Computability 2012
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Formal Languages
Midterm
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1.)

a.) total, one to one, not onto

$f(x) = x+1$

b.) total, onto, not one to one

$f(0) = 0$
$f(x) = x-1$

c.) total, one to one, onto

$f(x) = \begin{cases} \text{if } x \text{ is even then } x+1 \\ \text{otherwise } x-1 \end{cases}$

d.) not total, onto

$f(0,1)$: undefined
$f(x) = x-2$

2.) $(a,b) \in LT$ when $a < b$

is not reflexive : $(1,1)$; $1 < 1 ==$ false

is not symmetric $(2,3)$; $2<3$ is true while $(3,2)$; $3<2$ is false

3.) GT on $N \times N$

(i) $(s(0), 0) \in GT$

(ii) if $(a,b) \in GT$ then $(s(a), s(b)) \in GT$
$(s(a), b) \in GT$

(iii) $(a,b) \in GT$ iff
it can be reached by a finite # of applications of (ii) to (i)

4.) $2+5+8+\ldots+(3n-1) = n(3n+1)/2$ for $n>0$

base case: $n=1$; $3\cdot1-1=2$ $\quad 1(3\cdot1+1)/2=2$ ✓

inductive case:

assume: $\sum_{n=1}^{k} 3n-1 = k(3k+1)/2$ prove: $\sum_{n=1}^{k+1} 3n-1 = (k+1)(3(k+1)+1)/2$

$$\frac{k(3k+1)}{2} + 3(k+1)-1 = (k+1)(3(k+1)+1)/2$$

$$\frac{k(3k+1)+2(3k+2)}{2} \qquad (k+1)(3k+4)/2$$

$$\frac{3k^2+k+6k+4}{2} \qquad \frac{3k^2+7k+4}{2}$$

$$\frac{3k^2+7k+4}{2}$$

✓ □

5.) Let L = set of strings over $\{a,b\}$ where 2 times number of a's as b's

(i) λ ; a^0b^0

(ii) if $u \in L$ then {uabb, ubab, ubba,
aubb, buab, buba,
abub, baub, bbua,
abbu, babu, bbau} $\in L$

(iii) Some string $w \in L$ iff it can be reached by a finite # of applications of (ii) to (i).

6.) (i.)

$L_0 = \{\lambda\}$

$L_1 = \{aab\}$

$L_2 = \{aab, aaaabb, \lambda\}$

(ii.) $\{a^n b^m \mid n = 2m\}$

a.) $\lambda \in L$

b.) if $u \in L$ then $aaub \in L$

c.) closure

(iii) prove $n_a(u) = 2n_b(u)$

base case: $u = \lambda$; $n_a(u) = 0$, $n_b(u) = 0$; $0 = 2 \cdot 0$

inductive case:

Let u be some string generated by n applications of the recursive step b.) to a.). assume: $n_a(u) = 2n_b(u) \leftarrow$ I.H.

Let w be some string generated by $(n+1)$ applications of the recursive step b.) to the basis a.), then $w = aaub$

$n_a(w) = 2 + n_a(u)$

$n_b(w) = 1 + n_b(u)$

$2(n_b(w)) = 2(1 + n_b(u))$

$= 2 + 2n_b(u)$ — I.H.

$= 2 + n_a(u)$

$= n_a(w)$ $\square$

7.

a.) $\{(aa)^m b^n \mid m \geq 0, n \geq 0\}$

b.) $\{a^n c^m (bb)^n \mid n \geq 0, m > 0\}$

c.) $\{(ab)^m (cd)^n (ba)^n (dc)^m \mid m \geq 0, n \geq 0\}$

d.) $\{a^m c^l a^n b^n d^l b^m \mid m \geq 0, l > 0, n > 0\}$

e.) $\{a^n b^m \mid n > 0, n \leq m \leq 2n\}$

8.) $b^* (ab^+)^* aab^* (ab^+)^*$

8.

