Report on Classification and Regression

by Ruijie Rao

Library Import

In [3]:

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from numpy.random import RandomState
```

Data import

In [4]:

```
with open("linear-regression.txt", "r") as file:
    content = file.readlines()
r_data = np.array([line.strip("\n").split(",") for line in content] ).astype(float)
```

In [5]:

```
with open("classification.txt", "r") as file:
    content = file.readlines()
c_data = np.array([line.strip("\n").split(",") for line in content] ).astype(float)
```

Algorithm: Gradient Descent

Algorithm Introduction

Minimize the **RSS**:

$$\min_{\vec{w}} g(\vec{w})$$

by moving w towards the negative gradient direction:

$$w^{t+1} = w^t - \eta_t \frac{dg(w)}{dw}$$

with step size:

$$\eta_t = \frac{\alpha}{t}$$
$$\eta_t = \frac{\alpha}{\sqrt{t}}$$

until convergence:

$$|\frac{dg(w)}{dw}| < \epsilon$$

Algorithm Construct

Goal: Perform Gradient Descent on inputted function and training set.

Input:

- X: training set X
- y: training set y
- alpha: step size
- theta: threshold
- N: max number of iterations
- g: loss function
- dg: derivative of loss function

Output:

- w: weights

In [43]:

```
class Gradient_Descent:
    def __init__(self, X, y, g, dg):
        self.rs = RandomState(2022)
        self.X = X
        self.y = y
        self.g = g
        self.dg = dg

    def plot_loss(self, loss, c):
        ax = plt.gca()
        ax.set_title('Loss')
        ax.plot(np.arange(c), loss)
        plt.show()

    def run(self, alpha, theta, N):
        D = len(self.X[0]) # Dimension
        w = self.rs.rand(D) # initiate random weights 1xD vector
        dw = np.array([np.inf for i in range(D)]) # initiate large graident to preven
        c = 0 # counter
        loss = []
        while np.linalg.norm(dw)>theta and c<N: # Terminatates when gradient is less
            dw = self.dg(self.X, self.y, w)
            new_w = w - alpha*dw
            w = new_w
            c += 1
            l = self.g(self.X, self.y, w)
            loss.append(l)
        self.plot_loss(loss, c)
        print(np.linalg.norm(dw))
        print(f"Iteration: {c} MSE ends at {l}")
        return w
```

Model: Linear Regression

- **Goal**: Find best fit $f(x)$ that predicts y using $\vec{x}$ that minimizes the **MSE(Mean Square Error)/RSS(Residual Sum of Squares)**.
- **Function**: $h(x^{(i)}) = \hat{y^{(i)}} = \sum_{j=0}^{D} w_j x^{(i)}$
- **Loss**: Loss is the difference between true y and predicted y $y - \hat{y}$
  - **Residual Sum of Squares**: $\sum_{i=1}^{N}(y^{(i)} - h(x^{(i)}))^2$
  - **Mean Square Error**: $\frac{RSS}{N}$

Function

Input:

- X: X
- w: weights

Output:

- result: $h(x) = \hat{y} = Xw$

In [234...

```
def h(X, w): # X is NxD, w is 1xD
    result = X@w.T
    return result
```

Loss Function

Going to use Mean Square Error as the Loss Function

In [235...

```
def MSE(X, y, w):# X is NxD, w is 1xD, y is Nx1
    rss = np.sum(np.square(y-h(X, w)))
    mse = rss/len(X)
    return mse
```

Gradient of Loss Function

Loss Function:$g = \frac{1}{N}\sum_{i=1}^{N}(h(x^{(i)}) - y^{(i)})^2$

Where its partial derivative with respect to $j^{th}w$: $\frac{dg}{dw_j} = \frac{2}{N}\sum_{i=1}^{N}(h(x^{(i)}) - y^{(i)})x_j^{(i)}$

In [236...

```
def dMSE(X, y, w):
    m = len(X)
    d = len(X[0])
    dw = np.array([(2/m)*np.sum((h(X, w)-y)@X[:, j]) for j in range(d)])
    return dw
```

Application

Data Preparation

In [237...

```
m = len(r_data)
```

Insert a column of 1s at the beginning of X.

In [238...

