Ordinary Differential Equations -- Introduction: ODEs, PDEs and IVPs

Differential Equation (DE): an equation in which the unknown is a function (of one or more variables) and derivatives of the unknown function occur in the equation.

Ordinary Differential Equation (ODE): a DE in which the unknown is a function of one variable (and derivatives of the unknown function w.r.t. that variable occur in the equation).

Partial Differential Equation (PDE): a DE in which the unknown is a function of more than one variable (and partial derivatives of the unknown function w.r.t. the variables occur in the equation).

CSC436 studies ODEs. CSC446 (2310) focuses on PDEs.

Introduction: ODEs, PDEs and IVPs

Examples of ODEs:

$$\frac{dy}{dt} = \cos t + \sin t \quad (1) \qquad \text{solution } y(t) = \sin t - \cos t + c$$

$$\frac{dy}{dt} = \lambda\, y(t) \quad (2) \qquad \text{solution } y(t) = \kappa\, e^{\lambda t}$$

In (1) and (2), the unknown (dependent variable) y is a function of (the independent variable) t. In (2), λ is a given scalar.

Short notation: $\dfrac{dy}{dt} = y'$. *Generic ODE*: $y' = f(t, y)$.

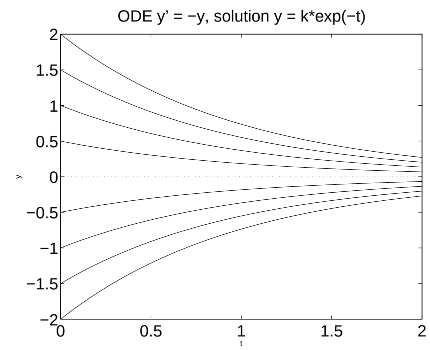
Notice that the solution of each of (1) and (2) involves a constant of integration (free parameters c and κ).

In order to uniquely define a solution of each of (1) and (2), we have to specify an additional condition, often a value of $y(t)$ at one point, say $t = t_0$.
Often the independent variable t represents time, and we are usually given the value of y at the initial state. The ODE then describes how $y(t)$ evolves as time increases.

Introduction: ODEs, PDEs and IVPs

Example: Consider the ODE (2) ($y' = \lambda\, y$), with λ = -1. The solution curves for various values of the constant κ are shown in the figure. Specifying $\kappa = y(0)$ fixes the solution curve.

Initial Value Problem (IVP) for an ODE: an ODE together with associated initial conditions (IC) at one point.

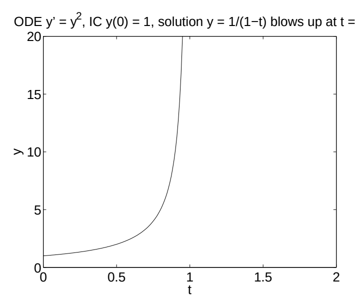


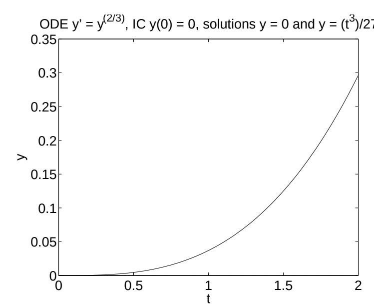
Existence and uniqueness of solution of an IVP for an ODE

IVPs for ODEs do not necessarily have solutions or do not necessarily have unique solutions.
Examples:

$y' = y^2$, $y(0) = 1$ (3) solution $y(t) = \dfrac{1}{1-t}$ for $t \in [0, 1-\varepsilon]$, $\varepsilon > 0$,
but no solution for $t \in [0, 2]$, for example.
($y(t)$ blows-up at $t = 1$.)

