ \hat{T}'_x \in \mathrm{Hom}(x, x)$ is again invertable.

C. Braiding and Twist Compatibility

Furthermore, the braiding and twist are required to satisfy the 'interwining' constraint

$$\text{Twist-Braid Compatibility:}\ \hat{T}_{X\otimes Y} = \hat{B}_{X\otimes Y} \circ (\hat{T}_X \otimes \hat{T}_Y) \circ \hat{B}_{Y\otimes X} \tag{5}$$

A braid and twist structure that satisfy the condition 5 is said to be a compatible braid and twist structure.

- Need a graphic here

D. Compatible Duality

Let $\mathbb{C}$ be a rigid category with braid and twist structure. The duality is said to be compatible with the braiding and twist if and only if

$$(\hat{T}_x \otimes \mathbb{I}_{x^\star}) \circ b_x = (\mathbb{I}_x \otimes \hat{T}_{x^\star}) \circ b_x$$

hold.

- Is this the only requirement?
- Maybe comment on the geometric picture here

V. RIBBON FUSION CATEGORIES

A ribbon category is a pivotal fusion category equipped with additional dual, braid and twist structure. Let $\mathcal{C}$ be a fusion category.

A. Frobinious-Schur Indicators

Linear representations of compact groups can be categorized into three categories, real, psuedo-real and complex. Frobinious-Schur indicators assign to each representation a $+1$ (real),0(complex) or -1(psuedo-real). Frobinious-Schur indicators can be defined for self-dual objects in fusion categories. For completeness, we review Frobinious-Schur indicators in finite groups.

B. Frobinious-Schur Indicator in Finite Groups

Let (ρ, V) be a self-dual unitary G-representation. Then there exists a matrix S such that

$$\bar{\rho}(g) = S\rho(g)S^{-1}$$

The matrix S is unitary and satisfies either $S^T = S$ (real representation) or $S^T = -S$ (psuedo-real representation). For finite groups, the quantity

$$\frac{1}{|G|}\sum_{g\in G}\chi_\rho(g^2)$$

is called the Frobinious-Schur indicator of a group representation. The Frobinous-Schur indicator is $+1$ when the representation ρ is real and -1 when the representation is psuedo-real. The Frobinious-Schur Indicators of a category is a generalization of the finite group case. A natural way to think about this is as follows: Let (ρ, V) be a self-dual unitary G-representation. We can then form the tensor product representation

$$(\rho\otimes\rho, V\otimes V) \cong (\rho\otimes\bar{\rho}, V\otimes V^\star) \cong (\bar{\rho}\otimes\rho, V^\star\otimes V) \cong (\bar{\rho}\otimes\bar{\rho}, V^\star\otimes V^\star)$$

Where $(\bar{\rho}, V^\star)$ is the dual representation. Let $\hat{S}$ be the swap operator which interchanges the V subspaces,

$$\forall v_1, v_2 \in V, \quad \hat{S}(v_1\otimes v_2) = v_2\otimes v_1$$

Note that the tensor product representation $(\rho, V)\otimes(\rho, V)$ commutes with the swap operator.

$$\forall g\in G, \quad \hat{S}\cdot[\rho(g)\otimes\rho(g)] = [\rho(g)\otimes\rho(g)]\cdot\hat{S}$$

Ergo, the 2-fold tensor product representation is a representation of the group $G\times\mathbb{Z}_2$.

C. Frobinious-Schur Indicator in Fusion Categories

Let $x = x^\star$ be a self-dual object in a Ribbon Fusion Category. Let $\Phi_x \in \mathrm{Hom}(x \otimes x, 1)$ be a morpism. Using the duality axiom, the dual morpism $\Phi_x^\star$ of Φ_x is an element of $\Phi_x^\star \in \mathrm{Hom}(1, x \otimes x)$. Now, using the fact that we are working with fusion categories,

$$\dim \mathrm{Hom}(x \otimes x^\star, 1) = \dim \mathrm{Hom}(x, x) = 1$$

Thus, the morpisms Φ_x and $\Phi_x^\star$ live in a one dimensional space and we must have that

$$\Phi_x = \theta_x \Phi_x^\star$$

for some scalar $\theta_x \in \mathbb{C}$. Applying this relation twice, we have that $\theta_x^2 = 1$ so $\theta_x = \pm 1$. The value of θ_x is called the Frobinious-Schur indicator of x.

VI. EXAMPLES OF FUSION CATEGORIES

We give some examples of Fusion Categories frequently found in the literature.

Tambara-Yamagami RFC Let G be a finite group. Define $\mathbb{C}[G]$ as the group algebra generated by elements of G. The algebra $\mathbb{C}[G]$ is a $\mathbb{C}$-vector space. Define the label set $L = G \bigcup m$. Then, the Tambara-Yamagami fusion rule is defined as

$$g \otimes h = gh, \quad m \otimes g = g \otimes m = m, \quad m \otimes m = \bigoplus_{g \in G} g$$

For the special case $G = \mathbb{Z}_2$, the Tambara-Yamagami fusion category is called the Ising FC with elements $\{1, \tau, \sigma\}$ and non-trivial fusion rules give by

$$\sigma \otimes \sigma = 1 \quad \tau \otimes \tau = 1 \oplus \sigma \quad \sigma \otimes \tau = \tau = \tau \otimes \sigma$$

The duals of elements are given by $\hat{\tau} = \tau$ and $\hat{\sigma} = \sigma$. The Ising FC is a commutative, multiplicity-free fusion category. The corresponding quantum dimensions are given by...

- Do some calculations here, dimensions basis ect.

Fermionic Moore-Read FC

Let $L = \{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$. The elements $\{1, \alpha, \psi, \alpha'\}$ are identified with the group $\mathbb{Z}_4$. The other non-trivial fusion rules are given by

$$\begin{aligned}
&\sigma \otimes \sigma' = \sigma' \otimes \sigma = 1 \oplus \psi \quad \sigma \otimes \sigma = \sigma' \otimes \sigma' = \alpha \oplus \alpha' \\
&\sigma \otimes \psi = \psi \otimes \sigma = \sigma \quad \sigma' \otimes \psi = \psi \otimes \sigma' = \sigma' \quad \sigma \otimes \alpha = \sigma \otimes \alpha' = \alpha \otimes \sigma = \alpha' \otimes \sigma = \sigma' \\
&\sigma' \otimes \alpha = \sigma' \otimes \alpha' = \alpha \otimes \sigma' = \alpha' \otimes \sigma' = \sigma
\end{aligned}$$

- Do some calculations here, dimensions basis ect.

