Indeterminacy and Logical Atoms

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Indeterminacy and Logical Atoms

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Contents

Introduction

The beginnings of this project arose out of a seminar led by Professor Austen Clark on classic readings in the philosophy of perception. There I encountered a peculiar argument of W. H. F. Barnes (1944) which alleged that sense-data, if they exist, must disobey the Law of Excluded Middle. Barnes presumably did not endorse his argument's conclusion, instead offering it as an addition to the growing list of *reductiones ad absurdum* against theories of sense-data. My interest in the argument, however, was not primarily in the question of its soundness, but in a desire to grasp its rather provocative conclusion.

Approaching this from the standpoint of philosophical logic, the question becomes how to make formal sense of sense-data having peculiar logical features. What might a minimal revision of logic look like that countenanced the failure of the Law of Excluded Middle (LEM), while preserving as much as possible of the classical inferences?

A natural way to express the the failure of LEM is as follows:

$$B \nvdash A \vee \neg A$$

where $\vdash$ abbreviates logical consequence, and $\neg$ and $\vee$ are taken to express negation and disjunction, respectively. In this dissertation I formulate a three-valued logic that accommodates this failure, and expand on further desiderata unmet by other nearby systems on offer.

There is a range of various well-studied formal systems that have this feature (for discussion, see Chapter 2), and so further specifications are necessary to clarify the formal requirements of the desired system. In general, as above, the goal is a 'minimal' revision to classical logic such that LEM fails. This failure, however, occurs not as a result of general semantic properties of linguistic items (e.g. sentences or propositions), but because of the indeterministic character of a particular kind of metaphysical entity.

I take this general characterization to provide two constraints on the desired logic. The first, corresponding to the goal of minimal revision, is that the system maintain in every respect the Law of Noncontradiction (LNC). This amounts to three things, foremost that the system admit no case in which both a sentence and its negation are true. Second, the system should maintain the related classical principle of *Explosion*, or *Ex Falso Quodlibet*:

$$A \wedge \neg A \vdash B$$

That is, from a contradiction, anything follows. The final, and strongest, form of LNC I consider states that the negation of a contradiction is a tautology, or *logical truth*:

$$B \vdash \neg(A \wedge \neg A)$$

The logical revision, then, seeks at minimum to block the inference from the failure of LEM to the failure of LNC. The is the starting point for the system I develop here, called GO (for 'gappy objects').

The interpretation of GO allows one to accept Barnes's conclusion about sense-data without thereby accepting its dialectical force as an *absurdum*. For the majority of philosophical debates, determining what makes something 'absurd' in the sense required for a reductio is unproblematic. In classical logic, any 'violation' of a logical truth entails a contradiction, and so it suffices to identify an absurdum with any logical contradiction. However, once the debate turns to the logical principles themselves, the criteria for absurdity are often the very source of disagreement.

Here I will take it for granted that extracting a contradiction provides a sufficient demonstration of absurdity, despite this not being universally accepted. The Law of Noncontradiction, however, is only one among several logical principles. The Law of Excluded Middle seems pre-theoretically to be an independent, though arguably related, principle. Aristotle, at least, apparently felt the need to state the two as separate, independently motivated, principles. My suggestion here is that in order to understand the proposition that sense-data disobey LEM in a substantive way, one must attempt to conceive of such a circumstance without thereby entertaining a contradiction.

The second desideratum for the formal system corresponds to the idea that the failure of LEM is in some sense *restricted*, such that it occurs only for a peculiar type of object. In this case, sense-data are the target, though from the perspective of the formal system it is inconsequential what kinds of objects one has in mind. It is this idea of a 'restricted' failure of a logical principle that presents a bit of a puzzle. I will comment on this in later chapters, and so here I will make a few brief remarks.

On the surface it seems natural to assert that a particular kind of object or property has distinctive logical characteristics. The property of transitivity, for instance, applies to certain properties (being a descendant, for example), and not to others, and it seems in principle unproblematic to attribute such a logical characteristic to some kinds of properties and objects, and not to others. On the other hand, one could distinguish these facts about particular objects or properties as a posteriori, or topic-specific truths which, though certainly 'logical' in some sense, are not 'purely' logical inasmuch as they cannot be deduced on the basis of form alone. One must have particular knowledge about what it is to be a descendant that justifies abstracting transitivity as one of its logical features. Logic, it is generally thought, is topic-neutral, and thus what one might call 'logic proper' does not vary according to content, as it concerns everything there is. And since a principle of transitivity does not apply to *all* properties, it is not a principle of logic proper in this sense. LEM, however, is (or was supposed to be) a law of logic proper, and so it seems that even a 'restricted' failure would count as a total failure insofar as its application was thought to be topic-neutral and completely general.

Yet there seems to be a significant pull in the other direction, particularly if one allows for the possibility that classical logic might benefit from an upgrade. In this context, any revision of logic, if countenanced, is to 'fix' the logical system in order to accommodate unanticipated anomalies, since generally speaking the classical system is adequate. Mathematical proof, for instance, is in fine shape, and its deductive certainty remains unthreatened by revisions to solve truth-theoretic paradoxes; and the deviant behavior of sense-data, or some kind of peculiar object makes one's daily applications of, say, modus tollens no less valid. A minimal mutilation to logic, then, would do the minimum required to accommodate the anomaly, while leaving the rest intact.

A tension results. One intuition suggests that deductive logic is topic-neutral and purely formal. Validity is truth-preserving in *all* contexts, irrespective of the particular subject matter. Thus it seems impossible to 'isolate' the failure of a logical principle, since a putative logical law, if really a law, must hold irrespective of content. A logical principle that is said to 'hold' in only some contexts merely *resembles* a logical law, in the same way that transitivity, as above, resembles a law.

One suggestion to reconcile this is to adopt a logical pluralism which holds that different domains may require different logics. Mathematics may require classical logic, while the metaphysics of perception might require a logic without LEM, and the logic of a truth predicate may countenance the failure of Explosion. Which logic one's reasoning is subject to will depend on the subject matter about which one is reasoning. However, this attempt will fail to reconcile the perceived tension in one of two ways. Imagine that the different logics are linearly ordered from weakest to strongest, where a weaker consequence relation is a subset of a stronger one. If one holds that the weakest logic is 'logic proper', since it subsumes the rest, then it is hard to see how reasoning conducted in any of the stronger logics ought to be considered deductive. (Alternatively, if they are not linearly ordered, once could consider the strongest logic that is weaker than all the rest to be logic proper.) For the inferences in the stronger logics are only 'valid' under simplifying assumptions that are justified by relevant facts about the domain for which the stronger logic holds. This, however, is the primary characteristic of *non*-deductive reasoning, that it can always be *considered* deductive, once appropriate domain-specific premises are added.

This version of pluralism, then, seems to collapse into monism in an important respect, since logic proper is really just the weakest logic. If, on the other hand, a pluralist holds that there is no sense at all to be made of logic proper, and that the appropriate logic for a domain is wholly relative and independent of all others, then the revision of logic becomes far from minimal. For not only have classical principles of logic been revised, but the very nature of logic as wholly general has been abandoned. Topic-neutrality would have to be rejected outright, since logic would become by definition topic-specific.

The interpretation I offer for the formal system developed here offers one possible way this tension might be reconciled. One feature that aids in this is the restriction of truth-value gaps on the basis of logical complexity. As this develops, it will be most naturally interpreted as underwriting some form of logical atomism. This is most evident in the condition that only atomic propositions as well as their negations admit of truth-value gaps. This does not result from a direct specification in the semantics, but rather from a rather simple and straightforward reading of the binary connectives. I will say briefly, then, what I intend by 'logical atomism'.

Considering only the works Russell and Wittgenstein, one can find a great number of theses, different collections of which have at some time or other been referred to as Logical Atomism. (Bradley 1992), for instance, outlines fifteen distinct theses on which both Russell and Wittgenstein seem to agree.

I will not rehearse them all here, but most important for present purposes is the following from Russell:

> Particulars have this peculiarity, among the sort of objects that you have to take account of in an inventory of the world, that each of them stands alone and is completely self-subsistent. It has the sort of self-subsistence that used to belong to substance.... each particular that there is in the world does not depend upon any other particular. (1918, p.202)

As Russell gestures, this idea of the logical independence or distinctness of simple particulars compares to some degree with Aristotle's notion of substance. Armstrong also cites Hume for a similar idea as inspiration for his Combinatorialism, discussed in Chapter 3:

> This principle draws its inspiration from Hume's principle that there are no necessary connections between distinct existences. Any two distinct existences may be found together, or found one without the other, in a single world. (1989, p. 20)

A related thesis of atomism is, of course, that the world consists of atoms of some sort or other. For Wittgenstein this is expressed primarily in terms of simple facts, and likewise for the most part for Russell. In Chapter 3 I adopt the vocabulary of facts for purposes of expounding combinatorialism in concert with the modal system. Generally, however, one can think of atomism apart from the particular ontology, as what facts consist in are simple particulars, which is perhaps more natural to think of as atoms.

I take these two theses, that there are genuine atoms, which are logically independent of each other, as a minimal version of logical atomism, and it is this conception that will underwrite the formal system. Notably absent from this, for present purposes, is the claim that this is totality of what the world consists in. I return to some of these issues in later chapters.

Chapter summary

The first Chapter focuses on the 'atoms' of perception, beginning with a defense against arguments from later C. D. Broad, and ending with Barnes's argument that sense-data disobey LEM. Chapter 2 develops the propositional system with remarks on formal features and their interpretation. The modal system in Chapter 3 examines D. M. Armstrong's atomist combinatorialism. Chapter 4 presents a tableaux proof system and its adequacy results. Chapter 5 concludes with a survey of some outstanding issues as well as areas for future research. This includes a discussion of an extended 4-valued system and its application to 'atomless' metaphysical views.

Chapter 1

Remarks on Sense Data

1.1 Introduction

The two parts of this chapter each defend sense-data against a different argument. The first section defends against part of C.D. Broad's (1952) argument against *direct apprehension* accounts of perception. The second section takes on the charge that sense-data must in some sense disobey the Law of Excluded Middle.

Each of the arguments has the form of a reductio. Apart from questioning a premise, a defense of a philosophical thesis against a reductio has at least approaches available. One could argue that the alleged absurdum is not in fact absurd, perhaps on the grounds that it does not entail a logical contradiction. §1.2 takes this tack in examining C. D. Broad's (1952) argument against sense-data accounts of perception. I propose a way to block the argument by accepting the putative absurdum that it is logically possible that one may apprehend another's dreaming sensations.

Another, more drastic, defense against a reductio embraces the putative absurdum and accounts for its tenability by way of logical revision. §1.3 takes this strategy in response to W. H. F. Barnes's (1944) argument that sense data, if they do exist, must disobey the Law of Excluded Middle. I propose a minimal revision to classical logic that allows for such strange behavior for atomic objects, while blocking the inference to a contradiction (in this context, the failure of the Law of Non-contradiction).

Proposing a revision of logic to block a conclusion in many ways seems a doomsday response which could easily be avoided by simply rejecting the problematic theory it is intended to save. This may seem especially distasteful in the service of an out-of-favor theory of perception like sense-data. For even if Barnes's argument is blocked, one might claim, there are many other arguments more well-rehearsed that sense-data must overcome. And to be sure, a full defense of sense-data is not forthcoming.

The case of sense-data, however, bears an interesting parallel to better known arguments for logical revision. The truth-theoretic paradoxes, for instance, have driven many to propose logical revision as a result of the transparency of the truth predicate:

$$\mathbb{T}\ulcorner p \urcorner \text{ iff } p$$

Traditional realist theories that give an 'act-object' analysis of perception appeal to a relation of direct acquaintance or apprehension. In sense-data, or *indirect* realist, theories, this relation of direct apprehension provides a transparency principle for appearance, similar to that for truth. The nature of perception, according to the indirect realist, puts us in direct acquaintance with objects, such that we cannot be mistaken about appearances. This certainty cannot, however, be of the nature of the things we ordinarily take ourselves to perceive, for clearly the following principle of appearances is false:

$$\mathbb{A}\ulcorner F\alpha \urcorner \Rightarrow F\alpha$$

The converse is surely false, too. Indeed, §1.2.1 examines Broad's refutation of a weaker version of this naïve direct realist principle:

$$\mathbb{A}\ulcorner F\alpha \urcorner \Rightarrow \exists! \alpha$$

Sense-data theorists, then, account for the certainty of apprehension indirectly, as consisting in a correlation between an ordinary physical object's appearing F and something *or other* being F. This something or other is the immediate object of perception and it bears some close relation with the physical object we take ourselves to perceive. At minimum this suggests something akin to the following principle:

$$\mathbb{A}\ulcorner F\alpha \urcorner \Rightarrow \exists y F y$$

Thus something appearing F implies, not that *it* is F, but that some immediate object of perception is F. Further, there is a strong intuition that, given the role sense-data are to play in a theory of perception, not only can we not be mistaken about their positive features, but they also cannot have any positive phenomenological features that we do not directly apprehend. Some accounts of sense-data, then, accept the bi-conditional

$$\mathbb{A}\ulcorner F\alpha\urcorner \text{ iff } \exists y Fy$$

for some suitable restriction on F that only considers phenomenological properties. §1.3 examines these latter two principles and the potential resulting paradoxes in light of Barnes's argument that the first, weaker, principle entails that sense-data violate the Law of Excluded Middle.

The result, if not an enduring defense of sense-data, is a connection between paradoxes resulting from *semantic* theories and those resulting from *metaphysical* theories. Later chapters examine a similar parallel that might be made with paradoxes from *scientific* theories. The broader approach, then, is not necessarily to defend any particular theory, but to attempt to understand what features these disparate motivations for logical revision might share, and where they may differ.

1.2 Broad's argument against sense data

Broad argues against prehensive accounts of the epistemological character of perception. He has two arguments from dreams and hallucinations. The first refutes the view that sense-perception consists in the prehension of physical objects. The second argues against the view that sense-perception consists in the prehension of sense-data.

There are two ways one might speak of the 'character' of perception. The *epistemological* character accounts for the way in which our perception puts us in a relation with ordinary physical objects, such that we are able to gain knowledge of those objects. It describes the cognitive process of perception which puts us "in touch" with physical objects.

Just as the epistemological character of perception is the way in which perception *does* connect us with physical objects, the *phenomenological character* is the way in which perception *seems* to so connect us with objects. That is, the phenomenological character is how the epistemological character is presented to us by experience, such that *if* the phenomenological character of a given form of sense perception were accurate, it would be *identical* to its epistemological character.

The question then becomes, *Is* the common-sense description of phenomenological character accurate? Is the phenomenological character of vision a good guide to its epistemological character? Broad attacks two possible answers to these questions. The first is the *extreme view*, which holds that phenomenological character is a *perfect* guide to epistemological character. Visual experience presents itself as the direct apprehension of objects, so, the extreme view says, we must in fact directly apprehend objects. The second is the *moderate view*, which holds that, although phenomenological character is not a *perfect* guide to epistemological character, it is still *some* sort of a guide. We may be wrong to infer from phenomenology that we are directly apprehending ordinary physical objects, but we are correct to infer that we are directly apprehending *something or other*.

1.2.1 Broad's argument against the extreme view

The phenomenology of sense-perception seems to inform us that, when we look at objects, we are directly apprehending (or *prehending*) them. Physical objects *leap* out of the place they appear, such that we are directly aware of them. Since the extreme view holds that phenomenological character is a complete guide to epistemological character, the theory states that visual perception is the direct prehension of objects. This view is the target of Broad's first argument. Whatever else it may hold, such a theory is committed to the following:

($\mathrm{PR_O}$) A form of sense-perception φ (e.g. see, hear, or touch) has the epistemological character of *Prehension of Objects* only if an external object's being at a certain place p at a certain time t is a *necessary condition* for a person's having an experience which he would naturally describe as φ-ing an external object at p at t.

If visually perceiving is the direct prehension of ordinary physical objects, then in order to be in such a prehension relation with an object at p at t, there must be an object there and then with which to be in the relation.

There is a certain plausibility to this account with ordinary cases of veridical perception. When I veridically perceive a burrito in front of me at arms length, it seems that I am directly and immediately apprehending—"grasping" in some sense—the burrito. In this case the necessary condition stated in $\mathrm{PR_O}$ is satisfied.

Given, though, that dreams and hallucinations are phenomenologically indistinguishable from waking sense-perception, there is a problem for this account.[1] Broad writes:

> There are certain experiences, viz., dreams and waking hallucinations, which exactly resemble normal waking sense-perceptions in all their phenomenological characteristics (including that of being ostensibly prehensive of foreign bodies and external physical events), but which are certainly not in fact prehensions of any such objects. It seems most unlikely that experiences which exactly resemble these in all their phenomenological characteristics, as do normal waking sense perceptions, should be fundamentally unlike them in their epistemological character.
>
> ...On the whole, then, I see nothing for it but to draw the following conclusion. Our waking experiences of seeing, hearing, and touching are not, as they appear to us to be, prehensions of foreign bodies and physical events and of certain of their intrinsic qualities.(Broad 1952, p. 41)

Broad's argument against $\mathrm{PR_O}$ from dreams and hallucinations seems to be the following:

[1] Whether dreams are *phenomenologically* indistinguishable from waking sense perception I will not debate here. Surely, just because I cannot *know* whether I am dreaming or waking right now, it does not follow that dreams and waking perception are *phenomenologically* indistinguishable, for this is perhaps merely an epistemological concern. Broad points out, however, that just because we do distinguish between waking and dreaming we do not do so "by noting dissimilarities in their phenomenological character. We do so by considering the interrelations of experiences with the earlier and later experiences of the same person and the contemporary experiences of others" (Broad 1952, p. 41). This is not clearly true, and it is hard to imagine what evidence would count one way or the other, since it would require a suitable criterion for phenomenological indistinguishability. But it does seem plausible to assert that dreams definitely *seem* different.

1. Dreams and waking hallucinations have the same phenomenological character as normal waking sense-perceptions. [Premise]

2. If dreams/hallucinations and normal waking perceptions have the same phenomenological character, then they have the same epistemological character. [Premise]

3. Thus, dreams and waking hallucinations have the same epistemological character as normal waking sense-perceptions. [From (1) and (2)]

    4. Suppose that waking sense-perception has the epistemological character PR_{O}. [Reductio premise]

    5. Thus, dreams and waking hallucinations must have the epistemological character PR_{O}. [From (3) and (4)]

    6. However, one can have an experience during a dream that one would naturally describe as seeing an external object at some place and time, while there is no external object there and then. [Premise]

    7. Hence, the epistemological character of dreams is not PR_{O}. This contradicts (5). [From (5) and (6)]

8. Therefore, our supposition (4) is false. That is, waking sense-perception does not have the epistemological character PR_{O}. [From (4)-(7)]

Premise (1) is from the traditional assumption that dreams and hallucinations are phenomenologically indistinguishable from waking sense-perception. Premise (2) is an apparent commitment of the extreme view. If the extreme view infers epistemological character directly from phenomenological character, then it seems one must be willing to apply the inference for all forms of perception, waking or dreaming. (We will return to this in the next section.) This leads us to contradiction: in dreaming perception we seem to perceive objects when no objects are present (6), thus failing to meet the necessary condition in $\mathrm{PR_O}$.

1.2.2 Broad's argument against the moderate view

It seems we listened to our phenomenology too much: we do not directly prehend physical objects. It does seem, however, that we are directly prehending *something.* If it is not the ordinary objects of perception, there must be some other kind of thing that we prehend, in virtue of which we perceive ordinary objects. Since these sense data are not 'ordinary' objects, they must be something else. The moderate view, then, holds that, although phenomenology may not tell us anything about what *kinds* of objects we are directly aware of, we can still infer *something* about epistemological character from phenomenological character—namely, that the *relation* we are in is one of prehension.

Broad's argues against the moderate view by examining what follows merely from the existence of a prehension relation between a subject and a particular. As before, we need only focus on a necessary condition for the moderate view:

($\mathrm{PR_S}$) A form of sense-perception φ has the epistemological character of *Prehension of Sensa* only if a person S φ-ing consists in (or involves) a relation R—the target relation here being prehension—such that:

(a) There is some particular, x such that $\langle S, x\rangle \in R$.

(b) If $\langle S, x\rangle \in R$, then it is *logically possible* that for some S^* such that $S \neq S^*$, $\langle S^*, x\rangle \in R$.

Broad outlines other constraints on R which fall out of its being a prehension relation, but it is only (b) that will concern this next argument. Note that Broad is ultimately arguing that the epistemological character of sense-perception does not consist in a prehension relation of *any* kind. The first argument shows that the epistemological character of sense-perception is not that of $\mathrm{PR_O}$. If the only other available option is $\mathrm{PR_S}$, and Broad successfully refutes *that*, then he has shown that sense-perception does not consist in a subject being in a prehension relation with a particular.

The argument against $\mathrm{PR_S}$ follows a similar structure as before.

9. Dreams and hallucinations have the same phenomenological character as waking sense-perceptions. [Premise]

10. If dreams/hallucinations and normal waking perceptions have the same phenomenological character, then they have the same epistemological character. [Premise]

11. Thus, dreams and waking hallucinations have the same epistemological character as normal waking sense-perceptions. [From (9) and (10)]

    12. Suppose that waking sense-perception has the epistemological character $\mathrm{PR_S}$. [Reductio premise]

    13. Thus, dreams and waking hallucinations must have the epistemological character $\mathrm{PR_S}$. [From (11) and (12)]

14. However, in the case of S dream-seeing, it is not logically possible that some S^* (such that $S \neq S^*$) prehend the alleged particular of S's dream. [Premise]

15. Hence, the epistemological character of dreams is not $\mathrm{PR_S}$. This contradicts (13). [From (13) and (14)]

16. Therefore, our supposition (12) is false, and so waking sense-perception does not have the epistemological character $\mathrm{PR_S}$. [From (12)-(15)]

The form of this argument is similar to the previous one. Here I suggest that the sense-data theorist can deny (14), and accept the logical possibility of another person experiencing my dream sensing.

Broad takes this as absurd, but we should look more closely at the reasons behind accepting premises (2) and (10). As the premises are stated they are identical, but there are important differences in their respective justifications. Premise (2) seems to be an application of a more general principle:

(EP) For all forms of sense-perception φ_1 and φ_2, if φ_1 and φ_2 have the same phenomenological character, then φ_1 and φ_2 have the same epistemological character.

One may wonder why we should accept this. After all, Broad is claiming that the phenomenological character of sense-perception (viz. sight) is *not* a good guide to its epistemological character. If this is so, why think that two forms of sense-perception (or two modes of one form) which share phenomenological character will share epistemological character? If length is not a good guide to its width, then there is no reason to think that two things of the same length will have the same width.

Since Broad recognizes that (EP) is false, in order for his argument to go through, he will have to commit his opponents to it. He might be able to do so for the proponents of ($\mathrm{PR_O}$).[2] In the case of sight, the reason for thinking it consists in (or at least involves) the direct prehension of physical objects is simply that its phenomenological character seems to do just so. If this is the *only* phenomenological reason to think sight obeys ($\mathrm{PR_O}$), then it would seem to commit the proponents of ($\mathrm{PR_O}$) to (EP) and thus to (2).

But the commitments of proponents of ($\mathrm{PR_S}$) are not as straightforward. Note that an even more general principle Broad is challenging is:

(G) For all forms of sense-perception φ and all characters δ, if φ has phenomenological character δ, then φ has epistemological character δ.

This implies (EP). However, as Broad points out, proponents of ($\mathrm{PR_S}$) (i.e. sense-data theorists) do not wholly accept (G). He acknowledges that, to the extent that philosophers saw the inadequacy of ($\mathrm{PR_O}$), "they felt obliged to hold that the phenomenological character of [sense-experiences] is a misleading guide to their epistemological character" (42). It is clear, then, that proponents of ($\mathrm{PR_S}$) do not accept (G) outright. So, the sense-data theorists reject that the epistemological character of a given form of sense-perception is exactly like its phenomenological character. It is left to explain, however, whence comes the commitment to sense-data.

Though phenomenological character may not be a *perfect guide* to epistemological character, the moderate view maintains that it is at least *some* such

[2]Note, he may not need do this for *refuting* ($\mathrm{PR_O}$). He has a simpler argument against ($\mathrm{PR_O}$): we see objects in mirrors when none is present, and we see stars that are not there. Here we are concerned only with defending the argument from dreams and hallucinations.

guide. Phenomenology may deceive us as to exactly what kind of objects we are prehending, but it does inform us *that* we are prehending something or other. It seems the sense-data theorist accepts, then, that if the phenomenological character of sight is that of prehension of *something*, then the epistemological character of sight is that of prehension of *something*. We see here the commitment to the following principle:

(SR) For all φ, if φ has the phenomenological character of a relation R to some kind of object X, then φ has the epistemological character of relation R to some kind of object Y.

Note that, importantly, X need not be identical to Y. In fact, as the argument against ($\mathrm{PR_O}$) shows, X *cannot* be the same kind of thing as Y.

The sense-data theorist thus accepts (10) insofar as it follows from (SR). Perhaps, then, the following is a better formulation:

10′. If the phenomenological character of dreams/hallucinations and waking sense-perception both consist in relation R, then the epistemological character of dreams/hallucinations and waking sense-perception both consist in relation R.

