CS1231 Discrete Structures

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1 Propositional Logic

- Not $\neg p$, Or $p \vee q$, And $p \wedge q$
- Implies $p \to q \equiv \neg p \vee q$, Iff $p \leftrightarrow q$
- $P \equiv Q$ means two expressions equivalent
- Contrapositive $p \to q \equiv \neg q \to \neg p$
- De morgan's law $\neg(p \vee q) \equiv \neg p \wedge \neg q$

2 Predicate Logic

- Natural numbers start from 0
- For all $\forall$, Exists $\exists$
- $\neg \forall x\ P(x) \leftrightarrow \exists x\ \neg P(x)$, $\neg \exists x\ P(x) \leftrightarrow \forall x\ \neg P(x)$

3 Proofs

- To prove existence, produce a witness
- To prove implication $p \to q$, assume p and show q
- To prove $p \to q$, assume $\neg q$ then prove $\neg p$
- Mathematical Induction

4 Sets

- Roster notation $\{x_1, x_2, ..., x_n\}$
- Set-builder notation $\{x \in U : P(x)\}$
- Replacement notation $\{t(x) : x \in A\}$
- Power set is the set of all subsets
- Union $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- Intersect $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- Complement $A \backslash B$

5 Relations

- Composition $S \circ R = \{(x, z) \in A \times C : (x, y) \in R$ and $(y, z) \in S$ for some $y \in B\}$
- Inverse $R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}$
- A directed graph is a pair (V, D) where V is a set and D is a binary relation on V
- An undirected graph is a pair (V, E) where V is a set and E is a set of elements of the form $\{x, y\}$ for $x, y \in V$
- A loop is an edge from a vertex to itself

6 Equivalences and Orders

- Reflexive $\forall x \in A\ (xRx)$
- Symmetric $\forall x, y \in A\ (xRy \Rightarrow yRx)$
- Transitive $\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$
- An equivalence relation is a relation that is reflexive, symmetric and transitive
- Equivalence class $[x] = \{y \in A : x \sim y\}$
- Partition p if p is a set of nonempty subsets of A, every element of A is in some element of p and if two elements of p have a nonempty intersection, then they are equal
- $A/\sim = \{[x]_\sim : x \in A\}$
- Antisymmetric $\forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$
- Partial order is reflexive, antisymmetric, and transitive
- Total order means every pair of elements is comparable $\forall x, y \in A\ (xRy \vee yRx)$
- Well-ordering principle. Let $b \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}_{\geq b}$. If $S \neq \emptyset$, then S has a smallest element.

7 Functions

- $\forall x \in A\ \exists y \in B\ (x, y) \in f$
- $\forall x \in A\ \forall y_1, y_2 \in B$ $((x, y_1) \in f \wedge (x, y_2) \in f \Rightarrow y_1 = y_2)$
- $f(x)$ is the image of x under f
- $range(f) = \{f(x) : x \in A\}$
- $(g \circ f)(x) = g(f(x))$
- Surjective $\forall y \in B\ \exists x \in A\ y = f(x)$
- Injective $\forall x_1, x_2 \in A\ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$
- Bijective is both injective and surjective
- $g = f^{-1} \Leftrightarrow \forall x \in A\ \forall y \in B\ (g(y) = x \Leftrightarrow y = f(x))$

8 Cardinality

- Pigeonhole principle. If there is an injection from $A \to B$, then $|A| \leq |B|$. If there is a surjection, then $|A| \geq |B|$
- A has the same cardinality as B if there is a bijection $A \to B$
- Set is finite if it has same cardinality as $\{1, 2, ..., n\}$ for some $n \in \mathbb{N}$. A set is infinite if it is not finite.

9 Countable Sets

- A set is countable if it is finite or has the same cardinality as $\mathbb{N}$. A set is uncountable if it is not countable.
- $\mathbb{Z}, \mathbb{N} \times \mathbb{N}, \{0, 1\}^*$ are countable.
- Every infinite set B has a countable infinite subset.
- Suppose $A \subseteq B$, if B is finite that A is finite. If B is countable, then A is countable.
- No set A has the same cardinality as $\mathcal{P}(A)$

10 Counting

- $|A \cup B| = |A| + |B| - |A \cap B|$
- If A, B is finite then $A \times B$ is finite and $|A \times B| = |A| \times |B|$
- If A is finite, then $\mathcal{P}(A)$ is finite and $|\mathcal{P}(A)| = 2^{|A|}$
- $nPr = \frac{n!}{(n-r)!}$, $nCr = \frac{n!}{r!(n-r)!}$

11 Graphs

- A path from x_0 to x_l is a subgraph $(\{x_0, x_1, ..., x_l\}, \{x_0x_1, x_1x_2, ..., x_{l-1}x_l\})$ where all x's are different
- If there is a path from u to v in G and a path from v to w in G then there is a path from v to w in G.
- A cycle is a subgraph of the form $(\{x_1, x_2, ..., x_l\}, \{x_1x_2, x_2x_3, ..., x_{l-1}x_l, x_lx_1\})$
- An undirected graph is cyclic if it has a loop or a cycle, else it is acyclic
- An undirected graph with no loop is cyclic if and only if it has two vertices between which there are two distinct paths
- An undirected graph is connected if there is a path between any two vertices
- A connected component of G is a maximal connected subgraph of G
- Every vertex is in some connected component of G

12 Trees

- A tree is a connected acyclic undirected graph
- An undirected graph with no loop is a tree if and only iff between any two vertices there is exactly one path in the graph
- A connected undirected graph is a tree if and only if removing any edge disconnected G
- For a finite tree G, $|E(G)| = |V(G)| - 1$
- Let G be a connected cyclic finite undirected graph. Then $|E(G)| \geq |V(G)|$
- The height is the length of the longest path between the root and some vertex
- Terminology: parent, child, terminal vertex (leaf) and internal vertex