Computing Equilibrium Distributions Of Interacting Particles

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Joint work with
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- What drives large groups of bees/birds/bacteria/particles/etc. to form nice distributions?
- Attractive–Repulsive dynamics!
- Can we compute these distributions fast and accurately?
- Yes! Using the beautiful theory of orthogonal polynomials

- Applications: Biology, physics, and mathematics
- Classical problem: point charges in a potential well (repulsion only)
- Modern problem: power law attractive-repulsive interactions

Point dynamics → Measure minimisation → Low-dimensional root finding

Using orthogonal polynomials to discretise!

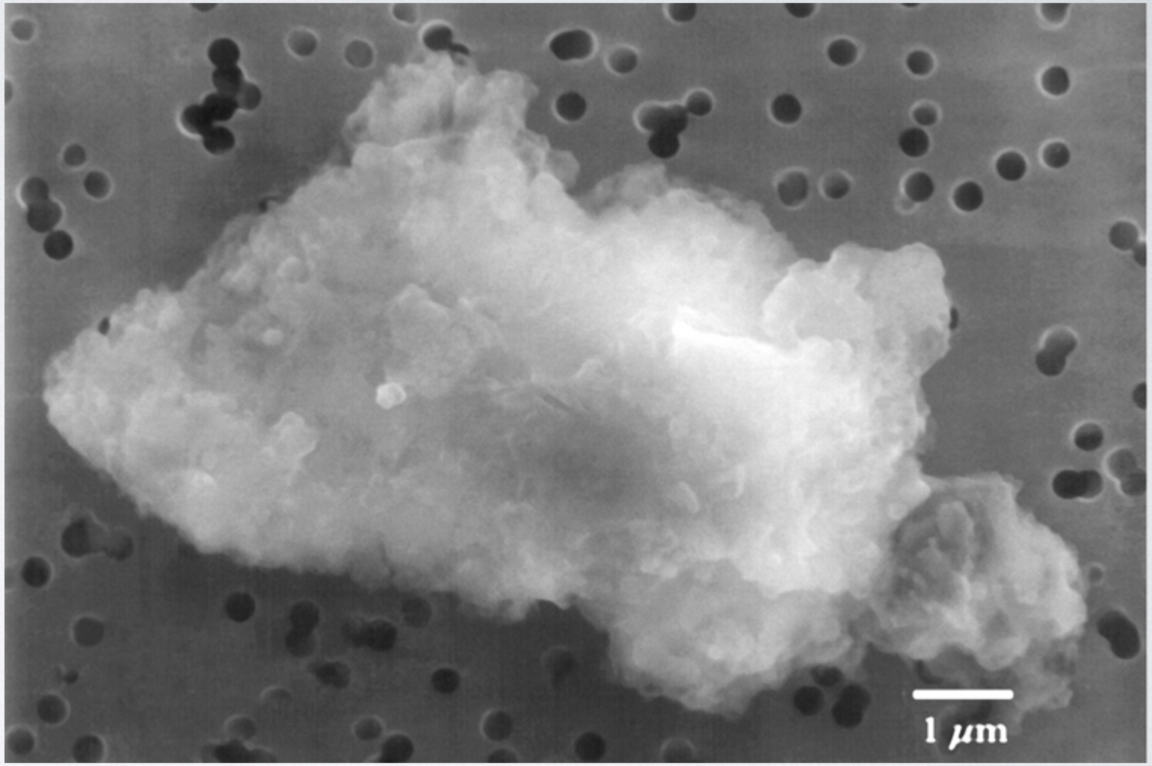
BIOLOGY: BIRDS

Birds want to fly together for protection (attractive) but not too close to avoid collisions (repulsive)

From WikiMedia, Rison Thumboor

Physics: Cosmic Dust

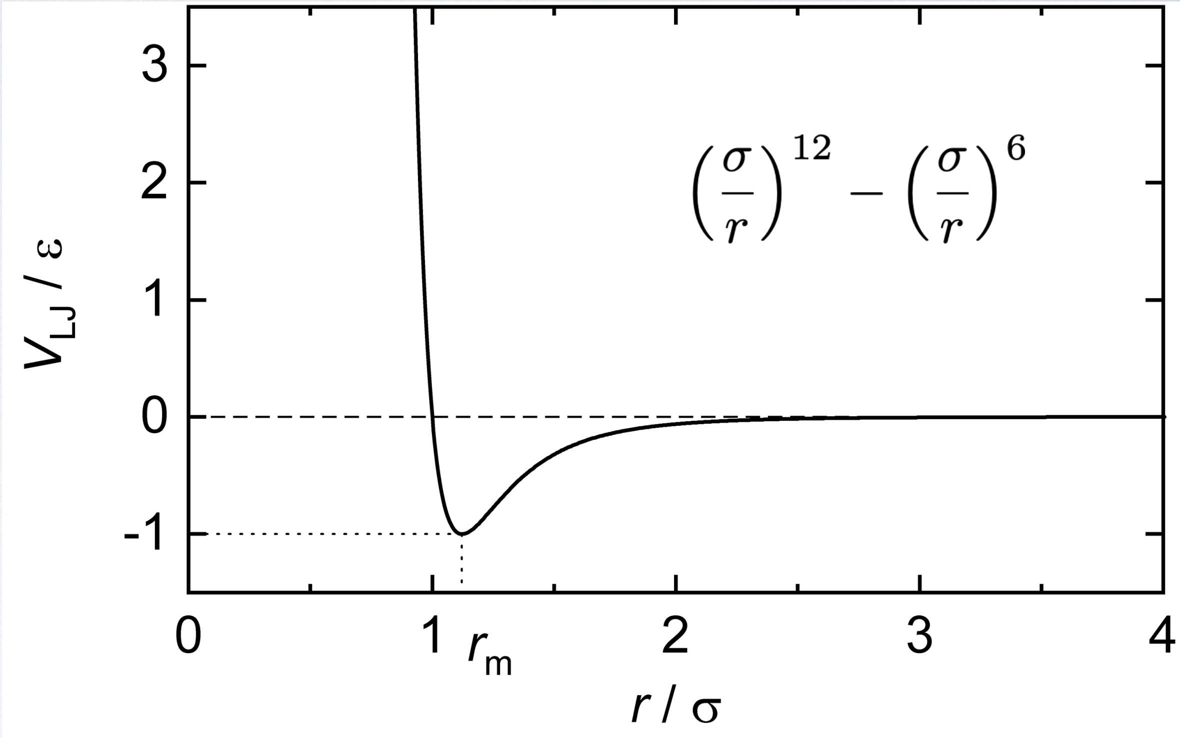
Gravity (attractive)
"Size" (repulsive)



From Wikipedia article on Cosmic Dust

PHYSICS: LENNARD–JONES POTENTIAL

Models interaction of molecules which repel at small distances due to overlapping electron orbitals and attractive dispersive force



From Wikipedia Lennard–Jones potential article

Point particle model

$$v_k = \frac{\mathrm{d}x_k}{\mathrm{d}t}$$

$$\frac{\mathrm{d}v_k}{\mathrm{d}t} = f\left(\|v_k\|\right) v_k - \frac{1}{N}\sum_{j\neq k} \nabla K(\|x_k - x_j\|) - \nabla V(x_k)$$

interaction potential

Background potential

self-propulsion/friction

Point particle model

$$v_k = \frac{\mathrm{d}x_k}{\mathrm{d}t}$$

$$\frac{\mathrm{d}v_k}{\mathrm{d}t} = f\left(\|v_k\|\right) v_k - \frac{1}{N}\sum_{j\neq k} \nabla K(\|x_k - x_j\|) - \nabla V(x_k)$$

interaction potential

Electrostatic, Newtonian,
power law,
Cucker–Smale, …

Classical example in 1D: Particles in electrostatic well

$$K(|x-y|) = \log|x-y|^{-1}$$

$$v_k = \frac{\mathrm{d}x_k}{\mathrm{d}t}$$

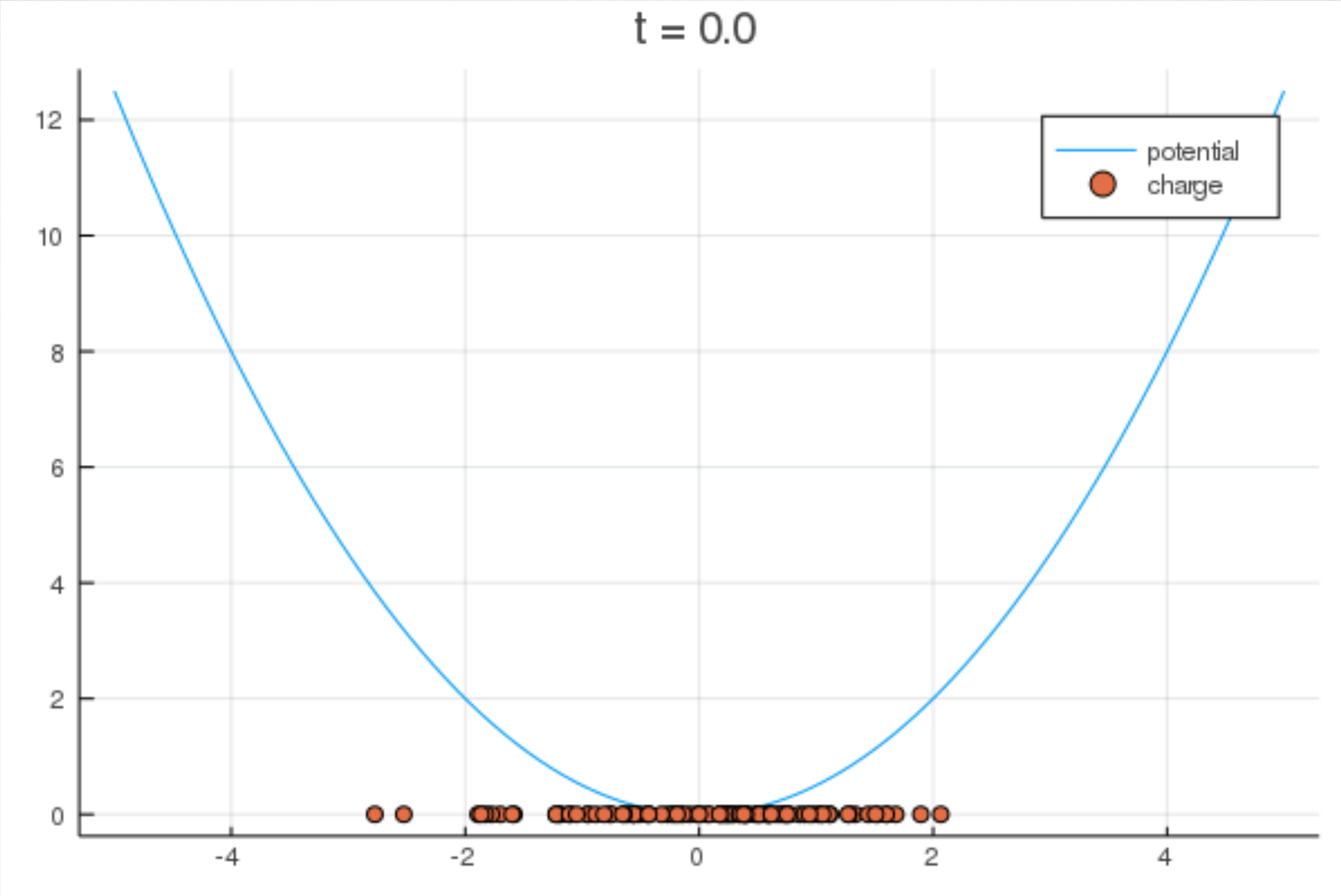
$$\frac{\mathrm{d}v_k}{\mathrm{d}t} = -\gamma v_k + \frac{1}{N}\sum_{j\neq k}\frac{1}{x_k - x_j} - V'(x_k)$$