$y' = y^{2/3}$, $y(0) = 0$ (4) solutions $y(t) = 0$, $y(t) = \dfrac{1}{27}\, t^3$





Existence and uniqueness of solution of an IVP for an ODE

Theorem: Let $f(t, y)$ be continuous for $t \in [a, b]$ and for $y \in (-\infty, \infty)$. Suppose that f satisfies a **Lipschitz condition in y**

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2| \qquad \text{(A)}$$

for all $t \in [a, b]$ and for all $y_1, y_2 \in (-\infty, \infty)$, where L is a constant independent of t, y_1 and y_2. Then for any y_0, there exists a unique solution $y(t)$ to the IVP for an ODE

$$y' = f(t, y),\ y(a) = y_0 \qquad \text{(B)}$$

in the interval $[a, b]$. The constant L is called the **Lipschitz constant** associated with the Lipschitz condition (A).

Notes:

- The theorem gives sufficient conditions for existence and uniqueness of solution to the IVP, not necessary ones.
- The Lipschitz condition is stronger than continuity of f w.r.t. y.
- The Lipschitz condition is weaker than differentiability of f w.r.t. y and boundedness of $|\partial f / \partial y|$.
- The Lipschitz condition can be interpreted geometrically:
  Let us graph f versus y. The slope of any chord joining any two points on the graph must be bounded (i.e. it should not tend to or be plus or minus infinity).

Existence and uniqueness of solution of an IVP for an ODE

Examples:

In ODE (1), $f(t, y) = \cos t + \sin t$ (i.e. f is independent of y),
thus $|f(t, y_1) - f(t, y_2)| = 0$,
thus the Lipschitz condition is satisfied (with $L = 0$, for example)
and a unique solution exists for any y_0, to the IVP
$y' = \cos t + \sin t,\ y(t_0) = y_0$.

In ODE (2), $f(t, y) = \lambda y(t)$,
thus $|f(t, y_1) - f(t, y_2)| = |\lambda||y_1 - y_2|$,
thus the Lipschitz condition is satisfied with $L = |\lambda|$,
and a unique solution exists for any y_0, to the IVP
$y' = \lambda y,\ y(t_0) = y_0$.

In IVP-ODE (3), $f(t, y) = y^2$,
thus $|f(t, y_1) - f(t, y_2)| = |y_1 + y_2||y_1 - y_2|$,
but $|y_1 + y_2|$ is not bounded for all $y_1, y_2 \in (-\infty, \infty)$,
therefore, the theorem is not applicable,
i.e. it does not tell us whether a unique solution exists.

Second order ODEs and BVPs

In all the previous examples, the highest order derivative of the unknown function occurring in the ODE was 1. This need not always be the case.

Example:
$y'' = 2 \qquad (5) \qquad$ solution $y(t) = t^2 + c_1 t + c_2$

Notice that the solution of (5) involves two constants of integration (free parameters c_1 and c_2).

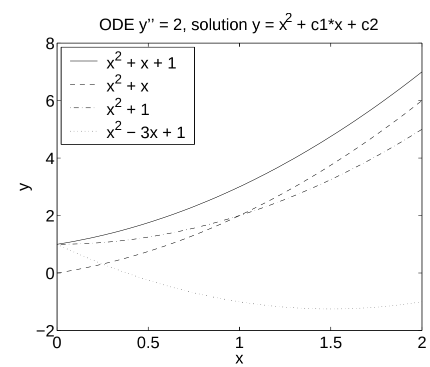
In order to pick one solution of (5) from the two-parameter family of solutions, we have to specify two additional conditions.
These may be the value of $y(t)$ at one point, say $t = t_0$, and the value of $y'(t)$ at t_0.
Or, alternatively,
these may be the values of $y(t)$ at two points, say $t = a$ and $t = b$.

In the latter case, often the independent variable t represents space and we are given the value of y at two boundary points. The ODE then describes a physical law that y satisfies within an interval (a, b), while the two conditions describe boundary constraints.

Second order ODEs and BVPs

Example: Consider the ODE (5). The solution curves for various values of the constants c_1 and c_2 are shown in the figure. Specifying c_1 and c_2 fixes the solution curve.

