ECE358S

Foundations of Computing

Homework 3 Solutions

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April 24, 2013

Problem 1

a) Given two different spanning trees T and T' obtained from the graph G=V,E, lets define:

- E_T: Edges in T
- $E_{T'}$: Edges in T'
- S=$E_T \cap E_{T'}$

To transform T into T':

1. Take edge $e_{i'} \in \{T' - S\}$
2. Add $e_{i'}$ to T
3. By properties of spanning tree, a cycle (lets call it C) will be formed in T.
4. Take edge $e_i \in C$, where also $e_i \in \{T - S\}$
   Note: e_i always exists because other way all the edges $\in C$ belong to T'. That would mean that there is a cycle in T'
5. Remove e_i from T. Now the difference between T and T' is one edge smaller.
6. Repeat steps 1 to 5 until S is empty.

By definition, a spanning tree has $|V| - 1$ edges. In the worst case, $|S| = |V| - 1$ (no edges shared by T and T'). That is the number of edge-swaps that would be required.

b) Using the algorithm from a) with T a spanning tree and T' an MST, lets add two restrictions:

1. $e_{i'}$ must be the edge with lowest weight in T'
2. $weight\{e_i\} \geq weight\{e_{i'}\}$

Now, to demonstrate that there is an e_i with weight greater or equal than that of $e_{i'}$, lets use contradiction and say that $e_i < e_{i'}$. If e_i is added to T', then $e_{i'}$ could be discarded. Doing this would reduce the total weight of the MST, which by definition is impossible.

With the mentioned restrictions, all the swaps will be non increasing and after the necessary iterations, the result will be T'.

c) (a) and (b) have shown that a MST is the result of the algorithm.

For the maximum number of iterations, the worst case would be when the algorithm has to create all the possible spanning trees from G before arriving to the MST. That is:

$$\binom{|E|}{|V|-1}$$

d) a) If all the bins are full:

$$S = \sum_{i=1}^{n} b_i = nB$$

If any of the bins is not fully used,

$$S \leq nB \rightarrow n \geq \frac{S}{B}$$

As we are dealing with integers:

$$n \geq \left\lceil \frac{S}{B} \right\rceil$$

b) By contradiction, lets suppose that two bins b_a and b_b are at most half full. The a_i elements in each bin will be $\leq \frac{b}{2}$. Then the sum of the contents of both bins will be $\leq B$. But FirstFit fills a bin as long as the current a_i fits in it, so b_a would have taken the elements in both b_a and b_b.

c) If all bins are at least half full:

$$S = \sum_{i=1}^{n} b_i \rightarrow S \geq \frac{Bn}{2} \rightarrow n \leq \left\lceil \frac{2S}{B} \right\rceil$$

If there is one bin less than half full:

$$S = \sum_{i=1}^{n} b_i = b_n + \sum_{i=1}^{n-1} b_i$$

where b_n is the bin that is less than half full.
As the rest of the b_i are $\geq \frac{B}{2}$:

$$S \geq b_n + \frac{B}{2}(n-1) \rightarrow (S - b_n)\frac{2}{B} \geq n-1 \rightarrow \frac{2S}{B} - \frac{2b_n}{B} \geq n-1$$

As $\frac{2b_n}{B}$ is positive:

$$\frac{2S}{B} > n-1 \rightarrow n < \frac{2S}{B} - 1 \rightarrow n \leq \left\lceil \frac{2S}{B} \right\rceil$$

As demonstrated in (b), these are all the possibilities.

d) The optimal number of bins is $\lceil \frac{S}{B} \rceil$. From (b), the worst case for FirstFit implies $N = \lceil \frac{2S}{B} \rceil$
The ratio R is then:

$$R = \frac{\lceil \frac{2S}{B} \rceil}{\lceil \frac{S}{B} \rceil} = 2$$

e) Variables (quantities of ingredients in salad)

T = Tomatoin salad

L = lettuce in salad

S = spinach

C = Carrot

F = oil

Constraints:

$0.85T + 1.62L + 12.78S + 98.39C \geq 15$

$0.33T + 0.2L + 1.58S + 1.39C + 100F \geq 2$

$0.33T + 0.2L + 1.58S + 1.39C + 100F \leq 6$

$4.64T + 2.37L + 74.69S + 80.7C \geq 4$

$9T + 8L + 7S + 508.2C \leq 100$

$L + S \leq T + C + F$

$T, L, S, C, F \geq 0$

Minimize $21T + 16L + 371S + 346C + 884F$

Result:

Cal = 232.51
T = 588.48 grams
L = 584.32 grams
S = 4.16 grams
C and F = 0 grams

f) Convert maximum absolute error equation to linear constraints:

$ax_i + by_i - c \leq e$

$ax_i + by_i - c \geq -e$

where e is the error.

Use the given points to define the constraints:

$a + 3b - c \leq d$
$a + 3b - c \geq -d$
$2a + 5b - c \leq d$
$2a + 5b - c \geq -d$

$3a + 7b - c \leq d$
$3a + 7b - c \geq -d$
$5a + 11b - c \leq d$
$5a + 11b - c \geq -d$
$7a + 14b - c \leq d$
$7a + 14b - c \geq -d$
$8a + 15b - c \leq d$
$8a + 15b - c \geq -d$
$10a + 19b - c \leq d$
$10a + 19b - c \geq -d$

If one of the variables a, b, or c is not fixed, the result that will be obtained is the trivial: a = 0, b = 0, c = 0.

If the used LP solver does not work with negative errors, consider the following points:

(a) If the given points are drawn in a plane, it is possible to see that the slope is positive.

(b) If the line equation is rewritten in its simplified form y = mx + b:
$y = (\frac{-a}{b})x + (c/b)$
It is possible to see that there are two options to obtain a positive slope: use a fixed negative a or a fixed negative b.

For a=-1 (could be any other negative number):

e=0.33
b=0.58
c=1.08

For b=-1

e=1.0
a=2.0
c=0.0

Then, a=-1 gives the best approximation.