Damping

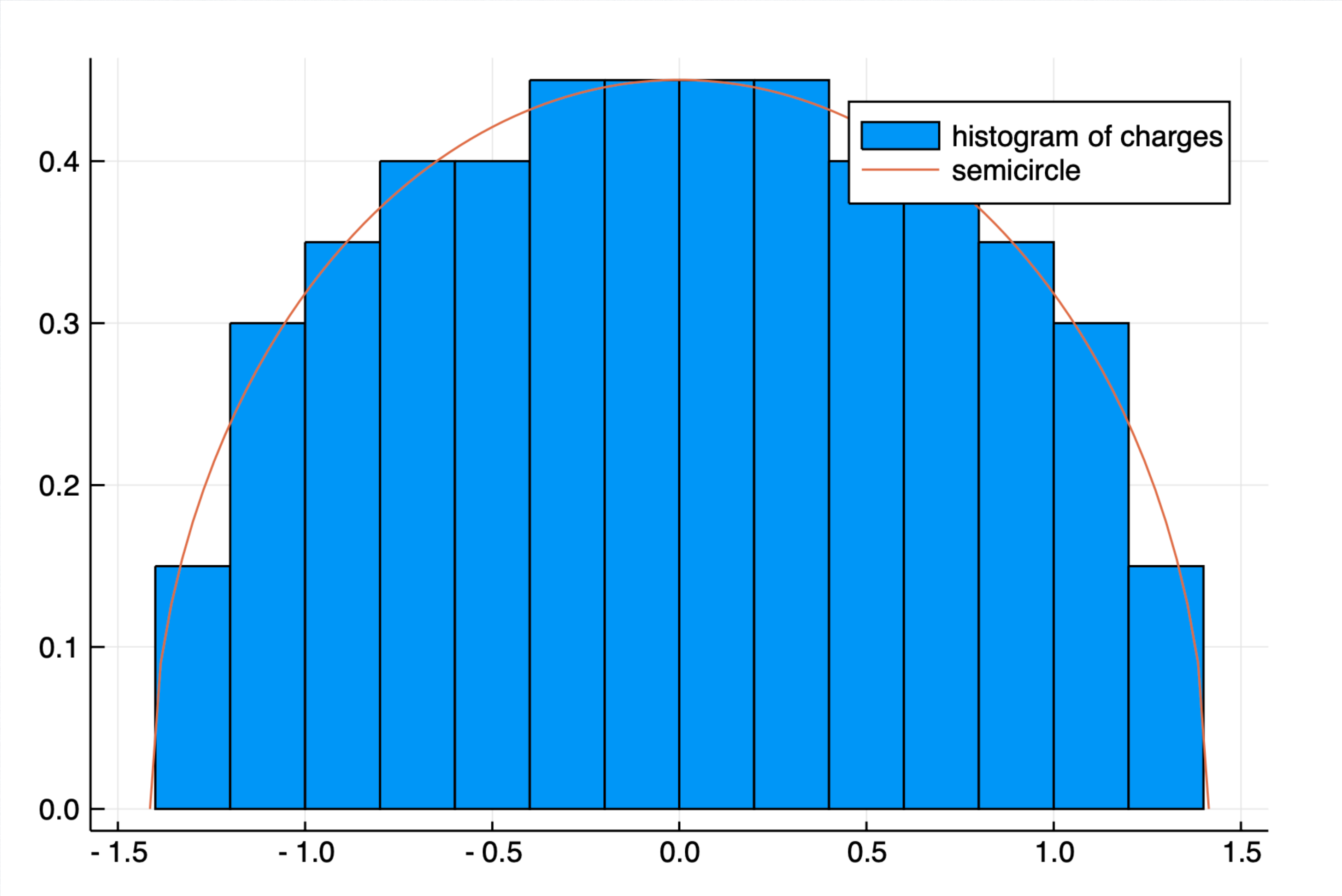
Repulsive

Background potential

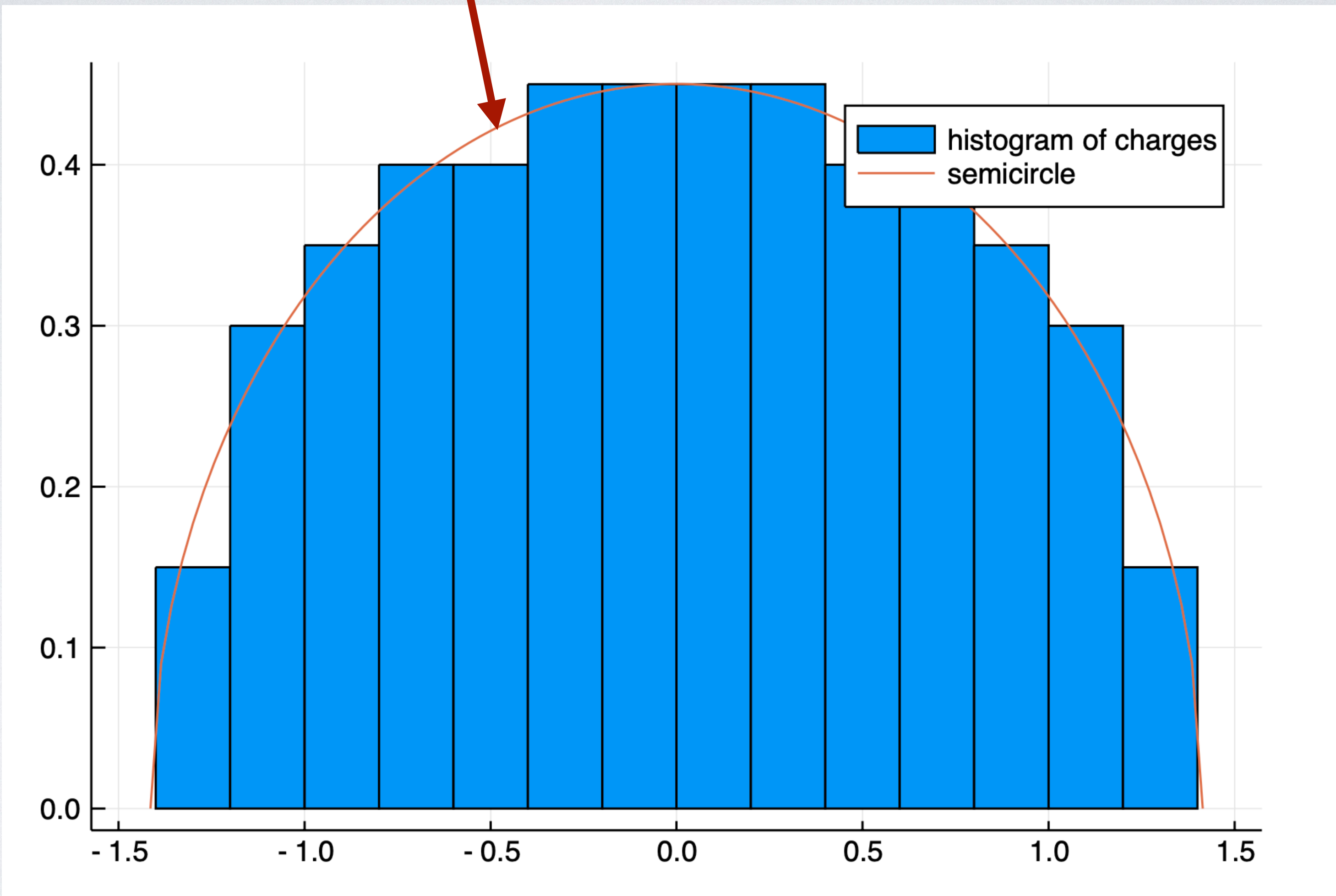
Electrostatic repulsion: $K(x) = -\log|x|, V(x) = x^2$



Charges binned and rescaled by $\sqrt{N}$



How do we compute this curve?



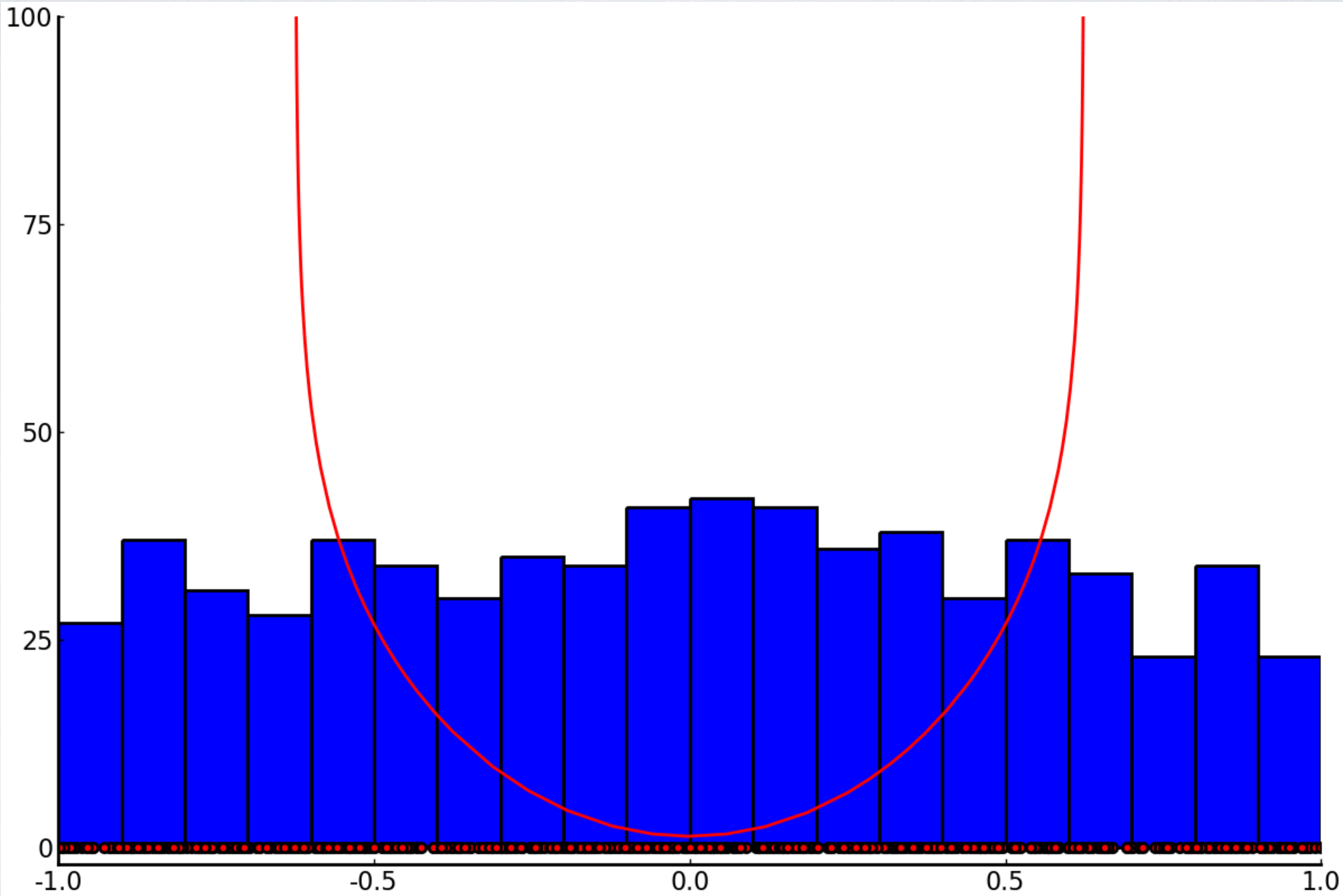
Modern example: Attractive-Replusive dynamics

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{1}{N}\sum_{j\neq i}\nabla\left[\frac{\|x_i - x_j\|^\alpha}{\alpha} - \frac{\|x_i - x_j\|^\beta}{\beta}\right]$$

