EE3731C Signal Analytics

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1 Discrete Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

1.1 Properties

$$\mathscr{F}(ax_1[n] + bx_2[n]) = aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

$$\mathscr{F}(x[n-n_0]) = e^{-j\omega n_0}X(e^{j\omega})$$

$$\mathscr{F}(e^{j\omega_0 n}x[n]) = X(e^{j(\omega-\omega_0)})$$

$$\mathscr{F}(x[n]*h[n]) = H(e^{j\omega})X(e^{j\omega})$$

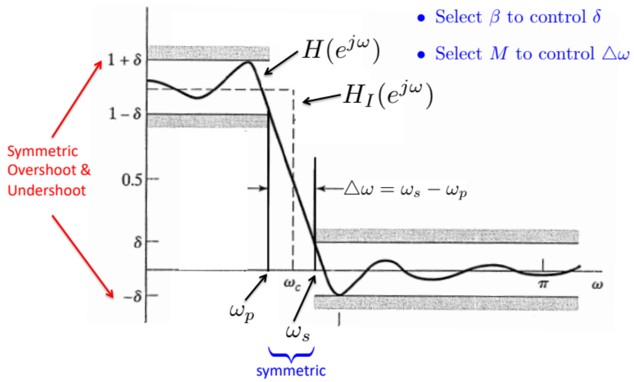
$$\mathscr{F}(x[n]w[n]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})\,d\theta$$

$$\mathscr{F}(u[n]) = U(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}$$

2 Filters

Causal filters: $h[n] = 0$ for $n < 0$
Stable filters: $\sum_n |h[n]| < \infty$
For linear phase, windowed filter is causal and symmetric about $M/2$, which has delay $M/2$
Type 1 filter has even M
Type 2 filter has odd M, not suitable for highpass or bandstop

2.1 Kaiser Window



$$w[n] = I_0\left(\beta\left(1-\left[\frac{n-M/2}{M/2}\right]^2\right)^{1/2}\right)/I_0(\beta)$$

Select β to control δ overshoot and undershoot
Select M to control $\Delta\omega$ (transition width)
Given δ and $\Delta\omega = \omega_s - \omega_p$,

$$A = -20\log_{10}\delta$$

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0.0, & A < 21 \end{cases}$$

$$M = \frac{A-8}{2.285\Delta\omega}$$

3 Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} \quad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n} \quad 0 \le n \le N-1$$

3.1 Convolution using FFT

Pad $x[n], y[n]$ to $x_p[n], y_p[n]$ with length $2N-1$
Compute $X_p[k], Y_p[k]$ using FFT
Compute $Z_p[k] = X_p[k]Y_p[k]$
Compute $z[n] = IFFT(Z_p)$

4 Wavelet Transform

$\downarrow$ freq should use $\uparrow$ time window, and be $\downarrow$ localized in time but $\uparrow$ localized in freq
$\uparrow$ freq can use $\downarrow$ time window, and be $\uparrow$ localized in time and $\downarrow$ localized in freq

4.1 Haar Wavelet Transform

For $*$ operations, linear convolution and drop last term
$(x*HP)\downarrow 2, HP = \sqrt{2}[-1/2, 1/2]$
To downsample by 2, keep odd samples only
$(x*LP)\downarrow 2$ and repeat, $LP = \sqrt{2}[1/2, 1/2]$

4.2 Inverse Haar Wavelet Transform

$\uparrow 2 * HP_0 + \uparrow 2 * LP_0$,
$LP_0 = \sqrt{2}[1/2, 1/2], HP_0 = \sqrt{2}[1/2, -1/2]$
To upsample by 2, insert 0 at odd indices

4.3 Overcomplete Transform

Circular shift k times before applying DWT
To invert, IDWT and invert the shift and average

5 Statistical Signal Processing

5.1 Probability

$$p(x|y) = \frac{p(x)p(y|x)}{\int p(x)p(y|x)dx}$$

$p(x|y)$ is the posterior, $p(x)$ is the prior, $p(y|x)$ is the likelihood, $p(y) = \int p(x)p(y|x)dx$ is the evidence.

