EE4302 Advanced Control Systems

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1 State Space Representation

$$\dot{x} = Fx + Gu \quad y = Hx + Ju$$

Consider $\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_1\ddot{u} + b_2\dot{u} + b_3 u$

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

1.1 Method 1 (Controller Can. Form)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \left[\begin{array}{c|cc} 0 & 1 & 0\\ 0 & 0 & 1\\ \hline -a_3 & -a_2 & -a_1\end{array}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u$$

$$y = \begin{bmatrix}b_3 & b_2 & b_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

To get this formulation, take $\frac{\xi(s)}{U(s)} = \frac{1}{s^3+a_1s^2+a_2s+a_3}$ so that $\dddot{\xi} + a_1\ddot{\xi} + a_2\dot{\xi} + a_3\xi = u(t)$ and $y(t) = b_1\ddot{\xi} + b_2\dot{\xi} + b_3\xi$ and start with $x_1 = \xi, x_2 = \dot{\xi}, ...$

1.2 Method 2 (Observer Canonical Form)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \left[\begin{array}{c|cc} -a_1 & 1 & 0\\ -a_2 & 0 & 1\\ \hline -a_3 & 0 & 0\end{array}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

To get this formulation, let $\iiint \dddot{y} = \iiint(-a_1\ddot{y} + b_1\ddot{w}) + \iiint(-a_2\dot{y} + b_2\dot{u}) + \iiint(-a_3 y + b_3 u)$ so that $y(t) = \int(-a_1 y + b_1 u) + \iint(-a_2 y + b_2 u) + \iiint(-a_3 y + b_3 u) = \int(-a_1 y + b_1 u + \int(-a_2 y + b_2 u + \int(-a_3 y + b_3 u)))$ and set $x_1 = y, \dot{x}_1 = -a_1 y + b_1 u + x_2, x_2 = \int ....$

1.3 State Transformations

Suppose $\dot{p} = Fp + Gu, y = Hp + Ju$ and $p = Tx$,

$$\dot{x} = (T^{-1}FT)x + (T^{-1}G)u$$

$$y = (HT)x + Ju$$
$$T = \mathcal{C}_2\mathcal{C}_1^{-1}$$

1.4 Transfer Function

$$Y(s) = \left[H\left\{sI - F\right\}^{-1}G + J\right]U(s)$$

1.5 Direct Poles and Zeroes

Poles of the system are eigenvalues of F,

$$\det\{\lambda_i I - F\} = 0$$

Zeroes of the system are

$$\det\left[\begin{array}{c|c} sI - F & -G\\ \hline H & J\end{array}\right] = 0$$

2 State Feedback

Suppose we use state feedback $u = -kx$,

$$\dot{x} = Fx + Gu = [F - Gk]x$$

Then closed-loop poles are given by

$$\det\left[sI - \{F - Gk\}\right] = 0$$

Let our desired closed-loop poles be given by

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + ... + \alpha_n$$

We can expand the determinant expression and equate to our desired closed-loop pole equation, or used closed form solution:

$$k = \begin{bmatrix}0 & 0 & ... & 0 & 1\end{bmatrix}\mathcal{C}^{-1}\alpha_c(F)$$
$$\mathcal{C} = \begin{bmatrix}G & FG & F^2G & ... & F^{n-1}G\end{bmatrix}$$
$$\alpha_c(F) = F^n + \alpha_1 F^{n-1} + ... + \alpha_n I$$

Controllability condition is thus that $\mathcal{C}$ is non-singular.

2.1 Selection of Pole Locations

Use standard tables or use LQR

3 Changing Set-Point

3.1 Include term in control U

$$u = -kx + k_s r, \quad y = Hx$$
$$\dot{x} = Fx + Gu = [F - Gk]x + Gk_s r$$
$$\frac{Y(s)}{R(s)} = H[sI - (F - Gk)]^{-1}Gk_s$$
$$\lim_{s\to 0} H[sI - (F - Gk)]^{-1}Gk_s = 1$$
$$k_s = \frac{1}{\left(H[sI - (F - Gk)]^{-1}G\right)\Big|_{s=0}}$$

3.2 Augmentation with Error Term

Let $\dot{x}_I = y - r = Hx - r$

$$\dot{\bar{x}} = \begin{bmatrix}\dot{x}\\ \dot{x}_I\end{bmatrix} = \begin{bmatrix}F & 0\\ H & 0\end{bmatrix}\begin{bmatrix}x\\ x_I\end{bmatrix} + \begin{bmatrix}G\\0\end{bmatrix}u + \begin{bmatrix}0\\-1\end{bmatrix}r$$

$$= \bar{F}\bar{x} + \bar{G}\bar{u} + \begin{bmatrix}0\\-1\end{bmatrix}r, \quad \bar{u} = -\bar{k}\bar{x}$$

$$y = \begin{bmatrix}H & 0\end{bmatrix}\begin{bmatrix}x\\ x_I\end{bmatrix} = \bar{H}\bar{x}$$

4 Nonlinear Systems

4.1 Root Locus of Nonlinear System

Useful if the nonlinearity has no dynamics (memoryless) and can be approximated as a gain that varies in the size of its input signal

4.2 Phase Plane Analysis

Equilibrium points are where system states can stay forever.

