COMP 251: Algorithms & Data Structures

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1 Overview of Graph Theory

1.1 Definitions

A graph $G = (V, E)$ is a set V of vertices (a.k.a. nodes) and a set E of edges (denoting vertex pairs). We set $n = |V|$, and $m = |E|$. A graph is said to be *undirected* when for any edge $(u, v) \in E$ there exists an edge $(v, u) \in E$ for some nodes u, and v. A graph is said to be *directed* if it is not undirected. In other words, the edge set of a directed graph consists of ordered pairs where the edge set of an undirected graph consists of unordered pairs.

A *walk* is a set of vertices $\{v_0, v_1, \ldots, v_\ell\}$ such that $(v_i, v_{i+1}) \in E$, $\forall\ 0 \le i \le \ell$. A walk where $v_0 = v_\ell$ is said to be a *circuit* or a *closed walk*. A circuit where every edge in the graph is used exactly once is known as an *Eulerian circuit*. A *cycle* is a walk $\{v_0, v_1, \ldots, v_\ell\}$ such that every vertex is distinct except $v_0 = v_\ell$. A cycle where every vertex of the graph is used exactly once is known as a *Hamiltonian cycle*. A walk where every vertex is distinct is said to be a *path*.

A graph is said to be *connected* if for each $u, v \in V$ there exists a walk from u to v. A graph is said to be *disconnected* id it is not connected. Each connected subgraph of a graph is called a *component*. A connected graph therefore has exactly one component.

A connected component with no cycles is called a *tree*. A graph whose components are all trees is said to be a *forrest*. A tree is said to be *spanning* if it contains every vertex in the graph. A vertex in a tree with at most one neighbour is called a *leaf*.

A *matching* is a set of vertex-disjoint edges i.e. each edge is incident to at most one other edge in a matching. A matching is said to be *perfect* if every vertex is incident to exactly one edge in the matching.

A *clique* is a set of pairwise adjacent vertices. In *independent set* (a.k.a. a *stable set*) is set of pairwise non-adjacent vertices.

A *bipartite graph* is a graph such that the vertex set V can be partitioned as $V = X \cup Y$ where each edge has one node in X and the other node in Y. Note that X and Y are necessarily independent sets.

1.2 Some Theorems for Undirected Graphs

Theorem: (Handshaking Lemma) Let $G = (V, E)$ be an undirected graph, let $\Gamma(v) := \{u : (u, v) \in E\}$ be the set of neighbours of a node v, and let the *degree* $\deg(v)$ of a vertex v equal the cardinality of $\Gamma(v)$. Then there are an even number of vertices with odd degree.
Proof: First note that since we're double-counting the number of pairs where

(v, e) is an edge incident to v

$$2 \cdot |E| = \sum_{v \in V} \deg(v)$$

Since the degree of a vertex is either even or odd, we can partition V into a set of odd-degree vertices $\mathcal{O}$, and a set of even-degree vertices $\mathcal{E}$. This gives us

$$\sum_{v \in V} \deg(v) = \sum_{v \in \mathcal{O}} \deg(v) + \sum_{v \in \mathcal{E}} \deg(v)$$

which implies

$$\sum_{v \in \mathcal{O}} \deg(v) = 2 \cdot |E| - \sum_{v \in \mathcal{E}} \deg(v)$$

since both the $2 \cdot |E|$ term is even (obvious) and the $\sum_{v \in \mathcal{E}} \deg(v)$ term is even (sum of even numbers) then the $\sum_{v \in \mathcal{O}} \deg(v)$ term must also be even. □

Theorem: (Euler's Theorem) If G is an undirected graph then G contains an Eulerian circuit if and only if every vertex has even degree.
Proof: Easy proof by induction

Lemma: A tree T with $n \geq 2$ vertices has at least one leaf vertex.
Proof: Trees are connected so there exists no vertices with degree 0 when $n \geq 2$. Suppose each vertex has degree of at least 2. Then consider the longest path $P \subseteq T$, $P = \{v_1, v_2, \ldots, v_{\ell-1}, v_\ell\}$. Since $\deg(v_\ell) \geq 2$, $\exists$ a neighbour (of v_ℓ) $x \in P$ with $x \neq v_{\ell-1}$. If $x = v_{\ell+1}$ then P is not the longest path, a contradiction. Therefore, for P to be the longest path, x must be somewhere else in P, but this creates a cycle, another contradiction. Thus there must exist at least one node v such that $0 < \deg(v) < 2$ – a leaf. □