E_6 Fusion Rule Both the Tambara-Yamagami RFC and Fermionic Moore-Read RFC are muplicity free. Consider the label set $L = \{1, x, y\}$ with non-trivial fusion rules

$$x \otimes x = 1 \oplus 2x \oplus y, \quad x \otimes y = y \otimes x = x, \quad y \otimes y = 1$$

Both x and y are self-dual.

- Do some calculations here, dimensions basis ect.

VII. UNITARY RIBBION FUSION CATEGORIES

Unitary Ribbon Fusion Categories are defined similarly to unitary fusion categories. Let $\mathcal{R}$ be a Ribbon fusion category. Let σ be an involution on morpisms of $\mathcal{R}$ satisfying

$$\sigma(f \otimes g) = \sigma(f) \otimes \sigma(g) \quad \sigma(f \circ g) = \sigma(g) \circ \sigma(f)$$

such as σ is called a conjugation. A RFC is Hermitian if there exists a conjugation σ on $\mathcal{R}$ satisfying

Right Death/Birth: $\exists b'_x \in \mathrm{Hom}(x \otimes x^\star, 1)$ and $\exists d'_x \in \mathrm{Hom}(1, x \otimes x^\star)$
Right-Left Birth-Death: $\sigma(b_x) = d'_x$, and $\sigma(d_x) = b'_x$
Braid: $\sigma(B_{x,y}) = B^{-1}_{x,y}$
Twist: $\sigma(T_x) = T_x^{-1}$

The pair $(\mathcal{R}, \sigma)$ is called a Hermitian RFC. A Hermitian RFC $(\mathcal{R}, \sigma)$ is called unitary if for every $f \in \mathrm{Hom}(x, x)$, the traces

$$\mathrm{Tr}_L[\sigma(f)f] \geq 0, \quad \mathrm{Tr}_R[\sigma(f)f] \geq 0$$

of the morpism $\sigma(f) \circ f \in \mathrm{Hom}(x, x)$ are positive semi-definite.

1. S-Matrix

The S-matrix of a Ribbon Fusion Category as the trace of the braid operator

$$\forall a, b \in L, \quad s_{ab} = \mathrm{Tr}[\hat{B}_{ab}\hat{B}_{ba}]$$

Note that because $\hat{B}_{ab} = \hat{B}^\star_{ba}$ Is this true? Check this?, the trace of an object $\mathrm{Tr}[\hat{B}_{ab}\hat{B}_{ba}] = \mathrm{Tr}[\hat{B}^\dagger_{ab}\hat{B}_{ab}] \geq 0$ is positive semi-definite. The quantum dimension of an object in a unitary ribbon fusion category is defined as the trace of the identity operator

$$d_x = \mathrm{Tr}[1]$$

It is conjectured [14], that every Ribbon Fusion Category with all objects of positive quantum dimension is unitary.

Conjecture: Every Ribbon Fusion Category with all objects of positive quantum dimension is unitary. (Need to find citation here, what is the orginal paper that proposed this?)

VIII. UNITARY MODULAR TENSOR CATEGORIES (UMTC)

A unitary modular tensor category is a unitary ribbon fusion category $(\mathcal{R}, \sigma)$ with with a full rank S-matrix. unitary modular tensor categories have incredibly rich structure. Define the fusion rule matrices $(N_i)_{jk} = N^k_{ij}$. There is one fusion rule matrix for each object. Let $T = T_i \delta_{ij}$ be the diagonal twist matrix (can this always be done? not sure).

- Verlinde formula for unitary modular tensor categories
- Vafa formula

Amazingly, for specific choices of central charge, unitary modular tensor categories form representations of the modular group. Specifically, the S and T matrix satisfy

$$(ST)^3 = \exp(\frac{i\pi c}{4})\mathbb{I}$$
$$S^2 = \mathbb{I}$$

which is a finite dimensional unitary representation of the modular group $SL(2, \mathbb{Z})$ when $8c$ is an integer.

- When $8c$ is not an integer does this form a projective representation of the modular group?

IX. FUSION CATEGORIES FROM FINITE GROUPS: G-UMTCS

Let G be a finite group. Let $\mathbb{C}[G]$ be the group algebra over $\mathbb{C}$. The standard group multiplication law generates a monodial fusion category. In this context, the objects of $\mathcal{C}$ correspond to the elements of G, and the morphisms are defined via $\mathrm{Hom}(g,h) = \mathbb{C}\delta_{g^{-1}h}$. This implies that morphisms exist only between identical objects, and each such morphism space is isomorphic to the complex numbers $\mathbb{C}$, representing scalar multiples of the identity morphism for each object. Specifically, define the fusion rule

$$\forall g,h \in G, \quad g \otimes h = \bigoplus_{k\in G} \delta_{gh,k} k$$

The Fusion Category generated by the finite group G will be called a G-category.

1. *Semisimplicity and Simple Objects*

The category $\mathcal{C}$ is *semisimple*, meaning that every object can be decomposed into a direct sum of simple objects. In this case, each object X_g is simple, as there are no non-trivial morphisms between different objects, and the only endomorphisms are scalar multiples of the identity.

2. *Monoidal Structure*

The monoidal unit in this category is the object corresponding to the identity element $e \in G$, denoted by X_e. For any object X_g, the following holds:

$$X_e \otimes X_g = X_g \otimes X_e = X_g.$$

This property aligns with the identity element's role in the group G.

3. *Rigid Structure*

The category $\mathcal{C}$ is also *rigid*, meaning that each object has a dual. For an object X_g, the dual object is $X_{g^{-1}}$, satisfying:

$$X_g \otimes X_{g^{-1}} = X_{g^{-1}} \otimes X_g = X_e.$$

X. FUSION CATEGORIES FROM IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS: $\hat{G}$-UMTCS

Some of the most interesting fusion categories are generated by representations of finite groups. Many of the conjectures for general fusion categories are inspired by statements from fusion categories generated by representations of finite groups. Let G be a finite group. The set of irreducible representations of G form a finite set $\hat{G}$. The tensor product representation of two irreducible G-representations is again a G-representation, which is itself not necessarily irreducible. The tensor product representation then breaks down into irreducible representations. Thus, for all irreducible representation $\rho, \sigma \in \hat{G}$ we have a decomposition of the tensor product representation into an integer number of irreducible representations given by

$$\rho \otimes \sigma \cong \bigoplus_{\tau\in\hat{G}} N^{\tau}_{\rho\sigma} \tau$$

where the integers $N^{\tau}_{\rho\sigma}$ count the number of copies of $\tau \in \hat{G}$ irreducible in the tensor product representation $\rho \otimes \sigma$. The tensor product of representations is an commutative operation so

$$\forall \rho, \sigma, \tau \in \hat{G}, \quad N^{\tau}_{\rho\sigma} = N^{\tau}_{\sigma\rho}$$

holds. Furthermore, irreducible representations are simple objects. By Schur's lemma B 1, the endomorpism space

$$\forall \rho \in \hat{G}, \quad \mathrm{Hom}(\rho, \rho) \cong \mathbb{C}\mathbb{I}_{d_\rho}$$

so the tensor category generated by irreducible representations is semi-simple. For any representations ρ and σ, the morpism space is a $\mathbb{C}$-linear vector space as

$$\forall w, z \in \mathbb{C}, \quad f, g \in \mathrm{Hom}(\rho, \sigma) \implies wf + zg \in \mathrm{Hom}(\rho, \sigma)$$

