# MA2104 Multivariable Calculus

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## 1 Vectors

$$a \cdot (b+c) = a \cdot b + a \cdot c, a \cdot b = |a||b|\cos\theta$$

$$\text{proj}_a b = \frac{a \cdot b}{a \cdot a} a = (\hat{a} \cdot b)\hat{a}$$

$$a \times b = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = -b \times a \quad |a \times b| = |a||b|\sin\theta$$

$$a \times (b+c) = a \times b + a \times c \quad (a+b) \times c = a \times c + b \times c$$

Area of parallelogram is given by $|a \times b|$
Volume of parallelepiped is $|a \cdot (b \times c)|$

## 2 Lines

Parametric eqn $r = r_0 + tv$, direction $v$
Symmetric eqn $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$
If (WLOG) $c = 0$, $\frac{x-x_0}{a} = \frac{y-y_0}{b}, y = y_0$
Lines skew if non-$\parallel$ and non-intersecting
Plane is $n \cdot (r - r_0) = 0 \implies n \cdot r = n \cdot r_0$

## 3 Vector Functions (Curves)

$$r(t) = \langle x(t), y(t), z(t) \rangle \quad r'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\frac{d}{dt}(r(t) \cdot s(t)) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$$

$$\frac{d}{dt}(r(t) \times s(t)) = r'(t) \times s(t) + r(t) \times s'(t)$$

Arc length is $\int_a^b |r'(t)| dt$

## 4 Functions of Two Variables

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \to f(x, y)$$

Visualize using level curves
Cylinder has a plane P such that all planes parallel to P intersect surface in the same curve
Quadric is $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

| | |
|---|---|
| Elliptic paraboloid | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ |
| Hyperbolic paraboloid | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ |
| Ellipsoid | $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ |
| Elliptic cone | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ |
| Hyperboloid of one sheet | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ |
| Hyperboloid of two sheets | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ |

## 5 Limits

$$\lim_{(X,y)\to(a,b)} f(x,y) = L$$

if for any number $\epsilon > 0$, $\exists \delta > 0$ such that

$$|f(x,y) - L| < \epsilon \quad \forall\, 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

### 5.1 Non existence

Can show that limit does not exist by demonstrating different values when approaching from different lines.

### 5.2 Existence

$$\lim_{(x,y)\to(a,b)} (f(x,y) \pm g(x,y)) = \lim f(x,y) \pm \lim g(x,y)$$

$$\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = (\lim f(x,y))(\lim g(x,y))$$

$$\lim_{\to(a,b)} g(x,y) \neq 0 \implies \lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{\to(a,b)} f(x,y)}{\lim_{\to(a,b)} g(x,y)}$$

### 5.3 Existence (Squeeze)

If $|f(x,y) - L| \leq g(x,y)\ \forall (x,y)$ interior of circle centered at $(a,b)$, except at $(a,b)$, then

$$\lim_{(x,y)\to(a,b)} g(x,y) = 0 \implies \lim_{(x,y)\to(a,b)} f(x,y) = L$$

## 6 Continuity

Function is continuous at the point if the limit is equal to the function value.
$\pm, \cdot, /$ preserve continuity supposing not dividing by 0
Polynomial, trigonometric, exponential, and rational functions are continuous in its domain.

## 7 Partial Derivatives

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)} = f_x(a,b) = g'(a) \text{ where } g(x) = f(x,b)$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$$

Clairaut's Theorem: if $f_{xy}$ and $f_{yx}$ are continuous, then $f_{xy} = f_{yx}$
Heat equation $u_t(x,t) = u_{xx}(x,t)$
Laplace equation $u_{xx}(x,y) + u_{yy}(x,y) = 0$

## 8 Tangent Planes

$$n = \langle -f_x(a,b), -f_y(a,b), 1 \rangle$$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is the linearisation of $f$ at $(a,b)$
$f(x,y)$ is differentiable at $(a,b)$ if $f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

## 9 Chain Rule

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$\frac{d}{dt} f(r(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f(r(t)) \cdot r'(t)$$

## 10 Implicit Differentiation

Treat $z$ as $z(x,y)$ and hold $y$ constant, then take $\frac{\partial}{\partial x}$

## 11 Directional Derivatives

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$

$\nabla f$ points in the direction of increasing $f$.
$\nabla f$ is perpendicular to the level set

## 12 Extrema

Point is a critical point if $\nabla f = 0$ or if one of the partial derivatives does not exist.
Second derivative test $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$. If $D < 0$, saddle point. If $D > 0$, then if $f_{xx} > 0$, local minimum, if $f_{xx} < 0$, local maximum. If $D = 0$, test is inconclusive.
Extreme value theorem: any continuous function $f$ attains it absolute max and min on a closed (contains all boundary points) and bounded set (contained within some large box).

## 13 Lagrange Multipliers

Under the constraint $g(x,y) = c$, $f(x,y)$ achieves local extrema at points satisfying $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$

## 14 Double Integrals

Fubini's theorem

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

if $a, b, c, d$ are constants.
Type-I region $a \le x \le b$, $g_1(x) \le y \le g_2(x)$
Type-II region $c \le y \le d$, $h_1(y) \le x \le h_2(y)$
Change order of integration by sketching domain.

$$\int_D f(x,y)dA = \int_\alpha^\beta \int_{h(\theta)}^{g(\theta)} f(r\cos\theta, r\sin\theta)\ rdrd\theta$$

Spherical coordinates $\rho$ is the distance from origin, $0 \le \theta \le 2\pi$ is angle between projection of P onto xy-plane with positive x-axis, $0 \le \phi \le \pi$ is angle between OP and positive z-axis.

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)}$$

$$f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

## 15 Change of Variables

$$\int_a^b f(x)\ dx = \int_c^d f(g(u))g'(u)\ du = \int_c^d f(x(u))\frac{dx}{du}\ du$$

where $x = g(u)$ and $a = g(c), b = g(d)$

$$\int_{[a,b]} f(x)\ dx = \int_{[c,d]} f(g(u))|g'(u)|\ du = \int_{[c,d]} f(x(u))|\frac{dx}{du}|\ du$$

Let $T$ be a transformation $(x,y) = T(u,v)$ differentiable and injective.

$$\iint_R f(x,y)dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|\ du\ dv$$

## 16 Line Integral

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\|\mathbf{r}'(t)\|dt$$

Let $\mathbf{F} = \langle P, Q, R\rangle$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz$$

If $\mathbf{F} = \nabla f$,

$$\int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

Field is conservative if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ or in 3D case $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

## 17 Green's Theorem

Counterclockwise traversal of $C$ is positive orientation

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

## 18 Surface Integral

$$\iint_S f(x,y,z)dS = \iint_D f(x(u,v), y(u,v), z(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|dA$$

If surface is given by $z = g(x,y)$, then let $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + g(x,y)\mathbf{k}$,

$$\iint_S f(x,y,z)dS =$$

$$\iint_D f(x,y,g(x,y)) \left( \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \right) dA$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS =$$

$$\iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)dA = \iint_D \left( -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

## 19 Divergence Theorem and Gauss' Theorem

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\ dV$$

## 20 Curl and Stoke's Theorem

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Let $S$ be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive orientation.

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$