Theorem: A tree with n vertices has exactly $n - 1$ edges.
Proof: Simple proof by induction.
Base case: A tree with one vertex trivially has 0 edges.
Induction Hypothesis: Assume any tree with $n - 1$ vertices has $n - 2$ edges.
Inductive Step: Take a tree with $n \geq 2$ vertices. By the previous lemma this tree contains a leaf vertex v. This implies that $T \setminus \{v\}$ is a tree with $n - 1$ vertices and by the induction hypothesis $T \setminus \{v\}$ is a tree with $n - 2$ edges, which imples that T is a tree with $n - 1$ edges. □

Theorem: (Hall's Theorem) Let $G = (X \cup Y, E)$ with $|X| = |Y|$ be a bipartite graph. G contains a perfect matching if and only if $\forall\ B \subseteq X$, $|\Gamma(B)| \geq |B|$ (Hall's condition).
Proof: Firstly, the ($\Rightarrow$) direction is fairly obvious. If $B \subseteq X$ with $\Gamma(B) < |B|$ then the graph can't have a perfect matching. The ($\Leftarrow$) direction is a bit trickier. Suppose Hall's condition is satisfied. Then, take the maximum cardinality

matching M is the graph. If M is perfect then we are done. Otherwise there must exist an unmached vertex b_0.

- Since Hall's condition holds, we have $|\Gamma(\{b_0\})| \geq |\{b_0\}| = 1$ so b_0 must have at least one neighbour s_0.
- Suppose s_0 is matched in M to b_1.
- Since Hall's condition holds, we have $|\Gamma(\{b_0, b_1\})| \geq |\{b_0, b_1\}| = 2$ so $\{b_0, b_1\}$ must have at least one neighbour $s_1 \neq s_0$.
- Suppose s_1 is matched in M to b_2.
- Since Hall's condition holds, we have $|\Gamma(\{b_0, b_1, b_2\})| \geq |\{b_0, b_1, b_2\}| = 3$ so $\{b_0, b_1, b_2\}$ must have at least one neighbour $s_2 \notin \{s_0, s_1\}$.
- ...

we repeat this argument as long as we can. Since the graph contains a finite number of vertices this process must terminate, but it can only terminate when we reach an unmatched node s_k. Using the edges we've formed in M we can create a path P from b_0 to s_k that alternates between using non-matching edges and using matching edges. Swapping the matching edges with the non-matching edges gives us one more matching edge (as we have an odd number of edges.) This is still a valid matching as the internal nodes of P are still incident to exactly one matching edge. Also, the end nodes, b_0 and s_k were previously unmatched but are now incident to exactly one edge in the new matching. Thus M isn't the maximum capacity matching – a contradiction. □

1.3 """"Data Structures"""" for Representing Graphs

1.3.1 Adjacency Matrices

For a graph, an *adjacency matrix* M is a matrix suxh that

1. There is a row for each vertex
2. There is a column for each vertex
3. The $ij - th$ entry is defined as $M_{ij} = \begin{cases} 1, (i,j) \in E \\ 0, (i,j) \notin E \end{cases}$

Note that in an undirected graph the matric is symmetric around the diagonal because $(i, j) \equiv (j, i)$. Of course this is not necessarily true of directed graphs.

1.3.2 Adjacency Lists

An *adjacency list* of an undirected graph is such that for each vertex v of V we store a list of its neighbours. For a directed graph we have two lists: one in which we store the in-neighbours of v and one in which we store the out-neighbours of v.

1.3.3 Adjacency Matrices vs. Adjacency Lists

The main difference between the two is the amount of storage required to implement them.

- An adjacency matrix requires we store $\Theta(n^2)$ numbers
- An adjacency list requires we store $\Theta(m)$ numbers

In any graph $m = O(n^2)$. This means that for a sparse graph adjacency lists are highly favourable in terms of space complexity.

Verifying whether an edge exists, however, is much faster in an adjacency matrix – when using the array representation of a matrix it takes $O(1)$ time, where verifying the existance of an edge takes $O(\log n)$ time for an ordered adjacency list (using binary search), and $O(n)$ time if the adjacency list is not ordered (using sequential search).