Boundary Value Problem (BVP) for an ODE (or **two-point BVP**): an ODE together with associated boundary conditions (BC) at two points.



nth order ODEs and IVPs for ODEs -- Systems of ODEs

Order of an ODE (or **order** of an IVP for an ODE): the order of the highest derivative occurring in the ODE.

The general nth order IVP for an ODE takes the form

$$y^{(n)} = f(t, y, y', y'', \cdots, y^{(n-1)}), \qquad \text{(Ia)}$$

$$y(t_0) = y_0,\ y'(t_0) = y'_0,\ y''(t_0) = y''_0,\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}, \qquad \text{(Ib)}$$

where f is a given function of the independent variable t and of the unknowns $y, y', y'', \cdots, y^{(n-1)}$, while y_0, y'_0, y''_0, ... , $y_0^{(n-1)}$ are given numbers.

A nth order IVP for an ODE, such as (Ia)-(Ib), can be converted into a mathematically equivalent first order IVP for a **system of *n* ODEs** in the following way. Let

$$\begin{array}{lcl}
u_1(t) \equiv y(t) & & u'_1(t) = u_2(t), \\
u_2(t) \equiv y'(t) & & u'_2(t) = u_3(t), \\
u_3(t) \equiv y''(t) & & u'_3(t) = u_4(t), \\
\ldots & \text{Then} & \ldots \\
u_{n-1}(t) \equiv y^{(n-2)}(t) & & u'_{n-1}(t) = u_n(t), \\
u_n(t) \equiv y^{(n-1)}(t) & & u'_n(t) = f(t, u_1, u_2, u_3, \cdots, u_n)
\end{array}$$

nth order ODEs and IVPs for ODEs -- Systems of ODEs

The above gives rise to the following system of n first order ODEs:

$$\bar{u}'(t) = \bar{f}(t, \bar{u}), \text{ where } \bar{u}(t) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdots \\ u_{n-1} \\ u_n \end{bmatrix},\ \bar{f}(t, \bar{u}(t)) = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \cdots \\ u_n \\ f(t, u_1, u_2, u_3, \cdots, u_n) \end{bmatrix} \qquad \text{(IIa)}$$

The unknown $\bar{u}(t)$ is a column vector of n functions and so is the right side $\bar{f}(t, \bar{u}(t))$. Then, write the initial conditions associated with (IIa):

$$u_1(t_0) = y_0,\ u_2(t_0) = y'_0,\ u_3(t_0) = y''_0,\ \cdots,\ u_n(t_0) = y_0^{(n-1)} \qquad \text{(IIb)}$$

Thus (Ia)-(Ib) is converted into (IIa)-(IIb).
It is easy to see that $u_1(t)$ satisfies (Ia)-(Ib), i.e. $u_1(t) = y(t)$.

Example: Consider the second order ODE (5) ($y'' = 2$) and two initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$\begin{array}{l} u_1 = y \\ u_2 = y' \end{array} \Rightarrow \begin{array}{l} u'_1 = u_2 \\ u'_2 = 2 \end{array} \Rightarrow \bar{u}' = \bar{f} \text{ where } \bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ \bar{f} = \begin{bmatrix} u_2 \\ 2 \end{bmatrix} \text{ with } \begin{array}{l} u_1(0) = 1 \\ u_2(0) = 0 \end{array}$$

nth order ODEs and IVPs for ODEs -- Systems of ODEs

A general system of n first order ODEs takes the form

$$\bar{y}'(t) = \bar{f}(t, \bar{y}), \text{ where } \bar{y}(t) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_n \end{bmatrix},\ \bar{f}(t, \bar{y}(t)) = \begin{bmatrix} f_1(t, y_1, y_2, y_3, \cdots, y_n) \\ f_2(t, y_1, y_2, y_3, \cdots, y_n) \\ f_3(t, y_1, y_2, y_3, \cdots, y_n) \\ \cdots \\ f_n(t, y_1, y_2, y_3, \cdots, y_n) \end{bmatrix} \qquad \text{(IIIa)}$$

The initial conditions associated with (IIIa) take the form

$$y_1(t_0) = y_{1,0},\ y_2(t_0) = y_{2,0},\ y_3(t_0) = y_{3,0},\ \cdots,\ y_n(t_0) = y_{n,0} \qquad \text{(IIIb)}$$

Equations (IIIa) and (IIIb) form a **first order IVP for a system of *n* (first order) ODEs**.