The important point here is this: Just as the sense-data theorist maintains that *seeming* to prehend a certain kind of object (i.e. a physical object) is no guide to what kind of objects we *actually* prehend (i.e. sense-data), he can also consistently maintain that the kind of objects we actually prehend in *waking* sense-perception is no guide to what kind of objects we prehend in *dream* sense-perception.

This returns us to constraint (b) in (PR_S). Russell, Moore, and here Broad, are careful to emphasize this constraint on waking sense-data. But one wonders, Why think it is logically possible, in the case of waking sense-data, that someone else could have prehended the very same sense-data that I prehend? One fairly straightforward answer is that the relation of prehension is a relation between a subject and a non-identical particular, and for any such relation, it is logically possible that each of the relata be in the same relation with a distinct relatum.

This suggests one of several reasons for the possibility of prehending another's waking sense-data. Moore, for instance, writes:

> I think, then, that the term 'sensation' is liable to be misleading, because it may be used in two different senses, which it is very important to distinguish from one another. It may be used *either* for the colour which I saw or for the experience which consisted in my seeing it. And it is, I think very important, for several reasons, to distinguish these two things. In the first place, it is, I think, quite conceivable (I do not say it is actually true) but *conceivable* that the patch of colour which I saw may have continued to exist after I saw it: whereas, of course, when I ceased to see it, *my seeing* of it ceased to exist (1953, p. 31).

Russell distinguishes between sensibilia and sense-data, the latter being sensibilia that are apprehended. He writes:

> We cannot ask, 'Can sense-data exist without being given?' for that is like asking, 'Can husbands exist without being married?'... Unless

> we have the word *sensibile* as well as the word 'sense-datum', such questions are apt to entangle us in trivial logical puzzles.
>
> It will be seen that all sense-data are *sensibilia*. It is a metaphysical question whether all *sensibilia* are sense-data, and an epistemological question whether there exist means of inferring *sensibilia* which are not data from those that are (Russell 1917, pp. 110-11).

Perhaps we should treat this constraint as sufficient for logical possibility as required by (b):

(L) For all x and y, xRy where $x \neq y$, only if it is logically possible that there is some z such that zRy where $x \neq z$.

It is hard to know exactly what Broad thought logical possibility amounts to, so I will suggest constraints that give a fairly broad conception of logical possibility.

With the development of non-standard logics, the question is no longer clearly whether something is logically possible simpliciter, but whether it is logically possible *according to a certain logic*. Or, if one thinks of possibility as governed by logical *laws*, we must consider which logic the laws of which we should inspect. For a logical monist who thinks there is one "true logic," this is tantamount to asking what the *real* logical laws are.

However, even if it is clear which logic to choose, formulating a criterion of logical possibility is not straightforward. In classical propositional logic, our resources are scarce. The obvious candidate there is to define possibility as truth in some model, and necessity as truth in all models. Here any sentence but the negation of a logical truth will be possible. Perhaps this is all that logical possibility amounts to. On this account we have limited ability to express inferences with respect to possibility in the object language. But if this is all that is meant by logical possibility, then since (14) is not straightforwardly a denial of a classical logical truth, it follows that accepting (14) brings no contradiction.

If one takes standard Kripke modal logic (e.g. $\mathsf{S5}$) to in some sense model logical possibility, things are perhaps less clear. It may be unproblematic on this account to talk of *valid inferences* with respect to possibility, but to formulate a criterion of logical possibility requires filling in the schema:

(S) p is logically possible iff ...

Note that (L) can be formulated several ways, including:

(L*) $\forall x \forall y((xRy \wedge x \neq y) \supset \Diamond \exists z(zRy \wedge z \neq x))$

and an inferential reading:

(L**) $\exists x \exists y(xRy \wedge x \neq y) \vdash \exists x \exists y((xRy \wedge x \neq y) \wedge \Diamond \exists z(zRy \wedge x \neq z))$

Under the assumption that $\mathsf{S5}$ or a similar logic correctly models logical possibility, there are several options available for filling in (S). We can talk of truth in *some* model, truth in *all* models, or perhaps truth in *the correct* model. The first two options seem not to give us what we want. If we adopt $\mathsf{S5}$ as our paradigm, and define logical possibility in terms of truth in some model, we get the following:

(S$_1$) p is logically possible iff $\Diamond p$ is true at w_0 in some model.

Here the only logical impossibilities are the negations of classical logical truths with a few additions, such as $\Box\neg A \wedge \Box A$. Similarly, if we opt for

(S_2) p is logically possible iff $\Diamond p$ is true at w_0 in all models.

the only logical possibilities are the logical necessities. Clearly, neither option lends much support to (14). The last seems to be our best candidate. Thus:

(S_3) p is logically possible iff $\Diamond p$ is true at w_0 in the correct model of the universe.

This is of course contentious for anyone who believes that the 'models' at issue cannot completely represent the universe. But under the assumption that S5 correctly models logical possibility, it seems the most eligible candidate for a criterion. Of course, in order to argue that a given claim is or is not logically possible, it is hard to see how one can make non-question begging assumptions about the correct model. Here our target p, without any further constraints, comes out contingent, i.e. (L^*) is true at w_0 in some models but not in others, and (L^{**}) comes out invalid, but the set consisting of the premise and conclusion is satisfiable. Plausibly, though, if we are looking for the correct model, we are not looking at validity across all models, but truth in *the* model—and thus not truth in all or merely some models. However, putting the requisite constraints on *the* model (i.e. picking out a set of candidate models) such that a form of (L) comes out true will only serve to beg the question.

But to get a hold on the dialectic, consider that Broad takes (14) to be an absurdum for the sense-data theorist. Though we have not shown the truth of (14), it is hard to see how we should be convinced to find an absurdum therein.

We can, however, rule out reasons for accepting (14) relating to considerations about R. One can perfectly well say that during dreams one is in a prehension relation with some particular or other for which it is *logically possible* that someone else be in a prehension relation to that same particular. To see what this is asserting, consider what it does not assert. The following two considerations are *not* reasons for asserting (14):

(i) Waking sense-data are causally dependent on physical objects, whereas dreaming sense-data are not.

(ii) Though waking sense-data may be existentially dependent on there being some mind or other that prehends them, they are not existentially dependent on a particular mind, as they are in the case of dreaming sense-data.

(i) is irrelevant to the consideration of (b). As Broad notes, the point here is not about causal or nomological possibility, but logical possibility. More importantly, given what (10′) asserts, there is no reason to hold that the nature of waking sense-data tells us anything whatever about the nature of dreaming sense-data. Thus, it seems that (ii) —or any similar consideration—gives us no reason to deny that it is logically possible for one to prehend another's dreaming sense-data.

We might be drawn to believe (14) because it must be the very *nature* of these supposed dream sense-data that they are private and they belong essentially to the particular dreamer. But we have shown that the moderate view can hold that dream sense-data are essentially private, while still holding that it is logically possible that another prehend one's dream sense-data. The sense of logical possibility at issue here is the absence of logical contradiction. It is no problem to coherently think about or model any subject prehending a particular. And this point is all that the sense data theorists like Moore and Russell need. It remains open at this point to talk about the natures of these particulars and whether their natural features preclude in any more narrow sense prehension by others. The sense-data theorist can make the requisite inferences from phenomenology without commitment to a particular conception of the nature of these particulars, nor to there being only one such nature. For the inferences drawn from phenomenology only tell us about the nature of the relation constitutive of perception, but not about the natures of the relata.

1.3 Sense data and LEM

In this section, we are faced with another argument that seems to be a reductio against sense-data theory. The absurdum in this case is that, if sense-data do exist, they disobey the Law of Excluded Middle (herein, LEM). I will examine several things one might say in response to the argument. The argument is in several ways unclear, and I will look at several possible clarifications. However, it will seem that none of the responses conclusively block the argument. Thus, I will embrace the putative absurdum, and accept the conclusion that sense-data disobey (in some sense) LEM.

1.3.1 Barnes's argument

W. H. F. Barnes (1944) raises several objections to sense-data theories. I would like to focus on just one particular argument, which charges sense-data (if they exist) with disobeying the LEM. Barnes writes:

> If I contemplate an object at some distance, it often happens that I am uncertain whether it is circular or polygonal. It is necessary for me to approach close before I can determine the matter with certainty. On the [sense-data] theory, the mode in which the object appeared to me at first is a sensum, every sensum is what it appears to be. Now this sensum appears neither circular nor non-circular. Therefore it is neither circular nor non-circular. (1944: 145)

Barnes's argument seems to be the following:

i. If a sense-datum appears F, then it is F.

ii. Sense-datum s appears neither circular nor non-circular.

---

iii. Therefore, s is neither circular nor non-circular.

If this argument is successful, it seems to show that sense-data in some disobey the Law of Excluded Middle. Barnes apparently takes this to be a reductio of sense-data theories that accept some form of (i). Indeed this would commit the sense-data theorist to some peculiarly behaving entities. However, I intend to show that it need not follow from (iii) that sense-data theories are incoherent.

The first premise is the assumption of incorrigibility from the target sense-data theories. The relation of apprehension is such that I cannot be mistaken about features of my own sense-data of which I am aware. The second premise comes from Barnes's proposed example of an object that appears neither to have nor lack a certain feature.

There are few initial ways one might respond to the argument. One could (a) deny across the board the occurrence of phenomena of the type Barnes proposes, thus denying that (ii) is ever true, (b) question the inference to (iii), or (c) embrace the conclusion. Though I will ultimately go for (c), let us first consider (a) and (b).

One possible objection to make is to question the scope of 'appears' in (ii). Is it the case that an object determinately *appears* a certain way, where that way is neither circular nor non-circular? Or, is it rather that the object does not determinately appear to be circular nor determinately appear to be non-circular? I am not sure how to gather evidence for the truth of one reading over the other, but Barnes insists that an instance of a sense-datum appearing neither F nor non-F, as (ii) intends to pick out, is not merely a case of an object failing to appear a certain way.

At this point, however, it is unclear how to take (ii). From the form of the argument, it seems that 'neither circular nor non-circular' must be a property that s positively appears to have, in order to detach (i), thus:

F: is neither circular nor non-circular

Now, it would be fine (for the purposes of detaching (i)) if F were a conjunctive property, for instance:

G: is non-circular

H: is non-non-circular

$F*$: is G and H

In which case we would restate (ii):

ii′. s appears non-circular and s appears non-non circular.

However, $F*$ seems to be more like a "contradictory" property, and not some sort of "gappy" property. But if we take (ii) like the following:

ii″. s does not appear circular and s does not appear non-circular.

then the argument as stated does not go through. In order for (ii″) to work, we would need to read (i) as:

i′. A sensum appears F iff it is F.

This would amount to claiming that sense-data are all and *only* what they appear to be. Broad (1927) and others argue against this principle, though, and it does not seem that Barnes is assuming it. As the argument is stated, it is unclear how to take (ii) with respect to F. (ii′) does not seem to get us (iii), but instead to something like:

iii′. s is both non-circular and non-non-circular.

And (ii″) requires (i′) and not merely (i) in order to get to (iii).

One might conclude that this argument trades on some confusion. But if so, it is difficult to precisely locate. Instead one might accept (iii) and take it that it expresses a failure of sense-data to obey LEM.

Although Barnes's argument serves as the initial motivation for the logic, there are other possible applications as well. With respect to sense-data, one might treat such phenomena as the waterfall illusion with truth-value gaps. Further, Putnam (1957) noted that quantum physics might require a gappy logic that rejects LEM. The target logic would allow for such a model that keeps the gaps at the quantum level. We will discuss many of these issues in subsequent chapters.

1.3.2 LEM and LNC

There are several ways to formulate LEM and LNC precisely. Broadly speaking, LEM states that every sentence[3] is either true or false, and LNC states that no sentence is both true and false. One might, for instance, distinguish between *not accepting* LEM and *denying* it. Formally, this might involve introducing two types of negation. Or one might formulate the laws using modal operators. For our purposes, however, it will suffice to forgo such distinctions, and formulate the two laws in strong form:

(LEM) $B \Vdash A \vee \neg A$

(LNC) $A, \neg A \Vdash B$

(LNC*) $B \Vdash \neg(A \wedge \neg A)$

Note that in classical propositional logic, the following inference is valid:

$$\neg(A \vee \neg A) \Vdash A \wedge \neg A$$

[3] I use the term 'sentence' here and throughout to refer to a meaningful, declarative sentence.

Thus, to deny LEM is to deny LNC.[4] This is just an instance of DeMorgan. Note, many standard gappy logics, including Kleene, maintain the above inference. The aim, then, is to develop a logic that does two things. First, it blocks this inference, and second, it allows for failures of LEM.

When discussing our general idea of a triangle—i.e. our idea of a triangle *in general*, not specifically a right triangle, equilateral triangle, scalene, or other specific kind of triangle—Locke writes:

> ... Does it not require some pains and skill to form the *general Idea* of a *Triangle*, (which is yet none of the most abstract, comprehensive, and difficult,) for it must be neither Oblique, nor Rectangle, neither Equilateral, Equicrural, nor Scalenon; but all and none of these at once. (IV.VII.9 p. 596)

Locke does make the inference from our general idea of a triangle being none of these, to it being all of them. But why make this inference? Barnes tries to show that sense-data are deficient in a certain way—they are gappy. Why think because of this that they are deficient in some other way—e.g. contradictory? Now, there are *other* phenomenon that might lead one to believe that sense-data are contradictory (e.g. waterfall illusion). But these, I think, are different concerns, and depending on one's specific theory of sense-data, one could have explanations for these contradictory phenomena, independent of accepting the 'gappy' phenomena. My point is that we should not let Barnes's argument show that sense-data are contradictory.

[4]By 'deny' here I mean 'assert the negation of', and, in turn, that to deny LEM is to assert something equivalent to the denial on LNC.

1.3.3 Resulting Motivation

The resulting picture is as follows. Let us suppose that our language is, by and large, classical. That is, our language is, for the most part, modeled correctly by classical logic. Many of the inferences we think are valid are the classical inferences. In fact, our language might be completely classical, if it were not for certain misbehaved objects. That is, let us further suppose that there are these ill-mannered objects that in some sense ‘disobey’ the LEM. As a result certain atomic sentences about those objects—and likewise their negations—are neither true nor false.

But it is nothing about our *language* per se that makes this the case. That is, we have, suppose, no reason to think that every sentence whatsoever could be gappy. It is just these deficient ‘gappy’ objects. Other than that, as before, our language is ‘classical’. In other words, let us suppose that we do not think that we have some special semantic predicate or funny sort of negation—both of which would be features of an essentially non-classical *language*—that is designed specifically to handle these cases. The next chapter develops the logic.

Chapter 2

GO: A Basic Picture

In this chapter I present the propositional system GO, a three-valued *gappy* logic that restricts the assignment of gaps to literals.[1] The logic results from combining a familiar treatment of three-valued negation with a less-famliar treatment of conjunction and disjunction. The result has a natural interpretation as an *atomistic* logic, i.e. one that assumes some form of Logical Atomism.

§1 covers some brief background of many-valued logics. §2 gives an informal sketch of the basic GO semantics. §3 highlights features of the formal system, and §4 covers the philosophical interpretation.

[1]Literals are typically defined as atomic or negated atomic sentences. Here we also mean an atomic parameter preceded by any number of negation signs, such that $\neg\neg p$, $\neg\neg\neg p$, and so forth, are treated as literals.

2.1 Background

This section briefly reviews a few logical systems relevant to the current discussion. The reader quite familiar with many-valued logics may wish to skip ahead to §2.

2.1.1 Classical Logic

Today's standard system of classical propositional logic (herein CPL) comes from Frege and has been refined over decades by many others. The recognition of CPL as the standard is a relatively recent development, however, as for centuries the standard was Aristotle's syllogistic logic, and there are good reasons for believing that it conflicts with CPL in a few, yet significant ways.[2] Other reasons abound for restricting (or expanding, depending on one's viewpoint) CPL, and they are well-rehearsed in the recent literature, and so we will not survey them here.[3]

A familiar presentation of CPL begins by recursively defining a set of sentence elements, which includes a base of atomic elements $\{p, q, r, \ldots\}$ as well as all possible combinations built in the usual way from a set of connectives $\{\neg, \wedge, \vee\}$, standing for negation, conjunction and disjunction, respectively.[4] Logical consequence is then defined in terms of models, or valuation functions ν that assign each sentence a single value, 1 or 0. A conclusion A is a logical consequence of premises $B_0, \ldots, B_n$ iff for every valuation where each of

[2] See for instance (Łukasiewicz 1957) and (Corcoran 1972).
[3] For a useful survey, see (Rescher 1969) and (Beall and van Fraassen 2003).
[4] For present purposes, we take both $\wedge$ and $\vee$ as primitive, disregarding redundancy.

$\nu(B_0), \ldots, \nu(B_n)$ is 1, it is also the case that $\nu(A) = 1$. Our valuations are restricted to those that accord with the following diagrams:

$$
\begin{array}{c|c}
\neg & \\
\hline
0 & 1 \\
1 & 0
\end{array}
\qquad
\begin{array}{c|cc}
\wedge & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1
\end{array}
\qquad
\begin{array}{c|cc}
\vee & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1
\end{array}
$$

Two additional connectives, the *material conditional* $\supset$ and *material equivalence* $\equiv$ are defined, where $A \supset B$ abbreviates $\neg A \vee B$, and $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

2.1.2 Many-valued logics

A many-valued logic, somewhat confusingly, is one that has more than two values. Here we consider systems that have exactly three values, $\{0, \nicefrac{1}{2}, 1\}$.

Expanding the set of values naturally requires modifications in other parts of the system. In some cases this includes modifying the definition of logical consequence. In *paraconsistent* logics, which may countenance true contradictions $(A \wedge \neg A)$, the intermediate value functions as an additional way of being *true*. A typical strategy for this is to define logical consequence in terms of a set $\mathcal{D}$ of *designated* values, where an argument is valid iff every valuation that assigns each of the premises some value in $\mathcal{D}$, also assigns the conclusion a value in $\mathcal{D}$. Thus in CPL, $\mathcal{D}=\{1\}$, and in the Logic of Paradox (Priest 1979), LP, $\mathcal{D}=\{\nicefrac{1}{2}, 1\}$. In this system, the value $\nicefrac{1}{2}$ can be thought of as *both* true and false, and a sentence receiving this value is said to be *glutty*.

The current discussion, on the other hand, focuses on systems that treat $\nicefrac{1}{2}$ as in some sense *neither* true nor false. A sentence receiving the value $\nicefrac{1}{2}$ is said to be *gappy*, the idea being that it falls within a gap between truth and falsity. Precisely what this amounts to depends on the particular philosophical issue for which the system is used. The label 'indeterminate' is common, which in some cases means 'unprovable', and in others something a bit more metaphysical. We return to these issues in §4. The upshot in any case is that, as in classical logic, $\mathcal{D}=\{1\}$, and so our set of *un*designated values is $\{0, \frac{1}{2}\}$. As a result, for present purposes, we need not alter our CPL definition of logical consequence.

It remains to say how the connectives behave with respect to this new value. Naturally, there are several different ways one might go, although some are no doubt more interesting than others. A review of all these ways is too large a task, but we will have a brief look at some systems of Kleene, Łukasiewicz and Bochvar.

Kleene

The *Strong Kleene* system (herein K_3) treats the connectives in the following way:

$\neg$	
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

With respect to the classical values 1 and 0, the connectives preserve their treatment in CPL. What of the value $\frac{1}{2}$? Negation behaves similar to the intuitionistic mode. Intuitively, if a sentence is gappy, so too is its negation.

The tables for $\wedge$ and $\vee$ also have somewhat intuitive readings. Here we can think of a conjunction as true just when both conjuncts are true, false when one or more conjuncts is false, and gappy in all other cases. Likewise, a disjunction is true just when one or more disjuncts is true, false when both disjuncts are false, and gappy in all other cases. (Incidentally, it is worth noting that LP is the glutty dual of K_3, having the same readings of the connectives but treating ½ as designated.)

The *Weak Kleene* system keeps $\neg$ the same (likewise for all systems discussed herein, including GO), but treats the binary connectives differently:

$\vee$	0	½	1
0	0	½	1
½	½	½	½
1	1	½	1

$\wedge$	0	½	1
0	0	½	0
½	½	½	½
1	0	½	1

With the interpretation of ½ as *meaningless*, the thought here is that any statement built from one or more meaningless statements is itself meaningless. (Or, as the saying goes, one bit of rat's dung spoils the soup.)

Consider now the valuation that assigns each atomic element the value ½. In the Kleene systems, the tables show that this valuation must also assign the value ½ to every sentence element whatsoever. As a result, though there are certainly valid arguments, there are no tautologies—or *logical truths*—in these systems. Consequently, it is often remarked that the Kleene systems lack a genuine conditional, since *Identity* ('if A then A') and *Equivalence* ('A iff A') fail as logical truths, not only for $\supset$ and $\equiv$, but for any definable connective.

Łukasiewicz

The Łukasiewicz three-valued system $\mathsf{Ł}_3$ preserves the tables for K_3, but has an additional connective $\rightarrow$ intended as an adequate conditional, one for which identity and equivalence hold:[5]

$\rightarrow$	0	½	1
0	1	1	1
½	½	1	1
1	0	½	1

Intuitively, if the consequent is at least as strong as the antecedent, a conditional is true (thinking of ½ as "in between" true and false). Otherwise, the 'value' of the conditional corresponds to the how much 'stronger' the antecedent is than the consequent.

Bochvar

Lastly, we consider the system of D. A. Bochvar (1937). Bochvar's so-called 'internal' system B_3 is equivalent to Weak Kleene, while the full system $\mathsf{B}_3^{\mathsf{E}}$ includes an additional 'assertion operator' $\succ$ with the following table:

$\succ$	
0	0
½	0
1	1

In turn, $\mathsf{B}_3^{\mathsf{E}}$ defines 'external' versions of the binary connectives:

[5]The $\mathsf{Ł}_3$ $\rightarrow$ connective is not definable in terms of $\neg$, $\wedge$ and $\vee$. However, Łukasiewicz took the connectives $\neg$ and $\rightarrow$ as the only primitives, and defined $\wedge$ and $\vee$ from them.

$$\begin{aligned} &\underset{\cdot}{\neg} A := \neg \succ A \\ &A \mathbin{\dot{\vee}} B := \succ A \vee \succ B \\ &A \mathbin{\underset{\cdot}{\wedge}} B := \succ A \wedge \succ B \end{aligned}$$

The GO system developed here bears a close relation to $\mathsf{B}_3^{\mathsf{E}}$, which we will explore in more detail in later sections. It is worth remarking, though, that these external connectives are in a clear sense fully *classical*, and thus the $\mathsf{B}_3^{\mathsf{E}}$ logical consequence relation contains CPL logical consequence as a fragment.

2.2 GO semantics

Common to these systems is the preservation of CPL's treatment of the connectives with respect to the classical values. In fact, it is widely assumed that this is a requirement for any 'acceptable' deviation from CPL, insofar as one construes the values of the system as *truth*-values. It is evident, however, that this requirement underdetermines the behavior of many-valued systems with respect to the intermediate value.

It is therefore natural to consider the connectives of each of the above systems as faithful expansions of their classical counterparts. Their philosophical legitimacy, so to speak, is sufficiently grounded in the pre-theoretical plausibility of the respective truth tables for each connective. This is perhaps most clear when a more or less precise reading is given to the intermediate value, although a precise reading is not in general necessary.

Taking the standard treatment of three-valued negation, an intuitive reading of the GO connectives also holds true for the CPL connectives. A conjunction is true just when both conjuncts are true, otherwise it is false, and a disjunction is true just when at least one disjunct is true, and false otherwise. This reading gives the following tables:

$\neg$	
0	1
½	½
1	0

$\wedge$	0	½	1
0	0	0	0
½	0	0	0
1	0	0	1

$\vee$	0	½	1
0	0	0	1
½	0	0	1
1	1	1	1

It is fitting that an intuitive reading of the connectives can easily be given, especially for an alternative logic with an eye toward a 'conservative' deviation from CPL. The system's main significance, however, comes from its characteristic features and their philosophical interpretation, which we review in §3 and §4, respectively.

As above, logical consequence is defined in the usual way, where A is a logical consequence of $B_1, \ldots, B_n$ iff all valuations that assign each of $B_1, \ldots, B_n$ value 1 assign A value 1.

Notation. We write $B \vdash A$ for 'A is a logical consequence of B' and $B \nvdash A$ when A is *not* a consequence of B. Tautologies are either explicitly indicated as such, or written as a consequence of an arbitrary sentence, for example $B \vdash A \to A$.