2 Divide & Conquer

A *divide and conquer* algorithm ideally breaks up a problem of size n into smaller sub-problems such that:

- There are exactly a sub-problems
- Each sub-problem has a size of at most $\dfrac{1}{b} \cdot n$
- Once solved, the solutions to the sub-problems must be combined in $O(n^d)$ time to produce a solution to the original problem

Therefore the time-complexity of a divide and conquer algorithm satisfies a recurrence relation given by

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$$

2.1 The Master Theorem

Lemma 1: $\displaystyle\sum_{k=0}^{\ell} \tau^k = \frac{1-\tau^{\ell+1}}{1-\tau}$, for any $\tau \neq 1$.
Proof:

$$\begin{aligned}(1-\tau)\sum_{k=0}^{\ell} \tau^k &= \sum_{k=0}^{\ell} \tau^k - \sum_{k=1}^{\ell+1} \tau^k \\ &= \tau^0 - \tau^{\ell+1} \\ &= 1 - \tau^{\ell+1} \\ \sum_{k=0}^{\ell} \tau^k &= \frac{1-\tau^{\ell+1}}{1-\tau}\end{aligned}$$

□

Lemma 2: $x^{\log_b y} = y^{\log_b x}$ for any base $b \in \mathbb{R}$.
Proof: From the power rule of logarithms we have

$$\log_b x \cdot \log_b y = \log_b \left(y^{\log_b x}\right)$$

similarily, we have

$$\log_b x \cdot \log_b y = \log_b \left(x^{\log_b y}\right)$$

therefore

$$\log_b \left(x^{\log_b y}\right) = \log_b \left(y^{\log_b x}\right)$$

thus

$$x^{\log_b y} = y^{\log_b x}$$

$\square$

Theorem: (The Master Theorem) If a recurrence relation is of the form $T(n) = a \cdot T\Big(\frac{n}{b}\Big) + O(n^d)$, for constants $a > 0$, $b > 1$, and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d) & \text{, if } a < b^d \text{ [Case I]} \\ O(n^d \cdot \log n) & \text{, if } a = b^d \text{ [Case II]} \\ O(n^{log_b a}) & \text{, if } a > b^d \text{ [Case III]} \end{cases}$$

Proof: By adding dummy numbers, we may assume that n is a power of b, i.e. $n = b^\ell$, for some $\ell \in \mathbb{N}_0$. Then

$$T(n) = n^d + a\Big(\frac{n}{b}\Big)^d + a^2\Big(\frac{n}{b^2}\Big)^d + \cdots + a^\ell\Big(\frac{n}{b^\ell}\Big)^d$$

simlifying, we get

$$T(n) = n^d\bigg(1 + \frac{a}{b^d} + \Big(\frac{a}{b^d}\Big)^2 + \cdots + \Big(\frac{a}{b^d}\Big)^\ell\bigg)$$

we now have three cases:

Case I: $\frac{a}{b^d} < 1$
Set $\tau := \frac{a}{b^d}$ then we have

$$T(n) = n^d \sum_{k=0}^{\ell} \tau^k$$

applying lemma 1 we get

$$T(n) = n^d\bigg(\frac{1 - \tau^{\ell+1}}{1 - \tau}\bigg) \leq n^d\Big(\frac{1}{1-\tau}\Big)$$

but $\dfrac{1}{1-\tau}$ is a constant so $T(n) \leq n^d$ and thus

$$T(n) = O(n^d)$$

Case II: $\frac{a}{b^d} = 1$
Then we have that

$$T(n) = n^d\big(1 + 1 + 1^2 + \cdots + 1^\ell\big) = n^d\big(\ell + 1\big)$$

but $n = b^\ell \Rightarrow \ell = \log_b n$, thus

$$T(n) = O(n^d \cdot \log n)$$

Case III: $\frac{a}{b^d} > 1$
Set $\tau := \frac{a}{b^d}$ then we have

$$T(n) = n^d \sum_{k=0}^{\ell} \tau^k$$

applying lemma 1 we get

$$T(n) = n^d \left(\frac{\tau^{\ell+1} - 1}{\tau - 1} \right) \leq n^d \left(\frac{\tau^{\ell+1}}{\tau - 1} \right)$$

but since $\tau - 1$ is a constant we get

$$\begin{aligned} T(n) &= n^d O(\tau^{\ell+1}) \\ &= O(n^d \tau^{\ell+1}) \\ &= O(n^d \tau^{\ell}) \\ &= O\left(\left(\frac{a}{b^d} \right)^{\ell} n^d \right) \\ &= O\left(\left(\frac{n}{b^{\ell}} \right)^{d} a^{\ell} \right) \end{aligned}$$

but $n = b^{\ell}$ so

$$T(n) = O(a^{\ell})$$

and $\ell = \log_b a$ so

$$T(n) = O(a^{\log_b n})$$

and applying lemma 2 gives

$$T(n) = O(n^{\log_b a})$$

This completes the proof. □

2.2 MergeSort

MergeSort is an algorithm to sort n numbers into non-decreasing order.

```
MergeSort($x_1, x_2, \dots, x_n$)
    if $n = 1$
        return $x_1$
    else
        return Merge(MergeSort($x_1, \dots, x_{\lfloor \frac{n}{2} \rfloor}$), MergeSort($x_{\lfloor \frac{n}{2} \rfloor + 1}, \dots, x_n$))
```

Where the Merge function is a linear time algorithm combining two sorted lists (by comparing the first element in each list and moving the smaller element of the two into our new list.)

