Every Grothendieck topos T (over some site W) has a canonical geometric morphism $f^* \dashv f_* : T \to Set$ and $N = f^*\mathbb{N}$ is a natural number object of T for the the natural number object $\mathbb{N}$ of Set. The question is: is it possible to have $PNN \sim N$ or more generally can there be a mono $m : PN \to N$ in a Grothendieck topos ? We could reframe this question: what are the conditions that we need to impose on an elementary topos T so that we can generalize Cantor's diagonal argument ? Does Cantor's argument carry over to any Boolean topos ? Consider $N \sim f^*\mathbb{N}$ in the case of a category of sheaves over a topological space X. Now f^* (part of the canonical geometric morphism to Set) arises as the composition of the functor yielding the constant presheaf and the sheafication functor. f_* on the other hand corresponds to taking global sections. Thus there is an open subset $U \subset X$ such that $N(U) \sim \mathbb{N}$. So that if we had a mono $m : PN \to N$ then PN would be isomorphic to a subsheaf of N ([1][p.72]) and so taking sections we would have, for a suitable U, a set theoretic inclusion $PN(U) \to \mathbb{N}$. Now let's look at $PN(U) \sim \Omega^N(U)$. According to [1][pp. 97-98] we have that $\Omega^N(U)$ can be identified with the set of natural transformations i.e. morphisms of presheaves $Hom(N_{|U}, \Omega_{|U})$. But by our choice of U $N_{|U}$ is just the constant presheaf with stalk $\mathbb{N}$. So these natural transformations f correspond to families of maps $f_W : \mathbb{N} \to \{V : V \subseteq W\}$ satisfying the condition : if $W_1 \subset W_2$ then $f_{W_2}(n) \cap W_1 = f_{W_1}(n)$ for any $n \in \mathbb{N}$. So clearly the cardinality of $Hom(N_{|U}, \Omega_{|U})$ is at least $\mathcal{P}\mathbb{N}$ so we obtain a contradiction. It seems likely that this result extends to Grothendieck toposes...

References

[1] S. MacLane, I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction To Topos Theory*, Springer, 1992.