Furthermore, compositions of G-intertwiners are again G-intertwiners as

$$f \in \mathrm{Hom}(\rho, \sigma), \;\; g \in \mathrm{Hom}(\sigma, \tau) \implies g \circ f \in \mathrm{Hom}(\rho, \tau)$$

Ergo, irreducible representations of finite groups generate commutative fusion tensor categories. The Fusion Category generated by the irreducible representations of the finite group G will be called a $\hat{G}$-category.

Fusion categories generated by representations of compact groups satisfy both 1 and an additional set of integer equations. Specifically, each irreducible representation $\rho \in \hat{G}$ has an integer dimension d_ρ so

$$d_\rho d_\sigma = \sum_{\tau \in \hat{G}} N^\tau_{\rho\sigma} d_\tau$$

must hold. This places additional restrictions on the allowed N^c_{ab} so that irreducible representations of finite groups generate a subset of all allowed fusion categories.

Now, let us give some examples of the interesting structures inherent in fusion categories generated by finite groups. To begin, note that tensor products of irreducible representations require a change of basis to be block diagonal so that,

$$\rho \otimes \sigma = U_{\rho\sigma}[\bigoplus_{\tau \in \hat{G}} N^\tau_{\rho\sigma}\tau]U^\dagger_{\rho\sigma}$$

where the matrices $U_{\rho\sigma}$ are unitary matrices of dimension $d_\rho d_\sigma \times d_\rho d_\sigma$. An incredible amount of structure is contained in the matrices $U_{\rho\sigma}$. For ease of notation, we will define the triple tensor product decomposition

$$\rho \otimes \sigma \otimes \tau = U_{\rho\sigma\tau}[\bigoplus_{\delta \in \hat{G}} N^\delta_{\rho\sigma\tau}\delta]U^\dagger_{\rho\sigma\tau}$$

and so on for any tensor product of irreducible representations. All higher order tensor product terms can be written in terms of the two irreducible tensor product decompositions.

A. Unitary of $\hat{G}$-Categories

Fusion categories generated by representations of finite groups always have unitary structure. Specifically, let $\Phi \in \mathrm{Hom}(\rho, \sigma)$ be a G intertwiner,

$$\forall g \in G, \quad \Phi\rho(g) = \sigma(g)\Phi$$

Taking the conjugate, we have that

$$\forall g \in G, \quad \rho^\dagger(g)\Phi^\dagger = \Phi^\dagger\sigma^\dagger(g) \implies \rho(g^{-1})\Phi^\dagger = \Phi^\dagger\sigma(g^{-1})$$

so that $\Phi^\dagger \in \mathrm{Hom}(\sigma, \rho)$. Furthermore, if $\Phi^\dagger \circ \Phi = 0$, then $\Phi = 0$ must hold identically.

B. Braiding of $\hat{G}$-Categories

Furthermore, fusion categories generated by representations of finite groups always admit a compatible braiding. Specifically, consider the tensor product decompositions

$$\rho \otimes \sigma = U_{\rho\sigma}[\bigoplus_{\tau \in \hat{G}} N^\tau_{\rho\sigma}\tau]U^\dagger_{\rho\sigma} \quad \sigma \otimes \rho = U_{\sigma\rho}[\bigoplus_{\tau \in \hat{G}} N^\tau_{\rho\sigma}\tau]U^\dagger_{\sigma\rho}$$

The matrices $U_{\rho\sigma}$ and $U_{\sigma\rho}$ are related by partial conjugation, but are not in general equal. Thus, using the extended Schur's lemma B 1 a, any element

$$\hat{B}_{\rho\sigma} \in \mathrm{Hom}(\rho \otimes \sigma, \sigma \otimes \rho)$$

can always be written as

$$\hat{B}_{\rho\sigma} = U_{\rho\sigma}[\bigoplus_{\tau\in\hat{G}} M^{\tau}_{\rho\sigma}\mathbb{I}_{d_\tau}]U^{\dagger}_{\sigma\rho}$$

where $M^{\tau}_{\rho\sigma}$ is a matrix of size $N^{\tau}_{\rho\sigma} \times N^{\tau}_{\rho\sigma}$. The morpism $\hat{B}_{\rho\sigma}$ is a braid element if and only if the matrices $M^{\tau}_{\rho\sigma}$ are invertable matrices. The tensor product representations are associative

$$\rho \otimes \sigma \otimes \tau = \rho \otimes (\sigma \otimes \tau) = (\rho \otimes \sigma) \otimes \tau$$

Let us define the matrices $U_{\rho\sigma\tau}$ as the unitary matrix that diagonlizes the tensor product representation,

$$\rho \otimes \sigma \otimes \tau = U_{\rho\sigma\tau}[\bigoplus_{\delta\in\hat{G}} N^{\delta}_{\rho\sigma\tau}\delta]U^{\dagger}_{\rho\sigma\tau}$$

Now, using the associativity of the tensor product we have that

$$\rho \otimes (\sigma \otimes \tau) = \rho \otimes U_{\sigma\tau}[\bigoplus_{\delta\in\hat{G}} N^{\delta}_{\sigma\tau}\delta]U^{\dagger}_{\sigma\tau} = (\mathbb{I}_{d_\rho} \otimes U_{\sigma\tau})[\rho \otimes \bigoplus_{\delta\in\hat{G}} N^{\delta}_{\sigma\tau}\delta](\mathbb{I}_{d_\rho} \otimes U_{\sigma\tau})^{\dagger}$$

again using the tensor product rules, we have that

$$\rho \otimes \bigoplus_{\delta\in\hat{G}} N^{\delta}_{\sigma\tau}\delta = \bigoplus_{\delta\in\hat{G}} N^{\delta}_{\sigma\tau}[\rho \otimes \delta] = \bigoplus_{\delta\in\hat{G}} N^{\delta}_{\sigma\tau}U_{\rho\delta}[\bigoplus_{\epsilon} N^{\epsilon}_{\rho\delta}\epsilon]U^{\dagger}_{\rho\delta}$$

