Kelley-Morse Set Theory in PyLog

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Abstract

We present the first section (theorems 4 to 100) of our PyLog formalisation of Set Theory as presented in the famous appendix of Kelley's General Topology.

We use here the new $ShowProof2()$ command which attempts to eliminate trivial or obvious steps.

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Welcome to PyLog 1.0

Natural Deduction Proof Assistant and Proof Checker

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>>> Test()

Th4. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))

0. z ϵ (x ∪ y)  Hyp
2. z ϵ {z: ((z ϵ x) v (z ϵ y))}  EqualitySub 0 1
3. Set(z) & ((z ϵ x) v (z ϵ y))  ClassElim 2
5. (z ϵ (x ∪ y)) -> ((z ϵ x) v (z ϵ y))  ImpInt 4
6. (z ϵ x) v (z ϵ y)  Hyp
7. z ϵ x  Hyp
8. ∃x.(z ϵ x)  ExistsInt 7
9. Set(z)  DefSub 8
10. z ϵ y  Hyp
11. ∃y.(z ϵ y)  ExistsInt 10
12. Set(z)  DefSub 11
13. Set(z)  OrElim 6 7 9 10 12
14. Set(z) & ((z ϵ x) v (z ϵ y))  AndInt 13 6
15. z ϵ {z: ((z ϵ x) v (z ϵ y))}  ClassInt 14
17. z ϵ (x ∪ y)  EqualitySub 15 16
18. ((z ϵ x) v (z ϵ y)) -> (z ϵ (x ∪ y))  ImpInt 17
19. ((z ϵ (x ∪ y)) -> ((z ϵ x) v (z ϵ y))) & (((z ϵ x) v (z ϵ y)) -> (z ϵ (x ∪ y)))  AndInt 5 18
21. z ϵ (x ∩ y)  Hyp
23. z ϵ {z: ((z ϵ x) & (z ϵ y))}  EqualitySub 21 22
24. Set(z) & ((z ϵ x) & (z ϵ y))  ClassElim 23
26. (z ϵ (x ∩ y)) -> ((z ϵ x) & (z ϵ y))  ImpInt 25
27. (z ϵ x) & (z ϵ y)  Hyp
29. ∃x.(z ϵ x)  ExistsInt 28
30. Set(z)  DefSub 29
31. Set(z) & ((z ϵ x) & (z ϵ y))  AndInt 30 27
32. z ϵ {z: ((z ϵ x) & (z ϵ y))}  ClassInt 31
34. z ϵ (x ∩ y)  EqualitySub 32 33
35. ((z ϵ x) & (z ϵ y)) -> (z ϵ (x ∩ y))  ImpInt 34
36. ((z ϵ (x ∩ y)) -> ((z ϵ x) & (z ϵ y))) & (((z ϵ x) & (z ϵ y)) -> (z ϵ (x ∩ y)))  AndInt 26 35
38. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  AndInt 20 37 Qed

Used Theorems
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Th5. ((x ∪ x) = x) & ((x ∩ x) = x)

0. z ϵ (x ∪ x)  Hyp
1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
6. (z ϵ (x ∪ x)) -> ((z ϵ x) v (z ϵ x))  ForallElim 5
7. (z ϵ x) v (z ϵ x)  ImpElim 0 6
8. z ϵ x  Hyp
9. z ϵ x  Hyp
10. z ϵ x  OrElim 7 8 8 9 9
11. (z ϵ (x ∪ x)) -> (z ϵ x)  ImpInt 10
12. z ϵ x  Hyp
13. (z ϵ x) v (z ϵ x)  OrIntL 12
16. ((z ϵ x) v (z ϵ x)) -> (z ϵ (x ∪ x))  ForallElim 15
17. z ϵ (x ∪ x)  ImpElim 13 16
18. (z ϵ x) -> (z ϵ (x ∪ x))  ImpInt 17
19. ((z ϵ (x ∪ x)) -> (z ϵ x)) & ((z ϵ x) -> (z ϵ (x ∪ x)))  AndInt 11 18
21. ∀z.((z ϵ (x ∪ x)) <-> (z ϵ x))  ForallInt 20
22. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
23. ∀y.(((x ∪ x) = y) <-> ∀z.((z ϵ (x ∪ x)) <-> (z ϵ y)))  ForallElim 22
24. ((x ∪ x) = x) <-> ∀z.((z ϵ (x ∪ x)) <-> (z ϵ x))  ForallElim 23
27. (x ∪ x) = x  ImpElim 21 26
28. z ϵ (x ∩ x)  Hyp
33. (z ϵ (x ∩ x)) -> ((z ϵ x) & (z ϵ x))  ForallElim 32
34. (z ϵ x) & (z ϵ x)  ImpElim 28 33
36. (z ϵ (x ∩ x)) -> (z ϵ x)  ImpInt 35
37. z ϵ x  Hyp
38. (z ϵ x) & (z ϵ x)  AndInt 37 37
41. ((z ϵ x) & (z ϵ x)) -> (z ϵ (x ∩ x))  ForallElim 40
42. z ϵ (x ∩ x)  ImpElim 38 41
43. (z ϵ x) -> (z ϵ (x ∩ x))  ImpInt 42
44. ((z ϵ (x ∩ x)) -> (z ϵ x)) & ((z ϵ x) -> (z ϵ (x ∩ x)))  AndInt 36 43
46. ∀y.(((x ∩ x) = y) <-> ∀z.((z ϵ (x ∩ x)) <-> (z ϵ y)))  ForallElim 22
47. ((x ∩ x) = x) <-> ∀z.((z ϵ (x ∩ x)) <-> (z ϵ x))  ForallElim 46
50. ∀z.((z ϵ (x ∩ x)) <-> (z ϵ x))  ForallInt 45
51. (x ∩ x) = x  ImpElim 50 49
52. ((x ∪ x) = x) & ((x ∩ x) = x)  AndInt 27 51 Qed

Used Theorems

1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))

Th6. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))

0. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
4. z ϵ (x ∪ y)  Hyp
5. (z ϵ x) v (z ϵ y)  ImpElim 4 3
6. (A v B) -> (B v A)  TheoremInt
7. ((z ϵ x) v B) -> (B v (z ϵ x))  PolySub 6
8. ((z ϵ x) v (z ϵ y)) -> ((z ϵ y) v (z ϵ x))  PolySub 7
9. (z ϵ y) v (z ϵ x)  ImpElim 5 8
16. ((z ϵ y) v (z ϵ x)) -> (z ϵ (y ∪ x))  ForallElim 15
17. z ϵ (y ∪ x)  ImpElim 9 16
18. (z ϵ (x ∪ y)) -> (z ϵ (y ∪ x))  ImpInt 17
26. (z ϵ (y ∪ x)) -> (z ϵ (x ∪ y))  ForallElim 25
27. ((z ϵ (x ∪ y)) -> (z ϵ (y ∪ x))) & ((z ϵ (y ∪ x)) -> (z ϵ (x ∪ y)))  AndInt 18 26
28. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
29. ∀e.(((x ∪ y) = e) <-> ∀z.((z ϵ (x ∪ y)) <-> (z ϵ e)))  ForallElim 28
30. ((x ∪ y) = (y ∪ x)) <-> ∀z.((z ϵ (x ∪ y)) <-> (z ϵ (y ∪ x)))  ForallElim 29
34. ∀z.((z ϵ (x ∪ y)) <-> (z ϵ (y ∪ x)))  ForallInt 33
35. (x ∪ y) = (y ∪ x)  ImpElim 34 32
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36. z ϵ (x ∩ y)  Hyp
40. (z ϵ x) & (z ϵ y)  ImpElim 36 39
41. (A & B) -> (B & A)  TheoremInt
42. ((z ϵ x) & B) -> (B & (z ϵ x))  PolySub 41
43. ((z ϵ x) & (z ϵ y)) -> ((z ϵ y) & (z ϵ x))  PolySub 42
44. (z ϵ y) & (z ϵ x)  ImpElim 40 43
46. ∀w.(((z ϵ w) & (z ϵ y)) -> (z ϵ (w ∩ y)))  ForallInt 45
48. ∀w.(((z ϵ w) & (z ϵ x)) -> (z ϵ (w ∩ x)))  ForallElim 47
49. ((z ϵ y) & (z ϵ x)) -> (z ϵ (y ∩ x))  ForallElim 48
50. z ϵ (y ∩ x)  ImpElim 44 49
51. (z ϵ (x ∩ y)) -> (z ϵ (y ∩ x))  ImpInt 50
52. ∀v.((z ϵ (v ∩ y)) -> (z ϵ (y ∩ v)))  ForallInt 51
54. ∀v.((z ϵ (v ∩ x)) -> (z ϵ (x ∩ v)))  ForallElim 53
55. (z ϵ (y ∩ x)) -> (z ϵ (x ∩ y))  ForallElim 54
56. ((z ϵ (x ∩ y)) -> (z ϵ (y ∩ x))) & ((z ϵ (y ∩ x)) -> (z ϵ (x ∩ y)))  AndInt 51 55
57. ∀g.(((x ∩ y) = g) <-> ∀z.((z ϵ (x ∩ y)) <-> (z ϵ g)))  ForallElim 28
58. ((x ∩ y) = (y ∩ x)) <-> ∀z.((z ϵ (x ∩ y)) <-> (z ϵ (y ∩ x)))  ForallElim 57
62. ∀z.((z ϵ (x ∩ y)) <-> (z ϵ (y ∩ x)))  ForallInt 61
63. (x ∩ y) = (y ∩ x)  ImpElim 62 60
64. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))  AndInt 35 63 Qed
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Used Theorems

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2. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
1. (A v B) -> (B v A)
3. (A & B) -> (B & A)
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Th7. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))

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0. w ϵ ((x ∪ y) ∪ z)  Hyp
1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
12. (w ϵ ((x ∪ y) ∪ z)) -> ((w ϵ (x ∪ y)) v (w ϵ z))  ForallElim 11
13. (w ϵ (x ∪ y)) v (w ϵ z)  ImpElim 0 12
14. w ϵ (x ∪ y)  Hyp
15. (w ϵ x) v (w ϵ y)  ImpElim 14 6
16. ((w ϵ x) v (w ϵ y)) v (w ϵ z)  OrIntR 15
17. w ϵ z  Hyp
18. ((w ϵ x) v (w ϵ y)) v (w ϵ z)  OrIntL 17
19. ((w ϵ x) v (w ϵ y)) v (w ϵ z)  OrElim 13 14 16 17 18
20. ((A v B) v C) <-> (A v (B v C))  TheoremInt
21. (((w ϵ x) v B) v C) <-> ((w ϵ x) v (B v C))  PolySub 20
22. (((w ϵ x) v (w ϵ y)) v C) <-> ((w ϵ x) v ((w ϵ y) v C))  PolySub 21
23. (((w ϵ x) v (w ϵ y)) v (w ϵ z)) <-> ((w ϵ x) v ((w ϵ y) v (w ϵ z)))  PolySub 22
26. (w ϵ x) v ((w ϵ y) v (w ϵ z))  ImpElim 19 25
35. ((w ϵ y) v (w ϵ z)) -> (w ϵ (y ∪ z))  ForallElim 34
36. (w ϵ y) v (w ϵ z)  Hyp
37. w ϵ (y ∪ z)  ImpElim 36 35
38. (w ϵ x) v (w ϵ (y ∪ z))  OrIntL 37
42. ((w ϵ x) v (w ϵ (y ∪ z))) -> (w ϵ (x ∪ (y ∪ z)))  ForallElim 41
43. w ϵ (x ∪ (y ∪ z))  ImpElim 38 42
44. w ϵ x  Hyp
45. (w ϵ x) v (w ϵ (y ∪ z))  OrIntR 44
49. ((w ϵ x) v (w ϵ (y ∪ z))) -> (w ϵ (x ∪ (y ∪ z)))  ForallElim 48
50. w ϵ (x ∪ (y ∪ z))  ImpElim 45 49
51. w ϵ (x ∪ (y ∪ z))  OrElim 26 44 50 36 43
52. (w ϵ ((x ∪ y) ∪ z)) -> (w ϵ (x ∪ (y ∪ z)))  ImpInt 51
53. w ϵ (x ∪ (y ∪ z))  Hyp
57. (w ϵ (x ∪ (y ∪ z))) -> ((w ϵ x) v (w ϵ (y ∪ z)))  ForallElim 56
58. (w ϵ x) v (w ϵ (y ∪ z))  ImpElim 53 57
59. w ϵ x  Hyp
60. (w ϵ x) v ((w ϵ y) v (w ϵ z))  OrIntR 59
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61. w ϵ (y ∪ z)  Hyp
63. (w ϵ (y ∪ z)) -> ((w ϵ y) v (w ϵ z))  ForallElim 11
64. (w ϵ y) v (w ϵ z)  ImpElim 61 63
65. (w ϵ x) v ((w ϵ y) v (w ϵ z))  OrIntL 64
66. (w ϵ x) v ((w ϵ y) v (w ϵ z))  OrElim 58 59 60 61 65
68. ((w ϵ x) v (w ϵ y)) v (w ϵ z)  ImpElim 66 67
69. (w ϵ x) v (w ϵ y)  Hyp
71. ((w ϵ x) v (w ϵ y)) -> (w ϵ (x ∪ y))  ForallElim 28
72. w ϵ (x ∪ y)  ImpElim 69 71
73. (w ϵ (x ∪ y)) v (w ϵ z)  OrIntR 72
74. w ϵ z  Hyp
75. (w ϵ (x ∪ y)) v (w ϵ z)  OrIntL 74
76. (w ϵ (x ∪ y)) v (w ϵ z)  OrElim 68 69 73 74 75
78. ((w ϵ (x ∪ y)) v (w ϵ z)) -> (w ϵ ((x ∪ y) ∪ z))  ForallElim 34
79. w ϵ ((x ∪ y) ∪ z)  ImpElim 76 78
80. (w ϵ (x ∪ (y ∪ z))) -> (w ϵ ((x ∪ y) ∪ z))  ImpInt 79
81. ((w ϵ ((x ∪ y) ∪ z)) -> (w ϵ (x ∪ (y ∪ z)))) & ((w ϵ (x ∪ (y ∪ z))) -> (w ϵ ((x ∪ y) ∪ z)))  AndInt 52 80
83. w ϵ ((x ∩ y) ∩ z)  Hyp
94. (w ϵ ((x ∩ y) ∩ z)) <-> ((w ϵ (x ∩ y)) & (w ϵ z))  ForallElim 93
97. (w ϵ (x ∩ y)) & (w ϵ z)  ImpElim 83 96
101. (w ϵ x) & (w ϵ y)  ImpElim 98 100
105. (w ϵ y) & (w ϵ z)  AndInt 104 102
111. ((w ϵ y) & (w ϵ z)) -> (w ϵ (y ∩ z))  ForallElim 110
112. w ϵ (y ∩ z)  ImpElim 105 111
113. (w ϵ x) & (w ϵ (y ∩ z))  AndInt 103 112
117. ((w ϵ x) & (w ϵ (y ∩ z))) -> (w ϵ (x ∩ (y ∩ z)))  ForallElim 116
118. w ϵ (x ∩ (y ∩ z))  ImpElim 113 117
119. (w ϵ ((x ∩ y) ∩ z)) -> (w ϵ (x ∩ (y ∩ z)))  ImpInt 118
120. w ϵ (x ∩ (y ∩ z))  Hyp
124. ∀b.((w ϵ (x ∩ b)) -> ((w ϵ x) & (w ϵ b)))  ForallInt 123
126. (w ϵ (x ∩ (y ∩ z))) -> ((w ϵ x) & (w ϵ (y ∩ z)))  ForallElim 124
127. (w ϵ x) & (w ϵ (y ∩ z))  ImpElim 120 126
133. (w ϵ (y ∩ z)) -> ((w ϵ y) & (w ϵ z))  ForallElim 132
134. (w ϵ y) & (w ϵ z)  ImpElim 128 133
137. (w ϵ x) & (w ϵ y)  AndInt 129 135
139. w ϵ (x ∩ y)  ImpElim 137 138
140. (w ϵ (x ∩ y)) & (w ϵ z)  AndInt 139 136
141. ∀a.((w ϵ (a ∩ b)) -> ((w ϵ a) & (w ϵ b)))  ForallInt 121
145. ((w ϵ (x ∩ y)) & (w ϵ z)) -> (w ϵ ((x ∩ y) ∩ z))  ForallElim 144
146. w ϵ ((x ∩ y) ∩ z)  ImpElim 140 145
147. (w ϵ (x ∩ (y ∩ z))) -> (w ϵ ((x ∩ y) ∩ z))  ImpInt 146
148. ((w ϵ ((x ∩ y) ∩ z)) -> (w ϵ (x ∩ (y ∩ z)))) & ((w ϵ (x ∩ (y ∩ z))) -> (w ϵ ((x ∩ y) ∩ z)))  AndInt 119 147
150. ((w ϵ ((x ∪ y) ∪ z)) <-> (w ϵ (x ∪ (y ∪ z)))) & ((w ϵ ((x ∩ y) ∩ z)) <-> (w ϵ (x ∩ (y ∩ z))))  AndInt 82 14
152. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
153. ∀h.((((x ∩ y) ∩ z) = h) <-> ∀i.((i ϵ ((x ∩ y) ∩ z)) <-> (i ϵ h)))  ForallElim 152
155. ∀w.((w ϵ ((x ∩ y) ∩ z)) <-> (w ϵ (x ∩ (y ∩ z))))  ForallInt 151
158. ((x ∩ y) ∩ z) = (x ∩ (y ∩ z))  ImpElim 155 157
159. ∀j.((((x ∪ y) ∪ z) = j) <-> ∀k.((k ϵ ((x ∪ y) ∪ z)) <-> (k ϵ j)))  ForallElim 152
160. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) <-> ∀k.((k ϵ ((x ∪ y) ∪ z)) <-> (k ϵ (x ∪ (y ∪ z))))  ForallElim 159
164. ∀w.((w ϵ ((x ∪ y) ∪ z)) <-> (w ϵ (x ∪ (y ∪ z))))  ForallInt 163
165. ((x ∪ y) ∪ z) = (x ∪ (y ∪ z))  ImpElim 164 162
166. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))  AndInt 165 158 Qed
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Used Theorems

```
3. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
1. ((A v B) v C) <-> (A v (B v C))
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Th8. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))
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0. w ϵ (x ∩ (y ∪ z))  Hyp
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1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
5. ((w ϵ (x ∪ a)) <-> ((w ϵ x) v (w ϵ a))) & ((w ϵ (x ∩ a)) <-> ((w ϵ x) & (w ϵ a)))  ForallElim 4
10. (w ϵ (x ∩ (y ∪ z))) -> ((w ϵ x) & (w ϵ (y ∪ z)))  ForallElim 9
11. (w ϵ x) & (w ϵ (y ∪ z))  ImpElim 0 10
20. (w ϵ (y ∪ z)) <-> ((w ϵ y) v (w ϵ z))  ForallElim 19
23. (w ϵ y) v (w ϵ z)  ImpElim 12 22
24. (w ϵ x) & ((w ϵ y) v (w ϵ z))  AndInt 13 23
25. (A & (B v C)) <-> ((A & B) v (A & C))  TheoremInt
26. ((w ϵ x) & (B v C)) <-> (((w ϵ x) & B) v ((w ϵ x) & C))  PolySub 25
27. ((w ϵ x) & ((w ϵ y) v C)) <-> (((w ϵ x) & (w ϵ y)) v ((w ϵ x) & C))  PolySub 26
28. ((w ϵ x) & ((w ϵ y) v (w ϵ z))) <-> (((w ϵ x) & (w ϵ y)) v ((w ϵ x) & (w ϵ z)))  PolySub 27
31. ((w ϵ x) & (w ϵ y)) v ((w ϵ x) & (w ϵ z))  ImpElim 24 30
32. (w ϵ x) & (w ϵ y)  Hyp
36. w ϵ (x ∩ y)  ImpElim 32 35
37. (w ϵ (x ∩ y)) v (w ϵ (x ∩ z))  OrIntR 36
38. (w ϵ x) & (w ϵ z)  Hyp
40. ((w ϵ x) & (w ϵ z)) -> (w ϵ (x ∩ z))  ForallElim 39
41. w ϵ (x ∩ z)  ImpElim 38 40
42. (w ϵ (x ∩ y)) v (w ϵ (x ∩ z))  OrIntL 41
43. (w ϵ (x ∩ y)) v (w ϵ (x ∩ z))  OrElim 31 32 37 38 42
49. ((w ϵ (x ∩ y)) v (w ϵ (x ∩ z))) -> (w ϵ ((x ∩ y) ∪ (x ∩ z)))  ForallElim 48
50. w ϵ ((x ∩ y) ∪ (x ∩ z))  ImpElim 43 49
51. (w ϵ (x ∩ (y ∪ z))) -> (w ϵ ((x ∩ y) ∪ (x ∩ z)))  ImpInt 50
52. w ϵ ((x ∩ y) ∪ (x ∩ z))  Hyp
57. (w ϵ ((x ∩ y) ∪ (x ∩ z))) -> ((w ϵ (x ∩ y)) v (w ϵ (x ∩ z)))  ForallElim 56
58. (w ϵ (x ∩ y)) v (w ϵ (x ∩ z))  ImpElim 52 57
62. (w ϵ (x ∩ z)) -> ((w ϵ x) & (w ϵ z))  ForallElim 9
63. w ϵ (x ∩ y)  Hyp
64. (w ϵ x) & (w ϵ y)  ImpElim 63 60
66. (w ϵ y) v (w ϵ z)  OrIntR 65
71. ((w ϵ y) v (w ϵ z)) -> (w ϵ (y ∪ z))  ForallElim 70
72. w ϵ (y ∪ z)  ImpElim 66 71
74. (w ϵ x) & (w ϵ (y ∪ z))  AndInt 73 72
77. ((w ϵ x) & (w ϵ (y ∪ z))) -> (w ϵ (x ∩ (y ∪ z)))  ForallElim 76
78. w ϵ (x ∩ (y ∪ z))  ImpElim 74 77
79. w ϵ (x ∩ z)  Hyp
80. (w ϵ x) & (w ϵ z)  ImpElim 79 62
83. (w ϵ y) v (w ϵ z)  OrIntL 82
84. w ϵ (y ∪ z)  ImpElim 83 71
85. (w ϵ x) & (w ϵ (y ∪ z))  AndInt 81 84
86. w ϵ (x ∩ (y ∪ z))  ImpElim 85 77
87. w ϵ (x ∩ (y ∪ z))  OrElim 58 63 78 79 86
88. (w ϵ ((x ∩ y) ∪ (x ∩ z))) -> (w ϵ (x ∩ (y ∪ z)))  ImpInt 87
89. ((w ϵ (x ∩ (y ∪ z))) -> (w ϵ ((x ∩ y) ∪ (x ∩ z)))) & ((w ϵ ((x ∩ y) ∪ (x ∩ z))) -> (w ϵ (x ∩ (y ∪ z)
)))  AndInt 51 88
91. w ϵ (x ∪ (y ∩ z))  Hyp
96. ((w ϵ (x ∪ (y ∩ z))) -> ((w ϵ x) v (w ϵ (y ∩ z)))) & (((w ϵ x) v (w ϵ (y ∩ z))) -> (w ϵ (x ∪ (y ∩ z)
)))  ForallElim 95
98. (w ϵ x) v (w ϵ (y ∩ z))  ImpElim 91 97
99. w ϵ x  Hyp
100. (w ϵ x) v (w ϵ y)  OrIntR 99
105. ((w ϵ x) v (w ϵ y)) -> (w ϵ (x ∪ y))  ForallElim 104
106. w ϵ (x ∪ y)  ImpElim 100 105
107. (w ϵ x) v (w ϵ z)  OrIntR 99
109. ((w ϵ x) v (w ϵ z)) -> (w ϵ (x ∪ z))  ForallElim 104
110. w ϵ (x ∪ z)  ImpElim 107 109
111. (w ϵ (x ∪ y)) & (w ϵ (x ∪ z))  AndInt 106 110
113. (w ϵ (b ∩ a)) <-> ((w ϵ b) & (w ϵ a))  ForallElim 112
119. ((w ϵ (x ∪ y)) & (w ϵ (x ∪ z))) -> (w ϵ ((x ∪ y) ∩ (x ∪ z)))  ForallElim 118
120. w ϵ ((x ∪ y) ∩ (x ∪ z))  ImpElim 111 119
121. w ϵ (y ∩ z)  Hyp
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126. (w ϵ (y ∩ z)) -> ((w ϵ y) & (w ϵ z))  ForallElim 125
127. (w ϵ y) & (w ϵ z)  ImpElim 121 126
130. (w ϵ x) v (w ϵ y)  OrIntL 128
131. (w ϵ x) v (w ϵ z)  OrIntL 129
132. w ϵ (x ∪ z)  ImpElim 131 109
137. ((w ϵ x) v (w ϵ y)) -> (w ϵ (x ∪ y))  ForallElim 136
138. w ϵ (x ∪ y)  ImpElim 130 137
139. (w ϵ (x ∪ y)) & (w ϵ (x ∪ z))  AndInt 138 132
140. w ϵ ((x ∪ y) ∩ (x ∪ z))  ImpElim 139 119
141. w ϵ ((x ∪ y) ∩ (x ∪ z))  OrElim 98 99 120 121 140
142. (w ϵ (x ∪ (y ∩ z))) -> (w ϵ ((x ∪ y) ∩ (x ∪ z)))  ImpInt 141
143. w ϵ ((x ∪ y) ∩ (x ∪ z))  Hyp
148. ((w ϵ ((x ∪ y) ∩ (x ∪ z))) -> ((w ϵ (x ∪ y)) & (w ϵ (x ∪ z)))) & (((w ϵ (x ∪ y)) & (w ϵ (x ∪ z))) ->
(w ϵ ((x ∪ y) ∩ (x ∪ z))))  ForallElim 147
150. (w ϵ (x ∪ y)) & (w ϵ (x ∪ z))  ImpElim 143 149
157. (w ϵ (x ∪ z)) -> ((w ϵ x) v (w ϵ z))  ForallElim 156
158. (w ϵ x) v (w ϵ y)  ImpElim 151 155
159. (w ϵ x) v (w ϵ z)  ImpElim 152 157
160. w ϵ x  Hyp
161. (w ϵ x) v (w ϵ (y ∩ z))  OrIntR 160
165. ((w ϵ x) v (w ϵ (y ∩ z))) -> (w ϵ (x ∪ (y ∩ z)))  ForallElim 164
166. w ϵ (x ∪ (y ∩ z))  ImpElim 161 165
167. (w ϵ x) -> (w ϵ (x ∪ (y ∩ z)))  ImpInt 166
168. w ϵ y  Hyp
169. w ϵ x  Hyp
170. w ϵ (x ∪ (y ∩ z))  ImpElim 169 167
171. w ϵ z  Hyp
172. (w ϵ y) & (w ϵ z)  AndInt 168 171
176. ((w ϵ y) & (w ϵ z)) -> (w ϵ (y ∩ z))  ForallElim 175
177. w ϵ (y ∩ z)  ImpElim 172 176
178. (w ϵ x) v (w ϵ (y ∩ z))  OrIntL 177
179. w ϵ (x ∪ (y ∩ z))  ImpElim 178 165
180. w ϵ (x ∪ (y ∩ z))  OrElim 159 169 170 171 179
182. (w ϵ ((x ∪ y) ∩ (x ∪ z))) -> (w ϵ (x ∪ (y ∩ z)))  ImpInt 181
183. ((w ϵ (x ∪ (y ∩ z))) -> (w ϵ ((x ∪ y) ∩ (x ∪ z)))) & ((w ϵ ((x ∪ y) ∩ (x ∪ z))) -> (w ϵ (x ∪ (y ∩ z)
)))  AndInt 142 182
185. ((w ϵ (x ∩ (y ∪ z))) <-> (w ϵ ((x ∩ y) ∪ (x ∩ z)))) & ((w ϵ (x ∪ (y ∩ z))) <-> (w ϵ ((x ∪ y) ∩ (x ∪
z))))  AndInt 90 184
188. ∀w.((w ϵ (x ∪ (y ∩ z))) <-> (w ϵ ((x ∪ y) ∩ (x ∪ z))))  ForallInt 186
189. ∀w.((w ϵ (x ∩ (y ∪ z))) <-> (w ϵ ((x ∩ y) ∪ (x ∩ z))))  ForallInt 187
190. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
191. ∀j.(((x ∩ (y ∪ z)) = j) <-> ∀k.((k ϵ (x ∩ (y ∪ z))) <-> (k ϵ j)))  ForallElim 190
192. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) <-> ∀k.((k ϵ (x ∩ (y ∪ z))) <-> (k ϵ ((x ∩ y) ∪ (x ∩ z))))  ForallEl
195. (x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))  ImpElim 189 194
196. ∀l.(((x ∪ (y ∩ z)) = l) <-> ∀m.((m ϵ (x ∪ (y ∩ z))) <-> (m ϵ l)))  ForallElim 190
197. ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))) <-> ∀m.((m ϵ (x ∪ (y ∩ z))) <-> (m ϵ ((x ∪ y) ∩ (x ∪ z))))  ForallEl
200. (x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))  ImpElim 188 199
201. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))  AndInt 195 200 Qed
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Used Theorems

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1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
2. (A & (B v C)) <-> ((A & B) v (A & C))
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Th11. ~~x = x

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0. z ϵ ~~x  Hyp
3. ~~x = {y: ¬(y ϵ ~x)}  ForallElim 2
4. z ϵ {y: ¬(y ϵ ~x)}  EqualitySub 0 3
5. Set(z) & ¬(z ϵ ~x)  ClassElim 4
7. ¬(z ϵ x)  Hyp
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9. Set(z) & ¬(z ϵ x)  AndInt 8 7
10. z ϵ {y: ¬(y ϵ x)}  ClassInt 9
12. z ϵ ~x  EqualitySub 10 11
13. _|_  ImpElim 12 6
14. ¬¬(z ϵ x)  ImpInt 13
15. D <-> ¬¬D  TheoremInt
16. (z ϵ x) <-> ¬¬(z ϵ x)  PolySub 15
19. z ϵ x  ImpElim 14 18
20. (z ϵ ~~x) -> (z ϵ x)  ImpInt 19
21. z ϵ x  Hyp
23. ¬¬(z ϵ x)  ImpElim 21 22
24. z ϵ ~x  Hyp
25. z ϵ {y: ¬(y ϵ x)}  EqualitySub 24 1
26. Set(z) & ¬(z ϵ x)  ClassElim 25
28. _|_  ImpElim 27 23
29. ¬(z ϵ ~x)  ImpInt 28
30. ∃y.(z ϵ y)  ExistsInt 21
31. Set(z)  DefSub 30
32. Set(z) & ¬(z ϵ ~x)  AndInt 31 29
33. z ϵ {y: ¬(y ϵ ~x)}  ClassInt 32
35. z ϵ ~~x  EqualitySub 33 34
36. (z ϵ x) -> (z ϵ ~~x)  ImpInt 35
37. ((z ϵ ~~x) -> (z ϵ x)) & ((z ϵ x) -> (z ϵ ~~x))  AndInt 20 36
39. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
40. ∀y.((~~x = y) <-> ∀z.((z ϵ ~~x) <-> (z ϵ y)))  ForallElim 39
41. (~~x = x) <-> ∀z.((z ϵ ~~x) <-> (z ϵ x))  ForallElim 40
44. ∀z.((z ϵ ~~x) <-> (z ϵ x))  ForallInt 38
45. ~~x = x  ImpElim 44 43 Qed
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Used Theorems

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1. D <-> ¬¬D
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Th12. (~(x ∪ y) = (~x ∩ ~y)) & (~(x ∩ y) = (~x ∪ ~y))

0. z ϵ ~(x ∪ y)  Hyp
3. ~(x ∪ y) = {t: ¬(t ϵ (x ∪ y))}  ForallElim 2
4. z ϵ {t: ¬(t ϵ (x ∪ y))}  EqualitySub 0 3
5. Set(z) & ¬(z ϵ (x ∪ y))  ClassElim 4
6. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
10. (A -> B) -> (¬B -> ¬A)  TheoremInt
11. (((z ϵ x) v (z ϵ y)) -> B) -> (¬B -> ¬((z ϵ x) v (z ϵ y)))  PolySub 10
12. (((z ϵ x) v (z ϵ y)) -> (z ϵ (x ∪ y))) -> (¬(z ϵ (x ∪ y)) -> ¬((z ϵ x) v (z ϵ y)))  PolySub 11
13. ¬(z ϵ (x ∪ y)) -> ¬((z ϵ x) v (z ϵ y))  ImpElim 9 12
15. ¬((z ϵ x) v (z ϵ y))  ImpElim 14 13
16. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
17. (¬((z ϵ x) v B) <-> (¬(z ϵ x) & ¬B)) & (¬((z ϵ x) & B) <-> (¬(z ϵ x) v ¬B))  PolySub 16
18. (¬((z ϵ x) v (z ϵ y)) <-> (¬(z ϵ x) & ¬(z ϵ y))) & (¬((z ϵ x) & (z ϵ y)) <-> (¬(z ϵ x) v ¬(z ϵ y))))
PolySub 17
22. ¬(z ϵ x) & ¬(z ϵ y)  ImpElim 15 21
26. Set(z) & ¬(z ϵ y)  AndInt 23 25
27. z ϵ {z: ¬(z ϵ y)}  ClassInt 26
28. Set(z) & ¬(z ϵ x)  AndInt 23 24
29. z ϵ {z: ¬(z ϵ x)}  ClassInt 28
32. z ϵ ~x  EqualitySub 29 31
34. ~y = {x_0: ¬(x_0 ϵ y)}  ForallElim 33
36. z ϵ ~y  EqualitySub 27 35
37. (z ϵ ~x) & (z ϵ ~y)  AndInt 32 36
44. ((z ϵ ~x) & (z ϵ ~y)) -> (z ϵ (~x ∩ ~y))  ForallElim 43
45. z ϵ (~x ∩ ~y)  ImpElim 37 44
46. (z ϵ ~(x ∪ y)) -> (z ϵ (~x ∩ ~y))  ImpInt 45
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47. z ϵ (~x ∩ ~y)  Hyp
51. (z ϵ (~x ∩ ~y)) <-> ((z ϵ ~x) & (z ϵ ~y))  ForallElim 50
54. (z ϵ ~x) & (z ϵ ~y)  ImpElim 47 53
57. z ϵ {y: ¬(y ϵ x)}  EqualitySub 56 30
58. z ϵ {x_0: ¬(x_0 ϵ y)}  EqualitySub 55 34
59. Set(z) & ¬(z ϵ x)  ClassElim 57
60. Set(z) & ¬(z ϵ y)  ClassElim 58
63. ¬(z ϵ x) & ¬(z ϵ y)  AndInt 61 62
65. ¬((z ϵ x) v (z ϵ y))  ImpElim 63 64
66. z ϵ (x ∪ y)  Hyp
68. (z ϵ x) v (z ϵ y)  ImpElim 66 67
69. _|_  ImpElim 68 65
70. ¬(z ϵ (x ∪ y))  ImpInt 69
72. Set(z) & ¬(z ϵ (x ∪ y))  AndInt 71 70
73. z ϵ {w: ¬(w ϵ (x ∪ y))}  ClassInt 72
75. {x_0: ¬(x_0 ϵ (x ∪ y))} = ~(x ∪ y)  ForallElim 74
76. z ϵ ~(x ∪ y)  EqualitySub 73 75
77. (z ϵ (~x ∩ ~y)) -> (z ϵ ~(x ∪ y))  ImpInt 76
78. ((z ϵ ~(x ∪ y)) -> (z ϵ (~x ∩ ~y))) & ((z ϵ (~x ∩ ~y)) -> (z ϵ ~(x ∪ y)))  AndInt 46 77
80. z ϵ ~(x ∩ y)  Hyp
82. ~(x ∩ y) = {x_0: ¬(x_0 ϵ (x ∩ y))}  ForallElim 81
83. z ϵ {x_0: ¬(x_0 ϵ (x ∩ y))}  EqualitySub 80 82
84. Set(z) & ¬(z ϵ (x ∩ y))  ClassElim 83
86. (((z ϵ x) & (z ϵ y)) -> B) -> (¬B -> ¬((z ϵ x) & (z ϵ y)))  PolySub 10
87. (((z ϵ x) & (z ϵ y)) -> (z ϵ (x ∩ y))) -> (¬(z ϵ (x ∩ y)) -> ¬((z ϵ x) & (z ϵ y)))  PolySub 86
88. ¬(z ϵ (x ∩ y)) -> ¬((z ϵ x) & (z ϵ y))  ImpElim 85 87
90. ¬((z ϵ x) & (z ϵ y))  ImpElim 89 88
92. ¬((z ϵ x) & B) <-> (¬(z ϵ x) v ¬B)  PolySub 91
93. ¬((z ϵ x) & (z ϵ y)) <-> (¬(z ϵ x) v ¬(z ϵ y))  PolySub 92
96. ¬(z ϵ x) v ¬(z ϵ y)  ImpElim 90 95
97. ¬(z ϵ x)  Hyp
99. Set(z) & ¬(z ϵ x)  AndInt 98 97
100. z ϵ {w: ¬(w ϵ x)}  ClassInt 99
101. (z ϵ {w: ¬(w ϵ x)}) v (z ϵ {w: ¬(w ϵ y)})  OrIntR 100
104. {x_1: ¬(x_1 ϵ y)} = ~y  ForallElim 103
105. (z ϵ ~x) v (z ϵ {w: ¬(w ϵ y)})  EqualitySub 101 102
106. (z ϵ ~x) v (z ϵ ~y)  EqualitySub 105 104
110. ((z ϵ ~x) v (z ϵ ~y)) -> (z ϵ (~x ∪ ~y))  ForallElim 109
111. z ϵ (~x ∪ ~y)  ImpElim 106 110
112. ¬(z ϵ y)  Hyp
113. Set(z) & ¬(z ϵ y)  AndInt 98 112
114. z ϵ {z: ¬(z ϵ y)}  ClassInt 113
115. (z ϵ {z: ¬(z ϵ x)}) v (z ϵ {z: ¬(z ϵ y)})  OrIntL 114
116. (z ϵ ~x) v (z ϵ {z: ¬(z ϵ y)})  EqualitySub 115 102
117. (z ϵ ~x) v (z ϵ ~y)  EqualitySub 116 104
118. z ϵ (~x ∪ ~y)  ImpElim 117 110
119. z ϵ (~x ∪ ~y)  OrElim 96 97 111 112 118
120. (z ϵ ~(x ∩ y)) -> (z ϵ (~x ∪ ~y))  ImpInt 119
121. z ϵ (~x ∪ ~y)  Hyp
122. ∃w.(z ϵ w)  ExistsInt 121
123. Set(z)  DefSub 122
131. (z ϵ (~x ∪ ~y)) -> ((z ϵ ~x) v (z ϵ ~y))  ForallElim 130
132. (z ϵ ~x) v (z ϵ ~y)  ImpElim 121 131
133. z ϵ ~x  Hyp
134. z ϵ {y: ¬(y ϵ x)}  EqualitySub 133 30
135. Set(z) & ¬(z ϵ x)  ClassElim 134
137. z ϵ ~y  Hyp
139. ~y = {x_3: ¬(x_3 ϵ y)}  ForallElim 138
140. z ϵ {x_3: ¬(x_3 ϵ y)}  EqualitySub 137 139
141. Set(z) & ¬(z ϵ y)  ClassElim 140
143. ¬(z ϵ x) v ¬(z ϵ y)  OrIntR 136
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144. ¬(z ϵ x) v ¬(z ϵ y) OrIntL 142
145. ¬(z ϵ x) v ¬(z ϵ y) OrElim 132 133 143 137 144
149. (¬(z ϵ x) v ¬B) -> ¬((z ϵ x) & B) PolySub 148
150. (¬(z ϵ x) v ¬(z ϵ y)) -> ¬((z ϵ x) & (z ϵ y)) PolySub 149
151. ¬((z ϵ x) & (z ϵ y)) ImpElim 145 150
155. ((z ϵ (x ∩ y)) -> B) -> (¬B -> ¬(z ϵ (x ∩ y))) PolySub 10
156. ((z ϵ (x ∩ y)) -> ((z ϵ x) & (z ϵ y))) -> (¬((z ϵ x) & (z ϵ y)) -> ¬(z ϵ (x ∩ y))) PolySub 155
157. ¬((z ϵ x) & (z ϵ y)) -> ¬(z ϵ (x ∩ y)) ImpElim 154 156
158. ¬(z ϵ (x ∩ y)) ImpElim 151 157
159. Set(z) DefSub 122
160. Set(z) & ¬(z ϵ (x ∩ y)) AndInt 159 158
161. z ϵ {w: ¬(w ϵ (x ∩ y))} ClassInt 160
163. {x_5: ¬(x_5 ϵ (x ∩ y))} = ~(x ∩ y) ForallElim 162
164. z ϵ ~(x ∩ y) EqualitySub 161 163
165. (z ϵ (~x ∪ ~y)) -> (z ϵ ~(x ∩ y)) ImpInt 164
166. ((z ϵ ~(x ∩ y)) -> (z ϵ (~x ∪ ~y))) & ((z ϵ (~x ∪ ~y)) -> (z ϵ ~(x ∩ y))) AndInt 120 165
168. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
169. ∀x_6.((~(x ∪ y) = x_6) <-> ∀z.((z ϵ ~(x ∪ y)) <-> (z ϵ x_6))) ForallElim 168
171. ∀z.((z ϵ ~(x ∪ y)) <-> (z ϵ (~x ∩ ~y))) ForallInt 79
174. ~(x ∪ y) = (~x ∩ ~y) ImpElim 171 173
175. ∀x_7.((~(x ∩ y) = x_7) <-> ∀z.((z ϵ ~(x ∩ y)) <-> (z ϵ x_7))) ForallElim 168
176. (~(x ∩ y) = (~x ∪ ~y)) <-> ∀z.((z ϵ ~(x ∩ y)) <-> (z ϵ (~x ∪ ~y))) ForallElim 175
179. ∀z.((z ϵ ~(x ∩ y)) <-> (z ϵ (~x ∪ ~y))) ForallInt 167
180. ~(x ∩ y) = (~x ∪ ~y) ImpElim 179 178
181. (~(x ∪ y) = (~x ∩ ~y)) & (~(x ∩ y) = (~x ∪ ~y)) AndInt 174 180 Qed

Used Theorems

2. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
3. (A -> B) -> (¬B -> ¬A)
1. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))

Th14. (x ∩ (y ~ z)) = ((x ∩ y) ∩ ~z)

1. ∀a.((a ~ y) = (a ∩ ~y)) ForallInt 0
3. ∀a.((a ~ z) = (a ∩ ~z)) ForallElim 2
4. (y ~ z) = (y ∩ ~z) ForallElim 3
6. (x ∩ (y ~ z)) = (x ∩ (y ∩ ~z)) EqualitySub 5 4
7. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) TheoremInt
11. (x ∩ (y ∩ ~z)) = ((x ∩ y) ∩ ~z) ForallElim 10
12. (x ∩ (y ~ z)) = ((x ∩ y) ∩ ~z) EqualitySub 6 11 Qed

Used Theorems

4. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))

Th16. ¬(x ϵ 0)

0. x ϵ 0 Hyp
2. x ϵ {x: ¬(x = x)} EqualitySub 0 1
3. Set(x) & ¬(x = x) ClassElim 2
6. _|_ ImpElim 5 4
7. ¬(x ϵ 0) ImpInt 6 Qed

Used Theorems

Th17. ((0 ∪ x) = x) & ((0 ∩ x) = 0)

0. z ϵ (0 ∪ x) Hyp
5. (0 ∪ x) = {z: ((z ϵ 0) v (z ϵ x))} ForallElim 4

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6. z ϵ {z: ((z ϵ 0) v (z ϵ x))}  EqualitySub 0 5
7. Set(z) & ((z ϵ 0) v (z ϵ x))  ClassElim 6
9. z ϵ 0  Hyp
10. ¬(x ϵ 0)  TheoremInt
12. ¬(z ϵ 0)  ForallElim 11
13. _|_  ImpElim 9 12
14. z ϵ x  AbsI 13
15. z ϵ x  Hyp
16. z ϵ x  OrElim 8 9 14 15 15
17. (z ϵ (0 ∪ x)) -> (z ϵ x)  ImpInt 16
18. z ϵ x  Hyp
19. (z ϵ 0) v (z ϵ x)  OrIntL 18
20. ∃x.(z ϵ x)  ExistsInt 18
21. Set(z)  DefSub 20
22. Set(z) & ((z ϵ 0) v (z ϵ x))  AndInt 21 19
23. z ϵ {z: ((z ϵ 0) v (z ϵ x))}  ClassInt 22
25. z ϵ (0 ∪ x)  EqualitySub 23 24
26. (z ϵ x) -> (z ϵ (0 ∪ x))  ImpInt 25
27. ((z ϵ (0 ∪ x)) -> (z ϵ x)) & ((z ϵ x) -> (z ϵ (0 ∪ x)))  AndInt 17 26
29. ∀z.((z ϵ (0 ∪ x)) <-> (z ϵ x))  ForallInt 28
30. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
31. ∀y.(((0 ∪ x) = y) <-> ∀z.((z ϵ (0 ∪ x)) <-> (z ϵ y)))  ForallElim 30
32. ((0 ∪ x) = x) <-> ∀z.((z ϵ (0 ∪ x)) <-> (z ϵ x))  ForallElim 31
35. (0 ∪ x) = x  ImpElim 29 34
36. z ϵ (0 ∩ x)  Hyp
41. (0 ∩ x) = {z: ((z ϵ 0) & (z ϵ x))}  ForallElim 40
42. z ϵ {z: ((z ϵ 0) & (z ϵ x))}  EqualitySub 36 41
43. Set(z) & ((z ϵ 0) & (z ϵ x))  ClassElim 42
46. (z ϵ (0 ∩ x)) -> (z ϵ 0)  ImpInt 45
47. z ϵ 0  Hyp
48. _|_  ImpElim 47 12
49. z ϵ (0 ∩ x)  AbsI 48
50. (z ϵ 0) -> (z ϵ (0 ∩ x))  ImpInt 49
51. ((z ϵ (0 ∩ x)) -> (z ϵ 0)) & ((z ϵ 0) -> (z ϵ (0 ∩ x)))  AndInt 46 50
54. ∀y.(((0 ∩ x) = y) <-> ∀z.((z ϵ (0 ∩ x)) <-> (z ϵ y)))  ForallElim 30
55. ((0 ∩ x) = 0) <-> ∀z.((z ϵ (0 ∩ x)) <-> (z ϵ 0))  ForallElim 54
58. (0 ∩ x) = 0  ImpElim 53 57
59. ((0 ∪ x) = x) & ((0 ∩ x) = 0)  AndInt 35 58 Qed
```