2.2.1 Alternative semantics

A mathematically concise way to give the semantics, which will be useful for subsequent chapters, makes use of the following two arithmetic functions:

$$g(x) = \mathsf{min}\,\{x, 1 - x\}$$

$$c(x) = x - g(x)$$

Note that, with respect to the three values $\{0, \frac{1}{2}, 1\}$ the function g returns 0 for the classical values, which gives the latter function c the following behavior:

$$c(x) = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ otherwise.} \end{cases}$$

One might call this a 'classical cruncher,' since its output is always a classical value. For the connectives, our valuation function ν behaves as follows:

» *Negation.* $\nu(\neg A) = 1 - \nu(A)$

» *Conjunction.* $\nu(A \wedge B) = \mathsf{min}\{c(\nu(A)), c(\nu(B))\}$

» *Disjunction.* $\nu(A \vee B) = \mathsf{max}\{c(\nu(A)), c(\nu(B))\}$

This version of the semantics allows for convenient symmetry in giving the semantics of the modal connectives in Chapter 3. It is also useful for the discussion on expanding the set of values for GO in Chapter 5. It is unnecessary, however, to see it as anything more than mathematical convenience, since the truth tables for the connectives invite independently plausible readings.

2.3 Logical features

This section highlights some aspects of the GO system. We return to the philosophical interpretation in §4.

2.3.1 Characteristic Inferences

One can see that any counterexamples in GO to a CPL validity will be valuations involving the value ½. Among these, of course, is the failure of LEM:

$$B \nvDash A \vee \neg A$$

A contrast between GO and the other systems canvassed above is that counterexamples to this tautology assign $A \vee \neg A$ the value 0 as opposed to ½. This generalizes to any failure of a classical tautology in GO, since every CPL tautology involves either $\vee$ or $\wedge$ (or connectives defined from these). Of course, this does not hold true of failures of CPL *validities* in general, for example:

$$\neg(A \vee B) \nvDash \neg A$$

For this fails exactly when the conclusion is gappy. This in turn shows that some classical DeMorgan inferences must also fail.

$$\neg(A \vee B) \nvDash \neg A \wedge \neg B$$

$$\neg(A \wedge B) \nvDash \neg A \vee \neg B$$

But the other direction of the DeMorgan inferences do hold:

$$\neg A \vee \neg B \vdash \neg(A \wedge B)$$

$$\neg A \wedge \neg B \vdash \neg(A \vee B)$$

The failure of some DeMorgan inferences is precisely what is in order for a strong separation between LEM and LNC. We can see this independence in several ways. Most apparent is that a contradiction of the form $A \wedge \neg A$ never receives value 1, and in light of the above, it uniformly assumes value 0. A result is that the negation of a contradiction is a tautology.

$$B \vdash \neg(A \wedge \neg A)$$

Consequently *Explosion*, or *Ex Falso Quodlibet*, holds.

$$A \wedge \neg A \vdash B$$

The failure of one direction of DeMorgan, however, allows one to *negate* a case of excluded middle without inferring a failure of LNC. Hence:

$$\neg(A \vee \neg A) \nvdash A \wedge \neg A$$

This occurs because valuations where A is gappy must assign $A \vee \neg A$ value 0, and hence the premise is satisfiable.

Thus we see that, in cases where A is gappy, $\nu(A) \neq \nu(A \vee A)$. Similarly for the case of conjunction, since $\nu(A \vee A) = \nu(A \wedge A)$. We might be tempted to consider these inequalities a failure of an important substitution principle. Here, though, one must be somewhat careful. It is true that there are models where A and $A \wedge A$ do not receive the same value. However, there is a sense in which each can be substituted for the other *salva veritate*, since the result of any such substitution into a *true* sentence will never be *untrue*, and vice versa. Furthermore, the following principles hold:

$$A \vdash A \wedge A \text{ and } A \wedge A \vdash A$$

$$B \vdash A \text{ iff } B \vdash A \wedge A$$

Hence A and $A \wedge A$ can be substituted for each other *salva validate*. Similarly for $A \vee A$ (mutatis mutandis).

Material connectives

We define the standard *material* connectives:

$$A \supset B := \neg A \vee B$$
$$A \equiv B := (A \supset B) \wedge (B \supset A)$$

These connectives behave according to the following tables:

$\supset$	0	½	1
0	1	1	1
½	0	0	1
1	0	0	1

$\equiv$	0	½	1
0	1	0	0
½	0	0	0
1	0	0	1

Many classical inferences hold for the material conditional, for instance:

Modus Ponens. $A, A \supset B \vdash B$

Modus Tollens. $\neg B, A \supset B \vdash \neg A$

Contraction. $A \supset (A \supset B) \vdash A \supset B$

Contraposition. $A \supset B \vdash \neg B \supset \neg A$

We have neither *Material Identity* nor *Material Equivalence*.

$$B \nvdash A \supset A$$

$$B \nvdash A \equiv A$$

This is no surprise, since $\supset$ merely abbreviates a disjunction. We define a more adequate conditional, where *Identity* and *Equivalence* hold, in §2.3.2.

'Restricted' classical inferences

Note that, although LEM does not hold generally, we do have

$$C \vdash (A \vee B) \vee \neg(A \vee B)$$

of which the following is an instance:

$$B \vdash (A \vee A) \vee \neg(A \vee A)$$

The idea here is that, since determinacy arises at a certain level of complexity, one would expect classical 'behavior' to emerge at some level. Indeed, this is given a general treatment in a later section, but it is useful to see the unabbreviate forms of some inferences. Note that the following also holds:

$$B \vdash \neg(A \vee A) \vee A$$

But not this:

$$B \nvdash (A \vee A) \vee \neg A$$

This occurs despite the validity of communtation, association, idempotence and distribution.

With respect to the failures of DeMorgan, one can formulate the following restricted versions:

$$\neg((A \vee B) \vee (C \vee D)) \vdash \neg(A \vee B) \wedge \neg(C \vee D)$$
$$\neg((A \vee B) \wedge (C \vee D)) \vdash \neg(A \vee B) \vee \neg(C \vee D)$$

Equivalently, one can replace the first and third occurences of $\vee$ with $\wedge$ in each premise and conclusion above *salva validate*.

Material Identity and Equivalence have the following restricted forms:

Restricted Identity. $B \vdash (A \wedge A) \supset A$

Restricted Equivalence. $B \vdash (A \wedge A) \equiv (A \wedge A)$

These restricted versions of classical inferences illustrate the basis of the interpretation of GO as distinctively *atomistic*. We return to this in a §2.4.3. They also reflect the idea that determinacy, and thus classicality, result as a matter of form at a level of complexity, which is discussed in §2.3.3.

2.3.2 Expressibility

We have already seen a definable predicate for a strong notion of truth, or 'determinate truth'.

$$\mathbb{T}A := A \wedge A$$

That is, when A is false or gappy, $\mathbb{T}A$ is false. We get from this so-called *Release* ($\mathbb{T}A \vdash A$) and *Capture* ($A \vdash \mathbb{T}$).

From this we can see that $\vee$ and $\wedge$ are interdefinable, along the following lines:

$$A \vee B := \neg(\neg\mathbb{T}A \wedge \neg\mathbb{T}B)$$

Furthermore, we can define a connective expressing a 'gap' operator:

$$\circ A := \neg(A \vee \neg A)$$

The expressibility of these operators shows that GO is expressively complete with respect to the values $\{1, 0\}$. That is, every operator represented by a truth table consisting of any combination of only classical values is expressible in GO through a definable connective. This is not difficult to see, given that we have defined unary connectives whose respective operators evaluate to 1 for each respective value, and in turn the operator for $\neg$ returns 0 for 1. We should not concern ourselves here with the technicalities of a rigorous proof of this, but an illustrative sketch for binary operators is perhaps helpful. For this, we can label the positions on a truth table for a binary connective as follows:

	0	½	1
0	a	b	c
½	d	e	f
1	g	h	i

Let Table_X be the truth table (operator) with value 1 at all positions in X, and 0 everywhere else. In turn, we define the following array of connectives corresponding to each position on the truth table.

$A \otimes_a B \;:=\; \neg A \wedge \neg B$	$A \otimes_b B \;:=\; \neg A \wedge \circ B$	$A \otimes_c B \;:=\; \neg A \wedge B$
$A \otimes_d B \;:=\; \circ A \wedge \neg B$	$A \otimes_e B \;:=\; \circ A \wedge \circ B$	$A \otimes_f B \;:=\; \circ A \wedge B$
$A \otimes_g B \;:=\; A \wedge \neg B$	$A \otimes_h B \;:=\; A \wedge \circ B$	$A \otimes_i B \;:=\; A \wedge B$

We can see that each connective $\otimes_x$ expresses $\mathsf{Table}_{\{x\}}$. Let Y be the set of table positions $\{a, \ldots, i\}$. Hence $\{\mathsf{Table}_X \mid X \subseteq Y\}$ is the set of all possible truth tables containing only 1s and 0s. We can express $\mathsf{Table}_\varnothing$ (the all-0 table) with $A \wedge \neg A$. Now if a sentence form A expresses Table_α and B expresses Table_β, then given the semantics of disjunction, $A \vee B$ expresses $\mathsf{Table}_{\{\alpha,\beta\}}$. Hence by induction we can express every Table_x such that $x \in Y$.

Conditional

Given the above, it is easy to see that we can define a conditional connective $\rightarrow$ as follows.

» $A \rightarrow B := (A \supset B) \vee (\circ A \wedge \circ B)$

» $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$

The connective $\rightarrow$ is defined in terms of a disjunction. The first disjunct is just the material conditional. The second disjunct makes use of our defined *gap* operator, so that $\circ A \wedge \circ B$ is true just when both A and B are gappy. These connectives accord with the following tables:

$\rightarrow$	0	½	1
0	1	1	1
½	0	1	1
1	0	0	1

$\leftrightarrow$	0	½	1
0	1	0	0
½	0	1	0
1	0	0	1

Thus, we have identity and equivalence:

$$B \vdash A \rightarrow A$$

$$B \vdash A \leftrightarrow B$$

2.3.3 CPL containment

An important aspect of a non-classical system is its relation to CPL. This is for various reasons. Primarily, there is often a trade-off between the greater expressive power of a many-valued system, and the stronger provability powers of CPL. The desire to preserve the strong proof features of CPL thus pushes one to search for ways to express some sort of 'containment' of CPL, albeit within obvious limits.

One concept relevant here is that of an *extension* of a system's consequence relation. Simply stated, a system S' is an extension of S iff the S consequence relation is a subset of the S' consequence relation; that is, iff anything valid in S is valid in S'. In this sense, CPL is an extension of all systems canvassed so far, GO included. However, this relationship is quite weak, as it merely rules out the validity of anything not valid in CPL, which holds, for example, of an *empty* consequence relation.

A further relationship between CPL and many gappy systems is often expressed in noting that failures of classical inferences are preserved under an assumption of bivalence for the sentences at issue. Thus if we add to our premises that LEM holds for $B_1, \ldots, B_n$, then every classical inference from these sentences obtains. For example:

$$A \vee \neg A \vdash A \equiv A$$

Once the appropriate premises are added, the logic essentially 'collapses' into CPL. This is certainly the case for K_3, $\mathsf{Ł}_3$ and B_3, and for GO a similar principle also holds. Here, this principle requires that for every propositional parameter p occurring in the premises, $p \vee \neg p$ is included in the premises. This condition is sufficient for K_3 and the other systems, since it will guarantee classicality for complex sentences. In GO, however, complex sentences are determinate even if their propositional parameters receive indeterminate values, and so it is insufficient merely to require that excluded middle for premises be added to guarantee classicality, since the premises might include complex sentences.

The intuitive idea here is that, in most 'domains' in which actual reasoning takes place, the assumption of bivalence is justified for the domain.[6] One might consider this, too, a rather weak *logical* relationship, since it requires topic-specific information relevant to a particular subclass of sentences—information which is independent of logical form.

In the present case, one can recognize a stronger 'containment' of CPL. More precisely, there is a translation schema that shows the CPL consequence relation contained in GO consequence. Define the following GO connectives:

$$A \curlyvee B := \mathbb{T}A \vee \mathbb{T}B$$

$$A \curlywedge B := \mathbb{T}A \wedge \mathbb{T}B$$

For each argument from sentences $B_1, \ldots, B_n$ to A there is an argument $B_1^*, \ldots, B_n^*$ to A^* which is the result of translating each occurence of $\wedge$ and $\vee$ in the former

[6] For a discussion, see (Beall and Restall 2006)

with $\curlywedge$ and $\curlyvee$ respectively, such that:

$$A \text{ is a CPL consequence of } B_1, \ldots, B_n \text{ iff } B_1^*, \ldots, B_n^* \vdash A^*$$

We see that the CPL consequence relation is in a sense ‘contained’ in GO consequence. Indeed, the same holds for $\mathsf{B}_3^{\mathsf{E}}$, since GO is isomorphic to the fragment of $\mathsf{B}_3^{\mathsf{E}}$ containing only external connectives and $\neg$.

However, the characteristic of GO that determinacy arises as a matter of *form* results in an additional sense in which CPL is contained therein. This formal or *syntactic* relationship can be given a precise characterization in the following way. Define a complexity function κ as follows, where $\odot$ is the main connective of a sentence A ($\varnothing$ if A is atomic), and A_n is the nth operand of A:

$$\kappa(A) = \begin{cases} 0 \text{ if } \odot = \varnothing \\ \kappa(A_1) \ \text{ if } \odot = \neg \\ 1 + \kappa(A_1) + \cdots + \kappa(A_n) \ \text{ otherwise} \end{cases}$$

We immediately see that if $\kappa(A) > 0$, then $\nu(A) \in \{0, 1\}$, and hence:

$$B \vdash A \vee \neg A$$

And if $\kappa(A) + \kappa(B) > 1$

$$\neg(A \vee B) \vdash \neg A \wedge \neg B$$

$$\neg(A \wedge B) \vdash \neg A \vee \neg B$$

More generally, all classical inferences hold for sentences A where $\kappa(A) > 0$.

2.3.4 Another way to GO?

It is interesting to consider what other systems might be said to restrict indeterminacy on the the basis of form. A somewhat similar phenomenon might be seen, for instance, in the systems of Emil Post (1921), where $\vee$ is defined familiarly as:

$$\nu(A \vee B) = \max\{\nu(A), \nu(B)\}$$

Conjunction in turn is defined in a standard way:

$$A \wedge B := \neg(\neg A \vee \neg B)$$

Negation, however, behaves along less standard lines, which in turn gives $\wedge$ a peculiar behavior. Post's 3-valued system P_3 has the following tables:

$\neg$	
0	1
½	0
1	½

$\vee$	0	½	1
0	0	½	1
½	½	½	1
1	1	1	1

$\wedge$	0	½	1
0	½	½	½
½	½	1	0
1	½	0	0

The characteristic mode of negation in Post's systems in effect *shifts* the values. This is seen more clearly when we consider $\neg$ in the 4-valued version P_4:

$\neg$	
0	1
⅓	0
⅔	⅓
1	⅔

We will return to discuss 4-valued systems in Chapter 5. For present purposes, we need only consider P_3. Note that double-negation fails in both directions:

$A \nvDash \neg\neg A$

$\neg\neg A \nvDash A$

Consequently, while $A \vee \neg A$ is clearly not a tautology, one can see that the following *is* a tautology:

$$(A \vee \neg A) \vee \neg\neg A$$

One might consider this a 'restricted' form of LEM, in comparison to the GO's restricted version of LEM above. Besides the obvious differences in their respective consequence relations, it is helpful to see two general ways in which P_3 contrasts with GO. First, the restricted version of LEM occurs in P_3 as a result of the 'rotational' mode of ¬, whereas restricted versions of classical inferences occur as the result of the behavior of the binary connectives.

Perhaps more interestingly, one might, in concert with our interpretation of GO, consider determinacy arising in P_3 as a matter of *form*. However, in P_3 there does not occur the same logical separation between literals and non-literals. Post did not offer such an interpretation, though it may be useful to consider at least parts of his system in this context.

2.4 Interpretation

We have already noted in the previous chapter some motivation for a system that preserves the Law of Noncontradiction (LNC) in all of its formulations, while allowing for restricted failures of the Law of Excluded Middle (LEM).

Here we briefly expand on the philosophical interpretation of some particular features of the GO system. Chapter 5 will return to some of the issues raised in this section in greater detail. At present we concern ourselves with outlining the general features of a philosophical theory that would favor this logic.

Such a theory might accept the failure of the LEM, but hold that this failure can only occur for atomics sentences, or a certain subclass of them. Such a failure, however, would not affect a qualified form of all classical inferences.

2.4.1 Indeterminacy

A challenge of making clear an interpretation of a gappy system like the one proposed is to give a perspicuous interpretation of the intermediate value. One can avoid some philosophical headaches by shifting focus to *applications* of the system by demonstrating the system's usefulness to particular philophical questions. This often involves a shift from a notion of *alethic* indeterminacy, or *truth*-value gaps, to a notion of *epistemic* indeterminacy, or gaps in our knowledge. In some cases, this shift occurs as a sleight-of-hand, but the resulting applications are no less applicable, and the philosophical terrain no more forgiving.

Epistemic Indeterminacy

Thus we first consider how, in general, one might construe a notion of epistemic indeterminacy, insofar as many-valued systems could apply to it. Most notably for present purposes is the application of many-valued systems to areas of science. Many such applications are well-known, and their development deserves more attention than we can give here.[7]

The broad idea of a science *requiring* a many-valued logic, can be characterized as the failure in principle of necessary conditions for the dispatchment of some classical inferences. Thus a domain is rightly said to require a logic weaker than CPL when it is in principle impossible, at least in some cases, for the objects in the domain to admit of determinate measurement with respect to a basic property or quantity of the operative scientific theory.

In such cases, the interpretation of a logical system views the semantic values as something other than truth values, as the inquiry to which the system is applied precludes determinate knowledge (viz. measurability) of its basic subject matter. This preclusion may result from the impossibility of observing or measuring two basic quantities of the domain at the same time.

In modeling social help among higher animals, for example, (Weingartner 2004) notes the following incommensurability:

> If the proposition $p(I)$ represents (describes) the states of affairs that the measurable rate I [the growth-rate for the propagation of genes] has the value i and the proposition $q(L)$ represents(describes) the states of affairs that the measurable rate L [the loss-rate] has the value l then the proposition $p(I) \wedge q(I)$ does not represent the state of affairs that the measurable rate of both I and L has a certain value; because there is no such measurable rate: I and L cannot be measured simultaneously with a specific (sharp) value (p. 234).

[7] For a cross-disciplinary persepective, see (Weingartner 2004).

2.4.2 Bivalence

One should note that, strictly speaking, the theory requires only the failure of bivalence (again, only for atomics). In many systems this brings with it the failure of LEM, and this is certainly true in GO, at least for $\vee$. One could, however, insist that disjunction is best interpreted as the defined connective $\veebar$, for which LEM holds:

$$A \veebar B := (A \wedge A) \vee (B \wedge B)$$

Yet treating $\veebar$ as disjunction suggests treating conjunction along similar "external" lines. At this point, however, one is dealing only with the classical portion of the consequence relation, and absent a suitable interpretation of $\wedge$ or $\vee$, it is unclear what the failure of bivalence amounts to in this context.[8] For our purposes, it is thereby appropriate to focus on the failure of bivalence.

Theories of truth

Bivalence is about truth, and so a reason for rejecting bivalence might naturally come from a philosophical theory of truth. One might have a *metaphysical* theory of what makes a sentence true which allows for the possibility that the 'making' relationship is indeterminate for at least one sentence. If truth consists in a relation of correspondence between propositions and facts, for instance, a theory might countenance the possibility of a proposition neither determiniately corresponding nor determinately failing to correspond to a par-

[8]There are non-bivalent systems that preserve LEM, most notable van Fraassen's supervaluational system. See (Bencivwenga, Lambert, and van Fraassen 1991, pp40-47).

ticular fact. The explanation for this kind of indeterminacy will doubtless vary depending on details of the theory of truth. It may, for instance, result from a view about the nature of propositions, specifically, one that holds that some propositions themselves are 'incomplete' in a suitable sense. Alternatively, a theory of relations might allow for some sort of 'ontic vagueness' which results in indeterminacy with respect to the relata.

In contrast to metaphysical theories of truth, one might hold a *formal* theory of truth where the logical features of the truth predicate, together with basic compositional features of language, require that the predicate 'true' neither determinately applies nor determinately does not apply to some sentences. So-called *deflationist* or *minimalist* theories of truth claim that the 'nature' of truth consists only in the logical properties of the predicate 'true'. The schema [α *iff* 'α' *is true*] captures the whole meaning of 'true' and thus of truth.

The formal truth paradoxes that result in CPL are famous. Given certain basic features of a system, the existence of paradoxical sentences, e.g. Liar sentences and Curry sentences, is guaranteed. In CPL this induces triviality ($B \vdash A$). Thus the logical nature of 'true' requires a system weaker than CPL in order to remain faithful to it, since CPL does not validate every argument.

In this case, indeterminacy is accepted as an implication of a general theory that the truth predicate must be indeterminate (this in practice usually comes from minimalist responses to truth paradoxes). We cover more in chapter 5.

Theories of truths

Many reasons for rejecting bivalence, however, come not from a theory of truth in general, but from a theory of what makes some *particular* sentences true.

This could be because of a *physical* theory which contains at least one sentence that quantifies over objects to which a (precise) predicate of the theory neither determinately applies nor determinately does not apply. In this case, again this reduces to indeterminacy in the application of a predicate, with the caveats that the predicate is 'precise' and that the theory is likely true.

Another possibility is a *linguistic* theory which recognizes that at least one language contains some meaningful declarative sentence that is neither true nor false. In the linguistic case we might consider presupposition failures, certain theories of fictional objects, etc., or even an aggressively empirical approach to the meaning of 'true' that countenances uses that are traditionally considered instances of imprecision or incompetence.

2.4.3 Logical Atomism

The logic is distinctly *atomistic*. Literals are special in the sense that they exhibit a different *logical* behavior from all other sentences. This sense goes beyond the ordinary fact that, in standard model-theoretic semantics, atomic sentences are given separate truth conditions from moleculars. This by itself will not make the logical consequence relation vary depending on which sentences a class of interpretations assigns as atomic. Any reassignment (or retranslation) that preserves truth conditions for every sentence will result in the same consequence relation. Not so for the current logic: a different choice for atoms yields a different consequence relation. Though this makes atomism of one sort or other a natural friend for the logic, it does not strictly *bind* the logic to atomism. However, without such a commitment, in absence of principled distinction between the *truly* atomic and all other sentences, this logic would seem to point to a rather unique view of logical relativity or conventionalism indexed to an arbitrary *choice* of atoms.

Chapter 3

GO Modal: A Combinatorial Approach

This chapter presents the modal extension of the GO semantics, by way of Armstrong's combinatorialist analysis of possibility. §3.3 considers a combinatorialist approach to possibility detached from the thesis of naturalism. §3.4 discusses an Armstrongian interpretation of GO, specifically with respect to negative states of affairs. §3.5 presents the formal semantics, and §3.6 briefly highlights some significant logical features.

3.1 Introduction

Given the propositional GO semantics, one naturally wonders what happens in a modal setting. Here we explore the $\mathsf{GO}_{\mathsf{MODAL}}$ system in which the modal operators behave classically. This extends from the common quantifier approach to necessity and possibility as generalized conjunction and disjunction, respectively.

As we have seen, GO has a natural interpretation as a system for logical atomism. It is fitting, then, to look at the Combinatorialism of D. M. Armstrong, since it is a thoroughgoing logical atomist theory. Our interpretation of $\mathsf{GO}_{\mathsf{MODAL}}$ will center around its philosophical implications on a combinatorialist framework.

It should be emphasized, however that Armstrong's theory is not the only possible interpretation of $\mathsf{GO}_{\mathsf{MODAL}}$. This holds in two respects. First, one can have a combinatorialist view of possibility—or a certain type of possibility—that is divorced from central aspects of Armstrong's metaphysics. We remark on one broad approach to this in §3.3. Second, one could develop a view where the points over which the modal operators range are not worlds produced from combinatorialist principles, but rather some other sort of thing. In order to develop a system neutral to some of these issues, as well as for the sake of simplicity, we restrict the formal semantics to the propositional case. First-order semantics are given in Chapter 5.

3.2 Combinatorialism

Taking the standard quantifier approach to possibility, the modal operators range over points, or 'worlds'. The essential feature of a combinatorial account, though, is that these worlds are not taken as primitive a la (Lewis 1973), but are "built up" from base constituents—in Armstrong's case, those of the actual world—according to one or several combinatorial principles. Here we follow

Armstrong's (1989) presentation, not because it is the origin of the idea,[1] but because it is widely known and straightforward, and its metaphysical interpretation provides an intuitive picture that demonstrates combinatorialism's philosophical significance. As Armstrong himself points out, a combinatorial account of possibility is not essentially tied to his particular version of naturalism, or even to naturalism at all (p. 37). §3.3 will discuss an alternative approach to a combinatorialist account of possibility.

The stage is set with an ontology consisting of simple individuals, properties and relations. A 'simple' individual is one with no proper parts, in the usual mereological sense. What kinds of things these individuals actually are is an empirical matter, one left for a total science. He indicates point-instants as potential candidates (so long as they can bear properties), but this is only to be thought of heuristically.

A simple property, similarly, is one that has no other property as a constituent. Here, 'constituent' is a universal's analogue to a part, but it is some non-mereological relation. We can think of having a constituent in much the same way as having a part, though not identically.[2] Armstrong's view is that properties and relations are universals, and that universals and simple individuals have no being apart from the states of affairs they enter into. This follows an essentially Tractarian line of thought, where simples are thought of as abstractions from the states of affairs of which they are constituents. The important combinatorial step, however, is from the notion of a state of affairs to that of a *possible* state of affairs. These are introduced via the representa-

[1] See (Wittgenstein 1961), (Cresswell 1979), and (Skyrms 1981).