$S \rightarrow bS \mid aA$
$A \rightarrow bS \mid aB$
$B \rightarrow bC \mid \lambda$
$C \rightarrow bC \mid aD \mid \lambda$
$D \rightarrow bC \mid \lambda$

9.) Let $G = \begin{array}{l} S \rightarrow aSb \mid B \\ B \rightarrow bB \mid b \end{array}$ prove $L(G) = \{a^n b^m \mid 0 \leq n < m\}$

First, show $L \subseteq G$: every string in L is derivable in G :

$S \overset{n}{\Rightarrow} a^n S b^n$	$S \rightarrow aSb$	$n \geq 0$
$\Rightarrow a^n B b^n$	$S \rightarrow B$	$m = k + n + 1 > n$
$\overset{k}{\Rightarrow} a^n b^k B b^n$	$B \rightarrow bB$	
$\Rightarrow a^n b^k b b^n$	$B \rightarrow b$	

Second, show $G \subseteq L$: every string derivable in G has the form $\{a^n b^m \mid 0 \leq n < m\}$

(i) Let $n_a(u)$ = # of a's in u for some string u in <u>sentential form</u>
$n_b(u)$ = # of b's $\underbrace{+B+S}_{\text{optional occurances}}$ in u derived from $\overset{n}{\underset{G}{\Rightarrow}}$: $n_a(u) \geq 0 < n_b(u)$

(ii) the a^n preceed the optional S preceeds the b^m & optional B

Base case : let $u = aSb$ or B

$S \Rightarrow aSb$ $\quad 0 \leq n_a(u) = 1 < n_b(u) = 2$ (i) ✓

$S \Rightarrow B$ $\quad 0 \leq n_a(u) = 0 < n_b(u) = 1$ (ii) ✓

a^1 preceeds S preceeds b^1 ; a^0 preceeds b^0 B

... cont.

9. (continued)

assume: $0 \le n_a(u) < n_b(u)$

Let $\overset{n}{\underset{G}{\Rightarrow}} u \underset{G}{\Rightarrow} w$

	$n_a(w)$	$n_b(w)$
$S \to aSb$	$n_a(u)+1$	$n_b(u)+2$
$S \to B$	$n_a(u)$	$n_b(u)$
$B \to bB$	$n_a(u)$	$n_b(u)+1$
$B \to b$	$n_a(u)$	$n_b(u)$

In all case the relationship between $n_a(w)$ and $n_b(w)$ remains the same as $n_a(u)$ and $n_b(u)$ or $n_b(w)$ is increased more than $n_a(w)$

$0 \le n_a(w) < n_b(w)$

Let w_1 = 0 or more a's

w_2 = 1 or more b's

then $u = w_1 S w_2$ or

$w_1 w_2 B$

$w = w_1 a S b w_2$ $\quad w_1 w_2 b B$

$w_1 B w_2$ $\quad w_1 w_2 b$

in all cases ordering of (ii) is preserved $\square$

10.)

a.) a^+b^+

b.)

$S \Rightarrow Sb$

$\Rightarrow aSb$

$\Rightarrow aabb$

$S \Rightarrow aS$

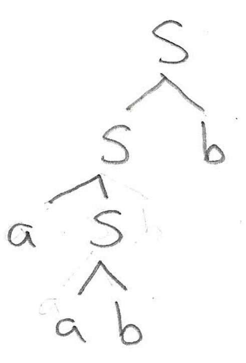
$\Rightarrow aSb$

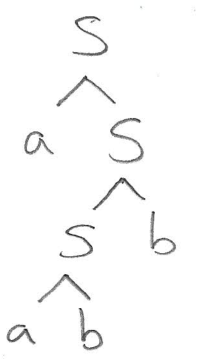
$\Rightarrow aabb$

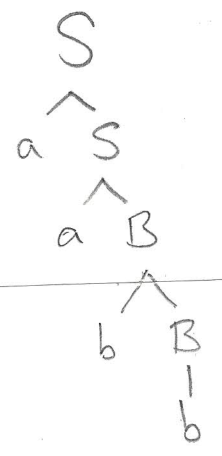
d.) $S \to aS \mid aB$

$B \to bB \mid b$

c.)