```
r_data = np.c_[np.ones((m, 1)), r_data]
```

Split data into X and y.

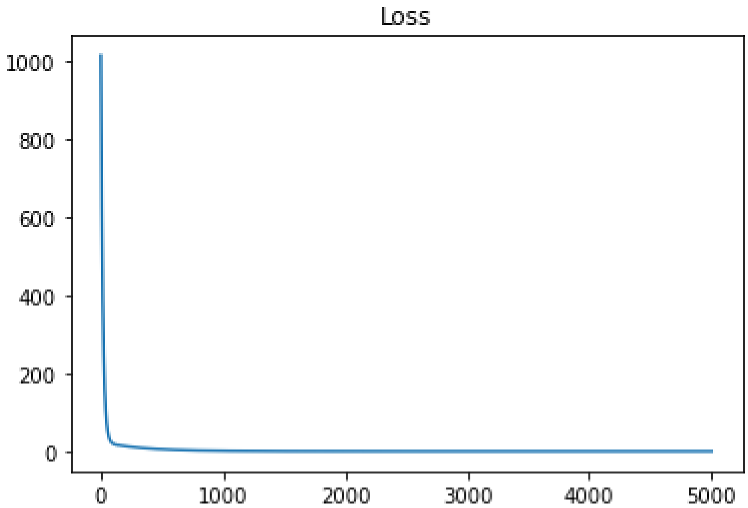
In [239...

```
X = r_data[:, :-1]
y = r_data[:, -1]
```

Run

In [240…
```
LR_train = Gradient_Descent(X[:2400], y[:2400], MSE, dMSE)
```

In [242…
```
w = LR_train.run(alpha=1e-2, theta=1e-4, N=5000)
```




```
Iteration: 5000 MSE ends at 0.039617908799452956
```

In [243…
```
w
```

Out[243…
```
array([-0.00710663,  1.1054901 ,  4.01405851])
```

Test

In [244…
```
def predict_LNR(X, w):
    result = X@w
    return result
```

In [245…
```
predict_LNR(X, w)
```

Out[245…
```
array([3.55984163, 3.07020817, 1.83274012, ..., 1.88111492, 4.12165041,
       0.69212904])
```

In [246…
```
MSE(X[2400:], y[2400:], w)
```

Out[246…
```
0.03854573333534492
```

Model: Logistic Regression

- **Goal**: Find best fit $f(x)$ that predicts y using $\vec{x}$ that maximizes the probability reward.
- **Function**: $h(x^{(i)}) = \hat{y^{(i)}} = \theta(\sum_{j=0}^{D} w_j x^{(i)}) = \theta(w^T x^{(i)})$
- **Loss**: Loss is the negative reward given to the prediction probability.
  - if $y = 1$, reward is $\theta(w^T x^{(i)})$.
  - if $y = -1$, reward is $-\theta(w^T x^{(i)}) = \theta(-w^T x^{(i)})$.

Which is equal to $\theta(y^{(i)}w^Tx^{(i)})$. Reward is then aggregated into:

$$\prod_{i=1}^{N}\theta(y^{(i)}w^Tx^{(i)}) = \frac{1}{N}\sum_{i=1}^{N}log\theta(y^{(i)}w^Tx^{(i)})$$

or

$$\frac{1}{N}\sum_{i=1}^{N}[y^{(i)}log(\theta(w^Tx^{(i)}))] + [(1-y^{(i)})log(1-\theta(w^Tx^{(i)}))]$$

Function

Sigmoid function is used to generalize the sum function to a probability score within the range of 1 and 0:

$$\theta(x) = \frac{1}{1+e^{-x}}$$

Input:

- X: X
- w: weights

Output:

- result: $h(x) = \hat{y} = \theta(Xw)$

In [7]:

```
def sigmoid(x): # input x: Nx1
    result = 1/(1+np.exp(-x))
    return result

def h(X,w): # X is NxD, w is 1xD
    result = X@w.T
    return result
```

Loss Function

Going to use Mean Square Error as the Loss Function

In [65]:

```
# for -1/1 Classification
def MLE(X,y,w):# X is NxD, w is 1xD, y is Nx1
    m = len(y)
    reward = -np.sum(np.log(sigmoid(y*h(X,w)))) / m
    return reward
```

In [9]:

```
# for 0/1 Classification
def MLE(X,y,w):# X is NxD, w is 1xD, y is Nx1
    s = sigmoid(h(X,w))
    if s.any()<= 0:
        print(s)
    m = len(y)
    reward = np.sum(-(y * np.log(s) + (1 - y) * np.log(1 - s))) / m
    return reward
```

Gradient of Loss Function

Loss Function:$g = \frac{1}{N}\sum_{i=1}^{N} log\frac{1}{1+e^{-(y_i w^T x(i))}}$

Gradient:

$$\frac{dg}{dw} = -\frac{1}{N}\sum_{i=1}^{N}\frac{y^{(i)}x^{(i)}}{1+e^{-(y_i w^T x(i))}}$$

or

$$\frac{dg}{dw} = -\frac{1}{N}(\theta(Xw) - y)X$$

In [100…

```
# for -1/1 Classification
def dMLE(X,y,w):
    m = len(X)
    d = len(X[0])
    temp = []
    '''for i in range(len(X)):
        xy = y[i]*X[i]
        s = sigmoid(y[i]*h(X[i],w))
        temp.append(xy*s)
    temp = np.array(temp)'''
    temp = (y.reshape(-1,1)*X)*sigmoid(y*h(X,w)).reshape(-1,1)
    dw = (1/m)*np.sum(temp,axis=0)
    return dw
```

In [57]:

```
# for 0/1 Classification
def dMLE(X,y,w):
    m = len(X)
    d = len(X[0])
    s = sigmoid(h(X,w))
    dw = (1/m)*(X.T@(s-y))
    return dw
```

Application

Data Preparation

In [90]:

```
m = len(c_data)
```

Insert a column of 1s at the beginning of X.

In [91]:

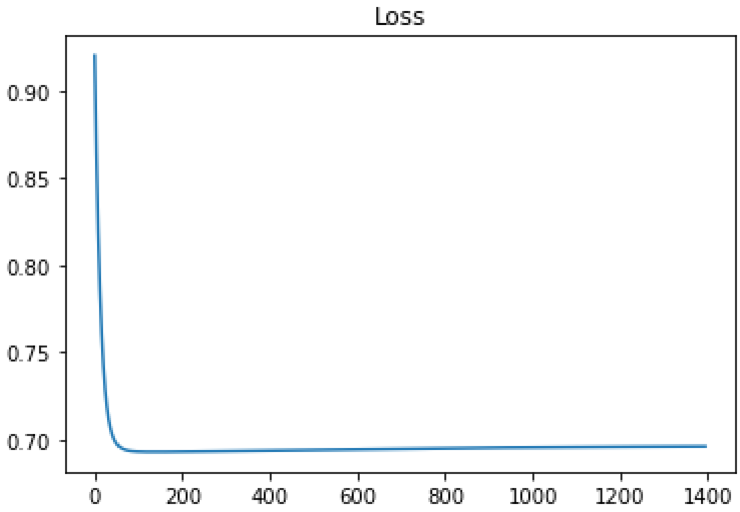
```
X = np.c_[np.ones((m,1)),c_data[:,:3]]
y = c_data[:,-1]
#y = np.array([int(i>0) for i in y])
```

Run

In [101…

```
training_size = int(m*0.8)
LGR_train = Gradient_Descent(X[:training_size],y[:training_size],MLE,dMLE)
```

In [116…
```
w = LGR_train.run(alpha=1e-1,theta=1e-3,N=7000)
```