$$Cov(x,y) = E[(x-\mu_x)(y-\mu_y)] = E(xy) - E(x)E(y)$$

Maximum-A-Posteriori (MAP) Estimate:

$$x^* = \arg\max_x p(x|y)$$

Maximum Likelihood (ML) Estimate:

$$x_{ML} = \arg\max_x p(y|x)$$

Minimum Mean Square Error (MMSE) Estimate:

$$x_{MMSE} = \arg\min_{\hat{x}} E_{p(x|y)}(x-\hat{x})^2 = E_{p(x|y)}(x)$$

5.2 Random Processes

Markov Process: future and past are conditionally independent given present

$$p(x_1, ..., x_{n-1}, x_{n+1}, ..., x_N|x_n) =$$
$$p(x_1, ..., x_{n-1}|x_n)p(x_{n+1}, ..., x_N|x_n)$$

Transition probability matrix. $T_{ij} = p(x_{n+1} = s_j|x_n = s_i)$ is the probability of moving from state i to state j. Let π_n be a row vector representing probabilities of being in a given state, then

$$\pi_{n+1} = \pi_n T$$

Stationary distribution is where $\pi^* = \pi^* T$ (left eigenvector of T with eigenvalue 1)
Fundamental theorem of markov chains: If there is n_0 such that $T^n(i,j) > 0$ for all i, j and $n > n_0$, then markov chain has unique stationary distribution π^*.

5.3 Markov Chain Monte Carlo (MCMC)

5.3.1 Metropolis Algorithm

- Begin with $x = x_0$
- Sample new x' using proposal distribution $q(x'|x)$ (constraint $q(x'|x) = q(x|x')$)
- If $\pi(x') \geq \pi(x)$ then keep x', otherwise replace x with x' with probability $\frac{\pi(x')}{\pi(x)}$
- Then $x' \sim \pi(X)$

6 Pattern Recognition

6.1 Dimensionality Reduction

Principal component analysis
Assumes data lies on a linear subspace

- N data points, each of dimension D
- Subtract mean from each data point
- For the $D \times N$ matrix, $X = [\bar{x}^{(1)}, ..., \bar{x}^{(N)}]$
- Compute sample covariance matrix $\Sigma = \frac{1}{N} X X^T$
- Find top K eigenvalues and corresponding eigenvalues, $v_1, ..., v_K$, which will be used as the basis for the new coordinates

6.2 Non-parametric Density Estimation

6.2.1 Parzen Window

Fix V volume, estimate k

$$\phi(u) = \begin{cases} 1 & |u_j| \leq 0.5, j = 1, ..., d \\ 0 & \text{otherwise} \end{cases}$$

$$k(x) = \sum_i \phi(\frac{x - x_i}{h})$$

$$p(x) = \frac{k(x)/N}{V} = \frac{\sum_i \phi(\frac{x-x_i}{h})/N}{h^d} = \frac{1}{N}\sum_i \frac{1}{h^d}\phi(\frac{x - x_i}{h})$$

Convergence conditions:

- $\sup_u \phi(u) < \infty$
- $\lim_{||u|| \to \infty} \phi(u) \prod_{i=1}^d u_i = 0$
- $\lim_{N\to\infty} V_N = 0$
- $\lim_{N\to\infty} N V_N = \infty$

To set window size h, can use cross-validation

6.2.2 K-nearest Neighbours

Fix k (number of samples), determine V, so grow volume around x until $k_N(x)$ samples
If $p(x)$ is high, then volume will be small, if $p(x)$ is low, then volume will be big
$p(y = c|x)$ is fraction of K neighbours that are from class c

7 Extras

$$e^{jx} = \cos x + j \sin x$$
$$e^{jx} + e^{-jx} = 2\cos x$$
$$e^{jx} - e^{-jx} = j2 \sin x$$
$$\sum_{k=0}^{n-1} ar^k = a\left(\frac{1 - r^n}{1 - r}\right)$$