MA2108 Mathematical Analysis

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1 The Real Numbers

- $\mathbb{N} := \{1, 2, 3, ...\}$
- $\mathbb{Z} := \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- $\mathbb{Q} := \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$
- $\mathbb{R}$ real numbers
- Every nonempty subset S of $\mathbb{N}$ as a least element
- Let $S \subseteq \mathbb{N}$. If $1 \in S$ and $\forall k \in \mathbb{N}$ $(k \in S \Rightarrow k+1 \in S)$, then $S = \mathbb{N}$
- Bernoulli's inequality. For $x \geq -1$, $(1+x)^n \geq 1 + nx$
- AM-GM-HM inequality. $\frac{\sum a_n}{n} \leq (\prod a_n)^{\frac{1}{n}} \leq \frac{n}{\sum \frac{1}{a_n}}$
- Triangle inequality. $|a + b| \leq |a| + |b|$

2 Completeness of Real Numbers

- Upper bound. $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$
- Bounded means bounded above and below
- Maximum. $M \in S$ and $\forall x \in S$ $(M \geq x)$
- Supremum (Least upper bound). M is an upper bound of E and if M' is an upper bound of E then $M' \geq M$.
- Infimum (Greatest lower bound). m is a lower bound of E and if m' is a lower bound of E, then $m' \leq m$.
- Completeness property of $\mathbb{R}$. Every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$
- Archimedean property of $\mathbb{R}$. For any $x \in \mathbb{R}$, there exists $n_x \in \mathbb{N}$ such that $x < n_x$.
- Density theorem. For any real numbers $x < y$, $\exists r \in \mathbb{Q}$ $(x < r < y)$.

3 Sequences

- A sequence is an infinite ordered list of numbers, or a real-valued function $X : \mathbb{N} \to \mathbb{R}$, written as (x_n)
- x is the limit of (x_n) if for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq K$, then $\lim_{n\to\infty} x_n = x$
- Sequence is bounded if $|x_n| \leq M$ for all $n \in \mathbb{N}$
- Every convergent sequence is bounded.
- Squeeze theorem. If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, then $\lim_{n\to\infty} y_n = a$
- $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$
- $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$
- $\lim_{n\to\infty} c^{\frac{1}{n}} = 1, c > 0$
- Monotone is either increasing or decreasing $\geq$ or $\leq$
- Monotone convergence theorem. If (x_n) is monotone and bounded, then (x_n) converges.
- If $(y_k) = (x_{n_k})$ is a subsequence of (x_n), then $n_k \geq k$
- If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x.
- Monotone subsequence theorem. Every sequence has a monotone subsequence.
- Bolzano-Weierstrass theorem. Every bounded sequence has a convergent subsequence.
- If $S(x_n)$ is the set of all subsequential limits of (x_n), then $\limsup x_n := \sup S(x_n)$ and $\liminf x_n := \inf S(x_n)$
- Alternative definition:
  $\limsup x_n = \lim_{n\to\infty}(\sup\{x_k : k \geq n\})$.
- Cauchy sequence. For every $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $n > m \geq K$
- Every convergent sequence is cauchy.
- Every cauchy sequence is convergent.
- A sequence is contractive if there exists $0 < C < 1$ such that $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$
- Every contractive sequence is cauchy.
- Sequence tends to ∞ if for every $M > 0$, there exists $K = K(M) \in \mathbb{N}$ such that $x_n > M$ for all $n \geq K$
- Properly divergent if it tends to ∞ or $-\infty$
- $x_n = o(y_n)$ or $x_n << y_n$ if $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$

4 Infinite Series

- $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty}(\sum_{k=1}^{n} a_k)$
- Geometric series. $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$ if $|r| < 1$
- Telescoping series.
- Harmonic series diverges
- N-th term divergence test. If $\lim_{n\to\infty} a_n \neq 0$ then series diverges.
- Series is eventually non-negative if each term $a_k \geq 0$ eventually.
- Eventually non-negative series converges if and only if the sequence of partial sums is bounded above.
- If $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
- Comparison test. For two eventually non-negative series, if $0 \leq a_k \leq b_k$ for all $k \geq K$, if $\sum b_k$ converges, then $\sum a_k$ converges, and if $\sum a_k$ diverges then $\sum b_k$ diverges.
- Limit comparison test. $\rho = \lim_{n\to\infty} \frac{a_n}{b_n}$, if $\rho > 0$ both series converge or both diverge. If $\rho = 0$, if $\sum b_n$ converges then $\sum a_n$ converges.
- Ratio test. $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, if $\rho < 1$ then the series converges, if $\rho > 1$ then the series diverges, if $\rho = 1$ then no conclusion
- Limit comparison test is applicable to series that look like geometric series or p-series.
- Ratio test is applicable to series that look like geometric series or containing factorials or recursively defined.
- Root test. $\rho = \limsup a_n^{\frac{1}{n}}$ or $\rho = \lim_{n\to\infty} a_n^{\frac{1}{n}}$, if $\rho < 1$ the series converges, if $\rho > 1$ the series diverges, if $\rho = 1$ no conclusion.
- Root test is applicable to series involving n-th powers.
- Alternating series test. For $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and $a_n \geq 0$ and (a_n) is decreasing and $\lim_{n\to\infty} a_n = 0$ then the series is convergent.
- $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
- $\sum a_n$ converges conditionally if $\sum |a_n|$ diverges.