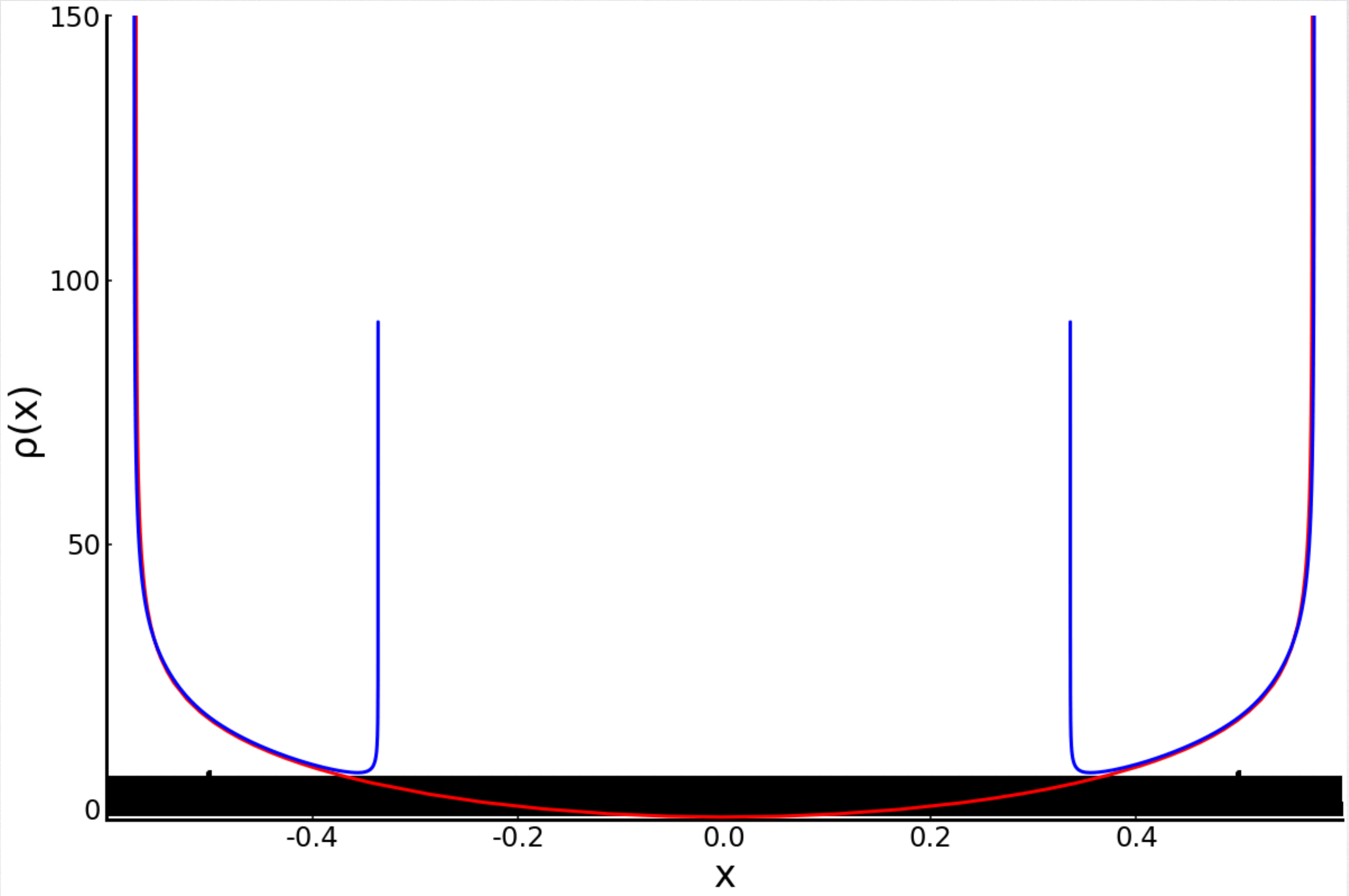
Repulsive

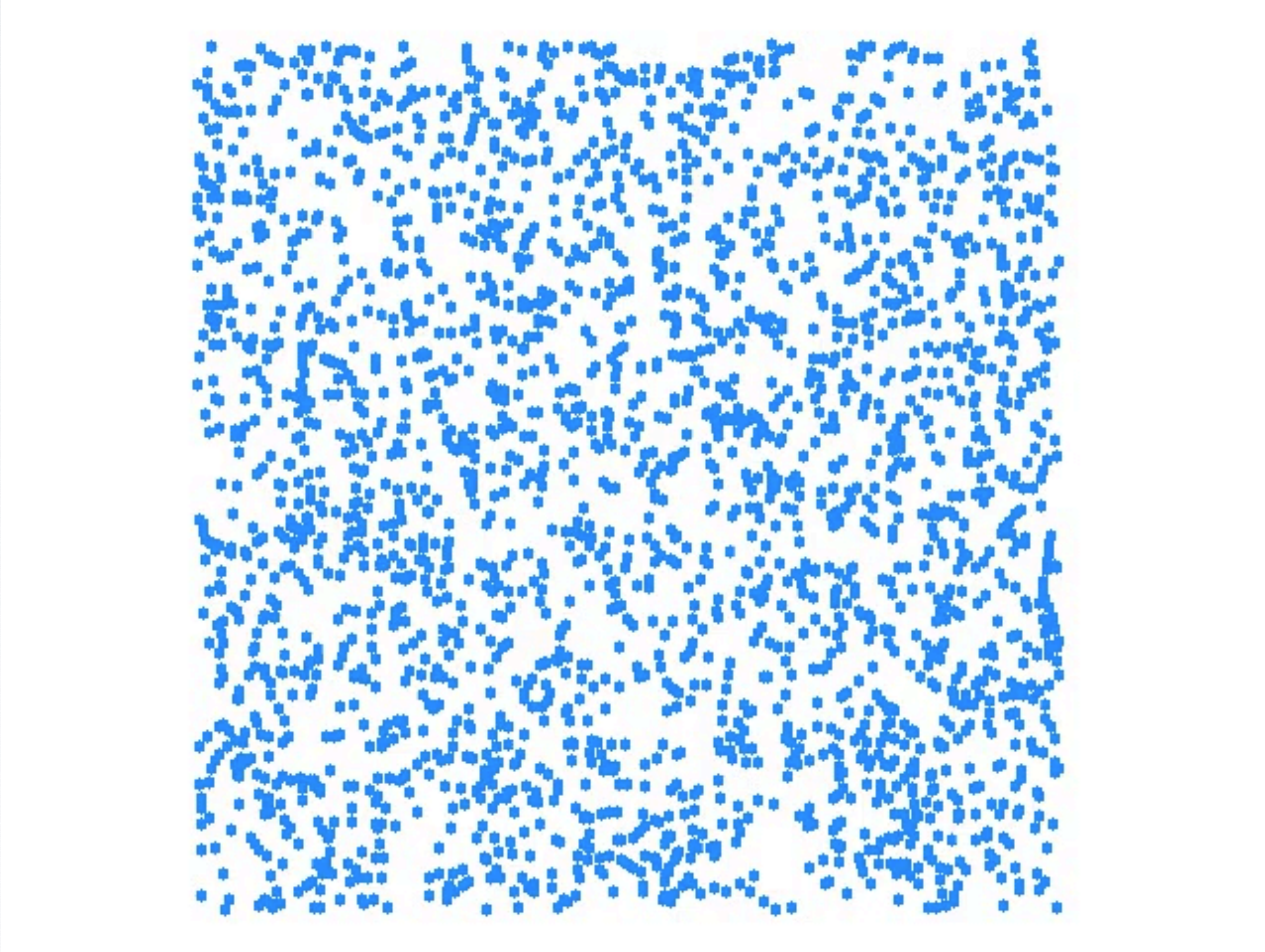
Attractive

Attractive-repulsive: $\alpha = 4, \beta = 1.45$



Attractive-repulsive: $a = 4,\ \beta = 1.61$





$\alpha = 4, \beta = 0.85$

STOCHASTIC MODEL

Brownian motion

$$\mathrm{d}x_k = \left[-\frac{1}{N} \sum_{j \neq k} \nabla K(\|x_k - x_j\|) - \nabla V(x_k) \right] \mathrm{d}t + \mathrm{d}W_k$$

interaction potential

Background potential

STOCHASTIC MODEL

$$\mathrm{d}x_k = \left[-\frac{1}{N} \sum_{j \neq k} \nabla K(\|x_k - x_j\|) - \nabla V(x_k) \right] \mathrm{d}t + \mathrm{d}W_k$$

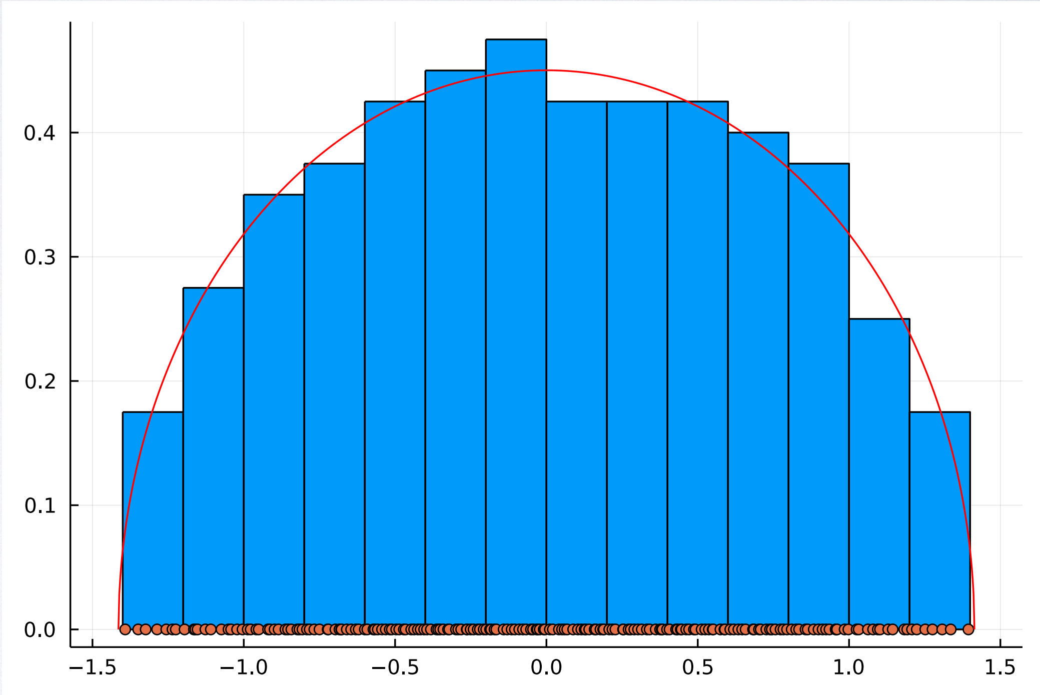
interaction potential

Electrostatic (eg. Dyson brownian motion),
Connected to eigenvalues of random matrices