Jacobian of a system of ODEs: the matrix J associated with the right side of (IIIa) and defined by $J_{ij} = \dfrac{\partial f_i}{\partial y_j}$, or

$$J = \begin{bmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} & \cdot & \dfrac{\partial f_1}{\partial y_n} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} & \cdot & \dfrac{\partial f_2}{\partial y_n} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial f_n}{\partial y_1} & \dfrac{\partial f_n}{\partial y_2} & \cdot & \dfrac{\partial f_n}{\partial y_n} \end{bmatrix}$$

nth order ODEs and IVPs for ODEs -- Systems of ODEs

Example: Consider the system of 2 first order ODEs

$$y_1' = (1 - \alpha y_2) y_1,\ \ y_2' = (-1 + \beta y_1) y_2$$

where α and β are given scalars. The Jacobian is

$$J = \begin{bmatrix} 1 - \alpha y_2 & -\alpha y_1 \\ \beta y_2 & -1 + \beta y_1 \end{bmatrix}$$

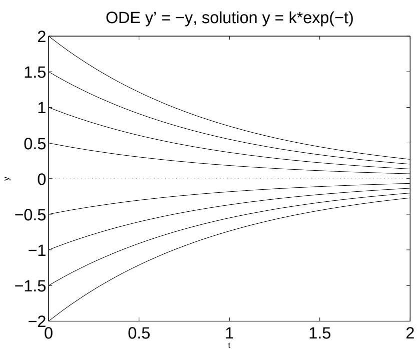
Note that the entries of the Jacobian are functions of $y_1(t)$ and $y_2(t)$ and (indirectly) of t.

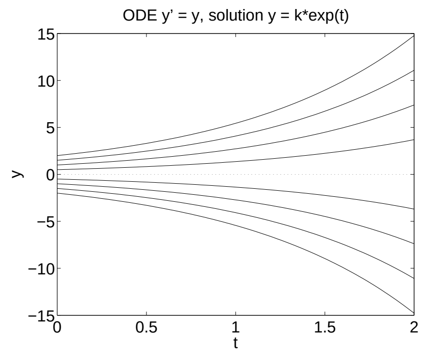
Stability of ODEs -- The flow field and the Jacobian

Consider again the ODE

$\frac{dy}{dt} = \lambda y(t)$ (2) solution $y(t) = \kappa e^{\lambda t}$

for the cases $\lambda = -1$ and $\lambda = 1$. The families of solutions for each case are shown in the figures.





Stability of ODEs -- The flow field and the Jacobian

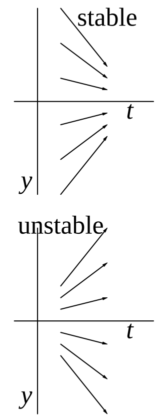
Notice that for $\lambda = -1$ the solution curves fan-in / converge, while for $\lambda = 1$ they fan-out / diverge. In the former case, we say that the ODE is **stable**, while in the latter **unstable**. For an unstable ODE, a small move away from a solution curve will get us much further away from it, as we proceed in time.

The *flow field* (or *direction field*) allows us to visualize how the solution curves behave and how the error propagates in time. The flow field is drawn by drawing, at several points in time, the directions (tangent arrows) towards which the solution curves proceed.

Since $y' = f(t, y)$, the quantity $f(t^*, y^*)$ shows the slope of the solution curve that passes from (t^*, y^*).

For a fixed t, the rate at which the slopes vary with y is given by $\frac{\partial f}{\partial y} \equiv f_y \equiv J$. This quantity is called the **Jacobian**.