MergeSort is correct!

- It calls itself on smaller instances until the division process terminates when it reaches a base case where each list has size 1
- MergeSort works trivially on the base cases
- This sort of strong induction-type proof works for just about all divide and conquer algorithms

MergeSort is efficient!

- The recurrence relation that we can construct to model the running time of MergeSort is given by

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

  as we break the problem into two sub-problems of size $\frac{n}{2}$, plus a linear time sub-routine to combine the solutions (Merge.)
- This recurrence relation is in the proper form to use the Master Theorem and indeed we're in its Case II.
- By the master theorem the running time of MergeSort is $O(n \cdot \log n)$.
- The running time can also be proved by unwinding the recurrence.

2.3 Binary Search

BInary Search is an algorithm to search for whether a key k exists in a given sorted list.

```
BinarySearch($a_1, a_2, a_n; k$)
    while $n > 0$
        if $a_{\lfloor \frac{n}{2} \rfloor} = k$
            return true
        if $a_{\lfloor \frac{n}{2} \rfloor} > k$
            return BinarySearch($a_1, \ldots, a_{\lceil \frac{n}{2} \rceil - 1}; k$)
        if $a_{\lfloor \frac{n}{2} \rfloor} < k$
            return BinarySearch($a_{\lceil \frac{n}{2} \rceil + 1}, \ldots, a_n; k$)
    return false
```

Binary Search works!

- It works for the same strong induction argument as MergeSort

Binary Search is efficient!

- The recurrence relation that we can construct to model the running time of Binary Search is given by
$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$
as we break the problem into one sub-problem of size $\frac{n}{2}$.
- This recurrence relation is case II of the master theorem.
- By the master theorem the running time of MergeSort is $O(\log n)$.
- Again, the running time can also be proved by unwinding the recurrence.

2.4 Fast Multiplication

In COMP 250 we found an $O(n^2)$ algorithm to multiply 2 n-digit numbers (in any base). Let's try to do better using divide and conquer. Consider the example:

Let $\mathbf{x} := x_n x_{n-1} \dots x_{\frac{n}{2}+1} x_{\frac{n}{2}} \dots x_2 x_1$
Let $\mathbf{y} := y_n y_{n-1} \dots y_{\frac{n}{2}+1} y_{\frac{n}{2}} \dots y_2 y_1$

Then $\mathbf{x} = 10^{\frac{n}{2}}\mathbf{x}_L + \mathbf{x}_R$, where $\mathbf{x}_L = x_n \dots x_{\frac{n}{2}+1}$, and $\mathbf{x}_R = x_{\frac{n}{2}} \dots x_1$
Similarly $\mathbf{y} = 10^{\frac{n}{2}}\mathbf{y}_L + \mathbf{y}_R$, where $\mathbf{y}_L = y_n \dots y_{\frac{n}{2}+1}$, and $\mathbf{y}_R = y_{\frac{n}{2}} \dots y_1$

Thus

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &= (10^{\frac{n}{2}}\mathbf{x}_L + \mathbf{x}_R)(10^{\frac{n}{2}}\mathbf{y}_L + \mathbf{y}_R)\\ &= 10^n \cdot \mathbf{x}_L\mathbf{y}_L + 10^{\frac{n}{2}}(\mathbf{x}_L\mathbf{y}_R + \mathbf{x}_R\mathbf{y}_L) + \mathbf{x}_R\mathbf{y}_R\end{aligned}$$

So we've found a divide and conquer algorithm to multiply two n-digit numbers. Here we're breaking the problem down into four sub-problems of size $\frac{n}{2}$, with a few additions thrown in there (can be done in linear time). Therefore, the recurrence for this algorithm is given by

$$T(n) = 4\cdot T\left(\frac{n}{2}\right) + O(n)$$

then, by the master theorem, the running time of this algorithm is $O(n^2)$. Hmmm. This isn't better. Well thanks to our good friend Carl "G-Money" Gauss, all is not lost. Gauss was studying the product of complex numbers

$$(a+bi)(c+di) = ac - bd + (bc+ad)i$$

which seemingly involved 4 multiplications, when he noticed that actually we can do it with only 3. Note that

$$(bc+ad) = (a+b)(c+d)$$

Now we can use this exact same trick but we'll replace the i with $10^{\frac{n}{2}}$. This gives

$$(\mathbf{x}_L\mathbf{y}_R + \mathbf{x}_R\mathbf{y}_L) = (\mathbf{x}_R + \mathbf{x}_L)(\mathbf{y}_R + \mathbf{y}_L) - \mathbf{x}_R\mathbf{y}_R - \mathbf{x}_L\mathbf{y}_L$$