Thus, we have that

$$\rho \otimes (\sigma \otimes \tau) = (\mathbb{I}_{d_\rho} \otimes U_{\sigma\tau})[\bigoplus_{\delta\in\hat{G}} N^{\delta}_{\sigma\tau}U_{\rho\delta}[\bigoplus_{\epsilon\in\hat{G}} N^{\epsilon}_{\rho\delta}\epsilon]U^{\dagger}_{\rho\delta}](\mathbb{I}_{d_\rho} \otimes U_{\sigma\tau})^{\dagger} \tag{6}$$

Via a similar computation for $(\rho \otimes \sigma) \otimes \tau$ we have that

$$(\rho \otimes \sigma) \otimes \tau = (U_{\rho\sigma} \otimes \mathbb{I}_{d_\tau})[\bigoplus_{\delta\in\hat{G}} N^{\delta}_{\rho\sigma}U_{\delta\tau}[\bigoplus_{\epsilon\in\hat{G}} N^{\epsilon}_{\delta\tau}\epsilon]U^{\dagger}_{\rho\delta}](U_{\rho\sigma} \otimes \mathbb{I}_{d_\tau})^{\dagger}$$

Thus, associativity of the tensor product (ref) demands that we have that

$$\forall\epsilon \in \hat{G}, \quad (\mathbb{I}_{d_\rho} \otimes U_{\sigma\tau})[\bigoplus_{\delta\in\hat{G}} N^{\delta}_{\sigma\tau}N^{\epsilon}_{\rho\delta}U_{\rho\delta}] = (U_{\rho\sigma} \otimes \mathbb{I}_{d_\tau})[\bigoplus_{\delta\in\hat{G}} N^{\delta}_{\rho\sigma}N^{\epsilon}_{\delta\tau}U_{\delta\tau}]$$

This gives a non-trivial set of relations on the unitary matrices $U_{\rho\sigma}$.

C. Twists of $\hat{G}$-Categories

Let $\rho \in \hat{G}$ be an irreducible G-representation. Consider a twist element $\hat{T}_\rho \in \mathrm{Hom}(\rho, \rho)$. Using the Schur lemma B 1, the twist can always be written as $\hat{T}_\rho = T_\rho \mathbb{I}_{d_\rho}$ which is an isomorphism if and only $T_\rho = \mathbb{C}/\{0\}$. The twist is thus a proportional to the identity so that braid and twist compatibility is automatically enforced. Specifically, we have that

$$\hat{T}_{\rho\otimes\sigma} = \hat{B}_{\rho\sigma}(\hat{T}_\rho \otimes \hat{T}_\sigma)\hat{B}_{\sigma,\rho}$$

holds automatically as

$$\hat{T}_\rho \otimes \hat{T}_\sigma = T_\rho T_\sigma \mathbb{I}_{d_\rho} \otimes \mathbb{I}_{d_\sigma} = T_\rho T_\sigma \mathbb{I}_{d_\rho d_\sigma}$$

so that

$$\hat{B}_{\rho\sigma}(\hat{T}_\rho \otimes \hat{T}_\sigma)\hat{B}_{\sigma,\rho} = T_\rho T_\sigma \hat{B}_{\rho\sigma}\hat{B}_{\sigma,\rho} = T_\rho T_\sigma \mathbb{I}_{d_\rho d_\sigma}$$

holds. Using the fact that $\hat{T}_{\rho\otimes\sigma} = T_{\rho\sigma}\mathbb{I}_{d_\rho d_\sigma}$, we see that braid-twist consistency is always satisfied if the twist coefficients satisfy

$$T_{\rho\sigma} = T_\rho T_\sigma$$

Thus, the twist coefficients are completely specified by a map

$$T_\rho : \hat{G} \to \mathbb{C}/\{0\}$$

Additionally, note that the twist is natural. Let $\hat{T}_\rho$ be a twist element. We have that

$$\Phi \circ \hat{T}_\rho = \hat{T}_\rho \circ \Phi$$

for all $\Phi \in \mathrm{Hom}(\rho, \rho)$. Fusion categories generated by representations of finite groups have inherent additional structures.

- Compute the S-matrix of $\hat{G}$-categories

XI. ERROR CORRECTING CODES AND LEVIN-WEN MODELS

The theory of error correcting codes is one of the triumphs of complexity theory. Both the classical and quantum theory of error correcting codes is well understood. The definition of quantum error correcting code is given in (cite gottsman). Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{W}$ be a Hilbert space containing $\mathcal{H}$. The inclusion map $\mathcal{H} \hookrightarrow \mathcal{W}$ is then well defined. Using the (cite gottsman), a code is said to be error correcting if and only if

$$\mathcal{H} \hookrightarrow \mathcal{W} \underbrace{\to}_{\hat{O}_k} \mathcal{W} \underbrace{\to}_{\pi} \mathcal{H}$$

is the identity mapping where $\pi : \mathcal{W} \to \mathcal{H}$ is the projection operator and $\hat{O}_k$ denotes any k-local operator. It is well understood that the toric code (cite kiteav), which corresponds to the Levin-Wen model with the group $G = \mathbb{Z}_4$ is error correcting. It is conjectured in [14] that the Levin-Wen model is error correcting for any finite group G.

A. Toric Code

- Maybe write a bit about how the toric code is error correcting, this is common knowledge maybe?

XII. BRAID GROUP THEORY

Braid group representations are an integral component in topological quantum computing, where trajectories of anyonic quasi-particles generate projective representations of braid groups. Topological quantum computing has generated a interesting set of new conjectures about braid group representations. In a similar vein, the braid group has recently been applied to understanding chaotic fluid flows (cite). One very interesting use of braid groups is the development of braid group public key cryptography [5, 8]. In braid group cryptography, a public key is encoded into a product of braid group elements. Of course, the main use of braid group theory is in the study of knots. Alexander's theorem states that every knot or link is in one-to-one correspondence with a braid group element. Specifically, by identifying string start and end positions, elements of the braid group on n-strings can be mapped to knots consisting of n-links.

A. Yang-Baxter Equation

Braid group representations also appear directly in chaos theory. Specifically, in integrability theory the Yang-Baxter (YB) equation is a factorization equation that integrabile systems satisfy. The Yang-Baxter equation arises naturally in the Bethe Ansatz (cite), which is a method for exactly solving one-dimensional quantum systems.