Used Theorems

```
2. ¬(x ϵ 0)
```

Th19. (x ϵ U) <-> Set(x)

```
0. x ϵ U  Hyp
2. x ϵ {x: (x = x)}  EqualitySub 0 1
3. Set(x) & (x = x)  ClassElim 2
5. (x ϵ U) -> Set(x)  ImpInt 4
6. Set(x)  Hyp
8. Set(x) & (x = x)  AndInt 6 7
9. x ϵ {x: (x = x)}  ClassInt 8
11. x ϵ U  EqualitySub 9 10
12. Set(x) -> (x ϵ U)  ImpInt 11
13. ((x ϵ U) -> Set(x)) & (Set(x) -> (x ϵ U))  AndInt 5 12
14. (x ϵ U) <-> Set(x)  EquivConst 13 Qed
```

Used Theorems

Th20. ((x ∪ U) = U) & ((x ∩ U) = x)

```
0. z ϵ (x ∪ U)  Hyp
1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
4. (z ϵ (x ∪ U)) <-> ((z ϵ x) v (z ϵ U))  ForallElim 3
7. (z ϵ x) v (z ϵ U)  ImpElim 0 6
8. z ϵ x  Hyp
9. ∃y.(z ϵ y)  ExistsInt 8
10. Set(z)  DefSub 9
11. (x ϵ U) <-> Set(x)  TheoremInt
15. Set(z) -> (z ϵ U)  ForallElim 14
16. z ϵ U  ImpElim 10 15
17. z ϵ U  Hyp
18. z ϵ U  OrElim 7 8 16 17 17
19. (z ϵ (x ∪ U)) -> (z ϵ U)  ImpInt 18
20. z ϵ U  Hyp
21. (z ϵ x) v (z ϵ U)  OrIntL 20
23. z ϵ (x ∪ U)  ImpElim 21 22
24. (z ϵ U) -> (z ϵ (x ∪ U))  ImpInt 23
25. ((z ϵ (x ∪ U)) -> (z ϵ U)) & ((z ϵ U) -> (z ϵ (x ∪ U)))  AndInt 19 24
27. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
28. ∀y.(((x ∪ U) = y) <-> ∀z.((z ϵ (x ∪ U)) <-> (z ϵ y)))  ForallElim 27
30. ∀z.((z ϵ (x ∪ U)) <-> (z ϵ U))  ForallInt 26
33. (x ∪ U) = U  ImpElim 30 32
34. z ϵ (x ∩ U)  Hyp
37. (z ϵ (x ∩ U)) <-> ((z ϵ x) & (z ϵ U))  ForallElim 36
40. (z ϵ x) & (z ϵ U)  ImpElim 34 39
42. (z ϵ (x ∩ U)) -> (z ϵ x)  ImpInt 41
43. z ϵ x  Hyp
44. ∃y.(z ϵ y)  ExistsInt 43
45. Set(z)  DefSub 44
46. z ϵ U  ImpElim 45 15
47. (z ϵ x) & (z ϵ U)  AndInt 43 46
49. z ϵ (x ∩ U)  ImpElim 47 48
50. (z ϵ x) -> (z ϵ (x ∩ U))  ImpInt 49
51. ((z ϵ (x ∩ U)) -> (z ϵ x)) & ((z ϵ x) -> (z ϵ (x ∩ U)))  AndInt 42 50
54. ∀y.(((x ∩ U) = y) <-> ∀z.((z ϵ (x ∩ U)) <-> (z ϵ y)))  ForallElim 27
55. ((x ∩ U) = x) <-> ∀z.((z ϵ (x ∩ U)) <-> (z ϵ x))  ForallElim 54
58. (x ∩ U) = x  ImpElim 53 57
59. ((x ∪ U) = U) & ((x ∩ U) = x)  AndInt 33 58 Qed
```

Used Theorems

```
1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
2. (x ϵ U) <-> Set(x)
```

Th21. (~0 = U) & (~U = 0)

```
0. z ϵ ~0  Hyp
2. ∀x.(~x = {y: ¬(y ϵ x)})  ForallInt 1
4. ~0 = {y: ¬(y ϵ 0)}  ForallElim 3
5. z ϵ {y: ¬(y ϵ 0)}  EqualitySub 0 4
6. Set(z) & ¬(z ϵ 0)  ClassElim 5
8. (x ϵ U) <-> Set(x)  TheoremInt
12. Set(z) -> (z ϵ U)  ForallElim 11
13. z ϵ U  ImpElim 7 12
14. (z ϵ ~0) -> (z ϵ U)  ImpInt 13
15. z ϵ U  Hyp
18. (z ϵ U) -> Set(z)  ForallElim 17
19. Set(z)  ImpElim 15 18
20. ¬(x ϵ 0)  TheoremInt
22. ¬(z ϵ 0)  ForallElim 21
```

23. Set(z) & ¬(z ϵ 0) AndInt 19 22
24. z ϵ {y: ¬(y ϵ 0)} ClassInt 23
26. z ϵ ~0 EqualitySub 24 25
27. (z ϵ U) -> (z ϵ ~0) ImpInt 26
28. ((z ϵ ~0) -> (z ϵ U)) & ((z ϵ U) -> (z ϵ ~0)) AndInt 14 27
30. ∀z.((z ϵ ~0) <-> (z ϵ U)) ForallInt 29
31. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
32. ∀y.((~0 = y) <-> ∀z.((z ϵ ~0) <-> (z ϵ y))) ForallElim 31
33. (~0 = U) <-> ∀z.((z ϵ ~0) <-> (z ϵ U)) ForallElim 32
36. ~0 = U ImpElim 30 35
37. z ϵ ~U Hyp
39. ~U = {y: ¬(y ϵ U)} ForallElim 38
40. z ϵ {y: ¬(y ϵ U)} EqualitySub 37 39
41. Set(z) & ¬(z ϵ U) ClassElim 40
44. z ϵ U ImpElim 43 12
45. _|_ ImpElim 44 42
46. z ϵ 0 AbsI 45
47. (z ϵ ~U) -> (z ϵ 0) ImpInt 46
48. z ϵ 0 Hyp
50. z ϵ {x: ¬(x = x)} EqualitySub 48 49
51. Set(z) & ¬(z = z) ClassElim 50
55. _|_ ImpElim 54 53
56. z ϵ ~U AbsI 55
57. (z ϵ 0) -> (z ϵ ~U) ImpInt 56
58. ((z ϵ ~U) -> (z ϵ 0)) & ((z ϵ 0) -> (z ϵ ~U)) AndInt 47 57
61. ∀y.((~U = y) <-> ∀z.((z ϵ ~U) <-> (z ϵ y))) ForallElim 31
62. (~U = 0) <-> ∀z.((z ϵ ~U) <-> (z ϵ 0)) ForallElim 61
65. ~U = 0 ImpElim 60 64
66. (~0 = U) & (~U = 0) AndInt 36 65 Qed

Used Theorems

1. (x ϵ U) <-> Set(x)
2. ¬(x ϵ 0)

Th24. (∩0 = U) & (∪0 = 0)

0. x ϵ ∩0 Hyp
3. ∩0 = {z: ∀y.((y ϵ 0) -> (z ϵ y))} ForallElim 2
4. x ϵ {z: ∀y.((y ϵ 0) -> (z ϵ y))} EqualitySub 0 3
5. Set(x) & ∀y.((y ϵ 0) -> (x ϵ y)) ClassElim 4
7. (x ϵ U) <-> Set(x) TheoremInt
10. x ϵ U ImpElim 6 9
11. (x ϵ ∩0) -> (x ϵ U) ImpInt 10
12. x ϵ U Hyp
13. y ϵ 0 Hyp
14. ¬(x ϵ 0) TheoremInt
16. ¬(y ϵ 0) ForallElim 15
17. _|_ ImpElim 13 16
18. x ϵ y AbsI 17
19. (y ϵ 0) -> (x ϵ y) ImpInt 18
20. ∀y.((y ϵ 0) -> (x ϵ y)) ForallInt 19
22. Set(x) ImpElim 12 21
23. Set(x) & ∀y.((y ϵ 0) -> (x ϵ y)) AndInt 22 20
24. x ϵ {z: ∀y.((y ϵ 0) -> (z ϵ y))} ClassInt 23
26. x ϵ ∩0 EqualitySub 24 25
27. (x ϵ U) -> (x ϵ ∩0) ImpInt 26
28. ((x ϵ ∩0) -> (x ϵ U)) & ((x ϵ U) -> (x ϵ ∩0)) AndInt 11 27
30. ∀z.((z ϵ ∩0) <-> (z ϵ U)) ForallInt 29
31. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
32. ∀y.((∩0 = y) <-> ∀z.((z ϵ ∩0) <-> (z ϵ y))) ForallElim 31

33. (∩0 = U) <-> ∀z.((z ϵ ∩0) <-> (z ϵ U)) ForallElim 32
36. ∩0 = U ImpElim 30 35
37. z ϵ ∪0 Hyp
40. ∪0 = {z: ∃y.((y ϵ 0) & (z ϵ y))} ForallElim 39
41. z ϵ {z: ∃y.((y ϵ 0) & (z ϵ y))} EqualitySub 37 40
42. Set(z) & ∃y.((y ϵ 0) & (z ϵ y)) ClassElim 41
44. (a ϵ 0) & (z ϵ a) Hyp
46. ¬(a ϵ 0) ForallElim 45
48. _|_ ImpElim 47 46
49. z ϵ 0 AbsI 48
50. z ϵ 0 ExistsElim 43 44 49
51. (z ϵ ∪0) -> (z ϵ 0) ImpInt 50
52. z ϵ 0 Hyp
54. ¬(z ϵ 0) ForallElim 53
55. _|_ ImpElim 52 54
56. z ϵ ∪0 AbsI 55
57. (z ϵ 0) -> (z ϵ ∪0) ImpInt 56
58. ((z ϵ ∪0) -> (z ϵ 0)) & ((z ϵ 0) -> (z ϵ ∪0)) AndInt 51 57
61. ∀y.((∪0 = y) <-> ∀z.((z ϵ ∪0) <-> (z ϵ y))) ForallElim 31
62. (∪0 = 0) <-> ∀z.((z ϵ ∪0) <-> (z ϵ 0)) ForallElim 61
65. ∪0 = 0 ImpElim 60 64
66. (∩0 = U) & (∪0 = 0) AndInt 36 65 Qed

Used Theorems

1. (x ϵ U) <-> Set(x)
2. ¬(x ϵ 0)

Th26. (0 ⊂ x) & (x ⊂ U)

0. z ϵ 0 Hyp
1. ¬(x ϵ 0) TheoremInt
3. ¬(z ϵ 0) ForallElim 2
4. _|_ ImpElim 0 3
5. z ϵ x AbsI 4
6. (z ϵ 0) -> (z ϵ x) ImpInt 5
7. ∀z.((z ϵ 0) -> (z ϵ x)) ForallInt 6
8. 0 ⊂ x DefSub 7
9. z ϵ x Hyp
10. ∃y.(z ϵ y) ExistsInt 9
11. Set(z) DefSub 10
12. (x ϵ U) <-> Set(x) TheoremInt
16. Set(z) -> (z ϵ U) ForallElim 15
17. z ϵ U ImpElim 11 16
18. (z ϵ x) -> (z ϵ U) ImpInt 17
19. ∀z.((z ϵ x) -> (z ϵ U)) ForallInt 18
20. x ⊂ U DefSub 19
21. (0 ⊂ x) & (x ⊂ U) AndInt 8 20 Qed

Used Theorems

1. ¬(x ϵ 0)
2. (x ϵ U) <-> Set(x)

Th27. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

0. a = b Hyp
1. z ϵ a Hyp
2. z ϵ b EqualitySub 1 0
3. (z ϵ a) -> (z ϵ b) ImpInt 2
4. ∀z.((z ϵ a) -> (z ϵ b)) ForallInt 3

5. a ⊂ b DefSub 4
6. z ϵ b Hyp
8. z ϵ a EqualitySub 6 7
9. (z ϵ b) -> (z ϵ a) ImpInt 8
10. ∀z.((z ϵ b) -> (z ϵ a)) ForallInt 9
11. b ⊂ a DefSub 10
12. (a ⊂ b) & (b ⊂ a) AndInt 5 11
13. (a = b) -> ((a ⊂ b) & (b ⊂ a)) ImpInt 12
14. (a ⊂ b) & (b ⊂ a) Hyp
17. z ϵ a Hyp
18. ∀z.((z ϵ a) -> (z ϵ b)) DefExp 15
19. (z ϵ a) -> (z ϵ b) ForallElim 18
20. z ϵ b ImpElim 17 19
21. (z ϵ a) -> (z ϵ b) ImpInt 20
22. z ϵ b Hyp
23. ∀z.((z ϵ b) -> (z ϵ a)) DefExp 16
24. (z ϵ b) -> (z ϵ a) ForallElim 23
25. z ϵ a ImpElim 22 24
26. (z ϵ b) -> (z ϵ a) ImpInt 25
27. ((z ϵ a) -> (z ϵ b)) & ((z ϵ b) -> (z ϵ a)) AndInt 21 26
29. ∀z.((z ϵ a) <-> (z ϵ b)) ForallInt 28
30. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
31. ∀y.((a = y) <-> ∀z.((z ϵ a) <-> (z ϵ y))) ForallElim 30
32. (a = b) <-> ∀z.((z ϵ a) <-> (z ϵ b)) ForallElim 31
35. a = b ImpElim 29 34
36. ((a ⊂ b) & (b ⊂ a)) -> (a = b) ImpInt 35
37. ((a = b) -> ((a ⊂ b) & (b ⊂ a))) & (((a ⊂ b) & (b ⊂ a)) -> (a = b)) AndInt 13 36
42. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) ForallElim 41 Qed

Used Theorems

Th28. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)

0. (a ⊂ b) & (b ⊂ c) Hyp
3. ∀z.((z ϵ b) -> (z ϵ c)) DefExp 1
4. ∀z.((z ϵ a) -> (z ϵ b)) DefExp 2
5. (z ϵ b) -> (z ϵ c) ForallElim 3
6. (z ϵ a) -> (z ϵ b) ForallElim 4
7. z ϵ a Hyp
8. z ϵ b ImpElim 7 6
9. z ϵ c ImpElim 8 5
10. (z ϵ a) -> (z ϵ c) ImpInt 9
11. ∀z.((z ϵ a) -> (z ϵ c)) ForallInt 10
12. a ⊂ c DefSub 11
13. ((a ⊂ b) & (b ⊂ c)) -> (a ⊂ c) ImpInt 12
19. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z) ForallElim 18 Qed

Used Theorems

Th29. (x ⊂ y) <-> ((x ∪ y) = y)

0. a ⊂ b Hyp
1. z ϵ (a ∪ b) Hyp
2. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
8. ((z ϵ (a ∪ b)) -> ((z ϵ a) v (z ϵ b))) & (((z ϵ a) v (z ϵ b)) -> (z ϵ (a ∪ b))) ForallElim 7
10. (z ϵ a) v (z ϵ b) ImpElim 1 9
11. z ϵ a Hyp
12. ∀z.((z ϵ a) -> (z ϵ b)) DefExp 0
13. (z ϵ a) -> (z ϵ b) ForallElim 12

14. z ϵ b ImpElim 11 13
15. z ϵ b Hyp
16. z ϵ b OrElim 10 11 14 15 15
17. (z ϵ (a ∪ b)) -> (z ϵ b) ImpInt 16
18. z ϵ b Hyp
19. (z ϵ a) v (z ϵ b) OrIntL 18
21. z ϵ (a ∪ b) ImpElim 19 20
22. (z ϵ b) -> (z ϵ (a ∪ b)) ImpInt 21
23. ((z ϵ (a ∪ b)) -> (z ϵ b)) & ((z ϵ b) -> (z ϵ (a ∪ b))) AndInt 17 22
25. ∀z.((z ϵ (a ∪ b)) <-> (z ϵ b)) ForallInt 24
26. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
27. ∀y.(((a ∪ b) = y) <-> ∀z.((z ϵ (a ∪ b)) <-> (z ϵ y))) ForallElim 26
28. ((a ∪ b) = b) <-> ∀z.((z ϵ (a ∪ b)) <-> (z ϵ b)) ForallElim 27
31. (a ∪ b) = b ImpElim 25 30
32. (a ⊂ b) -> ((a ∪ b) = b) ImpInt 31
33. (a ∪ b) = b Hyp
34. z ϵ a Hyp
35. (z ϵ a) v (z ϵ b) OrIntR 34
37. z ϵ (a ∪ b) ImpElim 35 36
38. z ϵ b EqualitySub 37 33
39. (z ϵ a) -> (z ϵ b) ImpInt 38
40. ∀z.((z ϵ a) -> (z ϵ b)) ForallInt 39
41. a ⊂ b DefSub 40
42. ((a ∪ b) = b) -> (a ⊂ b) ImpInt 41
43. ((a ⊂ b) -> ((a ∪ b) = b)) & (((a ∪ b) = b) -> (a ⊂ b)) AndInt 32 42
48. (x ⊂ y) <-> ((x ∪ y) = y) ForallElim 47 Qed

Used Theorems

1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))

Th30. (x ⊂ y) <-> ((x ∩ y) = x)

0. a ⊂ b Hyp
1. z ϵ (a ∩ b) Hyp
2. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
7. (z ϵ (a ∩ b)) <-> ((z ϵ a) & (z ϵ b)) ForallElim 6
10. (z ϵ a) & (z ϵ b) ImpElim 1 9
12. (z ϵ (a ∩ b)) -> (z ϵ a) ImpInt 11
13. z ϵ a Hyp
14. ∀z.((z ϵ a) -> (z ϵ b)) DefExp 0
15. (z ϵ a) -> (z ϵ b) ForallElim 14
16. z ϵ b ImpElim 13 15
17. (z ϵ a) & (z ϵ b) AndInt 13 16
19. z ϵ (a ∩ b) ImpElim 17 18
20. (z ϵ a) -> (z ϵ (a ∩ b)) ImpInt 19
21. ((z ϵ (a ∩ b)) -> (z ϵ a)) & ((z ϵ a) -> (z ϵ (a ∩ b))) AndInt 12 20
23. ∀z.((z ϵ (a ∩ b)) <-> (z ϵ a)) ForallInt 22
24. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
25. ∀y.(((a ∩ b) = y) <-> ∀z.((z ϵ (a ∩ b)) <-> (z ϵ y))) ForallElim 24
26. ((a ∩ b) = a) <-> ∀z.((z ϵ (a ∩ b)) <-> (z ϵ a)) ForallElim 25
29. (a ∩ b) = a ImpElim 23 28
30. (a ⊂ b) -> ((a ∩ b) = a) ImpInt 29
31. (a ∩ b) = a Hyp
32. z ϵ a Hyp
34. z ϵ (a ∩ b) EqualitySub 32 33
35. (z ϵ a) & (z ϵ b) ImpElim 34 9
37. (z ϵ a) -> (z ϵ b) ImpInt 36
38. ∀z.((z ϵ a) -> (z ϵ b)) ForallInt 37
39. a ⊂ b DefSub 38
40. ((a ∩ b) = a) -> (a ⊂ b) ImpInt 39

```
41. ((a ⊂ b) -> ((a ∩ b) = a)) & (((a ∩ b) = a) -> (a ⊂ b))  AndInt 30 40
46. (x ⊂ y) <-> ((x ∩ y) = x)  ForallElim 45 Qed
```

Used Theorems

```
1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
```

Th31. (x ⊂ y) -> ((∪x ⊂ ∪y) & (∩y ⊂ ∩x))

```
0. a ⊂ b  Hyp
1. z ϵ ∪a  Hyp
4. ∪a = {z: ∃y.((y ϵ a) & (z ϵ y))}  ForallElim 3
5. z ϵ {z: ∃y.((y ϵ a) & (z ϵ y))}  EqualitySub 1 4
6. Set(z) & ∃y.((y ϵ a) & (z ϵ y))  ClassElim 5
8. (y ϵ a) & (z ϵ y)  Hyp
9. ∀z.((z ϵ a) -> (z ϵ b))  DefExp 0
10. (y ϵ a) -> (y ϵ b)  ForallElim 9
12. y ϵ b  ImpElim 11 10
14. (y ϵ b) & (z ϵ y)  AndInt 12 13
15. ∃y.((y ϵ b) & (z ϵ y))  ExistsInt 14
17. Set(z) & ∃y.((y ϵ b) & (z ϵ y))  AndInt 16 15
18. z ϵ {z: ∃y.((y ϵ b) & (z ϵ y))}  ClassInt 17
20. ∪b = {z: ∃y.((y ϵ b) & (z ϵ y))}  ForallElim 19
22. z ϵ ∪b  EqualitySub 18 21
23. z ϵ ∪b  ExistsElim 7 8 22
24. (z ϵ ∪a) -> (z ϵ ∪b)  ImpInt 23
25. ∀z.((z ϵ ∪a) -> (z ϵ ∪b))  ForallInt 24
26. ∪a ⊂ ∪b  DefSub 25
27. z ϵ ∩b  Hyp
30. ∩b = {z: ∀y.((y ϵ b) -> (z ϵ y))}  ForallElim 29
31. z ϵ {z: ∀y.((y ϵ b) -> (z ϵ y))}  EqualitySub 27 30
32. Set(z) & ∀y.((y ϵ b) -> (z ϵ y))  ClassElim 31
35. (y ϵ b) -> (z ϵ y)  ForallElim 34
36. y ϵ a  Hyp
37. y ϵ b  ImpElim 36 10
38. z ϵ y  ImpElim 37 35
39. (y ϵ a) -> (z ϵ y)  ImpInt 38
40. ∀y.((y ϵ a) -> (z ϵ y))  ForallInt 39
41. Set(z) & ∀y.((y ϵ a) -> (z ϵ y))  AndInt 33 40
42. z ϵ {z: ∀y.((y ϵ a) -> (z ϵ y))}  ClassInt 41
44. ∩a = {z: ∀y.((y ϵ a) -> (z ϵ y))}  ForallElim 43
46. z ϵ ∩a  EqualitySub 42 45
47. (z ϵ ∩b) -> (z ϵ ∩a)  ImpInt 46
48. ∀z.((z ϵ ∩b) -> (z ϵ ∩a))  ForallInt 47
49. ∩b ⊂ ∩a  DefSub 48
50. (∪a ⊂ ∪b) & (∩b ⊂ ∩a)  AndInt 26 49
51. (a ⊂ b) -> ((∪a ⊂ ∪b) & (∩b ⊂ ∩a))  ImpInt 50
55. (x ⊂ y) -> ((∪x ⊂ ∪y) & (∩y ⊂ ∩x))  ForallElim 54 Qed
```

Used Theorems

Th32. (x ϵ y) -> ((x ⊂ ∪y) & (∩y ⊂ x))

```
0. a ϵ b  Hyp
1. x ϵ a  Hyp
2. (a ϵ b) & (x ϵ a)  AndInt 0 1
4. ∃y.(x ϵ y)  ExistsInt 1
5. Set(x)  DefSub 4
6. Set(x) & ∃y.((y ϵ b) & (x ϵ y))  AndInt 5 3
7. x ϵ {z: ∃y.((y ϵ b) & (z ϵ y))}  ClassInt 6
```

11. {z: $\exists$y.((y ϵ b) & (z ϵ y))} = $\cup$b ForallElim 10
12. x ϵ $\cup$b EqualitySub 7 11
13. (x ϵ a) -> (x ϵ $\cup$b) ImpInt 12
14. $\forall$z.((z ϵ a) -> (z ϵ $\cup$b)) ForallInt 13
15. a $\subset$ $\cup$b DefSub 14
16. x ϵ $\cap$b Hyp
19. $\cap$b = {z: $\forall$y.((y ϵ b) -> (z ϵ y))} ForallElim 18
20. x ϵ {z: $\forall$y.((y ϵ b) -> (z ϵ y))} EqualitySub 16 19
21. Set(x) & $\forall$y.((y ϵ b) -> (x ϵ y)) ClassElim 20
23. (a ϵ b) -> (x ϵ a) ForallElim 22
24. x ϵ a ImpElim 0 23
25. (x ϵ $\cap$b) -> (x ϵ a) ImpInt 24
26. $\forall$z.((z ϵ $\cap$b) -> (z ϵ a)) ForallInt 25
27. $\cap$b $\subset$ a DefSub 26
28. (a $\subset$ $\cup$b) & ($\cap$b $\subset$ a) AndInt 15 27
29. (a ϵ b) -> ((a $\subset$ $\cup$b) & ($\cap$b $\subset$ a)) ImpInt 28
33. (x ϵ y) -> ((x $\subset$ $\cup$y) & ($\cap$y $\subset$ x)) ForallElim 32 Qed

Used Theorems

Th33. (Set(x) & (y $\subset$ x)) -> Set(y)

0. Set(a) & (b $\subset$ a) Hyp
1. Set(x) -> $\exists$y.(Set(y) & $\forall$z.((z $\subset$ x) -> (z ϵ y))) AxInt
3. Set(a) -> $\exists$y.(Set(y) & $\forall$z.((z $\subset$ a) -> (z ϵ y))) ForallElim 2
5. $\exists$y.(Set(y) & $\forall$z.((z $\subset$ a) -> (z ϵ y))) ImpElim 4 3
6. Set(w) & $\forall$z.((z $\subset$ a) -> (z ϵ w)) Hyp
8. (b $\subset$ a) -> (b ϵ w) ForallElim 7
10. b ϵ w ImpElim 9 8
11. $\exists$z.(b ϵ z) ExistsInt 10
12. Set(b) DefSub 11
13. Set(b) ExistsElim 5 6 12
14. (Set(a) & (b $\subset$ a)) -> Set(b) ImpInt 13
18. (Set(x) & (y $\subset$ x)) -> Set(y) ForallElim 17 Qed

Used Theorems

Th34. (0 = $\cap$U) & (U = $\cup$U)

0. z ϵ 0 Hyp
2. z ϵ {x: $\neg$(x = x)} EqualitySub 0 1
3. Set(z) & $\neg$(z = z) ClassElim 2
6. _|_ ImpElim 5 4
7. z ϵ $\cap$U AbsI 6
8. (z ϵ 0) -> (z ϵ $\cap$U) ImpInt 7
9. z ϵ $\cap$U Hyp
13. $\cap$U = {z: $\forall$y.((y ϵ U) -> (z ϵ y))} ForallElim 12
14. z ϵ {z: $\forall$y.((y ϵ U) -> (z ϵ y))} EqualitySub 9 13
15. Set(z) & $\forall$y.((y ϵ U) -> (z ϵ y)) ClassElim 14
17. (0 ϵ U) -> (z ϵ 0) ForallElim 16
18. (0 $\subset$ x) & (x $\subset$ U) TheoremInt
19. (Set(x) & (y $\subset$ x)) -> Set(y) TheoremInt
26. (Set(z) & (0 $\subset$ z)) -> Set(0) ForallElim 25
28. Set(z) & (0 $\subset$ z) AndInt 27 22
29. Set(0) ImpElim 28 26
30. (x ϵ U) <-> Set(x) TheoremInt
34. Set(0) -> (0 ϵ U) ForallElim 33
35. 0 ϵ U ImpElim 29 34
36. z ϵ 0 ImpElim 35 17

37. (z ϵ ∩U) -> (z ϵ 0) ImpInt 36
38. ((z ϵ 0) -> (z ϵ ∩U)) & ((z ϵ ∩U) -> (z ϵ 0)) AndInt 8 37
40. ∀z.((z ϵ 0) <-> (z ϵ ∩U)) ForallInt 39
41. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
42. ∀y.((0 = y) <-> ∀z.((z ϵ 0) <-> (z ϵ y))) ForallElim 41
43. (0 = ∩U) <-> ∀z.((z ϵ 0) <-> (z ϵ ∩U)) ForallElim 42
46. 0 = ∩U ImpElim 40 45
47. z ϵ U Hyp
50. ∪U = {z: ∃y.((y ϵ U) & (z ϵ y))} ForallElim 49
51. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ϵ y))) AxInt
54. (z ϵ U) -> Set(z) ForallElim 53
55. Set(z) ImpElim 47 54
57. Set(z) -> ∃y.(Set(y) & ∀i.((i ⊂ z) -> (i ϵ y))) ForallElim 56
58. ∃y.(Set(y) & ∀i.((i ⊂ z) -> (i ϵ y))) ImpElim 55 57
59. Set(a) & ∀i.((i ⊂ z) -> (i ϵ a)) Hyp
61. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
65. (z = z) <-> ((z ⊂ z) & (z ⊂ z)) ForallElim 64
68. (z ⊂ z) & (z ⊂ z) ImpElim 60 67
71. (z ⊂ z) -> (z ϵ a) ForallElim 70
72. z ϵ a ImpElim 69 71
75. Set(a) -> (a ϵ U) ForallElim 74
76. a ϵ U ImpElim 73 75
77. (a ϵ U) & (z ϵ a) AndInt 76 72
78. ∃y.((y ϵ U) & (z ϵ y)) ExistsInt 77
79. ∃y.((y ϵ U) & (z ϵ y)) ExistsElim 58 59 78
80. Set(z) & ∃y.((y ϵ U) & (z ϵ y)) AndInt 55 79
81. z ϵ {y: ∃j.((j ϵ U) & (y ϵ j))} ClassInt 80
83. z ϵ ∪U EqualitySub 81 82
84. (z ϵ U) -> (z ϵ ∪U) ImpInt 83
85. z ϵ ∪U Hyp
86. ∃y.(z ϵ y) ExistsInt 85
87. Set(z) DefSub 86
89. Set(z) -> (z ϵ U) ForallElim 88
90. z ϵ U ImpElim 87 89
91. (z ϵ ∪U) -> (z ϵ U) ImpInt 90
92. ((z ϵ U) -> (z ϵ ∪U)) & ((z ϵ ∪U) -> (z ϵ U)) AndInt 84 91
95. ∀y.((U = y) <-> ∀z.((z ϵ U) <-> (z ϵ y))) ForallElim 41
96. (U = ∪U) <-> ∀z.((z ϵ U) <-> (z ϵ ∪U)) ForallElim 95
99. U = ∪U ImpElim 94 98
100. (0 = ∩U) & (U = ∪U) AndInt 46 99 Qed

Used Theorems

1. (0 ⊂ x) & (x ⊂ U)
2. (Set(x) & (y ⊂ x)) -> Set(y)
3. (x ϵ U) <-> Set(x)
4. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

Th35. ¬(x = 0) -> Set(∩x)