So now we can break our problem into only 3 sub-problems! Our new recurrence is

$$T(n) = 3 \cdot T\left(\frac{n}{2}\right) + O(n)$$

so finally, by the master theorem, $T(n) = O(n^{\log_2 3}) \approx O(n^{1.59})$

2.5 Fast Matrix Multiplication

Let's try to efficiently multiply two $n \times n$ matrices. Let

$$X := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ and } Y := \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

where X, and Y are two $n \times n$ matrices and $A, B, \ldots H$ are eight $\frac{n}{2} \times \frac{n}{2}$ matrices. Then their product Z is given by

$$Z = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}$$

Therefore multiplying two $n \times n$ matrices involves eight products of $\frac{n}{2} \times \frac{n}{2}$ matrices, and n^2 additions. Therefore the recurrence relation is given by

$$T(n) = 8 \cdot T\left(\frac{n}{2}\right) + O(n^2)$$

By the master theorem, this then takes $O(n^3)$ time. That's ok but again, we can do better. To do better however, we need a little algebraic trick. To do this, we need to define 7 matrices, $S_1, \ldots, S_7$ given by

$$\begin{array}{cc} S_1 = (B - D) \cdot (G + H) & S_2 = (A + D) \cdot (E + H) \\ S_3 = (A - C) \cdot (E + F) & S_4 = (A + B) \cdot H \\ S_5 = A \cdot (F - H) & S_6 = D \cdot (G - E) \\ S_7 = (C + D) \cdot E & \end{array}$$

then the product of X, and Y is given by

$$Z = \begin{pmatrix} S_1 + S_2 - S_4 + S_6 & S_4 + S_5 \\ S_6 + S_7 & S_2 - S_3 + S_5 - S_7 \end{pmatrix}$$

This means that we only need to do 7 multiplications! Therefore our new recurrence is

$$T(n) = 7 \cdot T\left(\frac{n}{2}\right) + O(n^2)$$

and thus by the master theorem $T(n) = O(n^{\log_2 7}) \approx O(n^{2.81})$

2.6 Fast Exponentiation

Consider the following algorithm to quickly calculate x^n:

```
FastExp($x, n$)
    if $n = 1$
        return $x_1$
    else if $n$ is even
        return FastExp($x, \lfloor \frac{n}{2} \rfloor$)$^2$
    else if $n$ is odd
        return $x\cdot$FastExp($x, \lfloor \frac{n}{2} \rfloor$)$^2$
```

Assuming that n is a power of two, we get that recurrence is given by

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

and thus by the master theorem this algorithm has time complexity of $O(\log n)$.

2.7 The Selection Problem

Suppose we're given a list of number and we want to find the k^{th} smallest number in that list. We can use the following divide and conquer algorithm:

```
Select($\mathcal{S}, k$)
    if $|\mathcal{S}| = 1$
        return $x_1$
    else
        set $\mathcal{S}_L := \{x_i \in \mathcal{S} : x_i < x_1\}$
        set $\mathcal{S}_R := \{x_i \in \mathcal{S} : x_i \geq x_1\}$

        if $|\mathcal{S}_L| = k - 1$
            return $x_1$
        else if $|\mathcal{S}_L| > k - 1$
            return Select($\mathcal{S}_L, k$)
        else if $|\mathcal{S}_L| < k - 1$
            return Select($\mathcal{S}_R, k - 1 - |\mathcal{S}_L|$)
```

Well.. this actually sucks because it runs in $\Omega(n^2)$ time. Note that it would actually be faster to just sort the list and then pull the k^{th} element (this can of course be done in $O(n \cdot \log n)$ time.) One idea that we can exploit to improve this is to use a random pivot rather than deterministically selecting x_1 as a pivot.

We will define a *good pivot* to be such neither $\mathcal{S}_L$, nor $\mathcal{S}_R$ contain more than $\dfrac{3}{4}$ of $\mathcal{S}$, and a *bad pivot* to be otherwise. Note that by this definition the probability that a pivot will be either good or bad (respectively) is 0.5.

