CSC436 Tutorial 7 – Numerical Integration III

QUESTION 1 *Consider $I = \int_0^1 \frac{e^x}{\sqrt{x}}dx$. Use appropriate tricks and Simpson's rule to approximate I within a given tolerance tol.*
How many panels (assuming uniform points) are needed?

SOLUTION:

The given integral has a singularity at the left endpoint, thus Simpson's rule cannot be applied directly to it. What alternatives do we have?

1. We can use truncation, but since the integrand tends to infinity as $x \to 0$, it will most likely take several panels to reach the tolerance.
2. We can also use some appropriate transformation that gets rid of the singularity, without introducing more problems.
3. We could also use an open rule, but here we are asked to somehow use Simpson's.

Here is a trick that is applicable to integrals of the form $\int_a^b \frac{g(x)}{(x-a)^p}dx$, where $0 < p < 1$, and $g \in C[a,b]$.

Note: Such integrals are known to exist (i.e. have a finite value).

Consider the Taylor polynomial $t_k(x)$ of degree k approximating $g(x)$ about $x = a$. Then,

$$I = \int_a^b \frac{g(x)}{(x-a)^p}dx = \int_a^b \frac{t_k(x)}{(x-a)^p}dx + \int_a^b \frac{g(x)-t_k(x)}{(x-a)^p}dx.$$

Consider the two integrals in the right side of the above relation. The first can be evaluated analytically, since it simplifies to integrals of terms of the form $(x-a)^m$, and the second simplifies to the integral of a function in $C^k[a,b]$.

We explain the details using the example $g(x) = e^x$, $p = 1/2$, $a = 0$, $b = 1$ and Simpson's rule. Let $k = 4$. We have

$$t_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

and

$$I = \int_0^1 \frac{e^x}{x^{1/2}}dx = \int_0^1 \frac{t_4(x)}{x^{1/2}}dx + \int_0^1 \frac{e^x - t_4(x)}{x^{1/2}}dx \equiv I_1 + I_2.$$

Note that $I_1 = \int_0^1 \frac{t_4(x)}{x^{1/2}}dx = \int_0^1 x^{-1/2} + x^{1/2} + \frac{1}{2}x^{3/2} + \frac{1}{6}x^{5/2} + \frac{1}{24}x^{7/2}dx = \Big[2x^{1/2} + \frac{2}{3}x^{3/2} + \frac{1}{5}x^{5/2} + \frac{1}{21}x^{7/2} + \frac{1}{108}x^{9/2}\Big]_0^1 = 2 + \frac{2}{3} + \frac{1}{5} + \frac{1}{21} + \frac{1}{108} \approx 2.9253034918143632176$. (We can obtain the value of the above integral to arbitrary precision using Maple, or to machine precision using Matlab.)

We now consider $I_2 = \int_0^1 \frac{e^x - t_4(x)}{x^{1/2}}dx$. While from the first blink it seems that Simpson's isn't applicable to I_2, with a careful look we find that it actually is. Consider the function

$$G(x) = \begin{cases} \frac{e^x - t_4(x)}{x^{1/2}} & 0 < x \le 1 \\ 0 & x = 0. \end{cases}$$

Since $t_4(x)$ agrees with e^x and its first 4 derivatives at $x = 0$, we have $G \in C^4[0,1]$. Also $I_2 = \int_0^1 \frac{e^x - t_4(x)}{x^{1/2}}dx = \int_0^1 G(x)dx$,
since the value of the function at one point does not change the value of the integral.

We now apply Simpson's to I_2 with tolerance tol. To find how many panels we need to reach the tolerance, we find a bound for $G^{(4)}$.
It is easy to see that $G^{(4)}(x) = (\frac{x^{9/2}}{5!} + \frac{x^{11/2}}{6!} + \cdots)^{(4)}$ is an increasing function in $[0,1]$ thus $|G^{(4)}(x)| \le G^{(4)}(1) = \cdots = \frac{41}{16}e^1 - \frac{771}{128} \approx 0.9422 < 1$.
Thus, Simpson's error satisfies $|E| = \frac{|G^{(4)}(\eta)|}{2880}\frac{1}{s^4} \le \frac{1}{2880 s^4}$.
We want $|E| \le tol$. It suffices to have $\frac{1}{2880s^4} \le tol$, or equivalently $s \ge \frac{1}{(tol\ 2880)^{1/4}}$. For example, if $tol = 10^{-3}$, $s = 1$ panel suffices.

Thus, by applying Simpson's to I_2 we get an approximation to I_2 within the given tolerance, and by adding it to the value of I_1 (which is exact or of appropriate precision), we get an

approximation to I within the given tolerance.
It is important that $G \in C^4[0,1]$, otherwise we would not be able to apply the Simpson's rule error formula and work out a bound for s.
Also note that $k = 4$ is the minimum degree for a Taylor polynomial $t_k(x)$ so that $G \in C^4[0,1]$. For example, $t_3(x)$ agrees with e^x and its first 3 derivatives at $x = 0$, but not in its 4th derivative.
(Recall that the Taylor polynomial $t_k(x)$ of degree k obtained by expanding $f(x)$ about $x = a$ agrees with $f(x)$ and its derivatives up to order k at the point $x = a$ about which the Taylor expansion is taken.)