EIGENVALUES OF RANDOM MATRICES

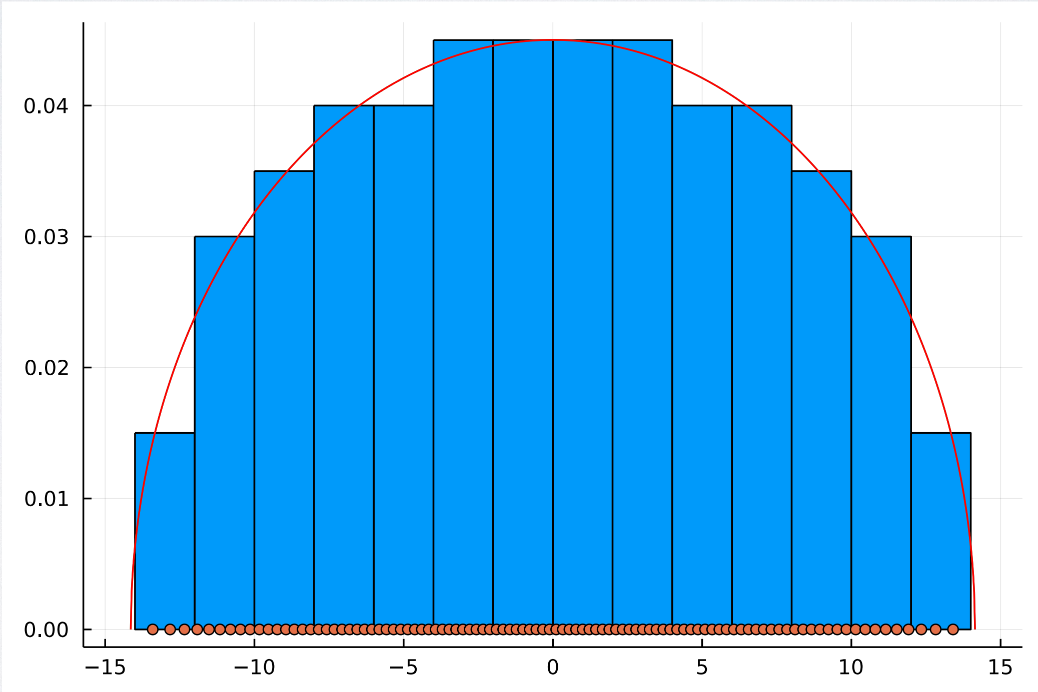
Eigenvalues of random matrices whose distributions are invariant under unitary conjugation are distributed according to

$$\mathrm{d}x_k = \left[\frac{1}{N}\sum_{j\neq k}\frac{1}{x_k - x_j} - \frac{V'(x_k)}{2}\right]\mathrm{d}t + \sqrt{\frac{2}{\beta N}}\,\mathrm{d}W_k$$



Gaussian Unitary Ensemble (GUE) eigenvalues

ZEROS OF ORTHOGONAL POLYNOMIALS



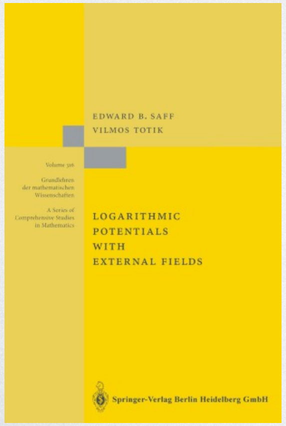
Zeros of Hermite polynomials

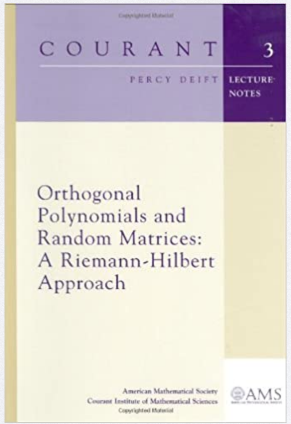
Equilibrium Measure

Given an external field $V : \mathbb{R} \to \mathbb{R}$, the equilibrium measure is the unique Borel measure $\mathrm{d}\mu = \psi(x)\,\mathrm{d}x$ such that

$$\iint K(|t-s|)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s) + \int V(s)\,\mathrm{d}\mu(s)$$

is minimal.





Electrostatic case: [Saff and Totik 1997], [Deift 1999]
Power-law case: [Balagué, Carrillo, Laurent and Raoul 2003]

Equilibrium Measure

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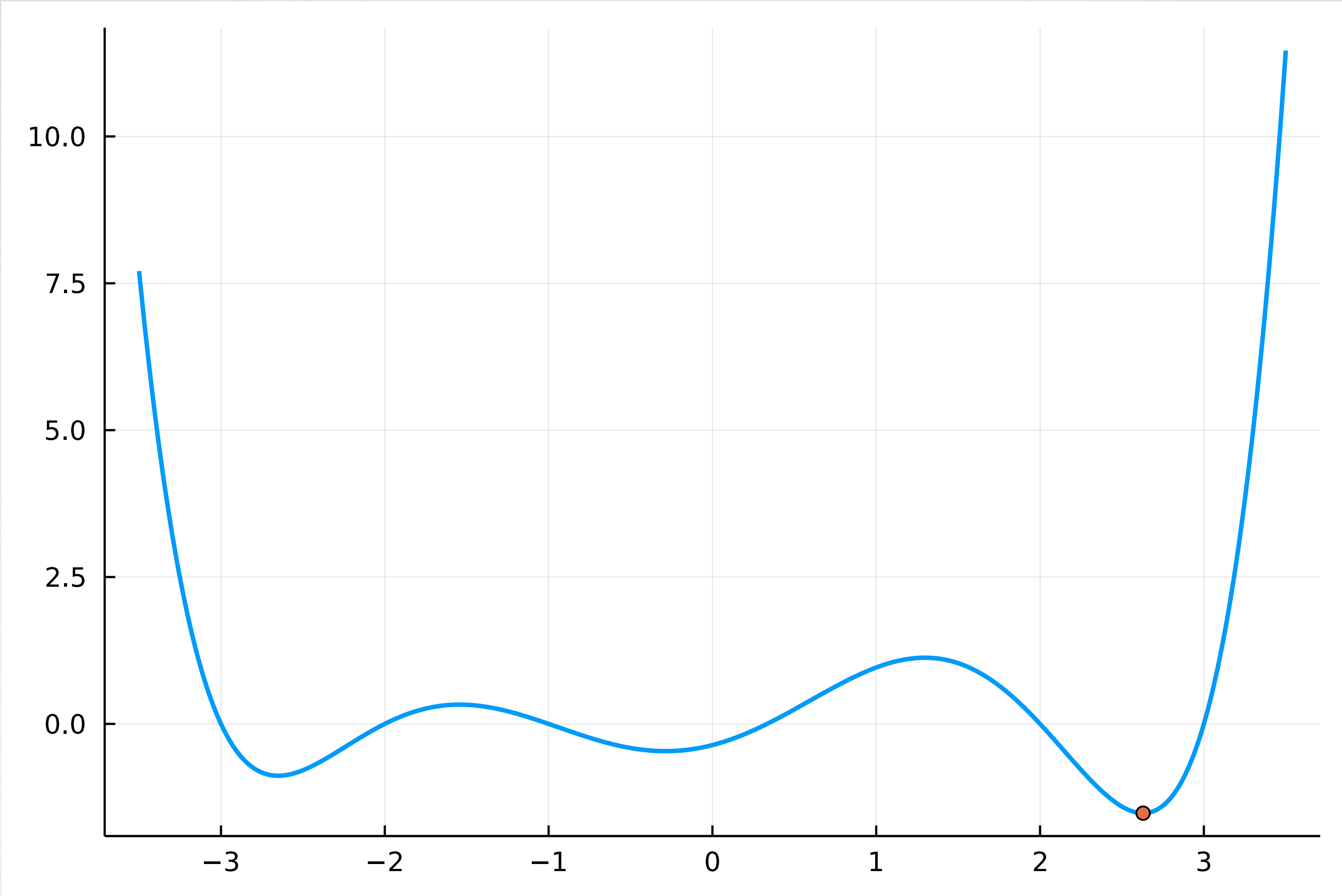
is minimal.

How to compute μ?

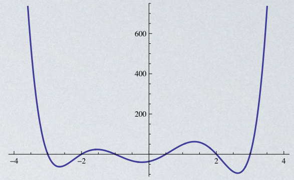
Electrostatic case: [Saff and Totik 1997], [Deift 1999]
Power-law case: [Balagué, Carrillo, Laurent and Raoul 2003]

Potential

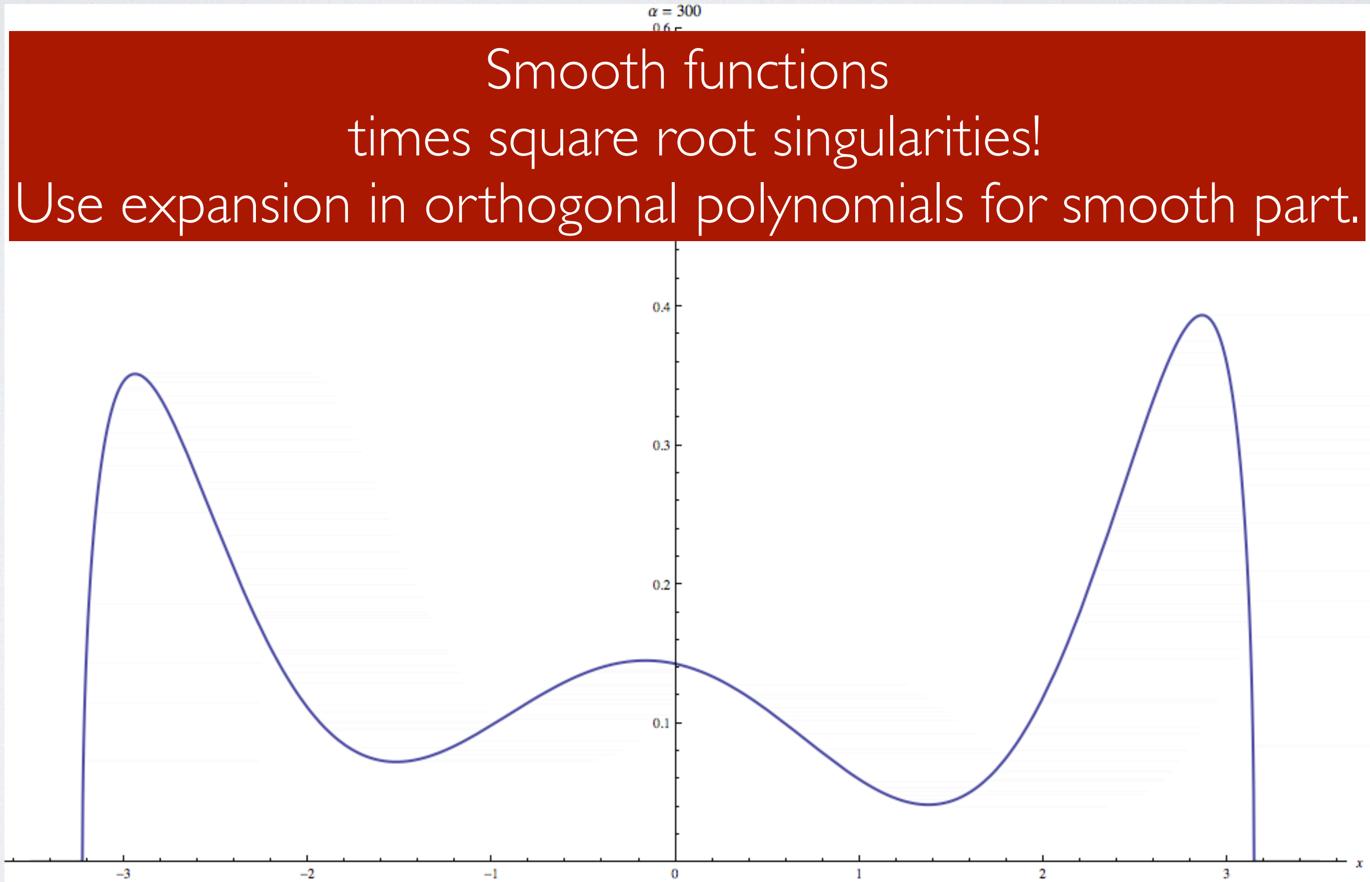


One charge minimum

$$V(x) = \frac{(-3+x)(-2+x)(1+x)(2+x)(3+x)(-1+2x)}{\alpha} = \frac{1}{\alpha}$$



Smooth functions
times square root singularities!
Use expansion in orthogonal polynomials for smooth part.



Outline of Rest of Talk

- Orthogonal polynomials and sparse discretisations
- Log and Cauchy kernel singular integral equations
- Potential theory equilibrium measures
- Power-law equilibriums
- Applications to PDEs

Orthogonal Polynomials In 1D

- We work with *orthogonal polynomials*, which are polynomials

$$p_n(x) = k_n x^n + O(x^{n-1}),$$

($\kappa_n \neq 0$) orthogonal w.r.t. a weight $w(x)$:

$$\int p_n(x) p_m(x) w(x) \, \mathrm{d}x = 0$$

if $n \neq m$.