- Put in the YB graphic

Let V be a vector space over the field $\mathbb{F}$ of characteristic zero. An automorpism $R : V \otimes V \to V \otimes V$ is said to be an R-matrix if it satisfies the equation

$$(R \otimes \mathbb{I})(\mathbb{I} \otimes R)(R \otimes \mathbb{I}) = (\mathbb{I} \otimes R)(R \otimes \mathbb{I})(\mathbb{I} \otimes R)$$

R-matrices furnish unitary braid group representations. Specifically, consider the map

$$\rho_R : B_n \to \mathrm{Hom}(V^{\otimes n}, V^{\otimes n})$$

defined on generators as

$$\rho_R(\sigma_i) = \mathbb{I}^{\otimes(i-1)} \otimes R \otimes \mathbb{I}^{\otimes(n-i-1)}$$

and extended in the natural way. Then, by definition of R, we have that

$$\rho_R(\sigma_i)\rho_R(\sigma_{i+1})\rho_R(\sigma_i) = \rho_R(\sigma_{i+1})\rho_R(\sigma_i)\rho_R(\sigma_{i+1})$$

and for $|i-j| \geq 2$, $\rho_R(\sigma_i)\rho_R(\sigma_j) = \rho_R(\sigma_j)\rho_R(\sigma_i)$ is commuting. Thus, $(\rho_R, V^{\otimes n})$ forms a representation of B_n. Thus, every solution to the Yang-Baxter equation generates a braid group representation. It should be noted that many of these solutions will be reducible. To see this note that if R_1 and R_2 are solutions to the Yang-Baxter equation the direct sum $R_1 \oplus R_2$ is also a solution to the Yang-Baxter equation, which will generate a reducible representation of the Braid group. One can ask if the converse statement is true. Do braid group representations generate solution to the Yang-Baxter equation. I don't think so. I.e there exist braid group representations which do not generate solutions to the Yang-Baxter equation.

B. Unitary Representations of the Braid Group

See [1] for a good introduction to representations of braid groups.

The braid group on n strands B_n describes the intertwining of n strings with fixed start and end positions. The braid group on n strands has the following presentation

$$B_n = \langle \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \forall |i-j| \geq 2, \;\; \sigma_i\sigma_j = \sigma_j\sigma_i \rangle$$

Non-identity elements of the braid group are torsion free. It is usually easier to work with finite torsion subgroups of B_n. We can define the s-torsion braid group on n elements as

$$B_n^s = \langle \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \forall |i-j| \geq 2, \;\; \sigma_i\sigma_j = \sigma_j\sigma_i, \quad (\sigma_i)^s = 1 \rangle$$

The s-torsion analogue B_n^s of B_n is created by forcing any s-twist of a generator element to be the identity. Of special interest is the 2-torsion braid group.

C. Group Cohomology and Braid Groups

The braid group carries a highly non-trivial set of cohomology. Understanding the group cohomology of B_n is essential to categorizing projective braid group representations. Specifically, via a standard result of representation theory (Cite), the homology groups $H(B_n, U(1))$ determine the allowed projective representations **??** of B_n. There is a natural group homemoprism $f : B_n \to \mathbb{Z}$ of the braid group into the integers. Define the function f as assigning one to the generator σ_i and negative one to the generator σ_i^{-1}, so that

$$f(\prod_{i=1}^{m} \sigma_i^{s_i}) = \sum_{i=1}^{m} s_i$$

The kernel of f is then the maximal commuting subgroup of B_n. The map f is $\mathbb{Z}_2$-equivariant. If we define the group automorpism $T : B_n \to B_n$ sending a generator to its inverse

$$T\sigma_i T^{-1} = \sigma_i^{-1}$$

Then we have that

$$f(T \cdot b) = -f(b)$$

Let $g : \mathbb{Z} \to U(1)$ be the standard group homomorpism

$$\forall k \in \mathbb{Z}, \quad g(k)(\phi) = \exp(2\pi i k\phi)$$

Then the composition of maps $g \circ f : B_n \to U(1)$. Note that this mapping is $\mathbb{Z}_2$-equivariant.

- Can any map be written in this way?
- I think cohomology of B_n has already been computed.

XIII. TEMPERLY-LIEB-JONES ALGEBRAS

Some of the most important examples of fusion categories are the Temperly-Lieb and Temperly-Lieb-Jones algebras. The Jones polynomials are well understood within the context of TLJ algebras.

A. Temperly-Lieb Algebras

Let A be an free element of $\mathbb{C}$. Define the twist number as $d = -A^2 - A^{-2}$. The Temperly-Lieb algebras are spanned by the set of continuous (non-crossing) elements. We define an equivalence relation on diagrams as follows: Any closed loop in a diagram D_0 can be deleted to form the diagram D_1. The two diagrams $D_0 \sim dD_1$ are said to be d-isotopic. Two TL diagrams can be composed to form an additional TL diagram in the natural way. This composition operation serves as the multiplication operation of the TL algebra. For any TL diagram D, there is a natural involution operation $\sigma(D)$ which reflects the diagram D about the x-axis.

The Temperly-Lieb algebra inherits a natural trace operation $\mathrm{Tr} : \mathrm{TL}(A) \to \mathbb{C}[A]$. Specifically, for any diagram X the Markov trace is defined by connecting objects. The resulting number of loops is an integer n. The Markov trace of X is then given by $\mathrm{Tr}[X] = d^n$. The Markov trace induces an inner product on $\mathrm{TL}(A)$. Specifically, for any two TL diagrams D_1 and D_2, define the inner product

$$\langle D_1, D_2 \rangle = \mathrm{Tr}[\sigma(D_1)D_2]$$

and extended to general TL diagrams in the natural way.

- What are the properties of this inner product?
- Non-degenerate? PSD?

B. Kaufman Bracket

The Temperly-Lieb algebras can naturally be motivated by the Kaufman bracket. Specifically, let $\mathbb{C}[B_n]$ be the group algebra of the braid group. Then, the Kaufman bracket decomposes a link via the graphical rule (show image).

Thus, for each knot L, the Kaufman bracket maps $L \to \langle L \rangle \in \mathrm{TL}_n(A)$ where n is the number of crossings of L. The Jones polynomial is defined as

$$J(A) = (-A^3)^{w(L)} \langle L \rangle$$

where $w(L)$ is the writhe of L. Colored Jones polynomials are defined similarly.

- Define the colored jones polynomials here

The action of the Kaufman bracket on braid group elements defines a representation of B_n. Specifically, the map $\rho_A : B_n \to \mathrm{TL}_n(A)$ is called the generic Jones representation (cite) of B_n. It is unknown if the generic Jones representation maps non-trivial links to the identity matrix. We have the following famous conjecture,

Conjecture: The generic Jones representation ρ_A of B_n is faithful.

This conjecture, which was stated in 1981 by Jones (cite) is still unproven.