4.2.1 Sketching

To sketch the Phase portrait, solve differential equations for x as a function of t, or use

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

(method of isoclines)
A closed curve is a limit cycle.

4.2.2 Time from a phase portrait

- $\dot{x} = \frac{\Delta x}{\Delta t} \implies \Delta t \approx \frac{\Delta x}{\dot{x}}$ break into small Δx segments
- $\int dt = \int \frac{1}{\dot{x}} dx$ so plot phase portrait of x and $\frac{1}{\dot{x}}$ then find area under curve

4.3 Describing Function Analysis

Equivalent gain, but frequency dependent.
Output of nonlinear component can be approximated as

$$c(t) \approx a_1 \cos(\omega t) + b_1 \sin(\omega t) = M \sin(\omega t + \phi)$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} c(t) \cos(\omega t) d(\omega t)$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} c(t) \sin(\omega t) d(\omega t)$$

$$M = \sqrt{a_1^2 + b_1^2} \quad \phi = \arctan\left(\frac{a_1}{b_1}\right)$$

$$N(A, \omega) = \frac{M\angle\phi}{A\angle 0} - \frac{1}{A}(b_1 + ja_1)$$

Limit cycling is obtained when $G(j\omega)N(A,\omega) = -1$

4.4 Linearization

Select operating points, and at each operating point, make a linear approximation to plant dynamics and design a linear controller.

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial g}{\partial u} \\ \frac{\partial h}{\partial u} \end{bmatrix} \Delta u = F\Delta x + G\Delta u$$

4.5 Sliding Control

Problem formulation is

$$\frac{d^n y}{dt^n} = f_1(y, \frac{dy}{dt}, ..., \frac{d^{n-1}y}{dt^{n-1}}) + g_1(y, \frac{dy}{dt}, ..., \frac{d^{n-1}y}{dt^{n-1}})u$$

$$x = \begin{bmatrix} \frac{d^{n-1}y}{dt^{n-1}} & ... & \frac{dy}{dt} & y \end{bmatrix}^T$$

$$\dot{x} = \begin{bmatrix} f_1(x) \\ x_1 \\ ... \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} g_1(x) \\ 0 \\ ... \\ 0 \end{bmatrix} u = f(x) + g(x)u$$

Define the switching surface

$$\sigma(x) = p^T x = 0$$

which is stable if $P(s) = p_1 s^{n-1} + p_2 s^{n-2} + ... + p_n$ has all roots in the left-half plane.
Lyapunov function $V(x) = \frac{\sigma^2(x)}{2}$

$$\frac{dV}{dt} = \sigma(x)\dot{\sigma}(x) = \sigma(x) p^T \dot{x} = \sigma(x)(p^T f(x) + p^T g(x) u(t))$$

$$u(t) = -\frac{p^T f}{p^T g} - \frac{\mu}{p^T g} \mathrm{sign}(\sigma(x))$$

$$\frac{dV}{dt} = -\mu\sigma(x)\mathrm{sign}(\sigma(x)) = -\mu|\sigma(x)| < 0$$

The switching surface is reached in approximately

$$t_\rho = \frac{\sigma_0}{\mu}$$

$\mathrm{sat}(\sigma, \epsilon)$, which rises linearly from -1 to 1 between $[-\epsilon, \epsilon]$ can be used in place of sign to prevent chattering.

5 Extras

5.1 Laplace Transform

$$\mathcal{L}(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

f	$\mathcal{L}(f)$	
$u(t)$	$\frac{1}{s}$	unit step
$\delta(t)$	1	delta function
$\sin(bt)$	$\frac{b}{s^2+b^2}$	sine
$\cos(bt)$	$\frac{s}{s^2+b^2}$	cosine
$f'(t)$	$sF(s) - f(0^-)$	differentiation
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	integration
$tf(t)$	$-F'(s)$	multiplying by t
$f(t-t_0)$	$e^{-st_0}F(s)$	time shifting
$e^{-at}f(t)$	$F(s+a)$	frequency shifting
$f(at)$	$\frac{1}{\lvert a \rvert}F(\frac{s}{a})$	time scaling

5.2 Mathematics Identities

- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$