  – Specifying κ_n is one way to uniquely define

- All orthogonal polynomials satisfy a *three-term recurrence*

$$\begin{aligned} x p_0(x) &= a_0 p_0(x) + b_0 p_1(x) \\ x p_n(x) &= c_{n-1} p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x) \end{aligned}$$

Name	Chebyshev polynomials (1st kind)	Chebyshev polynomials (2nd kind)	Ultraspherical
Notation	$T_n(x)$	$U_n(x) \equiv C_n^{(1)}(x)$	$C_n^{(\lambda)}(x)$
Explicit formula	$\cos n \operatorname{acos} x$	$\dfrac{T'_{n+1}(x)}{n+1}$	$\dfrac{C_{n+1}^{(\lambda-1)'}(x)}{2\lambda - 2}$
Weight	$\dfrac{1}{\sqrt{1-x^2}}$	$\sqrt{1-x^2}$	$(1-x^2)^{\lambda-1/2}$
k_n	$\max(1,2^{n-1})$	2^n	$\dfrac{2^n(\lambda)_n}{n!}$

Expansion of functions with singularities

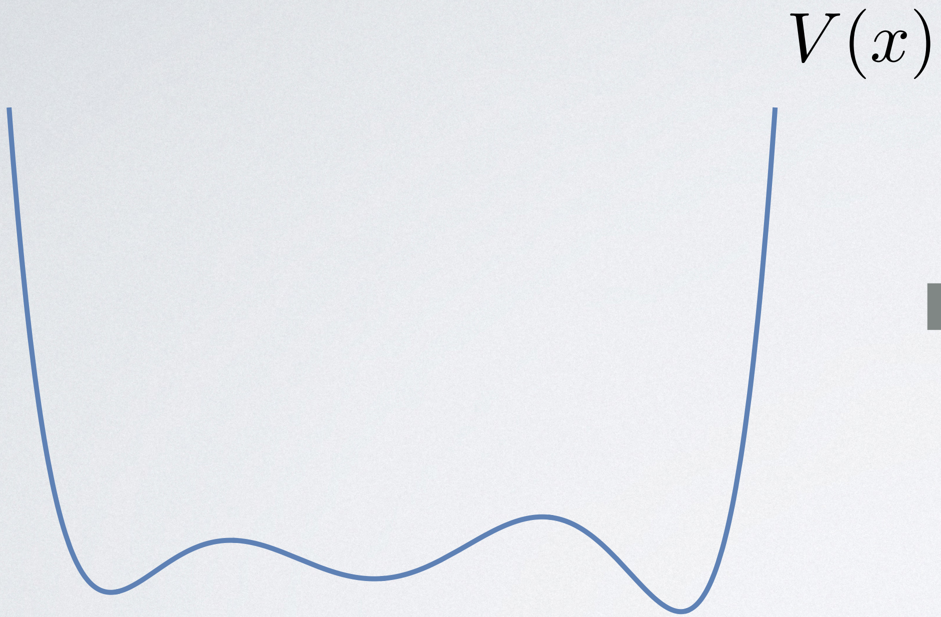
- We saw numerically that equilibrium measures with log kernel interactions tend to have square-root singularities at the edge of their support.
- It is therefore natural to expand on each interval of support $[a, b]$

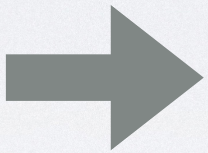
$$\mathrm{d}\mu(x) = \sqrt{b-x}\sqrt{x-a} \underbrace{\sum_{n=0}^{\infty} c_n U_n \left(\frac{2x-a-b}{b-a} \right)}_{\text{polynomial}} \mathrm{d}x$$

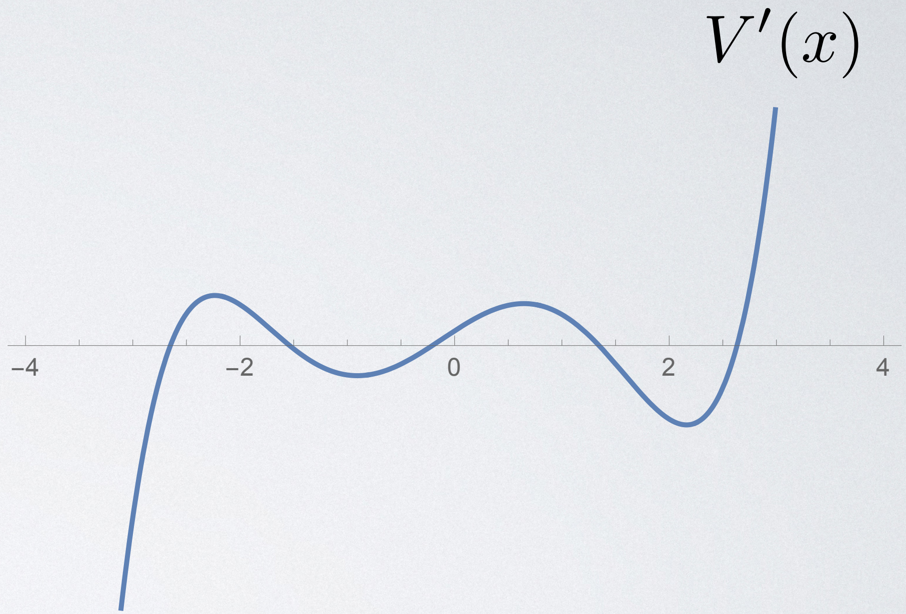
- Power law kernels will involve ultraspherical weights $(b-x)^{\lambda-1/2}(x-a)^{\lambda-1/2}$
- Goal: express the energy minimisation problem as an easy problem on a, b and c_n

MINIMISATION

ROOT-FINDING







(Stochastic) Gradient descent

Newton iteration,
Deflation for multiple solutions

Minimize

$$\iint K(|t-s|)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s) + \int V(s)\,\mathrm{d}\mu(s)$$

$\Big\Downarrow$ Euler–Lagrange

$$\int K(|x-t|)\,\mathrm{d}\mu(t) + V(x) = \ell$$

$\Big\Downarrow$ Differentiation

$$\int K'(|x-t|)\,\mathrm{d}\mu(t) + V'(x) = 0$$

Linear in c_n

If we know a and b!

- *Ultraspherical spectral methods* [Olver & Townsend 2012] reduce ordinary differential equations to sparse linear systems
  - Adapted to partial differential equations on rectangles, disks, and triangles
- Our goal: adapt this to equations involving Power-law, log kernels

$$\mathcal{L}_\alpha u(x) = \int_a^b |x-t|^\alpha u(t)\,\mathrm{d}t \qquad \mathcal{L}u(x) = \int_a^b \log|x-t| u(t)\,\mathrm{d}t$$

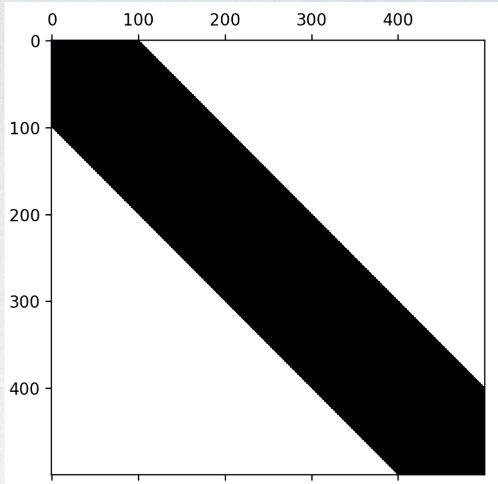
and Hilbert transforms (the derivative of $\mathcal{L}u(x)$)

$$\mathcal{H}u(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}u(x) = \fint_a^b \frac{u(t)}{x-t}\,\mathrm{d}t = \lim_{\epsilon\to 0^+}\left[\int_{x-\epsilon} + \int_{x+\epsilon}^b\right]\frac{u(t)}{x-t}\,\mathrm{d}t$$

$$T_n'(x) = nU_{n-1}(x)$$

$$T_n(x) \;=\; \frac{U_n(x) - U_{n-2}(x)}{2}$$

$$u'' - xu = f$$

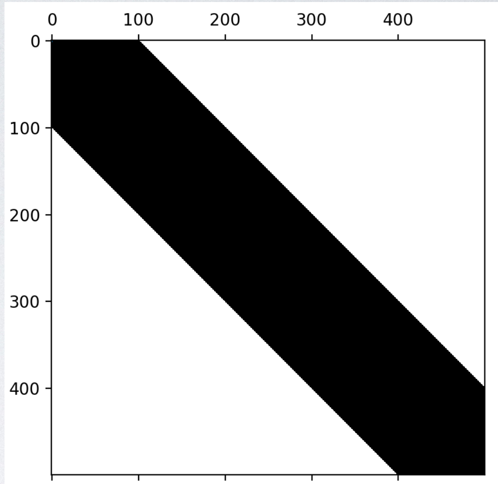


Recurrence relationships
For Differential Equations

Sparse discretisations
Of ODEs

$$T_n'(x) = nU_{n-1}(x)$$

$$T_n(x) = \frac{U_n(x) - U_{n-2}(x)}{2}$$



Can we do this for singular integral equations?

Theme 1:
nonlocal in physical space ,
Local (banded) in coefficient space

Theme 2:
Don't discretise integral operators

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Log kernel	$\mathscr{L}$	$\frac{1}{\pi}\int_{-1}^{1} \log\lvert t - x\rvert\, u(t)\mathrm{d}t$
Hilbert	$\mathscr{H}$	$\frac{1}{\pi}\fint_{-1}^{1} \frac{u(t)}{x - t}\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\mathscr{L}$
Fractional Laplacian	$\sqrt{\lvert\Delta\rvert}$	$\frac{-1}{\pi}⨎_{-1}^{1} \frac{u(t)}{(x - t)^2}\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\mathscr{H}$

- Denote $p_n(x)$ a family of orthogonal polynomials with respect to a weight $w(x)$ which satisfies

$$xp_n(x) = c_{n-1}p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x)$$

- $\mathcal{H}wP_n$ satisfies a 3-term recurrence:

$$\begin{aligned} x\mathcal{H}[wp_n](x) &= \frac{1}{\pi}\int_a^b \frac{x-t}{x-t} w(t) p_n(t)\,\mathrm{d}t + \mathcal{H}[twp_n](x) \\ &= \frac{1}{\pi}\int_a^b w(x)\,\mathrm{d}x \delta_{n0} + c_{n-1}\mathcal{H}[wp_{n-1}] + a_n \mathcal{H}[wp_n] + b_n \mathcal{H}[wp_{n+1}] \end{aligned}$$

- In the special case where $w(x) = \frac{1}{\sqrt{1-x^2}}$ we have $\mathcal{H}w(x) = 0$ and $\int_{-1}^1 w(x)\,\mathrm{d}x = \pi$. The recurrence becomes exactly that of the Chebyshev U polynomials and we find

$$\begin{aligned} \mathcal{H}\left[\frac{T_n}{\sqrt{1-t^2}}\right](x) &= -U_{n-1}(x) \\ \mathcal{H}\left[U_n\sqrt{1-t^2}\right](x) &= -T_{n+1}(x) \end{aligned}$$

Sparse recurrence!

- $\mathcal{H}$ and $\frac{\mathrm{d}}{\mathrm{d}x}$ commute (for "nice" u).
- In particular, using a recurrence for weighted differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{H}\sqrt{1-x^2}U_n(x) = \mathcal{H}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1-x^2}U_n(x) = -\mathcal{H}(n+1)\frac{T_{n+1}(x)}{\sqrt{1-x^2}}$$

- Integrating we get

$$\mathcal{L}\frac{1}{\sqrt{1-x^2}} = -\log 2$$

$$\mathcal{L}\frac{T_n(x)}{\sqrt{1-x^2}} = -\frac{T_n(x)}{n}$$

Diagonal!