1. Jones-Wenzl Projectors

TL algebras have a canonical basis. Let U_i denote the loop diagrams at position $(i, i+1)$ with U_0 denoting the trivial diagram. The U_i generate the TL algebra and satisfy the relations

$$\begin{aligned}&d\text{-idempotent: } U_i^2 = dU_i\\&\text{Braiding: } U_i U_{i\pm1} U_i = U_{i\pm1}\\&\text{Commutativity: } U_i U_j = U_j U_i \text{ for } |i-j| \geq 2\end{aligned}$$

Understanding the structure of the generic Jones representation is an important question. The Jones-Wenzl projector P_n of $\mathrm{TL}_n(A)$ is a unique operator satisfying

$$\begin{aligned}&\text{Projector: } P_n^2 = P_n \text{ and } P_n \neq 0\\&\text{Commutativity: } \forall i,\ \ U_i P_n = P_n U_i\\&\text{Simplicity: } P_n = \sum_{i=0}^{n-1} c_{ni} U_i\end{aligned}$$

where c_{ni} are complex numbers.

C. Jones Algebroids

For particular choices of the indeterminate variable A, the Temperly-Lieb algebras are not semi-simple. Quotients of these degenerate TL algebras are called Jones algebroids. Jones algebroids are denoted as J_k where k is an integer.
Suppose that A is chosen such that d (denomintor of JWP) is zero. Then, the P_n is undefined. Let $\ker(P_n)$ be the kernel of the n-th JWP. We can then define the morpisms of the Jones algebroids as...

- Define the Jones Algebroid here

XIV. CONCLUSION

Ribbon Fusion Categories are an important object in topological quantum computing. Fusion categories are generalizations of finite groups. Ribbon Fusion Categories are fusion categories equipped with additional duality, twist and braiding structures. A complete classification of groups has been completed [2], but a complete classification of Ribbon Fusion Categories does not yet exist. Modular tensor categories are unitary ribbon fusion categories with an additional requirement on the S-matrix. In this work, we try and resolve some open questions about ribbon fusion categories.

XV. ACKNOWLEDGMENTS

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Appendix A: Group Theory

A group G is a non-empty set combined with a associative binary operation $\cdot : G \times G \to G$ that satisfies the following properties

$$\begin{aligned}&\text{existence of identity: } e \in G, \text{ s.t. } \forall g \in G, \;\; e \cdot g = g \cdot e = g\\&\text{existence of inverse: } \forall g \in G, \implies \exists g^{-1} \in G, \;\; g \cdot g^{-1} = g^{-1} \cdot g = e\end{aligned}$$

Appendix B: Representation Theory

We review the basics of representation theory, see [4, 15] for a comprehensive exposition. Let V be a vector space over the field $\mathbb{C}$. A representation (ρ, V) of a group G consists of V and a group homomorphism $\rho : G \to \mathrm{Hom}[V, V]$. By definition, the ρ map satisfies

$$\forall g, g' \in G, \;\; \forall v \in V, \;\; \rho(g)\rho(g')v = \rho(gg')v$$

Heuristically, a group can be thought of as the embedding of an group (which is an abstract mathematical object) into a set of matrices. Two representation (ρ, V) and (σ, W) are said to be equivalent representations if there exists a unitary matrix U

$$\forall g \in G, \quad U\rho(g) = \sigma(g)U$$

The linear map U is said to be a G-intertwiner of the (ρ, V) and (σ, W) representations. The space of all G-intertwiners is denoted as $\mathrm{Hom}_G[(\rho, V), (\sigma, W)]$. Specifically,

$$\mathrm{Hom}_G[(\rho, V), (\sigma, W)] = \{\Phi : V \to W| \;\; \forall g \in G, \;\; \Phi\rho(g) = \sigma(g)\Phi, \;\; \Phi \text{ is linear}\}$$

The sum of two G-intertwiner is a G-intertwiner and $\mathrm{Hom}_G[(\rho, V), (\sigma, W)]$ forms a vector space over $\mathbb{C}$. The vector space of of G-intertwiners from a representation to itself is called the G endomorpism space of the representation (ρ, W),

$$\mathrm{End}_G[(\rho, W)] = \mathrm{Hom}_G[(\rho, W), (\rho, W)]$$

which is termed the endomorpism space of (ρ, W). Much of classical group theory studies the structure of the intertwiners of representations [3]. The unitary theorem in representation theory [3] says that all representations of compact groups are equivalent to a unitary representation. A representation is said to be reducible if it breaks into a direct sum of smaller representations. Specifically, a unitary representation ρ is reducible if there exists an unitary matrix U such that

$$\forall g \in G, \quad \rho(g) = U[\bigoplus_{i=1}^{k} \sigma_i(g)]U^\dagger$$

where $k \geq 2$ and σ_i are smaller irreducible representations of G. The set of all non-equivalent representations of a group G will be denoted as $\hat{G}$. All representations of compact groups G can be decomposed into direct sums of irreducible representations. Specifically, if (σ, V) is a G-representation,

$$(\sigma, V) = U[\bigoplus_{\rho \in \hat{G}} m_\sigma^\rho (\rho, V_\rho)]U^\dagger$$

where U is a unitary matrix and the integers m_σ^ρ denote the number of copies of the irreducible (ρ, V_ρ) in the representation (σ, V). At a high level, irreducible representations are the ‘nucleic acids’ that are the building blocks of generic representations.

1. Schur's Lemma

Schur's lemma is one of the fundamental results in representation theory [15]. Let G be a compact group. Let (ρ, V) and (σ, W) be irreducible representations of G. Then, Schur's lemma states the following: Let $\Phi : V \to W$ be an intertwiner of (ρ, V) and (σ, W). Then, Φ is either zero or the proportional to the identity map. In other words,

$$\text{if } \forall g \in G,\ \ \Phi\rho(g) = \sigma(g)\Phi \implies \begin{cases} \Phi \propto \mathbb{I} \text{ if } (\rho, V) = (\sigma, W) \\ \Phi = 0 \text{ if else} \end{cases}$$

Equivalently, if (ρ, V) and (σ, W) are irreducible representations, the space of intertwiners of representations satisfies

$$\mathrm{Hom}_G[(\rho, V), (\sigma, W)] \cong \begin{cases} \mathbb{C} \text{ if } (\rho, V) = (\sigma, W) \\ 0 \text{ if else} \end{cases}$$