```
0.0009995473464741238
Iteration: 1395 MSE ends at 0.6959498888941883
```

Test

In [117…
```
def predict_LGR(X,w):
    prob = sigmoid(h(X,w))
    pred = np.array([-2+int(p>=0) for p in prob])
    return pred
```

In [118…
```
def accuracy_LGR(X,y,w):
    pred = predict_LGR(X,w)
    accuracy = sum(pred==y)/len(y)
    return accuracy
```

In [119…
```
accuracy_LGR(X,y,w.reshape(1,-1))
```

Out[119…
```
0.506
```

In [120…
```
predict_LGR(X[:20],w)
```

Out[120…
```
array([-1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1,
       -1, -1, -1])
```

The accuracy is very low. Going to try SKLearn.

In [562…
```
from sklearn.linear_model import LogisticRegression
```

In [563…
```
LGR = LogisticRegression(random_state=0).fit(X, y)
```

In [564…
```
LGR.score(X, y)
```

Out[564…
```
0.5295
```

Neural Network - Perceptron Learning

Goal: Use Multiple Perceptrons to accomplish more complex classifications.

1. Dynamic Programming for efficiently computing the gradient.
   - Storing the outputs of subproblems and avoid re-calculating.
2. Stochastic Gradient Descent

Iterations:

1. Compute $x_j^{(l)}$ for all $1 \leq l \leq L$,$1 \leq j \leq d^{(l)}$ in the forward direction:

$$x_j^{(l)} = \theta(\sum_{i=0}^{d^{l-1}} w_{ij}^{(l)} x_i^{(l-1)})$$

2. Compute $\delta_i^{(l)}$ for all $1 \leq l \leq L$,$1 \leq i \leq d^{(l)}$ in the reverse direction:

$$\delta_i^{(l-1)} = (1 - \theta(x_i^{(l-1)})^2)(\sum_{j=0}^{d^{l-1}} w_{ij}^{(l)} \delta_j^{(l)})$$

   Base case:

$$\delta_i^{(L)} = 2(\theta(x_i^{(L)}) - y)(1 - \theta(x_i^{(L)})^2)$$

3. For each $w_{ij}^{(l)}$:

$$w_{ij}^{(l)} = w_{ij}^{(l)} - \eta \delta_j^{(l)} x_i^{(l-1)}$$

Model Script

Data Prep

In [16]:

```
m = len(c_data)
X = np.c_[np.ones((m,1)),c_data[:,:3]]
y = c_data[:,-2]
rs = RandomState(0)
```

Initialization

Will be using 1 hidden layer with a number of neurons equals to the mean of input and output neuron numbers.

In [7]:

```
def gen_d_list(Ni,No,Nn,Nh):
    d_list = [Nn for l in range(Nh)]
    d_list.insert(0,Ni)
    d_list.append(No)
    return d_list

def init_w(d_list):
    w_list = [rs.rand(d_list[i+1],d_list[i]) for i in range(len(d_list)-1)]
    w_list.insert(0,None)
    return w_list
```

Iteration Updates

In [27]:

```
theta = np.tanh
```

In [74]:

```
# Compute all x in forward direction
def update_x(x_in,w_list):
    x_list= [x_in.reshape(1,4)]
    for l in range(1,L+1):
        z = x_list[l-1]@w_list[l].T # each x array should be 1xdi, w should be djxdi
        hx = theta(z)
        x_list.append(hx)
    return x_list # output list of 1xdj
```

In [73]:

```
# Compute all delta in forward direction
def update_delta(y_in,x_list,w_list):
    delta_list = [0 for i in x_list]
    delta_list[L] = 2*(x_list[L]-y_in)*(1-np.square(x_list[L]))
    for l in reversed(range(1,L+1)):
        delta = (1-(np.square(x_list[l-1])))*(delta_list[l]@w_list[l]) # delta should
        delta_list[l-1] = delta
    return delta_list # output list of 1xdi
```

In [33]:

```
# Update w
def update_w(x_list,w_list,delta_list,alpha):
    for l in range(1,L+1):
        w = w_list[l] # djxdj
        x = x_list[l-1] # 1xdi
        delta = delta_list[l] # 1xdj
        w = w-alpha*(delta.T*x)
        w_list[l] = w
    return w_list
```

In [95]:

```
def predict_NNP(X,w_list):
    hx = X
    for l in range(1,L+1):
        z = hx@w_list[l].T # X should be mxdi, w should be djxdi, result mxdj
        hx = theta(z)
    pred = np.array([-1+2*(int(p>=0)) for p in hx])
    return pred
```

In [76]:

```
def accuracy_NNP(X,y,w_list):
    pred = predict_NNP(X,w_list)
    accuracy = sum(pred==y)/len(y)
    return accuracy
```

In [121…