$S \Rightarrow aS$

$\Rightarrow aaB$

$\Rightarrow aabB$

$\Rightarrow aabb$

Let $G_1 = S \to aS \mid Sb \mid ab$

$G_2 = S \to aS \mid aB$ prove $G_1 = G_2$

$B \to bB \mid b$ prove $G_1 = L$

$G_2 = L$

where $L = \{a^n b^m \mid n>0, m>0\}$

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show any string derivable by G_1 has:

(i) the form: a^n preceeds optional S preceeds b^m

(ii) there are 1 or more a's and 1 or more b's

Let $n_a(u)$ be the # of a's | Let $\overset{n}{\underset{G_1}{\Rightarrow}} u \Rightarrow w$

$n_b(u)$ be the # of b's

base case:

$u = aS$ or (i) and (ii) hold

Sb or

ab

Let w_1 = 1 or more a's

w_2 = 1 or more b's

Inductive case: assume $n_a(u) > 0$, $n_b(u) > 0$ and (i)

in all case were w can be derived from u, (i) & (ii) hold

	$n_a(w)$	$n_b(w)$
$S \to aS$	$n_a(u)+1$	$n_b(u)$
$S \to Sb$	$n_a(u)$	$n_b(u)+1$
$S \to ab$	$n_a(u)+1$	$n_b(u)+1$

$u = w_1 S w_2$

$w = w_1 ab w_2$ or

$w_1 aS w_2$ or

$w_1 Sb w_2$

$S \overset{n}{\Rightarrow} a^n S$ $\quad S \to aS$

$\overset{m}{\Rightarrow} a^n S b^m$ $\quad S \to Sb$

$\Rightarrow a^n ab b^m$ $\quad S \to ab$

or

$S \overset{m}{\Rightarrow} Sb^m$ $\quad S \to Sb$

$\overset{n}{\Rightarrow} a^n S b^m$ $\quad S \to aS$

$\Rightarrow a^n ab b^m$ $\quad S \to ab$

...cont

... cont for $G_2 = S \to aS \mid aB$ show $G_2 = L$
$B \to bB \mid b$

a schema showing that any string $\in L$ can be derived by G_2

$$S \overset{n}{\Rightarrow} a^n S \qquad S \to aS$$
$$\Rightarrow a^n aB \qquad S \to aB$$
$$\overset{k}{\Rightarrow} a^n ab^k B \qquad B \to bB$$
$$\Rightarrow a^n ab^k b \qquad B \to b$$

where $n \geq 0$
$k \geq 0$

any string with a's preceding b's and at least 1 a + 1 b can be derived by G_2

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next show that all strings derivable in G_2 have the conform to L in that they:

(i) a's preceed the b's ; in sentential form: a^n preceeds optional S preceeds b^m preceeds optional B

(ii) $n_a(u) > 0$ and $n_b(u) > 0$
where $n_a(u) = \#$ of a's and $n_b(u) = \#$ of b's + 1 if there is S
+ 1 if there is B

Let $\overset{n}{\underset{G_2}{\Rightarrow}} u \Rightarrow w$ where u is in sentential form

assume (i) + (ii) for u ; show for w

	$n_a(w)$	$n_b(w)$
$S \to aS$	$n_a(u)+1$	$n_b(u)$
$S \to aB$	$n_a(u)+1$	$n_b(u)$
$B \to bB$	$n_a(u)$	$n_b(u)+1$
$B \to b$	$n_a(u)$	$n_b(u)$

shows (ii)

w_1 = 1 or more a's
w_2 = 1 or more b's

$u = w_1 S$ or
$w_1 w_2 B$

$$\underbrace{w = \{w_1 aS, w_1 aB\} \text{ or } \{w_1 w_2 bB, w_1 w_2 b\}}_{\text{shows (i)}}$$

out of order:

base case: aS or aB (i) and (ii) hold