0. ∀z.¬(z ϵ a) Hyp
1. z ϵ a Hyp
2. ¬(z ϵ a) ForallElim 0
3. _|_ ImpElim 1 2
4. z ϵ 0 AbsI 3
5. (z ϵ a) -> (z ϵ 0) ImpInt 4
6. z ϵ 0 Hyp
8. z ϵ {x: ¬(x = x)} EqualitySub 6 7
9. Set(z) & ¬(z = z) ClassElim 8
12. _|_ ImpElim 11 10
13. z ϵ a AbsI 12

```
14. (z ϵ 0) -> (z ϵ a)  ImpInt 13
15. ((z ϵ a) -> (z ϵ 0)) & ((z ϵ 0) -> (z ϵ a))  AndInt 5 14
17. ∀z.((z ϵ a) <-> (z ϵ 0))  ForallInt 16
18. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
19. ∀y.((a = y) <-> ∀z.((z ϵ a) <-> (z ϵ y)))  ForallElim 18
20. (a = 0) <-> ∀z.((z ϵ a) <-> (z ϵ 0))  ForallElim 19
23. a = 0  ImpElim 17 22
24. ∀z.¬(z ϵ a) -> (a = 0)  ImpInt 23
25. (A -> B) -> (¬B -> ¬A)  TheoremInt
26. (∀z.¬(z ϵ a) -> B) -> (¬B -> ¬∀z.¬(z ϵ a))  PolySub 25
27. (∀z.¬(z ϵ a) -> (a = 0)) -> (¬(a = 0) -> ¬∀z.¬(z ϵ a))  PolySub 26
28. ¬(a = 0) -> ¬∀z.¬(z ϵ a)  ImpElim 24 27
29. ¬∀z.¬(z ϵ a)  Hyp
30. ¬∃z.(z ϵ a)  Hyp
31. z ϵ a  Hyp
32. ∃z.(z ϵ a)  ExistsInt 31
33. _|_  ImpElim 32 30
34. ¬(z ϵ a)  ImpInt 33
35. ∀z.¬(z ϵ a)  ForallInt 34
36. ¬∃z.(z ϵ a) -> ∀z.¬(z ϵ a)  ImpInt 35
37. (A -> B) -> (¬B -> ¬A)  TheoremInt
38. (¬∃z.(z ϵ a) -> B) -> (¬B -> ¬¬∃z.(z ϵ a))  PolySub 37
39. (¬∃x_0.(x_0 ϵ a) -> ∀z.¬(z ϵ a)) -> (¬∀z.¬(z ϵ a) -> ¬¬∃x_0.(x_0 ϵ a))  PolySub 38
40. ¬∀z.¬(z ϵ a) -> ¬¬∃x_0.(x_0 ϵ a)  ImpElim 36 39
41. D <-> ¬¬D  TheoremInt
42. ∃l.(l ϵ a) <-> ¬¬∃l.(l ϵ a)  PolySub 41
45. ¬(a = 0)  Hyp
46. ¬∀z.¬(z ϵ a)  ImpElim 45 28
47. ¬¬∃x_0.(x_0 ϵ a)  ImpElim 46 40
48. ∃l.(l ϵ a)  ImpElim 47 44
49. ¬(a = 0) -> ∃l.(l ϵ a)  ImpInt 48
50. ∃l.(l ϵ a)  Hyp
51. b ϵ a  Hyp
52. (x ϵ y) -> ((x ⊂ ∪y) & (∩y ⊂ x))  TheoremInt
56. (b ϵ a) -> ((b ⊂ ∪a) & (∩a ⊂ b))  ForallElim 55
57. (b ⊂ ∪a) & (∩a ⊂ b)  ImpElim 51 56
59. ∃y.(b ϵ y)  ExistsInt 51
60. Set(b)  DefSub 59
61. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
65. (Set(b) & (∩a ⊂ b)) -> Set(∩a)  ForallElim 64
66. Set(b) & (∩a ⊂ b)  AndInt 60 58
67. Set(∩a)  ImpElim 66 65
68. Set(∩a)  ExistsElim 50 51 67
69. ∃l.(l ϵ a) -> Set(∩a)  ImpInt 68
70. ¬(a = 0)  Hyp
71. ∃l.(l ϵ a)  ImpElim 70 49
72. Set(∩a)  ImpElim 71 69
73. ¬(a = 0) -> Set(∩a)  ImpInt 72
75. ¬(x = 0) -> Set(∩x)  ForallElim 74 Qed
```

Used Theorems

```
1. (A -> B) -> (¬B -> ¬A)
2. D <-> ¬¬D
4. (x ϵ y) -> ((x ⊂ ∪y) & (∩y ⊂ x))
5. (Set(x) & (y ⊂ x)) -> Set(y)
```

Th37. U = PU

```
0. x ϵ U  Hyp
1. (0 ⊂ x) & (x ⊂ U)  TheoremInt
```

5. PU = {y: (y ⊂ U)} ForallElim 4
6. ∃y.(x ϵ y) ExistsInt 0
7. Set(x) DefSub 6
8. Set(x) & (x ⊂ U) AndInt 7 2
9. x ϵ {y: (y ⊂ U)} ClassInt 8
11. x ϵ PU EqualitySub 9 10
12. (x ϵ U) -> (x ϵ PU) ImpInt 11
13. x ϵ PU Hyp
14. ∃y.(x ϵ y) ExistsInt 13
15. Set(x) DefSub 14
16. (x ϵ U) <-> Set(x) TheoremInt
19. x ϵ U ImpElim 15 18
20. (x ϵ PU) -> (x ϵ U) ImpInt 19
21. ((x ϵ U) -> (x ϵ PU)) & ((x ϵ PU) -> (x ϵ U)) AndInt 12 20
23. ∀z.((z ϵ U) <-> (z ϵ PU)) ForallInt 22
24. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
25. ∀y.((U = y) <-> ∀z.((z ϵ U) <-> (z ϵ y))) ForallElim 24
26. (U = PU) <-> ∀z.((z ϵ U) <-> (z ϵ PU)) ForallElim 25
29. U = PU ImpElim 23 28 Qed

Used Theorems

1. (0 ⊂ x) & (x ⊂ U)
2. (x ϵ U) <-> Set(x)

Th38. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ϵ Px)))

0. Set(a) Hyp
1. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ϵ y))) AxInt
3. Set(a) -> ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ϵ y))) ForallElim 2
4. ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ϵ y))) ImpElim 0 3
5. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
7. (Set(x) & (Pa ⊂ x)) -> Set(Pa) ForallElim 6
8. Set(b) & ∀z.((z ⊂ a) -> (z ϵ b)) Hyp
10. (Set(b) & (Pa ⊂ b)) -> Set(Pa) ForallElim 9
11. z ϵ Pa Hyp
14. Pa = {y: (y ⊂ a)} ForallElim 13
15. z ϵ {y: (y ⊂ a)} EqualitySub 11 14
16. Set(z) & (z ⊂ a) ClassElim 15
19. (z ⊂ a) -> (z ϵ b) ForallElim 17
20. z ϵ b ImpElim 18 19
21. (z ϵ Pa) -> (z ϵ b) ImpInt 20
22. ∀z.((z ϵ Pa) -> (z ϵ b)) ForallInt 21
23. Pa ⊂ b DefSub 22
25. Set(b) & (Pa ⊂ b) AndInt 24 23
26. Set(Pa) ImpElim 25 10
27. Set(Pa) ExistsElim 4 8 26
28. z ⊂ a Hyp
29. Set(a) & (z ⊂ a) AndInt 0 28
33. (Set(a) & (z ⊂ a)) -> Set(z) ForallElim 32
34. Set(z) ImpElim 29 33
35. Set(z) & (z ⊂ a) AndInt 34 28
36. z ϵ {y: (y ⊂ a)} ClassInt 35
38. z ϵ Pa EqualitySub 36 37
39. (z ⊂ a) -> (z ϵ Pa) ImpInt 38
40. z ϵ Pa Hyp
41. z ϵ {y: (y ⊂ a)} EqualitySub 40 14
42. Set(z) & (z ⊂ a) ClassElim 41
44. (z ϵ Pa) -> (z ⊂ a) ImpInt 43
45. ((z ⊂ a) -> (z ϵ Pa)) & ((z ϵ Pa) -> (z ⊂ a)) AndInt 39 44
47. Set(Pa) & ((z ⊂ a) <-> (z ϵ Pa)) AndInt 27 46

48. Set(a) -> (Set(Pa) & ((z ⊂ a) <-> (z ϵ Pa))) ImpInt 47
52. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ϵ Px))) ForallElim 51 Qed

Used Theorems

1. (Set(x) & (y ⊂ x)) -> Set(y)

Th39. ¬Set(U)

1. rus ϵ rus Hyp
2. rus ϵ {z: ¬(z ϵ z)} EqualitySub 1 0
3. Set(rus) & ¬(rus ϵ rus) ClassElim 2
5. _|_ ImpElim 1 4
6. ¬Set(rus) AbsI 5
7. ¬(rus ϵ rus) Hyp
8. Set(rus) Hyp
9. Set(rus) & ¬(rus ϵ rus) AndInt 8 7
10. rus ϵ {z: ¬(z ϵ z)} ClassInt 9
12. rus ϵ rus EqualitySub 10 11
13. _|_ ImpElim 12 7
14. ¬Set(rus) ImpInt 13
15. A v ¬A TheoremInt
16. (rus ϵ rus) v ¬(rus ϵ rus) PolySub 15
17. ¬Set(rus) OrElim 16 1 6 7 14
18. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
19. (0 ⊂ x) & (x ⊂ U) TheoremInt
21. Set(U) Hyp
23. rus ⊂ U ForallElim 22
24. Set(U) & (rus ⊂ U) AndInt 21 23
28. (Set(U) & (rus ⊂ U)) -> Set(rus) ForallElim 27
29. Set(rus) ImpElim 24 28
30. _|_ ImpElim 29 17
31. ¬Set(U) ImpInt 30 Qed

Used Theorems

1. A v ¬A
2. (Set(x) & (y ⊂ x)) -> Set(y)
3. (0 ⊂ x) & (x ⊂ U)

Th41. Set(x) -> ((y ϵ {x}) <-> (y = x))

0. Set(x) Hyp
1. y ϵ {x} Hyp
3. y ϵ {z: ((x ϵ U) -> (z = x))} EqualitySub 1 2
4. Set(y) & ((x ϵ U) -> (y = x)) ClassElim 3
5. (x ϵ U) <-> Set(x) TheoremInt
8. x ϵ U ImpElim 0 7
10. y = x ImpElim 8 9
11. (y ϵ {x}) -> (y = x) ImpInt 10
12. y = x Hyp
14. Set(y) EqualitySub 0 13
15. y = x Hyp
16. x ϵ U Hyp
17. (x ϵ U) -> (y = x) ImpInt 15
18. (y = x) -> ((x ϵ U) -> (y = x)) ImpInt 17
19. (x ϵ U) -> (y = x) ImpElim 12 18
20. Set(y) & ((x ϵ U) -> (y = x)) AndInt 14 19
21. y ϵ {z: ((x ϵ U) -> (z = x))} ClassInt 20
23. y ϵ {x} EqualitySub 21 22
24. (y = x) -> (y ϵ {x}) ImpInt 23

```
25. ((y ϵ {x}) -> (y = x)) & ((y = x) -> (y ϵ {x}))  AndInt 11 24
27. Set(x) -> ((y ϵ {x}) <-> (y = x))  ImpInt 26 Qed
```

Used Theorems

```
1. (x ϵ U) <-> Set(x)
```

Th42. Set(x) -> Set({x})

```
0. Set(x)  Hyp
1. z ϵ {x}  Hyp
3. z ϵ {z: ((x ϵ U) -> (z = x))}  EqualitySub 1 2
4. Set(z) & ((x ϵ U) -> (z = x))  ClassElim 3
6. (x ϵ U) <-> Set(x)  TheoremInt
10. x ϵ U  ImpElim 0 9
11. z = x  ImpElim 10 5
12. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
18. (z = x) -> ((z ⊂ x) & (x ⊂ z))  ForallElim 17
19. (z ⊂ x) & (x ⊂ z)  ImpElim 11 18
21. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ϵ Px)))  TheoremInt
22. Set(Px) & ((y ⊂ x) <-> (y ϵ Px))  ImpElim 0 21
27. (z ⊂ x) -> (z ϵ Px)  ForallElim 26
28. z ϵ Px  ImpElim 20 27
29. (z ϵ {x}) -> (z ϵ Px)  ImpInt 28
30. ∀z.((z ϵ {x}) -> (z ϵ Px))  ForallInt 29
31. {x} ⊂ Px  DefSub 30
32. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
36. (Set(Px) & ({x} ⊂ Px)) -> Set({x})  ForallElim 35
38. Set(Px) & ({x} ⊂ Px)  AndInt 37 31
39. Set({x})  ImpElim 38 36
40. Set(x) -> Set({x})  ImpInt 39 Qed
```

Used Theorems

```
3. (x ϵ U) <-> Set(x)
2. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
1. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ϵ Px)))
4. (Set(x) & (y ⊂ x)) -> Set(y)
```

Th43. ({x} = U) <-> ¬Set(x)

```
0. Set(x)  Hyp
1. Set(x) -> Set({x})  TheoremInt
2. Set({x})  ImpElim 0 1
3. ¬Set(U)  TheoremInt
4. {x} = U  Hyp
5. Set(U)  EqualitySub 2 4
6. _|_  ImpElim 5 3
7. ¬({x} = U)  ImpInt 6
8. ¬Set(x)  Hyp
9. x ϵ U  Hyp
10. ∃y.(x ϵ y)  ExistsInt 9
11. Set(x)  DefSub 10
12. _|_  ImpElim 11 8
13. ¬(x ϵ U)  ImpInt 12
14. x ϵ U  Hyp
15. _|_  ImpElim 14 13
16. y = x  AbsI 15
17. (x ϵ U) -> (y = x)  ImpInt 16
18. y ϵ U  Hyp
19. (x ϵ U) <-> Set(x)  TheoremInt
```

```
23. (y ∈ U) -> Set(y)  ForallElim 22
24. Set(y)  ImpElim 18 23
25. Set(y) & ((x ∈ U) -> (y = x))  AndInt 24 17
26. y ∈ {z: ((x ∈ U) -> (z = x))}  ClassInt 25
29. y ∈ {x}  EqualitySub 26 28
30. (y ∈ U) -> (y ∈ {x})  ImpInt 29
31. ∀z.((z ∈ U) -> (z ∈ {x}))  ForallInt 30
32. U ⊂ {x}  DefSub 31
33. (0 ⊂ x) & (x ⊂ U)  TheoremInt
35. (0 ⊂ {x}) & ({x} ⊂ U)  ForallElim 34
37. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
41. ({x} = U) <-> (({x} ⊂ U) & (U ⊂ {x}))  ForallElim 40
45. ({x} ⊂ U) & (U ⊂ {x})  AndInt 36 32
46. {x} = U  ImpElim 45 44
47. ¬Set(x) -> ({x} = U)  ImpInt 46
48. Set(x) -> ¬({x} = U)  ImpInt 7
49. (A -> B) -> (¬B -> ¬A)  TheoremInt
50. (Set(x) -> B) -> (¬B -> ¬Set(x))  PolySub 49
51. (Set(x) -> ¬({x} = U)) -> (¬¬({x} = U) -> ¬Set(x))  PolySub 50
52. ¬¬({x} = U) -> ¬Set(x)  ImpElim 48 51
53. D <-> ¬¬D  TheoremInt
56. ({x} = U) -> ¬¬({x} = U)  PolySub 55
57. {x} = U  Hyp
58. ¬¬({x} = U)  ImpElim 57 56
59. ¬Set(x)  ImpElim 58 52
60. ({x} = U) -> ¬Set(x)  ImpInt 59
61. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U))  AndInt 60 47
62. ({x} = U) <-> ¬Set(x)  EquivConst 61 Qed
```

Used Theorems

```
1. Set(x) -> Set({x})
2. ¬Set(U)
3. (x ∈ U) <-> Set(x)
4. (0 ⊂ x) & (x ⊂ U)
6. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
10. (A -> B) -> (¬B -> ¬A)
9. D <-> ¬¬D
```

Th44. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))

```
0. z ∈ ∩{x}  Hyp
3. ∩{x} = {z: ∀y.((y ∈ {x}) -> (z ∈ y))}  ForallElim 2
4. z ∈ {z: ∀y.((y ∈ {x}) -> (z ∈ y))}  EqualitySub 0 3
5. Set(z) & ∀y.((y ∈ {x}) -> (z ∈ y))  ClassElim 4
7. Set(x)  Hyp
8. Set(x) -> ((y ∈ {x}) <-> (y = x))  TheoremInt
9. (y ∈ {x}) <-> (y = x)  ImpElim 7 8
13. (x = x) -> (x ∈ {x})  ForallElim 12
15. x ∈ {x}  ImpElim 14 13
16. (x ∈ {x}) -> (z ∈ x)  ForallElim 6
17. z ∈ x  ImpElim 15 16
18. (z ∈ ∩{x}) -> (z ∈ x)  ImpInt 17
19. z ∈ x  Hyp
20. y ∈ {x}  Hyp
22. y = x  ImpElim 20 21
24. z ∈ y  EqualitySub 19 23
25. (y ∈ {x}) -> (z ∈ y)  ImpInt 24
26. ∀y.((y ∈ {x}) -> (z ∈ y))  ForallInt 25
27. ∃x.(z ∈ x)  ExistsInt 19
28. Set(z)  DefSub 27
```

```
29. Set(z) & ∀y.((y ϵ {x}) -> (z ϵ y))  AndInt 28 26
30. z ϵ {z: ∀y.((y ϵ {x}) -> (z ϵ y))}  ClassInt 29
32. z ϵ ∩{x}  EqualitySub 30 31
33. (z ϵ x) -> (z ϵ ∩{x})  ImpInt 32
34. ((z ϵ ∩{x}) -> (z ϵ x)) & ((z ϵ x) -> (z ϵ ∩{x}))  AndInt 18 33
36. ∀z.((z ϵ ∩{x}) <-> (z ϵ x))  ForallInt 35
37. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
38. ∀y.((∩{x} = y) <-> ∀z.((z ϵ ∩{x}) <-> (z ϵ y)))  ForallElim 37
39. (∩{x} = x) <-> ∀z.((z ϵ ∩{x}) <-> (z ϵ x))  ForallElim 38
42. ∩{x} = x  ImpElim 36 41
43. z ϵ ∪{x}  Hyp
46. ∪{x} = {z: ∃y.((y ϵ {x}) & (z ϵ y))}  ForallElim 45
47. z ϵ {z: ∃y.((y ϵ {x}) & (z ϵ y))}  EqualitySub 43 46
48. Set(z) & ∃y.((y ϵ {x}) & (z ϵ y))  ClassElim 47
50. (a ϵ {x}) & (z ϵ a)  Hyp
52. (a ϵ {x}) -> (a = x)  ForallElim 51
54. a = x  ImpElim 53 52
56. z ϵ x  EqualitySub 55 54
57. z ϵ x  ExistsElim 49 50 56
58. (z ϵ ∪{x}) -> (z ϵ x)  ImpInt 57
59. z ϵ x  Hyp
62. (x = x) -> (x ϵ {x})  ForallElim 61
63. x ϵ {x}  ImpElim 14 62
64. (x ϵ {x}) & (z ϵ x)  AndInt 63 59
66. ∃y.(z ϵ y)  ExistsInt 59
67. Set(z)  DefSub 66
68. Set(z) & ∃y.((y ϵ {x}) & (z ϵ y))  AndInt 67 65
69. z ϵ {z: ∃y.((y ϵ {x}) & (z ϵ y))}  ClassInt 68
71. z ϵ ∪{x}  EqualitySub 69 70
72. (z ϵ x) -> (z ϵ ∪{x})  ImpInt 71
73. ((z ϵ ∪{x}) -> (z ϵ x)) & ((z ϵ x) -> (z ϵ ∪{x}))  AndInt 58 72
76. ∀y.((∪{x} = y) <-> ∀z.((z ϵ ∪{x}) <-> (z ϵ y)))  ForallElim 37
77. (∪{x} = x) <-> ∀z.((z ϵ ∪{x}) <-> (z ϵ x))  ForallElim 76
80. ∪{x} = x  ImpElim 75 79
81. (∩{x} = x) & (∪{x} = x)  AndInt 42 80
82. Set(x) -> ((∩{x} = x) & (∪{x} = x))  ImpInt 81
83. ¬Set(x)  Hyp
84. ({x} = U) <-> ¬Set(x)  TheoremInt
87. {x} = U  ImpElim 83 86
88. (0 = ∩U) & (U = ∪U)  TheoremInt
90. (0 = ∩{x}) & (U = ∪{x})  EqualitySub 88 89
95. (∩{x} = 0) & (∪{x} = U)  AndInt 93 94
96. ¬Set(x) -> ((∩{x} = 0) & (∪{x} = U))  ImpInt 95
97. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))  AndInt 82 96 Qed
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Used Theorems

```
1. Set(x) -> ((y ϵ {x}) <-> (y = x))
2. ({x} = U) <-> ¬Set(x)
3. (0 = ∩U) & (U = ∪U)
```

Th46. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))