```
class Neural_Network_Perceptron:
    def __init__(self,X,y,Nh=1):
        def gen_d_list(Ni,No,Nn,Nh):
            d_list = [Nn for l in range(Nh)]
            d_list.insert(0,Ni)
            d_list.append(No)
            return d_list

        def init_w(d_list):
            w_list = [self.rs.rand(d_list[i+1],d_list[i]) for i in range(len(d_list)-
            w_list.insert(0,None)
            return w_list
```

```python
        self.theta = np.tanh
        self.m = X.shape[0]
        self.rs = RandomState(2022)
        self.L = Nh+1
        D = X.shape[1]
        Ni = D # Number of Neurons in input layer
        No = 1 # Number of Neurons in output layer
        Nn = int((1+D)/2) # Number of Neurons in hidden layey
        d_list = gen_d_list(Ni,No,Nn,Nh)
        self.w_list = init_w(d_list)
        self.X = X
        self.y = y

    def update_x(self,x_in): # Compute all x in forward direction
        x_list= [x_in.reshape(1,4)]
        for l in range(1,self.L+1):
            z = x_list[l-1]@self.w_list[l].T # each x array should be 1xdi, w should
            hx = self.theta(z)
            x_list.append(hx)
        return x_list # output list of 1xdj


    def update_delta(self,y_in,x_list): # Compute all delta in forward direction
        delta_list = [0 for i in x_list]
        delta_list[-1] = 2*(x_list[-1]-y_in)*(1-np.square(x_list[-1]))
        for l in reversed(range(1,self.L+1)):
            delta = (1-(np.square(x_list[l-1])))*(delta_list[l]@self.w_list[l]) # del
            delta_list[l-1] = delta
        return delta_list # output list of 1xdi

    def update_w(self,x_list,delta_list,alpha): # Update w
        for l in range(1,self.L+1):
            w = self.w_list[l] # djxdj
            x = x_list[l-1] # 1xdi
            delta = delta_list[l] # 1xdj
            w = w-alpha*(delta.T*x)
            self.w_list[l] = w

    def predict_NNP(self):
        hx = self.X
        for l in range(1,self.L+1):
            z = hx@self.w_list[l].T # X should be mxdi, w should be djxdi, result mxd
            hx = self.theta(z)
        pred = np.array([-1+2*(int(p>=0)) for p in hx])
        return pred

    def accuracy_NNP(self):
        pred = self.predict_NNP()
        accuracy = sum(pred==self.y)/self.m
        return accuracy

    def fit(self,alpha,N):
        for i in range(1,N+1):
            random_p = self.rs.randint(0,self.m-1)
            x_in = self.X[random_p] # random x point
            y_in = self.y[random_p]
            x_list = self.update_x(x_in)
            delta_list = self.update_delta(y_in,x_list)
            self.update_w(x_list,delta_list,alpha)
            if i%int(N/10) == 0 or i == N:
                print(f"Iteration:{i}, Accuracy:{self.accuracy_NNP()}")
            i += 1
        return self.w_list
```

Test

In [231…

```
my_NNP = Neural_Network_Perceptron(X,y,Nh=1)
```

In [232…

```
my_NNP.fit(alpha=4e-3,N=1000)
```

```
Iteration:100, Accuracy:0.8305
Iteration:200, Accuracy:0.82
Iteration:300, Accuracy:0.7815
Iteration:400, Accuracy:0.874
Iteration:500, Accuracy:0.8515
Iteration:600, Accuracy:0.8775
Iteration:700, Accuracy:0.869
Iteration:800, Accuracy:0.899
Iteration:900, Accuracy:0.9425
Iteration:1000, Accuracy:0.917
```

Out[232…

```
[None,
 array([[-0.26985413,  1.0839181 , -0.67227052, -0.4951232 ],
        [ 0.64970235,  0.49549587,  0.8729425 ,  0.62238197]]),
 array([[1.64984871, 0.01440497]])]
```

Pocket Algorithm

By butting the best solution in the pocket during stochastic GD.

In [215…

```
def pocket(w,w_best,acc,acc_best):
    if acc>acc_best:
        w_best = w
        acc_best = acc
    return w_best, acc_best
```

In [122…