For randomized algorithms we're always interested in the *expected runtime* of the algorithm rather than the worst case runtime. The recurrence relation, $\bar{T}(n)$, for the expected runtime for the randomized version of the selection is then given by

$$\bar{T}(n) \leq \frac{1}{2} \cdot \bar{T}\left(\frac{3n}{4}\right) + \frac{1}{2} \cdot \bar{T}(n) + O(n)$$

where the $\dfrac{1}{2} \cdot \bar{T}\left(\dfrac{3n}{4}\right)$ term represents the recursive call given a good pivot, and the $\dfrac{1}{2} \cdot \bar{T}(n)$ term represents the recursive call on a bad pivot. It's less than or equal to because we will only ever use one of those terms at once but not both. We can clean this recurrence up a bit to find the expected runtime.

$$\begin{aligned}
\bar{T}(n) &\leq \frac{1}{2} \cdot \bar{T}\left(\frac{3n}{4}\right) + \frac{1}{2} \cdot \bar{T}(n) + O(n) \\
\frac{1}{2} \cdot \bar{T}(n) &\leq \frac{1}{2} \cdot \bar{T}\left(\frac{3n}{4}\right) + O(n) \\
\bar{T}(n) &\leq \bar{T}\left(\frac{3n}{4}\right) + O(n)
\end{aligned}$$

Applying the master theorem to that recurrence gives $\bar{T}(n) = O(n)$

This is good but let's try to do better, i.e. let's see if we can find a deterministic linear time algorithm. It turns out we can but it requires a little trick called the *median of medians.*

It's best to illustrate this concept with an example. Suppose we have a list, $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$, from which we want to find the median of medians. To do this we partition the list into $\left\lceil \dfrac{n}{5} \right\rceil$ groups of cardinality 5. Denote each such group by $G_1, \ldots, G_{\lceil \frac{n}{5} \rceil}$ Then we'll sort each group and let z_i be the median of each group G_i. Let m be the median of the set $\mathcal{Z} = \{z_1, \ldots, z_{\lceil \frac{n}{5} \rceil}\}$. Predictably, we call m the *median of medians.* This is significant because m will always be a good pivot. This is because there will always be at least $\dfrac{3n}{10} - 1$ numbers in $\mathcal{S}$ less than m, and $\dfrac{3n}{10} - 1$ numbers in $\mathcal{S}$ at least as large as m. Therefore $|\mathcal{S}_R| \leq \dfrac{7n}{10}$, and $|\mathcal{S}_L| \leq \dfrac{7n}{10}$.

Now that we have a deterministic way to guarntee we have a good pivot, all that's left is to bake it into a selection algorithm. Luckily, our algorithm won't vary too much since all we have to do is add a few steps at the beginning.

```
DetSelect($\mathcal{S}, k$)
    if $|\mathcal{S}| = 1$
        return $x_1$
    else
        partition $\mathcal{S}$ into $\lceil \frac{n}{5} \rceil$ groups of 5
        for $j$ in range $\left(1, \lceil \frac{n}{5} \rceil\right)$
            let $z_j$ be the median of the group $G_j$.
        let $\mathcal{Z} = \{z_1, \ldots, z_{\lceil \frac{n}{5} \rceil}\}$
        set $m :=$ DetSelect$\left(\mathcal{Z}, \lceil \frac{n}{10} \rceil\right)$

        set $\mathcal{S}_L := \{x_i \in \mathcal{S} : x_i < m\}$
        set $\mathcal{S}_R := \{x_i \in \mathcal{S} : x_i \geq m\}$

        if $|\mathcal{S}_L| = k - 1$
            return $m$
        if $|\mathcal{S}_L| > k - 1$
            return DetSelect($\mathcal{S}_L, k$)
        if $|\mathcal{S}_L| < k - 1$
            return DetSelect($\mathcal{S}_R, k - 1 - |\mathcal{S}_L|$)
```

The recursive formula for the running time of this algorithm is

$$T(n) \leq T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + O(n)$$

where the $T\left(\frac{n}{5}\right)$ term is the time to find the median of medians, the $T\left(\frac{7n}{10}\right)$ term is pivoting based on the median of medians. Unfortunately, this doesn't work with the master theorem and we can't simplify it down to a form that does (unlike with the randomized algorithm, we need both of the $T(\ldots)$ terms.) Since it doesn't work with the master theorem, we need to use the recusion tree method. From the recursion tree method, $T(n) = O(n)$. So there it is.. a linear time deterministic algorithm to select the k^{th} smallest element from a list!

2.8 The Closest Pair of Points in a Plane

Suppose we're given a set $\mathcal{P}$ of n points in a 2D plane, $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ and want to find the closest 2. For each $i \leq i \leq n$, let $p_i = (x_i, y_i)$.

The easiest (correct) solution to think of is an exhaustive search where we calculate every distance between any two points and return the smallest pair. There are $\binom{n}{2}$ possible pairs of points, and calculating the distance between any two can be done in constant time, so this means the runtime is $O(n^2)$. This is okay, it's certainly polynomial-time efficient, but let's do it faster.