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- Minimizing

$$-\iint \log|t-s| \, \mathrm{d}\mu(t)\, \mathrm{d}\mu(s) + \int V(s)\, \mathrm{d}\mu(s)$$

can be reduced via Euler–Lagrange reformulation to

$$-\int \log|x-t|\, \mathrm{d}\mu(t) + V(x) = \ell \quad \text{for} \quad x \in \operatorname{supp}\mu$$

$$-\int \log|x-t|\, \mathrm{d}\mu(t) + V(x) \geq \ell \quad \text{for all real } x$$

- Ignore the second condition: finite number of solutions to first condition
- Differentiating we get

$$⨍ \frac{\mathrm{d}\mu(t)}{x-t} = V'(x)$$

- If we know the support is $[a, b]$ then the problem is linear. How to find $[a, b]$?

- Condition (1): The solution must be a probability measure
- Condition (2): The linear system must be solvable:

$$\begin{pmatrix} 0 & & \\ 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} V_0 \\ V_1 \\ \vdots \end{pmatrix}$$

Mapped
Chebyshev U coefficients
of solution

Mapped
Chebyshev T coefficients
of RHS, depending on a, b

Newton + Auto Differentiation

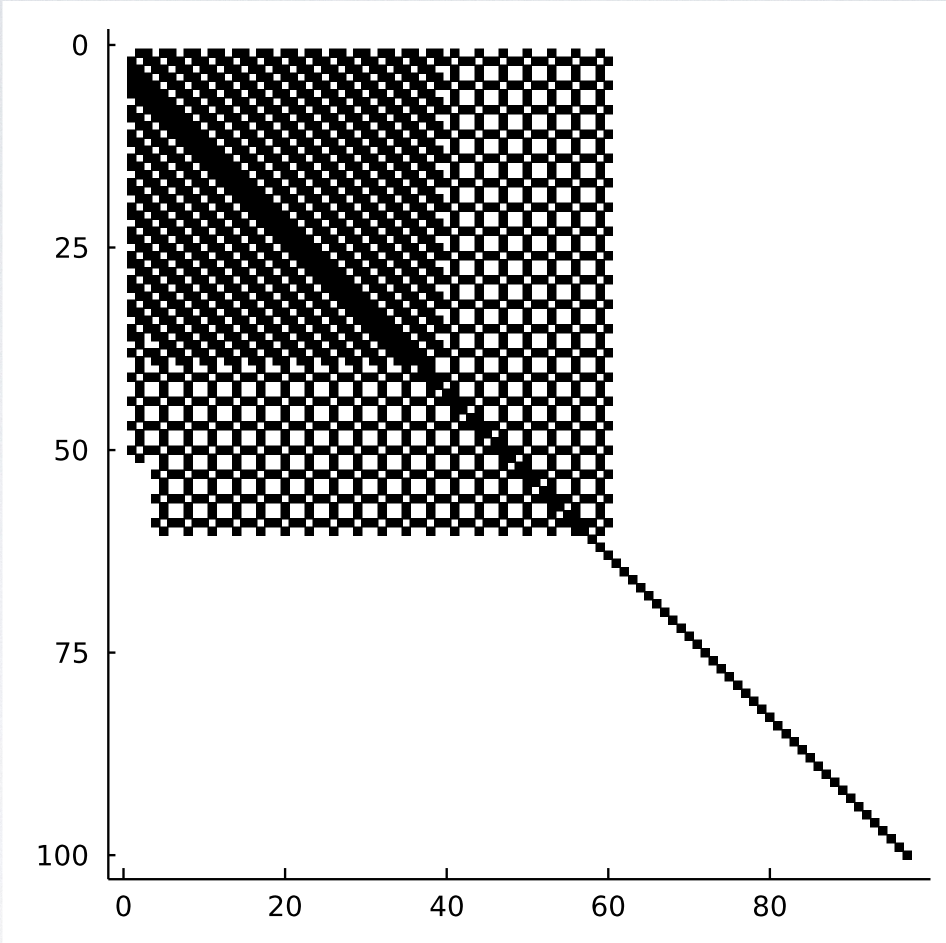
- Multiple intervals of support can be handled similarly: assume the support of the measure is a union of N intervals $(a_1, b_1), \ldots, (a_N, b_N)$ and take as an ansatz a weighted mapped Chebyshev expansion in each interval
- The interaction between intervals is compact
- Note: the constant ℓ in

$$\int \log|x - z| \, \mathrm{d}\mu + V(z) = \ell \quad \text{for} \quad z \in \text{supp}\, \mu$$

must be the same in each interval

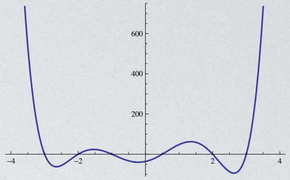
- Thus we get $2N$ conditions: N conditions to make the linear solvable, $N - 1$ conditions that ℓ must be the same, and finally the condition that the solution must be a probability measure

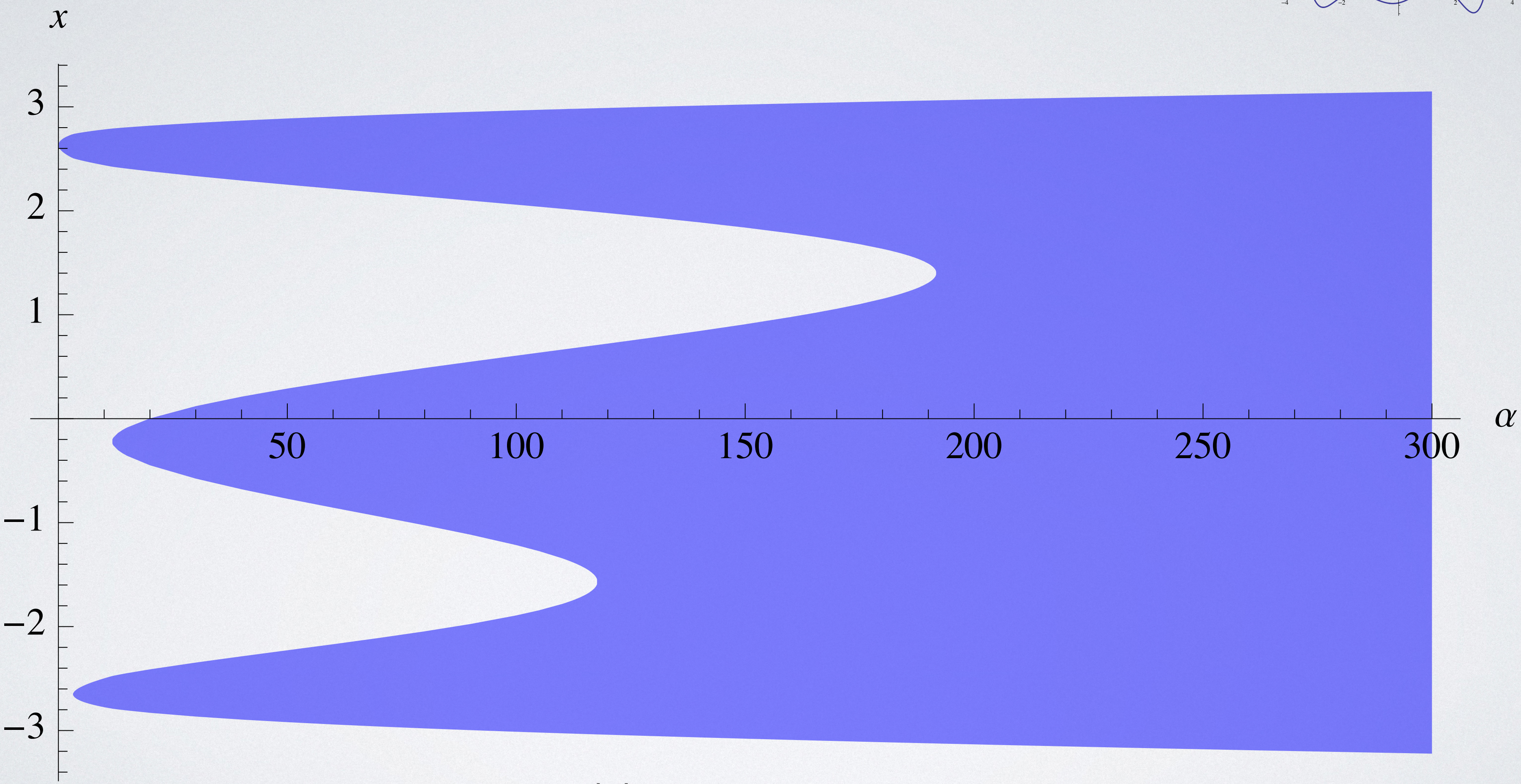
[Olver 2011]



Discretisation of Hilbert transform on 3 intervals

$$V(x) = \frac{(-3+x)(-2+x)(1+x)(2+x)(3+x)(-1+2x)}{\alpha} = \frac{1}{\alpha}$$





Measure support

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- To generalise this to power-law kernel

$$\mathcal{L}_\alpha u(x) = \int_{-1}^{1} |x-y|^\alpha u(t)\,\mathrm{d}t$$

we need to find a basis that we can apply the operator in closed form

- Probably weighted ultraspherical $(1-x^2)^{\lambda-1/2} C_n^{(\lambda)}(x)$ where λ depends on α
- Let's check the DLMF:

18.17.12 $$\frac{\Gamma(\lambda-\mu) C_n^{(\lambda-\mu)}\left(x^{-\frac{1}{2}}\right)}{x^{\lambda-\mu+\frac{1}{2}n}} = \int_x^\infty \frac{\Gamma(\lambda) C_n^{(\lambda)}\left(y^{-\frac{1}{2}}\right)}{y^{\lambda+\frac{1}{2}n}} \frac{(y-x)^{\mu-1}}{\Gamma(\mu)}\,\mathrm{d}y,$$

18.17.13 $$\frac{x^{\frac{1}{2}n}(x-1)^{\lambda+\mu-\frac{1}{2}}}{\Gamma\left(\lambda+\mu+\frac{1}{2}\right)} \frac{C_n^{(\lambda+\mu)}\left(x^{-\frac{1}{2}}\right)}{C_n^{(\lambda+\mu)}(1)} = \int_1^x \frac{y^{\frac{1}{2}n}(y-1)^{\lambda-\frac{1}{2}}}{\Gamma\left(\lambda+\frac{1}{2}\right)} \frac{C_n^{(\lambda)}\left(y^{-\frac{1}{2}}\right)}{C_n^{(\lambda)}(1)} \frac{(x-y)^{\mu-1}}{\Gamma(\mu)}\,\mathrm{d}y,$$

No good…but these “fractional integrals” give a hint

Left-fractional integral	$\mathcal{Q}_{\mathrm{L}}^{\alpha}$	$\frac{1}{\Gamma(\alpha)}\int_{-1}^{x} u(t)(x-t)^{\alpha-1}\mathrm{d}t$
Right-fractional integral	$\mathcal{Q}_{\mathrm{R}}^{\alpha}$	$\frac{1}{\Gamma(\alpha)}\int_{x}^{1} u(t)(t-x)^{\alpha-1}\mathrm{d}t$
Power-law	$\mathscr{L}_{\alpha}$	$\Gamma(\alpha+1)\left[\mathcal{Q}_{\mathrm{L}}^{\alpha+1}+\mathcal{Q}_{\mathrm{R}}^{\alpha+1}\right]$

Aside: Differentiating gives Caputo or Riemann–Liouville fractional derivatives
Which can be used for ultraspherical spectral method [Hale & Olver 2018]

- Let's see if fractional integrals of weighted OPs satisfy a recurrence by trying the same trick as what worked for Hilbert transforms:

$$x\mathcal{Q}_{\mathrm{L}}^{\alpha}[wp_n] = \frac{1}{\Gamma(\alpha)}\int_{-1}^{x}(x-t)^{\alpha}w(t)p_n(t)\,\mathrm{d}t + \mathcal{Q}_{\mathrm{L}}^{\alpha}[twp_n]$$
$$= \alpha\mathcal{Q}_{\mathrm{L}}^{\alpha+1}[wp_n] + c_{n-1}\mathcal{Q}_{\mathrm{L}}^{\alpha}[wp_{n-1}] + a_n\mathcal{Q}_{\mathrm{L}}^{\alpha}[wp_n] + b_n\mathcal{Q}_{\mathrm{L}}^{\alpha}[wp_{n+1}]$$

- Note that these fractional integrals behave like fractional versions of indefinite integrals:

$$\mathcal{Q}_{\mathrm{L}}^{\alpha+1} = \mathcal{Q}_{\mathrm{L}}^{\alpha}\mathcal{Q}_{\mathrm{L}} \qquad \text{for} \qquad \mathcal{Q}_{\mathrm{L}}u(x) = \int_{-1}^{x}u(t)\,\mathrm{d}t$$

- For *classical* ultraspherical polynomials we have

$$\mathcal{Q}_{\mathrm{L}}[(1-t^2)^{\lambda-1/2}C_n^{(\lambda)}](x) = A_n(1-x^2)^{\lambda-1/2}C_{n-1}^{(\lambda)}(x)+B_n(1-x^2)^{\lambda-1/2}C_{n+1}^{(\lambda)}(x)$$

- Thus $\mathcal{Q}_{\mathrm{L}}^{\alpha}[wp_n]$ and $\mathcal{Q}_{\mathrm{R}}^{\alpha}[wp_n]$ also satisfy a 3-term recurrence when w is ultraspherical weight!