```
class Neural_Network_Perceptron:
    def __init__(self,X,y,Nh=1):
        def gen_d_list(Ni,No,Nn,Nh):
            d_list = [Nn for l in range(Nh)]
            d_list.insert(0,Ni)
            d_list.append(No)
            return d_list

        def init_w(d_list):
            w_list = [self.rs.rand(d_list[i+1],d_list[i]) for i in range(len(d_list)-
            w_list.insert(0,None)
            return w_list
        self.theta = np.tanh
        self.w_best = None
        self.acc_best = 0
        self.m = X.shape[0]
        self.rs = RandomState(2022)
        self.L = Nh+1
        D = X.shape[1]
        Ni = D # Number of Neurons in input layer
        No = 1 # Number of Neurons in output layer
        Nn = int((1+D)/2) # Number of Neurons in hidden layey
        d_list = gen_d_list(Ni,No,Nn,Nh)
        self.w_list = init_w(d_list)
        self.acc = None
        self.X = X
        self.y = y
```

```
    def update_x(self,x_in): # Compute all x in forward direction
        x_list= [x_in.reshape(1,4)]
        for l in range(1,self.L+1):
            z = x_list[l-1]@self.w_list[l].T # each x array should be 1xdi, w should
            hx = self.theta(z)
            x_list.append(hx)
        return x_list # output list of 1xdj


    def update_delta(self,y_in,x_list): # Compute all delta in forward direction
        delta_list = [0 for i in x_list]
        delta_list[-1] = 2*(x_list[-1]-y_in)*(1-np.square(x_list[-1]))
        for l in reversed(range(1,self.L+1)):
            delta = (1-(np.square(x_list[l-1])))*(delta_list[l]@self.w_list[l]) # del
            delta_list[l-1] = delta
        return delta_list # output list of 1xdi

    def update_w(self,x_list,delta_list,alpha): # Update w
        for l in range(1,self.L+1):
            w = self.w_list[l] # djxdj
            x = x_list[l-1] # 1xdi
            delta = delta_list[l] # 1xdj
            w = w-alpha*(delta.T*x)
            self.w_list[l] = w

    def predict_NNP(self):
        hx = self.X
        for l in range(1,self.L+1):
            z = hx@self.w_list[l].T # X should be mxdi, w should be djxdi, result mxd
            hx = self.theta(z)
        pred = np.array([-1+2*(int(p>=0)) for p in hx])
        return pred

    def accuracy_NNP(self):
        pred = self.predict_NNP()
        accuracy = sum(pred==self.y)/self.m
        return accuracy

    def pocket(self):
        if self.acc>self.acc_best:
            self.w_best = self.w_list
            self.acc_best = self.acc

    def fit(self,alpha,N):
        for i in range(1,N+1):
            random_p = self.rs.randint(0,self.m-1)
            x_in = self.X[random_p] # random x point
            y_in = self.y[random_p]
            x_list = self.update_x(x_in)
            delta_list = self.update_delta(y_in,x_list)
            self.update_w(x_list,delta_list,alpha)
            self.acc = self.accuracy_NNP()
            self.pocket()
            if i%int(N/10) == 0 or i == N:
                print(f"Iteration:{i}, Accuracy:{self.acc}")
            i += 1
        return self.w_best, self.acc_best
```

In [228…

```
my_NNP_pocket = Neural_Network_Perceptron(X,y,Nh=1)
```

In [229…

```
my_NNP_pocket.fit(alpha=4e-3,N=7000)
```

```
Iteration:700, Accuracy:0.869
Iteration:1400, Accuracy:0.9625
Iteration:2100, Accuracy:0.972
Iteration:2800, Accuracy:0.9915
Iteration:3500, Accuracy:0.961
Iteration:4200, Accuracy:0.9635
Iteration:4900, Accuracy:0.9735
Iteration:5600, Accuracy:0.982
Iteration:6300, Accuracy:0.9735
Iteration:7000, Accuracy:0.9915
```

Out[229…

```
([None,
  array([[-0.03938579,  2.14097998, -1.67675918, -1.24077207],
         [ 0.64577848,  0.4923601 ,  0.87221983,  0.62140206]]),
  array([[ 3.1109982 , -0.04472031]])],
 0.999)
```

In [ ]: