EE3331C Feedback Control Systems

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1 Systems

1.1 Linear Approximation

Linear model for a small deviation is obtained by taking truncated 1st order Taylor Series expansion about the operating point. Define $y = y_0 + \delta y$

$$f(y) - f(y_0) \approx \frac{df}{dy}|_{y=y_0}(y - y_0)$$

1.2 System Types

- **Static** $y(t)$ depends only on $u(t)$
- **Dynamic** depends on past or future values or $u(t)$
- **Causal** only depends on past values

2 Laplace Transform

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

f	$\mathcal{L}(f)$	
$u(t)$	$\frac{1}{s}$	unit step
$\delta(t)$	1	delta function
$\sin(bt)$	$\frac{b}{s^2+b^2}$	sine
$\cos(bt)$	$\frac{s}{s^2+b^2}$	cosine
$f'(t)$	$sF(s) - f(0^-)$	differentiation
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	integration
$tf(t)$	$-F'(s)$	multiplying by t
$f(t-t_0)$	$e^{-st_0}F(s)$	time shifting
$e^{-at}f(t)$	$F(s+a)$	frequency shifting
$f(at)$	$\frac{1}{\lvert a\rvert}F(\frac{s}{a})$	time scaling

- **Final value theorem** $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ on the condition that the system is stable.
- **Initial value theorem** $f(0^+) = \lim_{s\to\infty} sF(s)$
- Convolution in the time domain is multiplication in the laplace domain.
- System transfer function = laplace transform of impulse response

3 Impulse Responses

3.1 First order transfer function $\frac{1}{s+\sigma}$ or $\frac{1}{\tau s+1}$

- Time constant $\tau = \frac{1}{\alpha}$ is the time when the impulse response is $\frac{1}{e} = 0.368$ of initial value
- Impulse response of $G(s) = \frac{1}{s+\sigma}$ is $g(t) = e^{-\sigma t}$

3.2 Second order transfer function $\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

- Undamped natural frequency ω_n
- Damping ratio ζ: 0 to 1 is underdamped, > 1 is overdamped (no oscillations like first order system), 1 is critically damped
- Poles at $s = -\sigma \pm j\omega_d$
- $\sigma = \zeta\omega_n$
- $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- Impulse response is $h(t) = \frac{K\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin(\omega_d t)$
- Step response is $y_s(t) = K - \frac{K}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin(\omega_d t + \cos^{-1}\zeta)$
- $\zeta = \sin(\theta)$ where θ is the angle from the imaginary axis. Real part of the pole determines the exponential envelope. Imaginary part of the pole determined the frequency of oscillation.
- An additional zero adds a derivative of the response scaled by $\frac{1}{z}$

4 System Parameters

- Rise time $t_r \approx \frac{1.8}{\omega_n}$ is the time is takes to rise from 10% to 90% of the set point
- Settling time is the time it takes for the transient to decay $t_s = \frac{4}{\sigma}$ for a 2% transient band and $\frac{3}{\sigma}$ for a 5% tolerance band.
- Overshoot $M_p = Ke^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$
- Overshoot $\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$
- Peak time $t_p = \frac{\pi}{\omega_d}$

5 Feedback

Unity feedback: standard formula is $G_{cl}(s) = \frac{G(s)K(s)}{1+G(s)K(s)}$
$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)K(s)}$
For $r(t) = rt^{n-1}$, $R(s) = \frac{Cr}{s^n}$, $C = (n-1)!$
Let m be the number of integrators in $G(s)K(s)$.

m	$r(t) = r$ $n = 1$	$r(t) = rt$ $n = 2$	$r(t) = rt^2$ $n = 3$
0	$\frac{r}{1+p}$	∞	∞
1	0	$\frac{r}{p}$	∞
2	0	0	$\frac{2r}{p}$
≥ 3	0	0	0

For $m > 0$, if $m > n-1$, $e_{ss} = 0$. If $m = n-1$, $e_{ss} = \frac{Cr}{p}$. If $m < n-1$, $e_{ss} = \infty$.

- Position error constant $K_p = \lim_{s\to 0} G(s)K(s)$
- Velocity error constant $K_v = \lim_{s\to 0} sG(s)K(s)$
- Acceleration error constant $K_a = \lim_{s\to 0} s^2G(s)K(s)$

6 Root Locus

For $G_{cl}(s) = \frac{GK}{1+GK}$, closed loop poles are $1 + GK = 0$. Express closed loop pole equation as $1 + KL(s) = 0$ with K being the parameter of interest. Let $L(s) = \frac{b(s)}{a(s)}$. For small K, closed loop poles are at $a(s) = 0$. For large K, closed loop poles are at $b(s) = 0$ or go to infinity. Use root locus to find conditions for stability.

7 Frequency Response

- Given a sinusoidal input of frequency ω_0
- Amplitude ratio is $|G(j\omega_0)|$
- Phase gain is $\angle G(j\omega_0)$
- Only for **steady-state** response

Plotting out magnitude and phase wrt input frequency gives us the Bode Plot. Polar Plot is the locus of as input frequency is varied from 0 to ∞.