A corollary of Schur's lemma is the following: Let (ρ, V) be a irreducible representation of G. Let $M \in \mathbb{C}^{d_\rho \times d_\rho}$ be a matrix. Suppose that

$$\forall g \in G, \quad \rho(g)M = M\rho(g)$$

holds. Then, M is proportional to the identity matrix. The constant of proportionally can be determined by taking traces. Specifically,

$$M = \frac{\mathrm{Tr}[M]}{d_\rho}\mathbb{I}_{d_\rho}$$

a. Extended Shur Lemma Schur's Lemma can be extended to reducible representations. Let (ρ, V_ρ) and (σ, W_ρ) be G representations which decompose into irriducibles as

$$(\rho, V_\rho) = U[\bigoplus_{\tau \in \hat{G}} m_\tau^\rho(\tau, W_\tau)]U^\dagger \quad (\sigma, V_\sigma) = V[\bigoplus_{\tau \in \hat{G}} m_\tau^\sigma(\tau, W_\tau)]V^\dagger$$

where U, V are fixed unitary matrices that diogonlize the ρ and σ representations, respectively. Then, the vector space of intertwiners between (ρ, V_ρ) and (σ, W_σ) has dimension

$$\dim \mathrm{Hom}_G[(\rho, V_\rho), (\sigma, V_\sigma)] = \sum_{\tau \in \hat{G}} m_\tau^\rho m_\tau^\sigma$$

Furthermore, elements of the space $\mathrm{Hom}_G[(\rho, V_\rho), (\sigma, V_\sigma)]$ have block structure. Specifically, any $\Phi \in \mathrm{Hom}_G[(\rho, V_\rho), (\sigma, V_\sigma)]$ can be parameterized in block diagonal form as

$$\Phi = U[\bigoplus_{\tau \in \hat{G}} \Phi^\tau]V^\dagger$$

and each block Φ^τ can be written as

$$\Phi^\tau = \begin{bmatrix} \Phi_{11}^\tau \mathbb{I}_{d_\tau} & \Phi_{12}^\tau \mathbb{I}_{d_\tau} & \dots & \Phi_{1m_\tau^\sigma}^\tau \mathbb{I}_{d_\tau} \\ \Phi_{21}^\tau \mathbb{I}_{d_\tau} & \Phi_{22}^\tau \mathbb{I}_{d_\tau} & \dots & \Phi_{2m_\tau^\sigma}^\tau \mathbb{I}_{d_\tau} \\ \dots & \dots & \dots & \dots \\ \Phi_{m_\tau^\rho 1}^\tau \mathbb{I}_{d_\tau} & \Phi_{m_\tau^\rho 2}^\tau \mathbb{I}_{d_\tau} & \dots & \Phi_{m_\tau^\rho m_\tau^\sigma}^\tau \mathbb{I}_{d_\tau} \end{bmatrix} = \begin{bmatrix} \Phi_{11}^\tau & \Phi_{12}^\tau & \dots & \Phi_{1m_\tau^\sigma}^\tau \\ \Phi_{21}^\tau & \Phi_{22}^\tau & \dots & \Phi_{2m_\tau^\sigma}^\tau \\ \dots & \dots & \dots & \dots \\ \Phi_{m_\tau^\rho 1}^\tau & \Phi_{m_\tau^\rho 2}^\tau & \dots & \Phi_{m_\tau^\rho m_\tau^\sigma}^\tau \end{bmatrix} \otimes \mathbb{I}_{d_\tau}$$

where each $\Phi_{ij}^\tau \in \mathbb{C}$ is a complex constant and $\mathbb{I}_{d_\tau}$ is the identity matrix of the (τ, V_τ) irreducible representation with dimension $d_\tau = \dim(\tau, W_\tau)$.

2. Frobenius–Schur Indicators

Let (ρ, V) be a representation of a compact group G. The complex conjugate representation of (ρ, V) is defined by taking the conjugate over $\mathbb{C}$. The conjugate representation $(\bar{\rho}, \bar{V})$ is defined as the action

$$\forall g \in G, \ \ \forall v \in V, \quad \rho^\dagger(g)v$$

A representation is said to be real if its complex conjugate is similar to a real matrix. A real representation satisfies the constraint

$$\rho(\bar{g}) = U\rho(g)U^{-1}$$

where U is a unitary matrix. Real representations split into two categories. In a proper real representation the matrix U satisfies $U^T = U$. In a pseudo-real representation, the matrix U satisfies the relation $U^T = -U$.

Appendix C: Induced and Restricted Representations

Induced and Restricted representations provide a method for generating representations of a group G given representations of a subgroup H and vice-versa.

1. Restricted Representation

Let $H \subseteq G$. Let (ρ, V) be a representation of G. The restricted representation of (ρ, V) from G to H is denoted as $\mathrm{Res}_H^G[(\rho, V)]$. Intuitively, $\mathrm{Res}_H^G[(\rho, V)]$ can be viewed as (ρ, V) evaluated on the subgroup H. Specifically,

$$\forall v \in V, \quad \mathrm{Res}_H^G[\rho](h)v = \rho(h)v$$

Note that the restricted representation and the original representation both live on the same vector space V.

2. Induced Representation

The induction representation is a way to construct representations of a larger group G out of representations of a subgroup $H \subseteq G$. Let (ρ, V) be a representation of H. The induced representation of (ρ, V) from H to G is denoted as $\mathrm{Ind}_H^G[(\rho, V)]$. Define the space of functions

$$\mathcal{F} = \{ \ f \ \mid \ f : G \to V, \ \ \forall h \in H, \ \ f(gh) = \rho(h^{-1})f(g) \ \}$$

Then the induced representation is defined as $(\pi, \mathcal{F}) = \mathrm{Ind}_H^G[(\rho, V)]$ where the induced action π acts on the function space $\mathcal{F}$ via

$$\forall g, g' \in G, \ \ \forall f \in \mathcal{F} \quad (\pi(g) \cdot f)(g') = f(g^{-1}g')$$

$$\begin{tikzcd}
(\rho, V) \arrow[r, "\Phi_\rho"] \arrow[dr, "\Psi"'] & \mathrm{Ind}_H^G(\rho, V) \arrow[d, "\Psi^\uparrow"] \\
& (\sigma, W)
\end{tikzcd}$$

FIG. 3: Commutative Diagram for Uniqueness Property of Induced Representations.

The induced representation is essentially unique. Specifically, a standard result in group theory establishes the following universal property of induced representations, as stated in [3]:

Theorem 1. *Let $H \subseteq G$. Let (ρ, V) be any H-representation. Let $\mathit{Ind}_H^G(\rho, V)$ be the induced representation of (ρ, V) from H to G. Then, there exists a unique H-equivariant linear map $\Phi_\rho : V \to \mathit{Ind}_H^G V$ such that for any G-representation (σ, W) and any H-equivariant linear map $\Psi : V \to W$, there is a unique G-equivariant map $\Psi^\uparrow : \mathit{Ind}_H^G V \to W$ such that the diagram 3 is commutative.*

Let (ρ, V) be a H-representation and let (σ, W) be a G-representation. Let $\Psi : V \to W$ where Ψ is an intertwiner of a the H-representation and the restriction of the G-representation to an H-representation so that

$$\forall h \in H, \quad \Psi\rho(h) = \mathrm{Res}_H^G[\sigma](h)\Psi$$

so that $\Psi \in \mathrm{Hom}_H[(\rho, V), \mathrm{Res}_H^G(\sigma, W)]$. The universal property of the induced representation allows us to write any such Ψ in a canonical form.

a. Induced Representation for Finite Groups

There is also an equivalent definition of the induced representation for finite groups that is slightly more intuitive [4]. Let G be a group and let $H \subseteq G$. The set of left cosets of G/H form a partition of G so that

$$G = \bigcup_{i=1}^{|G/H|} g_i H$$

where $\{g_i\}_{i=1}^{|G/H|}$ are a set of representatives of each unique left coset. Note that the choice of left coset representatives is not unique. Now, left multiplication by the element $g \in G$ is an automorphism of G. Left multiplication by $g \in G$ must thus permute left cosets of G/H so that

$$\forall g \in G, \quad g \cdot g_i = g_{j_g(i)} h_i(g)$$

where $j_g : \{1, 2, ..., m\} \to \{1, 2, ..., m\} \in S_m$ is a permutation of left coset representatives. The $h_i(g) \in H$ is an element of subgroup H. The map $j_g(i)$ and group element $h_i(g) \in H$ satisfy a compositional property. Specifically, we have that

$$\forall g, g' \in G, \quad j_{g'} \circ j_g = j_{g'g}, \quad h_i(g'g) = h_{j_g(i)}(g') \cdot h_i(g)$$

which can be seen by acting on the left cosets with g followed by g' versus acting on the left cosets with $g'g$. Note that

$$e \cdot g_i = g_i \cdot e = g_{j_e(i)} h_i(e)$$

holds so $j_e = e$ and $h_i(e) = e$ holds. Now, let (ρ, V) be a representation of the group H. Let us define the vector space W as

$$W = \bigoplus_{i=1}^{|G/H|} g_i V_{(i)}$$

where the (standard albeit somewhat confusing) notation $g_i V_{(i)}$ denotes an independent copy of the vector space V. This notation is simply a labeling and all copies of $g_i V_{(i)}^H$ are isomorphic to V^H,

$$V \cong g_1 V_1 \cong g_2 V_2 \cong ... \cong g_{|G/H|} V_{|G/H|}$$

so that the space $W \cong \bigoplus_{i=1}^{|G/H|} V$ is just $|G/H|$ independent copies of V. The induced representation lives on this vector space, $(\pi, W) = \mathrm{Ind}_H^G[(\rho, V)]$. The induced action $\pi = \mathrm{Ind}_H^G \rho$ acts on the vector space W via

$$\forall g \in G, \ \forall w = \sum_{i=1}^{|G/H|} g_i v_i \in W, \quad \pi(g) \cdot w = \sum_{i=1}^{|G/H|} \sigma(h_i(g)) v_{j_g(i)} \in W$$

where $v_i \in V_{(i)}$ is in the i-th independent copy of the vector space V. Using the compositional property of j_g and $h_i(g)$, it is easy to see that this is a valid group action so that $(\pi, W) = \mathrm{Ind}_H^G[(\rho, V)]$ is a valid representation. Note that the induced action π acts on the vector space W by permuting and left action by the H-representation $\rho(h)$.

Appendix D: Projective Representations

In quantum mechanics, states $|\Psi\rangle$ are defined only up to a global phase $|\Psi\rangle \sim \exp(i\phi)|\Psi\rangle$. Let V be a vector space over a field $\mathcal{F}$. A projective representation $(\rho, V, \mathcal{F})$ is consists of V, $\mathcal{F}$ and a group homeomorpism $\rho : G \to V/\mathcal{F}$

$$\forall g \in G, \quad \rho(g)\rho(g') = \rho(gg') + F$$

For the case $\mathcal{F} = \mathbb{C}$, which is the quantum mechanical case, the homeomorpism constraint can be written as

$$\forall g \in G, \quad \rho(gg') = \exp(i\alpha(g, g'))\rho(g)\rho(g')$$

where $\alpha : G \times G \to \mathbb{R}$ is called the Schur multiplier or Schur Phase. The phase is required to satisfy the cocycle condition

$$\forall g, h, k \in G, \quad \alpha(g, h)\alpha(gh, k) = \alpha(g, hk)\alpha(h, k)$$

Furthermore, the Schur-phase satisfies $\exp(i\alpha(g, e)) = 1 = \exp(i\alpha(e, g))$.

- discuss central extensions, covering spaces
- Important point: when is central extension unique?

A $U(1)$ central extension $\tilde{G}$ of the group G is a short exact sequence of groups

$$1 \to U(1) \to \tilde{G} \to G \to 1$$

where $Z(G)$ is the center of G. For compact (semi-simple?) Lie groups, the projective representations of G are in one-to-one correspondence with the unitary representations of the central extension $\tilde{G}$. This procedure is called de-projectivization.

De-Projectivization	
Group	Central Extension
$O(d)$	$\mathrm{Pin}_{\pm}(d)$
$U(d)$	$U(d)$

TABLE I: Common Groups and their central extensions.

The most famous example of projective representations is the $SO(3)$ in quantum mechanics. The central extension of $SO(3)$ is $SU(2)$ and projective representations of $SO(3)$ are in one-to-one correspondence with unitary representations of $SU(2)$. See [**?**] for a good review of central extensions for some common physics applications.

1. Wigner's Theorem

Wigner's theorem states that any projective representation on a Hilbert space $\mathcal{H}$ can be written as either a unitary or antiunitary representation. Specifically, if $(\rho, \mathcal{H})$ is a representation of G, then

$$\forall g \in G, \quad \rho(g) = U(g) \text{ or } \rho(g) = U(g)K$$

where $U(g)$ is a unitary matrix and K is the anti-unitary operator defined via the equation

$$Ki = -iK$$

i.e. K takes the complex conjugate of representations. Time reversal symmetry is probably the most important example of a anti-unitary representation. Specifically, under the time reversal transformation $\hat{T}$ the position operator is unchanged,

$$\hat{T}\hat{x}\hat{T}^{-1} = \hat{x}$$

Similarly, the momentum operator must transform as

$$\hat{T}\hat{p}\hat{T}^{-1} = -\hat{p}$$

Then, using the canonical commutation relations $[\hat{x}, \hat{p}] = i$, we have that

$$\hat{T}[\hat{x}, \hat{p}]\hat{T}^{-1} = [\hat{T}\hat{x}\hat{T}^{-1}, \hat{T}\hat{p}\hat{T}^{-1}] = -[\hat{x}, \hat{p}]$$

This can only hold if

$$\hat{T}i\hat{T}^{-1} = -i$$

holds. Thus, the time reversal operator $\hat{T}$ must be anti-unitary.