[2] This issue traces back to (Leonard 1930). For a useful discussion see (Rossberg 2009).

tion of states of affairs.

Assuming a is F, this state of affairs is represented Fa, and so on in the usual way. Supposing a is not F, Fa is false, and so it does not represent a state affairs. However, it has the right *form*, and so represents a *possible* state of affairs—one that does not exist. The generalization of this is Armstrong's basic Combinatorial Principle:

(CP) "The simple individuals, properties and relations may be combined in *all* ways to yield possible atomic states of affairs, provided only that the form of atomic facts is respected" (1989, p. 579).

The principle (CP) allows us to generate a set of worlds from our base ontology. There are, however, some conditions on this principle which Armstrong adds.

We can formulate the general picture in the following way. Let an ontology $o = \langle \mathcal{I}, \mathcal{P}, \mathcal{R} \rangle$ consisting of:

» A set of individuals $\mathcal{I} = \{a, b, c, \ldots\}$

» A set of properties $\mathcal{P} = \{F, G, H, \ldots\}$

» A set of n-adic relations $\mathcal{R} = \{R^n, S^n, T^n, \ldots\}$ for $n > 1$

Define a combination function Λ over ontologies:

$$\Lambda(o) = \{\Phi\alpha \mid \Phi \in \mathcal{P} \text{ and } \alpha \in \mathcal{I}\} \cup \{\Theta_n \alpha_1 \ldots \alpha_n \mid \Theta \in \mathcal{R} \text{ and } \alpha_1, \ldots, \alpha_n \in \mathcal{I}\}$$

The function Λ generates all atomic states of affairs, or 'facts', available to a given ontology. A world w with ontology o_w, then, is a non-empty subset $\mathcal{F}$ of $\Lambda(o_w)$ such that:

1. For each $\alpha \in \mathcal{I}$ there is some $\Phi \in \mathcal{P}$ such that $\Phi\alpha \in \mathcal{F}$
2. For each $\Phi \in \mathcal{P}$ there is some $\alpha \in \mathcal{I}$ such that $\Phi\alpha \in \mathcal{F}$
3. For each $\Theta^n \in \mathcal{R}$ there are some $\alpha_1, \ldots, \alpha_n \in \mathcal{I}$ such that $\Theta^n\alpha_1 \ldots \alpha_n \in \mathcal{F}$

Given a base world β, the set of possible worlds $\mathcal{W}$ contains all and only worlds in $\Lambda(o_\beta)$, where to be a *world*, it must meet the three constraints above.

The first condition requires that each individual have some non-relational property. This rules out so-called 'propertyless' individuals. Armstrong considers individuals to be abstractions from the states of affairs they enter into. Thus an individual that does not enter into at least one fact simply does not exist. This view rejects *Haecceitism*, the view that individuals have a unique 'inner essence' distinct from their properties.

The second and third conditions respectively ensure that each property and relation is instantiated. This prohibits uninstantiated, or *alien* universals, and it is motivated by Armstrong's naturalist account of universals. The rejection of alien universals poses some difficulty, as some have argued that their conceivability undermines Armstrong's analysis of possibility.[3] In any case, there is a logical significance in the rejection of alien universals. Since Armstrong will allow for worlds that contain fewer universals than the actual world, but not more, such a 'contracted' world w_1 will be 'accessible' from the actual world, though there is some universal, say F, which it does not contain. The actual world, however, will not be accessible from w_1, since from its perspective F is an alien universal. The result is an accessibility relation

[3] See, for instance (Schneider 2001).

that is reflexive and transitive, but not symmetric. Thus the logic corresponds to the standard S4. (More on 'contracted' worlds below.)

3.3 Naturalism

As the principle (CP) is formulated, it yields a somewhat limited collection of worlds, since it requires each individual in the ontology to appear in each world (as well for property and relations). This is the set of so-called *Wittgenstein* worlds. Armstrong proposes additions to the $\mathcal{W}$ to include those from *contraction* and *expansion*. Contraction allows worlds that contain a (non-empty) proper subset of the individuals in the base world, and expansion allows worlds that contain 'more' individuals. Armstrong, following Skyrms, proposes these principles via a slight departure from strict combinatorialism, by appeal to analogy. [4]

We can see combinatorialism's connection to logical atomism in the following way. Suppose we did not require logical atomism for combinatorialism and we allowed any predicate, whether atomic or not, to yield a representation of an acceptable recombination. Take it for granted that there are at least two objects a and b, and one universal F, such that Fa and $\neg Fb$. Thus, per the combinatorial principle and our hypothesis both F and $\neg F$ are predicates available for recombination. Thus $Fa \wedge \neg Fa$ represents a possible state of affairs. Hence, only genuinely atomic facts are available for recombination.

The restriction that the items available for recombination are genuinely

[4] This appeal also requires the abandonment of *Haecceitism* and the adoption of so-called weak anti-Haecceitism. For discussion, see (Armstrong 1986, pp. 580-4) .

atomic seems to presuppose that there are such things as genuine atoms. Thus, combinatorialism without atomism cannot rule out these unwanted recombinations. This might not be an essential connection, however, and Armstrong does gives an 'atomless' interpretation of combinatorialism that allows for so-called *relative* atoms.[5]

For Armstrong, naturalism is the thesis that all that exists is the space-time world: there are no transcendent, or 'other-worldly' entities. However, we might frame a general argument for combinatorialism irrespective of naturalism in the following way:

(i) Science requires an account of possibility with a naturalistically respectable basis.

(ii) The space-time world, logical constructions therefrom, and idealizations thereof are naturalistically respectable.

(iii) Combinatorialism provides an adequate account for scientific claims of possibility and necessity and is an idealization of or logical construction from the actual space-time world.

The demand for an account of possibility is pronounced in a naturalistic theory like Armstrong's, which rejects the existence of transcendent entities like abstract primitive modalities or possible worlds. With respect to (ii), Armstrong is a fictionalist about possible worlds. These useful fictions account for modal truths in science, in much the same way that ideal gases are useful fictions which ground truths about actual gases. Combinatorialism's base,

[5] For further discussion, see Chapter 5.

then, is as naturalistic as it gets, and so its constructed fictions must also be naturalistically respectable.

(i), however, is not a naturalist thesis per se. The combinatorial worlds provide a scientifically respectable account of possibility, whether or not naturalism as a philosophical thesis is true. For providing a 'respectable account', however, it is not sufficient that the base from which possible worlds are constructed are scientifically respectable. The account should further countenance a broad range of open scientific theories. Armstrong recognizes this when he considers whether there are no atoms, in essence treating the core assumption of logical atomism as an empirical hypothesis.

Thus a combinatorial theory should comport with future scientific discovery—not merely the discovery of new objects and properties, but also with broader changes in the theory of the structure of the space-time world. In this spirit it is natural to countenance scientific theories that allow for indeterminacy. But then what happens to our logical constructions from an indeterminate base? Combinatorialism is a hybrid logical-physical theory, and the GO logic attempts to take this at face value. It models indeterminacy from scientific theories as occurring at the atomic level, while the determinacy of logic is reflected at the level of combination.

Suppose then that recalcitrant evidence comes from science that signals the failure of bivalence. What are the options for responding that do not simply discredit the results?

At minimum, to accept the evidence is to admit that a bivalent framework is inadequate for a true scientific theory. An extreme option, then, is to accept that science has refuted—and hence, *can* refute—the Law of Excluded Middle, the very same law in every respect that was thought to hold for all reasoning in general. One thereby accepts the unequivocal failure of a putative logical truth. Logical 'principles', then, are in every important respect no different from scientific hypotheses, and in this case the evidence shows that classical logic simply got it wrong.

Perhaps this position is tenable, but it seems contrary to basic intuitions many hold about the nature and scope of logic. It implies that logic is subject to revision in a much more direct way than the usual Quinean picture suggests. Classical logic's central position in the web of belief is presumably not merely a result of our degree of confidence in its content (if logic has any content). Rather, logic's centrality is due to its totally general role in governing the acceptance and rejection of all propositions whatsoever in the web of belief. Subjecting logical principles to 'direct' refutation from science to this extent does more than holism demands, effectively shifting logic's *position* in the web to the edge.

It is doubtful that this extreme position is sincerely adopted by many logicians, even diehard Quinean sympathizers. Logic, after all, should contain only principles that are certain. A tamer position holds that the inadequacy of a bivalent framework for science does not show that bivalence fails in logic. This is close to the view proposed here, but one must be careful how wide the cleavage between logic and science is drawn. There are certainly many applications of formal systems to scientific problems that in no way 'threaten' logic. However, this is because a mere *application* of logic only credits interpreting the 'semantic' values as something strictly other than *truth* values.[6]

[6]For further discussion, see Chapter 5.

An electronic 'logic' circuit, for example, credits interpreting the values as stable voltage states, and the existence of three such states does not refute LEM. This is just as well, as this is not a scientific theory.

But this would undermine the original goal, as it is equivalent to rejecting the supposed evidence *as* evidence. To automatically dismiss any putative scientific evidence that challenges bivalence is to discredit the evidence, which fails to meet our initial challenge. Further, to do so on *principled* grounds seems to make logic completely immune to revision. This move would seem to suggest that scientific 'truth' is completely separate from what logic studies.

The middle road suggested here is to countenance the possibility that genuine indeterminacy occurs at the basic empirical level, while maintaining logic's central position in the web of belief. Such an occurrence would doubtless require a revision of 'logic proper', but would respect the intuition that logical principles are not wholly empirical principles, and they hold no matter what the domain.

This is the broader motivation for the $\mathsf{GO}_{\mathsf{MODAL}}$ framework for combinatorialism. A more specific advantage is in relation to 'negative facts'.

3.4 Negative Facts

For Armstrong, every truth has a truthmaker.[7] An atomic proposition that a is F is made true by the fact Fa. But what about the proposition that a is *not* F? Supposing it is true, what makes it so? It cannot be the fact Fa, since it does not exist. For Armstrong, negative atomic propositions are made true

[7] See (2000, p. 150) and (2004, pp. 5, 19).

by 'totality' facts, or what Russell (1918) calls *general* facts. What makes it the case that a is not F is all of the atomic facts together with the total fact that these are *all* the first-order facts.

An alternative to admitting total facts is to admit first-order 'negative' facts as truthmakers for negative atomic propositions.[8] However, Armstrong claims that Combinatorialism cannot admit negative facts. He says:

> Suppose we admit both a's being F and a's not being F as possible states of affairs. Our combinatorial scheme when the allow us to select *both* these states of affairs (1989, p. 48).

To avoid this, one might introduce additional constraints on our combinatorial principle in order to rule out out such 'contradictory' combinations. This is problematic for Armstrong, however, given that he wishes to provide an analysis of possibility.

> [T]hen we are using in our statement of constraints that very notion of modality which it was our hope to analyse. For contradictory states of affairs would be the ones for which one state of affairs *must* obtain and the other fail to obtain.

Even apart from the attempt to analyze possibility, it is a common charge of an atomist-combinatorialist framework to eschew logical connections among atomic states of affairs. It may be noted, however, that Armstrong's conditions (1-3) on (CP), as well as his appeals to analogy for contraction and

[8] For recent discussions on some advantages and difficulties of negative facts, see (Molnar 2000), (Priest 2000), (Beall 2000), (Simons 2005), (Mumford 2005), (Cheyne and Pidgen 2006), (Parsons 2006) and Armstrong (2000, 2005, 2006).

expansion, might already count as departures from strict analysis. One might argue that these principles can only be justified by appealing to a prior notion of possibility.

Allowing for indeterminacy within Armstrong's metaphysical framework does seem to require admitting negative facts, and along with it a commitment to an additional metaphysical constraint that rules out contradictory combinations. The non-existence of a truthmaker for p, together with a total fact, is not sufficient for the truth of $\neg p$. Indeed this is plausibly what the rejection of bivalence on scientific grounds must amount to for the truthmaker theorist. If we allow that science can give us reason to accept genuine indeterminacy, then we do not have a basis for the rejection of negative facts. A realism about scientific theories which countenances the failure of bivalence for scientific reasons presupposes that science is in the business of investigating negative facts.

Indeed, Armstrong later sees totality facts as essentially negative facts (Armstrong 2000, p. 153). One wonders, then, why he is so reluctant to admit them as corresponding to each positive fact. His answer is an appeal to parsimony: admitting negative facts for *every* positive fact is just too ontologically indulgent. With the GO logic, however, this worry dissipates to a large degree. As above, once negative facts are admitted for atomic propositions, no further negative facts are required for complexes.

Suppose an atomic proposition p is true, and q is gappy. Thus it follows:

1. There is a truthmaker for p.
2. There is no truthmaker for q.

3. There is no truthmaker for $\neg q$.

What is the status of $p \wedge q$? If we were to admit of the case where a conjunction is gappy, then we would have to admit of negative facts for their negations, and so on for every complex fact. But since the gappy case is ruled out by GO, the absence of a truthmaker for either p or q suffices for the falsity of $p \wedge q$ and thus the truth of $\neg(p \wedge q)$. Thus the only domain for which one must posit negative facts is that which is the source of indeterminacy. And so Armstrong's original condition for conjunction remains intact, where $p \wedge q$ has a truthmaker just when each of p and q has a truthmaker. Similarly for disjunction.

Thus, though not the original motivation for the GO logic, the truthmaker theory yields some natural payoffs from the combinatorialist interpretation of the logic. With these in mind, I now turn to model-theoretic semantics of $\mathsf{GO}_{\mathsf{MODAL}}$. Chapter 4 develops a tableaux proof system with its soundness and completeness results.

3.5 $\mathsf{GO}_{\mathsf{MODAL}}$

The formal semantics for $\mathsf{GO}_{\mathsf{MODAL}}$ is as follows. The syntax is the standard syntax for CPL, augmented with our modal connectives $\Box$ and $\Diamond$:

- A set of atomic formulas $\mathcal{A} = \{p_0, \ldots, p_n, q_0, \ldots, q_n\}$
- A set of unary connectives $\mathcal{C}_u = \{\neg, \Box, \Diamond\}$
- A set of binary connectives $\mathcal{C}_b = \{\wedge, \vee, \supset, \equiv, \rightarrow, \leftrightarrow\}$
- Let $\mathcal{C} = \mathcal{C}_u \cup \mathcal{C}_b$

» A set of punctuation marks $\mathcal{U} = \{(,)\}$

» A set of sentences $\mathcal{S}$:

(a) $\mathcal{A} \subset \mathcal{S}$.
(b) If $A \in \mathcal{S}$ and $\odot \in \mathcal{C}_u$, then $\odot A \in \mathcal{S}$.
(c) If A and B are in $\mathcal{S}$ and $\odot \in \mathcal{C}_b$, then $(A \odot B) \in \mathcal{S}$. [9]

We mark a division between unary and binary connectives solely for convenience for the Tableaux adequacy proofs in the next chapter.

Our semantics includes a constant set of values, and we will make use of two arithmetic functions:

» A set of values $\mathcal{V} = \{0, \frac{1}{2}, 1\}$

» Two convenient functions:

1. $g(x) = \mathsf{min}\,\{x, 1 - x\}$
2. $c(x) = x - g(x)$

As noted in Chapter 2, we can think of g as the 'distance' from a classical value: in this case, 0 for 1 and 0, and ½ for ½. The function c is our 'classical cruncher' which subtracts the distance from the value.

The propositional semantics formalize the presentation in Chapter 2. The semantics for the modal machinery resembles for the most part standard Kripke semantics. The difference, of course, comes in the clauses for the modal connectives.

A model $\mathfrak{M}$ is a triple $\langle \mathcal{W}, \mathcal{R}, \nu \rangle$ where:

» $\mathcal{W}$ is a non-empty set of worlds $\{w_0, w_1, \ldots, w_n\}$

[9] For readability, outer parentheses are dropped when no ambiguity results.

» $\mathcal{R} : \mathcal{W} \longrightarrow \wp(\mathcal{W})$. We abbreviate $w' \in \mathcal{R}(w)$ as $w\mathcal{R}w'$. $\mathcal{R}$ is:

(a) *Reflexive*: $w\mathcal{R}w$ for all $w \in \mathcal{W}$
(b) *Transitive*: If $w\mathcal{R}w'$ and $w'\mathcal{R}w''$ then $w\mathcal{R}w''$

» $\nu : \mathcal{S} \times \mathcal{W} \longrightarrow \mathcal{V}$. We abbreviate $\nu(\langle A, w\rangle)$ as $\nu_w(A)$. Lo:

(i) $\nu_w(\neg A) = 1 - \nu_w(A)$
(ii) $\nu_w(A \wedge B) = \mathsf{min}\,\{c(\nu_w(A)), c(\nu_w(B))\}$
(iii) $\nu_w(\Box A) = \mathsf{min}\,\{c(\nu_{w'}(A)) : w\mathcal{R}w'\})$

» Defined connectives:

1. $\nu_w(\Diamond A) = \nu_w(\neg\Box\neg(A \wedge A))$
2. $\nu_w(A \vee B) = \nu_w(\neg(\neg(A \wedge A) \wedge \neg(B \wedge B))))$
3. $\nu_w(A \supset B) = \nu_w(\neg A \vee B)$
4. $\nu_w(A \equiv B) = \nu_w((A \supset B) \wedge (B \supset A))$
5. $\nu_w(A \to B) = \nu_w((A \supset B) \vee (\neg(A \vee \neg A) \wedge \neg(B \vee \neg B)))$
6. $\nu_w(A \leftrightarrow B) = \nu_w((A \to B) \wedge (B \to A))$

The conditions of reflexivity and transitivity on $\mathcal{R}$ are those of standard $\mathsf{S4}$ logic. Note the following equivalences for some of the defined connectives:

» $\nu_w(\Diamond A) = \mathsf{max}\,\{c(\nu_{w'}(A)) : w\mathcal{R}w'\}$

» $\nu_w(A \vee B) = \mathsf{max}\,\{c(\nu_w(A)), c(\nu_w(B))\}$

» $\nu_w(A \to B) = c(\mathsf{max}\,\{\nu_w(\neg A), \nu_w(B), g(\nu_w(A)) + g(\nu_w(B))\})$

For $\Diamond$ the equivalence is as one would expect, given its treatment as generalized disjunction. The equivalence for $\to$ reflects its definition in terms of $\supset$ disjoined with the sum of the distance from a classical value of the antecedent and consequent.

Logical consequence is defined in the standard way.

Definition 5.1. *$X \Vdash A$ iff for all models $\mathfrak{M}$, for every world $w \in \mathcal{W}$, if $\nu_w(B) = 1$ for each $B \in X$, then $\nu_w(A) = 1$.*

The semantics have it that the truth-functional connectives are interpreted at each world in accordance with the following tables:

$\neg$	
0	1
½	½
1	0

$\vee$	0	½	1
0	0	0	1
½	0	0	1
1	1	1	1

$\wedge$	0	½	1
0	0	0	0
½	0	0	0
1	0	0	1

$\supset$	0	½	1
0	1	1	1
½	0	0	1
1	0	0	1

$\equiv$	0	½	1
0	1	0	0
½	0	0	0
1	0	0	1

$\rightarrow$	0	½	1
0	1	1	1
½	0	1	1
1	0	0	1

$\leftrightarrow$	0	½	1
0	1	0	0
½	0	1	0
1	0	0	1

3.6 Logical Features

§3.7 gives many notable inferences, but a few should be mentioned here. As one would anticipate, given that $\Box$ and $\Diamond$ are conceived in terms of generalized conjunction and disjunction, respectively, the characteristic rejection of certain DeMorgan inferences discussed in the previous chapter carry over from the propositional to the modal case. This manifests in a failure of the standard interdefinability of the modal connectives:

$$\neg\Diamond\neg A \nvdash \Box A$$

$$\neg\Box\neg A \nvdash \Diamond A$$

Take a model where A is gappy in all worlds. Thus $\neg A$ is gappy everywhere, and so $\Diamond A$ is false, though obviously so too is $\Box A$. Similarly for the second case.

The other directions for the standard interdefinability do hold:

$$\Diamond A \vdash \neg\Box\neg A$$

$$\Box A \vdash \neg\Diamond\neg A$$

As one would expect, with our $\mathbb{T}$ operator (see Chapter 2), we can define a modal connective

$$\diamondsuit A := \Diamond\mathbb{T}A$$

For $\diamondsuit$, both directions of the standard definability hold:

$$\diamondsuit A \dashv\vdash \neg\Box\neg A$$

$$\Box A \dashv\vdash \neg\diamondsuit\neg A$$

As with standard $\mathsf{S4}$, the so-called necessitation principle holds: If $B \vdash A$ then $B \vdash \Box A$.

3.6.1 Alternative Modal Semantics

In passing, one might consider an alternative approach to extended GO to a modal system that defines the modal connectives exactly as in standard many-valued modal systems.

1. $\nu_w(\Box A) = \mathsf{min}\,\{\nu_{w'}(A)) : w\mathcal{R}w'\})$
2. $\nu_w(\Diamond A) = \mathsf{max}\,\{\nu_{w'}(A)) : w\mathcal{R}w'\})$

This approach countenances gaps for sentences with our modal connectives. Roughly, if A is gappy at all accessible worlds, then so too will $\Diamond A$ and $\Box A$. This gives us the standard interdefinability of the modal connectives:

$$\Diamond A := \neg\Box\neg A$$

or

$$\Box A := \neg\Diamond\neg A$$

Given the motivation, though, this is unsatisfactory, as it countenances indeterminacy in purely 'logical' combinations.

3.7 Inferences

Conjunction / Disjunction

2. $B \nvdash A \vee \neg A$ *(LEM)*
3. $C \vdash (A \vee B) \vee \neg(A \vee B)$ *(LEM - Restricted)*

4. $A \wedge \neg A \vdash B$ *(LNC)*

5. $B \vdash \neg(A \wedge \neg A)$ *(LNC*)*

6. $\neg(A \vee \neg A) \nvdash A \wedge \neg A$

7. $B \nvdash (A \wedge A) \vee (\neg A \wedge \neg A)$

8. $B \nvdash (A \vee A) \vee (\neg A \vee \neg A)$

9. $A \wedge B \dashv\vdash B \wedge A$ *(Commutation - Conjunction)*

10. $A \vee B \dashv\vdash B \vee A$ *(Commutation - Disjunction)*

11. $A \wedge (B \wedge C) \dashv\vdash (A \wedge B) \wedge C$ *(Association - Conjunction)*

12. $A \vee (B \vee C) \dashv\vdash (A \vee B) \vee C$ *(Association - Disjunction)*

13. $A \dashv\vdash A \wedge A$ *(Idempotence - Conjunction)*

14. $A \dashv\vdash A \vee A$ *(Idempotence - Disjunction)*

15. $A \wedge (B \vee C) \dashv\vdash (A \wedge B) \vee (A \wedge C)$ *(Distribution$_1$)*

16. $A \vee (B \wedge C) \dashv\vdash (A \vee B) \wedge (A \vee C)$ *(Distribution$_2$)*

17. $\{A \vee B, \neg A\} \vdash B$ *(Disjunctive Syllogism)*

DeMorgan

18. $\neg(A \vee B) \nvdash \neg A \wedge \neg B$

19. $\neg A \wedge \neg B \vdash \neg(A \vee B)$

20. $\neg(A \wedge B) \nvdash \neg A \vee \neg B$

21. $\neg A \vee \neg B \vdash \neg(A \wedge B)$

22. $\neg((A \vee B) \vee (C \vee D)) \dashv\vdash \neg(A \vee B) \wedge \neg(C \vee D)$

23. $\neg((A \vee B) \wedge (C \vee D)) \dashv\vdash \neg(A \vee B) \vee \neg(C \vee D)$

Material Conditional

24. $B \nvdash A \supset A$ *(Identity)*

25. $B \vdash (A \wedge A) \supset A$ *(Identity - Restricted)*

26. $\{A, A \supset B\} \vdash B$ *(Modus Ponens)*

27. $C \vdash (A \wedge (A \supset B)) \supset B$ *(Pseudo Modus Ponens)*

28. $\{\neg B, A \supset B\} \vdash \neg A$ *(Modus Tollens)*

29. $C \vdash (\neg B \wedge (A \supset B)) \supset \neg A$ *(Pseudo Modus Tollens)*

30. $\{A \supset B, B \supset C\} \vdash A \supset C$ *(Hypothetical Syllogism)*

31. $A \supset (A \supset B) \dashv\vdash A \supset B$ *(Contraction)*

32. $C \vdash (A \supset (A \supset B)) \supset (A \supset B)$ *(Pseudo Contraction)*

33. $A \supset B \dashv\vdash \neg B \supset \neg A$ *(Contraposition)*

34. $C \vdash (A \supset B) \supset (\neg B \supset \neg A)$ *(Pseudo Contraposition)*

35. $A \supset (B \supset C) \dashv\vdash (A \wedge B) \supset C$ *(Exportation)*

36. $\neg A \vdash A \supset B$

37. $\neg(A \supset B) \nvdash \neg B$

38. $A \vdash B \supset A$

39. $A \supset B \vdash (A \wedge C) \supset B$

40. $(A \wedge B) \supset C \nvdash (A \supset C) \vee (B \supset C)$

Material Equivalence

41. $B \nvdash A \equiv A$

42. $D \nvdash ((A \equiv B) \vee (A \equiv C)) \vee (B \equiv C)$

Conditional

43. $B \vdash A \rightarrow A$ *(Identity)*

44. $B \vdash (A \wedge A) \rightarrow A$ *(Identity - Restricted)*

45. $\{A, A \rightarrow B\} \vdash B$ *(Modus Ponens)*

46. $C \vdash (A \wedge (A \rightarrow B)) \rightarrow B$ *(Pseudo Modus Ponens)*

47. $\{\neg B, A \rightarrow B\} \vdash \neg A$ *(Modus Tollens)*

48. $C \vdash (\neg B \wedge (A \rightarrow B)) \rightarrow \neg A$ *(Pseudo Modus Tollens)*

49. $\{A \rightarrow B, B \rightarrow C\} \vdash A \rightarrow C$ *(Hypothetical Syllogism)*

50. $A \rightarrow (A \rightarrow B) \dashv\vdash A \rightarrow B$ *(Contraction)*

51. $C \vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ *(Pseudo Contraction)*

52. $A \rightarrow B \dashv\vdash \neg B \rightarrow \neg A$ *(Contraposition)*

53. $C \vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ *(Pseudo Contraposition)*

54. $A \rightarrow (B \rightarrow C) \dashv\vdash (A \wedge B) \rightarrow C$ *(Exportation)*

55. $\neg A \vdash A \rightarrow B$

56. $\neg(A \rightarrow B) \nvdash \neg B$

57. $A \vdash B \rightarrow A$

58. $A \rightarrow B \vdash (A \wedge C) \rightarrow B$

59. $(A \wedge B) \rightarrow C \nvdash (A \rightarrow C) \vee (B \rightarrow C)$

Biconditional

60. $B \vdash A \leftrightarrow A$

61. $D \nvdash ((A \leftrightarrow B) \vee (A \leftrightarrow C)) \vee (B \leftrightarrow C)$

Modal

62. $\Box A \vdash \neg\Diamond\neg A$

63. $\Diamond A \vdash \neg\Box\neg A$

64. $\neg\Diamond\neg A \nvdash \Box A$

65. $\neg\Box\neg A \nvdash \Diamond A$

66. $\Box A \dashv\vdash \neg\Diamond\neg(A \wedge A)$

67. $\Diamond A \dashv\vdash \neg\Box\neg(A \wedge A)$

68. $\Diamond A \dashv\vdash \Diamond(A \wedge A)$

69. $\Box A \dashv\vdash \Box(A \wedge A)$

70. $\Box A \vdash A$

71. $A \vdash \Diamond A$

72. $\Box\Box A \vdash \Box A$

73. $\Box A \vdash \Box\Box A$

74. $B \vdash \Box A \to \Diamond A$

75. $B \vdash \Box A \to \Box\Box A$

76. $\Diamond A \nvdash \Box\Diamond A$

77. $B \vdash \Box A \to \Box\Diamond A$

78. $\Diamond(A \vee B) \vdash \Diamond A \vee \Diamond B$

79. $\Diamond(A \to B) \vdash \Box A \to \Diamond B$

80. $\Box\neg A \vdash \Box(A \to B)$

81. $\Diamond A \nvdash \Box\Diamond A$

82. $\Box A \vdash \Diamond\Box A$

83. $\Diamond A \vdash \Diamond\Diamond A$

84. $\Box A \vdash \Box\Box A$

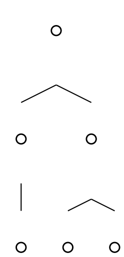
Chapter 4

$\mathsf{GO}_{\mathsf{MODAL}}$ Tableaux

4.1 Background

This section briefly covers some basics of Tableaux systems, focusing on the general expansion of classical tableaux to many-valued and modal versions. The presentation here owes much to (Beall and van Fraassen 2003), and also (Priest 2008).

For our purposes a tableaux system provides a mechanical procedure for determining the validity of arguments. A tableau has something like the following structure:



The $\circ$s are *nodes*, where the top node is the *root* and the bottom ones are the *leaves*. A *branch* consists of all nodes along a path from the root to a leaf.

In the tableaux system for CPL, each node consists of a single sentence. A tableau for an argument consists of an *initial list*, which is a branch starting with a node for each premises, and ending with a node for the conclusion preceded by the negation symbol. For an argument from $B_1, \ldots, B_n$ to A, then, the initial list is:

$$\begin{array}{c} B_1 \\ \vdots \\ B_n \\ \neg A \end{array}$$

The construction of the tableau proceeds by applying a set of rules to its branches. Rules come in two broad types: resolution rules and closure rules. When a closure rule is applied, it 'finishes' the branch, halting the application of any further rules. In CPL, there is exactly one closure rule, and it applies to a branch having both a node with some sentence A and a node with $\neg A$. A resolution rule, for our purposes, is any rule that is not a closure rule.

The rules of a tableaux system are constructed to reflect the semantics of the system for which they are developed. We can give an intuitive reading of how a tableau is so related. In CPL, the construction of each branch reflects the attempt to find a model where each of the nodes' sentences are true. Since the semantics for $\neg$ rules out any model where A and $\neg A$ are both true, any branch with nodes of those forms does not represent a model, and so the branch closes. Since the initial list is formed by listing all the premises and negating the conclusion, any completed open branch will represent a model where each of the premises are true and the negation of the conclusion is true—in short, a countermodel to the argument.

For other systems nodes may take various forms, and for different reasons. In tableaux systems for modal logic, for example, since models do not assign sentences values simpliciter, but only relative to a world, nodes must also contain some index that can be mapped to worlds. A simple way is to use natural numbers, and thus a node $A, 0$ on a completed open branch reflects a model where $\nu_w(A) = 1$.

Many-valued tableaux systems may require nodes with an additional element, sometimes called a *designation marker*. In the propositional case, the classical method for constructing the initial list for an argument will not work. Take, for example, an argument from B to A where there is a model that assigns B the value 1, and A value ½. Given our definition of logical consequence, this model is a counterexample to the argument. However, $\neg A$ might not be designated (depending on the semantics), and so the initial list will not produce a branch that reflects this model. A standard solution is to construct the initial list as follows:

$$\begin{array}{c} B_1\oplus \\ \vdots \\ B_n\oplus \\ A\ominus \end{array}$$

Intuitively, $\oplus$ means that the sentence gets the value 1, and $\ominus$ means that the sentence gets some other value, e.g. 0 or ½. In this case, the mere 'occurrence' of a sentence on a completed open branch does not reflect that the sentence is true, unless it is followed by $\oplus$. As a result, the closure rules might need revision, and the resolution rules may increase, since we now have an additional element in our nodes. Here we combine this approach with the modal version above.

4.2 Some Definitions and Lemmas

Note that the function $c(x)$ behaves accordingly:

$$c(x) = \begin{cases} 1 \text{ if } x = 1 \\ \\ 0 \text{ if } 0 \leq x < 1 \end{cases}$$

Given $\mathcal{V} = \{0, ½, 1\}$, this gives us the following lemmas.

LEMMA 2.2. *For all $\mathfrak{M}$ and all $w \in \mathcal{W}$, and each connective $\odot \in \{\Box, \Diamond\}$:*

(i) if $\nu_w(\odot A) \neq 1$, then $\nu_w(\odot A) = 0$, and

(ii) if $\nu_w(\neg \odot A) \neq 1$, then $\nu_w(\neg \odot A) = 0$.

LEMMA 2.3. *For all $\mathfrak{M}$ and all $w \in \mathcal{W}$, and each $\odot_b \in \mathcal{C}_b$, and $\odot_u \in \mathcal{C}_u$:*

(i) if $\nu_w(A \odot_b B) \neq 1$, then $\nu_w(A \odot B) = 0$, and

(ii) if $\nu_w(\odot_u(A \odot_b B)) \neq 1$, then $\nu_w(\odot_u(A \odot_b B)) = 0$.

The proofs are trivial, and they are left as trivial exercises.

Logical consequence is defined in the standard way.

DEFINITION 2.4. *$X \Vdash A$ iff for all models $\mathfrak{M}$, for every world $w \in \mathcal{W}$, if $\nu_w(B) = 1$ for each $B \in X$, then $\nu_w(A) = 1$.*

4.3 $\mathsf{GO}_{\mathsf{MODAL}}$ Tableaux

4.3.1 Nodes

Every node on a $\mathsf{GO}_{\mathsf{MODAL}}$ tableau has one of the following forms, where $A \in \mathcal{S}$, and $i, j \in \mathbb{N}$:

» $A, i\oplus$

» $A, i\ominus$

» $i\mathcal{R}j$

4.3.2 Initial List

The *initial list* of a $\mathsf{GO}_{\mathsf{MODAL}}$ tableau for the argument from $B_1, \ldots, B_n$ to A is formed as follows:

$$\begin{array}{c} B_1, 0\oplus \\ \vdots \\ B_n, 0\oplus \\ A, 0\ominus \end{array}$$

4.3.3 Closure & Completion

Definition 3.5. *A branch b of a* $\mathsf{GO}_{\mathsf{MODAL}}$ *tableau is* closed *iff either* $A, i\oplus$ *and* $A, i\ominus$ *appear on b, or* $A, i\oplus$ *and* $\neg A, i\oplus$ *appear on b.*

We mark the closure of a branch with $\otimes$. Witness:

$$\begin{array}{cc}
\vdots & \vdots \\
A, i\oplus & A, i\oplus \\
A, i\ominus & \neg A, i\oplus \\
\otimes & \otimes
\end{array}$$

Definition 3.6. *A* $\mathsf{GO}_{\mathsf{MODAL}}$ *tableau is* closed *iff all branches on the tableau are closed.*

Definition 3.7. *A* $\mathsf{GO}_{\mathsf{MODAL}}$ *tableau is* complete *iff all* $\mathsf{GO}_{\mathsf{MODAL}}$ *resolution rules that can be applied have been applied.*

Definition 3.8. $X \vdash A$ *iff there is a closed tableau for the argument from all members* $B_1, ..., B_n$ *of* X *to* A.

4.3.4 Resolution Rules

Define the following convenient functions:

» $\text{ß}_b(x) = \{y \mid x\mathcal{R}y \text{ is on } b\}$

» $z(b) = \{n \in \mathbb{N} \mid \text{ for some } A \in \mathcal{S},\ A, n\oplus \text{ or } A, n\ominus \text{ is on } b\}$

$\Box$ rules

For rule $\Box\oplus$, $A, j\oplus$ must not already occur on b. For $\neg\Box\oplus$, $j = \mathsf{max}\{z(b)\}+1$.

$$
\begin{array}{c}
\Box\oplus \\ \hline \\
\Box A, i\oplus \\
i\mathcal{R}j \\
| \\
A, j\oplus
\end{array}
\qquad
\begin{array}{c}
\neg\Box\oplus \\ \hline \\
\boxed{\neg\Box A, i\oplus} \\
| \\
i\mathcal{R}j \\
A, j\ominus
\end{array}
\qquad
\begin{array}{c}
\Box\ominus \\ \hline \\
\boxed{\Box A, i\ominus} \\
| \\
\neg\Box A, i\oplus
\end{array}
\qquad
\begin{array}{c}
\neg\Box\ominus \\ \hline \\
\boxed{\neg\Box A, i\ominus} \\
| \\
\Box A, i\oplus
\end{array}
$$

$\Diamond$ rules

For rule $\neg\Diamond\oplus$, $A, j\ominus$ must not already occur on b. For $\Diamond\oplus$, $j = \mathsf{max}\{z(b)\}+1$.

$$
\begin{array}{c}
\Diamond\oplus \\ \hline \\
\boxed{\Diamond A, i\oplus} \\
| \\
i\mathcal{R}j \\
A, j\oplus
\end{array}
\qquad
\begin{array}{c}
\neg\Diamond\oplus \\ \hline \\
\neg\Diamond A, i\oplus \\
i\mathcal{R}j \\
| \\
A, j\ominus
\end{array}
\qquad
\begin{array}{c}
\Diamond\ominus \\ \hline \\
\boxed{\Diamond A, i\ominus} \\
| \\
\neg\Diamond A, i\oplus
\end{array}
\qquad
\begin{array}{c}
\neg\Diamond\ominus \\ \hline \\
\boxed{\neg\Diamond A, i\ominus} \\
| \\
\Diamond A, i\oplus
\end{array}
$$

$\mathcal{R}$ rules

For rule $\mathcal{R}_{\text{REFL}}$, i is any $n \in z(b) \smallsetminus \text{ß}_b(n)$. For rule $\mathcal{R}_{\text{TRAN}}$, $k \notin \text{ß}_b(i)$

$$
\begin{array}{c}
\mathcal{R}_{\text{REFL}} \\ \hline \\
\vdots \\
| \\
i\mathcal{R}i
\end{array}
\qquad
\begin{array}{c}
\mathcal{R}_{\text{TRAN}} \\ \hline \\
i\mathcal{R}j \\
j\mathcal{R}k \\
| \\
i\mathcal{R}k
\end{array}
$$

$\neg$ **Rules**

$$\begin{array}{c}\neg\neg\oplus\\ \hline \\ \boxed{\neg\neg A, i\oplus}\\ | \\ A, i\oplus\end{array} \qquad \begin{array}{c}\neg\neg\ominus\\ \hline \\ \boxed{\neg\neg A, i\ominus}\\ | \\ A, i\ominus\end{array}$$

$\wedge$ **Rules**

$$\begin{array}{c}\wedge\oplus\\ \hline \\ \boxed{A\wedge B, i\oplus}\\ | \\ A, i\oplus\\ B, i\oplus\end{array} \qquad \begin{array}{c}\neg\wedge\oplus\\ \hline \\ \boxed{\neg(A\wedge B), i\oplus}\\ \diagup \quad \diagdown \\ A, i\ominus \quad B, i\ominus\end{array} \qquad \begin{array}{c}\wedge\ominus\\ \hline \\ \boxed{A\wedge B, i\ominus}\\ | \\ \neg(A\wedge B), i\oplus\end{array} \qquad \begin{array}{c}\neg\wedge\ominus\\ \hline \\ \boxed{\neg(A\wedge B), i\ominus}\\ | \\ A\wedge B, i\oplus\end{array}$$

$\vee$ **Rules**

$$\begin{array}{c}\vee\oplus\\ \hline \\ \boxed{A\vee B, i\oplus}\\ \diagup \quad \diagdown \\ A, i\oplus \quad B, i\oplus\end{array} \qquad \begin{array}{c}\neg\vee\oplus\\ \hline \\ \boxed{\neg(A\vee B), i\oplus}\\ | \\ A, i\ominus\\ B, i\ominus\end{array} \qquad \begin{array}{c}\vee\ominus\\ \hline \\ \boxed{A\vee B, i\ominus}\\ | \\ \neg(A\vee B), i\oplus\end{array} \qquad \begin{array}{c}\neg\vee\ominus\\ \hline \\ \boxed{\neg(A\vee B), i\ominus}\\ | \\ A\vee B, i\oplus\end{array}$$

$\supset$ **Rules**

$$\begin{array}{c}\supset\oplus\\ \hline \\ \boxed{A\supset B, i\oplus}\\ \diagup \quad \diagdown \\ \neg A, i\oplus \quad B, i\oplus\end{array} \qquad \begin{array}{c}\neg\supset\oplus\\ \hline \\ \boxed{\neg(A\supset B), i\oplus}\\ | \\ \neg A, i\ominus\\ B, i\ominus\end{array} \qquad \begin{array}{c}\supset\ominus\\ \hline \\ \boxed{A\supset B, i\ominus}\\ | \\ \neg(A\supset B), i\oplus\end{array} \qquad \begin{array}{c}\neg\supset\ominus\\ \hline \\ \boxed{\neg(A\supset B), i\ominus}\\ | \\ A\supset B, i\oplus\end{array}$$

$\equiv$ Rules

$$\begin{array}{c} \equiv\oplus \\ \hline \boxed{A \equiv B, i\oplus} \\ \begin{array}{cc} \neg A, i\oplus & B, i\oplus \\ \neg B, i\oplus & A, i\oplus \end{array} \end{array} \qquad \begin{array}{c} \neg \equiv\oplus \\ \hline \boxed{\neg(A \equiv B), i\oplus} \\ \begin{array}{cc} \neg A, i\ominus & \neg B, i\ominus \\ B, i\ominus & A, i\ominus \end{array} \end{array} \qquad \begin{array}{c} \equiv\ominus \\ \hline \boxed{A \equiv B, i\ominus} \\ | \\ \neg(A \equiv B), i\oplus \end{array} \qquad \begin{array}{c} \neg \equiv\ominus \\ \hline \boxed{\neg(A \equiv B), i\ominus} \\ | \\ A \equiv B, i\oplus \end{array}$$

$\rightarrow$ Rules

$$\begin{array}{c} \rightarrow\oplus \\ \hline \boxed{A \rightarrow B, i\oplus} \\ \begin{array}{cc} \neg A \vee B, i\oplus & A, i\ominus \\ & B, i\ominus \\ & \neg A, i\ominus \\ & \neg B, i\ominus \end{array} \end{array} \qquad \begin{array}{c} \neg \rightarrow\oplus \\ \hline \boxed{\neg(A \rightarrow B), i\oplus} \\ \begin{array}{cc} A, i\oplus & \neg B, i\oplus \\ B, i\ominus & \neg A, i\ominus \end{array} \end{array} \qquad \begin{array}{c} \rightarrow\ominus \\ \hline \boxed{A \rightarrow B, i\ominus} \\ | \\ \neg(A \rightarrow B), i\oplus \end{array} \qquad \begin{array}{c} \neg \rightarrow\ominus \\ \hline \boxed{\neg(A \rightarrow B), i\ominus} \\ | \\ A \rightarrow B, i\oplus \end{array}$$

$\leftrightarrow$ Rules

$$\begin{array}{c} \leftrightarrow\oplus \\ \hline \boxed{A \leftrightarrow B, i\oplus} \\ | \\ A \rightarrow B, i\oplus \\ B \rightarrow A, i\oplus \end{array} \qquad \begin{array}{c} \neg \leftrightarrow\oplus \\ \hline \boxed{\neg(A \leftrightarrow B), i\oplus} \\ \begin{array}{cc} \neg(A \rightarrow B), i\oplus & \neg(B \rightarrow A), i\oplus \end{array} \end{array} \qquad \begin{array}{c} \leftrightarrow\ominus \\ \hline \boxed{A \leftrightarrow B, i\ominus} \\ | \\ \neg(A \leftrightarrow B), i\oplus \end{array} \qquad \begin{array}{c} \neg \leftrightarrow\ominus \\ \hline \boxed{\neg(A \leftrightarrow B), i\ominus} \\ | \\ A \leftrightarrow B, i\oplus \end{array}$$

4.4 Adequacy

We now demonstrate soundness and completeness of the $\mathsf{GO}_{\mathsf{MODAL}}$ tableaux system with respect to the semantics of $\mathcal{L}_{\mathsf{GO}_{\mathsf{MODAL}}}$.

4.4.1 Soundness

Faithful Model

DEFINITION 4.9. *A map $f : \mathbb{N} \longrightarrow \mathcal{W}$ shows a model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \nu \rangle$ of $\mathcal{L}_{\mathsf{GO}_{\mathsf{MODAL}}}$ to be* faithful *to a tableau branch b iff:*

» *for each $A \in \mathcal{S}$:*

  – *if $A, i\oplus$ is on b, then $\nu_{f(i)}(A) = 1$ and*

  – *if $A, i\ominus$ is on b, then $\nu_{f(i)}(A) \neq 1$.*

» *for each $n \in \mathbb{N}$, $ß_b(n) \subseteq \mathcal{R}(f(n))$.*

Soundness Lemma

With Definition 4.9. in mind, we formulate the Soundness Lemma to state that the $\mathsf{GO}_{\mathsf{MODAL}}$ resolutions rules are "faithfulness preserving."

LEMMA 4.10. *For any branch b of any $\mathsf{GO}_{\mathsf{MODAL}}$ tableau: If f shows $\mathfrak{M}$ to be faithful to b, and we apply any $\mathsf{GO}_{\mathsf{MODAL}}$ resolution rule to b, then there is an f' that shows $\mathfrak{M}$ to be faithful to at least one extension b' of b.*

Proof. The proof shows that Lemma 4.10. holds for each $\mathsf{GO}_{\mathsf{MODAL}}$ rule.

1. $\Box\oplus$. Suppose $\Box A, i\oplus$ is on b. Since f shows $\mathfrak{M}$ to be faithful to b, $\nu_{f(i)}(\Box A) = 1$. Further, $\text{ß}_b(i) \subseteq \mathcal{R}(f(i))$ and hence, by the semantics of $\Box$, $\nu_w(A) = 1$ for each w such that $f(i)\mathcal{R}w$. When we apply the $\Box\oplus$ rule to b, it produces an extension b' with a node of the form $A, j\oplus$ where $j \in \text{ß}_b(i)$. Since we've shown that $\nu_{f(j)}(A) = 1$, there is an f', namely f, that shows $\mathfrak{M}$ to be faithful to b'. $\Box$

2. $\neg\Box\oplus$. Suppose $\neg\Box A, i\oplus$ is on b. Since f shows $\mathfrak{M}$ to be faithful to b, $\nu_{f(i)}(\neg\Box A) = 1$, and so $\nu_{f(i)}(\Box A) = 0$. By the semantics of $\Box$, for some $w \in \mathcal{R}(f(i))$, $\nu_w(A) \neq 1$. When we apply the $\neg\Box\oplus$ rule to b, it produces an extension b' with nodes $i\mathcal{R}j$ and $A, j\ominus$ for some $j \notin z(b)$. Let f' be the same as f except $f'(j) = w$. Since f' differs from f only wrt j, and $j \notin z(b)$, f' shows $\mathfrak{M}$ to be faithful to b. Therefore, since $\nu_{f'(j)}(A) \neq 1$, f' shows $\mathfrak{M}$ to be faithful to b'. $\Box$

3. $\Box\ominus$. Suppose $\Box A, i\ominus$ is on b. Since f shows $\mathfrak{M}$ to be faithful to b, $\nu_{f(i)}(\Box A) \neq 1$, and thus by Lemma 2.2.(i), $\nu_{f(i)}(\Box A) = 0$. Hence, by the semantics of negation, $\nu_{f(i)}(\neg\Box A) = 1$. When we apply the $\Box\ominus$ rule, it produces an extension b' with a node $\neg\Box A, i\oplus$. Thus there is an f', namely f, that shows $\mathfrak{M}$ to be faithful to b'. $\Box$

4. $\neg\Box\ominus$. Suppose $\neg\Box A, i\ominus$ is on b. Since f shows $\mathfrak{M}$ to be faithful to b, $\nu_{f(i)}(\neg\Box A) \neq 1$. Hence, since by Lemma 2.2.(ii) $\nu_{f(i)}(\neg\Box A) = 0$, it follows that $\nu_{f(i)}(\Box A) = 1$. When we apply the $\neg\Box\ominus$ rule, it produces an extension b' with a node $\Box A, i\oplus$. Thus there is an f', namely f, that shows $\mathfrak{M}$ to be faithful to b'. $\Box$

5. $\Diamond\oplus$. Suppose $\Diamond A, i\oplus$ is on b. Thus $\nu_{f(i)}(\Diamond A) = 1$. Thus there is some world $w \in \mathcal{R}(f(i))$ such that $\nu_w(A) = 1$. When we apply the $\Diamond\oplus$ rule, it produces an extension b' with nodes $i\mathcal{R}j$ and $A, j\oplus$ for some $j \notin z(b)$. Let f' be the same as f except $f'(j) = w$. Since f' differs from f only wrt j, and $j \notin z(b)$, f' shows $\mathfrak{M}$ to be faithful to b. Therefore, since $\nu_{f'(j)}(A) = 1$, f' shows $\mathfrak{M}$ to be faithful to b'. $\square$

6. $\neg\Diamond\oplus$. Suppose $\neg\Diamond A, i\oplus$ is on b. Thus $\nu_{f(i)}(\neg\Diamond A) = 1$, and hence $\nu_{f(i)}(\Diamond A) = 0$. When we apply the $\neg\Diamond\oplus$ rule, it produces as extension b' with a node $A, j\ominus$ where $j \in \text{ß}_b(i)$. Since by definition $\text{ß}_b(i) \subseteq \mathcal{R}(f(i))$, by the semantics of $\Diamond$, $\nu_{f(j)}(A) \neq 1$. So there is an f', namely f, that shows $\mathfrak{M}$ to be faithful to b'. $\square$

7. $\Diamond\ominus$. Suppose $\Diamond A, i\ominus$ is on b. So $\nu_{f(i)}(\Diamond A) \neq 1$, and thus by Lemma 2.2.(i), $\nu_{f(i)}(\Diamond A) = 0$, and so $\nu_{f(i)}(\neg\Diamond A) = 1$. When we apply the $\Diamond\ominus$ rule, it produces an extension b' with a node $\neg\Diamond A, i\oplus$. Thus there is an f', namely f, that shows $\mathfrak{M}$ to be faithful to b'. $\square$

8. $\neg\Diamond\ominus$. Suppose $\neg\Diamond A, i\ominus$ is on b. Thus $\nu_{f(i)}(\neg\Diamond A) \neq 1$. Hence, since by Lemma 2.2.(ii) $\nu_{f(i)}(\neg\Diamond A) = 0$, $\nu_{f(i)}(\Diamond A) = 1$. When we apply the $\neg\Diamond\ominus$ rule, it produces an extension b' with a node $\Diamond A, i\oplus$. Thus there is an f', namely f, that shows $\mathfrak{M}$ to be faithful to b'. $\square$

For each of the remaining proofs, let w stand for the value of $f(i)$. Also, we speak of $\mathfrak{M}$ as faithful to b, dropping particular reference to f, since $f' = f$ throughout.

9. $\wedge\oplus$. Suppose $A \wedge B, i\oplus$ is on b. Since $\mathfrak{M}$ is faithful to b, $\nu_w(A \wedge B) = 1$.

Thus $\nu_w(A) = 1 = \nu_w(B)$. When we apply the $\wedge\oplus$ rule, it produces an extension b' with nodes $A, i\oplus$ and $B, i\oplus$. Hence $\mathfrak{M}$ is faithful to b'. $\square$

10. $\neg\wedge\oplus$. Suppose $\neg(A \wedge B), i\oplus$ is on b. So $\nu_w(\neg(A \wedge B)) = 1$. When we apply the $\neg\wedge\oplus$ rule, it produces two extensions, one with $A, i\ominus$, and the other with $B, i\ominus$. Since $\nu_w(A \wedge B) = 0$, either $\nu_w(A) \neq 1$ or $\nu_B(\neq)1$. In the first case, $\mathfrak{M}$ is faithful to the first extension, and in the other case, $\mathfrak{M}$ is faithful to the second extension. $\square$

11. $\wedge\ominus$. Suppose $A \wedge B, i\ominus$ is on b. Thus $\nu_w(A \wedge B) \neq 1$, and so by Lemma 2.3.(i), $\nu_w(A \wedge B) = 0$. Hence, $\nu_w(\neg(A \wedge B)) = 1$. Since $\wedge\ominus$ rule extends b only with $\neg(A \wedge B), i\oplus$, $\mathfrak{M}$ is faithful to the extension. $\square$

12. $\neg\wedge\ominus$. Suppose $\neg(A \wedge B), i\ominus$ is on b. So $\nu_w(\neg(A \wedge B)) \neq 1$, and hence by Lemma 2.3.(ii) $\nu_w(A \wedge B) = 1$. Since the $\neg\wedge\ominus$ rule extends b only with $A \wedge B, i\oplus$, $\mathfrak{M}$ is faithful to the extension. $\square$

13. $\vee\oplus$. Suppose $A \vee B, i\oplus$ is on b. Since $\mathfrak{M}$ is faithful, $\nu_w(A \vee B) = 1$. Applying the $\vee\oplus$ rule produces two extensions, one with $A, i\oplus$ and the other with $B, i\oplus$. By the semantics of $\vee$, $\nu_w(A) = 1$ or $\nu_w(B) = 1$. In the first case, $\mathfrak{M}$ is faithful to one extension, and to the other in the second case. $\square$

14. $\neg\vee\oplus$. Suppose $\neg(A \vee B), i\oplus$ is on b. Hence $\nu_w(\neg(A \vee B)) = 1$, and so $\nu_w(A \vee B) = 0$. The $\neg\vee\oplus$ rule extends b with $A, i\ominus$ and $B, i\ominus$. By the semantics of $\vee$, $\nu_w(A) \neq 1$ and $\nu_w(B) \neq 1$. Hence $\mathfrak{M}$ is faithful to the extension. $\square$

15. $\vee\ominus$. Suppose $A \vee B, i\ominus$ is on b. Since $\mathfrak{M}$ is faithful, $\nu_w(A \vee B) \neq 1$, and by Lemma 2.3.(i), $\nu_w(A \vee B) = 0$. Hence, $\nu_w(\neg(A \vee B)) = 1$. Applying the $\vee\ominus$ rule extends b only with $\neg(A \vee B), i\oplus$. Thus $\mathfrak{M}$ is faithful to the extension. $\square$

16. $\neg\vee\ominus$. Suppose $\neg(A \vee B), i\ominus$ is on b. So $\nu_w(\neg(A \vee B)) \neq 1$, hence by Lemma 2.3.(ii), $\nu_w(\neg(A \vee B)) = 0$. Hence, $\nu_w(A \vee B) = 1$. Applying the $\neg\vee\ominus$ rule extends b only with $A \vee B, i\oplus$, and so $\mathfrak{M}$ is faithful to the extension. $\square$

17. $\rightarrow\oplus$. Suppose $A \rightarrow B, i\oplus$ is on b. Hence $\nu_w(A \rightarrow B) = 1$. So,

$$\nu_w((A \supset B) \vee (\neg(A \vee \neg A) \wedge \neg(B \vee \neg B))) = 1$$

Thus either:

(i) $\nu_w(A \supset B) = 1$, or

(ii) $\nu_w(\neg(A \vee \neg A) \wedge \neg(B \vee \neg B)) = 1$.

When we apply the $\rightarrow\oplus$ rule, it produces two extensions. The first has $A \supset B, i\oplus$, to which $\mathfrak{M}$ is faithful in case (i). The second has the following:

$$A, i\ominus$$
$$B, i\ominus$$
$$\neg A, i\ominus$$
$$\neg B, i\ominus$$

To which $\mathfrak{M}$ is faithful iff

$$\nu_w(A) = {}^1\!/\!_2 = \nu_w(B)$$

In case (ii), it follows that

$$\nu_w(\neg(A \vee \neg A)) = 1$$

and so,

$$\nu_w(A \vee \neg A) = 0$$

This guarantees that $\nu_w(A) = {}^1\!/\!_2$. Similar reasoning on the second conjunct shows that $\nu_w(B) = {}^1\!/\!_2$. Hence $\mathfrak{M}$ must be faithful to at least one extension of b. □

18. $\neg \rightarrow \oplus$. Suppose $\neg(A \rightarrow B), i\oplus$ is on b. Thus $\nu_w(\neg(A \rightarrow B)) = 1$, and so $\nu_w(A \rightarrow B) = 0$. So,

$$\nu_w((A \supset B) \vee (\neg(A \vee \neg A) \wedge \neg(B \vee \neg B))) = 0$$

Hence by the semantics of $\vee$ and Lemma 2.3.(ii):

$$\nu_w(A \supset B) = 0$$

and

$$\nu_w(\neg(A \vee \neg A) \wedge \neg(B \vee \neg B)) = 0$$

And so by the semantics of $\wedge$ and Lemma 2.3.(ii) either:

(i) $\nu_w(\neg(A \vee \neg A)) = 0$, or

(ii) $\nu_w(\neg(B \vee \neg B)) = 0$

The semantics for $\neg$ show the following to hold in their respective cases:

(i) $\nu_w(A \vee \neg A) = 1$, or

(ii) $\nu_w(B \vee \neg B) = 1$

and consequently,

(i) $\nu_w(A) \in \{0, 1\}$, or

(ii) $\nu_w(B) \in \{0, 1\}$

Given that $\nu_w(A \supset B) = 0$ we have it that

(i) $\nu_w(A) = 1$, or

(ii) $\nu_w(B) = 0$

One extension from the $\neg\to\oplus$ rule has $A, i\oplus$ and $B, i\ominus$. In case (i), since $\nu_w(A \supset B) = 0$, $\nu_w(B) \neq 1$, and thus $\mathfrak{M}$ is faithful to this extension

The other extension has $\neg B, i\oplus$ and $\neg A, i\ominus$. In case (ii), $\nu_w(\neg B) = 1$, and since $\nu_w(A \supset B) = 0$, it follows that $\nu_w(A) \neq 1$, and thus $\mathfrak{M}$ is faithful to this extension. □

19. $\to\ominus$. Suppose $A \to B, i\ominus$ is on b. Thus $\nu_w(A \to B) \neq 1$, and hence by Lemma 2.3.(i), $\nu_w(A \to B) = 0$. So, $\nu_w(\neg(A \to B)) = 1$. The $\to\ominus$ rule extends b only with $\neg(A \to B), i\oplus$. Thus $\mathfrak{M}$ is faithful to the extension. □

20. $\neg\rightarrow\ominus$. Suppose $\neg(A \rightarrow B), i\ominus$ is on b. So $\nu_w(\neg(A \rightarrow B)) \neq 1$, and by Lemma 2.3.(ii), $\nu_w(\neg(A \rightarrow B)) = 0$. Hence $\nu_w(A \rightarrow B) = 1$, and the $\neg\rightarrow\ominus$ extends b only with $A \rightarrow B, i\oplus$. Thus $\mathfrak{M}$ is faithful to the extension. $\square$

21. $\leftrightarrow\oplus$. Suppose $A \leftrightarrow B, i\oplus$ is on b. Hence $\nu_w(A \leftrightarrow B) = 1$, and so

$$\nu_w(A \rightarrow B) = 1$$

and

$$\nu_w(B \rightarrow A) = 1$$

And since the $\leftrightarrow\oplus$ rule extends b only with $A \rightarrow B, i\oplus$ and $B \rightarrow A, i\oplus$, $\mathfrak{M}$ is faithful to the extension. $\square$

22. $\neg\leftrightarrow\oplus$. Suppose $\neg(A \leftrightarrow B), i\oplus$ is on b. Hence $\nu_w(\neg(A \leftrightarrow B)) = 1$, and so

$$\nu_w(\neg((A \rightarrow B) \wedge (B \rightarrow A))) = 1$$

Thus

$$\nu_w((A \rightarrow B) \wedge (B \rightarrow A) = 0)$$

and hence either:

(i) $\nu_w(A \rightarrow B) \neq 1$, or

(ii) $\nu_w(B \rightarrow A) \neq 1$

and so, respectively, either:

(i) $\nu_w(A \rightarrow B) = 0$, or

(ii) $\nu_w(B \to A) = 0$

Consequently:

(i) $\nu_w(\neg(A \to B)) = 1$, or

(ii) $\nu_w(\neg(B \to A)) = 1$

The $\neg \leftrightarrow\oplus$ rule produces two extensions. The first has $\neg(A \to B), i\oplus$, to which $\mathfrak{M}$ is faithful in case (i). In case (ii) $\mathfrak{M}$ is faithful to the second extension, which has $\neg(B \to A), i\oplus$. □

23. $\leftrightarrow\ominus$. Suppose $A \leftrightarrow B, i\ominus$ is on b. Hence $\nu_w(A \leftrightarrow B) \neq 1$, and by Lemma 2.3.(i), $\nu_w(A \leftrightarrow B) = 0$, and so $\nu_w(\neg(A \leftrightarrow B))$. The $\leftrightarrow\ominus$ rule gives a unique extension with $\neg(A \leftrightarrow B), i\oplus$, and thus $\mathfrak{M}$ is faithful to it. □

24. $\neg \leftrightarrow\ominus$. Suppose $\neg(A \leftrightarrow B), i\ominus$ is on b. So $\nu_w(\neg(A \leftrightarrow B)) \neq 1$, and by Lemma 2.3.(ii), $\nu_w(\neg(A \leftrightarrow B)) = 0$. Hence $\nu_w(A \leftrightarrow B) = 1$. The $\neg \leftrightarrow\ominus$ rule extends b with only $A \leftrightarrow B, i\oplus$, and thus $\mathfrak{M}$ is faithful to it. □

25. $\supset\oplus$. Suppose $A \supset B, i\oplus$ is on b. Hence $\nu_w(A \supset B) = 1$, and by definition $\nu_w(\neg A \vee B) = 1$. Thus either:

(i) $\nu_w(\neg A) = 1$, or

(ii) $\nu_w(B) = 1$

The $\supset\oplus$ rule produces two extensions. In case (i), $\mathfrak{M}$ is faithful to the extension with $\neg A, i\oplus$, and in case (ii), $\mathfrak{M}$ is faithful to the other extension with $B, i\oplus$. □

26. $\neg\supset\oplus$. Suppose $\neg(A \supset B), i\oplus$ is on b. Thus $\nu_w(\neg(A \supset B)) = 1$, and so $\nu_w(A \supset B) = 0$. Hence

$$\nu_w(\neg A \vee B) = 0$$

and so

$$\nu_w(\neg A) \neq 1$$

and

$$\nu_w(B) \neq 1$$

The $\neg\supset\oplus$ rule extends b with only $\neg A, i\ominus$ and $B, i\ominus$. Thus $\mathfrak{M}$ is faithful to the extension. □

27. $\supset\ominus$. Suppose $A \supset B, i\ominus$ is on b. Since $\nu_w(A \supset B) \neq 1$, by Lemma 2.3.(i), $\nu_w(A \supset B) = 0$, and so $\nu_w(\neg(A \supset B)) = 1$. The $\supset\ominus$ rule extends b only with $\neg(A \supset B), i\oplus$, and thus $\mathfrak{M}$ is faithful to the extension. □

28. $\neg\supset\ominus$. Suppose $\neg(A \supset B), i\ominus$ is on b. Since $\nu_w(\neg(A \supset B)) \neq 1$, by Lemma 2.3.(ii), $\nu_w(\neg(A \supset B)) = 0$, and so $\nu_w(A \supset B) = 1$. Since the $\neg\supset\ominus$ rule extends b only with $A \supset B, i\oplus$, $\mathfrak{M}$ is faithful to the extension. □

29. $\equiv\oplus$. Suppose $A \equiv B, i\oplus$ is on b. Hence $\nu_w(A \equiv B) = 1$. Thus

$$\nu_w(A) \neq 1/2$$

and

$$\nu_w(A) = \nu_w(B)$$

Thus either:

(i) $\nu_w(A) = 1 = \nu_w(B)$, or

(ii) $\nu_w(\neg A) = 1 = \nu_w(\neg B)$.

The $\equiv\oplus$ rule produces two extensions. The first has $\neg A, i\oplus$ and $\neg B, i\oplus$, to which, in case (i) $\mathfrak{M}$ is faithful. The second extension has $A, i\oplus$ and $B, i\oplus$, to which $\mathfrak{M}$ is faithful in case (ii). $\square$

30. $\neg\equiv\oplus$. Suppose $\neg(A \equiv B), i\oplus$ is on b. So $\nu_w(\neg(A \equiv B)) = 1$, and thus $\nu_w(A \equiv B) = 0$. By the semantics of $\equiv$ and Lemma 2.3.(i), either:

(i) $\nu_w(A \supset B) = 0$, or

(ii) $\nu_w(B \supset A) = 0$.

Consequently, in the respective cases:

(i) $\nu_w(\neg A \vee B) = 0$, or

(ii) $\nu_w(\neg B \vee A) = 0$.

And so

(i) $\nu_w(\neg A) \neq 1$ and $\nu_w(B) \neq 1$, or

(ii) $\nu_w(\neg B) \neq 1$ and $\nu_w(A) \neq 1$.

The $\neg\equiv\oplus$ produces two extensions. The first has $\neg A, i\ominus$ and $B, i\ominus$, to which case (i) shows $\mathfrak{M}$ to be faithful. The second extension has $\neg B, i\ominus$ and $A, i\ominus$, to which case (ii) shows $\mathfrak{M}$ to be faithful. $\square$

31. $\equiv\ominus$. Suppose $\boxed{A \equiv B, i\ominus}$ is on b. Hence $\nu_w(A \equiv B) \neq 1$, and so by Lemma 2.3.(i) and the semantics of $\neg$, $\nu_w(\neg(A \equiv B)) = 1$. Thus $\mathfrak{M}$ is faithful to the extension produced by the $\equiv\ominus$ rule, which has $\boxed{\neg(A \equiv B), i\oplus}$. $\square$

32. $\neg\equiv\ominus$. Suppose $\boxed{\neg(A \equiv B), i\ominus}$ is on b. So $\nu_w(\neg(A \equiv B)) \neq 1$, and so by Lemma 2.3.(ii) and the semantics of $\neg$, $\nu_w(A \equiv B) = 1$. Thus $\mathfrak{M}$ is faithful to the extension produced by the $\neg\equiv\ominus$ rule, which has $\boxed{A \equiv B, i\oplus}$. $\square$

33. $\neg\neg\oplus$. Suppose $\boxed{\neg\neg A, i\oplus}$ is on b. Since $\mathfrak{M}$ is faithful to b, $\nu_w(\neg\neg A) = 1$, and so $\nu_w(A) = 1$. The $\neg\neg\oplus$ extends b only with $\boxed{A, i\oplus}$, and hence $\mathfrak{M}$ is faithful to the extension. $\square$

34. $\neg\neg\ominus$. Suppose $\boxed{\neg\neg A, i\ominus}$ is on b. Since $\mathfrak{M}$ is faithful to b, $\nu_w(\neg\neg A) \neq 1$ and so $\nu_w(\neg A) \in \{1, \frac{1}{2}\}$, and consequently $\nu_w(A) \in \{0, \frac{1}{2}\}$. The $\neg\neg\ominus$ extends b only with $\boxed{A, i\ominus}$. Whence $1 \notin 0, \frac{1}{2}$, $\mathfrak{M}$ is faithful to the extension. $\square$

Soundness Theorem

THEOREM 4.11. *If* $\Sigma \vdash A$ *then* $\Sigma \Vdash A$*.*

Proof. We prove the contrapositive. Assume for conditional proof that $\Sigma \nVdash A$. Thus there is some world w in some model $\mathfrak{M}$ such that $\nu_w(B) = 1$ for all $B \in \Sigma$, and $\nu_w(A) \neq 1$. Suppose for reductio that there is a closed tableau for the argument from Σ to A. Let f show $\mathfrak{M}$ to be faithful to the initial list and let $f(0) = w$. When we apply any resolution rule to our tableau, by Lemma 4.10., there is an f' that shows $\mathfrak{M}$ to be faithful to at least one extension b' of b. Since b' is closed, for some $C \in \mathcal{S}$ and some $i \in \mathbb{N}$, either (a): both $C, i\oplus$ and $C, i\ominus$ are on b', or (b): both $C, i\oplus$ and $\neg C, i\oplus$ are on b'. In case (a), $\nu_{f'(i)}(C) = 1$ and $\nu_{f'(i)}(C) \neq 1$. Impossible. In case (b), $\nu_{f'(i)}(C) = 1$ and $\nu_{f'(i)}(\neg C) = 1$. Impossible. Hence, there is no closed tableau for the argument from Σ to A. Therefore, $\Sigma \nVdash A$. □

4.4.2 Completeness

Induced Model

DEFINITION 4.12. *A complete open branch b of a $\mathsf{GO}_{\mathsf{MODAL}}$ tableau* induces *a model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \nu \rangle$ iff*

» $\mathcal{W} = \{w_i \mid i \in z(b)\}$

» *For all $p \in \mathcal{A}$ on b and $i \in \mathbb{N}$:*

(i) $\mathcal{R}(w_i) = ß_b(i)$.

(ii) $\nu_{w_i}(p) = 1$ *iff* $p, i\oplus$ *is on b.*

(iii) $\nu_{w_i}(p) = ½$ *iff* $p, i\ominus$ *and* $\neg p, i\ominus$ *are on b.*

(iv) $\nu_{w_i}(p) = 0$ *iff either*

(a) $\neg p, i\oplus$ *is on b, or*

(b) $p, i\ominus$ *is on b and* $\neg p, i\ominus$ *is not on b.*

» *For any $p \in \mathcal{A}$ not on b, and all $w \in \mathcal{W}$, $\nu_w(p) = 1$.*

Completeness Lemma

LEMMA 4.13. *Given a branch b of a completed open tableau and its induced model $\mathfrak{M}$:*

(a) If $A, i\oplus$ *is on* b*, then* $\nu_{w_i}(A) = 1$*, and*

(b) If $\neg A, i\oplus$ *is on* b*, then* $\nu_{w_i}(A) = 0$*, and*

(c) If $A, i\ominus$ *and* $\neg A, i\ominus$ *are on* b*, then* $\nu_{w_i}(A) = \frac{1}{2}$*, and*

(d) If $A, i\ominus$ *is on* b*, then* $\nu_{w_i}(A) \neq 1$*, and*

(e) If $\neg A, i\ominus$ *is on* b*, then* $\nu_{w_i}(A) \neq 0$*.*

Proof. The proof shows that Lemma 4.13. holds for each $\mathcal{L}_{\mathsf{GO}_{\mathsf{MODAL}}}$ sentence. We start with our base case, and prove that each condition of Lemma 4.13. holds for atomic sentences.

1. Proof of (a) for atomics.

   Suppose $p, i\oplus$ is on b. By Definition 4.12.(ii), $\nu_{w_i}(p) = 1$. □

2. Proof of (b) for atomics.

   Suppose $\neg p, i\oplus$ is on b. By Definition 4.12.(iv), $\nu_{w_i}(p) = 0$. □

3. Proof of (c) for atomics.

   Suppose $p, i\ominus$ and $\neg p, i\ominus$ are on b. Hence, by Definition 4.12.(iii), $\nu_{w_i}(p) = \frac{1}{2}$. □

4. Proof of (d) for atomics.

   Suppose $p, i\ominus$ is on b. It is either the case or not the case that $\neg p, i\ominus$ is on b. If it is, then by Definition 4.12.(iii), $\nu_{w_i}(p) = \frac{1}{2}$. If it is not, then by Definition 4.12.(iv), $\nu_{w_i}(p) = 0$. In either case, $\nu_{w_i}(p) \neq 1$. □

5. Proof of (e) for atomics.

   Suppose $\neg p, i\ominus$ is on b. Since b is open, $\neg p, i\oplus$ is not on b. Thus by Definition 4.12.(iv), $\nu_{w_i}(p) \neq 0$. □

We proceed by induction for non-atomic cases. We start with the binary connectives. Our induction hypothesis, then, is that each condition of the Lemma holds for sentences A and B. Since Lemma 2.3. tells us that no binary sentence is assigned value ½, condition (c) should never apply. Thus we prove (c) for all binary connectives in one step.

6. Proof of (c) for each $\odot \in \mathcal{C}_b$.

   The rule for $A \odot B, i\ominus$ for each $\odot \in \mathcal{C}_b$ produces exactly one node of the form $\neg(A \odot B), i\oplus$.

   Suppose $A \odot B, i\ominus$ is on b. Since b is complete, $\neg(A \odot B), i\oplus$ is on b. Since b is open, $\neg(A \odot B), i\ominus$ is not on b, and so (c) is vacuously true. □

Next we prove conditions (a) and (b) hold for each binary connective. Once these are shown, we will prove that the remaining conditions, (d) and (e), must also hold for all binary connectives.

7. Proof of (a) for $\wedge$.

   Suppose $A \wedge B, i\oplus$ is on b. Since b is complete, $A, i\oplus$ and $B, i\oplus$ are on b. By the induction hypothesis, then, $\nu_{w_i}(A) = 1 = \nu_{w_i}(B)$. Hence, $\nu_{w_i}(A \wedge B) = 1$. □

8. Proof of (b) for $\wedge$.

   Suppose $\neg(A \wedge B), i\oplus$ is on b. Thus either $A, i\ominus$ or $B, i\ominus$ is on b. By the induction hypothesis, then either $\nu_{w_i}(A) \neq 1$ or $\nu_{w_i}(B) \neq 1$. In either case, $\nu_{w_i}(A \wedge B) = 0$. □

9. Proof of (a) for $\vee$.

   Suppose $A \vee B, i\oplus$ is on b. Thus either $A, i \oplus i$ or $B, i\oplus$ is on b. By the induction hypothesis, either $\nu_{w_i}(A) = 1$ or $\nu_{w_i}(B) = 1$. In each case, it follows that $\nu_{w_i}(A \vee B) = 1$. □

10. Proof of (b) for $\vee$.

    Suppose $\neg(A \vee B), i\oplus$ is on b. Thus both $A, i\ominus$ and $B, i\ominus$ are on b. By the induction hypothesis,

    $$\nu_{w_i}(A) \neq 1$$

    $$\nu_{w_i}(B) \neq 1$$

    Hence $\nu_{w_i}(A \vee B) = 0$. □

11. Proof of (a) for $\supset$.

    Suppose $A \supset B, i\oplus$ is on b. Since b is complete, either $\neg A, i\oplus$ or $B, i\oplus$ is on b. In the first case, it follows from the induction hypothesis that $\nu_{w_i}(A) = 0$, and hence $\nu_{w_i}(A \supset B) = 1$. Similarly, in the second case, $\nu_{w_i}(B) = 1$, and so $\nu_{w_i}(A \supset B) = 1$. $\square$

12. Proof of (b) for $\supset$.

    Suppose $\neg(A \supset B), i\oplus$ is on b. Thus $\neg A, i\ominus$ and $B, i\ominus$ are on b. By the induction hypothesis, $\nu_{w_i}(B) \neq 1$ and $\nu_{w_i}(\neg A) \neq 0$. Hence, by the semantics of $\supset$, $\nu_{w_i}(A \supset B) = 0$. $\square$

13. Proof of (a) for $\equiv$.

    Suppose $A \equiv B, i\oplus$ is on b. Thus either

    (I) $A, i\oplus$ and $B, i\oplus$ are on b, or

    (II) $\neg A, i\oplus$ and $\neg B, i\oplus$ are on b.

    In case (I), by the induction hypothesis,

    $$\nu_{w_i}(A) = 1 = \nu_{w_i}(B)$$

    and similarly in case (II),

    $$\nu_{w_i}(A) = 0 = \nu_{w_i}(B)$$

    In each case it follows that $\nu_{w_i}(A \equiv B) = 1$. $\square$

14. Proof of (b) for $\equiv$.

Suppose $\neg(A \equiv B), i\oplus$ is on b. Thus either

(I) $\neg A, i\ominus$ and $B, i\ominus$ are on b, or

(II) $A, i\ominus$ and $\neg B, i\ominus$ are on b.

In case (I), it follows that

$$\nu_{w_i}(A) \neq 0$$

$$\nu_{w_i}(B) \neq 1$$

and thus

$$\nu_{w_i}(A \supset B) = 0$$

In case (II), similarly:

$$\nu_{w_i}(A) \neq 1$$

$$\nu_{w_i}(B) \neq 0$$

whence

$$\nu_{w_i}(B \supset A) = 0$$

In each case it follows that $\nu_{w_i}(A \equiv B) = 0$. □

15. Proof of (a) for $\to$.

Suppose $A \to B, i\oplus$ is on b. Thus either:

(I) $\neg A \vee B, i\oplus$ is on b, or

(II) All of these are on b :

$$A, i\ominus$$

$$B, i\ominus$$

$$\neg A, i\ominus$$

$$\neg B, i\ominus$$

Case (I): Since b is complete, either $\neg A, i\oplus$ or $B, i\oplus$ is on b. Thus by the induction hypothesis, either $\nu_{w_i}(A) = 0$ or $\nu_{w_i}(B) = 1$. So $\nu_{w_i}(\neg A \vee B) = 1$, and hence $\nu_{w_i}(A \to B) = 1$.

Case (II): By the induction hypothesis (c),

$$\nu_{w_i}(A) = \tfrac{1}{2} = \nu_{w_i}(B)$$

Hence, $\nu_{w_i}(A \to B) = 1$. □

16. Proof of (b) for $\to$.

Suppose $\neg(A \to B), i\oplus$ is on b. Thus either:

(I) $A, i\oplus$ and $B, i\ominus$ are on b, or

(II) $\neg A, i\ominus$ and $\neg B, i\oplus$ are on b.

Case (I). By the induction hypothesis,

$$\nu_{w_i}(A) = 1$$

$$\nu_{w_i}(B) \neq 1$$

Thus $\nu_{w_i}(A \to B) = 0$.

Case (II). By the induction hypothesis,

$$\nu_{w_i}(A) \neq 0$$

$$\nu_{w_i}(B) = 0$$

Hence $\nu_{w_i}(A \to B) = 0$. □

17. Proof of (a) for $\leftrightarrow$.

Suppose $A \leftrightarrow B, i\oplus$ is on b. Thus $A \to B, i\oplus$ and $B \to A, i\oplus$ are on b, and so by the above proof (15)

$$\nu_{w_i}(A \to B) = 1$$

$$\nu_{w_i}(B \to A) = 1$$

Hence $\nu_{w_i}(A \leftrightarrow B) = 1$. □

18. Proof of (b) for $\leftrightarrow$.

Suppose $A \leftrightarrow B, i\oplus$ is on b. Thus either:

(I) $\neg(A \to B), i\oplus$ is on b, or

(II) $\neg(B \to A), i\oplus$ is on b

In either case, by the above proof (16), $\nu_{w_i}(A \leftrightarrow B) = 0$. □

Given that (a) and (b) hold for the binary connectives, we show that (d) and (e) must also hold

19. Proof of (d) for each $\odot \in \mathcal{C}_b$.

    The rule for $A \odot B, i\ominus$ for each $\odot \in \mathcal{C}_b$ produces exactly one node of the form $\neg(A \odot B), i\oplus$.

    Since (b) holds for $\odot$, $\nu_{w_i}(A \odot B) = 0$. Hence $\nu_{w_i}(A \odot B) \neq 1$. □

20. Proof of (e) for each $\odot \in \mathcal{C}_b$.

    The rule for $\neg(A \odot B), i\ominus$ for each $\odot \in \mathcal{C}_b$ produces exactly one node of the form $A \odot B, i\oplus$.

    Since (a) holds for $\odot$, $\nu_{w_i}(A \odot B) = 1$. Hence $\nu_{w_i}(A \odot B) \neq 0$. □

What are left are the unary connectives. For $\Box$ and $\Diamond$, we proceed in a fashion similar to the binary connectives. Our induction hypothesis is that Lemma 4.13. holds for sentence A. First we show that conditions (a) and (b) hold for $\Box$ and $\Diamond$.

21. Proof of (a) for $\Box$.

    Suppose $\Box A, i\oplus$ is on b. Since b is complete, for each $j \in ß_b(i)$, $A, j\oplus$ is on b, and by the induction hypothesis, $\nu_{w_j}(A) = 1$. By Definition 4.12.(i), it follows that $ß_b(i) = \mathcal{R}(w_i)$, and thus $\nu_{w_i}(\Box A) = 1$. □

22. Proof of (b) for $\Box$.

    Suppose $\neg\Box A, i\oplus$ is on b. Since b is complete, $i\mathcal{R}j$ and $A, j\ominus$ are on b. Thus by the induction hypothesis, $\nu_{w_j}(A) \neq 1$, and by the Definition 4.12.(i), $w_i\mathcal{R}w_j$. Hence by the semantics of $\Box$, $\nu_{w_i}(\Box A) = 0$. $\square$

23. Proof of (a) for $\Diamond$.

    Suppose $\Diamond A, i\oplus$ is on b. Since b is complete, $A, j\oplus$ and $i\mathcal{R}j$ are on b. By Definition 4.12.(i), $w_i\mathcal{R}w_j$, and by the induction hypothesis $\nu_{w_j}(A) = 1$, and thus it follows that $\nu_{w_i}(\Diamond A) = 1$. $\square$

24. Proof of (b) for $\Diamond$.

    Suppose $\neg\Diamond A, i\oplus$ is on b. Thus for each $j \in ß_b(i)$, $A, j\ominus$ is on b, and by the induction hypothesis, $\nu_{w_j}(A) \neq 1$. By Definition 4.12.(i), $ß_b(i) = \mathcal{R}(w_i)$, and thus $\nu_{w_i}(\Diamond A) = 0$. $\square$

The proofs for conditions (d) and (e) for $\Box$ and $\Diamond$ are similar to those for the binary connectives, and they are left as an exercise. This leaves $\neg$. We proceed with each condition individually.

25. Proof of (a) for $\neg$.

    Suppose $\neg A, i\oplus$ is on b. By the induction hypothesis, $\nu_{w_i}(A) = 0$, and thus $\nu_{w_i}(\neg A) = 1$. $\square$

26. Proof of (b) for $\neg$.

    Suppose $\neg\neg A, i\oplus$ is on b. Hence, $A, i\oplus$ is on b, and so by the induction hypothesis $\nu_{w_i}(A) = 1$. Thus $\nu_{w_i}(\neg A) = 0$. $\square$

27. Proof of (c) for $\neg$.

    Suppose $\neg A, i\ominus$ and $\neg\neg A, i\ominus$ are on b. Since b is complete, $A, i\ominus$ is also on b, and so by the induction hypothesis $\nu_{w_i}(A) = 1/2$. Hence, $\nu_{w_i}(\neg A) = 1/2$. $\square$

28. Proof of (d) for $\neg$.

    Suppose $\neg A, i\ominus$ is on b. By the induction hypothesis, $\nu_{w_i}(A) \neq 0$, and so $\nu_{w_i}(A) \in \{1/2, 1\}$. Thus, $\nu_{w_i}(\neg A) \in \{0, 1/2\}$, and hence $\nu_{w_i}(\neg A) \neq 1$. $\square$

29. Proof of (e) for $\neg$.

    Suppose $\neg\neg A, i\ominus$ is on b. Thus, $A, i\ominus$ is on b, and so by the induction hypothesis, $\nu_{w_i}(A) \neq 1$. Whence $\nu_{w_i}(A) \in \{0, 1/2\}$, thence $\nu_{w_i}(\neg A) \in \{1/2, 1\}$. In either case, $\nu_{w_i}(\neg A) \neq 0$. $\square$

Completeness Theorem

THEOREM 4.14. *If* $\Sigma \Vdash A$ *then* $\Sigma \vdash A$.

Proof. We prove the contrapositive. Assume for conditional proof that $\Sigma \nvdash A$. Thus there is a completed open tableau with an initial list of $B, 0\oplus$ for all $B \in \Sigma$ and $A, 0\ominus$. Let $\mathfrak{M}$ be a model induced by b. By Lemma 4.13., $\nu_{w_0}(B) = 1$ for each $B \in \Sigma$, and $\nu_{w_0}(A) \neq 1$. Therefore, by Definition 2.4., $\Sigma \nVdash A$. □

Chapter 5

Further Issues

This chapter briefly surveys a few further expansions of the GO system, as well as possible further applications. §5.1 gives a full first-order semantics. §5.2 considers the GO system in light of the rejection of mereological atomism. For this, §5.3 develops a 4-valued system.

5.1 Quantification

Here we expand our propositional language GO to a full first-order language with quantification.

We add to the syntax:

» A set of predicate symbols of any arity $n \in \mathbb{N}$ where $n > 0$,
$\mathcal{P} = \{F_1^n, F_2^n, \ldots, G_1^n, G_2^n, \ldots, H_1^n, H_2^n, \ldots\}$

» A set of constants $\mathcal{O} = \{a_1, b_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots\}$

We also add special k-constants to $\mathcal{O}$ to ensure each object in the domain has a name.

» $k_d \in \mathcal{O}$, for each $d \in \mathcal{D}$. ($\mathcal{D}$ here is our domain of objects, defined below.)

» A set of variables $\mathcal{B} = \{x_1, x_2, \dots, y_1, y_2, \dots, z_1, z_2, \dots\}$

» New connectives $\forall, \exists \in \mathcal{C}$

» New atomic formulas:

(a) If $t \in \mathcal{O}$ or $t \in \mathcal{B}$, then t is a term.
(b) If $t_1, \dots, t_n$ are terms and F^n an n-ary predicate then $F^n t_1, \dots, t_n \in \mathcal{A}$
(c) If A is a formula, then $\forall x A$ and $\exists x A \in \mathcal{S}$

A model $\mathfrak{M}$ comprises the following:

» A non-empty domain of objects, $\mathcal{D}$

» Our interpretation function ν which we augment in the following way:

(a) For each $a \in \mathcal{O}$, for some $d \in \mathcal{D}$, $\nu(a) = d$.
(b) The following constraint applies: For each k-constant $k_d \in \mathcal{O}$, $\nu(k_d) = d$.

For each n-place predicate P^n:

(c) $\nu^+(P^n) \subseteq \mathcal{D}^n$
(d) $\nu^-(P^n) \subseteq \mathcal{D}^n$
(e) The following constraint applies: $\nu^+(P^n) \cup \nu^-(P^n) = \varnothing$.

» Truth conditions for closed atomics:

(a) $\nu(P^n a_1, \dots, a_n) = 1$ iff $\langle \nu(a_1), \dots, \nu(a_n) \rangle \in \nu^+(P^n)$
(b) $\nu(P^n a_1, \dots, a_n) = 0$ iff $\langle \nu(a_1), \dots, \nu(a_n) \rangle \in \nu^-(P^n)$
(c) $\nu(P^n a_1, \dots, a_n) = \frac{1}{2}$ iff $\langle \nu(a_1), \dots, \nu(a_n) \rangle \notin \nu^+(P^n) \cup \nu^-(P^n)$

» Truth conditions for quantifiers[1]:

[1] $A(c/x)$ is the formula resulting from replacing c for each free occurrence of x in A.

(a) $\nu(\forall xA) = 1$ iff for all $d \in \mathcal{D}$, $\nu(A(k_d/x)) = 1$, else $\nu(\forall xA) = 0$

(b) $\nu(\exists xA) = 1$ iff for some $d \in \mathcal{D}$, $\nu(A(k_d/x)) = 1$, else $\nu(\exists xA) = 0$

» Our conditions for $\vee, \wedge, \neg$ are the same as before. Satisfaction and semantic consequence are defined similarly.

Similar to the treatment of conjunction and disjunction, we have it that no quantified sentence nor its negation gets value ½. This gives us the following lemmas:

Lemma 1.15. *If* $\nu(\forall xA) \neq 1$ *then* $\nu(\forall xA) = 0$.

Lemma 1.16. *If* $\nu(\neg\forall xA) \neq 1$ *then* $\nu(\neg\forall xA) = 0$.

Lemma 1.17. *If* $\nu(\exists xA) \neq 1$ *then* $\nu(\exists xA) = 0$.

Lemma 1.18. *If* $\nu(\neg\exists xA) \neq 1$ *then* $\nu(\neg\exists xA) = 0$.

Many classical inferences hold, for instance:

$$\forall x(Px \supset Qx), \forall x(Qx \supset Sx) \vdash \forall x(Px \supset Sx)$$

$$B \vdash \forall xA \supset \exists xA$$

Proof. By reductio. Suppose $\nu(\forall xA \supset \exists xA) \neq 1$. By Lemma 1.15., $\nu(\forall xA \supset \exists xA) = 0$. Thus $\nu(\neg\forall xA \vee \exists xA) = 0$. Hence $\nu(\neg\forall xA) \neq 1$ and $\nu(\exists xA) \neq 1$. By Lemmas 1.16. and 1.17., $\nu(\neg\forall xA) = 0 = \nu(\exists xA)$. Thus $\nu(\forall xA) = 1$. So, for all $d \in \mathcal{D}$, $\nu(A(k_d/x)) = 1$. But since, $\nu(\exists xA) = 0$, there is no $d \in \mathcal{D}$ such that $\nu(A(k_d/x)) = 1$. Since our domain is non-empty, a contradiction follows. Thus for all ν, $\nu(\forall xA \supset \exists xA) = 1$. Thus $B \vdash \forall xA \supset \exists xA$. □

The following hold, where C is any closed formula.

$$\forall xC \vdash C$$

Proof. Suppose $\nu(\forall xC) = 1$. Thus $\nu(C(k_d/x)) = 1$ for any $d \in \mathcal{D}$. Since C is closed, there are no unbound occurrences of x. Thus $\nu(C(k_d/x)) = \nu(C)$ for all $d \in \mathcal{D}$. Thus $\nu(C) = 1 = \nu(\forall xC)$. A similar proof follows for the others. □

$$C \vdash \forall xC$$

$$\exists xC \vdash C$$

$$C \vdash \exists xC$$

Restricted generality inferences such as the following hold:

$$Fa, \forall x(Fx \supset Gx) \vdash Ga$$

We do not have the classical bi-entailment of the quantifiers:

$$\neg\exists x\neg A \nvdash \forall xA$$

We give a countermodel as follows. Take F^1x for A. Let:

$$\mathcal{D} = \{\alpha, \beta\}$$

$$\nu^+(F^1) = \{\alpha\}$$

$$\nu^-(F^1) = \varnothing$$

Thus:

$$\begin{array}{ll}
\nu(\neg F^1_{k_\alpha}) = 0 & \nu(F^1_{k_\alpha}) = 1 \\
\nu(\neg F^1_{k_\beta}) = \tfrac{1}{2} & \nu(F^1_{k_\beta}) = \tfrac{1}{2} \\
\nu(\exists x\neg F^1x) = 0 & \nu(\forall xF^1x) = 0 \\
\nu(\neg\exists x\neg F^1x) = 1 &
\end{array}$$

This is to be expected, given our treatment of the quantifiers as generalized conjunction and disjunction, with conjunction and disjunction interpreted analogously to our $\wedge$ and $\vee$. We should expect these to result from features of GO analogous to those features that brought about a failure of some classical DeMorgan transformations. We might also expect to maintain the other direction:

$$\forall x A \vdash \neg\exists x \neg A$$

Proof. Assume $\nu(\forall x A) = 1$. Thus for all $d \in \mathcal{D}$, $\nu(A(k_d/x)) = 1$. Hence for all $d \in \mathcal{D}$, $\nu(\neg A(k_d/x)) = 0$. Thus $\nu(\exists x \neg A) = 0$, and hence, $\nu(\neg\exists x \neg A) = 1$. □

It follows by a similar counterexample that we do not have straightforward interdefinability of the quantifiers in the standard way:

It is not that case that for any sentence A, $\nu(\forall x A) = \nu(\neg\exists x \neg A)$.

5.2 What if there are no atoms?

Throughout we have assumed that some form of atomism is necessary for the interpretation of GO; specifically, that there is a fact of the matter as to which sentences or proposition are genuinely atomic. Given the motivation that logical principles can be subject to minimal revision for reasons that are 'extra-logical' (metaphysical, or perhaps physical reasons), one wonders what becomes of the system if, for some reason or other, one abandons the commitment to atomism.

It might certainly be the case that the world does consist of mereological atoms; that there is a bottom level at which the tiniest particles that compose everything in world themselves contain no proper parts, and cannot themselves be divided. However, this is a substantive assumption about the true makeup of the physical world, and it is by no means certain. Schaffer (2003), for instance, presents a compelling case for thinking that the epistemic possibility of infinite descent is not only an open question, but that there is indeed no evidence for the existence of a ‘fundamental’ level.

Toward the goal of making his account suitable for a broad range of scientific theories, Armstrong considers the doxastic possibility that the world contains no simple individuals, no ‘genuine atoms’. If this were the case, it appears on the surface that combinatorialism is doomed, since it explicitly assumes that there are genuinely atomic individuals and properties from which the combinatorial principle constructs possible worlds.

A potential solution that Armstrong considers is that, even if the world contains no genuine atoms, it still might contain *relative* atoms. The idea here is that any mereological level can be taken as ‘relatively’ atomic.

An individual at a mereological level l is an l-relative atom if it is ‘wholly distinct’ from every other individual at level l. An individual is wholly distinct from another if the two share no individual as a part. A similar notion of distinctness applies for universals, where, as before in Chapter 3, instead of ‘part’ in the mereological sense, a universal is wholly distinct from another if the two have no ‘constituent’ in common.

If the world does contain genuine atoms, then there is a fundamental level l_0 such that no individual at l_0 has a proper part. In this case, an l_0-relative atom is a genuine atom. If, however, there is no fundamental level, Armstrong's suggestion is that, at any given level l, the l-relative atoms generates a set of possible worlds, and at each lower level, a new set of worlds is revealed. If the world is infinitely divisible, then this process would continue ad infinitum.

This proposal raises an immediate question: Why atomism? If there aren't any genuine atoms, then what reason is there to maintain an atomistic framework? One answer is that a combinatorial framework in the context of atomlessness preserves the structure reflected in the principle of Hume Distinctness. Even if two distinct objects are infinitely divisible, they remain logically distinct, and so the mutual compatibility of the 'relative' recombinations is guaranteed by the condition that the individuals and universals be wholly distinct.

Perhaps the most straightforward picture of an infinite descent is one that is stratified into well-organized levels. That is, the properties of entities at each level supervene on properties of entities at the level below it, whose properties are determined by those of the things below it, and so forth. Without speaking of supervenience, we might instead say that, on this picture, the *logic of things* remains the same throughout all levels of descent.

One notable feature of this picture of infinite descent is that there seems to be no non-arbitrary way to distinguish any level as more or less "fundamental" as another. As Schaffer argues, for any supposed ontologically privileged cutoff level l_n, since l_{n-1} provides a supervenience base for l and entities at l are composed of those at l_{n-1}, l_{n-1} provides a supervenience base for all levels above l (2003, p. 507). This poses a problem for philosophical theses such as physicalism, which seem to presuppose that there is a fundamental level, and upon whose proponents it is incumbent to identify such a privileged base.

In the GO system, determinacy arises as a matter of form. While an atomic sentence p may be indeterminate, each of its combinations (e.g., $p \wedge q$) is determinate. The result is a strict logical line between literal[2] and complex sentences. One can think of this general idea of a "logical line" as a result of determinacy arising at a particular level of logical complexity, and remaining at all higher levels. For the GO logic, this level is directly above the literal level.

However, one might hold the view that, while determinacy arises as a matter of form, it does so at a higher level of complexity. Given an indeterminate atomic sentence p, then, one of its combinations $p \wedge q$ is also indeterminate, but at some level of complexity, say $(p \wedge q) \wedge r$, determinacy arises. For disjunction this might be either value 1 or 0, while for conjunction it will inevitably lead to 0.

What sort of philosophical view might such a logic interpret? A natural focus here is a mereological view according to which the world is stratified into levels. Thus far, the GO logic restricts indeterminacy to the lowest level, but prevents these *gaps* from percolating upward, such that the determinacy of any combination is unaffected. One might, however, hold that, though gaps do not percolate up *all the way*, indeterminacy does continue to some level higher than atomics.

A simplified example of such a view holds that indeterminacy occurs at the quantum level, and this in turn allows indeterminacy to occur all the way up to, say, the chemical level, but at all levels beyond (the biological, sociological, etc.) everything is determinate. There is an intuitive appeal to this idea, reflected in the commonsense view that at the macro level, things

[2]Literals are standardly defined as atomics or negated atomics. Here we mean "literals" to include any sentence featuring just an atomic sentence and any number of negation symbols, since if p is indeterminate, so too are $\neg p$, $\neg\neg p$, $\neg\neg\neg p$, and so forth.

behave determinately, even if what goes on at the most microscopic level is indeterminate; that the logic of all but perhaps the tiniest esoteric particles is bivalent and is not held hostage to quantum considerations.

There are several ways one might expand the GO logic to model the continuation of indeterminacy to higher levels. Here we briefly consider an expansion of GO to a 4-valued logic, GO^4, which allows for indeterminacy at one level higher than literals.

5.3 GO^4

Keeping the syntax of GO, the semantics for GO^4 expands our set of values with an additional indeterminate (i.e. undesignated) value.

$$\mathcal{V} = \left\{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\right\}$$

Logical consequence, as before, is defined in the usual way. The truth tables expand to accommodate the additional semantic value.

$\neg$	
0	1
$\frac{1}{3}$	$\frac{1}{3}$
$\frac{2}{3}$	$\frac{2}{3}$
1	0

$\wedge$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	0	0	0	0
$\frac{1}{3}$	0	0	0	0
$\frac{2}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	0	$\frac{1}{3}$	1

$\vee$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	0	0	$\frac{1}{3}$	1
$\frac{1}{3}$	0	0	$\frac{1}{3}$	1
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1
1	1	1	1	1

Using our functions g and c, we keep our definitions for $\wedge$ and $\vee$ the same.

$$g(x) = \mathsf{min}\,\{x, 1 - x\}$$

$$c(x) = x - g(x)$$

$$\nu(A \wedge B) = \mathsf{min}\,\{c(\nu(A)), c(\nu(B))\}$$

$$\nu(A \vee B) = \mathsf{max}\,\{c(\nu(A)), c(\nu(B))\}$$

Negation behaves similar to before, toggling classical values while holding indeterminate values fixed.

The binary connectives of GO^4 "push" toward classical values, though less immediately than the GO connectives. As before, LEM fails, though now with an additional counterexample which counts the disjunction as indeterminate:

$$B \nvdash A \vee \neg A$$

If $\nu(A) = ⅓$, then as expected $\nu(A \vee \neg A) = 0$. When $\nu(A) = ⅔$ however, the disjunction is indeterminate, although less than $\nu(A)$, as $\nu(A \vee \neg A) = ⅓$. As a result, at the next level of complexity, LEM still fails, and thus we lose GO's restricted versions of LEM:

$$C \nvdash (A \vee B) \vee \neg(A \vee B)$$

$$B \nvdash A \vee \neg(A \vee A)$$

The counterexamples here, however, assign the respective sentences value 0, and so, as one would expect, determinacy now arises. As a result, this allows yet weaker restricted versions of LEM:

$$C \vdash ((A \vee B) \vee \neg(A \vee B)) \vee \neg((A \vee B) \vee \neg(A \vee B))$$

$$B \vdash ((A \vee A) \vee \neg((A \vee A) \vee (A \vee A)))$$

What of DeMorgan transformations? The results for standard DeMorgan inferences remain the same:

$$\neg A \wedge \neg B \vdash \neg(A \vee B)$$
$$\neg A \vee \neg B \vdash \neg(A \wedge B)$$
$$\neg(A \vee B) \nvdash \neg A \wedge \neg B$$
$$\neg(A \wedge B) \nvdash \neg A \vee \neg B$$

As before, the failure of DeMorgan blocks the inference from the negation of an LEM instance to the failure of LNC:

$$\neg(A \vee B) \nvdash \neg A \wedge \neg B$$

As one would expect, though, the restricted GO versions of distributive DeMorgan also fail:

$$\neg((A \vee B) \vee (C \vee D)) \nvdash \neg(A \vee B) \wedge \neg(C \vee D)$$

$$\neg((A \vee B) \wedge (C \vee D)) \nvdash \neg(A \vee B) \vee \neg(C \vee D)$$

Replacing these are weaker versions:

$$\neg((A \vee B) \vee (C \vee D) \vee (E \vee F) \vee (G \vee H)) \vdash$$
$$\neg((A \vee B) \vee (C \vee D)) \wedge \neg((E \vee F) \vee (G \vee H))$$

$$\neg(((A \vee B) \vee (C \vee D)) \wedge ((E \vee F) \vee (G \vee H))) \vdash$$
$$\neg((A \vee B) \vee (C \vee D)) \vee \neg((E \vee F) \vee (G \vee H))$$

Though LNC still holds, the allowance for gaps in conjunctions brings the failure of GO's stronger form of LNC:

$$B \nvdash \neg(A \wedge \neg A)$$

Counterexamples to this inference arise when $\nu(A) = \nicefrac{2}{3}$. Our replacement in this case is only slightly weaker:

$$B \vdash \neg((A \wedge A) \wedge \neg A)$$

We have, for every sentence A, a sentence that is true iff A is gappy (i.e. iff $\nu(A) \in \{\nicefrac{1}{3}, \nicefrac{2}{3}\}$):

$$\circ A := \neg((A \vee A) \vee \neg A)$$

We also have one that is true iff $\nu(A) = \nicefrac{1}{3}$:

$$\text{⁊}A := \neg(A \vee \neg A)$$

Additionally there is a sentence that is true iff $\nu(A) = \nicefrac{2}{3}$:

$$\ddot{\text{⁊}}A := \neg((A \vee \neg A) \vee \neg(A \vee \neg A))$$

This expressive power does come at a slight cost, though, as we lose the interdefinability of $\wedge$ and $\vee$, and so take each as primitive. However, the advantages of the ability to isolate each value in the object language are most clear when it comes to conditionals.