THEOREM NEW?

For $x \in \mathbb{R}$, the integral operator

$$\mathcal{L}_\alpha u(x) = \int_{-1}^{1} |x-t|^\alpha u(t)\mathrm{d}t$$

satisfies a two-term recurrence relationship when acting on the ultraspherical polynomials $C_n^{(\lambda)}(x)$ with weight $w(y) = (1-x^2)^{\lambda-\frac{1}{2}}$ such that

$$x\mathcal{L}_\alpha\left[w(y)C_n^{(\lambda)}(y)\right](x) = \kappa_1\mathcal{L}_\alpha\left[w(y)C_{n-1}^{(\lambda)}(y)\right](x) + \kappa_2\mathcal{L}_\alpha\left[w(y)C_{n+1}^{(\lambda)}(y)\right](x),$$

where $n \geq 2$ and with the constants

$$\kappa_1 = \frac{(n-\alpha-1)(2\lambda+n-1)}{2n(\lambda+n)},$$

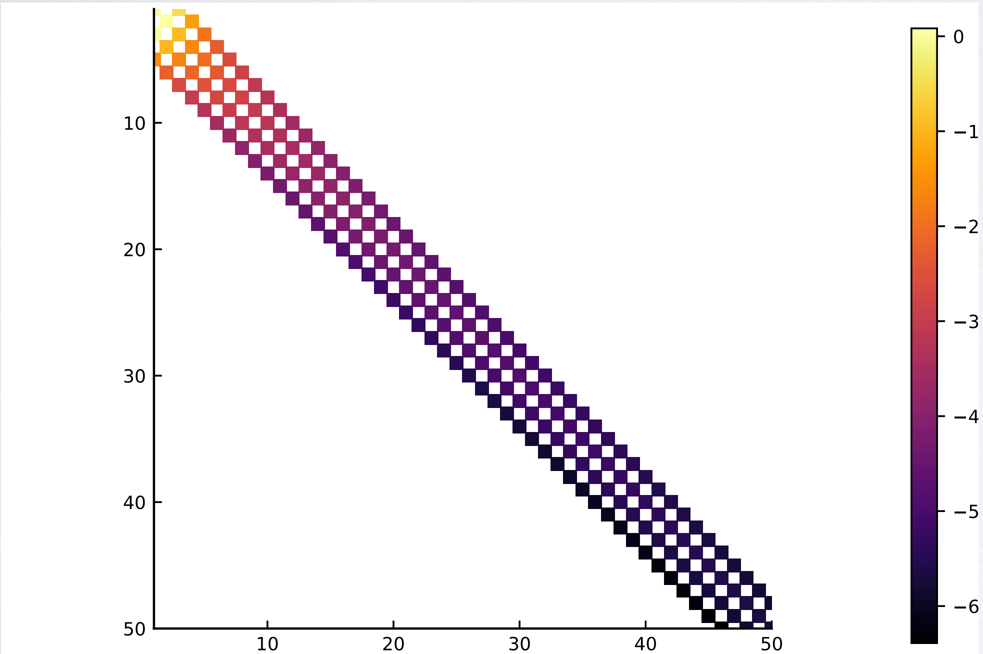
$$\kappa_2 = \frac{(n+1)(2\lambda+n+\alpha+1)}{2(\lambda+n)(2\lambda+n)}.$$

[Gutleb, Carrillo & Olver 2020]

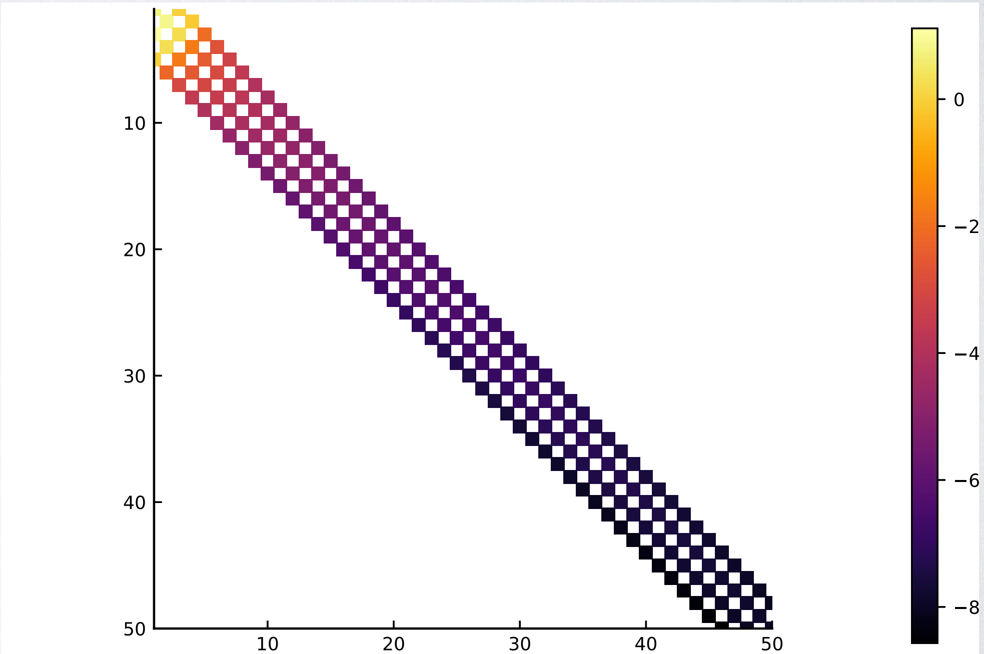
- We choose λ so that we map to polynomials

$$\lambda = \begin{cases} \left\lfloor \frac{\alpha}{2} \right\rfloor - \frac{\alpha}{2}, & \text{if } \left\lfloor \frac{\alpha}{2} \right\rfloor - \frac{\alpha}{2} > -\frac{1}{2} \\ \left\lceil \frac{\alpha}{2} \right\rceil - \frac{\alpha}{2}, & \text{otherwise} \end{cases}$$

The recurrence than becomes a banded operator



$\alpha = 3.9$



$\alpha = 2.5$

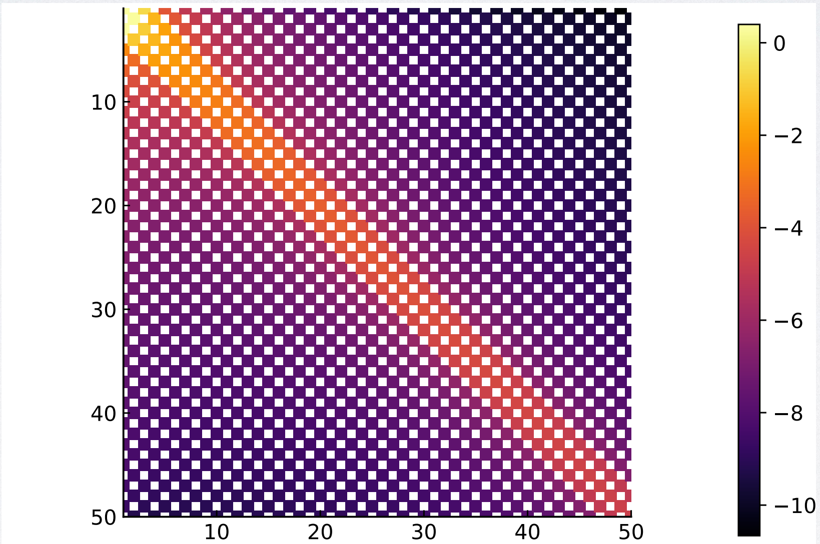
- For attractive-repulsive kernels

$$\frac{\mathcal{L}_\alpha}{\alpha} - \frac{\mathcal{L}_\beta}{\beta}$$

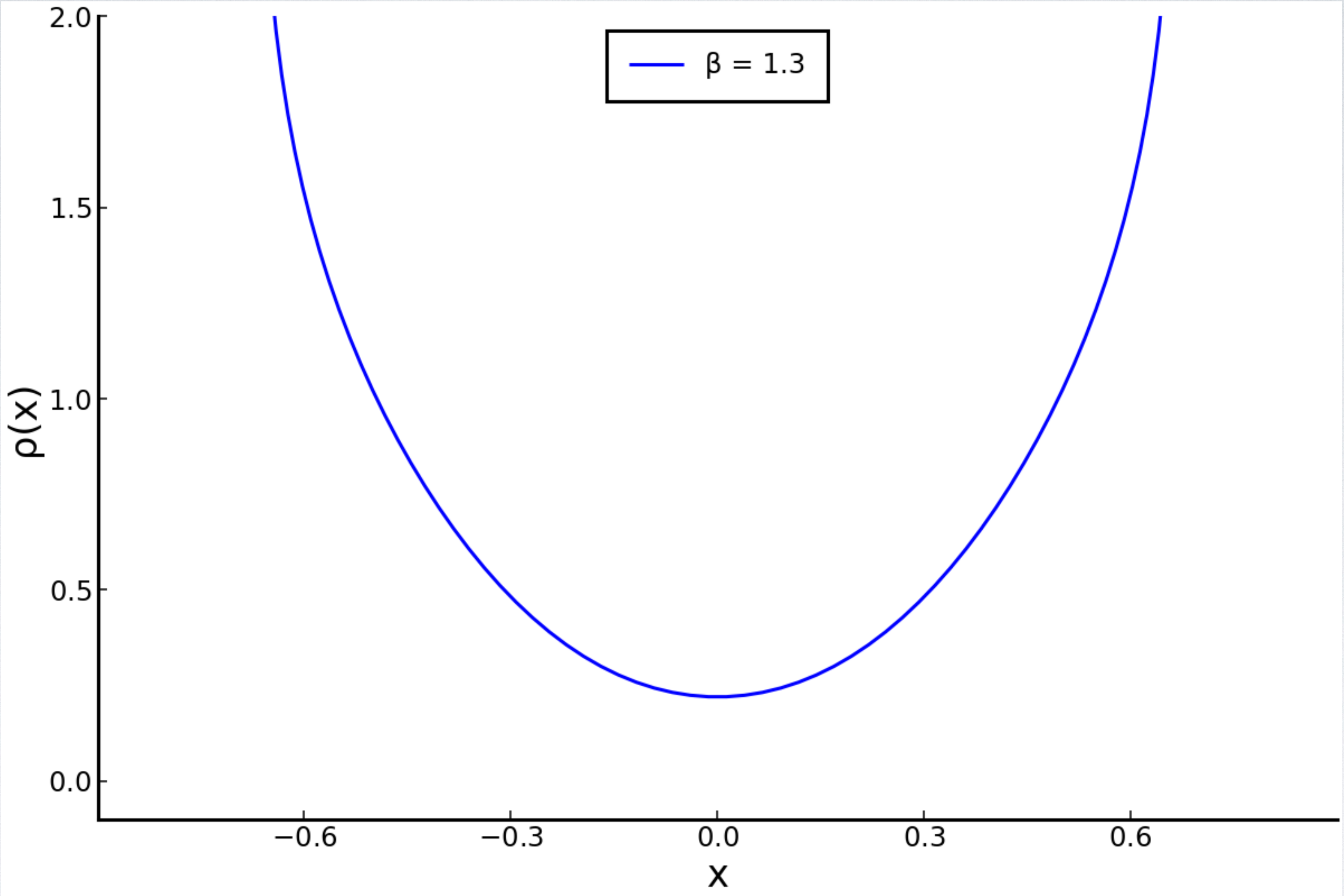
  we cannot (in general) send both to polynomials. So we choose to make $\mathcal{L}_\alpha$ banded

- Fortunately, we can work out $\mathcal{L}_\beta$ in the same basis and it is almost banded!