5 Limits of Functions

- $\epsilon - \delta$ definition of a limit. $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in A$ satisfying $|x - c| < \delta$
- Intuition: we can find a region where all points near to c will be mapped to near to L.
- Sequential criterion for limits. For every sequence (x_n) in $A\backslash\{c\}$ satisfying $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} f(x_n) = L$
- To prove the limit does not exist, find a sequence (x_n) such that $x_n \to c$ but the sequence $(f(x_n))$ diverges, or find two sequences $(x_n), (y_n)$ such that both converge to c but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$.
- Squeeze theorem. $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$
- One-sided limit. Let c be a cluster point of $A \cap (c, \infty)$, then L is a right-hand limit of f at c if for any $\epsilon > 0$ there exists $\delta > 0$ such that $x \in A$ and $c < x < c + \delta \Rightarrow |f(x) - L| < \epsilon$, then $\lim_{x\to c^+} f(x) = L$
- Sequential criterion for right-hand limits. For every sequence (x_n) in $A \cap (c, \infty)$ satisfying $x_n \to c$, $\lim_{n\to\infty} f(x_n) = L$
- Squeeze theorem for one-sided limit.
- $f(x)$ tends to ∞ as $x \to c$ if for every $M > 0$, there exists δ such that $x \in A$ and $|x - c| < \delta \Rightarrow f(x) > M$
- Limit at infinity. L is the limit of f as $x \to \infty$ if for any given $\epsilon > 0$, there exists $M = M(\epsilon) > 0$ such that $x \in A$ and $x > M \Rightarrow |f(x) - L| < \epsilon$

6 Continuous Functions

- $\epsilon - \delta$ definition of continuity. A function is said to be continuous at $x = a$ if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, a) > 0$ such that $|f(x) - f(a)| < \epsilon$ for all $x \in A$ satisfying $|x - a| < \delta$.
- f is continuous at $x = a \Leftrightarrow \lim_{x\to a} f(x) = f(a)$
- Sequential criterion for continuity. For every sequence (x_n) in A satisfying $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} f(x_n) = f(a)$
- Composition of continuous functions is continuous.
- Extreme value theorem. A continuous function on a closed bounded interval attains absolute maximum and absolute minimum. $f : [a, b] \to \mathbb{R}$ then there exists $c_1, c_2 \in [a, b]$ such that $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a, b]$
- Intermediate value theorem. continuous function $f : [a, b] \to \mathbb{R}$ then for any number $f(a) < L < f(b)$ there exists $c \in (a, b)$ such that $f(c) = L$
- If I is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is a continuous function then $f(I)$ is an interval.
- Jump of f $= \lim_{x\to c^+} f(x) - \lim_{x\to c^-} f(x)$
- If f is strictly monotone and continuous on I and $J = f(I)$ then its inverse is strictly monotone and continuous on J.
- A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $x, u \in A$ and $|x - u| < \delta \Rightarrow |f(x) - f(u)| < \epsilon$
- Sequential criterion for uniform continuity. For sequences $(x_n), (y_n)$ such that $(x_n - y_n) \to 0$, $(f(x_n) - f(y_n)) \to 0$
- If $f : [a, b] \to \mathbb{R}$ is continuous on a closed bounded interval then f is uniformly continuous on $[a, b]$.

7 Metric Spaces

- Metric $d : S \times S \to \mathbb{R}$ satisfies $d(x, y) \geq 0$, $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$
- Metric space (S, d) is a set S with a metric d.
- Cauchy-Schwarz inequality $|\sum_{i=1}^n a_i b_i| \leq (\sum_{i=1}^n a_i^2)^{\frac{1}{2}} (\sum_{i=1}^n b_i^2)^{\frac{1}{2}}$
- ϵ-neighbourhood is $V_\epsilon(c) := \{x \in S : d(x, c) < \epsilon\}$
- Open set G is an open set in S if for each $x \in G$, there exists a neighbourhood V of x such that $V \subseteq G$
- Arbitrary union of open sets is open, finite intersection of open sets is open
- Empty set and S are open.
- Set is closed if the complement is open in S.
- Empty set and S are closed.
- Arbitrary intersection of closed sets is closed, finite union of open sets is closed.
- Function f from one metric space to another is continuous at a point $c \in A$ if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, c) > 0$ such that $d_2(f(x), f(c)) < \epsilon$ for all $x \in A$ satisfying $d_1(x, c) < \delta$
- Global continuity theorem. f is continuous on A $\Leftrightarrow$ for every open set $G \subseteq S_2$, there exists an open set $H \subseteq S_1$ such that $f^{-1}(G) = A \cap H$
- Sequential criterion for continuity. f is continuous at c $\Leftrightarrow$ for every sequence (x_n) in A satisfying $x_n \to c$, $\lim_{n\to\infty} f(x_n) = f(c)$
- Subset $A \subseteq S$ is bounded if there exists $x_0 \in S$ and $M > 0$ such that $d(x, x_0) \leq M$ for all $x \in A$.
- Subset $A \subseteq S$ is sequentially compact if every sequence in A has a convergent subsequence whose limit is in A.
- If A is sequentially compact, then A is closed and bounded in S.
- If A is sequentially compact in (S_1, d_1), then the image $f(A)$ is sequentially compact in (S_2, d_2) if f is continuous.
- Extreme value theorem. Suppose A is a sequentially compact set and $f : A \to \mathbb{R}$ is a continuous function on A then there exists $x_1, x_2 \in A$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in A$.
- Heine-Borel theorem. $A \subseteq \mathbb{R}^K$ is compact if and only if A is closed and bounded in $\mathbb{R}^K$
- If $F \subseteq S$ is closed, every convergent sequence in F has its limit in F.