CSCI 341 - Fall 2024:
Homework 1

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Due: **On gradescope** Wednesday, Sep. 11, 2024
Based on problems in Sipser's *Introduction to the Theory of Computation*

Problems

Problem 1 (Sets)**.**

(a) What is $\mathcal{P}(\{\emptyset, \{\emptyset\}\})$?

> **Solution 1:**
>
> [

breakable, title=Solution] $\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(b) What is $\mathcal{P}(\emptyset)$?

(c) What is $\mathcal{P}(\{1,2\} \times \{1,2\})$?

(d) What is $\{x \mid x = 2k, k \in \mathbb{Z}\} \cap \{x \mid x \text{ is prime}\}$?

Problem 2 (Set Operators)**.** If $A = \{1,2,3,4\}, B = \{3,5,7,9\}$, and $C = \{2,3,5,7\}$, what is the following set?

$$D = (\{x | x \in A \text{ OR } (x \in B \text{ AND } x \in C)\} \times (C \cap A)) - (A \times (B \cup C))$$

Problem 3 (Relations)**.** Let $f : A \to B$ be a function. Define a relation on A where $x \in A$ is related to $y \in A$ if $f(x) = f(y)$. Show that this is an equivalence relation.

Recall that an *equivalence class* is a set of elements which are related by an equivalence relation. Equivalence relations partition their domains into a collection of equivalence classes. Write down the equivalence classes on $\mathbb{N}$ generated by the above relation with the function $f : \mathbb{N} \to \{0,1,2\}$ given by $f(n) = n \pmod 3$.

Problem 4 (Proofs)**.** Find the error in the following proof that $1 = 2$.

Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Factor each side: $(a+b)(a-b) = b(a-b)$, and divide each side by $(a-b)$, to get $a + b = b$. Finally, let a and b equal 1, which shows that $2 = 1$.

Problem 5 (Proofs-Induction)**.** Find the error in the following proof that all horses are the same color. *(Hint: Consider the concrete steps hidden in induction.)*

Claim 1. *In any set of h horses, all horses are the same color.*

Proof. By induction on h.

- Base Case: Let $h = 1$. In any set containing just one horse, all horses are clearly the same color.
- Inductive Hypothesis: For $k \geq 1$ assume that the claim is true for $h = k$.
- Inductive Step: Prove that the claim is true for $h = k + 1$. Take any set H of $k + 1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the inductive hypotheses, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all the horses in H_2 are the same color. Therefore all the horses in H must be the same color, and the proof is complete.

$\square$

Problem 6 (FSAs)**.** Draw Finite State Automata for the following languages over the alphabet $\Sigma = \{a, b, c\}$:

(a) $L = \{w \mid w[1] = w[-2]\}$ (python-style indexing)

(b) $L = \{w \mid w \text{ contains at least two } a\text{s and at least one } b\}$