- Support of measure can therefore be determined by Newton + Auto-differentiation

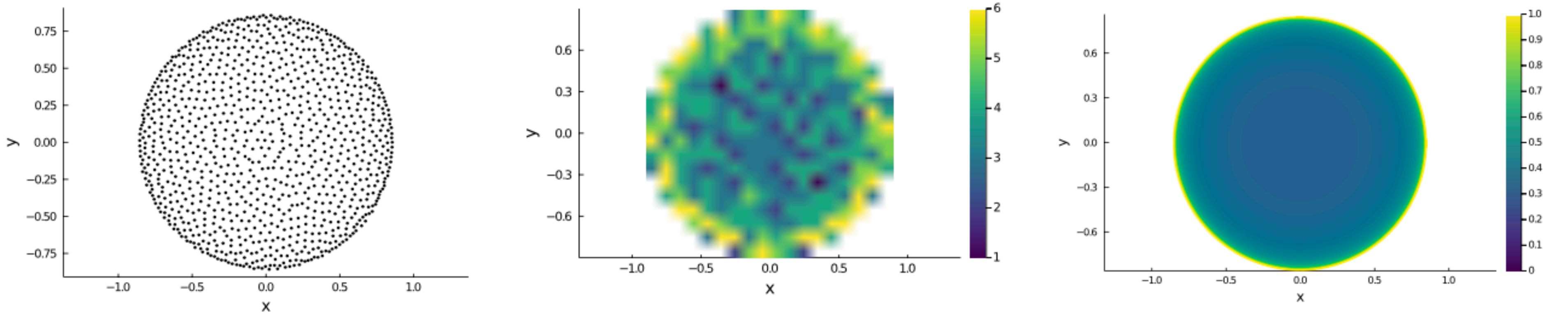


$$\alpha = 3.8, \beta = 1.7$$



$\alpha = 4$

Extends to higher dimensions, Radially symmetric



(a) $(\alpha, \beta, d) = (1.3, 1.1, 2)$ (b) 2D histogram based on (a) (c) computed measure

Using generalised Zernike polynomials, multivariate OPs in a ball using Meijer G functions to determine recurrence

Outline of Rest of Talk

- Orthogonal polynomials and sparse discretisations
- Log and Cauchy kernel singular integral equations
- Potential theory equilibrium measures
- Power-law equilibriums
- Applications to PDEs

PDE

$$(\Delta + 20 + y)u = 0$$
$$u|\Gamma = 0$$
$$u \sim \Phi(\mathbf{x}, \mathbf{x}_0)$$

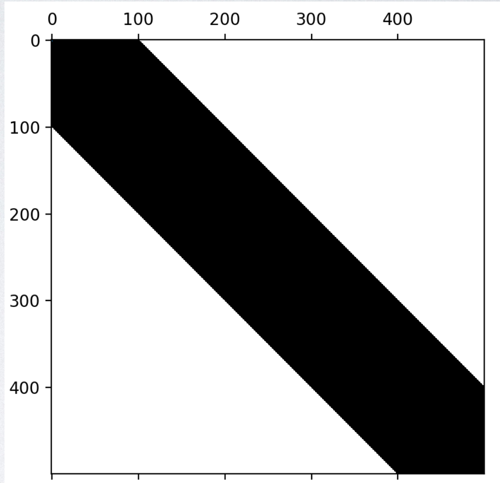
Fundamental solution
given as integral

Boundary
Reformulation

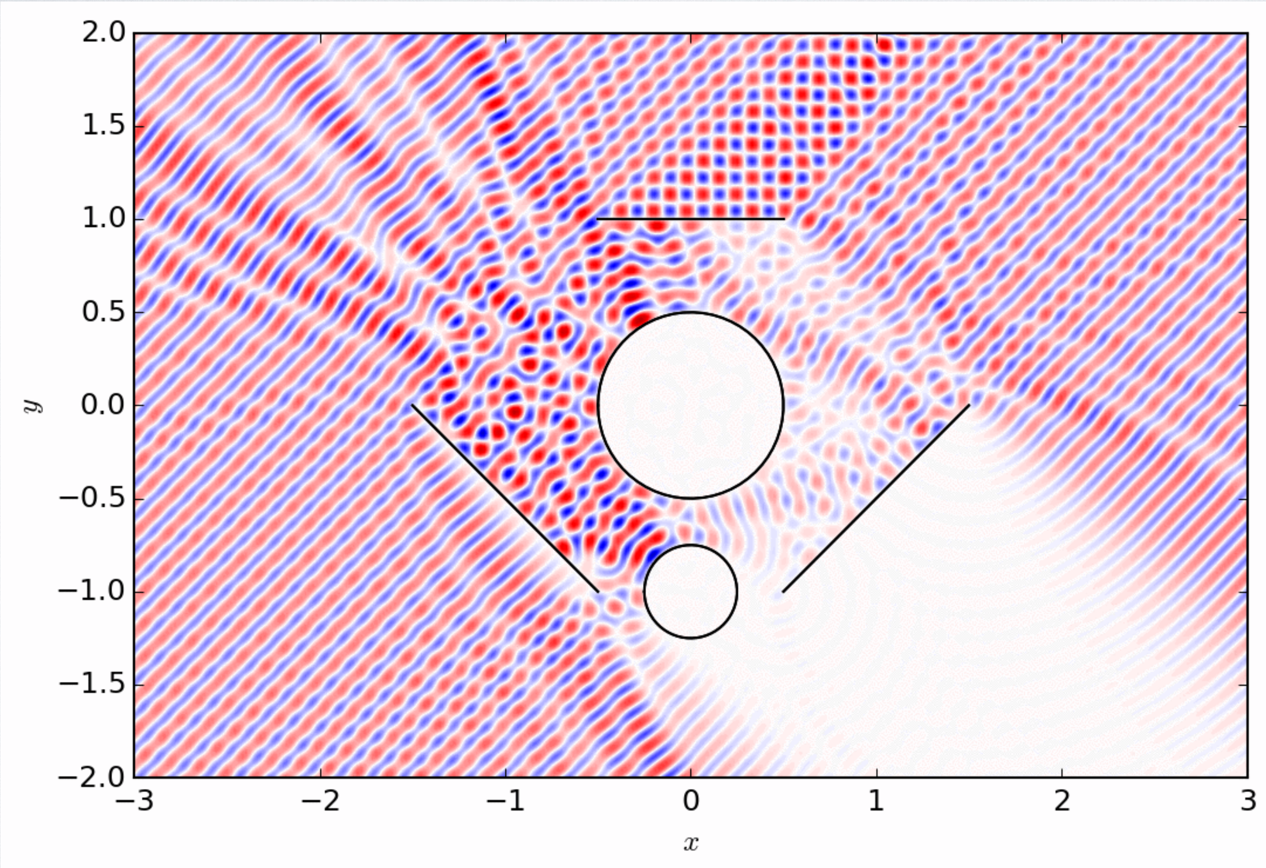
$$\int_\Gamma \Phi(\mathbf{x}, \mathbf{t})u(\mathbf{t})\mathrm{d}\mathbf{t} = f(\mathbf{x})$$

Use basis built
From weighted
Chebyshev polynomials

Sparse
linear system



Acoustic Scattering



Any interesting boundary reformulation examples with power-law kernels?

FRACTIONAL DIFFERENTIAL EQUATIONS

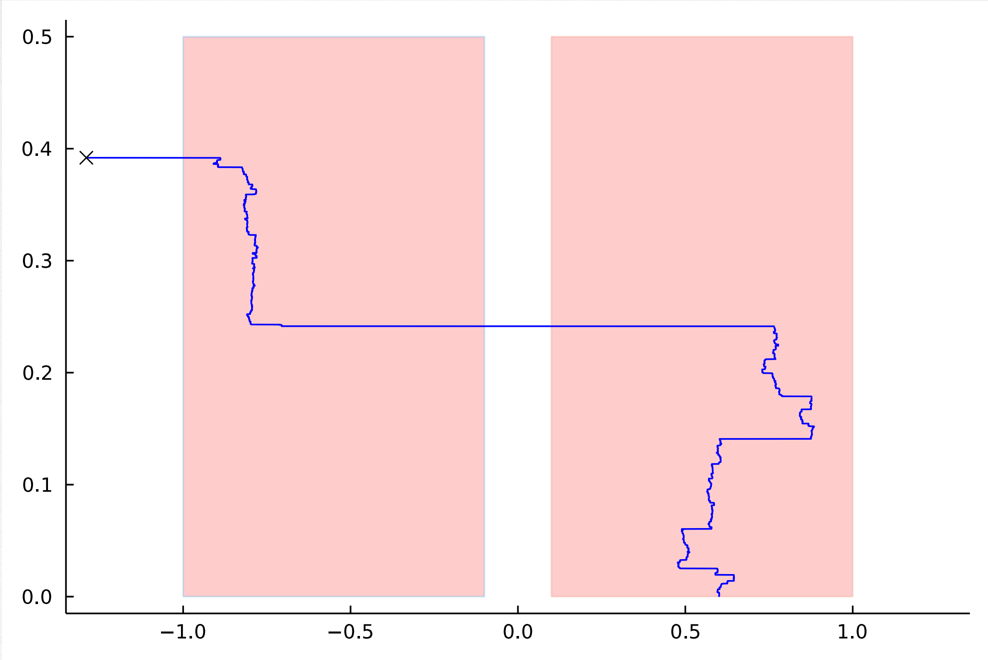
- Fractional Laplacians are the inverse of power law kernels:

$$\mathcal{L}_\alpha = |\Delta|^{-\alpha}.$$

- Heavy tailed statistics: the average exit time for a Levy flight is

$$|\Delta|^\alpha w = 1$$

- Other applications:
  - visco-elastic fluids
  - ultrasound
  - Landau equations for plasmas



Levy flight in two intervals

Now Hiring!

Job Details

Postdoctoral Research Associate in the Numerical Analysis of Nonlocal PDEs

Mathematical Institute, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford

Grade 7: £33,309 - £40,927 per annum

We invite applications for a Postdoctoral Research Associate to work with Professor Jose A. Carrillo on an exciting new project in the numerical analysis of nonlinear PDEs. This is a two-year, fixed-term position, and the post-holder will be based here at the Mathematical Institute, University of Oxford. The position is funded by a research grant from the Engineering and Physical Sciences Research Council.

The successful candidate will be incorporated into one of the main strands of the proposal concerning the scientific computing and numerical analysis of stationary states and solutions to nonlinear PDEs with fractional derivatives. This includes the numerical methods for gradient flows of interaction energies; dissipative quasi-geostrophic equations; Landau equations for plasmas; and Keller-Segel models with fractional derivatives. They will write up their results for publication, present papers at conferences and contribute ideas for new research projects, whilst also managing their own academic research and administrative activities.