CSC436 Tutorial 5 – Numerical Integration I

Question 1 *Approximate $I = \int_1^2 x^3 dx$ using midpoint, trapezoid, Simpson's and corrected trapezoid with 1 and 2 panels. Compute the error for each rule.*

Solution:

Exact value: $I = \int_1^2 x^3 dx = \frac{x^4}{4}\Big|_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4} = 3.75$

With 1 panel:

Midpoint: $Q_{M,0} = (2-1)f(\frac{3}{2}) = (\frac{3}{2})^3 = \frac{27}{8} = 3.375$

Trapezoid: $Q_{T,0} = \frac{2-1}{2}\big[f(1)+f(2)\big] = \frac{1}{2}(1+8) = 4.5$

Simpson's: $Q_{S,0} = \frac{2-1}{6}\big[f(1)+4f(\frac{3}{2})+f(2)\big] = \frac{1}{6}\big[1+4(\frac{3}{2})^3+8\big] = \frac{1}{6}\big[1+13.5+8\big] = 3.75$

Corrected trapezoid: $Q_{CT,0} = Q_T + \frac{(2-1)^2}{12}\big[f'(1)-f'(2)\big] = Q_T + \frac{1}{12}\big[3-12\big] = 4.5 - \frac{3}{4} = 3.75$

Error for midpoint: $3.75 - 3.375 = 0.375$

Error for trapezoid: $3.75 - 4.5 = -0.75$

Errors for rest of rules are equal to 0.

With 2 panels:

Midpoint: $Q_{M,1} = \frac{2-1}{2}(f(1.25) + f(1.75)) = 1.25^3 + 1.75^3 = 3.65625$

Trapezoid: $Q_{T,1} = \frac{2-1}{4}\big[f(1) + 2f(1.5) + f(2)\big] = \frac{1}{2}(Q_{T,0} + f(1.5)) = 3.9375$

Simpson's: $Q_{S,1} = \frac{2-1}{12}\big[f(1) + 4f(1.25) + 2(f(1.5) + 4f(1.75) + f(2)\big] = \cdots = 3.75$

Corrected trapezoid: $Q_{CT,0} = Q_T + \frac{(2-1)^2}{2^2 \times 12}\big[f'(1) - f'(2)\big] = 3.9375 - \frac{9}{4 \times 12} = 3.75$

Error for midpoint: $3.75 - 3.65625 = 0.09375$

Error for trapezoid: $3.75 - 3.9375 = -0.1875$ Errors for rest of rules are equal to 0.

Results for $\int_1^2 x^3 dx$ with various rules:

rule	error formula	error	poly. deg.
$s=1$			
Midpoint	$f''(\eta)\frac{(b-a)^3}{24}$	0.375	1
Trapezoid	$-f''(\eta)\frac{(b-a)^3}{12}$	-0.75	1
Simpson's	$-f^{(4)}(\eta)\frac{(b-a)^5}{2880}$	0	3
Corr.Trap	$f^{(4)}(\eta)\frac{(b-a)^5}{720}$	0	3
$s=2$			
Midpoint	$f''(\eta)\frac{(b-a)h^2}{24}$	0.09375	1
Trapezoid	$-f''(\eta)\frac{(b-a)h^2}{12}$	-0.1875	1
Simpson's	$-f^{(4)}(\eta)\frac{(b-a)h^4}{2880}$	0	3
Corr.Trap	$f^{(4)}(\eta)\frac{(b-a)h^4}{720}$	0	3

Comments:

- The sign of error calculated in the numerical experiment agrees with that of the error formula. Since $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$. Note that $f''(x) > 0$ in $[1, 2]$. Midpoint rule error formula is $f''(\eta)\frac{(b-a)^3}{24}$ (or $f''(\eta)\frac{(b-a)h^2}{24}$), which is positive for all $\eta \in [1, 2]$, so we expect the midpoint rule error to be positive and it is so. Similarly, we expect the trapezoid rule error to be negative and it is so.
- The magnitude of the error of trapezoid with $s = 1$ and $s = 2$ panels is exactly twice the magnitude of the midpoint rule error with $s = 1$ an $s = 2$ panels, respectively. By the error formulae, we know that, if $f''(x)$ does not vary a lot in $[1, 2]$, we expect the trapezoid rule error to be approximately twice in magnitude compared to the midpoint rule error. For this integral, it turns out to be exactly twice. This is because, for this integral, the η of the midpoint rule happens to be the same as the η of the trapezoid rule. Note that $f''(\eta)$ varies linearly with η (for this f), thus, if the η's were different, we would not be getting the exact factor of 2.
- The error of the midpoint and trapezoid rules with $s = 1$ panel is four times as large as those of the respective rules with $s = 2$ panels. Again, as above, if $f''(x)$

does not vary a lot in $[1, 2]$, we expect an improvement of approximately a factor of four when doubling the number of panels. For this integral, the improvement factor turns out to be exactly four. This is because, for this integral, the η's of the midpoint and trapezoid rules with $s = 1$ panel happen to be the same as the η's of the respective rules with $s = 2$ panels.

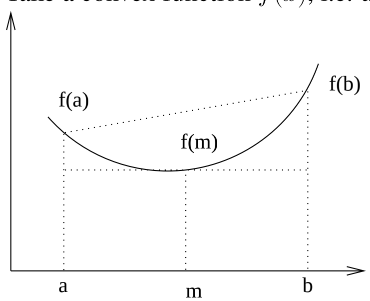
- Rules with polynomial degree 3 and up are exact as expected.
- If the function was x^4 or a higher degree polynomial, Simpson's and corrected trapezoid would not be exact.

QUESTION 2 *Can we obtain an estimate for the error of a rule, if we do not know the exact value of $I = \int_a^b f(x)dx$?*

SOLUTION:

Yes, we can use an error estimator. Error estimators use the results of the rule with s and $2s$ panels and the knowledge of the order of the rule (the exponent of h in the error formula). For example, for the midpoint rule, we know $E_{M,0} = I - Q_{M,0} \approx 4(I - Q_{M,1}) = 4E_{M,1}$, thus $E_{M,0} - E_{M,1} = I - Q_{M,0} - (I - Q_{M,1}) = Q_{M,1} - Q_{M,0} \approx 3E_{M,1}$, thus $E_{M,1} \approx (Q_{M,1} - Q_{M,0})/3$.
For the previous question, with $f(x) = x^3$,
$E_{M,1} \approx (Q_{M,1} - Q_{M,0})/3 = (3.65625 - 3.375)/3 = 0.09375$.
This estimate of the error turns out to be exactly equal to the error, as calculated in the previous question.
In general, the estimate of the error is an approximation to the error. But for this f, it turns out that the η of the midpoint rule with 1 panel is the same as the the η of the midpoint rule with 2 panels, thus the relation $I - Q_{M,0} \approx 4(I - Q_{M,1})$ becomes $I - Q_{M,0} = 4(I - Q_{M,1})$ and all subsequent relations above become equalities.

QUESTION 3 *How to remember the sign of error of the trapezoid and midpoint rules?*
SOLUTION:
Take a convex function $f(x)$, i.e. a function for which $f''(x) > 0$ in $[a, b]$.



The trapezoid rule joins $(a, f(a)), (b, f(b))$ by a straight line and computes the area beneath it. Since $f(x)$ is convex, its graph is below the $f(a)$, $f(b)$ line, therefore the trapezoid rule overestimates the integral. Thus the error is negative ($E = I - Q_T < 0$). The opposite happens with midpoint rule. For most values of x from a to b, the line $y = f(m)$ is below the graph of $f(x)$, thus we expect the midpoint rule to underestimate the area/integral. Thus the error is positive ($E = I - Q_M > 0$).

QUESTION 4 *Consider computing $\int_a^b(\alpha x^3 + \beta x^2 + \gamma x + \delta)dx$, $\alpha \neq 0$*
(a) by Simpson's rule (simple or composite)
(b) by midpoint rule (simple or composite)
on a computer with machine epsilon $\varepsilon = 10^{-6}$. How many function evaluations are required to compute the integral with 0 error (up to machine epsilon) using each rule?

SOLUTION:
(a) Simpson's has polynomial degree 3 so it is exact (up to round-off error) for cubic polynomials. So 1 application of Simpson's suffices. Thus 3 function evaluations suffice.
(b) Midpoint has polynomial degree 1, so 1 application of midpoint rule may not be enough.
Assume we have s subintervals/applications of midpoint rule. Note that the error of midpoint rule is $E = f''(\eta)\frac{(b-a)h^2}{24}$ with η in $[a, b]$. In our case, $E = f''(\eta)\frac{(b-a)h^2}{24} = \frac{(b-a)^3}{24s^2}(6\alpha\eta + 2\beta)$.
We want $\max\{|E|\} \leq \varepsilon$. Let's find $\max\limits_{a \leq x \leq b} |6\alpha x + 2\beta|$.
Let $g(x) = 6\alpha x + 2\beta$, $g'(x) = 6\alpha \neq 0$, thus max. and min. of g in $[a, b]$ are on $x = a$

and $x = b$. Let $M = \max\{|6\alpha a + 2\beta|, |6\alpha b + 2\beta|\}$. Then $|E| \leq \varepsilon \Rightarrow \frac{(b-a)^3}{24s^2}M \leq 10^{-6} \Rightarrow s^2 \geq \frac{10^6(b-a)^3M}{24} \Rightarrow s \geq \frac{10^3(b-a)\sqrt{b-a}\sqrt{M}}{2\sqrt{6}} = s_0$. Applying the midpoint rule to at least s_0 subintervals guarantees $|E| \leq \varepsilon$.
Nothing precludes that fewer than s_0 subintervals may result in $|E| \leq \varepsilon$, but s_0 or more guarantee it.

Example: $a = 0, b = 1, \alpha = 1, \beta = 2$.
In this case, $M = \max\{|2.2|, |6 + 2.2|\} = 10$. Thus $s \geq \frac{10^3 \times 1 \times 1 \times \sqrt{10}}{2\sqrt{6}} \approx 645$. So 645 function evaluations are required to guarantee that $|E| \leq \varepsilon$.
(Note: Clearly, the midpoint rule is very inefficient for this function. Composite rules are intended for difficult functions, not for low degree polynomials.)

Aside: Remember the integration by parts formula: $\int udv = uv - \int vdu$.