7.1 Bode Plot of s^n

Magnitude plot is straight line with gradient of $n \times 20$ dB/decade. Phase plot is horizontal line $\phi = n \times 90°$.

7.2 Bode Plot of a zero $\tau s + 1$

Magnitude plot is 0 dB/decade at low frequencies.
$\omega = \frac{1}{\tau}$ is the break point where magnitude increases at 20 dB/decade. Phase starts at $0°$, at $\frac{0.1}{\tau}$ rises to $45°$ at $\frac{1}{\tau}$ then to $90°$ at $\frac{10}{\tau}$.

7.3 Bode Plot of pole $\frac{1}{\tau s+1}$

Magnitude plot is 0 dB/decade at low frequencies.
$\omega = \frac{1}{\tau}$ is the break point where magnitude changes by -20 dB/decade. Phase starts at $0°$, at $\frac{0.1}{\tau}$ rises to $-45°$ at $\frac{1}{\tau}$ then to $-90°$ at $\frac{10}{\tau}$.

7.4 Bode Plot of second order pole

Behaves like two first order poles at ω_n with resonance occurring at $\omega_r = \omega_n\sqrt{1-2\zeta^2}$. Resonance peak $M_r = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

7.5 Effect of transport lag e^{-st_d}

Contributes a phase of $-\omega t_d$ rad. Does not affect magnitude plot.

7.6 General characteristics

- High frequency phase determined by pole excess and transport lag
- Low frequency magnitude shows presence of integrators

7.7 Polar plots

For low frequency check whether it approaches from infinity or from a point. If approaching from infinity, check which axis it comes from and which side of the axis. For high frequency, check what direction it approaches zero from. Transport lag leads to spirals.

8 Nyquist Stability

Generate the map of $G(s)K(s)$ along the D-contour.

$$Z = N + P$$

- Number of closed loop poles in RHP, Z
- Number of clockwise encirclements of -1 point, N
- Number of open loop poles in RHP
- Nyquist plot will always be symmetric about the real axis
- Modify to exclude poles or zeros on the imaginary axis

9 Stability Margins

9.1 Gain Margin

Gain Margin is the factor by which gain can be raised before instability results.

$$\text{GM} = k = \frac{1}{|L(j\omega_{cp})|}, \angle L(j\omega_{cp}) = -180°$$

where ω_{cp} is the phase crossover frequency.
To read off the bode plot, find where the phase plot crosses the $-180°$ point and look for the corresponding magnitude. Gain margin is negative of that value in dB.

9.2 Phase Margin

Phase Margin is the amount of phase lag $L(j\omega)$ can tolerate before instability results.

$$\text{PM} = 180° + \angle L(j\omega_{cg}), |L(j\omega_{cg})| = 1$$

where ω_{cg} is the gain crossover frequency.
To read off the bode plot, find where the gain plot crosses the 0 dB point and look for the corresponding phase. The phase margin is how much the phase is above $-180°$ by.

10 Dynamic Compensation

10.1 Bode's Gain-Phase Relationship

For system without poles/zeros in RHP, if slope of magnitude plot is constant for a decade before and after with slope n, then $\angle G(j\omega) \approx n \times 90°$.
Aim to control the slope of the magnitude plot of open loop transfer function around the gain crossover frequency.

10.2 PD Compensation

PD controller $K(T_D s + 1)$. Issue is that high frequency signals are amplified.

10.3 Lead Compensation

$$D_c(s) = K\frac{T_D s + 1}{\alpha T_D s + 1}, \alpha < 1$$

Lead compensation works by adding phase to the system.

- Estimate $\alpha = \frac{1-\sin\phi}{1+\sin\phi}$ where ϕ is the amount of phase to be added
- Find ω_{cg} such that $|KD_c(j\omega_{cg})G(j\omega_{cg})| = 1$
- $|D_c(s)|$ can be estimated as $\frac{1}{\sqrt{\alpha}}$
- Estimate $T_D = \frac{1}{w_{cg}\sqrt{\alpha}}$

10.4 Lag Compensation

$$D_c(s) = K\frac{T_l s + 1}{\alpha T_l s + 1}, \alpha > 1$$

Lag compensator works by moving the gain crossover frequency to the left.

- First find where desired gain crossover frequency ω should be, where $\angle KG(j\omega) = \text{PM} - 180°$
- Find $|D_c(j\omega)|$ so that $|KD_c(j\omega)G(j\omega)| = 1$
- Estimate $|D_c(j\omega)| \approx \frac{1}{\alpha}$
- Set $T_l = \frac{10}{\omega}$

10.5 Prefilter

A prefilter may be added in front of the closed loop system to eliminate the zero of the closed loop transfer function. This is usually done for lead compensation or PI compensator.

11 Mathematics Identities

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$
- $\cos(A+B) = \cos A\cos B - \sin A\sin B$
- $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$
- $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$