Observe that under the standard definition of the material conditional $A \supset$

B as $\neg A \vee B$, its behavior is slightly atypical:

$\supset$	0	⅓	⅔	1
0	1	1	1	1
⅓	0	0	⅓	1
⅔	⅓	⅓	⅓	1
1	0	0	⅓	1

As such, it may be unintuitive to consider this as a conditional. Consider the case where $\nu(A) = ⅓$ and $\nu(B) = 0$, and so $\nu(A \supset B) = 0$. This in itself is no surprise, as the situation is similar with values .5 and 0 in GO. What is counterintuitive is when the antecedent is strengthened to ⅔, the value of $A \supset B$ is also strengthened.

Note that modus ponens, modus tollens and transitivity hold:

$$A, A \supset B \vdash B$$

$$\neg B, A \supset B \vdash \neg A$$

$$A \supset B, B \supset C \vdash A \supset C$$

Although the anomalous $\supset$ does not seem to affect these important inferences, one might be inclined to look for a solution. From the outset, it is worth noting that, depending on the particular philosophical view with which one interprets this logic, it may be misleading to think of value ⅔ as *stronger* than ⅓ in any important sense.

A possible alteration is to return to GO's original semantic clause for $\neg$, where $\nu(\neg A) = 1 - \nu(A)$. The resulting tables would be:

$\neg$	
0	1
⅓	⅔
⅔	⅓
1	0

$\supset$	0	⅓	⅔	1
0	1	1	1	1
⅓	⅓	⅓	⅓	1
⅔	0	0	⅓	1
1	0	0	⅓	1

Either way, however, identity ($A \supset A$) and thereby equivalence ($A \equiv A$) fail, and so it remains a stretch to consider $\supset$ a conditional.

Most importantly, however, given the ability to isolate each value, we can define a range of stronger, suitable conditionals, for which identity and equivalence hold, in the following ways:

$$A \rightarrow B := (A \supset B) \vee ((\dot{\urcorner} A \wedge \dot{\urcorner} B) \vee (\ddot{\urcorner} A \wedge \ddot{\urcorner} B))$$

$$A \Rightarrow B := (A \supset B) \vee (\circ A \wedge \ddot{\urcorner} B)$$

$$A \Rrightarrow B := (A \supset B) \vee (\circ A \wedge \circ B)$$

The resulting tables are as follows:

$\rightarrow$	0	⅓	⅔	1
0	1	1	1	1
⅓	0	1	0	1
⅔	0	0	1	1
1	0	0	0	1

$\Rightarrow$	0	⅓	⅔	1
0	1	1	1	1
⅓	0	1	1	1
⅔	0	0	1	1
1	0	0	0	1

$\Rrightarrow$	0	⅓	⅔	1
0	1	1	1	1
⅓	0	1	1	1
⅔	0	1	1	1
1	0	0	0	1

In general, we can define any truth table which assigns only values 0 or 1.

Schaffer's attack on physicalism relies on the inability of the physicalist to locate a non-arbitrary mereological level to serve as the fundamental base level. The defining thesis of physicalism is that the fundamental physical level is the privileged base whose existence and properties underwrites those of all the higher levels. If the picture of infinite descent is correct, however, then any level that the physicalist might choose will be underwritten by an infinite descent of lower levels. To privilege a higher level over a lower one, or to maintain that no level is privileged, is to abandon the physicalist program.

Schaffer notes that there is one route the fundamentalist could take to block his conclusion. The resulting view holds that the fundamental level, though not mereologically simple, provides a supervenience base for all levels above it. The parts at this 'fundamental' level, though they admit of decomposition, decompose is a 'boring' sense, in that the properties of their parts supervene on the properties of the whole, and so on down. Schaffer notes that this sort of fundamentality is evidentially in the best shape, and "metaphysically speaking, more palatable" (Schaffer 2003, p. 510).

Besides a picture of 'boring' descent, though, there is another way to take this general route to pick out a non-arbitrary base level. The interpretation for a GO system affords a logical distinction between the fundamental level, which is determinate, and the 'sub-fundamental' levels, which allow for indeterminacy. The three-valued GO models an even cutoff at the fundamental level, while GO^4 models a stepped cutoff. Infinitely-valued cases are left for future research. Apart from supervenience, in a mereological application, the intermediate value models an indeterminate composition relation; similarly with sense-data and indeterminate representation.

Appendix: Tableaux Examples

This appendix provides proofs using the tableaux system developed in Chapter 4. Many of the inferences referenced in Chapters 2 and 3 are included.

(1) $B \nvdash A \vee \neg A$

$$
\begin{array}{c}
B, 0\oplus \\
\boxed{A \vee \neg A, 0\ominus} \\
0\mathcal{R}0 \\
\boxed{\neg(A \vee \neg A), 0\oplus} \\
A, 0\ominus \\
\neg A, 0\ominus
\end{array}
$$

Counter Model:

$\mathcal{W} = \{w_0\}$
$\mathcal{R} = \{\langle w_0, w_0\rangle\}$
$\nu_{w_0}(B) = 1$
$\nu_{w_0}(A) = .5$

(2) $C \vdash (A \vee B) \vee \neg(A \vee B)$

$$
\begin{array}{c}
C, 0\oplus \\
\boxed{(A \vee B) \vee \neg(A \vee B), 0\ominus} \\
0\mathcal{R}0 \\
\boxed{\neg((A \vee B) \vee \neg(A \vee B)), 0\oplus} \\
\boxed{A \vee B, 0\ominus} \\
\neg(A \vee B), 0\ominus \\
\neg(A \vee B), 0\oplus \\
\otimes
\end{array}
$$

(3) $A \wedge \neg A \vdash B$

$$\begin{array}{c} \boxed{A \wedge \neg A, 0\oplus} \\ B, 0\ominus \\ 0\mathcal{R}0 \\ A, 0\oplus \\ \neg A, 0\oplus \\ \otimes \end{array}$$

(4) $B \vdash \neg(A \wedge \neg A)$

$$\begin{array}{c} B, 0\oplus \\ \boxed{\neg(A \wedge \neg A), 0\ominus} \\ 0\mathcal{R}0 \\ \boxed{A \wedge \neg A, 0\oplus} \\ A, 0\oplus \\ \neg A, 0\oplus \\ \otimes \end{array}$$

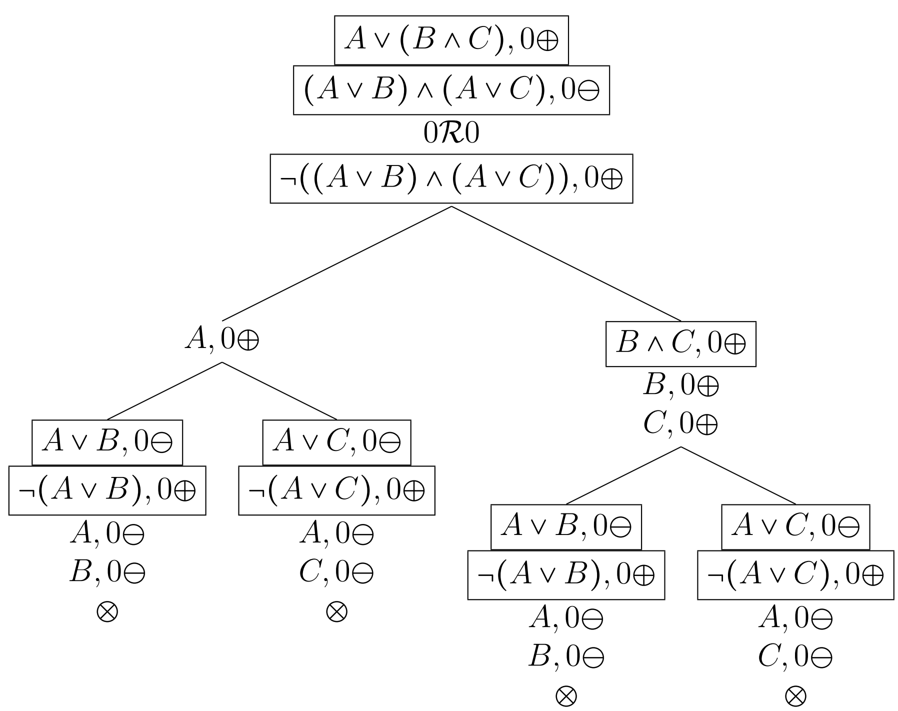
(5) $\neg(A \vee \neg A) \nvdash A \wedge \neg A$

$$\begin{array}{c} \boxed{\neg(A \vee \neg A), 0\oplus} \\ \boxed{A \wedge \neg A, 0\ominus} \\ 0\mathcal{R}0 \\ \boxed{\neg(A \wedge \neg A), 0\oplus} \\ A, 0\ominus \\ \neg A, 0\ominus \\ \swarrow \qquad \searrow \\ A, 0\ominus \qquad \neg A, 0\ominus \end{array}$$

Counter Model:

$\mathcal{W} = \{w_0\}$
$\mathcal{R} = \{\langle w_0, w_0 \rangle\}$
$\nu_{w_0}(A) = .5$

(15a) $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$



(15b) $(A \vee B) \wedge (A \vee C) \vdash A \vee (B \wedge C)$

$$\boxed{(A \vee B) \wedge (A \vee C), 0\oplus}$$
$$\boxed{A \vee (B \wedge C), 0\ominus}$$
$$0\mathcal{R}0$$
$$\boxed{A \vee B, 0\oplus}$$
$$\boxed{A \vee C, 0\oplus}$$
$$\boxed{\neg(A \vee (B \wedge C)), 0\oplus}$$
$$A, 0\ominus$$
$$\boxed{B \wedge C, 0\ominus}$$
$$\boxed{\neg(B \wedge C), 0\oplus}$$
$$\begin{array}{cc} A, 0\oplus & B, 0\oplus \\ \otimes & \begin{array}{cc} A, 0\oplus & C, 0\oplus \\ \otimes & \begin{array}{cc} B, 0\ominus & C, 0\ominus \\ \otimes & \otimes \end{array} \end{array} \end{array}$$

(30a) $A \supset (A \supset B) \vdash A \supset B$

$$\boxed{A \supset (A \supset B), 0\oplus}$$
$$\boxed{A \supset B, 0\ominus}$$
$$0\mathcal{R}0$$
$$\boxed{\neg(A \supset B), 0\oplus}$$
$$\neg A, 0\ominus$$
$$B, 0\ominus$$
$$\begin{array}{cc} \neg A, 0\oplus & A \supset B, 0\oplus \\ \otimes & \otimes \end{array}$$

(30b) $A \supset B \vdash A \supset (A \supset B)$

$$\begin{array}{c}
A \supset B, 0\oplus \\
\boxed{A \supset (A \supset B), 0\ominus} \\
0\mathcal{R}0 \\
\boxed{\neg(A \supset (A \supset B)), 0\oplus} \\
\neg A, 0\ominus \\
A \supset B, 0\ominus \\
\otimes
\end{array}$$

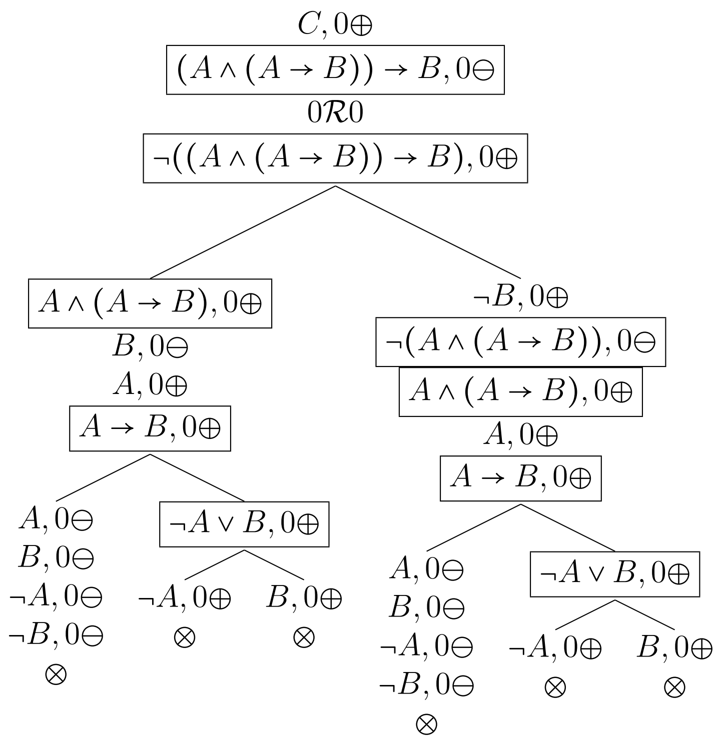
(42) $B \vdash A \to A$

$$\begin{array}{c}
B, 0\oplus \\
\boxed{A \to A, 0\ominus} \\
0\mathcal{R}0 \\
\boxed{\neg(A \to A), 0\oplus} \\
\begin{array}{cc}
A, 0\oplus & \neg A, 0\oplus \\
A, 0\ominus & \neg A, 0\ominus \\
\otimes & \otimes
\end{array}
\end{array}$$

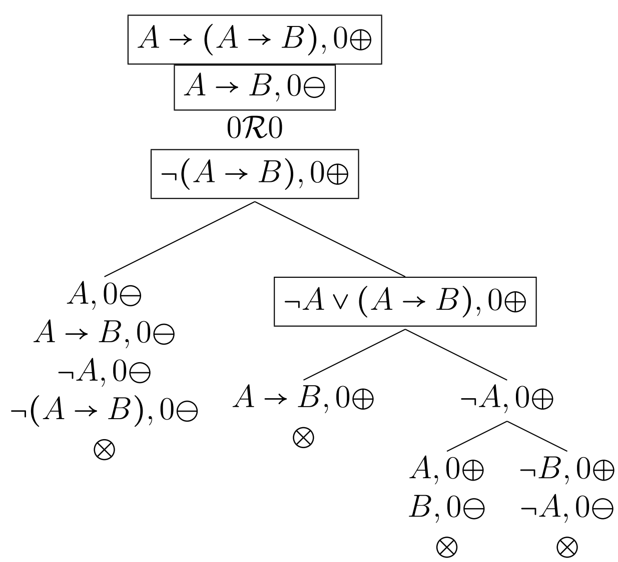
(43) $B \vdash (A \wedge A) \to A$

$$\begin{array}{c}
B, 0\oplus \\
\boxed{(A \wedge A) \to A, 0\ominus} \\
0\mathcal{R}0 \\
\boxed{\neg((A \wedge A) \to A), 0\oplus} \\
\begin{array}{cc}
\boxed{A \wedge A, 0\oplus} & \neg A, 0\oplus \\
A, 0\ominus & \boxed{\neg(A \wedge A), 0\ominus} \\
A, 0\oplus & \boxed{A \wedge A, 0\oplus} \\
\otimes & A, 0\oplus \\
 & \otimes
\end{array}
\end{array}$$

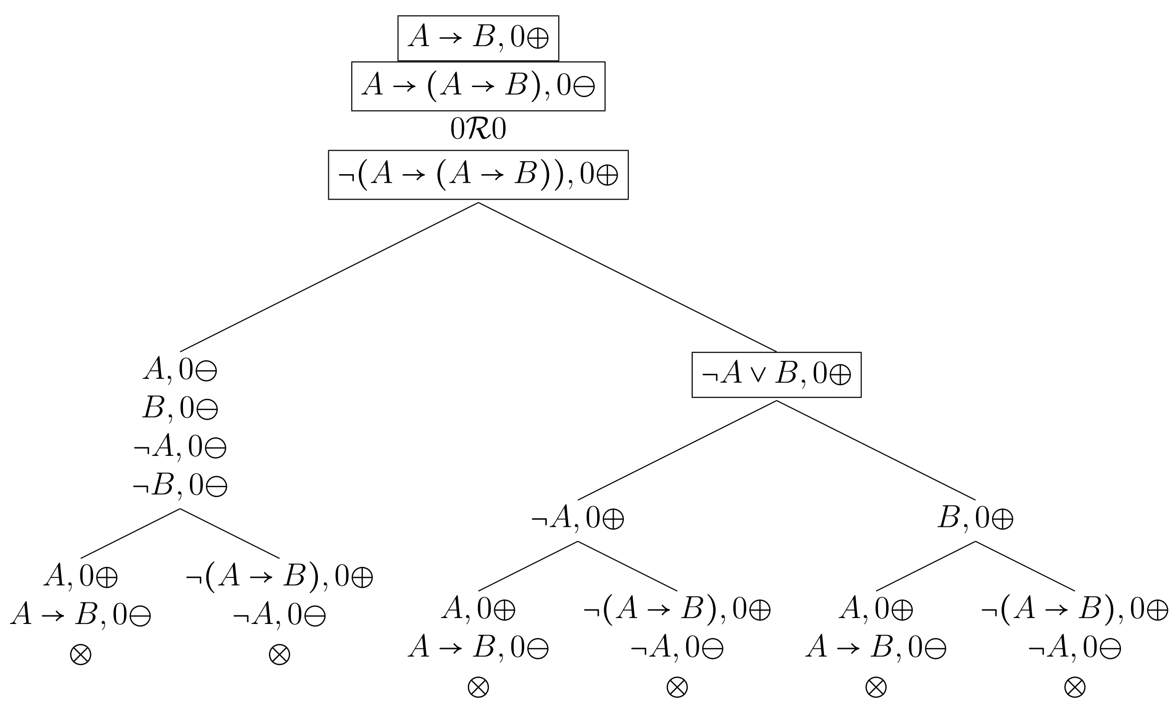
(45) $C \vdash (A \wedge (A \to B)) \to B$



(49a) $A \to (A \to B) \vdash A \to B$



(49b) $A \to B \vdash A \to (A \to B)$



(61) $\Box A \vdash \neg\Diamond\neg A$

$$\Box A, 0\oplus$$
$$\boxed{\neg\Diamond\neg A, 0\ominus}$$
$$0\mathcal{R}0$$
$$A, 0\oplus$$
$$\boxed{\Diamond\neg A, 0\oplus}$$
$$\neg A, 1\oplus$$
$$0\mathcal{R}1$$
$$1\mathcal{R}1$$
$$A, 1\oplus$$
$$\otimes$$

(62) $\Diamond A \vdash \neg\Box\neg A$

$$\boxed{\Diamond A, 0\oplus}$$
$$\boxed{\neg\Box\neg A, 0\ominus}$$
$$0\mathcal{R}0$$
$$\Box\neg A, 0\oplus$$
$$\neg A, 0\oplus$$
$$A, 1\oplus$$
$$0\mathcal{R}1$$
$$1\mathcal{R}1$$
$$\neg A, 1\oplus$$
$$\otimes$$

(63) $\neg\Diamond\neg A \nvdash \Box A$

$$\neg\Diamond\neg A, 0\oplus$$
$$\boxed{\Box A, 0\ominus}$$
$$0\mathcal{R}0$$
$$\boxed{\neg\Box A, 0\oplus}$$
$$A, 1\ominus$$
$$0\mathcal{R}1$$
$$1\mathcal{R}1$$
$$\neg A, 1\ominus$$
$$\neg A, 0\ominus$$

Counter Model:

$\mathcal{W} = \{w_0, w_1\}$
$\mathcal{R} = \{\langle w_0, w_0\rangle, \langle w_0, w_1\rangle, \langle w_1, w_1\rangle\}$
$\nu_{w_1}(A) = .5$

(74) $B \vdash \Box A \to \Box\Box A$

$$\begin{array}{c}
B, 0\oplus \\
\boxed{\Box A \to \Box\Box A, 0\ominus} \\
0\mathcal{R}0 \\
\boxed{\neg(\Box A \to \Box\Box A), 0\oplus}
\end{array}$$

$$\begin{array}{cc}
\Box A, 0\oplus & \boxed{\neg\Box\Box A, 0\oplus} \\
\boxed{\Box\Box A, 0\ominus} & \boxed{\neg\Box A, 0\ominus} \\
A, 0\oplus & \boxed{\Box A, 1\ominus} \\
\boxed{\neg\Box\Box A, 0\oplus} & 0\mathcal{R}1 \\
\boxed{\Box A, 1\ominus} & 1\mathcal{R}1 \\
0\mathcal{R}1 & \boxed{\neg\Box A, 1\oplus} \\
1\mathcal{R}1 & A, 2\ominus \\
A, 1\oplus & 1\mathcal{R}2 \\
\boxed{\neg\Box A, 1\oplus} & 2\mathcal{R}2 \\
A, 2\ominus & 0\mathcal{R}2 \\
1\mathcal{R}2 & \Box A, 0\oplus \\
2\mathcal{R}2 & A, 2\oplus \\
0\mathcal{R}2 & \otimes \\
A, 2\oplus & \\
\otimes &
\end{array}$$

(75) $\Diamond A \nvdash \Box\Diamond A$

$$\begin{array}{c}
\boxed{\Diamond A, 0\oplus} \\
\boxed{\Box\Diamond A, 0\ominus} \\
0\mathcal{R}0 \\
\boxed{\neg\Box\Diamond A, 0\oplus} \\
\boxed{\Diamond A, 1\ominus} \\
0\mathcal{R}1 \\
1\mathcal{R}1 \\
A, 2\oplus \\
0\mathcal{R}2 \\
2\mathcal{R}2 \\
\neg\Diamond A, 1\oplus \\
A, 1\ominus
\end{array}$$

Counter Model:

$\mathcal{W} = \{w_0, w_1, w_2\}$

$\mathcal{R} = \{\langle w_0, w_0\rangle, \langle w_0, w_1\rangle, \langle w_1, w_1\rangle, \langle w_0, w_2\rangle, \langle w_2, w_2\rangle\}$

$\nu_{w_2}(A) = 1$

$\nu_{w_1}(A) = 0$

(78) $\Diamond(A \to B) \vdash \Box A \to \Diamond B$

$$\boxed{\Diamond(A \to B), 0\oplus}$$
$$\boxed{\Box A \to \Diamond B, 0\ominus}$$
$$0\mathcal{R}0$$
$$\boxed{\neg(\Box A \to \Diamond B), 0\oplus}$$
$$\boxed{A \to B, 1\oplus}$$
$$0\mathcal{R}1$$
$$1\mathcal{R}1$$

- $A, 1\ominus$; $B, 1\ominus$; $\neg A, 1\ominus$; $\neg B, 1\ominus$
  - $\Box A, 0\oplus$; $\Diamond B, 0\ominus$; $A, 1\oplus$; $\otimes$
  - $\neg\Diamond B, 0\oplus$; $\boxed{\neg\Box A, 0\ominus}$; $\Box A, 0\oplus$; $A, 1\oplus$; $\otimes$
- $\boxed{\neg A \vee B, 1\oplus}$
  - $\neg A, 1\oplus$
    - $\Box A, 0\oplus$; $\Diamond B, 0\ominus$; $A, 1\oplus$; $\otimes$
    - $\neg\Diamond B, 0\oplus$; $\boxed{\neg\Box A, 0\ominus}$; $\Box A, 0\oplus$; $A, 1\oplus$; $\otimes$
  - $B, 1\oplus$
    - $\neg\Diamond B, 0\oplus$; $\boxed{\neg\Box A, 0\ominus}$; $\Box A, 0\oplus$; $A, 1\oplus$; $A, 0\oplus$; $B, 1\ominus$; $\otimes$
    - $\Box A, 0\oplus$; $\boxed{\Diamond B, 0\ominus}$; $A, 1\oplus$; $A, 0\oplus$; $\neg\Diamond B, 0\oplus$; $B, 1\ominus$; $\otimes$

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