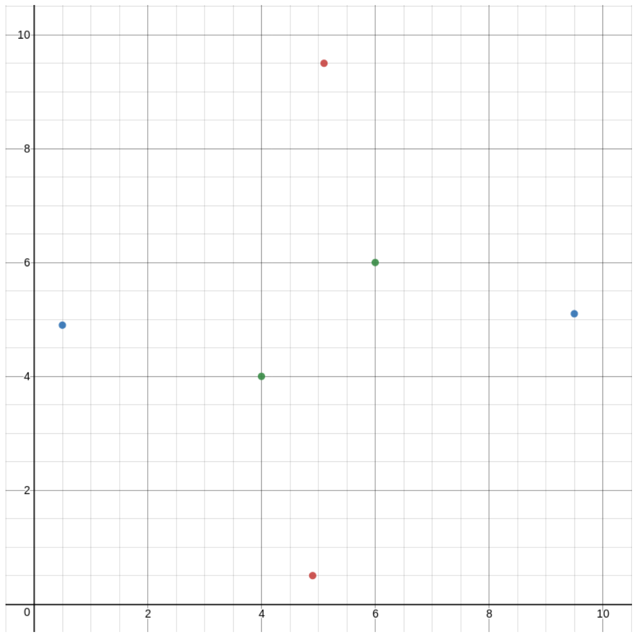
First, perhaps it will be handy to consider the 1D case (the closest two pairs of points on a line.) Note that in this case, the closest pair of point must be adjacent in their x-ordering. Therefore, assuming the list is already sorted according to their x-ordering, all we need to do is calculate $n-1$ pairwise distances, which can be done in $O(n)$ time. Of course if the list is not sorted we have to do that first, so the closest pair can be found in $O(n \cdot \log n)$ time. If we try to mimic that idea in the 2D case we come up with the following algorithm:

```
Pair2D($\mathcal{P}$)
    sort the pairs by their $x$-coordinate
    sort the pairs by their $y$-coordinate

    find the closest pair of points with respect to their $x$-coordinates
    find the closest pair of points with respect to their $y$-coordinates

    return the pair from the above 2 with the smallest distance between them
```

Sorting the lists takes $O(n \cdot \log n)$ time, and finding the closest points takes $O(n)$ time, so this algorithm take $O(n \cdot \log n)$ time. Does it work though? Nope. Consider this example:



It's clear here that the two green points are the closest together, but the algorithm would have returned either the pair of red points or the pair of blue points.

Let's try to come up with a divide and conquer algorithm to solve this problem. We can start by partitioning $\mathcal{P}$ into two sets, $\mathcal{P}_L$, and $\mathcal{P}_R$, of cardinality $\frac{n}{2}$. This gives us three possible outcomes:

1. The closest pair of points is in $\mathcal{P}_L$
2. The closest pair of points is in $\mathcal{P}_R$
3. The closest pair of points has one point in $\mathcal{P}_L$, and the other point in $\mathcal{P}_R$

To summarize this idea so far, we have the algorithm

```
DCPair2D($\mathcal{P}$)
    find the point $q$ with the median $x$-coordinate
    partition $\mathcal{P}$ into $\mathcal{P}_L$, and $\mathcal{P}_R$

    find the closest pair of points in $\mathcal{P}_L$
    find the closest pair of points in $\mathcal{P}_R$
    find the closest pair of points with one point in $\mathcal{P}_L$ and the other in $\mathcal{P}_R$

    return the pair from the above 3 with the smallest distance between them
```

The recursive formula for this algorithm is given by

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n^2)$$

The $O(n^2)$ term comes from the work needed to compare all $\frac{n}{2}$ points in $\mathcal{P}_L$ with all $\frac{n}{2}$ points in $\mathcal{P}_R$. $\left(\frac{n}{2} \times \frac{n}{2} = \frac{n^2}{4}\text{ comparisons.}\right)$ Therefore, by the master theorem, the running time is $O(n^2)$.

This isn't any better than exhaustive search because there's a bottleneck step where we compare the points in $\mathcal{P}_L$ with those in $\mathcal{P}_R$. Indeed we're pretty much just doing exhaustive search but on only $\frac{n}{2}$ elements. With this in mind, let's try to slim this step down. Note that since we've solved both the left and right sub-problems, there's a minimum pairwise distance δ, given by

$$\delta = \min(\delta_L, \delta_R)$$

for which the distance between the closest two points will not exceed, even if the two points are not in the same partition. Therefore, when looking for pairs of points with one point in each partition, we only need to look for points within δ of the dividing line! From now on we will denote the set of points within δ of the dividing line by $\mathcal{P}_M$. This trick does work, but not trivially, so it has to be proved.