Stability of ODEs -- The flow field and the Jacobian



Assume, for simplicity, that J is real.
If $f_y < 0$, then f is a decreasing function w.r.t. y, i.e., as y ranges from $y^* - \epsilon$ to $y^* + \epsilon$, the slopes are decreasing. (For example, for increasing positive y's, they get steeper and steeper pointing downward.) This means the solution curves fan-in and the ODE is stable.

If $f_y > 0$, the ODE is unstable.

For the ODE (2), and $\lambda = -1$, $(y' = -y)$, $f_y = -1 < 0$.
For the ODE (2), and $\lambda = +1$, $(y' = +y)$, $f_y = +1 > 0$.

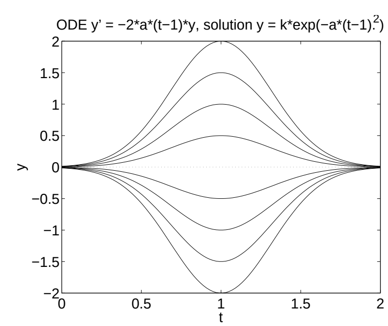
If $J \ll 0$, i.e. if J is very large in absolute value and negative, the ODE is stable, but **stiff**. (Stiff ODEs will be revisited later when studying numerical methods for ODEs.)

For a system of n ODEs, we define the Jacobian to be the $n \times n$ matrix $J_{ij} = \frac{\partial f_i}{\partial y_j}$.
(This relates the Jacobian of a system of ODEs to the Jacobian of a system of nonlinear equations.)

Stability of ODEs -- The flow field and the Jacobian

The stability of a system of ODEs is a more complicated issue, but a simplified view is the following: If all the eigenvalues of J have negative real parts, the system of ODEs is stable. Positive real parts often correspond to unstable systems of ODEs.

In the general case, f_y varies with t and y, so an ODE may be stable in a region and unstable in another region.
Example: $y' = -2\alpha(t-1)y$ (6), solution $y(t) = ce^{-\alpha(t-1)^2}$.



Let $\alpha = 5$ and consider the range $0 \le t \le 2$. We have $f_y = -2\alpha(t-1)$, thus, $f_y > 0$ for $t < 1$, and $f_y < 0$ for $t > 1$. Thus the ODE is unstable in $[0, 1)$ and stable in $(1, 2]$. Changing from $y(0) = 0.01$ to $y(0) = 0.1$ changes the solution at $t = 1$ from approximately 1.5 to approximately 15. But the solutions at $t = 2$ only differ by 0.09.

Stiff ODEs

We have seen that, for a single ODE, when the Jacobian J is real and very negative, the ODE is called *stiff*. For example, the ODE $y' = -\lambda y$ (solution $y = ce^{-\lambda t}$) is stiff, when λ is large.
Stiff ODEs will be revisited later when studying numerical methods for ODEs. We will see that, for a stiff ODE, certain numerical methods require a lot of effort to compute an accurate solution.

For the general case of a system of n ODEs, stiffness is a more complicated issue, thus we present a simplified view.

Consider the Jacobian (matrix) J (with $J_{ij} = \dfrac{\partial f_i}{\partial y_j}$) of the system of ODEs. Let λ_i, $i = 1, \cdots, n$, be the eigenvalues of J. Stiffness has to do with (a) some (or all) eigenvalues of the Jacobian having very negative real parts and/or (b) a large ratio $\dfrac{\max_i |Re(\lambda_i)|}{\min_i |Re(\lambda_i)|}$, where $Re(\lambda_i)$ is the real part of λ_i.

Stiff ODEs

Another view at stiffness calls stiff the ODEs whose exact solution has a term of the form $e^{-\kappa t}$, where κ is a large positive constant. Often, this is only a part of the solution, called the **transient** part. The other part is called the **steady-state** part. Note that the transient part decays to zero rapidly as t increases (thus it eventually reaches a steady state in short time), however the derivatives of it do not decay as fast. More specifically, the nth derivative $\kappa^n e^{-\kappa t}$ may increase as t increases.