Lemma: Let $p_i \in \mathcal{P}_L$, and let $p_j \in \mathcal{P}_R$. If $d(p_i, p_j) \leq \delta$ then $\{p_i, p_j\} \subseteq \mathcal{P}_M$.

Proof: WLOG assume that $p_i \notin \mathcal{P}_M$. Then $d(p_i, p_j) > \delta$. Similarily, if $p_j \notin \mathcal{R}_M$. Then $d(p_i, p_j) > \delta$. Thus if the pair of points is not within one of the two original sub-problems, it must be contained within $\mathcal{P}_M$. □

Let's implement this trick in the last algorithm we wrote up.

```
DCPair2D($\mathcal{P}$)
    find the point $q$ with the median $x$-coordinate
    partition $\mathcal{P}$ into $\mathcal{P}_L$, and $\mathcal{P}_R$

    find the closest pair of points in $\mathcal{P}_L$
    find the closest pair of points in $\mathcal{P}_R$
    find the closest pair of points in $\mathcal{P}_M$

    return the pair from the above 3 with the smallest distance between them
```

There's still a problem. $\mathcal{P}_M$ can possibly contain all (or most) of the points! This means that our algorithm can *still* take $\Omega(n^2)$ time to measure the distances between the points in $\mathcal{P}_L \cap \mathcal{P}_M$, and the points in $\mathcal{P}_R \cap \mathcal{P}_M$. Once again, however, this was fortunately not all for nothing. Since $\mathcal{P}_M$ covers only a narrow band of the x-axis, we'll consider the area of the plane within δ of the dividing line (i.e. the area pf $\mathcal{P}_M$.) Suppose we divide this area into small squares of size $\frac{\delta}{2}$ then we have the following lemma:

Lemma: No two points from $\mathcal{P}$ will lie within the same square of width $\frac{\delta}{2}$.

Proof: WLOG take two points in a square in $\mathcal{P}_L$. Then their pairwise distance d is

$$d \leq \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^2} = \frac{\delta}{\sqrt{2}} < \delta$$

which is a contradiction as δ is the smalles distance between two points. □

Theorem: Let $p_i \in \mathcal{P}_L$, and let $p_j \in \mathcal{P}_R$. If $d(p_i, p_j) \leq \delta$ then p_i, and p_j have at most 10 points between them in $\mathcal{P}_M$ with respect to their y-ordering.

Proof: WLOG suppose that p_i is below p_j in the y-order. Then p_j is in the same row of squares as p_i, or it is somewhere in the next two higher rows, or else they won't be within δ of eachother (a contradiction). Therefore we are working in a rectangle that is at most three rows of squares high and four columns of squares wide. By the previous lemma we can have at most one point in any square and therefore in our 4×3 rectangle there can be at most 12 points. Thus, by choosing the the points on either extreme of the y-ordering of our rectangle, there can be at most 10 points between. $\square$

This result implies that the idea underlying the algorithm from the 1D case can be applied after all! We just have to tweak it slightly so as to look for points that are 11 apart rather than just adjacent, i.e. we'll have to calculate at most $11n$ pairwise distances rather than at most $n-1$. Thus we either find a pair of points in $\mathcal{P}_M$ closer than δ or we don't, but if we don't then we can conclude that no such pair of points exists, and we can do so in $O(n)$ time!

Let's update our algorithm to reflect this.

```
DCPair2D($\mathcal{P}$)
    find the point $q$ with the median $x$-coordinate
    partition $\mathcal{P}$ into $\mathcal{P}_L$, and $\mathcal{P}_R$

    find the closest pair of points in $\mathcal{P}_L$
    find the closest pair of points in $\mathcal{P}_R$
    find the closest pair of points in $\mathcal{P}_M$ using the (updated) 1D algorithm

    return the pair from the above 3 with the smallest distance between them
```

Now that we've gotten the bottleneck step down to $O(n)$, the new recurrence relation for this algorithm is

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

where the $O(n)$ term comes from finding the median with respect to the x-coordinates (using algorithm from previous lecture), partitioning $\mathcal{P}$, finding $\mathcal{P}_M$ and applying the 1D algorithm. Thus, by the master theorem, the time complexity of this algorithm is $O(n \cdot \log n)$. (If you think about it, the new most expensive step is just sorting the points in $\mathcal{P}_M$ with respect to the y-coordinates! If you don't think that's hella cool then you have no soul.)

Like most divide and conquer algorithms, this one is correct and the proof follows immediately from strong induction, where the proof that the base cases are correct follows from the proofs of the above various lemmas and theorems.

3 Graph Algorithms

4 Greedy Algorithms

5 Dynamic Programming

6 Network Flows

7 Data Structures