```
0. Set(x) & Set(y)  Hyp
1. Set(x) -> Set({x})  TheoremInt
4. Set({x})  ImpElim 2 1
6. Set(y) -> Set({y})  ForallElim 5
7. Set({y})  ImpElim 3 6
8. (Set(x) & Set(y)) -> Set((x ∪ y))  AxInt
12. (Set({x}) & Set({y})) -> Set(({x} ∪ {y}))  ForallElim 11
```

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13. Set({x}) & Set({y})  AndInt 4 7
14. Set(({x} ∪ {y}))  ImpElim 13 12
17. Set({x,y})  EqualitySub 14 16
18. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
20. z ϵ {x,y}  Hyp
21. z ϵ ({x} ∪ {y})  EqualitySub 20 15
27. (z ϵ ({x} ∪ {y})) -> ((z ϵ {x}) v (z ϵ {y}))  ForallElim 26
28. (z ϵ {x}) v (z ϵ {y})  ImpElim 21 27
29. z ϵ {x}  Hyp
30. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
34. Set(y) -> ((z ϵ {y}) <-> (z = y))  ForallElim 33
35. (z ϵ {x}) <-> (z = x)  ImpElim 2 32
38. z = x  ImpElim 29 37
39. (z = x) v (z = y)  OrIntR 38
40. z ϵ {y}  Hyp
41. (z ϵ {y}) <-> (z = y)  ImpElim 3 34
44. z = y  ImpElim 40 43
45. (z = x) v (z = y)  OrIntL 44
46. (z = x) v (z = y)  OrElim 28 29 39 40 45
47. (z ϵ {x,y}) -> ((z = x) v (z = y))  ImpInt 46
48. (z = x) v (z = y)  Hyp
49. z = x  Hyp
51. z ϵ {x}  ImpElim 49 50
52. (z ϵ {x}) v (z ϵ {y})  OrIntR 51
57. ((z ϵ {x}) v (z ϵ {y})) -> (z ϵ ({x} ∪ {y}))  ForallElim 56
58. z ϵ ({x} ∪ {y})  ImpElim 52 57
59. z = y  Hyp
61. z ϵ {y}  ImpElim 59 60
62. (z ϵ {x}) v (z ϵ {y})  OrIntL 61
63. z ϵ ({x} ∪ {y})  ImpElim 62 57
64. z ϵ ({x} ∪ {y})  OrElim 48 49 58 59 63
65. ((z = x) v (z = y)) -> (z ϵ ({x} ∪ {y}))  ImpInt 64
66. ((z = x) v (z = y)) -> (z ϵ {x,y})  EqualitySub 65 16
67. ((z ϵ {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ϵ {x,y}))  AndInt 47 66
69. Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y)))  AndInt 17 68
70. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))  ImpInt 69
71. {x,y} = U  Hyp
72. ({x} ∪ {y}) = U  EqualitySub 71 15
73. ¬Set(U)  TheoremInt
75. ¬Set(({x} ∪ {y}))  EqualitySub 73 74
76. (Set(x) & Set(y)) -> Set((x ∪ y))  AxInt
77. (A -> B) -> (¬B -> ¬A)  TheoremInt
78. ((Set(x) & Set(y)) -> B) -> (¬B -> ¬(Set(x) & Set(y)))  PolySub 77
79. ((Set(x) & Set(y)) -> Set((x ∪ y))) -> (¬Set((x ∪ y)) -> ¬(Set(x) & Set(y)))  PolySub 78
80. ¬Set((x ∪ y)) -> ¬(Set(x) & Set(y))  ImpElim 76 79
84. ¬Set(({x} ∪ {y})) -> ¬(Set({x}) & Set({y}))  ForallElim 83
85. ¬(Set({x}) & Set({y}))  ImpElim 75 84
86. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
88. ¬(Set({x}) & B) <-> (¬Set({x}) v ¬B)  PolySub 87
89. ¬(Set({x}) & Set({y})) <-> (¬Set({x}) v ¬Set({y}))  PolySub 88
92. ¬Set({x}) v ¬Set({y})  ImpElim 85 91
93. ¬Set({x})  Hyp
94. Set(x) -> Set({x})  TheoremInt
95. (Set(x) -> B) -> (¬B -> ¬Set(x))  PolySub 77
96. (Set(x) -> Set({x})) -> (¬Set({x}) -> ¬Set(x))  PolySub 95
97. ¬Set({x}) -> ¬Set(x)  ImpElim 94 96
98. ¬Set(x)  ImpElim 93 97
99. ¬Set({x}) -> ¬Set(x)  ImpInt 98
100. ∀a.(¬Set({a}) -> ¬Set(a))  ForallInt 99
101. ¬Set({y})  Hyp
102. ¬Set({y}) -> ¬Set(y)  ForallElim 100
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103. ¬Set(y) ImpElim 101 102
104. ¬Set(x) v ¬Set(y) OrIntR 98
105. ¬Set(x) v ¬Set(y) OrIntL 103
106. ¬Set(x) v ¬Set(y) OrElim 92 93 104 101 105
107. ({x,y} = U) -> (¬Set(x) v ¬Set(y)) ImpInt 106
108. ¬Set(x) v ¬Set(y) Hyp
109. ¬Set(x) Hyp
110. ({x} = U) <-> ¬Set(x) TheoremInt
113. {x} = U ImpElim 109 112
114. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
117. ({y} ∪ U) = U ForallElim 116
119. ({y} ∪ {x}) = U EqualitySub 117 118
120. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
125. ({x} ∪ {y}) = ({y} ∪ {x}) ForallElim 124
127. ({x} ∪ {y}) = U EqualitySub 119 126
128. {x,y} = U EqualitySub 127 16
129. ¬Set(x) -> ({x,y} = U) ImpInt 128
130. ∀a.(¬Set(a) -> ({a,y} = U)) ForallInt 129
131. ∀b.∀a.(¬Set(a) -> ({a,b} = U)) ForallInt 130
132. ¬Set(y) Hyp
133. ∀a.(¬Set(a) -> ({a,z} = U)) ForallElim 131
144. {y,x} = ({y} ∪ {x}) ForallElim 143
145. {y,x} = ({x} ∪ {y}) EqualitySub 144 126
146. {y,x} = {x,y} EqualitySub 145 16
147. ¬Set(y) -> ({x,y} = U) EqualitySub 136 146
148. {x,y} = U ImpElim 132 147
149. {x,y} = U OrElim 108 109 128 132 148
150. (¬Set(x) v ¬Set(y)) -> ({x,y} = U) ImpInt 149
151. (({x,y} = U) -> (¬Set(x) v ¬Set(y))) & ((¬Set(x) v ¬Set(y)) -> ({x,y} = U)) AndInt 107 150
153. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) AndInt 70 152 Qed

Used Theorems

1. Set(x) -> Set({x})
2. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
3. Set(x) -> ((y ϵ {x}) <-> (y = x))
4. ¬Set(U)
5. (A -> B) -> (¬B -> ¬A)
6. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
7. ({x} = U) <-> ¬Set(x)
8. ((x ∪ U) = U) & ((x ∩ U) = x)
10. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))

Th47. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})))

0. Set(x) & Set(y) Hyp
1. z ϵ ∩{x,y} Hyp
4. ∩{x,y} = {z: ∀x_0.((x_0 ϵ {x,y}) -> (z ϵ x_0))} ForallElim 3
5. z ϵ {z: ∀x_0.((x_0 ϵ {x,y}) -> (z ϵ x_0))} EqualitySub 1 4
6. Set(z) & ∀x_0.((x_0 ϵ {x,y}) -> (z ϵ x_0)) ClassElim 5
8. (x ϵ {x,y}) -> (z ϵ x) ForallElim 7
9. (y ϵ {x,y}) -> (z ϵ y) ForallElim 7
10. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
12. Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))) ImpElim 0 11
19. ((y = x) v (y = y)) -> (y ϵ {x,y}) ForallElim 18
22. (x = x) v (x = y) OrIntR 20
23. x ϵ {x,y} ImpElim 22 17
24. z ϵ x ImpElim 23 8

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25. (y = x) v (y = y)  OrIntL 21
26. y ϵ {x,y}  ImpElim 25 19
27. z ϵ y  ImpElim 26 9
28. (z ϵ x) & (z ϵ y)  AndInt 24 27
29. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
33. z ϵ (x ∩ y)  ImpElim 28 32
34. (z ϵ ∩{x,y}) -> (z ϵ (x ∩ y))  ImpInt 33
35. z ϵ (x ∩ y)  Hyp
37. (z ϵ x) & (z ϵ y)  ImpElim 35 36
38. c ϵ {x,y}  Hyp
41. (c ϵ {x,y}) -> ((c = x) v (c = y))  ForallElim 40
42. (c = x) v (c = y)  ImpElim 38 41
43. c = x  Hyp
46. z ϵ c  EqualitySub 44 45
47. c = y  Hyp
50. z ϵ c  EqualitySub 48 49
51. z ϵ c  OrElim 42 43 46 47 50
52. (c ϵ {x,y}) -> (z ϵ c)  ImpInt 51
53. ∀c.((c ϵ {x,y}) -> (z ϵ c))  ForallInt 52
54. ∃c.(z ϵ c)  ExistsInt 35
55. Set(z)  DefSub 54
56. Set(z) & ∀c.((c ϵ {x,y}) -> (z ϵ c))  AndInt 55 53
57. z ϵ {c: ∀x_4.((x_4 ϵ {x,y}) -> (c ϵ x_4))}  ClassInt 56
59. z ϵ ∩{x,y}  EqualitySub 57 58
60. (z ϵ (x ∩ y)) -> (z ϵ ∩{x,y})  ImpInt 59
61. ((z ϵ ∩{x,y}) -> (z ϵ (x ∩ y))) & ((z ϵ (x ∩ y)) -> (z ϵ ∩{x,y}))  AndInt 34 60
63. ∀z.((z ϵ ∩{x,y}) <-> (z ϵ (x ∩ y)))  ForallInt 62
64. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
65. ∀x_6.((∩{x,y} = x_6) <-> ∀z.((z ϵ ∩{x,y}) <-> (z ϵ x_6)))  ForallElim 64
66. (∩{x,y} = (x ∩ y)) <-> ∀z.((z ϵ ∩{x,y}) <-> (z ϵ (x ∩ y)))  ForallElim 65
69. ∩{x,y} = (x ∩ y)  ImpElim 63 68
70. z ϵ ∪{x,y}  Hyp
73. ∪{x,y} = {z: ∃x_8.((x_8 ϵ {x,y}) & (z ϵ x_8))}  ForallElim 72
74. z ϵ {z: ∃x_8.((x_8 ϵ {x,y}) & (z ϵ x_8))}  EqualitySub 70 73
75. Set(z) & ∃x_8.((x_8 ϵ {x,y}) & (z ϵ x_8))  ClassElim 74
77. (u ϵ {x,y}) & (z ϵ u)  Hyp
79. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))  TheoremInt
81. Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y)))  ImpElim 0 80
86. (u ϵ {x,y}) -> ((u = x) v (u = y))  ForallElim 85
87. (u = x) v (u = y)  ImpElim 78 86
88. u = x  Hyp
90. z ϵ x  EqualitySub 89 88
91. (z ϵ x) v (z ϵ y)  OrIntR 90
92. u = y  Hyp
93. z ϵ y  EqualitySub 89 92
94. (z ϵ x) v (z ϵ y)  OrIntL 93
95. (z ϵ x) v (z ϵ y)  OrElim 87 88 91 92 94
96. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
100. z ϵ (x ∪ y)  ImpElim 95 99
101. z ϵ (x ∪ y)  ExistsElim 76 77 100
102. (z ϵ ∪{x,y}) -> (z ϵ (x ∪ y))  ImpInt 101
103. z ϵ (x ∪ y)  Hyp
105. (z ϵ x) v (z ϵ y)  ImpElim 103 104
106. z ϵ x  Hyp
110. ((x = x) v (x = y)) -> (x ϵ {x,y})  ForallElim 109
112. (x = x) v (x = y)  OrIntR 111
113. x ϵ {x,y}  ImpElim 112 110
114. (x ϵ {x,y}) & (z ϵ x)  AndInt 113 106
116. ∃y.(z ϵ y)  ExistsInt 106
117. Set(z)  DefSub 116
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118. Set(z) & ∃a.((a ϵ {x,y}) & (z ϵ a))  AndInt 117 115
119. z ϵ {b: ∃a.((a ϵ {x,y}) & (b ϵ a))}  ClassInt 118
121. z ϵ ∪{x,y}  EqualitySub 119 120
122. z ϵ y  Hyp
125. ((y = x) v (y = y)) -> (y ϵ {x,y})  ForallElim 124
126. (y = x) v (y = y)  OrIntL 123
127. y ϵ {x,y}  ImpElim 126 125
128. (y ϵ {x,y}) & (z ϵ y)  AndInt 127 122
130. ∃y.(z ϵ y)  ExistsInt 122
131. Set(z)  DefSub 130
132. Set(z) & ∃a.((a ϵ {x,y}) & (z ϵ a))  AndInt 131 129
133. z ϵ {b: ∃a.((a ϵ {x,y}) & (b ϵ a))}  ClassInt 132
134. z ϵ ∪{x,y}  EqualitySub 133 120
135. z ϵ ∪{x,y}  OrElim 105 106 121 122 134
136. (z ϵ (x ∪ y)) -> (z ϵ ∪{x,y})  ImpInt 135
137. ((z ϵ ∪{x,y}) -> (z ϵ (x ∪ y))) & ((z ϵ (x ∪ y)) -> (z ϵ ∪{x,y}))  AndInt 102 136
139. ∀z.((z ϵ ∪{x,y}) <-> (z ϵ (x ∪ y)))  ForallInt 138
140. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
141. ∀x_14.((∪{x,y} = x_14) <-> ∀z.((z ϵ ∪{x,y}) <-> (z ϵ x_14)))  ForallElim 140
142. (∪{x,y} = (x ∪ y)) <-> ∀z.((z ϵ ∪{x,y}) <-> (z ϵ (x ∪ y)))  ForallElim 141
145. ∪{x,y} = (x ∪ y)  ImpElim 139 144
146. (∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y))  AndInt 69 145
147. (Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))  ImpInt 146
148. ¬Set(x) v ¬Set(y)  Hyp
149. ({x} = U) <-> ¬Set(x)  TheoremInt
152. ¬Set(x)  Hyp
153. {x} = U  ImpElim 152 151
155. {x,y} = (U ∪ {y})  EqualitySub 154 153
156. ((x ∪ U) = U) & ((x ∩ U) = x)  TheoremInt
158. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))  TheoremInt
161. (x ∪ U) = (U ∪ x)  ForallElim 160
162. (U ∪ x) = U  EqualitySub 157 161
164. (U ∪ {y}) = U  ForallElim 163
165. {x,y} = U  EqualitySub 155 164
166. (0 = ∩U) & (U = ∪U)  TheoremInt
168. (0 = ∩{x,y}) & (U = ∪{x,y})  EqualitySub 166 167
169. ¬Set(y)  Hyp
171. ¬Set(y) -> ({y} = U)  ForallElim 170
172. {y} = U  ImpElim 169 171
173. {x,y} = ({x} ∪ U)  EqualitySub 154 172
175. ({x} ∪ U) = U  ForallElim 174
176. {x,y} = U  EqualitySub 173 175
178. (0 = ∩{x,y}) & (U = ∪{x,y})  EqualitySub 166 177
179. (0 = ∩{x,y}) & (U = ∪{x,y})  OrElim 148 152 168 169 178
180. (¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y}))  ImpInt 179
181. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x
,y}) & (U = ∪{x,y})))  AndInt 147 180 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))
2. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
3. ({x} = U) <-> ¬Set(x)
4. ((x ∪ U) = U) & ((x ∩ U) = x)
5. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
6. (0 = ∩U) & (U = ∪U)

Th49. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

0. Set(x) & Set(y)  Hyp
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2. Set(x) -> Set({x})  TheoremInt
3. Set({x})  ImpElim 1 2
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))  TheoremInt
6. Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y)))  ImpElim 0 5
11. (Set({x}) & Set({x,y})) -> (Set({{x},{x,y}}) & ((z ϵ {{x},{x,y}}) <-> ((z = {x}) v (z = {x,y}))))  ForallElim
12. Set({x}) & Set({x,y})  AndInt 3 7
13. Set({{x},{x,y}}) & ((z ϵ {{x},{x,y}}) <-> ((z = {x}) v (z = {x,y})))  ImpElim 12 11
17. Set((x,y))  EqualitySub 14 16
18. (Set(x) & Set(y)) -> Set((x,y))  ImpInt 17
19. ¬Set(x) v ¬Set(y)  Hyp
20. ¬Set(x)  Hyp
21. ({x} = U) <-> ¬Set(x)  TheoremInt
24. {x} = U  ImpElim 20 23
25. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))  TheoremInt
29. ¬Set(x) v ¬Set(y)  OrIntR 20
30. {x,y} = U  ImpElim 29 28
31. ¬Set(U)  TheoremInt
33. ¬Set({x})  EqualitySub 31 32
35. ¬Set({x}) -> ({{x}} = U)  ForallElim 34
36. {{x}} = U  ImpElim 33 35
41. {{x},{x,y}} = ({{x}} ∪ {{x,y}})  ForallElim 40
43. ¬Set({x,y})  EqualitySub 31 42
45. ¬Set({x,y}) -> ({{x,y}} = U)  ForallElim 44
46. {{x,y}} = U  ImpElim 43 45
47. {{x},{x,y}} = ({{x}} ∪ U)  EqualitySub 41 46
48. ((x ∪ U) = U) & ((x ∩ U) = x)  TheoremInt
51. ({{x}} ∪ U) = U  ForallElim 50
52. {{x},{x,y}} = U  EqualitySub 47 51
53. (x,y) = U  EqualitySub 15 52
55. ¬Set((x,y))  EqualitySub 31 54
56. ¬Set(y)  Hyp
57. ¬Set(x) v ¬Set(y)  OrIntL 56
58. {x,y} = U  ImpElim 57 28
60. ¬Set({x,y})  EqualitySub 31 59
61. {{x,y}} = U  ImpElim 60 45
62. {{x},{x,y}} = ({{x}} ∪ U)  EqualitySub 41 61
63. {{x},{x,y}} = U  EqualitySub 62 51
64. (x,y) = U  EqualitySub 15 63
66. ¬Set((x,y))  EqualitySub 31 65
67. ¬Set((x,y))  OrElim 19 20 55 56 66
68. (¬Set(x) v ¬Set(y)) -> ¬Set((x,y))  ImpInt 67
69. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
73. ¬(Set(x) & B) -> (¬Set(x) v ¬B)  PolySub 72
74. ¬(Set(x) & Set(y)) -> (¬Set(x) v ¬Set(y))  PolySub 73
75. ¬(Set(x) & Set(y))  Hyp
76. ¬Set(x) v ¬Set(y)  ImpElim 75 74
77. ¬Set((x,y))  ImpElim 76 68
78. ¬(Set(x) & Set(y)) -> ¬Set((x,y))  ImpInt 77
79. (A -> B) -> (¬B -> ¬A)  TheoremInt
80. (¬(Set(x) & Set(y)) -> B) -> (¬B -> ¬¬(Set(x) & Set(y)))  PolySub 79
81. (¬(Set(x) & Set(y)) -> ¬Set((x,y))) -> (¬¬Set((x,y)) -> ¬¬(Set(x) & Set(y)))  PolySub 80
82. ¬¬Set((x,y)) -> ¬¬(Set(x) & Set(y))  ImpElim 78 81
83. D <-> ¬¬D  TheoremInt
88. Set((x,y)) -> ¬¬Set((x,y))  PolySub 85
89. ¬¬(Set(x) & Set(y)) -> (Set(x) & Set(y))  PolySub 87
90. Set((x,y))  Hyp
91. ¬¬Set((x,y))  ImpElim 90 88
92. ¬¬(Set(x) & Set(y))  ImpElim 91 82
93. Set(x) & Set(y)  ImpElim 92 89
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94. Set((x,y)) -> (Set(x) & Set(y))  ImpInt 93
95. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  AndInt 18 94
97. ¬Set((x,y))  Hyp
98. ((Set(x) & Set(y)) -> B) -> (¬B -> ¬(Set(x) & Set(y)))  PolySub 79
99. ((Set(x) & Set(y)) -> Set((x,y))) -> (¬Set((x,y)) -> ¬(Set(x) & Set(y)))  PolySub 98
100. ¬Set((x,y)) -> ¬(Set(x) & Set(y))  ImpElim 18 99
101. ¬(Set(x) & Set(y))  ImpElim 97 100
102. ¬Set(x) v ¬Set(y)  ImpElim 101 74
103. ¬Set(x)  Hyp
104. {x} = U  ImpElim 103 23
106. ¬Set({x})  EqualitySub 31 105
107. {{x}} = U  ImpElim 106 35
108. {{x},{x,y}} = (U ∪ {{x,y}})  EqualitySub 41 107
109. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))  TheoremInt
114. (U ∪ {{x,y}}) = ({{x,y}} ∪ U)  ForallElim 113
115. {{x},{x,y}} = ({{x,y}} ∪ U)  EqualitySub 108 114
116. ((x ∪ U) = U) & ((x ∩ U) = x)  TheoremInt
119. ({{x,y}} ∪ U) = U  ForallElim 118
120. (U ∪ {{x,y}}) = U  EqualitySub 114 119
121. {{x},{x,y}} = U  EqualitySub 108 120
122. (x,y) = U  EqualitySub 15 121
123. ¬Set(y)  Hyp
127. ¬Set(x) v ¬Set(y)  OrIntL 123
128. {x,y} = U  ImpElim 127 126
130. ¬Set({x,y})  EqualitySub 31 129
131. {{x,y}} = U  ImpElim 130 45
132. {{x},{x,y}} = ({{x}} ∪ U)  EqualitySub 41 131
134. ({{x}} ∪ U) = U  ForallElim 133
135. {{x},{x,y}} = U  EqualitySub 132 134
136. (x,y) = U  EqualitySub 15 135
137. (x,y) = U  OrElim 102 103 122 123 136
138. ¬Set((x,y)) -> ((x,y) = U)  ImpInt 137
139. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  AndInt 96 138 Qed

Used Theorems

1. Set(x) -> Set({x})
2. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))
3. ({x} = U) <-> ¬Set(x)
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))
5. ¬Set(U)
6. ((x ∪ U) = U) & ((x ∩ U) = x)
9. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
7. (A -> B) -> (¬B -> ¬A)
8. D <-> ¬¬D
10. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))

Th50. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪
∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U))
& ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))

0. Set(x) & Set(y)  Hyp
1. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x
,y}) & (U = ∪{x,y})))  TheoremInt
3. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))  TheoremInt
5. Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y)))  ImpElim 0 4
7. Set(x) -> Set({x})  TheoremInt
9. Set({x})  ImpElim 8 7
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13. ((Set({x}) & Set({x,y})) -> ((∩{{x},{x,y}} = ({x} ∩ {x,y})) & (∪{{x},{x,y}} = ({x} ∪ {x,y})))) & ((¬Set({x}) v ¬Set({x,y})) -> ((0 = ∩{{x},{x,y}}) & (U = ∪{{x},{x,y}})))  ForallElim 12
14. Set({x}) & Set({x,y})  AndInt 9 6
16. (∩{{x},{x,y}} = ({x} ∩ {x,y})) & (∪{{x},{x,y}} = ({x} ∪ {x,y}))  ImpElim 14 15
18. (∩{{x},{x,y}} = ({x} ∩ ({x} ∪ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))  EqualitySub 16 17
19. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))  TheoremInt
25. (({x} ∩ ({x} ∪ {y})) = (({x} ∩ {x}) ∪ ({x} ∩ {y}))) & (({x} ∪ ({x} ∩ {y})) = (({x} ∪ {x}) ∩ ({x} ∪ {y})))  ForallElim 24
26. ((x ∪ x) = x) & ((x ∩ x) = x)  TheoremInt
28. (({x} ∪ {x}) = {x}) & (({x} ∩ {x}) = {x})  ForallElim 27
33. (∩{{x},{x,y}} = (({x} ∩ {x}) ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))  EqualitySub 18 31
34. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))  EqualitySub 33 30
35. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))  TheoremInt
42. (({x} ∪ {x}) ∪ {y}) = ({x} ∪ ({x} ∪ {y}))  ForallElim 41
44. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = (({x} ∪ {x}) ∪ {y}))  EqualitySub 34 43
45. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ {y}))  EqualitySub 44 29
46. z ϵ ({x} ∩ {y})  Hyp
47. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
54. (z ϵ ({x} ∩ {y})) -> ((z ϵ {x}) & (z ϵ {y}))  ForallElim 53
55. (z ϵ {x}) & (z ϵ {y})  ImpElim 46 54
57. (z ϵ ({x} ∩ {y})) -> (z ϵ {x})  ImpInt 56
58. ∀z.((z ϵ ({x} ∩ {y})) -> (z ϵ {x}))  ForallInt 57
62. ∀z.((z ϵ ({a} ∩ {b})) -> (z ϵ {a}))  ForallElim 61
63. ({a} ∩ {b}) ⊂ {a}  DefSub 62
64. (x ⊂ y) <-> ((x ∪ y) = y)  TheoremInt
68. (({a} ∩ {b}) ⊂ {a}) <-> ((({a} ∩ {b}) ∪ {a}) = {a})  ForallElim 67
71. (({a} ∩ {b}) ∪ {a}) = {a}  ImpElim 63 70
75. (({x} ∩ {y}) ∪ {x}) = {x}  ForallElim 74
76. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))  TheoremInt
83. (({x} ∩ {y}) ∪ {x}) = ({x} ∪ ({x} ∩ {y}))  ForallElim 82
84. ({x} ∪ ({x} ∩ {y})) = {x}  EqualitySub 75 83
85. (∩{{x},{x,y}} = {x}) & (∪{{x},{x,y}} = ({x} ∪ {y}))  EqualitySub 45 84
87. (∩{{x},{x,y}} = {x}) & (∪{{x},{x,y}} = {x,y})  EqualitySub 85 86
88. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))  TheoremInt
90. (∩{x} = x) & (∪{x} = x)  ImpElim 8 89
93. (∩(x,y) = {x}) & (∪(x,y) = {x,y})  EqualitySub 87 92
99. ∩∩(x,y) = x  EqualitySub 98 96
101. ∪∩(x,y) = x  EqualitySub 100 96
102. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})))  TheoremInt
104. (∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y))  ImpElim 0 103
107. ∩∪(x,y) = (x ∩ y)  EqualitySub 105 97
108. ∪∪(x,y) = (x ∪ y)  EqualitySub 106 97
110. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
114. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
118. (¬Set(x) v ¬B) -> ¬(Set(x) & B)  PolySub 117
119. (¬Set(x) v ¬Set(y)) -> ¬(Set(x) & Set(y))  PolySub 118
120. (A -> B) -> (¬B -> ¬A)  TheoremInt
121. (Set((x,y)) -> B) -> (¬B -> ¬Set((x,y)))  PolySub 120
122. (Set((x,y)) -> (Set(x) & Set(y))) -> (¬(Set(x) & Set(y)) -> ¬Set((x,y)))  PolySub 121
123. ¬(Set(x) & Set(y)) -> ¬Set((x,y))  ImpElim 113 122
125. ¬Set(x) v ¬Set(y)  Hyp
126. ¬(Set(x) & Set(y))  ImpElim 125 119
127. ¬Set((x,y))  ImpElim 126 123
128. (x,y) = U  ImpElim 127 124
130. (0 = ∩U) & (U = ∪U)  TheoremInt
131. (0 = ∩(x,y)) & (U = ∪(x,y))  EqualitySub 130 129
134. (∩0 = U) & (∪0 = 0)  TheoremInt
135. (0 = ∩∪(x,y)) & (U = ∪∪(x,y))  EqualitySub 130 132
136. (∩∩(x,y) = U) & (∪∩(x,y) = 0)  EqualitySub 134 133
141. (∪∪(x,y) = U) & (∩∪(x,y) = 0)  AndInt 140 139
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144. (∪∩(x,y) = 0) & (∩∩(x,y) = U) AndInt 143 142
145. ((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0)) AndInt 144 141
146. (¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))) ImpInt 145
147. (∪(x,y) = {x,y}) & (∩(x,y) = {x}) AndInt 95 94
148. (∪∩(x,y) = x) & (∩∩(x,y) = x) AndInt 101 99
149. (∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y)) AndInt 108 107
150. ((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x)) AndInt 147 148
151. (((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪
(x,y) = (x ∩ y))) AndInt 150 149
152. (Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪
(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y)))) ImpInt 151
153. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪
∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U))
& ((∪∪(x,y) = U) & (∩∪(x,y) = 0)))) AndInt 152 146 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x
,y}) & (U = ∪{x,y})))
2. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))
3. Set(x) -> Set({x})
4. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))
5. ((x ∪ x) = x) & ((x ∩ x) = x)
6. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))
7. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
9. (x ⊂ y) <-> ((x ∪ y) = y)
10. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
11. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))
12. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
13. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
14. (A -> B) -> (¬B -> ¬A)
15. (0 = ∩U) & (U = ∪U)
16. (∩0 = U) & (∪0 = 0)

Th53. proj2(U) = U

2. proj2(U) = (∩∪U ∪ (∪∪U ~ ∪∩U)) ForallElim 1
3. (0 = ∩U) & (U = ∪U) TheoremInt
4. (∩0 = U) & (∪0 = 0) TheoremInt
11. proj2(U) = (∩U ∪ (∪U ~ ∪∩U)) EqualitySub 2 10
12. proj2(U) = (0 ∪ (∪U ~ ∪0)) EqualitySub 11 9
13. proj2(U) = (0 ∪ (U ~ ∪0)) EqualitySub 12 10
14. proj2(U) = (0 ∪ (U ~ 0)) EqualitySub 13 8
15. ((0 ∪ x) = x) & ((0 ∩ x) = 0) TheoremInt
18. (0 ∪ (U ~ 0)) = (U ~ 0) ForallElim 17
19. proj2(U) = (U ~ 0) EqualitySub 14 18
24. (U ~ 0) = (U ∩ ~0) ForallElim 23
25. (~0 = U) & (~U = 0) TheoremInt
27. (U ~ 0) = (U ∩ U) EqualitySub 24 26
28. ((x ∪ x) = x) & ((x ∩ x) = x) TheoremInt
31. (U ∩ U) = U ForallElim 30
32. (U ~ 0) = U EqualitySub 27 31
33. proj2(U) = U EqualitySub 19 32 Qed

Used Theorems

1. (0 = ∩U) & (U = ∪U)
2. (∩0 = U) & (∪0 = 0)
3. ((0 ∪ x) = x) & ((0 ∩ x) = 0)
5. (~0 = U) & (~U = 0)

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6. ((x ∪ x) = x) & ((x ∩ x) = x)

Th54. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)))

0. Set(x) & Set(y)  Hyp
3. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))  TheoremInt
5. (((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y)))  ImpElim 0 4
9. ∀x.(proj1(x) = ∩∩x)  ForallInt 1
11. proj1((x,y)) = ∩∩(x,y)  ForallElim 10
12. proj1((x,y)) = x  EqualitySub 11 8
14. proj2((x,y)) = (∩∪(x,y) ∪ (∪∪(x,y) ~ ∪∩(x,y)))  ForallElim 13
19. proj2((x,y)) = (∩∪(x,y) ∪ ((x ∪ y) ~ ∪∩(x,y)))  EqualitySub 14 17
20. proj2((x,y)) = ((x ∩ y) ∪ ((x ∪ y) ~ ∪∩(x,y)))  EqualitySub 19 18
21. proj2((x,y)) = ((x ∩ y) ∪ ((x ∪ y) ~ x))  EqualitySub 20 15
22. z ϵ ((x ∪ y) ~ x)  Hyp
31. ((x ∪ y) ~ x) = ((x ∪ y) ∩ ~x)  ForallElim 30
32. z ϵ ((x ∪ y) ∩ ~x)  EqualitySub 22 31
33. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
44. (z ϵ ((x ∪ y) ∩ ~x)) -> ((z ϵ (x ∪ y)) & (z ϵ ~x))  ForallElim 43
45. (z ϵ (x ∪ y)) & (z ϵ ~x)  ImpElim 32 44
50. (z ϵ x) v (z ϵ y)  ImpElim 46 49
53. z ϵ {y: ¬(y ϵ x)}  EqualitySub 51 52
54. Set(z) & ¬(z ϵ x)  ClassElim 53
56. z ϵ x  Hyp
57. _|_  ImpElim 56 55
58. z ϵ (y ∩ ~x)  AbsI 57
59. z ϵ y  Hyp
60. (z ϵ y) & (z ϵ ~x)  AndInt 59 51
67. ∀a.(((z ϵ y) & (z ϵ a)) -> (z ϵ (y ∩ a)))  ForallInt 66
69. ((z ϵ y) & (z ϵ ~x)) -> (z ϵ (y ∩ ~x))  ForallElim 68
70. z ϵ (y ∩ ~x)  ImpElim 60 69
71. z ϵ (y ∩ ~x)  OrElim 50 56 58 59 70
72. (z ϵ ((x ∪ y) ~ x)) -> (z ϵ (y ∩ ~x))  ImpInt 71
73. z ϵ (y ∩ ~x)  Hyp
80. (z ϵ (y ∩ ~x)) -> ((z ϵ y) & (z ϵ ~x))  ForallElim 79
81. (z ϵ y) & (z ϵ ~x)  ImpElim 73 80
83. (z ϵ x) v (z ϵ y)  OrIntL 82
85. z ϵ (x ∪ y)  ImpElim 83 84
87. (z ϵ (x ∪ y)) & (z ϵ ~x)  AndInt 85 86
94. ((z ϵ (x ∪ y)) & (z ϵ ~x)) -> (z ϵ ((x ∪ y) ∩ ~x))  ForallElim 93
95. z ϵ ((x ∪ y) ∩ ~x)  ImpElim 87 94
97. z ϵ ((x ∪ y) ~ x)  EqualitySub 95 96
98. (z ϵ (y ∩ ~x)) -> (z ϵ ((x ∪ y) ~ x))  ImpInt 97
99. ((z ϵ ((x ∪ y) ~ x)) -> (z ϵ (y ∩ ~x))) & ((z ϵ (y ∩ ~x)) -> (z ϵ ((x ∪ y) ~ x)))  AndInt 72 98
101. ∀z.((z ϵ ((x ∪ y) ~ x)) <-> (z ϵ (y ∩ ~x)))  ForallInt 100
102. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
103. ∀o.((((x ∪ y) ~ x) = o) <-> ∀z.((z ϵ ((x ∪ y) ~ x)) <-> (z ϵ o)))  ForallElim 102
104. (((x ∪ y) ~ x) = (y ∩ ~x)) <-> ∀z.((z ϵ ((x ∪ y) ~ x)) <-> (z ϵ (y ∩ ~x)))  ForallElim 103
107. ((x ∪ y) ~ x) = (y ∩ ~x)  ImpElim 101 106
108. proj2((x,y)) = ((x ∩ y) ∪ (y ∩ ~x))  EqualitySub 21 107
109. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))  TheoremInt
111. proj2((x,y)) = ((y ∩ x) ∪ (y ∩ ~x))  EqualitySub 108 110
112. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))  TheoremInt
124. ((y ∩ x) ∪ (y ∩ ~x)) = (y ∩ (x ∪ ~x))  ForallElim 123
125. proj2((x,y)) = (y ∩ (x ∪ ~x))  EqualitySub 111 124
126. z ϵ U  Hyp
127. A v ¬A  TheoremInt
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128. (z ϵ x) v ¬(z ϵ x) PolySub 127
129. z ϵ x Hyp
130. (z ϵ x) v (z ϵ ~x) OrIntR 129
132. ((z ϵ x) v (z ϵ ~x)) -> (z ϵ (x ∪ ~x)) ForallElim 131
133. z ϵ (x ∪ ~x) ImpElim 130 132
134. ¬(z ϵ x) Hyp
135. ∃y.(z ϵ y) ExistsInt 126
136. Set(z) DefSub 135
137. ¬(z ϵ x) & Set(z) AndInt 134 136
138. z ϵ {z: ¬(z ϵ x)} ClassInt 137
140. z ϵ ~x EqualitySub 138 139
141. (z ϵ x) v (z ϵ ~x) OrIntL 140
142. z ϵ (x ∪ ~x) ImpElim 141 132
143. z ϵ (x ∪ ~x) OrElim 128 129 133 134 142
144. (z ϵ U) -> (z ϵ (x ∪ ~x)) ImpInt 143
145. ∀z.((z ϵ U) -> (z ϵ (x ∪ ~x))) ForallInt 144
146. U ⊂ (x ∪ ~x) DefSub 145
147. (0 ⊂ x) & (x ⊂ U) TheoremInt
150. (x ∪ ~x) ⊂ U ForallElim 149
151. (U ⊂ (x ∪ ~x)) & ((x ∪ ~x) ⊂ U) AndInt 146 150
152. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
158. ((U ⊂ (x ∪ ~x)) & ((x ∪ ~x) ⊂ U)) -> (U = (x ∪ ~x)) ForallElim 157
159. U = (x ∪ ~x) ImpElim 151 158
161. proj2((x,y)) = (y ∩ U) EqualitySub 125 160
162. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
165. (y ∩ U) = y ForallElim 164
166. proj2((x,y)) = y EqualitySub 161 165
167. (proj1((x,y)) = x) & (proj2((x,y)) = y) AndInt 12 166
168. (Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y)) ImpInt 167
169. ¬Set(x) v ¬Set(y) Hyp
171. ((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0)) ImpElim 169 170
174. proj1((x,y)) = U EqualitySub 11 173
179. proj2((x,y)) = (∩∪(x,y) ∪ (U ~ ∪∩(x,y))) EqualitySub 14 177
180. proj2((x,y)) = (∩∪(x,y) ∪ (U ~ 0)) EqualitySub 179 178
181. proj2((x,y)) = (0 ∪ (U ~ 0)) EqualitySub 180 176
182. ((0 ∪ x) = x) & ((0 ∩ x) = 0) TheoremInt
185. (0 ∪ (U ~ 0)) = (U ~ 0) ForallElim 184
186. proj2((x,y)) = (U ~ 0) EqualitySub 181 185
190. (U ~ 0) = (U ∩ ~0) ForallElim 189
191. proj2((x,y)) = (U ∩ ~0) EqualitySub 186 190
192. (~0 = U) & (~U = 0) TheoremInt
194. proj2((x,y)) = (U ∩ U) EqualitySub 191 193
195. ((x ∪ x) = x) & ((x ∩ x) = x) TheoremInt
198. (U ∩ U) = U ForallElim 197
199. proj2((x,y)) = U EqualitySub 194 198
200. (proj1((x,y)) = U) & (proj2((x,y)) = U) AndInt 174 199
201. (¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)) ImpInt 200
202. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U))) AndInt 168 201 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))
2. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
3. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
4. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))
0. A v ¬A
5. (0 ⊂ x) & (x ⊂ U)
6. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

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8. ((x ∪ U) = U) & ((x ∩ U) = x)
7. ((0 ∪ x) = x) & ((0 ∩ x) = 0)
9. (~0 = U) & (~U = 0)
10. ((x ∪ x) = x) & ((x ∩ x) = x)

Th55. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

0. (Set(x) & Set(y)) & ((x,y) = (u,v))  Hyp
1. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((
x,y)) = U) & (proj2((x,y)) = U)))  TheoremInt
4. (proj1((x,y)) = x) & (proj2((x,y)) = y)  ImpElim 3 2
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
9. Set((x,y))  ImpElim 3 8
11. Set((u,v))  EqualitySub 9 10
17. Set((u,v)) -> (Set(u) & Set(v))  ForallElim 16
18. Set(u) & Set(v)  ImpElim 11 17
22. (Set(u) & Set(v)) -> ((proj1((u,v)) = u) & (proj2((u,v)) = v))  ForallElim 21
23. (proj1((u,v)) = u) & (proj2((u,v)) = v)  ImpElim 18 22
28. proj1((u,v)) = x  EqualitySub 24 10
29. u = x  EqualitySub 28 26
30. proj2((u,v)) = y  EqualitySub 25 10
31. v = y  EqualitySub 30 27
34. (x = u) & (y = v)  AndInt 32 33
35. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  ImpInt 34 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((
x,y)) = U) & (proj2((x,y)) = U)))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th58. ((r∘s)∘t) = (r∘(s∘t))

0. z ϵ ((r∘s)∘t)  Hyp
5. ((r∘s)∘t) = {w: ∃x.∃y.∃z.((((x,y) ϵ t) & ((y,z) ϵ (r∘s))) & (w = (x,z)))}  ForallElim 4
6. z ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ t) & ((y,z) ϵ (r∘s))) & (w = (x,z)))}  EqualitySub 0 5
7. Set(z) & ∃x.∃y.∃x_1.((((x,y) ϵ t) & ((y,x_1) ϵ (r∘s))) & (z = (x,x_1)))  ClassElim 6
9. ∃y.∃x_1.((((x,y) ϵ t) & ((y,x_1) ϵ (r∘s))) & (z = (x,x_1)))  Hyp
10. ∃x_1.((((x,y) ϵ t) & ((y,x_1) ϵ (r∘s))) & (z = (x,x_1)))  Hyp
11. (((x,y) ϵ t) & ((y,c) ϵ (r∘s))) & (z = (x,c))  Hyp
17. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))}  ForallElim 16
18. (y,c) ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))}  EqualitySub 13 17
19. Set((y,c)) & ∃x.∃x_2.∃z.((((x,x_2) ϵ s) & ((x_2,z) ϵ r)) & ((y,c) = (x,z)))  ClassElim 18
21. ∃x_2.∃z.((((a,x_2) ϵ s) & ((x_2,z) ϵ r)) & ((y,c) = (a,z)))  Hyp
22. ∃z.((((a,b) ϵ s) & ((b,z) ϵ r)) & ((y,c) = (a,z)))  Hyp
23. (((a,b) ϵ s) & ((b,d) ϵ r)) & ((y,c) = (a,d))  Hyp
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
34. Set((y,c)) -> (Set(y) & Set(c))  ForallElim 33
36. Set(y) & Set(c)  ImpElim 35 34
37. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
45. ((Set(y) & Set(c)) & ((y,c) = (a,d))) -> ((y = a) & (c = d))  ForallElim 44
47. (Set(y) & Set(c)) & ((y,c) = (a,d))  AndInt 36 46
48. (y = a) & (c = d)  ImpElim 47 45
51. (x,a) ϵ t  EqualitySub 25 49
52. ((x,a) ϵ t) & ((a,b) ϵ s)  AndInt 51 26
54. g = (x,b)  Hyp
55. (((x,a) ϵ t) & ((a,b) ϵ s)) & (g = (x,b))  AndInt 52 54
59. ∃r.((b,d) ϵ r)  ExistsInt 53
60. Set((b,d))  DefSub 59
64. Set((b,d)) -> (Set(b) & Set(d))  ForallElim 63
65. Set(b) & Set(d)  ImpElim 60 64
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67. ∃t.((x,a) ϵ t)  ExistsInt 51
68. Set((x,a))  DefSub 67
70. Set((x,a)) -> (Set(x) & Set(a))  ForallElim 69
71. Set(x) & Set(a)  ImpElim 68 70
73. Set(x) & Set(b)  AndInt 72 66
77. (Set(x) & Set(b)) -> Set((x,b))  ForallElim 76
78. Set((x,b))  ImpElim 73 77
80. Set(g)  EqualitySub 78 79
81. Set(g) & ∃x.∃a.∃b.((((x,a) ϵ t) & ((a,b) ϵ s)) & (g = (x,b)))  AndInt 80 58
82. g ϵ {w: ∃x.∃a.∃b.((((x,a) ϵ t) & ((a,b) ϵ s)) & (w = (x,b)))}  ClassInt 81
86. (s∘t) = {w: ∃x.∃y.∃z.((((x,y) ϵ t) & ((y,z) ϵ s)) & (w = (x,z)))}  ForallElim 85
88. g ϵ (s∘t)  EqualitySub 82 87
89. (x,b) ϵ (s∘t)  EqualitySub 88 54
90. (g = (x,b)) -> ((x,b) ϵ (s∘t))  ImpInt 89
92. ((x,b) = (x,b)) -> ((x,b) ϵ (s∘t))  ForallElim 91
94. (x,b) ϵ (s∘t)  ImpElim 93 92
95. ((b,d) ϵ r) & ((x,b) ϵ (s∘t))  AndInt 53 94
98. ((x,b) ϵ (s∘t)) & ((b,d) ϵ r)  AndInt 94 53
99. (((x,b) ϵ (s∘t)) & ((b,d) ϵ r)) & (z = (x,c))  AndInt 98 97
100. (((x,b) ϵ (s∘t)) & ((b,c) ϵ r)) & (z = (x,c))  EqualitySub 99 96
103. ∃x.∃b.∃c.((((x,b) ϵ (s∘t)) & ((b,c) ϵ r)) & (z = (x,c)))  ExistsInt 102
105. Set(z) & ∃x.∃b.∃c.((((x,b) ϵ (s∘t)) & ((b,c) ϵ r)) & (z = (x,c)))  AndInt 104 103
106. z ϵ {w: ∃x.∃b.∃c.((((x,b) ϵ (s∘t)) & ((b,c) ϵ r)) & (w = (x,c)))}  ClassInt 105
110. (r∘(s∘t)) = {w: ∃x.∃y.∃z.((((x,y) ϵ (s∘t)) & ((y,z) ϵ r)) & (w = (x,z)))}  ForallElim 109
112. z ϵ (r∘(s∘t))  EqualitySub 106 111
113. z ϵ (r∘(s∘t))  ExistsElim 22 23 112
119. (z ϵ ((r∘s)∘t)) -> (z ϵ (r∘(s∘t)))  ImpInt 118
120. z ϵ (r∘(s∘t))  Hyp
124. (r∘(s∘t)) = {w: ∃x.∃y.∃z.((((x,y) ϵ (s∘t)) & ((y,z) ϵ r)) & (w = (x,z)))}  ForallElim 123
125. z ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ (s∘t)) & ((y,z) ϵ r)) & (w = (x,z)))}  EqualitySub 120 124
126. Set(z) & ∃x.∃y.∃x_7.((((x,y) ϵ (s∘t)) & ((y,x_7) ϵ r)) & (z = (x,x_7)))  ClassElim 125
128. ∃y.∃x_7.((((x,y) ϵ (s∘t)) & ((y,x_7) ϵ r)) & (z = (x,x_7)))  Hyp
129. ∃x_7.((((x,y) ϵ (s∘t)) & ((y,x_7) ϵ r)) & (z = (x,x_7)))  Hyp
130. (((x,y) ϵ (s∘t)) & ((y,c) ϵ r)) & (z = (x,c))  Hyp
135. (x,y) ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ t) & ((y,z) ϵ s)) & (w = (x,z)))}  EqualitySub 133 86
136. Set((x,y)) & ∃x_8.∃x_9.∃z.((((x_8,x_9) ϵ t) & ((x_9,z) ϵ s)) & ((x,y) = (x_8,z)))  ClassElim 135
139. ∃x_9.∃z.((((a,x_9) ϵ t) & ((x_9,z) ϵ s)) & ((x,y) = (a,z)))  Hyp
140. ∃z.((((a,b) ϵ t) & ((b,z) ϵ s)) & ((x,y) = (a,z)))  Hyp
141. (((a,b) ϵ t) & ((b,d) ϵ s)) & ((x,y) = (a,d))  Hyp
143. Set((a,d))  EqualitySub 137 142
148. Set((a,d)) -> (Set(a) & Set(d))  ForallElim 147
149. Set(a) & Set(d)  ImpElim 143 148
154. ((b,d) ϵ s) & ((y,c) ϵ r)  AndInt 153 134
155. Set(x) & Set(y)  ImpElim 137 144
156. (Set(x) & Set(y)) & ((x,y) = (a,d))  AndInt 155 142
157. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
161. ((Set(x) & Set(y)) & ((x,y) = (a,d))) -> ((x = a) & (y = d))  ForallElim 160
162. (x = a) & (y = d)  ImpElim 156 161
165. ((b,y) ϵ s) & ((y,c) ϵ r)  EqualitySub 154 164
166. h = (b,c)  Hyp
168. ∃w.((y,c) ϵ w)  ExistsInt 134
169. Set((b,d))  DefSub 167
170. Set((y,c))  DefSub 168
178. Set((y,c)) -> (Set(y) & Set(c))  ForallElim 177
179. Set(b) & Set(d)  ImpElim 169 174
180. Set(y) & Set(c)  ImpElim 170 178
187. (Set(b) & Set(c)) -> Set((b,c))  ForallElim 186
188. Set(b) & Set(c)  AndInt 181 182
189. Set((b,c))  ImpElim 188 187
191. Set(h)  EqualitySub 189 190
192. (((b,y) ϵ s) & ((y,c) ϵ r)) & (h = (b,c))  AndInt 165 166
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195. ∃b.∃y.∃c.((((b,y) ϵ s) & ((y,c) ϵ r)) & (h = (b,c))) ExistsInt 194
196. Set(h) & ∃b.∃y.∃c.((((b,y) ϵ s) & ((y,c) ϵ r)) & (h = (b,c))) AndInt 191 195
197. h ϵ {w: ∃b.∃y.∃c.((((b,y) ϵ s) & ((y,c) ϵ r)) & (w = (b,c)))} ClassInt 196
201. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))} ForallElim 200
203. h ϵ (r∘s) EqualitySub 197 202
204. (b,c) ϵ (r∘s) EqualitySub 203 166
205. (h = (b,c)) -> ((b,c) ϵ (r∘s)) ImpInt 204
207. ((b,c) = (b,c)) -> ((b,c) ϵ (r∘s)) ForallElim 206
209. (b,c) ϵ (r∘s) ImpElim 208 207
213. (x,b) ϵ t EqualitySub 210 212
214. ((x,b) ϵ t) & ((b,c) ϵ (r∘s)) AndInt 213 209
215. (((x,b) ϵ t) & ((b,c) ϵ (r∘s))) & (z = (x,c)) AndInt 214 131
218. ∃x.∃b.∃c.((((x,b) ϵ t) & ((b,c) ϵ (r∘s))) & (z = (x,c))) ExistsInt 217
220. Set(z) & ∃x.∃b.∃c.((((x,b) ϵ t) & ((b,c) ϵ (r∘s))) & (z = (x,c))) AndInt 219 218
221. z ϵ {w: ∃x.∃b.∃c.((((x,b) ϵ t) & ((b,c) ϵ (r∘s))) & (w = (x,c)))} ClassInt 220
225. ((r∘s)∘t) = {w: ∃x.∃y.∃z.((((x,y) ϵ t) & ((y,z) ϵ (r∘s))) & (w = (x,z)))} ForallElim 224
227. z ϵ ((r∘s)∘t) EqualitySub 221 226
228. z ϵ ((r∘s)∘t) ExistsElim 140 141 227
234. (z ϵ (r∘(s∘t))) -> (z ϵ ((r∘s)∘t)) ImpInt 233
235. ((z ϵ ((r∘s)∘t)) -> (z ϵ (r∘(s∘t)))) & ((z ϵ (r∘(s∘t))) -> (z ϵ ((r∘s)∘t))) AndInt 119 234
237. ∀z.((z ϵ ((r∘s)∘t)) <-> (z ϵ (r∘(s∘t)))) ForallInt 236
238. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
239. ∀y.((((r∘s)∘t) = y) <-> ∀z.((z ϵ ((r∘s)∘t)) <-> (z ϵ y))) ForallElim 238
240. (((r∘s)∘t) = (r∘(s∘t))) <-> ∀z.((z ϵ ((r∘s)∘t)) <-> (z ϵ (r∘(s∘t)))) ForallElim 239
243. ((r∘s)∘t) = (r∘(s∘t)) ImpElim 237 242 Qed

Used Theorems

2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
1. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th59. ((r∘(s ∪ t)) = ((r∘s) ∪ (r∘t))) & ((r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t)))

0. z ϵ (r∘(s ∪ t)) Hyp
5. (r∘(s ∪ t)) = {w: ∃x.∃y.∃z.((((x,y) ϵ (s ∪ t)) & ((y,z) ϵ r)) & (w = (x,z)))} ForallElim 4
6. z ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ (s ∪ t)) & ((y,z) ϵ r)) & (w = (x,z)))} EqualitySub 0 5
7. Set(z) & ∃x.∃y.∃x_1.((((x,y) ϵ (s ∪ t)) & ((y,x_1) ϵ r)) & (z = (x,x_1))) ClassElim 6
9. ∃y.∃x_1.((((x,y) ϵ (s ∪ t)) & ((y,x_1) ϵ r)) & (z = (x,x_1))) Hyp
10. ∃x_1.((((x,y) ϵ (s ∪ t)) & ((y,x_1) ϵ r)) & (z = (x,x_1))) Hyp
11. (((x,y) ϵ (s ∪ t)) & ((y,c) ϵ r)) & (z = (x,c)) Hyp
14. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
23. ((x,y) ϵ (s ∪ t)) -> (((x,y) ϵ s) v ((x,y) ϵ t)) ForallElim 22
24. ((x,y) ϵ s) v ((x,y) ϵ t) ImpElim 13 23
25. (x,y) ϵ s Hyp
27. ((x,y) ϵ s) & ((y,c) ϵ r) AndInt 25 26
29. (((x,y) ϵ s) & ((y,c) ϵ r)) & (z = (x,c)) AndInt 27 28
32. ∃x.∃y.∃c.((((x,y) ϵ s) & ((y,c) ϵ r)) & (z = (x,c))) ExistsInt 31
34. Set(z) & ∃x.∃y.∃c.((((x,y) ϵ s) & ((y,c) ϵ r)) & (z = (x,c))) AndInt 33 32
35. z ϵ {w: ∃x.∃y.∃c.((((x,y) ϵ s) & ((y,c) ϵ r)) & (w = (x,c)))} ClassInt 34
39. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))} ForallElim 38
41. z ϵ (r∘s) EqualitySub 35 40
42. (z ϵ (r∘s)) v (z ϵ (r∘t)) OrIntR 41
47. ((z ϵ (r∘s)) v (z ϵ (r∘t))) -> (z ϵ ((r∘s) ∪ (r∘t))) ForallElim 46
48. z ϵ ((r∘s) ∪ (r∘t)) ImpElim 42 47
49. (x,y) ϵ t Hyp
50. ((x,y) ϵ t) & ((y,c) ϵ r) AndInt 49 26
51. (((x,y) ϵ t) & ((y,c) ϵ r)) & (z = (x,c)) AndInt 50 28
54. ∃x.∃y.∃c.((((x,y) ϵ t) & ((y,c) ϵ r)) & (z = (x,c))) ExistsInt 53
55. Set(z) & ∃x.∃y.∃c.((((x,y) ϵ t) & ((y,c) ϵ r)) & (z = (x,c))) AndInt 33 54
56. z ϵ {w: ∃x.∃y.∃c.((((x,y) ϵ t) & ((y,c) ϵ r)) & (w = (x,c)))} ClassInt 55
60. (r∘t) = {w: ∃x.∃y.∃z.((((x,y) ϵ t) & ((y,z) ϵ r)) & (w = (x,z)))} ForallElim 59

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62. z ϵ (r∘t)  EqualitySub 56 61
63. (z ϵ (r∘s)) v (z ϵ (r∘t))  OrIntL 62
64. z ϵ ((r∘s) ∪ (r∘t))  ImpElim 63 47
65. z ϵ ((r∘s) ∪ (r∘t))  OrElim 24 25 48 49 64
66. z ϵ ((r∘s) ∪ (r∘t))  ExistsElim 10 11 65
69. (z ϵ (r∘(s ∪ t))) -> (z ϵ ((r∘s) ∪ (r∘t)))  ImpInt 68
70. z ϵ ((r∘s) ∪ (r∘t))  Hyp
74. (z ϵ ((r∘s) ∪ (r∘t))) -> ((z ϵ (r∘s)) v (z ϵ (r∘t)))  ForallElim 73
75. (z ϵ (r∘s)) v (z ϵ (r∘t))  ImpElim 70 74
76. z ϵ (r∘s)  Hyp
80. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))}  ForallElim 79
81. z ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))}  EqualitySub 76 80
82. Set(z) & ∃x.∃y.∃x_2.((((x,y) ϵ s) & ((y,x_2) ϵ r)) & (z = (x,x_2)))  ClassElim 81
84. ∃y.∃x_2.((((x,y) ϵ s) & ((y,x_2) ϵ r)) & (z = (x,x_2)))  Hyp
85. ∃x_2.((((x,y) ϵ s) & ((y,x_2) ϵ r)) & (z = (x,x_2)))  Hyp
86. (((x,y) ϵ s) & ((y,m) ϵ r)) & (z = (x,m))  Hyp
89. ((x,y) ϵ s) v ((x,y) ϵ t)  OrIntR 88
98. (((x,y) ϵ s) v ((x,y) ϵ t)) -> ((x,y) ϵ (s ∪ t))  ForallElim 97
99. (x,y) ϵ (s ∪ t)  ImpElim 89 98
100. ((x,y) ϵ (s ∪ t)) & ((y,m) ϵ r)  AndInt 99 90
102. (((x,y) ϵ (s ∪ t)) & ((y,m) ϵ r)) & (z = (x,m))  AndInt 100 101
105. ∃x.∃y.∃m.((((x,y) ϵ (s ∪ t)) & ((y,m) ϵ r)) & (z = (x,m)))  ExistsInt 104
107. Set(z) & ∃x.∃y.∃m.((((x,y) ϵ (s ∪ t)) & ((y,m) ϵ r)) & (z = (x,m)))  AndInt 106 105
108. z ϵ {w: ∃x.∃y.∃m.((((x,y) ϵ (s ∪ t)) & ((y,m) ϵ r)) & (w = (x,m)))}  ClassInt 107
110. z ϵ (r∘(s ∪ t))  EqualitySub 108 109
111. z ϵ (r∘(s ∪ t))  ExistsElim 85 86 110
114. z ϵ (r∘t)  Hyp
116. (r∘t) = {w: ∃x.∃y.∃z.((((x,y) ϵ t) & ((y,z) ϵ r)) & (w = (x,z)))}  ForallElim 115
117. z ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ t) & ((y,z) ϵ r)) & (w = (x,z)))}  EqualitySub 114 116
118. Set(z) & ∃x.∃y.∃x_4.((((x,y) ϵ t) & ((y,x_4) ϵ r)) & (z = (x,x_4)))  ClassElim 117
120. ∃y.∃x_4.((((x,y) ϵ t) & ((y,x_4) ϵ r)) & (z = (x,x_4)))  Hyp
121. ∃x_4.((((x,y) ϵ t) & ((y,x_4) ϵ r)) & (z = (x,x_4)))  Hyp
122. (((x,y) ϵ t) & ((y,e) ϵ r)) & (z = (x,e))  Hyp
125. ((x,y) ϵ s) v ((x,y) ϵ t)  OrIntL 124
126. (x,y) ϵ (s ∪ t)  ImpElim 125 98
128. ((x,y) ϵ (s ∪ t)) & ((y,e) ϵ r)  AndInt 126 127
130. (((x,y) ϵ (s ∪ t)) & ((y,e) ϵ r)) & (z = (x,e))  AndInt 128 129
133. ∃x.∃y.∃e.((((x,y) ϵ (s ∪ t)) & ((y,e) ϵ r)) & (z = (x,e)))  ExistsInt 132
135. Set(z) & ∃x.∃y.∃e.((((x,y) ϵ (s ∪ t)) & ((y,e) ϵ r)) & (z = (x,e)))  AndInt 134 133
136. z ϵ {w: ∃x.∃y.∃e.((((x,y) ϵ (s ∪ t)) & ((y,e) ϵ r)) & (w = (x,e)))}  ClassInt 135
137. z ϵ (r∘(s ∪ t))  EqualitySub 136 109
138. z ϵ (r∘(s ∪ t))  ExistsElim 121 122 137
141. z ϵ (r∘(s ∪ t))  OrElim 75 76 113 114 140
142. (z ϵ ((r∘s) ∪ (r∘t))) -> (z ϵ (r∘(s ∪ t)))  ImpInt 141
143. ((z ϵ (r∘(s ∪ t))) -> (z ϵ ((r∘s) ∪ (r∘t)))) & ((z ϵ ((r∘s) ∪ (r∘t))) -> (z ϵ (r∘(s ∪ t))))  AndInt 69 142
145. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
146. ∀y.(((r∘(s ∪ t)) = y) <-> ∀z.((z ϵ (r∘(s ∪ t))) <-> (z ϵ y)))  ForallElim 145
147. ((r∘(s ∪ t)) = ((r∘s) ∪ (r∘t))) <-> ∀z.((z ϵ (r∘(s ∪ t))) <-> (z ϵ ((r∘s) ∪ (r∘t))))  ForallElim 146
150. ∀z.((z ϵ (r∘(s ∪ t))) <-> (z ϵ ((r∘s) ∪ (r∘t))))  ForallInt 144
151. (r∘(s ∪ t)) = ((r∘s) ∪ (r∘t))  ImpElim 150 149
152. z ϵ (r∘(s ∩ t))  Hyp
156. (r∘(s ∩ t)) = {w: ∃x.∃y.∃z.((((x,y) ϵ (s ∩ t)) & ((y,z) ϵ r)) & (w = (x,z)))}  ForallElim 155
157. z ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ (s ∩ t)) & ((y,z) ϵ r)) & (w = (x,z)))}  EqualitySub 152 156
158. Set(z) & ∃x.∃y.∃x_5.((((x,y) ϵ (s ∩ t)) & ((y,x_5) ϵ r)) & (z = (x,x_5)))  ClassElim 157
160. ∃y.∃x_5.((((x,y) ϵ (s ∩ t)) & ((y,x_5) ϵ r)) & (z = (x,x_5)))  Hyp
161. ∃x_5.((((x,y) ϵ (s ∩ t)) & ((y,x_5) ϵ r)) & (z = (x,x_5)))  Hyp
162. (((x,y) ϵ (s ∩ t)) & ((y,e) ϵ r)) & (z = (x,e))  Hyp
171. ((x,y) ϵ (s ∩ t)) <-> (((x,y) ϵ s) & ((x,y) ϵ t))  ForallElim 170
174. ((x,y) ϵ s) & ((x,y) ϵ t)  ImpElim 164 173
177. ((x,y) ϵ s) & ((y,e) ϵ r)  AndInt 175 176
179. (((x,y) ϵ s) & ((y,e) ϵ r)) & (z = (x,e))  AndInt 177 178
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182. ∃x.∃y.∃e.((((x,y) ϵ s) & ((y,e) ϵ r)) & (z = (x,e))) ExistsInt 181
184. Set(z) & ∃x.∃y.∃e.((((x,y) ϵ s) & ((y,e) ϵ r)) & (z = (x,e))) AndInt 183 182
185. z ϵ {w: ∃x.∃y.∃e.((((x,y) ϵ s) & ((y,e) ϵ r)) & (w = (x,e)))} ClassInt 184
186. z ϵ (r∘s) EqualitySub 185 40
188. ((x,y) ϵ t) & ((y,e) ϵ r) AndInt 187 176
189. (((x,y) ϵ t) & ((y,e) ϵ r)) & (z = (x,e)) AndInt 188 178
192. ∃x.∃y.∃e.((((x,y) ϵ t) & ((y,e) ϵ r)) & (z = (x,e))) ExistsInt 191
193. Set(z) & ∃x.∃y.∃e.((((x,y) ϵ t) & ((y,e) ϵ r)) & (z = (x,e))) AndInt 183 192
194. z ϵ {w: ∃x.∃y.∃e.((((x,y) ϵ t) & ((y,e) ϵ r)) & (w = (x,e)))} ClassInt 193
195. z ϵ (r∘t) EqualitySub 194 61
196. (z ϵ (r∘s)) & (z ϵ (r∘t)) AndInt 186 195
202. ((z ϵ (r∘s)) & (z ϵ (r∘t))) -> (z ϵ ((r∘s) ∩ (r∘t))) ForallElim 201
203. z ϵ ((r∘s) ∩ (r∘t)) ImpElim 196 202
204. z ϵ ((r∘s) ∩ (r∘t)) ExistsElim 161 162 203
207. (z ϵ (r∘(s ∩ t))) -> (z ϵ ((r∘s) ∩ (r∘t))) ImpInt 206
208. ∀z.((z ϵ (r∘(s ∩ t))) -> (z ϵ ((r∘s) ∩ (r∘t)))) ForallInt 207
209. (r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t)) DefSub 208
210. ((r∘(s ∪ t)) = ((r∘s) ∪ (r∘t))) & ((r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t))) AndInt 151 209 Qed

Used Theorems

1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))

Th61. Relation(r) -> ($((r)^{-1})^{-1}$ = r)

0. z ϵ $((r)^{-1})^{-1}$ Hyp
3. $((r)^{-1})^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ $(r)^{-1}$) & (z = (y,x)))} ForallElim 2
4. z ϵ {z: ∃x.∃y.(((x,y) ϵ $(r)^{-1}$) & (z = (y,x)))} EqualitySub 0 3
5. Set(z) & ∃x.∃y.(((x,y) ϵ $(r)^{-1}$) & (z = (y,x))) ClassElim 4
7. ∃y.(((x,y) ϵ $(r)^{-1}$) & (z = (y,x))) Hyp
8. ((x,y) ϵ $(r)^{-1}$) & (z = (y,x)) Hyp
10. (x,y) ϵ {z: ∃x.∃y.(((x,y) ϵ r) & (z = (y,x)))} EqualitySub 9 1
11. Set((x,y)) & ∃x_0.∃x_2.(((x_0,x_2) ϵ r) & ((x,y) = (x_2,x_0))) ClassElim 10
13. ∃x_2.(((c,x_2) ϵ r) & ((x,y) = (x_2,c))) Hyp
14. ((c,d) ϵ r) & ((x,y) = (d,c)) Hyp
17. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
18. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
22. Set((y,x)) EqualitySub 16 15
28. Set((y,x)) -> (Set(y) & Set(x)) ForallElim 27
29. Set(y) & Set(x) ImpElim 22 28
32. Set(x) & Set(y) AndInt 31 30
36. ((Set(x) & Set(y)) & ((x,y) = (d,c))) -> ((x = d) & (y = c)) ForallElim 35
38. (Set(x) & Set(y)) & ((x,y) = (d,c)) AndInt 32 37
39. (x = d) & (y = c) ImpElim 38 36
45. (c,x) ϵ r EqualitySub 42 43
46. (y,x) ϵ r EqualitySub 45 44
47. (y,x) ϵ r ExistsElim 13 14 46
50. z ϵ r EqualitySub 48 49
51. z ϵ r ExistsElim 7 8 50
53. (z ϵ $((r)^{-1})^{-1}$) -> (z ϵ r) ImpInt 52
54. Relation(r) Hyp
55. z ϵ r Hyp
56. ∀z.((z ϵ r) -> ∃x.∃y.(z = (x,y))) DefExp 54
57. (z ϵ r) -> ∃x.∃y.(z = (x,y)) ForallElim 56
58. ∃x.∃y.(z = (x,y)) ImpElim 55 57
59. ∃y.(z = (x,y)) Hyp
60. z = (x,y) Hyp
61. f = (y,x) Hyp
62. (x,y) ϵ r EqualitySub 55 60
63. ((x,y) ϵ r) & (f = (y,x)) AndInt 62 61
64. Set((y,x)) EqualitySub 16 15

65. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
69. ∃w.(z ϵ w) ExistsInt 55
70. Set(z) DefSub 69
71. Set((x,y)) EqualitySub 70 60
72. Set(x) & Set(y) ImpElim 71 68
82. (Set(y) & Set(x)) -> Set((y,x)) ForallElim 81
83. Set(y) & Set(x) AndInt 74 73
84. Set((y,x)) ImpElim 83 82
86. Set(f) EqualitySub 84 85
88. ∃x.∃y.(((x,y) ϵ r) & (f = (y,x))) ExistsInt 87
89. Set(f) & ∃x.∃y.(((x,y) ϵ r) & (f = (y,x))) AndInt 86 88
90. f ϵ {w: ∃x.∃y.(((x,y) ϵ r) & (w = (y,x)))} ClassInt 89
92. f ϵ (r)$^{-1}$ EqualitySub 90 91
93. (y,x) ϵ (r)$^{-1}$ EqualitySub 92 61
94. (f = (y,x)) -> ((y,x) ϵ (r)$^{-1}$) ImpInt 93
96. ((y,x) = (y,x)) -> ((y,x) ϵ (r)$^{-1}$) ForallElim 95
98. (y,x) ϵ (r)$^{-1}$ ImpElim 97 96
99. ((y,x) ϵ (r)$^{-1}$) & (z = (x,y)) AndInt 98 60
101. ∃y.∃x.(((y,x) ϵ (r)$^{-1}$) & (z = (x,y))) ExistsInt 100
102. Set(z) & ∃y.∃x.(((y,x) ϵ (r)$^{-1}$) & (z = (x,y))) AndInt 70 101
103. z ϵ {w: ∃y.∃x.(((y,x) ϵ (r)$^{-1}$) & (w = (x,y)))} ClassInt 102
105. ((r)$^{-1}$)$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ (r)$^{-1}$) & (z = (y,x)))} ForallElim 104
107. z ϵ ((r)$^{-1}$)$^{-1}$ EqualitySub 103 106
108. z ϵ ((r)$^{-1}$)$^{-1}$ ExistsElim 59 60 107
110. (z ϵ r) -> (z ϵ ((r)$^{-1}$)$^{-1}$) ImpInt 109
111. ((z ϵ ((r)$^{-1}$)$^{-1}$) -> (z ϵ r)) & ((z ϵ r) -> (z ϵ ((r)$^{-1}$)$^{-1}$)) AndInt 53 110
113. ∀z.((z ϵ ((r)$^{-1}$)$^{-1}$) <-> (z ϵ r)) ForallInt 112
114. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
115. ∀y.((((r)$^{-1}$)$^{-1}$ = y) <-> ∀z.((z ϵ ((r)$^{-1}$)$^{-1}$) <-> (z ϵ y))) ForallElim 114
116. (((r)$^{-1}$)$^{-1}$ = r) <-> ∀z.((z ϵ ((r)$^{-1}$)$^{-1}$) <-> (z ϵ r)) ForallElim 115
119. ((r)$^{-1}$)$^{-1}$ = r ImpElim 113 118
120. Relation(r) -> (((r)$^{-1}$)$^{-1}$ = r) ImpInt 119 Qed

Used Theorems

1. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th62. ((r∘s))$^{-1}$ = ((s)$^{-1}$∘(r)$^{-1}$)

0. z ϵ ((r∘s))$^{-1}$ Hyp
3. ((r∘s))$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ (r∘s)) & (z = (y,x)))} ForallElim 2
4. z ϵ {z: ∃x.∃y.(((x,y) ϵ (r∘s)) & (z = (y,x)))} EqualitySub 0 3
5. Set(z) & ∃x.∃y.(((x,y) ϵ (r∘s)) & (z = (y,x))) ClassElim 4
11. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))} ForallElim 10
12. ∃y.(((x,y) ϵ (r∘s)) & (z = (y,x))) Hyp
13. ((x,y) ϵ (r∘s)) & (z = (y,x)) Hyp
15. (x,y) ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))} EqualitySub 14 11
16. Set((x,y)) & ∃x_0.∃x_1.∃z.((((x_0,x_1) ϵ s) & ((x_1,z) ϵ r)) & ((x,y) = (x_0,z))) ClassElim 15
18. ∃x_1.∃z.((((c,x_1) ϵ s) & ((x_1,z) ϵ r)) & ((x,y) = (c,z))) Hyp
19. ∃z.((((c,d) ϵ s) & ((d,z) ϵ r)) & ((x,y) = (c,z))) Hyp
20. (((c,d) ϵ s) & ((d,b) ϵ r)) & ((x,y) = (c,b)) Hyp
21. ∃w.((x,y) ϵ w) ExistsInt 14
22. Set((x,y)) DefSub 21
23. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
27. Set(x) & Set(y) ImpElim 22 26
29. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
33. ((Set(x) & Set(y)) & ((x,y) = (c,b))) -> ((x = c) & (y = b)) ForallElim 32
34. (Set(x) & Set(y)) & ((x,y) = (c,b)) AndInt 27 28
35. (x = c) & (y = b) ImpElim 34 33

40. (((x,d) ϵ s) & ((d,b) ϵ r)) & ((x,y) = (x,b)) EqualitySub 20 38
41. (((x,d) ϵ s) & ((d,y) ϵ r)) & ((x,y) = (x,y)) EqualitySub 40 39
43. h = (d,x) Hyp
45. ((x,d) ϵ s) & (h = (d,x)) AndInt 44 43
47. ∃x.∃d.(((x,d) ϵ s) & (h = (d,x))) ExistsInt 46
49. ∃w.((x,d) ϵ w) ExistsInt 48
50. Set((x,d)) DefSub 49
52. Set((x,d)) -> (Set(x) & Set(d)) ForallElim 51
53. Set(x) & Set(d) ImpElim 50 52
56. Set(x) & Set(d) AndInt 55 54
61. (Set(d) & Set(x)) -> Set((d,x)) ForallElim 60
62. Set(d) & Set(x) AndInt 54 55
63. Set((d,x)) ImpElim 62 61
65. Set(h) EqualitySub 63 64
66. Set(h) & ∃x.∃d.(((x,d) ϵ s) & (h = (d,x))) AndInt 65 47
67. h ϵ {w: ∃x.∃d.(((x,d) ϵ s) & (w = (d,x)))} ClassInt 66
69. (s)$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ s) & (z = (y,x)))} ForallElim 68
71. h ϵ (s)$^{-1}$ EqualitySub 67 70
72. (d,x) ϵ (s)$^{-1}$ EqualitySub 71 43
73. (h = (d,x)) -> ((d,x) ϵ (s)$^{-1}$) ImpInt 72
75. ((d,x) = (d,x)) -> ((d,x) ϵ (s)$^{-1}$) ForallElim 74
77. (d,x) ϵ (s)$^{-1}$ ImpElim 76 75
78. f = (y,d) Hyp
80. ((d,y) ϵ r) & (f = (y,d)) AndInt 79 78
82. ∃d.∃y.(((d,y) ϵ r) & (f = (y,d))) ExistsInt 81
84. Set(y) & Set(d) AndInt 83 54
88. (Set(y) & Set(d)) -> Set((y,d)) ForallElim 87
89. Set((y,d)) ImpElim 84 88
91. Set(f) EqualitySub 89 90
92. Set(f) & ∃d.∃y.(((d,y) ϵ r) & (f = (y,d))) AndInt 91 82
93. f ϵ {w: ∃d.∃y.(((d,y) ϵ r) & (w = (y,d)))} ClassInt 92
95. f ϵ (r)$^{-1}$ EqualitySub 93 94
96. (y,d) ϵ (r)$^{-1}$ EqualitySub 95 78
97. (f = (y,d)) -> ((y,d) ϵ (r)$^{-1}$) ImpInt 96
99. ((y,d) = (y,d)) -> ((y,d) ϵ (r)$^{-1}$) ForallElim 98
101. (y,d) ϵ (r)$^{-1}$ ImpElim 100 99
102. ((y,d) ϵ (r)$^{-1}$) & ((d,x) ϵ (s)$^{-1}$) AndInt 101 77
104. (((y,d) ϵ (r)$^{-1}$) & ((d,x) ϵ (s)$^{-1}$)) & (z = (y,x)) AndInt 102 103
107. ∃y.∃d.∃x.((((y,d) ϵ (r)$^{-1}$) & ((d,x) ϵ (s)$^{-1}$)) & (z = (y,x))) ExistsInt 106
109. Set(z) & ∃y.∃d.∃x.((((y,d) ϵ (r)$^{-1}$) & ((d,x) ϵ (s)$^{-1}$)) & (z = (y,x))) AndInt 108 107
110. z ϵ {w: ∃y.∃d.∃x.((((y,d) ϵ (r)$^{-1}$) & ((d,x) ϵ (s)$^{-1}$)) & (w = (y,x)))} ClassInt 109
114. ((s)$^{-1}$∘(r)$^{-1}$) = {w: ∃x.∃y.∃z.((((x,y) ϵ (r)$^{-1}$) & ((y,z) ϵ (s)$^{-1}$)) & (w = (x,z)))} ForallElim 113
116. z ϵ ((s)$^{-1}$∘(r)$^{-1}$) EqualitySub 110 115
117. z ϵ ((s)$^{-1}$∘(r)$^{-1}$) ExistsElim 19 20 116
118. (h = (d,x)) -> (z ϵ ((s)$^{-1}$∘(r)$^{-1}$)) ImpInt 117
120. ((d,x) = (d,x)) -> (z ϵ ((s)$^{-1}$∘(r)$^{-1}$)) ForallElim 119
122. z ϵ ((s)$^{-1}$∘(r)$^{-1}$) ImpElim 121 120
123. z ϵ ((s)$^{-1}$∘(r)$^{-1}$) ExistsElim 18 19 122
127. (z ϵ ((r∘s))$^{-1}$) -> (z ϵ ((s)$^{-1}$∘(r)$^{-1}$)) ImpInt 126
128. z ϵ ((s)$^{-1}$∘(r)$^{-1}$) Hyp
132. ((s)$^{-1}$∘(r)$^{-1}$) = {w: ∃x.∃y.∃z.((((x,y) ϵ (r)$^{-1}$) & ((y,z) ϵ (s)$^{-1}$)) & (w = (x,z)))} ForallElim 131
133. z ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ (r)$^{-1}$) & ((y,z) ϵ (s)$^{-1}$)) & (w = (x,z)))} EqualitySub 128 132
134. Set(z) & ∃x.∃y.∃x_9.((((x,y) ϵ (r)$^{-1}$) & ((y,x_9) ϵ (s)$^{-1}$)) & (z = (x,x_9))) ClassElim 133
137. ∃y.∃x_9.((((x,y) ϵ (r)$^{-1}$) & ((y,x_9) ϵ (s)$^{-1}$)) & (z = (x,x_9))) Hyp
138. ∃x_9.((((x,y) ϵ (r)$^{-1}$) & ((y,x_9) ϵ (s)$^{-1}$)) & (z = (x,x_9))) Hyp
139. (((x,y) ϵ (r)$^{-1}$) & ((y,a) ϵ (s)$^{-1}$)) & (z = (x,a)) Hyp
145. (s)$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ s) & (z = (y,x)))} ForallElim 144
146. (x,y) ϵ {z: ∃x.∃y.(((x,y) ϵ r) & (z = (y,x)))} EqualitySub 142 1
147. (y,a) ϵ {z: ∃x.∃y.(((x,y) ϵ s) & (z = (y,x)))} EqualitySub 143 145
148. Set((x,y)) & ∃x_10.∃x_11.(((x_10,x_11) ϵ r) & ((x,y) = (x_11,x_10))) ClassElim 146
149. Set((y,a)) & ∃x.∃x_12.(((x,x_12) ϵ s) & ((y,a) = (x_12,x))) ClassElim 147

154. ∃x_11.(((b,x_11) ϵ r) & ((x,y) = (x_11,b))) Hyp
155. ((b,c) ϵ r) & ((x,y) = (c,b)) Hyp
156. ∃x_12.(((d,x_12) ϵ s) & ((y,a) = (x_12,d))) Hyp
157. ((d,e) ϵ s) & ((y,a) = (e,d)) Hyp
162. Set(x) & Set(y) ImpElim 150 26
163. (Set(x) & Set(y)) & ((x,y) = (c,b)) AndInt 162 160
167. ((Set(x) & Set(y)) & ((x,y) = (c,b))) -> ((x = c) & (y = b)) ForallElim 166
168. (x = c) & (y = b) ImpElim 163 167
176. Set((y,a)) -> (Set(y) & Set(a)) ForallElim 175
177. Set(y) & Set(a) ImpElim 152 176
178. ((d,e) ϵ s) & ((b,c) ϵ r) AndInt 159 158
179. ((d,e) ϵ s) & ((b,x) ϵ r) EqualitySub 178 171
180. (Set(y) & Set(a)) & ((y,a) = (e,d)) AndInt 177 161
188. ((Set(y) & Set(a)) & ((y,a) = (e,d))) -> ((y = e) & (a = d)) ForallElim 187
189. (y = e) & (a = d) ImpElim 180 188
193. ((d,y) ϵ s) & ((b,x) ϵ r) EqualitySub 179 192
194. ((d,y) ϵ s) & ((y,x) ϵ r) EqualitySub 193 172
196. ((a,y) ϵ s) & ((y,x) ϵ r) EqualitySub 194 195
197. h = (a,x) Hyp
200. Set(a) & Set(x) AndInt 198 199
204. (Set(a) & Set(x)) -> Set((a,x)) ForallElim 203
205. Set((a,x)) ImpElim 200 204
207. Set(h) EqualitySub 205 206
208. (((a,y) ϵ s) & ((y,x) ϵ r)) & (h = (a,x)) AndInt 196 197
211. ∃a.∃y.∃x.((((a,y) ϵ s) & ((y,x) ϵ r)) & (h = (a,x))) ExistsInt 210
212. Set(h) & ∃a.∃y.∃x.((((a,y) ϵ s) & ((y,x) ϵ r)) & (h = (a,x))) AndInt 207 211
213. h ϵ {w: ∃a.∃y.∃x.((((a,y) ϵ s) & ((y,x) ϵ r)) & (w = (a,x)))} ClassInt 212
217. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ϵ s) & ((y,z) ϵ r)) & (w = (x,z)))} ForallElim 216
219. h ϵ (r∘s) EqualitySub 213 218
220. (a,x) ϵ (r∘s) EqualitySub 219 197
221. (h = (a,x)) -> ((a,x) ϵ (r∘s)) ImpInt 220
223. ((a,x) = (a,x)) -> ((a,x) ϵ (r∘s)) ForallElim 222
225. (a,x) ϵ (r∘s) ImpElim 224 223
226. f = (x,a) Hyp
228. Set((x,a)) EqualitySub 135 140
229. Set(f) EqualitySub 228 227
230. ((a,x) ϵ (r∘s)) & (f = (x,a)) AndInt 220 226
232. ∃a.∃x.(((a,x) ϵ (r∘s)) & (f = (x,a))) ExistsInt 231
233. Set(f) & ∃a.∃x.(((a,x) ϵ (r∘s)) & (f = (x,a))) AndInt 229 232
234. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ r) & (z = (y,x)))}) ForallInt 1
236. ((r∘s))$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ (r∘s)) & (z = (y,x)))} ForallElim 235
238. f ϵ {w: ∃a.∃x.(((a,x) ϵ (r∘s)) & (w = (x,a)))} ClassInt 233
239. f ϵ ((r∘s))$^{-1}$ EqualitySub 238 237
240. (x,a) ϵ ((r∘s))$^{-1}$ EqualitySub 239 226
241. (f = (x,a)) -> ((x,a) ϵ ((r∘s))$^{-1}$) ImpInt 240
243. ((x,a) = (x,a)) -> ((x,a) ϵ ((r∘s))$^{-1}$) ForallElim 242
245. (x,a) ϵ ((r∘s))$^{-1}$ ImpElim 244 243
246. f ϵ ((r∘s))$^{-1}$ EqualitySub 245 227
247. f ϵ ((r∘s))$^{-1}$ ExistsElim 156 157 246
252. (h = (a,x)) -> (f ϵ ((r∘s))$^{-1}$) ImpInt 251
253. ∀h.((h = (a,x)) -> (f ϵ ((r∘s))$^{-1}$)) ForallInt 252
255. ((a,x) = (a,x)) -> (f ϵ ((r∘s))$^{-1}$) ForallElim 254
257. f ϵ ((r∘s))$^{-1}$ ImpElim 256 255
258. (x,a) ϵ ((r∘s))$^{-1}$ EqualitySub 257 226
259. (f = (x,a)) -> ((x,a) ϵ ((r∘s))$^{-1}$) ImpInt 258
261. ((x,a) = (x,a)) -> ((x,a) ϵ ((r∘s))$^{-1}$) ForallElim 260
263. (x,a) ϵ ((r∘s))$^{-1}$ ImpElim 262 261
265. z ϵ ((r∘s))$^{-1}$ EqualitySub 263 264
266. z ϵ ((r∘s))$^{-1}$ ExistsElim 151 154 265
270. (z ϵ ((s)$^{-1}$∘(r)$^{-1}$)) -> (z ϵ ((r∘s))$^{-1}$) ImpInt 269
271. ((z ϵ ((r∘s))$^{-1}$) -> (z ϵ ((s)$^{-1}$∘(r)$^{-1}$))) & ((z ϵ ((s)$^{-1}$∘(r)$^{-1}$)) -> (z ϵ ((r∘s))$^{-1}$)) AndInt 127 270

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273. ∀z.((z ϵ ((r∘s))^-1) <-> (z ϵ ((s)^-1∘(r)^-1)))  ForallInt 272
274. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
275. ∀y.((((r∘s))^-1 = y) <-> ∀z.((z ϵ ((r∘s))^-1) <-> (z ϵ y)))  ForallElim 274
276. (((r∘s))^-1 = ((s)^-1∘(r)^-1)) <-> ∀z.((z ϵ ((r∘s))^-1) <-> (z ϵ ((s)^-1∘(r)^-1)))  ForallElim 275
279. ((r∘s))^-1 = ((s)^-1∘(r)^-1)  ImpElim 273 278 Qed
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Used Theorems

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1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
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Th64. (FUN(f) & FUN(g)) -> FUN((f∘g))

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0. FUN(f) & FUN(g)  Hyp
3. (a,b) ϵ (f∘g)  Hyp
4. (a,c) ϵ (f∘g)  Hyp
9. (f∘g) = {w: ∃x.∃y.∃z.((((x,y) ϵ g) & ((y,z) ϵ f)) & (w = (x,z)))}  ForallElim 8
10. (a,b) ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ g) & ((y,z) ϵ f)) & (w = (x,z)))}  EqualitySub 3 9
11. (a,c) ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ g) & ((y,z) ϵ f)) & (w = (x,z)))}  EqualitySub 4 9
12. Set((a,b)) & ∃x.∃y.∃z.((((x,y) ϵ g) & ((y,z) ϵ f)) & ((a,b) = (x,z)))  ClassElim 10
13. Set((a,c)) & ∃x.∃y.∃z.((((x,y) ϵ g) & ((y,z) ϵ f)) & ((a,c) = (x,z)))  ClassElim 11
15. ∃y.∃z.((((x,y) ϵ g) & ((y,z) ϵ f)) & ((a,b) = (x,z)))  Hyp
16. ∃z.((((x,y) ϵ g) & ((y,z) ϵ f)) & ((a,b) = (x,z)))  Hyp
17. (((x,y) ϵ g) & ((y,z) ϵ f)) & ((a,b) = (x,z))  Hyp
19. ∃y.∃z.((((u,y) ϵ g) & ((y,z) ϵ f)) & ((a,c) = (u,z)))  Hyp
20. ∃z.((((u,v) ϵ g) & ((v,z) ϵ f)) & ((a,c) = (u,z)))  Hyp
21. (((u,v) ϵ g) & ((v,w) ϵ f)) & ((a,c) = (u,w))  Hyp
22. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
29. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 28
31. Set(a) & Set(b)  ImpElim 30 29
37. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 36
39. Set(a) & Set(c)  ImpElim 38 37
42. (Set(a) & Set(b)) & ((a,b) = (x,z))  AndInt 31 41
44. (Set(a) & Set(c)) & ((a,c) = (u,w))  AndInt 39 43
45. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
53. ((Set(a) & Set(b)) & ((a,b) = (x,z))) -> ((a = x) & (b = z))  ForallElim 52
54. (a = x) & (b = z)  ImpElim 42 53
58. ((Set(a) & Set(c)) & ((a,c) = (u,w))) -> ((a = u) & (c = w))  ForallElim 57
59. (a = u) & (c = w)  ImpElim 44 58
70. x = u  EqualitySub 62 60
71. (u,y) ϵ g  EqualitySub 68 70
72. Relation(g) & ∀x.∀y.∀z.((((x,y) ϵ g) & ((x,z) ϵ g)) -> (y = z))  DefExp 2
74. ∀y.∀z.((((u,y) ϵ g) & ((u,z) ϵ g)) -> (y = z))  ForallElim 73
75. ∀z.((((u,y) ϵ g) & ((u,z) ϵ g)) -> (y = z))  ForallElim 74
76. (((u,y) ϵ g) & ((u,v) ϵ g)) -> (y = v)  ForallElim 75
77. ((u,y) ϵ g) & ((u,v) ϵ g)  AndInt 71 69
78. y = v  ImpElim 77 76
79. (v,z) ϵ f  EqualitySub 66 78
80. Relation(f) & ∀x.∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z))  DefExp 1
82. ∀y.∀z.((((v,y) ϵ f) & ((v,z) ϵ f)) -> (y = z))  ForallElim 81
83. ∀x_3.((((v,z) ϵ f) & ((v,x_3) ϵ f)) -> (z = x_3))  ForallElim 82
84. (((v,z) ϵ f) & ((v,w) ϵ f)) -> (z = w)  ForallElim 83
85. ((v,z) ϵ f) & ((v,w) ϵ f)  AndInt 79 67
86. z = w  ImpElim 85 84
87. b = w  EqualitySub 61 86
89. b = c  EqualitySub 87 88
90. b = c  ExistsElim 20 21 89
96. ((a,c) ϵ (f∘g)) -> (b = c)  ImpInt 95
97. ((a,b) ϵ (f∘g)) -> (((a,c) ϵ (f∘g)) -> (b = c))  ImpInt 96
98. A -> (B -> C)  Hyp
99. A & B  Hyp
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101. B -> C  ImpElim 100 98
103. C  ImpElim 102 101
104. (A & B) -> C  ImpInt 103
105. (A -> (B -> C)) -> ((A & B) -> C)  ImpInt 104
106. (((a,b) ϵ (f∘g)) -> (B -> C)) -> ((((a,b) ϵ (f∘g)) & B) -> C)  PolySub 105
107. (((a,b) ϵ (f∘g)) -> (((a,c) ϵ (f∘g)) -> C)) -> ((((a,b) ϵ (f∘g)) & ((a,c) ϵ (f∘g))) -> C)  PolySub 106
108. (((a,b) ϵ (f∘g)) -> (((a,c) ϵ (f∘g)) -> (b = c))) -> ((((a,b) ϵ (f∘g)) & ((a,c) ϵ (f∘g))) -> (b = c)
)  PolySub 107
109. (((a,b) ϵ (f∘g)) & ((a,c) ϵ (f∘g))) -> (b = c)  ImpElim 97 108
112. z ϵ (f∘g)  Hyp
113. z ϵ {w: ∃x.∃y.∃z.((((x,y) ϵ g) & ((y,z) ϵ f)) & (w = (x,z)))}  EqualitySub 112 9
114. Set(z) & ∃x.∃y.∃x_4.((((x,y) ϵ g) & ((y,x_4) ϵ f)) & (z = (x,x_4)))  ClassElim 113
116. ∃y.∃x_4.((((x,y) ϵ g) & ((y,x_4) ϵ f)) & (z = (x,x_4)))  Hyp
117. ∃x_4.((((x,y) ϵ g) & ((y,x_4) ϵ f)) & (z = (x,x_4)))  Hyp
118. (((x,y) ϵ g) & ((y,l) ϵ f)) & (z = (x,l))  Hyp
121. ∃x.∃l.(z = (x,l))  ExistsInt 120
122. ∃x.∃l.(z = (x,l))  ExistsElim 117 118 121
125. (z ϵ (f∘g)) -> ∃x.∃l.(z = (x,l))  ImpInt 124
126. ∀z.((z ϵ (f∘g)) -> ∃x.∃l.(z = (x,l)))  ForallInt 125
127. Relation((f∘g))  DefSub 126
128. ∀c.((((a,b) ϵ (f∘g)) & ((a,c) ϵ (f∘g))) -> (b = c))  ForallInt 109
129. ∀b.∀c.((((a,b) ϵ (f∘g)) & ((a,c) ϵ (f∘g))) -> (b = c))  ForallInt 128
130. ∀a.∀b.∀c.((((a,b) ϵ (f∘g)) & ((a,c) ϵ (f∘g))) -> (b = c))  ForallInt 129
131. Relation((f∘g)) & ∀a.∀b.∀c.((((a,b) ϵ (f∘g)) & ((a,c) ϵ (f∘g))) -> (b = c))  AndInt 127 130
132. FUN((f∘g))  DefSub 131
133. (FUN(f) & FUN(g)) -> FUN((f∘g))  ImpInt 132 Qed
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Used Theorems

```
1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
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Th67. (dom(U) = U) & (rg(U) = U)

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0. z ϵ dom(U)  Hyp
1. ∃w.(z ϵ w)  ExistsInt 0
2. Set(z)  DefSub 1
3. (x ϵ U) <-> Set(x)  TheoremInt
7. Set(z) -> (z ϵ U)  ForallElim 6
8. z ϵ U  ImpElim 2 7
9. (z ϵ dom(U)) -> (z ϵ U)  ImpInt 8
10. z ϵ U  Hyp
14. (z ϵ U) -> Set(z)  ForallElim 13
15. Set(z)  ImpElim 10 14
16. (0 ⊂ x) & (x ⊂ U)  TheoremInt
19. 0 ⊂ z  ForallElim 18
20. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
24. (Set(z) & (0 ⊂ z)) -> Set(0)  ForallElim 23
25. Set(z) & (0 ⊂ z)  AndInt 15 19
26. Set(0)  ImpElim 25 24
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
34. (Set(z) & Set(0)) -> Set((z,0))  ForallElim 33
36. Set(z) & Set(0)  AndInt 15 26
37. Set((z,0))  ImpElim 36 34
40. Set((z,0)) -> ((z,0) ϵ U)  ForallElim 39
41. (z,0) ϵ U  ImpElim 37 40
42. ∃w.((z,w) ϵ U)  ExistsInt 41
43. Set(z) & ∃w.((z,w) ϵ U)  AndInt 15 42
44. z ϵ {w: ∃i.((w,i) ϵ U)}  ClassInt 43
47. {x: ∃y.((x,y) ϵ U)} = dom(U)  ForallElim 46
48. z ϵ dom(U)  EqualitySub 44 47
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53. (Set(0) & Set(z)) -> Set((0,z)) ForallElim 52
54. Set(0) & Set(z) AndInt 26 15
55. Set((0,z)) ImpElim 54 53
57. Set((0,z)) -> ((0,z) ϵ U) ForallElim 56
58. (0,z) ϵ U ImpElim 55 57
59. ∃w.((w,z) ϵ U) ExistsInt 58
63. {y: ∃x.((x,y) ϵ U)} = rg(U) ForallElim 62
64. Set(z) & ∃w.((w,z) ϵ U) AndInt 15 59
65. z ϵ {w: ∃j.((j,w) ϵ U)} ClassInt 64
66. z ϵ rg(U) EqualitySub 65 63
67. (z ϵ U) -> (z ϵ dom(U)) ImpInt 48
68. (z ϵ U) -> (z ϵ rg(U)) ImpInt 66
69. z ϵ rg(U) Hyp
70. ∃w.(z ϵ w) ExistsInt 69
71. Set(z) DefSub 70
72. z ϵ U ImpElim 71 7
73. (z ϵ rg(U)) -> (z ϵ U) ImpInt 72
74. ((z ϵ dom(U)) -> (z ϵ U)) & ((z ϵ U) -> (z ϵ dom(U))) AndInt 9 67
76. ∀z.((z ϵ dom(U)) <-> (z ϵ U)) ForallInt 75
77. ((z ϵ rg(U)) -> (z ϵ U)) & ((z ϵ U) -> (z ϵ rg(U))) AndInt 73 68
79. ∀z.((z ϵ rg(U)) <-> (z ϵ U)) ForallInt 78
80. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
81. ∀y.((dom(U) = y) <-> ∀z.((z ϵ dom(U)) <-> (z ϵ y))) ForallElim 80
82. (dom(U) = U) <-> ∀z.((z ϵ dom(U)) <-> (z ϵ U)) ForallElim 81
85. dom(U) = U ImpElim 76 84
86. ∀y.((rg(U) = y) <-> ∀z.((z ϵ rg(U)) <-> (z ϵ y))) ForallElim 80
87. (rg(U) = U) <-> ∀z.((z ϵ rg(U)) <-> (z ϵ U)) ForallElim 86
90. rg(U) = U ImpElim 79 89
91. (dom(U) = U) & (rg(U) = U) AndInt 85 90 Qed

Used Theorems

1. (x ϵ U) <-> Set(x)
2. (0 ⊂ x) & (x ⊂ U)
3. (Set(x) & (y ⊂ x)) -> Set(y)
4. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th69. (¬(z ϵ dom(f)) -> ((f'z) = U)) & ((z ϵ dom(f)) -> ((f'z) ϵ U))

0. ¬(z ϵ dom(f)) Hyp
1. a ϵ {y: ((z,y) ϵ f)} Hyp
2. Set(a) & ((z,a) ϵ f) ClassElim 1
5. ∃v.((z,a) ϵ v) ExistsInt 3
6. Set((z,a)) DefSub 5
7. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
14. Set((z,a)) -> (Set(z) & Set(a)) ForallElim 13
15. Set(z) & Set(a) ImpElim 6 14
17. Set(z) & ∃w.((z,w) ϵ f) AndInt 16 4
18. z ϵ {w: ∃x_1.((w,x_1) ϵ f)} ClassInt 17
21. z ϵ dom(f) EqualitySub 18 20
22. _|_ ImpElim 21 0
23. ¬(a ϵ {y: ((z,y) ϵ f)}) ImpInt 22
24. ∀a.¬(a ϵ {y: ((z,y) ϵ f)}) ForallInt 23
25. b ϵ 0 Hyp
27. b ϵ {x: ¬(x = x)} EqualitySub 25 26
28. Set(b) & ¬(b = b) ClassElim 27
31. _|_ ImpElim 30 29
32. b ϵ {y: ((z,y) ϵ f)} AbsI 31
33. (b ϵ 0) -> (b ϵ {y: ((z,y) ϵ f)}) ImpInt 32
34. b ϵ {y: ((z,y) ϵ f)} Hyp
35. ¬(b ϵ {y: ((z,y) ϵ f)}) ForallElim 24

36. _|_ ImpElim 34 35
37. b ϵ 0 AbsI 36
38. (b ϵ {y: ((z,y) ϵ f)}) -> (b ϵ 0) ImpInt 37
39. ((b ϵ {y: ((z,y) ϵ f)}) -> (b ϵ 0)) & ((b ϵ 0) -> (b ϵ {y: ((z,y) ϵ f)})) AndInt 38 33
41. ∀b.((b ϵ {y: ((z,y) ϵ f)}) <-> (b ϵ 0)) ForallInt 40
42. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
43. ∀x_2.(({y: ((z,y) ϵ f)} = x_2) <-> ∀x_3.((x_3 ϵ {y: ((z,y) ϵ f)}) <-> (x_3 ϵ x_2))) ForallElim 42
44. ({y: ((z,y) ϵ f)} = 0) <-> ∀x_3.((x_3 ϵ {y: ((z,y) ϵ f)}) <-> (x_3 ϵ 0)) ForallElim 43
47. {y: ((z,y) ϵ f)} = 0 ImpElim 41 46
48. (∩0 = U) & (∪0 = 0) TheoremInt
51. ∩{y: ((z,y) ϵ f)} = U EqualitySub 49 50
54. (f'z) = ∩{y: ((z,y) ϵ f)} ForallElim 53
56. (f'z) = U EqualitySub 51 55
57. ¬(z ϵ dom(f)) -> ((f'z) = U) ImpInt 56
58. z ϵ dom(f) Hyp
59. z ϵ {x: ∃y.((x,y) ϵ f)} EqualitySub 58 19
60. Set(z) & ∃y.((z,y) ϵ f) ClassElim 59
63. {a: ((z,a) ϵ f)} = 0 Hyp
64. (z,y) ϵ f Hyp
65. ∃v.((z,y) ϵ v) ExistsInt 64
66. Set((z,y)) DefSub 65
67. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
72. Set((z,y)) -> (Set(z) & Set(y)) ForallElim 71
73. Set(z) & Set(y) ImpElim 66 72
75. Set(y) & ((z,y) ϵ f) AndInt 74 64
76. y ϵ {w: ((z,w) ϵ f)} ClassInt 75
77. y ϵ 0 EqualitySub 76 63
79. y ϵ {x: ¬(x = x)} EqualitySub 77 78
80. Set(y) & ¬(y = y) ClassElim 79
83. _|_ ImpElim 82 81
84. ¬({a: ((z,a) ϵ f)} = 0) ImpInt 83
85. ¬(x = 0) -> Set(∩x) TheoremInt
87. ¬({a: ((z,a) ϵ f)} = 0) -> Set(∩{a: ((z,a) ϵ f)}) ForallElim 86
88. Set(∩{a: ((z,a) ϵ f)}) ImpElim 84 87
91. (f'z) = ∩{y: ((z,y) ϵ f)} ForallElim 90
93. Set((f'z)) EqualitySub 88 92
94. (x ϵ U) <-> Set(x) TheoremInt
98. Set((f'z)) -> ((f'z) ϵ U) ForallElim 97
99. (f'z) ϵ U ImpElim 93 98
100. (f'z) ϵ U ExistsElim 62 64 99
101. (z ϵ dom(f)) -> ((f'z) ϵ U) ImpInt 100
102. (¬(z ϵ dom(f)) -> ((f'z) = U)) & ((z ϵ dom(f)) -> ((f'z) ϵ U)) AndInt 57 101 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. (∩0 = U) & (∪0 = 0)
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
4. ¬(x = 0) -> Set(∩x)
5. (x ϵ U) <-> Set(x)

Th70. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

0. FUN(f) Hyp
1. z ϵ f Hyp
2. Relation(f) & ∀x.∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z)) DefExp 0
4. ∀z.((z ϵ f) -> ∃x.∃y.(z = (x,y))) DefExp 3
5. (z ϵ f) -> ∃x.∃y.(z = (x,y)) ForallElim 4
6. ∃x.∃y.(z = (x,y)) ImpElim 1 5
7. ∃y.(z = (x,y)) Hyp
8. z = (x,y) Hyp

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11. a ϵ {y: ((x,y) ϵ f)}  Hyp
12. Set(a) & ((x,a) ϵ f)  ClassElim 11
14. ∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z))  ForallElim 9
15. ∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z))  ForallElim 14
16. (((x,y) ϵ f) & ((x,a) ϵ f)) -> (y = a)  ForallElim 15
17. (x,y) ϵ f  EqualitySub 1 8
18. ((x,y) ϵ f) & ((x,a) ϵ f)  AndInt 17 13
19. y = a  ImpElim 18 16
22. {y} = {z: ((y ϵ U) -> (z = y))}  ForallElim 21
23. (a ϵ {y: ((x,y) ϵ f)}) -> (y = a)  ImpInt 19
24. ∃w.(z ϵ w)  ExistsInt 1
25. Set(z)  DefSub 24
26. Set((x,y))  EqualitySub 25 8
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
31. Set(x) & Set(y)  ImpElim 26 30
33. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
37. Set(y) -> ((a ϵ {y}) <-> (a = y))  ForallElim 36
38. (a ϵ {y}) <-> (a = y)  ImpElim 32 37
42. a ϵ {y}  ImpElim 41 40
43. (a ϵ {y: ((x,y) ϵ f)}) -> (a ϵ {y})  ImpInt 42
44. a ϵ {y}  Hyp
47. a = y  ImpElim 44 46
49. (x,y) ϵ f  EqualitySub 1 8
50. (x,a) ϵ f  EqualitySub 49 48
51. Set(a)  EqualitySub 32 48
52. Set(a) & ((x,a) ϵ f)  AndInt 51 50
53. a ϵ {y: ((x,y) ϵ f)}  ClassInt 52
54. (a ϵ {y}) -> (a ϵ {y: ((x,y) ϵ f)})  ImpInt 53
55. ((a ϵ {y: ((x,y) ϵ f)}) -> (a ϵ {y})) & ((a ϵ {y}) -> (a ϵ {y: ((x,y) ϵ f)}))  AndInt 43 54
57. ∀a.((a ϵ {y: ((x,y) ϵ f)}) <-> (a ϵ {y}))  ForallInt 56
58. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
59. ∀x_5.(({y: ((x,y) ϵ f)} = x_5) <-> ∀z.((z ϵ {y: ((x,y) ϵ f)}) <-> (z ϵ x_5)))  ForallElim 58
60. ({x_6: ((x,x_6) ϵ f)} = {y}) <-> ∀z.((z ϵ {x_6: ((x,x_6) ϵ f)}) <-> (z ϵ {y}))  ForallElim 59
63. {x_6: ((x,x_6) ϵ f)} = {y}  ImpElim 57 62
64. (f'x) = ∩{y}  EqualitySub 10 63
65. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))  TheoremInt
68. Set(y) -> ((∩{y} = y) & (∪{y} = y))  ForallElim 67
69. (∩{y} = y) & (∪{y} = y)  ImpElim 32 68
71. (f'x) = y  EqualitySub 64 70
72. (z = (x,y)) & ((f'x) = y)  AndInt 8 71
74. ∃x.∃y.((z = (x,y)) & ((f'x) = y))  ExistsInt 73
75. Set(z) & ∃x.∃y.((z = (x,y)) & ((f'x) = y))  AndInt 25 74
76. z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ClassInt 75
77. z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ExistsElim 7 8 76
79. (z ϵ f) -> (z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  ImpInt 78
80. z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  Hyp
81. Set(z) & ∃x.∃y.((z = (x,y)) & ((f'x) = y))  ClassElim 80
84. ∃y.((z = (x,y)) & ((f'x) = y))  Hyp
85. (z = (x,y)) & ((f'x) = y)  Hyp
88. ∩{y: ((x,y) ϵ f)} = y  EqualitySub 87 10
89. Set((x,y))  EqualitySub 82 86
90. Set(x) & Set(y)  ImpElim 89 30
93. Set((f'x))  EqualitySub 91 92
94. (f'x) = U  Hyp
95. ¬Set(U)  TheoremInt
96. Set(U)  EqualitySub 93 94
97. _|_  ImpElim 96 95
98. ¬((f'x) = U)  ImpInt 97
99. (¬(z ϵ dom(f)) -> ((f'z) = U)) & ((z ϵ dom(f)) -> ((f'z) ϵ U))  TheoremInt
101. (A -> B) -> (¬B -> ¬A)  TheoremInt
102. (¬(z ϵ dom(f)) -> B) -> (¬B -> ¬¬(z ϵ dom(f)))  PolySub 101
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103. (¬(z ϵ dom(f)) -> ((f'z) = U)) -> (¬((f'z) = U) -> ¬¬(z ϵ dom(f)))  PolySub 102
104. ¬((f'z) = U) -> ¬¬(z ϵ dom(f))  ImpElim 100 103
105. D <-> ¬¬D  TheoremInt
108. ¬¬(z ϵ dom(f)) -> (z ϵ dom(f))  PolySub 107
109. ¬((f'z) = U)  Hyp
110. ¬¬(z ϵ dom(f))  ImpElim 109 104
111. z ϵ dom(f)  ImpElim 110 108
112. ¬((f'z) = U) -> (z ϵ dom(f))  ImpInt 111
114. ¬((f'x) = U) -> (x ϵ dom(f))  ForallElim 113
115. x ϵ dom(f)  ImpElim 98 114
117. x ϵ {x: ∃y.((x,y) ϵ f)}  EqualitySub 115 116
118. Set(x) & ∃y.((x,y) ϵ f)  ClassElim 117
120. (x,b) ϵ f  Hyp
121. e ϵ {b}  Hyp
122. ∃w.((x,b) ϵ w)  ExistsInt 120
123. Set((x,b))  DefSub 122
125. Set((x,b)) -> (Set(x) & Set(b))  ForallElim 124
126. Set(x) & Set(b)  ImpElim 123 125
128. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
130. Set(b) -> ((y ϵ {b}) <-> (y = b))  ForallElim 129
131. (y ϵ {b}) <-> (y = b)  ImpElim 127 130
133. (e ϵ {b}) <-> (e = b)  ForallElim 132
136. e = b  ImpElim 121 135
138. (x,e) ϵ f  EqualitySub 120 137
139. Set(e)  EqualitySub 127 137
140. Set(e) & ((x,e) ϵ f)  AndInt 139 138
141. e ϵ {y: ((x,y) ϵ f)}  ClassInt 140
142. e ϵ {y: ((x,y) ϵ f)}  Hyp
143. Set(e) & ((x,e) ϵ f)  ClassElim 142
145. Relation(f) & ∀x.∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z))  DefExp 0
147. (e ϵ {b}) -> (e ϵ {y: ((x,y) ϵ f)})  ImpInt 141
148. ((x,b) ϵ f) & ((x,e) ϵ f)  AndInt 120 144
149. ∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z))  ForallElim 146
150. ∀z.((((x,b) ϵ f) & ((x,z) ϵ f)) -> (b = z))  ForallElim 149
151. (((x,b) ϵ f) & ((x,e) ϵ f)) -> (b = e)  ForallElim 150
152. b = e  ImpElim 148 151
157. e ϵ {b}  ImpElim 156 155
158. (e ϵ {y: ((x,y) ϵ f)}) -> (e ϵ {b})  ImpInt 157
159. ((e ϵ {b}) -> (e ϵ {y: ((x,y) ϵ f)})) & ((e ϵ {y: ((x,y) ϵ f)}) -> (e ϵ {b}))  AndInt 147 158
161. ∀e.((e ϵ {b}) <-> (e ϵ {y: ((x,y) ϵ f)}))  ForallInt 160
162. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
163. ∀y.(({b} = y) <-> ∀z.((z ϵ {b}) <-> (z ϵ y)))  ForallElim 162
164. ({b} = {y: ((x,y) ϵ f)}) <-> ∀z.((z ϵ {b}) <-> (z ϵ {y: ((x,y) ϵ f)}))  ForallElim 163
167. {b} = {y: ((x,y) ϵ f)}  ImpElim 161 166
169. ∩{b} = y  EqualitySub 88 168
170. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))  TheoremInt
173. Set(b) -> ((∩{b} = b) & (∪{b} = b))  ForallElim 172
174. (∩{b} = b) & (∪{b} = b)  ImpElim 127 173
176. b = y  EqualitySub 169 175
177. (x,y) ϵ f  EqualitySub 120 176
178. (x,y) ϵ f  EqualitySub 120 176
180. z ϵ f  EqualitySub 178 179
182. z ϵ f  ExistsElim 119 120 180
185. (z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) -> (z ϵ f)  ImpInt 184
186. ((z ϵ f) -> (z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) & ((z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) =
y))}) -> (z ϵ f))  AndInt 79 185
188. ∀z.((z ϵ f) <-> (z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}))  ForallInt 187
189. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
190. ∀y.((f = y) <-> ∀z.((z ϵ f) <-> (z ϵ y)))  ForallElim 189
191. (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) <-> ∀z.((z ϵ f) <-> (z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x
) = y))}))  ForallElim 190
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194. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ImpElim 188 193
195. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  ImpInt 194 Qed

Used Theorems

2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. Set(x) -> ((y ϵ {x}) <-> (y = x))
4. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))
5. ¬Set(U)
6. (¬(z ϵ dom(f)) -> ((f'z) = U)) & ((z ϵ dom(f)) -> ((f'z) ϵ U))
7. (A -> B) -> (¬B -> ¬A)
8. D <-> ¬¬D

Th71. (FUN(f) & FUN(g)) -> ((f = g) <-> ∀z.((f'z) = (g'z)))

0. FUN(f) & FUN(g)  Hyp
1. ∀z.((f'z) = (g'z))  Hyp
2. e ϵ f  Hyp
3. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  TheoremInt
6. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ImpElim 4 3
7. e ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  EqualitySub 2 6
8. Set(e) & ∃x.∃y.((e = (x,y)) & ((f'x) = y))  ClassElim 7
11. ∃y.((e = (x,y)) & ((f'x) = y))  Hyp
12. (e = (x,y)) & ((f'x) = y)  Hyp
13. (f'x) = (g'x)  ForallElim 1
14. (e = (x,y)) & ((g'x) = y)  EqualitySub 12 13
16. ∃x.∃y.((e = (x,y)) & ((g'x) = y))  ExistsInt 15
17. Set(e) & ∃x.∃y.((e = (x,y)) & ((g'x) = y))  AndInt 9 16
18. e ϵ {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  ClassInt 17
20. FUN(g) -> (g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))})  ForallElim 19
21. g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  ImpElim 5 20
23. e ϵ g  EqualitySub 18 22
24. e ϵ g  ExistsElim 11 12 23
26. (e ϵ f) -> (e ϵ g)  ImpInt 25
27. e ϵ g  Hyp
28. e ϵ {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  EqualitySub 27 21
29. Set(e) & ∃x.∃y.((e = (x,y)) & ((g'x) = y))  ClassElim 28
32. ∃y.((e = (x,y)) & ((g'x) = y))  Hyp
33. (e = (x,y)) & ((g'x) = y)  Hyp
35. (e = (x,y)) & ((f'x) = y)  EqualitySub 33 34
37. ∃x.∃y.((e = (x,y)) & ((f'x) = y))  ExistsInt 36
38. Set(e) & ∃x.∃y.((e = (x,y)) & ((f'x) = y))  AndInt 30 37
39. e ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ClassInt 38
41. e ϵ f  EqualitySub 39 40
42. e ϵ f  ExistsElim 32 33 41
44. (e ϵ g) -> (e ϵ f)  ImpInt 43
45. ((e ϵ f) -> (e ϵ g)) & ((e ϵ g) -> (e ϵ f))  AndInt 26 44
47. ∀e.((e ϵ f) <-> (e ϵ g))  ForallInt 46
48. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
49. ∀y.((f = y) <-> ∀z.((z ϵ f) <-> (z ϵ y)))  ForallElim 48
50. (f = g) <-> ∀z.((z ϵ f) <-> (z ϵ g))  ForallElim 49
53. f = g  ImpElim 47 52
54. ∀z.((f'z) = (g'z)) -> (f = g)  ImpInt 53
55. f = g  Hyp
57. (f'z) = (g'z)  EqualitySub 56 55
58. ∀z.((f'z) = (g'z))  ForallInt 57
59. (f = g) -> ∀z.((f'z) = (g'z))  ImpInt 58
60. ((f = g) -> ∀z.((f'z) = (g'z))) & (∀z.((f'z) = (g'z)) -> (f = g))  AndInt 59 54
62. (FUN(f) & FUN(g)) -> ((f = g) <-> ∀z.((f'z) = (g'z)))  ImpInt 61 Qed

Used Theorems
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1. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

Th73. (Set(u) & Set(y)) -> Set(({u} X y))

0. Set(u) & Set(y)  Hyp
1. f = {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))}  Hyp
2. x ϵ dom(f)  Hyp
4. x ϵ {x: ∃y.((x,y) ϵ f)}  EqualitySub 2 3
5. Set(x) & ∃y.((x,y) ϵ f)  ClassElim 4
6. Set(x) & ∃x_0.((x,x_0) ϵ {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))})  EqualitySub 5 1
9. (x,c) ϵ {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))}  Hyp
10. Set((x,c)) & ∃w.∃z.(((x,c) = (w,z)) & ((w ϵ y) & (z = (u,w))))  ClassElim 9
13. ∃z.(((x,c) = (w,z)) & ((w ϵ y) & (z = (u,w))))  Hyp
14. ((x,c) = (w,z)) & ((w ϵ y) & (z = (u,w)))  Hyp
16. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
21. Set((x,c)) -> (Set(x) & Set(c))  ForallElim 20
22. Set(x) & Set(c)  ImpElim 11 21
23. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
29. ((Set(x) & Set(c)) & ((x,c) = (w,z))) -> ((x = w) & (c = z))  ForallElim 28
30. (Set(x) & Set(c)) & ((x,c) = (w,z))  AndInt 22 15
31. (x = w) & (c = z)  ImpElim 30 29
36. x ϵ y  EqualitySub 34 35
37. x ϵ y  ExistsElim 13 14 36
40. (x ϵ dom(f)) -> (x ϵ y)  ImpInt 39
41. x ϵ y  Hyp
42. z = (u,x)  Hyp
43. a = (x,z)  Hyp
44. (a = (x,z)) & (z = (u,x))  AndInt 43 42
47. ∃y.(x ϵ y)  ExistsInt 41
48. Set(x)  DefSub 47
50. Set(u) & Set(x)  AndInt 49 48
56. (Set(u) & Set(x)) -> Set((u,x))  ForallElim 55
57. Set((u,x))  ImpElim 50 56
59. Set(z)  EqualitySub 57 58
60. Set(x) & Set(z)  AndInt 48 59
61. ∀y.(((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))))  ForallInt 51
63. (Set(x) & Set(z)) -> Set((x,z))  ForallElim 62
64. Set((x,z))  ImpElim 60 63
66. Set(a)  EqualitySub 64 65
67. Set(a) & ∃x.∃z.((a = (x,z)) & (z = (u,x)))  AndInt 66 46
69. a ϵ {a: ∃x.∃z.((a = (x,z)) & (z = (u,x)))}  ClassInt 67
70. (x ϵ y) & (z = (u,x))  AndInt 41 42
71. (a = (x,z)) & ((x ϵ y) & (z = (u,x)))  AndInt 43 70
73. ∃x.∃z.((a = (x,z)) & ((x ϵ y) & (z = (u,x))))  ExistsInt 72
74. Set(a) & ∃x.∃z.((a = (x,z)) & ((x ϵ y) & (z = (u,x))))  AndInt 66 73
75. a ϵ {a: ∃x.∃z.((a = (x,z)) & ((x ϵ y) & (z = (u,x))))}  ClassInt 74
76. a ϵ f  EqualitySub 75 68
77. (x,z) ϵ f  EqualitySub 76 43
78. ∃z.((x,z) ϵ f)  ExistsInt 77
79. Set(x) & ∃z.((x,z) ϵ f)  AndInt 48 78
80. x ϵ {w: ∃z.((w,z) ϵ f)}  ClassInt 79
82. x ϵ dom(f)  EqualitySub 80 81
83. (a = (x,z)) -> (x ϵ dom(f))  ImpInt 82
85. ((x,z) = (x,z)) -> (x ϵ dom(f))  ForallElim 84
87. x ϵ dom(f)  ImpElim 86 85
88. (z = (u,x)) -> (x ϵ dom(f))  ImpInt 87
90. ((u,x) = (u,x)) -> (x ϵ dom(f))  ForallElim 89
92. x ϵ dom(f)  ImpElim 91 90
93. (x ϵ y) -> (x ϵ dom(f))  ImpInt 92
94. ((x ϵ dom(f)) -> (x ϵ y)) & ((x ϵ y) -> (x ϵ dom(f)))  AndInt 40 93
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96. ∀x.((x ϵ dom(f)) <-> (x ϵ y))  ForallInt 95
97. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
98. ∀y.((dom(f) = y) <-> ∀z.((z ϵ dom(f)) <-> (z ϵ y)))  ForallElim 97
99. (dom(f) = y) <-> ∀z.((z ϵ dom(f)) <-> (z ϵ y))  ForallElim 98
102. dom(f) = y  ImpElim 96 101
103. x ϵ rg(f)  Hyp
105. x ϵ {y: ∃x.((x,y) ϵ f)}  EqualitySub 103 104
106. Set(x) & ∃x_4.((x_4,x) ϵ f)  ClassElim 105
108. ∃x_4.((x_4,x) ϵ {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))})  EqualitySub 107 1
109. (c,x) ϵ {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))}  Hyp
110. Set((c,x)) & ∃w.∃z.(((c,x) = (w,z)) & ((w ϵ y) & (z = (u,w))))  ClassElim 109
112. ∃z.(((c,x) = (w,z)) & ((w ϵ y) & (z = (u,w))))  Hyp
113. ((c,x) = (w,z)) & ((w ϵ y) & (z = (u,w)))  Hyp
118. Set((c,x)) -> (Set(c) & Set(x))  ForallElim 117
119. Set(c) & Set(x)  ImpElim 114 118
127. ((Set(c) & Set(x)) & ((c,x) = (w,z))) -> ((c = w) & (x = z))  ForallElim 126
129. (Set(c) & Set(x)) & ((c,x) = (w,z))  AndInt 119 128
130. (c = w) & (x = z)  ImpElim 129 127
136. x = (u,w)  EqualitySub 133 135
139. Set(w)  EqualitySub 137 138
140. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
145. Set(u) -> ((u ϵ {u}) <-> (u = u))  ForallElim 144
146. (u ϵ {u}) <-> (u = u)  ImpElim 141 145
150. u ϵ {u}  ImpElim 149 148
151. (u ϵ {u}) & (w ϵ y)  AndInt 150 132
152. (x = (u,w)) & ((u ϵ {u}) & (w ϵ y))  AndInt 136 151
155. ∃b.∃w.((x = (b,w)) & ((b ϵ {u}) & (w ϵ y)))  ExistsInt 154
156. Set(x) & ∃b.∃w.((x = (b,w)) & ((b ϵ {u}) & (w ϵ y)))  AndInt 153 155
157. x ϵ {e: ∃b.∃w.((e = (b,w)) & ((b ϵ {u}) & (w ϵ y)))}  ClassInt 156
160. ({u} X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ϵ {u}) & (b ϵ y)))}  ForallElim 159
162. x ϵ ({u} X y)  EqualitySub 157 161
163. x ϵ ({u} X y)  ExistsElim 112 113 162
164. x ϵ ({u} X y)  Hyp
165. x ϵ {z: ∃a.∃b.((z = (a,b)) & ((a ϵ {u}) & (b ϵ y)))}  EqualitySub 164 160
166. Set(x) & ∃a.∃b.((x = (a,b)) & ((a ϵ {u}) & (b ϵ y)))  ClassElim 165
168. x ϵ ({u} X y)  ExistsElim 111 112 163
170. (x ϵ rg(f)) -> (x ϵ ({u} X y))  ImpInt 169
171. ∃b.((x = (a,b)) & ((a ϵ {u}) & (b ϵ y)))  Hyp
172. (x = (a,b)) & ((a ϵ {u}) & (b ϵ y))  Hyp
178. Set(u) -> ((a ϵ {u}) <-> (a = u))  ForallElim 177
180. (a ϵ {u}) <-> (a = u)  ImpElim 179 178
183. a = u  ImpElim 175 182
184. x = (u,b)  EqualitySub 173 183
185. c = (b,x)  Hyp
186. (b ϵ y) & (x = (u,b))  AndInt 176 184
187. (c = (b,x)) & ((b ϵ y) & (x = (u,b)))  AndInt 185 186
189. ∃b.∃x.((c = (b,x)) & ((b ϵ y) & (x = (u,b))))  ExistsInt 188
191. ∃y.(b ϵ y)  ExistsInt 176
192. Set(b)  DefSub 191
196. (Set(b) & Set(x)) -> Set((b,x))  ForallElim 195
197. Set(b) & Set(x)  AndInt 192 190
198. Set((b,x))  ImpElim 197 196
200. Set(c)  EqualitySub 198 199
201. Set(c) & ∃b.∃x.((c = (b,x)) & ((b ϵ y) & (x = (u,b))))  AndInt 200 189
202. c ϵ {w: ∃b.∃x.((w = (b,x)) & ((b ϵ y) & (x = (u,b))))}  ClassInt 201
204. c ϵ f  EqualitySub 202 203
205. (b,x) ϵ f  EqualitySub 204 185
206. ∃b.((b,x) ϵ f)  ExistsInt 205
207. Set(x) & ∃b.((b,x) ϵ f)  AndInt 190 206
208. x ϵ {w: ∃b.((b,w) ϵ f)}  ClassInt 207
210. x ϵ rg(f)  EqualitySub 208 209
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211. (c = (b,x)) -> (x ϵ rg(f))  ImpInt 210
213. ((b,x) = (b,x)) -> (x ϵ rg(f))  ForallElim 212
215. x ϵ rg(f)  ImpElim 214 213
216. x ϵ rg(f)  ExistsElim 171 172 215
218. (x ϵ ({u} X y)) -> (x ϵ rg(f))  ImpInt 217
219. ((x ϵ rg(f)) -> (x ϵ ({u} X y))) & ((x ϵ ({u} X y)) -> (x ϵ rg(f)))  AndInt 170 218
221. ∀x.((x ϵ rg(f)) <-> (x ϵ ({u} X y)))  ForallInt 220
222. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
223. ∀y.((rg(f) = y) <-> ∀z.((z ϵ rg(f)) <-> (z ϵ y)))  ForallElim 222
224. (rg(f) = ({u} X y)) <-> ∀z.((z ϵ rg(f)) <-> (z ϵ ({u} X y)))  ForallElim 223
227. rg(f) = ({u} X y)  ImpElim 221 226
228. (FUN(f) & Set(dom(f))) -> Set(rg(f))  AxInt
231. Set(dom(f))  EqualitySub 229 230
232. x ϵ f  Hyp
233. x ϵ {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))}  EqualitySub 232 1
234. Set(x) & ∃w.∃z.((x = (w,z)) & ((w ϵ y) & (z = (u,w))))  ClassElim 233
236. ∃z.((x = (w,z)) & ((w ϵ y) & (z = (u,w))))  Hyp
237. (x = (w,z)) & ((w ϵ y) & (z = (u,w)))  Hyp
240. ∃w.∃z.(x = (w,z))  ExistsInt 239
241. ∃w.∃z.(x = (w,z))  ExistsElim 236 237 240
243. (x ϵ f) -> ∃w.∃z.(x = (w,z))  ImpInt 242
244. ∀x.((x ϵ f) -> ∃w.∃z.(x = (w,z)))  ForallInt 243
245. Relation(f)  DefSub 244
246. (a,b) ϵ f  Hyp
247. (a,c) ϵ f  Hyp
248. (a,b) ϵ {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))}  EqualitySub 246 1
249. (a,c) ϵ {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))}  EqualitySub 247 1
250. Set((a,b)) & ∃w.∃z.(((a,b) = (w,z)) & ((w ϵ y) & (z = (u,w))))  ClassElim 248
251. Set((a,c)) & ∃w.∃z.(((a,c) = (w,z)) & ((w ϵ y) & (z = (u,w))))  ClassElim 249
254. ∃z.(((a,b) = (x1,z)) & ((x1 ϵ y) & (z = (u,x1))))  Hyp
255. ((a,b) = (x1,y1)) & ((x1 ϵ y) & (y1 = (u,x1)))  Hyp
256. ∃z.(((a,c) = (x2,z)) & ((x2 ϵ y) & (z = (u,x2))))  Hyp
257. ((a,c) = (x2,y2)) & ((x2 ϵ y) & (y2 = (u,x2)))  Hyp
260. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
271. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 270
272. Set(a) & Set(b)  ImpElim 264 269
273. Set(a) & Set(c)  ImpElim 265 271
274. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
282. ((Set(a) & Set(b)) & ((a,b) = (x1,y1))) -> ((a = x1) & (b = y1))  ForallElim 281
283. (Set(a) & Set(b)) & ((a,b) = (x1,y1))  AndInt 272 258
284. (a = x1) & (b = y1)  ImpElim 283 282
285. (Set(a) & Set(c)) & ((a,c) = (x2,y2))  AndInt 273 259
291. ((Set(a) & Set(c)) & ((a,c) = (x2,y2))) -> ((a = x2) & (c = y2))  ForallElim 290
292. (a = x2) & (c = y2)  ImpElim 285 291
297. x1 = x2  EqualitySub 296 295
301. y2 = (u,x1)  EqualitySub 299 300
303. y1 = y2  EqualitySub 298 302
304. (a,b) = (x2,y1)  EqualitySub 258 297
305. (a,b) = (x2,y2)  EqualitySub 304 303
307. (a,b) = (a,c)  EqualitySub 305 306
308. (Set(a) & Set(b)) & ((a,b) = (a,c))  AndInt 272 307
312. ((Set(a) & Set(b)) & ((a,b) = (a,c))) -> ((a = a) & (b = c))  ForallElim 311
313. (a = a) & (b = c)  ImpElim 308 312
315. b = c  ExistsElim 256 257 314
319. ((a,c) ϵ f) -> (b = c)  ImpInt 318
320. ((a,b) ϵ f) -> (((a,c) ϵ f) -> (b = c))  ImpInt 319
321. A -> (B -> C)  Hyp
322. A & B  Hyp
324. B -> C  ImpElim 323 321
326. C  ImpElim 325 324
327. (A & B) -> C  ImpInt 326
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328. (A -> (B -> C)) -> ((A & B) -> C)  ImpInt 327
329. (((a,b) ϵ f) -> (B -> C)) -> ((((a,b) ϵ f) & B) -> C)  PolySub 328
330. (((a,b) ϵ f) -> (((a,c) ϵ f) -> C)) -> ((((a,b) ϵ f) & ((a,c) ϵ f)) -> C)  PolySub 329
331. (((a,b) ϵ f) -> (((a,c) ϵ f) -> (b = c))) -> ((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  PolySub 330
332. (((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c)  ImpElim 320 331
333. ∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  ForallInt 332
334. ∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  ForallInt 333
335. ∀a.∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  ForallInt 334
336. Relation(f) & ∀a.∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  AndInt 245 335
337. FUN(f)  DefSub 336
338. FUN(f) & Set(dom(f))  AndInt 337 231
339. (FUN(f) & Set(dom(f))) -> Set(rg(f))  AxInt
340. Set(rg(f))  ImpElim 338 339
341. Set(({u} X y))  EqualitySub 340 227
342. (f = {a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))}) -> Set(({u} X y))  ImpInt 341
344. ({a: ∃w.∃z.((a = (w,z)) & ((w ϵ y) & (z = (u,w))))} = {x_8: ∃x_9.∃x_10.((x_8 = (x_9,x_10)) & ((x_9 ϵ y) & (x_10 = (u,x_9))))}) -> Set(({u} X y))  ForallElim 343
346. Set(({u} X y))  ImpElim 345 344
347. (Set(u) & Set(y)) -> Set(({u} X y))  ImpInt 346 Qed
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Used Theorems

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1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
3. Set(x) -> ((y ϵ {x}) <-> (y = x))
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Th74. (Set(x) & Set(y)) -> Set((x X y))

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0. f = {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))}  Hyp
1. c ϵ f  Hyp
2. c ϵ {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))}  EqualitySub 1 0
3. Set(c) & ∃u.∃z.((c = (u,z)) & ((u ϵ x) & (z = ({u} X y))))  ClassElim 2
5. ∃z.((c = (u,z)) & ((u ϵ x) & (z = ({u} X y))))  Hyp
6. (c = (u,z)) & ((u ϵ x) & (z = ({u} X y)))  Hyp
9. ∃u.∃z.(c = (u,z))  ExistsInt 8
10. ∃u.∃z.(c = (u,z))  ExistsElim 5 6 9
12. (c ϵ f) -> ∃u.∃z.(c = (u,z))  ImpInt 11
13. ∀c.((c ϵ f) -> ∃u.∃z.(c = (u,z)))  ForallInt 12
14. Relation(f)  DefSub 13
15. ((a,b) ϵ f) & ((a,c) ϵ f)  Hyp
18. (a,b) ϵ {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))}  EqualitySub 16 0
19. (a,c) ϵ {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))}  EqualitySub 17 0
20. Set((a,b)) & ∃u.∃z.(((a,b) = (u,z)) & ((u ϵ x) & (z = ({u} X y))))  ClassElim 18
21. Set((a,c)) & ∃u.∃z.(((a,c) = (u,z)) & ((u ϵ x) & (z = ({u} X y))))  ClassElim 19
24. ∃z.(((a,b) = (x1,z)) & ((x1 ϵ x) & (z = ({x1} X y))))  Hyp
25. ((a,b) = (x1,y1)) & ((x1 ϵ x) & (y1 = ({x1} X y)))  Hyp
26. ∃z.(((a,c) = (x2,z)) & ((x2 ϵ x) & (z = ({x2} X y))))  Hyp
27. ((a,c) = (x2,y2)) & ((x2 ϵ x) & (y2 = ({x2} X y)))  Hyp
30. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
39. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 38
40. Set(a) & Set(b)  ImpElim 28 37
41. Set(a) & Set(c)  ImpElim 29 39
42. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
45. ∀x.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 44
47. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))  ForallElim 46
53. ((Set(a) & Set(b)) & ((a,b) = (x1,y1))) -> ((a = x1) & (b = y1))  ForallElim 52
54. (Set(a) & Set(b)) & ((a,b) = (x1,y1))  AndInt 40 48
55. (a = x1) & (b = y1)  ImpElim 54 53
61. ((Set(a) & Set(c)) & ((a,c) = (x2,y2))) -> ((a = x2) & (c = y2))  ForallElim 60
62. (Set(a) & Set(c)) & ((a,c) = (x2,y2))  AndInt 41 49
63. (a = x2) & (c = y2)  ImpElim 62 61
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66. x2 = x1 EqualitySub 64 65
71. y2 = ({x1} X y) EqualitySub 70 66
73. y1 = y2 EqualitySub 69 72
76. b = y2 EqualitySub 74 73
78. c = b EqualitySub 75 77
79. c = b ExistsElim 26 27 78
84. (((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c) ImpInt 83
85. ∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c)) ForallInt 84
86. ∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c)) ForallInt 85
87. ∀a.∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c)) ForallInt 86
88. Relation(f) & ∀a.∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c)) AndInt 14 87
89. FUN(f) DefSub 88
90. a ϵ x Hyp
91. b = ({a} X y) Hyp
92. (a ϵ x) & (b = ({a} X y)) AndInt 90 91
93. c = (a,b) Hyp
94. (c = (a,b)) & ((a ϵ x) & (b = ({a} X y))) AndInt 93 92
96. ∃a.∃b.((c = (a,b)) & ((a ϵ x) & (b = ({a} X y)))) ExistsInt 95
97. Set(x) & Set(y) Hyp
98. ∃w.(a ϵ w) ExistsInt 90
99. Set(a) DefSub 98
100. Set(x) -> Set({x}) TheoremInt
102. Set(a) -> Set({a}) ForallElim 101
103. Set({a}) ImpElim 99 102
105. (Set(u) & Set(y)) -> Set(({u} X y)) TheoremInt
107. (Set(a) & Set(y)) -> Set(({a} X y)) ForallElim 106
108. Set(a) & Set(y) AndInt 99 104
109. Set(({a} X y)) ImpElim 108 107
111. Set(b) EqualitySub 109 110
112. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
119. (Set(a) & Set(b)) -> Set((a,b)) ForallElim 118
120. Set(a) & Set(b) AndInt 99 111
121. Set((a,b)) ImpElim 120 119
123. Set(c) EqualitySub 121 122
124. Set(c) & ∃a.∃b.((c = (a,b)) & ((a ϵ x) & (b = ({a} X y)))) AndInt 123 96
125. c ϵ {w: ∃a.∃b.((w = (a,b)) & ((a ϵ x) & (b = ({a} X y))))} ClassInt 124
126. (a,b) ϵ {w: ∃x_6.∃x_8.((w = (x_6,x_8)) & ((x_6 ϵ x) & (x_8 = ({x_6} X y))))} EqualitySub 125 93
128. (a,b) ϵ f EqualitySub 126 127
129. ∃b.((a,b) ϵ f) ExistsInt 128
130. Set(a) & ∃b.((a,b) ϵ f) AndInt 99 129
131. a ϵ {w: ∃b.((w,b) ϵ f)} ClassInt 130
134. a ϵ dom(f) EqualitySub 131 133
135. (c = (a,b)) -> (a ϵ dom(f)) ImpInt 134
137. ((a,b) = (a,b)) -> (a ϵ dom(f)) ForallElim 136
139. a ϵ dom(f) ImpElim 138 137
140. (b = ({a} X y)) -> (a ϵ dom(f)) ImpInt 139
142. (({a} X y) = ({a} X y)) -> (a ϵ dom(f)) ForallElim 141
144. a ϵ dom(f) ImpElim 143 142
145. (a ϵ x) -> (a ϵ dom(f)) ImpInt 144
146. a ϵ dom(f) Hyp
147. a ϵ {x: ∃y.((x,y) ϵ f)} EqualitySub 146 132
148. Set(a) & ∃y.((a,y) ϵ f) ClassElim 147
150. (a,b) ϵ f Hyp
151. (a,b) ϵ {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))} EqualitySub 150 0
152. Set((a,b)) & ∃u.∃z.(((a,b) = (u,z)) & ((u ϵ x) & (z = ({u} X y)))) ClassElim 151
155. ∃z.(((a,b) = (u,z)) & ((u ϵ x) & (z = ({u} X y)))) Hyp
156. ((a,b) = (u,z)) & ((u ϵ x) & (z = ({u} X y))) Hyp
157. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
164. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 163
165. Set(a) & Set(b) ImpElim 153 164
167. (Set(a) & Set(b)) & ((a,b) = (u,z)) AndInt 165 166

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168. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
174. ((Set(a) & Set(b)) & ((a,b) = (u,z))) -> ((a = u) & (b = z))  ForallElim 173
175. (a = u) & (b = z)  ImpElim 167 174
180. a ϵ x  EqualitySub 178 179
181. a ϵ x  ExistsElim 155 156 180
184. (a ϵ dom(f)) -> (a ϵ x)  ImpInt 183
185. ((a ϵ x) -> (a ϵ dom(f))) & ((a ϵ dom(f)) -> (a ϵ x))  AndInt 145 184
187. ∀a.((a ϵ x) <-> (a ϵ dom(f)))  ForallInt 186
188. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
189. ∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  ForallElim 188
190. (x = dom(f)) <-> ∀z.((z ϵ x) <-> (z ϵ dom(f)))  ForallElim 189
193. x = dom(f)  ImpElim 187 192
194. FUN(f) & (x = dom(f))  AndInt 89 193
195. (f = {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))}) -> (FUN(f) & (x = dom(f)))  ImpInt 194
196. ({a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))} = {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = (
{u} X y))))}) -> (FUN(f) & (x = dom(f)))  EqualitySub 195 0
198. FUN(f) & (x = dom(f))  ImpElim 197 196
201. Set(dom(f))  EqualitySub 200 199
203. FUN(f) & Set(dom(f))  AndInt 202 201
204. (FUN(f) & Set(dom(f))) -> Set(rg(f))  AxInt
205. Set(rg(f))  ImpElim 203 204
207. rg(f) = {x_10: ∃x_11.((x_11,x_10) ϵ {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))})}  EqualitySub 20
208. e ϵ rg(f)  Hyp
209. e ϵ {x_10: ∃x_11.((x_11,x_10) ϵ {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))})}  EqualitySub 208 20
210. Set(e) & ∃x_11.((x_11,e) ϵ {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))})  ClassElim 209
212. (c,e) ϵ {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))}  Hyp
213. Set((c,e)) & ∃u.∃z.(((c,e) = (u,z)) & ((u ϵ x) & (z = ({u} X y))))  ClassElim 212
215. ∃z.(((c,e) = (u,z)) & ((u ϵ x) & (z = ({u} X y))))  Hyp
216. ((c,e) = (u,z)) & ((u ϵ x) & (z = ({u} X y)))  Hyp
217. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
224. Set((c,e)) -> (Set(c) & Set(e))  ForallElim 223
226. Set(c) & Set(e)  ImpElim 225 224
227. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
231. ((Set(c) & Set(e)) & ((c,e) = (u,v))) -> ((c = u) & (e = v))  ForallElim 230
233. (Set(c) & Set(e)) & ((c,e) = (u,z))  AndInt 226 232
235. ((Set(c) & Set(e)) & ((c,e) = (u,z))) -> ((c = u) & (e = z))  ForallElim 234
236. (c = u) & (e = z)  ImpElim 233 235
241. e = ({u} X y)  EqualitySub 238 240
243. (u ϵ x) & (e = ({u} X y))  AndInt 242 241
244. ∃u.((u ϵ x) & (e = ({u} X y)))  ExistsInt 243
246. Set(e) & ∃u.((u ϵ x) & (e = ({u} X y)))  AndInt 245 244
247. e ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}  ClassInt 246
248. e ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}  ExistsElim 215 216 247
251. (e ϵ rg(f)) -> (e ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))})  ImpInt 250
252. e ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}  Hyp
253. Set(e) & ∃u.((u ϵ x) & (e = ({u} X y)))  ClassElim 252
256. (u ϵ x) & (e = ({u} X y))  Hyp
258. ((u,e) = (u,e)) & ((u ϵ x) & (e = ({u} X y)))  AndInt 257 256
260. ∃v.∃b.(((u,e) = (v,b)) & ((v ϵ x) & (b = ({v} X y))))  ExistsInt 259
262. ∃w.(u ϵ w)  ExistsInt 261
263. Set(u)  DefSub 262
264. Set(u) & Set(e)  AndInt 263 254
269. (Set(u) & Set(e)) -> Set((u,e))  ForallElim 268
270. Set((u,e))  ImpElim 264 269
271. Set((u,e)) & ∃v.∃b.(((u,e) = (v,b)) & ((v ϵ x) & (b = ({v} X y))))  AndInt 270 260
272. c = (u,e)  Hyp
274. Set(c) & ∃v.∃b.((c = (v,b)) & ((v ϵ x) & (b = ({v} X y))))  EqualitySub 271 273
275. c ϵ {w: ∃v.∃b.((w = (v,b)) & ((v ϵ x) & (b = ({v} X y))))}  ClassInt 274
276. (u,e) ϵ {w: ∃v.∃b.((w = (v,b)) & ((v ϵ x) & (b = ({v} X y))))}  EqualitySub 275 272
277. (c = (u,e)) -> ((u,e) ϵ {w: ∃v.∃b.((w = (v,b)) & ((v ϵ x) & (b = ({v} X y))))})  ImpInt 276
279. ((u,e) = (u,e)) -> ((u,e) ϵ {w: ∃v.∃b.((w = (v,b)) & ((v ϵ x) & (b = ({v} X y))))})  ForallElim 278
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281. (u,e) ϵ {w: ∃v.∃b.((w = (v,b)) & ((v ϵ x) & (b = ({v} X y))))}  ImpElim 280 279
283. (u,e) ϵ f  EqualitySub 281 282
284. ∃u.((u,e) ϵ f)  ExistsInt 283
285. ∃u.((u,e) ϵ f)  ExistsElim 255 256 284
286. Set(e) & ∃u.((u,e) ϵ f)  AndInt 254 285
287. e ϵ {w: ∃u.((u,w) ϵ f)}  ClassInt 286
290. e ϵ rg(f)  EqualitySub 287 289
291. (e ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}) -> (e ϵ rg(f))  ImpInt 290
292. ((e ϵ rg(f)) -> (e ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))})) & ((e ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}) -> (e ϵ rg(f)))  AndInt 251 291
294. ∀e.((e ϵ rg(f)) <-> (e ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}))  ForallInt 293
295. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
296. ∀y.((rg(f) = y) <-> ∀z.((z ϵ rg(f)) <-> (z ϵ y)))  ForallElim 295
297. (rg(f) = {w: ∃u.((u ϵ x) & (w = ({u} X y)))}) <-> ∀z.((z ϵ rg(f)) <-> (z ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}))  ForallElim 296
300. rg(f) = {w: ∃u.((u ϵ x) & (w = ({u} X y)))}  ImpElim 294 299
301. e ϵ ∪rg(f)  Hyp
302. e ϵ ∪{w: ∃u.((u ϵ x) & (w = ({u} X y)))}  EqualitySub 301 300
305. ∪rg(f) = {z: ∃y.((y ϵ rg(f)) & (z ϵ y))}  ForallElim 304
306. ∪rg(f) = {z: ∃x_13.((x_13 ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}) & (z ϵ x_13))}  EqualitySub 305 300
307. e ϵ {z: ∃x_13.((x_13 ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}) & (z ϵ x_13))}  EqualitySub 301 306
308. Set(e) & ∃x_13.((x_13 ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}) & (e ϵ x_13))  ClassElim 307
310. (x_5 ϵ {w: ∃u.((u ϵ x) & (w = ({u} X y)))}) & (e ϵ x_5)  Hyp
313. Set(x_5) & ∃u.((u ϵ x) & (x_5 = ({u} X y)))  ClassElim 312
316. (u ϵ x) & (x_5 = ({u} X y))  Hyp
318. e ϵ ({u} X y)  EqualitySub 311 317
321. ({u} X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ϵ {u}) & (b ϵ y)))}  ForallElim 320
322. e ϵ {z: ∃a.∃b.((z = (a,b)) & ((a ϵ {u}) & (b ϵ y)))}  EqualitySub 318 321
323. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ϵ {u}) & (b ϵ y)))  ClassElim 322
325. ∃b.((e = (a,b)) & ((a ϵ {u}) & (b ϵ y)))  Hyp
326. (e = (a,b)) & ((a ϵ {u}) & (b ϵ y))  Hyp
329. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
331. ∃w.(u ϵ w)  ExistsInt 330
332. Set(u)  DefSub 331
336. Set(u) -> ((a ϵ {u}) <-> (a = u))  ForallElim 335
337. (a ϵ {u}) <-> (a = u)  ImpElim 332 336
340. a = u  ImpElim 328 339
342. a ϵ x  EqualitySub 330 341
344. (a ϵ x) & (b ϵ y)  AndInt 342 343
346. (e = (a,b)) & ((a ϵ x) & (b ϵ y))  AndInt 345 344
348. ∃a.∃b.((e = (a,b)) & ((a ϵ x) & (b ϵ y)))  ExistsInt 347
350. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ϵ x) & (b ϵ y)))  AndInt 349 348
351. e ϵ {w: ∃a.∃b.((w = (a,b)) & ((a ϵ x) & (b ϵ y)))}  ClassInt 350
354. e ϵ (x X y)  EqualitySub 351 353
355. e ϵ (x X y)  ExistsElim 325 326 354
359. (e ϵ ∪rg(f)) -> (e ϵ (x X y))  ImpInt 358
360. e ϵ (x X y)  Hyp
361. e ϵ {z: ∃a.∃b.((z = (a,b)) & ((a ϵ x) & (b ϵ y)))}  EqualitySub 360 352
362. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ϵ x) & (b ϵ y)))  ClassElim 361
365. ∃b.((e = (a,b)) & ((a ϵ x) & (b ϵ y)))  Hyp
366. (e = (a,b)) & ((a ϵ x) & (b ϵ y))  Hyp
372. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 371
374. Set((a,b))  EqualitySub 363 373
375. Set(a) & Set(b)  ImpElim 374 372
380. Set(a) -> ((a ϵ {a}) <-> (a = a))  ForallElim 379
381. (a ϵ {a}) <-> (a = a)  ImpElim 376 380
385. a ϵ {a}  ImpElim 384 383
390. (a ϵ {a}) & (b ϵ y)  AndInt 385 389
391. (e = (a,b)) & ((a ϵ {a}) & (b ϵ y))  AndInt 386 390
393. ∃v.∃u.((e = (v,u)) & ((v ϵ {a}) & (u ϵ y)))  ExistsInt 392
394. Set(e) & ∃v.∃u.((e = (v,u)) & ((v ϵ {a}) & (u ϵ y)))  AndInt 363 393
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395. e ϵ {w: ∃v.∃u.((w = (v,u)) & ((v ϵ {a}) & (u ϵ y)))} ClassInt 394
397. ({a} X y) = {z: ∃x_15.∃b.((z = (x_15,b)) & ((x_15 ϵ {a}) & (b ϵ y)))} ForallElim 396
399. e ϵ ({a} X y) EqualitySub 395 398
400. g = ({a} X y) Hyp
402. (a ϵ x) & (g = ({a} X y)) AndInt 388 400
403. ∃a.((a ϵ x) & (g = ({a} X y))) ExistsInt 402
404. (Set(u) & Set(y)) -> Set(({u} X y)) TheoremInt
406. (Set(a) & Set(y)) -> Set(({a} X y)) ForallElim 405
408. Set(a) & Set(y) AndInt 376 407
409. Set(({a} X y)) ImpElim 408 406
410. Set(g) EqualitySub 409 401
411. Set(g) & ∃a.((a ϵ x) & (g = ({a} X y))) AndInt 410 403
412. g ϵ {w: ∃a.((a ϵ x) & (w = ({a} X y)))} ClassInt 411
413. e ϵ g EqualitySub 399 401
414. (g ϵ {w: ∃a.((a ϵ x) & (w = ({a} X y)))}) & (e ϵ g) AndInt 412 413
415. ∃g.((g ϵ {w: ∃a.((a ϵ x) & (w = ({a} X y)))}) & (e ϵ g)) ExistsInt 414
416. Set(e) & ∃g.((g ϵ {w: ∃a.((a ϵ x) & (w = ({a} X y)))}) & (e ϵ g)) AndInt 363 415
417. e ϵ {d: ∃g.((g ϵ {w: ∃a.((a ϵ x) & (w = ({a} X y)))}) & (d ϵ g))} ClassInt 416
419. e ϵ ∪rg(f) EqualitySub 417 418
420. (g = ({a} X y)) -> (e ϵ ∪rg(f)) ImpInt 419
422. (({a} X y) = ({a} X y)) -> (e ϵ ∪rg(f)) ForallElim 421
424. e ϵ ∪rg(f) ImpElim 423 422
425. e ϵ ∪rg(f) ExistsElim 365 366 424
427. (e ϵ (x X y)) -> (e ϵ ∪rg(f)) ImpInt 426
428. ((e ϵ ∪rg(f)) -> (e ϵ (x X y))) & ((e ϵ (x X y)) -> (e ϵ ∪rg(f))) AndInt 359 427
430. ∀e.((e ϵ ∪rg(f)) <-> (e ϵ (x X y))) ForallInt 429
431. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
432. ∀y.((∪rg(f) = y) <-> ∀z.((z ϵ ∪rg(f)) <-> (z ϵ y))) ForallElim 431
433. (∪rg(f) = (x X y)) <-> ∀z.((z ϵ ∪rg(f)) <-> (z ϵ (x X y))) ForallElim 432
436. ∪rg(f) = (x X y) ImpElim 430 435
437. Set(x) -> Set(∪x) AxInt
439. Set(rg(f)) -> Set(∪rg(f)) ForallElim 438
440. Set(∪rg(f)) ImpElim 205 439
441. Set((x X y)) EqualitySub 440 436
442. (Set(x) & Set(y)) -> Set((x X y)) ImpInt 441
443. (f = {a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))}) -> ((Set(x) & Set(y)) -> Set((x X y)))
ImpInt 442
445. ({a: ∃u.∃z.((a = (u,z)) & ((u ϵ x) & (z = ({u} X y))))} = {x_16: ∃x_17.∃x_18.((x_16 = (x_17,x_18)) &
((x_17 ϵ x) & (x_18 = ({x_17} X y))))}) -> ((Set(x) & Set(y)) -> Set((x X y))) ForallElim 444
447. (Set(x) & Set(y)) -> Set((x X y)) ImpElim 446 445 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
3. Set(x) -> Set({x})
4. (Set(u) & Set(y)) -> Set(({u} X y))
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
6. Set(x) -> ((y ϵ {x}) <-> (y = x))
7. (Set(u) & Set(y)) -> Set(({u} X y))

Th75. (FUN(f) & Set(dom(f))) -> (f ⊂ (dom(f) X rg(f)))

0. FUN(f) & Set(dom(f)) Hyp
1. z ϵ f Hyp
3. Relation(f) & ∀x.∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z)) DefExp 2
5. ∀z.((z ϵ f) -> ∃x.∃y.(z = (x,y))) DefExp 4
6. (z ϵ f) -> ∃x.∃y.(z = (x,y)) ForallElim 5
7. ∃x.∃y.(z = (x,y)) ImpElim 1 6
8. ∃y.(z = (x,y)) Hyp
9. z = (x,y) Hyp

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13. ∃f.(z ϵ f)  ExistsInt 1
14. Set(z)  DefSub 13
15. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
19. Set((x,y))  EqualitySub 14 9
20. Set(x) & Set(y)  ImpElim 19 18
22. (x,y) ϵ f  EqualitySub 1 9
23. ∃y.((x,y) ϵ f)  ExistsInt 22
24. Set(x) & ∃y.((x,y) ϵ f)  AndInt 21 23
25. x ϵ {w: ∃y.((w,y) ϵ f)}  ClassInt 24
27. x ϵ dom(f)  EqualitySub 25 26
28. ∃x.((x,y) ϵ f)  ExistsInt 22
30. Set(y) & ∃x.((x,y) ϵ f)  AndInt 29 28
31. y ϵ {w: ∃x.((x,w) ϵ f)}  ClassInt 30
33. y ϵ rg(f)  EqualitySub 31 32
34. (x ϵ dom(f)) & (y ϵ rg(f))  AndInt 27 33
35. (z = (x,y)) & ((x ϵ dom(f)) & (y ϵ rg(f)))  AndInt 9 34
37. ∃x.∃y.((z = (x,y)) & ((x ϵ dom(f)) & (y ϵ rg(f))))  ExistsInt 36
42. (dom(f) X rg(f)) = {z: ∃a.∃b.((z = (a,b)) & ((a ϵ dom(f)) & (b ϵ rg(f))))}  ForallElim 41
43. Set(z) & ∃x.∃y.((z = (x,y)) & ((x ϵ dom(f)) & (y ϵ rg(f))))  AndInt 14 37
44. z ϵ {w: ∃x.∃y.((w = (x,y)) & ((x ϵ dom(f)) & (y ϵ rg(f))))}  ClassInt 43
46. z ϵ (dom(f) X rg(f))  EqualitySub 44 45
47. z ϵ (dom(f) X rg(f))  ExistsElim 8 9 46
49. (z ϵ f) -> (z ϵ (dom(f) X rg(f)))  ImpInt 48
50. ∀z.((z ϵ f) -> (z ϵ (dom(f) X rg(f))))  ForallInt 49
51. f ⊂ (dom(f) X rg(f))  DefSub 50
52. (FUN(f) & Set(dom(f))) -> (f ⊂ (dom(f) X rg(f)))  ImpInt 51 Qed
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Used Theorems

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1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
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Th77. (Set(x) & Set(y)) -> Set(func(x,y))

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0. Set(x) & Set(y)  Hyp
1. f ϵ func(x,y)  Hyp
3. f ϵ {f: (FUN(f) & ((dom(f) = x) & (rg(f) = y)))}  EqualitySub 1 2
4. Set(f) & (FUN(f) & ((dom(f) = x) & (rg(f) = y)))  ClassElim 3
9. Relation(f) & ∀x.∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z))  DefExp 7
11. ∀z.((z ϵ f) -> ∃x.∃y.(z = (x,y)))  DefExp 10
12. z ϵ f  Hyp
13. (z ϵ f) -> ∃x.∃y.(z = (x,y))  ForallElim 11
14. ∃x.∃y.(z = (x,y))  ImpElim 12 13
15. ∃y.(z = (a,y))  Hyp
16. z = (a,b)  Hyp
18. (a,b) ϵ f  EqualitySub 12 16
19. ∃w.((a,w) ϵ f)  ExistsInt 18
22. ∃w.((a,b) ϵ w)  ExistsInt 18
23. Set((a,b))  DefSub 22
24. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
31. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 30
32. Set(a) & Set(b)  ImpElim 23 31
34. Set(a) & ∃w.((a,w) ϵ f)  AndInt 33 19
35. a ϵ {w: ∃x_5.((w,x_5) ϵ f)}  ClassInt 34
37. a ϵ dom(f)  EqualitySub 35 36
39. a ϵ x  EqualitySub 37 38
40. ∃w.((w,b) ϵ f)  ExistsInt 18
42. Set(b) & ∃w.((w,b) ϵ f)  AndInt 41 40
43. b ϵ {w: ∃x_8.((x_8,w) ϵ f)}  ClassInt 42
45. b ϵ rg(f)  EqualitySub 43 44
47. b ϵ y  EqualitySub 45 46
48. (a ϵ x) & (b ϵ y)  AndInt 39 47
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49. (z = (a,b)) & ((a ϵ x) & (b ϵ y)) AndInt 16 48
51. Set(z) EqualitySub 23 50
53. ∃a.∃b.((z = (a,b)) & ((a ϵ x) & (b ϵ y))) ExistsInt 52
54. Set(z) & ∃a.∃b.((z = (a,b)) & ((a ϵ x) & (b ϵ y))) AndInt 51 53
55. z ϵ {w: ∃a.∃b.((w = (a,b)) & ((a ϵ x) & (b ϵ y)))} ClassInt 54
57. z ϵ (x X y) EqualitySub 55 56
58. z ϵ (x X y) ExistsElim 15 16 57
60. (z ϵ f) -> (z ϵ (x X y)) ImpInt 59
61. ∀z.((z ϵ f) -> (z ϵ (x X y))) ForallInt 60
62. f ⊂ (x X y) DefSub 61
63. (Set(x) & Set(y)) -> Set((x X y)) TheoremInt
64. Set((x X y)) ImpElim 0 63
65. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ϵ Px))) TheoremInt
66. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
72. (Set((x X y)) & (f ⊂ (x X y))) -> Set(f) ForallElim 71
73. Set((x X y)) & (f ⊂ (x X y)) AndInt 64 62
74. Set(f) ImpElim 73 72
78. Set((x X y)) -> (Set(P(x X y)) & ((f ⊂ (x X y)) <-> (f ϵ P(x X y)))) ForallElim 77
79. Set(P(x X y)) & ((f ⊂ (x X y)) <-> (f ϵ P(x X y))) ImpElim 64 78
84. f ϵ P(x X y) ImpElim 62 83
85. (f ϵ func(x,y)) -> (f ϵ P(x X y)) ImpInt 84
86. ∀f.((f ϵ func(x,y)) -> (f ϵ P(x X y))) ForallInt 85
87. func(x,y) ⊂ P(x X y) DefSub 86
88. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
94. (Set(P(x X y)) & (func(x,y) ⊂ P(x X y))) -> Set(func(x,y)) ForallElim 93
95. Set(P(x X y)) & (func(x,y) ⊂ P(x X y)) AndInt 80 87
96. Set(func(x,y)) ImpElim 95 94
97. (Set(x) & Set(y)) -> Set(func(x,y)) ImpInt 96 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. (Set(x) & Set(y)) -> Set((x X y))
3. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ϵ Px)))
4. (Set(x) & (y ⊂ x)) -> Set(y)

Th88. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))

0. WO(r,x) Hyp
1. (u ϵ x) & ((v ϵ x) & (w ϵ x)) Hyp
2. ((u,v) ϵ r) & ((v,w) ϵ r) Hyp
3. z ϵ {u,v} Hyp
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
11. (Set(c) & Set(d)) -> (Set({c,d}) & ((e ϵ {c,d}) <-> ((e = c) v (e = d)))) ForallElim 10
15. ∃x.(u ϵ x) ExistsInt 12
16. Set(u) DefSub 15
17. ∃x.(v ϵ x) ExistsInt 14
18. Set(v) DefSub 17
22. (Set(u) & Set(v)) -> (Set({u,v}) & ((e ϵ {u,v}) <-> ((e = u) v (e = v)))) ForallElim 21
23. Set(u) & Set(v) AndInt 16 18
24. Set({u,v}) & ((e ϵ {u,v}) <-> ((e = u) v (e = v))) ImpElim 23 22
27. (z ϵ {u,v}) <-> ((z = u) v (z = v)) ForallElim 26
30. (z = u) v (z = v) ImpElim 3 29
31. z = u Hyp
34. z ϵ x EqualitySub 32 33
35. z = v Hyp
39. z ϵ x EqualitySub 37 38
40. z ϵ x OrElim 30 31 34 35 39
41. (z ϵ {u,v}) -> (z ϵ x) ImpInt 40
42. ∀z.((z ϵ {u,v}) -> (z ϵ x)) ForallInt 41

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43. {u,v} ⊂ x  DefSub 42
44. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  DefExp 0
46. (({u,v} ⊂ x) & ¬({u,v} = 0)) -> ∃z.First(r,{u,v},z)  ForallElim 45
48. (u = u) v (v = v)  OrIntR 47
52. ((u = u) v (u = v)) -> (u ϵ {u,v})  ForallElim 51
53. (u = u) v (u = v)  OrIntR 47
54. u ϵ {u,v}  ImpElim 53 52
55. {u,v} = 0  Hyp
56. u ϵ 0  EqualitySub 54 55
57. ¬(x ϵ 0)  TheoremInt
59. ¬(u ϵ 0)  ForallElim 58
60. _|_  ImpElim 56 59
61. ¬({u,v} = 0)  ImpInt 60
62. ({u,v} ⊂ x) & ¬({u,v} = 0)  AndInt 43 61
63. ∃z.First(r,{u,v},z)  ImpElim 62 46
64. First(r,{u,v},f)  Hyp
65. (f ϵ {u,v}) & ∀y.((y ϵ {u,v}) -> ¬((y,f) ϵ r))  DefExp 64
70. (f ϵ {u,v}) -> ((f = u) v (f = v))  ForallElim 69
71. (f = u) v (f = v)  ImpElim 66 70
73. (u ϵ {u,v}) -> ¬((u,f) ϵ r)  ForallElim 72
74. (v ϵ {u,v}) -> ¬((v,f) ϵ r)  ForallElim 72
75. f = u  Hyp
77. ((v = u) v (v = v)) -> (v ϵ {u,v})  ForallElim 76
79. (v = u) v (v = v)  OrIntL 78
80. v ϵ {u,v}  ImpElim 79 77
81. ¬((v,f) ϵ r)  ImpElim 80 74
82. ¬((v,u) ϵ r)  EqualitySub 81 75
83. ¬((v,u) ϵ r) v ¬((u,v) ϵ r)  OrIntR 82
84. f = v  Hyp
86. ((u = u) v (u = v)) -> (u ϵ {u,v})  ForallElim 85
88. (u = u) v (u = v)  OrIntR 87
89. u ϵ {u,v}  ImpElim 88 86
90. (u ϵ {u,v}) -> ¬((u,f) ϵ r)  ForallElim 72
91. ¬((u,f) ϵ r)  ImpElim 89 90
92. ¬((u,v) ϵ r)  EqualitySub 91 84
93. ¬((v,u) ϵ r) v ¬((u,v) ϵ r)  OrIntL 92
94. ¬((v,u) ϵ r) v ¬((u,v) ϵ r)  OrElim 71 75 83 84 93
95. ¬((v,u) ϵ r) v ¬((u,v) ϵ r)  ExistsElim 63 64 94
96. (B v ¬A) -> (A -> B)  TheoremInt
97. (¬((v,u) ϵ r) v ¬A) -> (A -> ¬((v,u) ϵ r))  PolySub 96
98. (¬((v,u) ϵ r) v ¬((u,v) ϵ r)) -> (((u,v) ϵ r) -> ¬((v,u) ϵ r))  PolySub 97
99. ((u,v) ϵ r) -> ¬((v,u) ϵ r)  ImpElim 95 98
100. ((u ϵ x) & ((v ϵ x) & (w ϵ x))) -> (((u,v) ϵ r) -> ¬((v,u) ϵ r))  ImpInt 99
102. ((u ϵ x) & ((v ϵ x) & (v ϵ x))) -> (((u,v) ϵ r) -> ¬((v,u) ϵ r))  ForallElim 101
103. (u ϵ x) & (v ϵ x)  Hyp
104. (u,v) ϵ r  Hyp
107. (v ϵ x) & (v ϵ x)  AndInt 106 106
108. (u ϵ x) & ((v ϵ x) & (v ϵ x))  AndInt 105 107
109. ((u,v) ϵ r) -> ¬((v,u) ϵ r)  ImpElim 108 102
110. ¬((v,u) ϵ r)  ImpElim 104 109
111. ((u,v) ϵ r) -> ¬((v,u) ϵ r)  ImpInt 110
112. ((u ϵ x) & (v ϵ x)) -> (((u,v) ϵ r) -> ¬((v,u) ϵ r))  ImpInt 111
113. ∀z.(((u ϵ x) & (z ϵ x)) -> (((u,z) ϵ r) -> ¬((z,u) ϵ r)))  ForallInt 112
114. ∀y.∀z.(((y ϵ x) & (z ϵ x)) -> (((y,z) ϵ r) -> ¬((z,y) ϵ r)))  ForallInt 113
115. Asymmetric(r,x)  DefSub 114
116. ¬TransIn(r,x)  Hyp
117. ¬∀u.∀v.∀w.(((u ϵ x) & ((v ϵ x) & (w ϵ x))) -> ((((u,v) ϵ r) & ((v,w) ϵ r)) -> ((u,w) ϵ r)))  DefExp 116
118. ¬∀i.P(i) -> ∃c.¬P(c)  TheoremInt
119. ¬∀i.∀v.∀w.(((i ϵ x) & ((v ϵ x) & (w ϵ x))) -> ((((i,v) ϵ r) & ((v,w) ϵ r)) -> ((i,w) ϵ r))) -> ∃c.¬∀
v.∀w.(((c ϵ x) & ((v ϵ x) & (w ϵ x))) -> ((((c,v) ϵ r) & ((v,w) ϵ r)) -> ((c,w) ϵ r)))  PredSub 118
120. ∃c.¬∀v.∀w.(((c ϵ x) & ((v ϵ x) & (w ϵ x))) -> ((((c,v) ϵ r) & ((v,w) ϵ r)) -> ((c,w) ϵ r)))  ImpElim 117 119
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121. ¬∀v.∀w.(((k ϵ x) & ((v ϵ x) & (w ϵ x))) -> ((((k,v) ϵ r) & ((v,w) ϵ r)) -> ((k,w) ϵ r)))  Hyp
122. ¬∀i.∀w.(((k ϵ x) & ((i ϵ x) & (w ϵ x))) -> ((((k,i) ϵ r) & ((i,w) ϵ r)) -> ((k,w) ϵ r))) -> ∃c.¬∀w.(
((k ϵ x) & ((c ϵ x) & (w ϵ x))) -> ((((k,c) ϵ r) & ((c,w) ϵ r)) -> ((k,w) ϵ r)))  PredSub 118
123. ∃c.¬∀w.(((k ϵ x) & ((c ϵ x) & (w ϵ x))) -> ((((k,c) ϵ r) & ((c,w) ϵ r)) -> ((k,w) ϵ r)))  ImpElim 121 122
124. ¬∀w.(((k ϵ x) & ((p ϵ x) & (w ϵ x))) -> ((((k,p) ϵ r) & ((p,w) ϵ r)) -> ((k,w) ϵ r)))  Hyp
125. ¬∀i.(((k ϵ x) & ((p ϵ x) & (i ϵ x))) -> ((((k,p) ϵ r) & ((p,i) ϵ r)) -> ((k,i) ϵ r))) -> ∃c.¬(((k ϵ
x) & ((p ϵ x) & (c ϵ x))) -> ((((k,p) ϵ r) & ((p,c) ϵ r)) -> ((k,c) ϵ r)))  PredSub 118
126. ∃c.¬(((k ϵ x) & ((p ϵ x) & (c ϵ x))) -> ((((k,p) ϵ r) & ((p,c) ϵ r)) -> ((k,c) ϵ r)))  ImpElim 124 125
127. ¬(((k ϵ x) & ((p ϵ x) & (q ϵ x))) -> ((((k,p) ϵ r) & ((p,q) ϵ r)) -> ((k,q) ϵ r)))  Hyp
128. (A -> B) -> (¬B -> ¬A)  TheoremInt
129. (A -> C) -> (¬C -> ¬A)  PolySub 128
130. ((B v ¬A) -> C) -> (¬C -> ¬(B v ¬A))  PolySub 129
131. ((B v ¬A) -> (A -> B)) -> (¬(A -> B) -> ¬(B v ¬A))  PolySub 130
132. (B v ¬A) -> (A -> B)  TheoremInt
133. ¬(A -> B) -> ¬(B v ¬A)  ImpElim 132 131
134. ¬(((k ϵ x) & ((p ϵ x) & (q ϵ x))) -> B) -> ¬(B v ¬((k ϵ x) & ((p ϵ x) & (q ϵ x))))  PolySub 133
135. ¬(((k ϵ x) & ((p ϵ x) & (q ϵ x))) -> ((((k,p) ϵ r) & ((p,q) ϵ r)) -> ((k,q) ϵ r))) -> ¬(((((k,p) ϵ r
) & ((p,q) ϵ r)) -> ((k,q) ϵ r)) v ¬((k ϵ x) & ((p ϵ x) & (q ϵ x))))  PolySub 134
136. ¬(((((k,p) ϵ r) & ((p,q) ϵ r)) -> ((k,q) ϵ r)) v ¬((k ϵ x) & ((p ϵ x) & (q ϵ x))))  ImpElim 127 135
137. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
139. ¬(A v C) <-> (¬A & ¬C)  PolySub 138
140. ¬(B v C) <-> (¬B & ¬C)  PolySub 139
141. ¬(B v ¬A) <-> (¬B & ¬¬A)  PolySub 140
144. D <-> ¬¬D  TheoremInt
147. ¬¬A -> A  PolySub 146
148. ¬(B v ¬A)  Hyp
149. ¬B & ¬¬A  ImpElim 148 143
152. A  ImpElim 151 147
153. ¬B & A  AndInt 150 152
154. ¬(B v ¬A) -> (¬B & A)  ImpInt 153
155. ¬(A -> B)  Hyp
156. ¬(B v ¬A)  ImpElim 155 133
157. ¬B & A  ImpElim 156 154
158. ¬(A -> B) -> (¬B & A)  ImpInt 157
159. ¬(((k ϵ x) & ((p ϵ x) & (q ϵ x))) -> B) -> (¬B & ((k ϵ x) & ((p ϵ x) & (q ϵ x))))  PolySub 158
160. ¬(((k ϵ x) & ((p ϵ x) & (q ϵ x))) -> ((((k,p) ϵ r) & ((p,q) ϵ r)) -> ((k,q) ϵ r))) -> (¬((((k,p) ϵ r
) & ((p,q) ϵ r)) -> ((k,q) ϵ r)) & ((k ϵ x) & ((p ϵ x) & (q ϵ x))))  PolySub 159
161. ¬((((k,p) ϵ r) & ((p,q) ϵ r)) -> ((k,q) ϵ r)) & ((k ϵ x) & ((p ϵ x) & (q ϵ x)))  ImpElim 127 160
164. ¬((((k,p) ϵ r) & ((p,q) ϵ r)) -> B) -> (¬B & (((k,p) ϵ r) & ((p,q) ϵ r)))  PolySub 158
165. ¬((((k,p) ϵ r) & ((p,q) ϵ r)) -> ((k,q) ϵ r)) -> (¬((k,q) ϵ r) & (((k,p) ϵ r) & ((p,q) ϵ r)))  PolySub 164
166. ¬((k,q) ϵ r) & (((k,p) ϵ r) & ((p,q) ϵ r))  ImpElim 162 165
172. ∀y.∀z.(((y ϵ x) & (z ϵ x)) -> ((y = z) v (((y,z) ϵ r) v ((z,y) ϵ r))))  DefExp 171
173. ∀z.(((k ϵ x) & (z ϵ x)) -> ((k = z) v (((k,z) ϵ r) v ((z,k) ϵ r))))  ForallElim 172
174. ((k ϵ x) & (q ϵ x)) -> ((k = q) v (((k,q) ϵ r) v ((q,k) ϵ r)))  ForallElim 173
175. (k ϵ x) & (q ϵ x)  AndInt 168 170
176. (k = q) v (((k,q) ϵ r) v ((q,k) ϵ r))  ImpElim 175 174
177. k = q  Hyp
179. ((q,p) ϵ r) & ((p,q) ϵ r)  EqualitySub 178 177
180. ∀z.(((q ϵ x) & (z ϵ x)) -> (((q,z) ϵ r) -> ¬((z,q) ϵ r)))  ForallElim 114
181. ((q ϵ x) & (p ϵ x)) -> (((q,p) ϵ r) -> ¬((p,q) ϵ r))  ForallElim 180
183. (q ϵ x) & (p ϵ x)  AndInt 170 182
184. ((q,p) ϵ r) -> ¬((p,q) ϵ r)  ImpElim 183 181
186. ¬((p,q) ϵ r)  ImpElim 185 184
188. _|_  ImpElim 187 186
189. (q,k) ϵ r  AbsI 188
190. ((k,q) ϵ r) v ((q,k) ϵ r)  Hyp
191. (k,q) ϵ r  Hyp
192. _|_  ImpElim 191 167
193. (q,k) ϵ r  AbsI 192
194. (q,k) ϵ r  Hyp
195. (q,k) ϵ r  OrElim 190 191 193 194 194
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197. ((q,k) ϵ r) & (((k,p) ϵ r) & ((p,q) ϵ r))  AndInt 196 178
198. cyc = {p,{q,k}}  Hyp
199. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))  TheoremInt
201. ∃w.(k ϵ w)  ExistsInt 200
202. Set(k)  DefSub 201
205. ∃w.(q ϵ w)  ExistsInt 204
206. Set(q)  DefSub 205
208. ∃w.(p ϵ w)  ExistsInt 207
209. Set(p)  DefSub 208
210. triad = ({p} ∪ ({q} ∪ {k}))  Hyp
211. z ϵ triad  Hyp
212. Set(x) -> Set({x})  TheoremInt
213. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
220. (z ϵ ({p} ∪ ({q} ∪ {k}))) -> ((z ϵ {p}) v (z ϵ ({q} ∪ {k})))  ForallElim 219
221. z ϵ ({p} ∪ ({q} ∪ {k}))  EqualitySub 211 210
222. (z ϵ {p}) v (z ϵ ({q} ∪ {k}))  ImpElim 221 220
223. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
224. z ϵ {p}  Hyp
226. Set(p) -> ((y ϵ {p}) <-> (y = p))  ForallElim 225
227. (y ϵ {p}) <-> (y = p)  ImpElim 209 226
231. (z ϵ {p}) -> (z = p)  ForallElim 230
232. z = p  ImpElim 224 231
234. z ϵ x  EqualitySub 207 233
235. z ϵ ({q} ∪ {k})  Hyp
239. (z ϵ ({q} ∪ {k})) -> ((z ϵ {q}) v (z ϵ {k}))  ForallElim 238
240. (z ϵ {q}) v (z ϵ {k})  ImpElim 235 239
241. z ϵ {q}  Hyp
243. Set(q) -> ((y ϵ {q}) <-> (y = q))  ForallElim 242
244. (y ϵ {q}) <-> (y = q)  ImpElim 206 243
248. (z ϵ {q}) -> (z = q)  ForallElim 247
249. z = q  ImpElim 241 248
251. z ϵ x  EqualitySub 204 250
252. z ϵ {k}  Hyp
254. Set(k) -> ((y ϵ {k}) <-> (y = k))  ForallElim 253
255. (y ϵ {k}) <-> (y = k)  ImpElim 202 254
259. (z ϵ {k}) -> (z = k)  ForallElim 258
260. z = k  ImpElim 252 259
262. z ϵ x  EqualitySub 200 261
263. z ϵ x  OrElim 240 241 251 252 262
265. (z ϵ triad) -> (z ϵ x)  ImpInt 264
266. ∀z.((z ϵ triad) -> (z ϵ x))  ForallInt 265
267. triad ⊂ x  DefSub 266
270. (p ϵ {p}) <-> (p = p)  ForallElim 269
274. p ϵ {p}  ImpElim 273 272
275. (p ϵ {p}) v (p ϵ ({q} ∪ {k}))  OrIntR 274
283. ((p ϵ {p}) v (p ϵ ({q} ∪ {k}))) -> (p ϵ ({p} ∪ ({q} ∪ {k})))  ForallElim 282
284. p ϵ ({p} ∪ ({q} ∪ {k}))  ImpElim 275 283
286. p ϵ triad  EqualitySub 284 285
287. ¬(x ϵ 0)  TheoremInt
288. triad = 0  Hyp
290. p ϵ 0  EqualitySub 286 288
292. ¬(p ϵ 0)  ForallElim 291
293. _|_  ImpElim 290 292
294. ¬(triad = 0)  ImpInt 293
295. (triad ⊂ x) & ¬(triad = 0)  AndInt 267 294
296. ∃z.First(r,triad,z)  ImpElim 295 268
297. First(r,triad,l)  Hyp
298. (l ϵ triad) & ∀y.((y ϵ triad) -> ¬((y,l) ϵ r))  DefExp 297
300. l ϵ ({p} ∪ ({q} ∪ {k}))  EqualitySub 299 210
302. (l ϵ ({p} ∪ ({q} ∪ {k}))) -> ((l ϵ {p}) v (l ϵ ({q} ∪ {k})))  ForallElim 301
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303. (l ϵ {p}) v (l ϵ ({q} ∪ {k}))  ImpElim 300 302
304. l ϵ {p}  Hyp
306. (l ϵ {p}) -> (l = p)  ForallElim 305
307. l = p  ImpElim 304 306
308. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
310. Set(k) -> ((y ϵ {k}) <-> (y = k))  ForallElim 309
312. (y ϵ {k}) <-> (y = k)  ImpElim 202 310
314. (k ϵ {k}) <-> (k = k)  ForallElim 313
317. k ϵ {k}  ImpElim 311 316
318. (k ϵ {q}) v (k ϵ {k})  OrIntL 317
324. ((k ϵ {q}) v (k ϵ {k})) -> (k ϵ ({q} ∪ {k}))  ForallElim 323
325. k ϵ ({q} ∪ {k})  ImpElim 318 324
326. (k ϵ {p}) v (k ϵ ({q} ∪ {k}))  OrIntL 325
332. ((k ϵ {p}) v (k ϵ ({q} ∪ {k}))) -> (k ϵ ({p} ∪ ({q} ∪ {k})))  ForallElim 331
333. k ϵ ({p} ∪ ({q} ∪ {k}))  ImpElim 326 332
335. k ϵ triad  EqualitySub 333 334
337. ∀y.((y ϵ triad) -> ¬((y,p) ϵ r))  EqualitySub 336 307
338. (k ϵ triad) -> ¬((k,p) ϵ r)  ForallElim 337
339. ¬((k,p) ϵ r)  ImpElim 335 338
342. _|_  ImpElim 341 339
343. l ϵ ({q} ∪ {k})  Hyp
350. (l ϵ ({q} ∪ {k})) -> ((l ϵ {q}) v (l ϵ {k}))  ForallElim 349
351. (l ϵ {q}) v (l ϵ {k})  ImpElim 343 350
352. l ϵ {q}  Hyp
356. Set(q) -> ((l ϵ {q}) <-> (l = q))  ForallElim 355
357. (l ϵ {q}) <-> (l = q)  ImpElim 206 356
360. l = q  ImpElim 352 359
362. ∀y.((y ϵ triad) -> ¬((y,q) ϵ r))  EqualitySub 361 360
363. (p ϵ triad) -> ¬((p,q) ϵ r)  ForallElim 362
364. ¬((p,q) ϵ r)  ImpElim 286 363
366. _|_  ImpElim 365 364
367. l ϵ {k}  Hyp
369. Set(k) -> ((y ϵ {k}) <-> (y = k))  ForallElim 368
370. (y ϵ {k}) <-> (y = k)  ImpElim 202 369
372. (l ϵ {k}) <-> (l = k)  ForallElim 371
375. l = k  ImpElim 367 374
376. ∀y.((y ϵ triad) -> ¬((y,k) ϵ r))  EqualitySub 361 375
379. Set(q) -> ((y ϵ {q}) <-> (y = q))  ForallElim 378
380. (y ϵ {q}) <-> (y = q)  ImpElim 206 379
382. (q ϵ {q}) <-> (q = q)  ForallElim 381
386. q ϵ {q}  ImpElim 383 385
387. (q ϵ {q}) v (q ϵ {k})  OrIntR 386
393. ((q ϵ {q}) v (q ϵ {k})) -> (q ϵ ({q} ∪ {k}))  ForallElim 392
394. q ϵ ({q} ∪ {k})  ImpElim 387 393
395. (q ϵ {p}) v (q ϵ ({q} ∪ {k}))  OrIntL 394
401. ((q ϵ {p}) v (q ϵ ({q} ∪ {k}))) -> (q ϵ ({p} ∪ ({q} ∪ {k})))  ForallElim 400
402. q ϵ ({p} ∪ ({q} ∪ {k}))  ImpElim 395 401
404. q ϵ triad  EqualitySub 402 403
405. ∀y.((y ϵ triad) -> ¬((y,k) ϵ r))  EqualitySub 361 375
406. (q ϵ triad) -> ¬((q,k) ϵ r)  ForallElim 405
407. ¬((q,k) ϵ r)  ImpElim 404 406
409. _|_  ImpElim 408 407
410. _|_  OrElim 351 352 366 367 409
412. _|_  ExistsElim 296 297 411
413. ¬(triad = ({p} ∪ ({q} ∪ {k})))  ImpInt 412
415. ¬(({p} ∪ ({q} ∪ {k})) = ({p} ∪ ({q} ∪ {k})))  ForallElim 414
417. _|_  ImpElim 416 415
418. _|_  ExistsElim 126 127 417
421. ¬¬TransIn(r,x)  ImpInt 420
422. D <-> ¬¬D  TheoremInt
425. ¬¬TransIn(r,x) -> TransIn(r,x)  PolySub 424
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426. TransIn(r,x)  ImpElim 421 425
427. Asymmetric(r,x) & TransIn(r,x)  AndInt 115 426
428. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  ImpInt 427 Qed
```

Used Theorems

```
1. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))
2. ¬(x ϵ 0)
3. (B v ¬A) -> (A -> B)
5. ¬∀i.P(i) -> ∃c.¬P(c)
7. (A -> B) -> (¬B -> ¬A)
6. (B v ¬A) -> (A -> B)
8. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
9. D <-> ¬¬D
10. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ϵ {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set
(x) v ¬Set(y)))
11. Set(x) -> Set({x})
12. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
13. Set(x) -> ((y ϵ {x}) <-> (y = x))
14. ¬(x ϵ 0)
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Th90. (¬(n = 0) & ∀y.((y ϵ n) -> Sec(r,x,y))) -> (Sec(r,x,∪n) & Sec(r,x,∩n))
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0. ¬(n = 0) & ∀y.((y ϵ n) -> Sec(r,x,y))  Hyp
1. z ϵ ∪n  Hyp
4. ∪n = {z: ∃y.((y ϵ n) & (z ϵ y))}  ForallElim 3
5. z ϵ {z: ∃y.((y ϵ n) & (z ϵ y))}  EqualitySub 1 4
6. Set(z) & ∃y.((y ϵ n) & (z ϵ y))  ClassElim 5
9. (m ϵ n) & (z ϵ m)  Hyp
10. (m ϵ n) -> Sec(r,x,m)  ForallElim 7
12. Sec(r,x,m)  ImpElim 11 10
13. ((m ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)) -> (u ϵ m))  DefExp 12
16. ∀z.((z ϵ m) -> (z ϵ x))  DefExp 15
17. (z ϵ m) -> (z ϵ x)  ForallElim 16
19. z ϵ x  ImpElim 18 17
20. z ϵ x  ExistsElim 8 9 19
21. (z ϵ ∪n) -> (z ϵ x)  ImpInt 20
22. ∀z.((z ϵ ∪n) -> (z ϵ x))  ForallInt 21
23. ∪n ⊂ x  DefSub 22
25. (u ϵ x) & ((v ϵ ∪n) & ((u,v) ϵ r))  Hyp
28. v ϵ {z: ∃y.((y ϵ n) & (z ϵ y))}  EqualitySub 27 4
29. Set(v) & ∃y.((y ϵ n) & (v ϵ y))  ClassElim 28
31. (m ϵ n) & (v ϵ m)  Hyp
33. (m ϵ n) -> Sec(r,x,m)  ForallElim 32
35. Sec(r,x,m)  ImpElim 34 33
36. ((m ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)) -> (u ϵ m))  DefExp 35
38. ∀v.((((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)) -> (u ϵ m))  ForallElim 37
39. (((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)) -> (u ϵ m)  ForallElim 38
44. (u ϵ x) & (v ϵ m)  AndInt 42 43
45. ((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)  AndInt 44 41
46. u ϵ m  ImpElim 45 39
47. (m ϵ n) & (u ϵ m)  AndInt 34 46
49. ∃w.(u ϵ w)  ExistsInt 46
50. Set(u)  DefSub 49
51. Set(u) & ∃m.((m ϵ n) & (u ϵ m))  AndInt 50 48
52. u ϵ {u: ∃m.((m ϵ n) & (u ϵ m))}  ClassInt 51
54. u ϵ ∪n  EqualitySub 52 53
55. u ϵ ∪n  ExistsElim 30 31 54
56. ((u ϵ x) & ((v ϵ ∪n) & ((u,v) ϵ r))) -> (u ϵ ∪n)  ImpInt 55
57. ((u ϵ x) & (v ϵ ∪n)) & ((u,v) ϵ r)  Hyp
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62. (v ϵ ∪n) & ((u,v) ϵ r)  AndInt 61 59
63. (u ϵ x) & ((v ϵ ∪n) & ((u,v) ϵ r))  AndInt 60 62
64. u ϵ ∪n  ImpElim 63 56
65. (((u ϵ x) & (v ϵ ∪n)) & ((u,v) ϵ r)) -> (u ϵ ∪n)  ImpInt 64
66. ∀v.((((u ϵ x) & (v ϵ ∪n)) & ((u,v) ϵ r)) -> (u ϵ ∪n))  ForallInt 65
67. ∀u.∀v.((((u ϵ x) & (v ϵ ∪n)) & ((u,v) ϵ r)) -> (u ϵ ∪n))  ForallInt 66
68. ∃w.(w ϵ n)  Hyp
69. a ϵ n  Hyp
71. (a ϵ n) -> Sec(r,x,a)  ForallElim 70
72. Sec(r,x,a)  ImpElim 69 71
73. ((a ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ a)) & ((u,v) ϵ r)) -> (u ϵ a))  DefExp 72
76. WO(r,x)  ExistsElim 68 69 75
77. ∃w.(w ϵ n) -> WO(r,x)  ImpInt 76
79. ¬∃i.P(i) -> ∀j.¬P(j)  TheoremInt
80. ¬∃w.(w ϵ n)  Hyp
81. ¬∃i.(i ϵ n) -> ∀j.¬(j ϵ n)  PredSub 79
82. ∀j.¬(j ϵ n)  ImpElim 80 81
83. b ϵ n  Hyp
84. ¬(b ϵ n)  ForallElim 82
85. _|_  ImpElim 83 84
86. b ϵ 0  AbsI 85
87. (b ϵ n) -> (b ϵ 0)  ImpInt 86
88. b ϵ 0  Hyp
90. b ϵ {x: ¬(x = x)}  EqualitySub 88 89
91. Set(b) & ¬(b = b)  ClassElim 90
94. _|_  ImpElim 93 92
95. b ϵ n  AbsI 94
96. (b ϵ 0) -> (b ϵ n)  ImpInt 95
97. ((b ϵ n) -> (b ϵ 0)) & ((b ϵ 0) -> (b ϵ n))  AndInt 87 96
99. ∀b.((b ϵ n) <-> (b ϵ 0))  ForallInt 98
100. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
101. ∀y.((n = y) <-> ∀z.((z ϵ n) <-> (z ϵ y)))  ForallElim 100
102. (n = 0) <-> ∀z.((z ϵ n) <-> (z ϵ 0))  ForallElim 101
105. n = 0  ImpElim 99 104
106. _|_  ImpElim 105 78
107. ¬¬∃w.(w ϵ n)  ImpInt 106
108. D <-> ¬¬D  TheoremInt
111. ¬¬∃w.(w ϵ n) -> ∃w.(w ϵ n)  PolySub 110
112. ∃w.(w ϵ n)  ImpElim 107 111
113. WO(r,x)  ImpElim 112 77
114. (∪n ⊂ x) & WO(r,x)  AndInt 23 113
115. ((∪n ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ ∪n)) & ((u,v) ϵ r)) -> (u ϵ ∪n))  AndInt 114 67
116. Sec(r,x,∪n)  DefSub 115
117. z ϵ ∩n  Hyp
120. ∩n = {z: ∀y.((y ϵ n) -> (z ϵ y))}  ForallElim 119
121. z ϵ {z: ∀y.((y ϵ n) -> (z ϵ y))}  EqualitySub 117 120
122. Set(z) & ∀y.((y ϵ n) -> (z ϵ y))  ClassElim 121
124. m ϵ n  Hyp
125. (m ϵ n) -> (z ϵ m)  ForallElim 123
126. z ϵ m  ImpElim 124 125
127. (m ϵ n) -> Sec(r,x,m)  ForallElim 7
128. Sec(r,x,m)  ImpElim 124 127
129. ((m ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)) -> (u ϵ m))  DefExp 128
132. ∀z.((z ϵ m) -> (z ϵ x))  DefExp 131
133. (z ϵ m) -> (z ϵ x)  ForallElim 132
134. z ϵ x  ImpElim 126 133
135. (z ϵ ∩n) -> (z ϵ x)  ImpInt 134
136. (z ϵ ∩n) -> (z ϵ x)  ExistsElim 112 124 135
137. ∀z.((z ϵ ∩n) -> (z ϵ x))  ForallInt 136
138. ∩n ⊂ x  DefSub 137
139. (∩n ⊂ x) & WO(r,x)  AndInt 138 113
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140. ((u ϵ x) & (v ϵ ∩n)) & ((u,v) ϵ r)  Hyp
143. v ϵ {z: ∀y.((y ϵ n) -> (z ϵ y))}  EqualitySub 142 120
144. Set(v) & ∀y.((y ϵ n) -> (v ϵ y))  ClassElim 143
146. (m ϵ n) -> (v ϵ m)  ForallElim 145
147. v ϵ m  ImpElim 124 146
149. ∀v.((((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)) -> (u ϵ m))  ForallElim 148
150. (((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)) -> (u ϵ m)  ForallElim 149
154. (u ϵ x) & (v ϵ m)  AndInt 153 147
155. ((u ϵ x) & (v ϵ m)) & ((u,v) ϵ r)  AndInt 154 151
156. u ϵ m  ImpElim 155 150
157. (m ϵ n) -> (u ϵ m)  ImpInt 156
158. ∀m.((m ϵ n) -> (u ϵ m))  ForallInt 157
159. ∃w.(u ϵ w)  ExistsInt 153
160. Set(u)  DefSub 159
161. Set(u) & ∀m.((m ϵ n) -> (u ϵ m))  AndInt 160 158
162. u ϵ {w: ∀m.((m ϵ n) -> (w ϵ m))}  ClassInt 161
164. u ϵ ∩n  EqualitySub 162 163
165. (((u ϵ x) & (v ϵ ∩n)) & ((u,v) ϵ r)) -> (u ϵ ∩n)  ImpInt 164
166. ∀v.((((u ϵ x) & (v ϵ ∩n)) & ((u,v) ϵ r)) -> (u ϵ ∩n))  ForallInt 165
167. ∀u.∀v.((((u ϵ x) & (v ϵ ∩n)) & ((u,v) ϵ r)) -> (u ϵ ∩n))  ForallInt 166
168. ((∩n ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ ∩n)) & ((u,v) ϵ r)) -> (u ϵ ∩n))  AndInt 139 167
169. Sec(r,x,∩n)  DefSub 168
170. Sec(r,x,∪n) & Sec(r,x,∩n)  AndInt 116 169
171. (¬(n = 0) & ∀y.((y ϵ n) -> Sec(r,x,y))) -> (Sec(r,x,∪n) & Sec(r,x,∩n))  ImpInt 170 Qed

Used Theorems

2. ¬∃i.P(i) -> ∀j.¬P(j)
3. D <-> ¬¬D

Th91. (Sec(r,x,y) & ¬(y = x)) -> ∃v.((v ϵ x) & (y = {u: ((u ϵ x) & ((u,v) ϵ r))}))

0. Sec(r,x,y) & ¬(y = x)  Hyp
3. ((y ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ y)) & ((u,v) ϵ r)) -> (u ϵ y))  DefExp 1
7. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
10. (A -> B) -> (¬B -> ¬A)  TheoremInt
11. (((x ⊂ y) & (y ⊂ x)) -> B) -> (¬B -> ¬((x ⊂ y) & (y ⊂ x)))  PolySub 10
12. (((x ⊂ y) & (y ⊂ x)) -> (x = y)) -> (¬(x = y) -> ¬((x ⊂ y) & (y ⊂ x)))  PolySub 11
13. ¬(x = y) -> ¬((x ⊂ y) & (y ⊂ x))  ImpElim 9 12
19. ¬(y = x) -> ¬((y ⊂ x) & (x ⊂ y))  ForallElim 18
20. ¬((y ⊂ x) & (x ⊂ y))  ImpElim 2 19
21. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
23. ¬((y ⊂ x) & B) <-> (¬(y ⊂ x) v ¬B)  PolySub 22
24. ¬((y ⊂ x) & (x ⊂ y)) <-> (¬(y ⊂ x) v ¬(x ⊂ y))  PolySub 23
27. ¬(y ⊂ x) v ¬(x ⊂ y)  ImpElim 20 26
28. ¬(y ⊂ x)  Hyp
29. _|_  ImpElim 5 28
30. ¬(x ⊂ y)  AbsI 29
31. ¬(x ⊂ y)  Hyp
32. ¬(x ⊂ y)  OrElim 27 28 30 31 31
33. ¬∀z.((z ϵ x) -> (z ϵ y))  DefExp 32
34. ¬∀i.P(i) -> ∃c.¬P(c)  TheoremInt
35. ¬∀i.((i ϵ x) -> (i ϵ y)) -> ∃c.¬((c ϵ x) -> (c ϵ y))  PredSub 34
36. ∃c.¬((c ϵ x) -> (c ϵ y))  ImpElim 33 35
37. (A -> B) -> (¬B -> ¬A)  TheoremInt
38. (C -> B) -> (¬B -> ¬C)  PolySub 37
39. (C -> D) -> (¬D -> ¬C)  PolySub 38
40. ((B v ¬A) -> D) -> (¬D -> ¬(B v ¬A))  PolySub 39
41. ((B v ¬A) -> (A -> B)) -> (¬(A -> B) -> ¬(B v ¬A))  PolySub 40
42. (B v ¬A) -> (A -> B)  TheoremInt
43. ¬(A -> B) -> ¬(B v ¬A)  ImpElim 42 41
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44. ¬((c ϵ x) -> (c ϵ y)) Hyp
45. ¬((c ϵ x) -> B) -> ¬(B v ¬(c ϵ x)) PolySub 43
46. ¬((c ϵ x) -> (c ϵ y)) -> ¬((c ϵ y) v ¬(c ϵ x)) PolySub 45
47. ¬((c ϵ y) v ¬(c ϵ x)) ImpElim 44 46
48. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B)) TheoremInt
52. ¬((c ϵ y) v B) -> (¬(c ϵ y) & ¬B) PolySub 51
53. ¬((c ϵ y) v ¬(c ϵ x)) -> (¬(c ϵ y) & ¬¬(c ϵ x)) PolySub 52
54. ¬(c ϵ y) & ¬¬(c ϵ x) ImpElim 47 53
57. D <-> ¬¬D TheoremInt
60. ¬¬(c ϵ x) -> (c ϵ x) PolySub 59
61. c ϵ x ImpElim 56 60
63. ∃w.(c ϵ w) ExistsInt 61
64. Set(c) DefSub 63
65. Set(c) & ¬(c ϵ y) AndInt 64 55
66. c ϵ {w: ¬(w ϵ y)} ClassInt 65
68. c ϵ {w: ¬(w ϵ y)} EqualitySub 66 67
70. {x_14: ¬(x_14 ϵ y)} = ~y ForallElim 69
71. c ϵ ~y EqualitySub 66 70
72. (c ϵ x) & (c ϵ ~y) AndInt 61 71
73. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
80. ((c ϵ x) & (c ϵ ~y)) -> (c ϵ (x ∩ ~y)) ForallElim 79
81. c ϵ (x ∩ ~y) ImpElim 72 80
83. c ϵ (x ~ y) EqualitySub 81 82
84. (x ~ y) = 0 Hyp
85. c ϵ 0 EqualitySub 83 84
86. ¬(x ϵ 0) TheoremInt
88. ¬(c ϵ 0) ForallElim 87
89. _|_ ImpElim 85 88
90. ¬((x ~ y) = 0) ImpInt 89
91. ¬((x ~ y) = 0) ExistsElim 36 44 90
92. ((y ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ y)) & ((u,v) ϵ r)) -> (u ϵ y)) DefExp 1
95. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 94
97. (((x ~ y) ⊂ x) & ¬((x ~ y) = 0)) -> ∃z.First(r,(x ~ y),z) ForallElim 96
99. z ϵ (x ~ y) Hyp
100. z ϵ (x ∩ ~y) EqualitySub 99 98
103. (z ϵ (x ∩ ~y)) -> ((z ϵ x) & (z ϵ ~y)) ForallElim 102
104. (z ϵ x) & (z ϵ ~y) ImpElim 100 103
106. (z ϵ (x ~ y)) -> (z ϵ x) ImpInt 105
107. ∀z.((z ϵ (x ~ y)) -> (z ϵ x)) ForallInt 106
108. (x ~ y) ⊂ x DefSub 107
109. ((x ~ y) ⊂ x) & ¬((x ~ y) = 0) AndInt 108 91
110. ∃z.First(r,(x ~ y),z) ImpElim 109 97
111. First(r,(x ~ y),v) Hyp
112. (v ϵ (x ~ y)) & ∀x_25.((x_25 ϵ (x ~ y)) -> ¬((x_25,v) ϵ r)) DefExp 111
113. z ϵ {u: ((u ϵ x) & ((u,v) ϵ r))} Hyp
114. Set(z) & ((z ϵ x) & ((z,v) ϵ r)) ClassElim 113
116. (z ϵ (x ~ y)) -> ¬((z,v) ϵ r) ForallElim 115
120. (A -> B) -> (¬B -> ¬A) TheoremInt
121. ((z ϵ (x ~ y)) -> B) -> (¬B -> ¬(z ϵ (x ~ y))) PolySub 120
122. ((z ϵ (x ~ y)) -> ¬((z,v) ϵ r)) -> (¬¬((z,v) ϵ r) -> ¬(z ϵ (x ~ y))) PolySub 121
123. ¬¬((z,v) ϵ r) -> ¬(z ϵ (x ~ y)) ImpElim 116 122
124. D <-> ¬¬D TheoremInt
127. ((z,v) ϵ r) -> ¬¬((z,v) ϵ r) PolySub 126
128. ¬¬((z,v) ϵ r) ImpElim 118 127
129. ¬(z ϵ (x ~ y)) ImpElim 128 123
130. ¬(z ϵ (x ∩ ~y)) EqualitySub 129 98
131. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
136. ((z ϵ x) & (z ϵ ~y)) -> (z ϵ (x ∩ ~y)) ForallElim 135
137. (((z ϵ x) & (z ϵ ~y)) -> B) -> (¬B -> ¬((z ϵ x) & (z ϵ ~y))) PolySub 120
138. (((z ϵ x) & (z ϵ ~y)) -> (z ϵ (x ∩ ~y))) -> (¬(z ϵ (x ∩ ~y)) -> ¬((z ϵ x) & (z ϵ ~y))) PolySub 137
139. ¬(z ϵ (x ∩ ~y)) -> ¬((z ϵ x) & (z ϵ ~y)) ImpElim 136 138

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140. ¬((z ϵ x) & (z ϵ ~y))  ImpElim 130 139
141. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
143. ¬((z ϵ x) & B) <-> (¬(z ϵ x) v ¬B)  PolySub 142
144. ¬((z ϵ x) & (z ϵ ~y)) <-> (¬(z ϵ x) v ¬(z ϵ ~y))  PolySub 143
147. ¬(z ϵ x) v ¬(z ϵ ~y)  ImpElim 140 146
148. ¬(z ϵ x)  Hyp
150. _|_  ImpElim 149 148
151. ¬(z ϵ ~y)  AbsI 150
152. ¬(z ϵ ~y)  Hyp
153. ¬(z ϵ ~y)  OrElim 147 148 151 152 152
155. Set(z) & ¬(z ϵ ~y)  AndInt 154 153
156. z ϵ {w: ¬(w ϵ ~y)}  ClassInt 155
159. ~~y = {x_26: ¬(x_26 ϵ ~y)}  ForallElim 158
161. z ϵ ~~y  EqualitySub 156 160
162. ~~x = x  TheoremInt
164. ~~y = y  ForallElim 163
165. z ϵ y  EqualitySub 161 164
166. (z ϵ {u: ((u ϵ x) & ((u,v) ϵ r))}) -> (z ϵ y)  ImpInt 165
167. z ϵ y  Hyp
168. ∀z.((z ϵ y) -> (z ϵ x))  DefExp 5
169. (z ϵ y) -> (z ϵ x)  ForallElim 168
170. z ϵ x  ImpElim 167 169
178. (v ϵ (x ∩ ~y)) -> ((v ϵ x) & (v ϵ ~y))  ForallElim 177
180. v ϵ (x ∩ ~y)  EqualitySub 179 6
181. (v ϵ x) & (v ϵ ~y)  ImpElim 180 178
183. (v,z) ϵ r  Hyp
185. ∀x_29.((((v ϵ x) & (x_29 ϵ y)) & ((v,x_29) ϵ r)) -> (v ϵ y))  ForallElim 184
186. (((v ϵ x) & (z ϵ y)) & ((v,z) ϵ r)) -> (v ϵ y)  ForallElim 185
188. (v ϵ x) & (z ϵ y)  AndInt 187 167
189. ((v ϵ x) & (z ϵ y)) & ((v,z) ϵ r)  AndInt 188 183
190. v ϵ y  ImpElim 189 186
193. ~y = {x_30: ¬(x_30 ϵ y)}  ForallElim 192
194. v ϵ {x_30: ¬(x_30 ϵ y)}  EqualitySub 182 193
195. Set(v) & ¬(v ϵ y)  ClassElim 194
197. _|_  ImpElim 190 196
198. ¬((v,z) ϵ r)  ImpInt 197
200. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  TheoremInt
201. Asymmetric(r,x) & TransIn(r,x)  ImpElim 199 200
203. ∀y.∀z.(((y ϵ x) & (z ϵ x)) -> (((y,z) ϵ r) -> ¬((z,y) ϵ r)))  DefExp 202
204. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  DefExp 199
206. ∀y.∀z.(((y ϵ x) & (z ϵ x)) -> ((y = z) v (((y,z) ϵ r) v ((z,y) ϵ r))))  DefExp 205
207. ∀z.(((v ϵ x) & (z ϵ x)) -> ((v = z) v (((v,z) ϵ r) v ((z,v) ϵ r))))  ForallElim 206
208. ((v ϵ x) & (z ϵ x)) -> ((v = z) v (((v,z) ϵ r) v ((z,v) ϵ r)))  ForallElim 207
209. (v ϵ x) & (z ϵ x)  AndInt 187 170
210. (v = z) v (((v,z) ϵ r) v ((z,v) ϵ r))  ImpElim 209 208
211. ∀z.(((v ϵ x) & (z ϵ x)) -> (((v,z) ϵ r) -> ¬((z,v) ϵ r)))  ForallElim 203
212. ((v ϵ x) & (z ϵ x)) -> (((v,z) ϵ r) -> ¬((z,v) ϵ r))  ForallElim 211
213. ((v,z) ϵ r) -> ¬((z,v) ϵ r)  ImpElim 209 212
214. v = z  Hyp
215. ¬(z ϵ y)  EqualitySub 196 214
216. _|_  ImpElim 167 215
217. (z,v) ϵ r  AbsI 216
218. ((v,z) ϵ r) v ((z,v) ϵ r)  Hyp
219. (v,z) ϵ r  Hyp
220. _|_  ImpElim 219 198
221. (z,v) ϵ r  AbsI 220
222. (z,v) ϵ r  Hyp
223. (z,v) ϵ r  OrElim 218 219 221 222 222
225. (z ϵ x) & ((z,v) ϵ r)  AndInt 170 224
226. ∃w.(z ϵ w)  ExistsInt 167
227. Set(z)  DefSub 226
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228. Set(z) & ((z ϵ x) & ((z,v) ϵ r))  AndInt 227 225
229. z ϵ {w: ((w ϵ x) & ((w,v) ϵ r))}  ClassInt 228
230. (z ϵ y) -> (z ϵ {w: ((w ϵ x) & ((w,v) ϵ r))})  ImpInt 229
231. ((z ϵ y) -> (z ϵ {w: ((w ϵ x) & ((w,v) ϵ r))})) & ((z ϵ {u: ((u ϵ x) & ((u,v) ϵ r))}) -> (z ϵ y))  AndInt 230
233. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y)))  AxInt
234. ∀x_38.((y = x_38) <-> ∀z.((z ϵ y) <-> (z ϵ x_38)))  ForallElim 233
235. (y = {u: ((u ϵ x) & ((u,v) ϵ r))}) <-> ∀z.((z ϵ y) <-> (z ϵ {u: ((u ϵ x) & ((u,v) ϵ r))}))  ForallElim 234
238. ∀z.((z ϵ y) <-> (z ϵ {w: ((w ϵ x) & ((w,v) ϵ r))}))  ForallInt 232
239. y = {u: ((u ϵ x) & ((u,v) ϵ r))}  ImpElim 238 237
240. (v ϵ x) & (y = {u: ((u ϵ x) & ((u,v) ϵ r))})  AndInt 187 239
241. ∃v.((v ϵ x) & (y = {u: ((u ϵ x) & ((u,v) ϵ r))}))  ExistsInt 240
242. ∃v.((v ϵ x) & (y = {u: ((u ϵ x) & ((u,v) ϵ r))}))  ExistsElim 110 111 241
243. (Sec(r,x,y) & ¬(y = x)) -> ∃v.((v ϵ x) & (y = {u: ((u ϵ x) & ((u,v) ϵ r))}))  ImpInt 242 Qed
```

Used Theorems

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1. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
2. (A -> B) -> (¬B -> ¬A)
4. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
3. ¬∀i.P(i) -> ∃c.¬P(c)
5. (B v ¬A) -> (A -> B)
6. D <-> ¬¬D
7. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
8. ¬(x ϵ 0)
9. ~~x = x
10. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
```

Th92. (Sec(r,z,a) & Sec(r,z,b)) -> ((a ⊂ b) v (b ⊂ a))

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0. Sec(r,z,a) & Sec(r,z,b)  Hyp
1. (Sec(r,x,y) & ¬(y = x)) -> ∃v.((v ϵ x) & (y = {u: ((u ϵ x) & ((u,v) ϵ r))}))  TheoremInt
7. (Sec(r,z,b) & ¬(b = z)) -> ∃v.((v ϵ z) & (b = {u: ((u ϵ z) & ((u,v) ϵ r))}))  ForallElim 6
8. ¬(a = z)  Hyp
9. ¬(b = z)  Hyp
12. Sec(r,z,a) & ¬(a = z)  AndInt 10 8
13. Sec(r,z,b) & ¬(b = z)  AndInt 11 9
14. ∃v.((v ϵ z) & (a = {u: ((u ϵ z) & ((u,v) ϵ r))}))  ImpElim 12 5
15. ∃v.((v ϵ z) & (b = {u: ((u ϵ z) & ((u,v) ϵ r))}))  ImpElim 13 7
16. (u ϵ z) & (a = {x_1: ((x_1 ϵ z) & ((x_1,u) ϵ r))})  Hyp
17. (v ϵ z) & (b = {u: ((u ϵ z) & ((u,v) ϵ r))})  Hyp
18. ((a ⊂ z) & WO(r,z)) & ∀u.∀v.((((u ϵ z) & (v ϵ a)) & ((u,v) ϵ r)) -> (u ϵ a))  DefExp 10
21. Connects(r,z) & ∀y.(((y ⊂ z) & ¬(y = 0)) -> ∃x_11.First(r,y,x_11))  DefExp 20
23. ∀y.∀x_14.(((y ϵ z) & (x_14 ϵ z)) -> ((y = x_14) v (((y,x_14) ϵ r) v ((x_14,y) ϵ r))))  DefExp 22
24. ∀x_14.(((u ϵ z) & (x_14 ϵ z)) -> ((u = x_14) v (((u,x_14) ϵ r) v ((x_14,u) ϵ r))))  ForallElim 23
25. ((u ϵ z) & (v ϵ z)) -> ((u = v) v (((u,v) ϵ r) v ((v,u) ϵ r)))  ForallElim 24
28. (u ϵ z) & (v ϵ z)  AndInt 26 27
29. (u = v) v (((u,v) ϵ r) v ((v,u) ϵ r))  ImpElim 28 25
30. u = v  Hyp
33. a = {x_1: ((x_1 ϵ z) & ((x_1,v) ϵ r))}  EqualitySub 31 30
35. b = a  EqualitySub 32 34
37. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
43. (a = b) -> ((a ⊂ b) & (b ⊂ a))  ForallElim 42
44. (a ⊂ b) & (b ⊂ a)  ImpElim 36 43
46. (a ⊂ b) v (b ⊂ a)  OrIntR 45
47. ((u,v) ϵ r) v ((v,u) ϵ r)  Hyp
48. (u,v) ϵ r  Hyp
49. x ϵ a  Hyp
50. x ϵ {x_1: ((x_1 ϵ z) & ((x_1,u) ϵ r))}  EqualitySub 49 31
51. Set(x) & ((x ϵ z) & ((x,u) ϵ r))  ClassElim 50
53. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  TheoremInt
55. WO(r,z) -> (Asymmetric(r,z) & TransIn(r,z))  ForallElim 54
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56. Asymmetric(r,z) & TransIn(r,z)  ImpElim 20 55
58. ∀u.∀v.∀w.(((u ϵ z) & ((v ϵ z) & (w ϵ z))) -> ((((u,v) ϵ r) & ((v,w) ϵ r)) -> ((u,w) ϵ r)))  DefExp 57
60. ∀v.∀w.(((x ϵ z) & ((v ϵ z) & (w ϵ z))) -> ((((x,v) ϵ r) & ((v,w) ϵ r)) -> ((x,w) ϵ r)))  ForallElim 58
61. ∀w.(((x ϵ z) & ((u ϵ z) & (w ϵ z))) -> ((((x,u) ϵ r) & ((u,w) ϵ r)) -> ((x,w) ϵ r)))  ForallElim 60
62. ((x ϵ z) & ((u ϵ z) & (v ϵ z))) -> ((((x,u) ϵ r) & ((u,v) ϵ r)) -> ((x,v) ϵ r))  ForallElim 61
63. (u ϵ z) & (v ϵ z)  AndInt 26 27
64. (x ϵ z) & ((u ϵ z) & (v ϵ z))  AndInt 59 63
65. (((x,u) ϵ r) & ((u,v) ϵ r)) -> ((x,v) ϵ r)  ImpElim 64 62
67. ((x,u) ϵ r) & ((u,v) ϵ r)  AndInt 66 48
68. (x,v) ϵ r  ImpElim 67 65
69. (x ϵ z) & ((x,v) ϵ r)  AndInt 59 68
70. ∃w.(x ϵ w)  ExistsInt 49
71. Set(x)  DefSub 70
72. Set(x) & ((x ϵ z) & ((x,v) ϵ r))  AndInt 71 69
73. x ϵ {w: ((w ϵ z) & ((w,v) ϵ r))}  ClassInt 72
75. x ϵ b  EqualitySub 73 74
76. (x ϵ a) -> (x ϵ b)  ImpInt 75
77. ∀x.((x ϵ a) -> (x ϵ b))  ForallInt 76
78. a ⊂ b  DefSub 77
79. (a ⊂ b) v (b ⊂ a)  OrIntR 78
80. (v,u) ϵ r  Hyp
81. x ϵ b  Hyp
82. x ϵ {u: ((u ϵ z) & ((u,v) ϵ r))}  EqualitySub 81 32
83. Set(x) & ((x ϵ z) & ((x,v) ϵ r))  ClassElim 82
86. ∀w.(((x ϵ z) & ((v ϵ z) & (w ϵ z))) -> ((((x,v) ϵ r) & ((v,w) ϵ r)) -> ((x,w) ϵ r)))  ForallElim 60
87. ((x ϵ z) & ((v ϵ z) & (u ϵ z))) -> ((((x,v) ϵ r) & ((v,u) ϵ r)) -> ((x,u) ϵ r))  ForallElim 86
88. (v ϵ z) & (u ϵ z)  AndInt 27 26
90. (x ϵ z) & ((v ϵ z) & (u ϵ z))  AndInt 89 88
91. (((x,v) ϵ r) & ((v,u) ϵ r)) -> ((x,u) ϵ r)  ImpElim 90 87
92. ((x,v) ϵ r) & ((v,u) ϵ r)  AndInt 85 80
93. (x,u) ϵ r  ImpElim 92 91
94. (x ϵ z) & ((x,u) ϵ r)  AndInt 89 93
95. ∃w.(x ϵ w)  ExistsInt 81
96. Set(x)  DefSub 95
97. Set(x) & ((x ϵ z) & ((x,u) ϵ r))  AndInt 96 94
98. x ϵ {w: ((w ϵ z) & ((w,u) ϵ r))}  ClassInt 97
100. x ϵ a  EqualitySub 98 99
101. (x ϵ b) -> (x ϵ a)  ImpInt 100
102. ∀x.((x ϵ b) -> (x ϵ a))  ForallInt 101
103. b ⊂ a  DefSub 102
104. (a ⊂ b) v (b ⊂ a)  OrIntL 103
105. (a ⊂ b) v (b ⊂ a)  OrElim 47 48 79 80 104
107. (a ⊂ b) v (b ⊂ a)  ExistsElim 15 17 106
109. b = z  Hyp
111. ((a ⊂ b) & WO(r,b)) & ∀u.∀v.((((u ϵ b) & (v ϵ a)) & ((u,v) ϵ r)) -> (u ϵ a))  EqualitySub 18 110
114. (a ⊂ b) v (b ⊂ a)  OrIntR 113
115. A v ¬A  TheoremInt
116. (b = z) v ¬(b = z)  PolySub 115
117. (a ⊂ b) v (b ⊂ a)  OrElim 116 109 114 9 108
118. a = z  Hyp
120. ((b ⊂ z) & WO(r,z)) & ∀u.∀v.((((u ϵ z) & (v ϵ b)) & ((u,v) ϵ r)) -> (u ϵ b))  DefExp 11
123. b ⊂ a  EqualitySub 122 119
124. (a ⊂ b) v (b ⊂ a)  OrIntL 123
125. (a = z) v ¬(a = z)  PolySub 115
126. (a ⊂ b) v (b ⊂ a)  OrElim 125 118 124 8 117
127. (Sec(r,z,a) & Sec(r,z,b)) -> ((a ⊂ b) v (b ⊂ a))  ImpInt 126 Qed
```

Used Theorems

```
1. (Sec(r,x,y) & ¬(y = x)) -> ∃v.((v ϵ x) & (y = {u: ((u ϵ x) & ((u,v) ϵ r))}))
2. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
```

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3. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
0. A v ¬A
```

```
FunctionApp. ((f ϵ func(x,y)) & (a ϵ x)) -> ((f'a) ϵ y)

0. (f ϵ func(x,y)) & (a ϵ x)  Hyp
3. f ϵ {f: (FUN(f) & ((dom(f) = x) & (rg(f) = y)))}  EqualitySub 1 2
4. Set(f) & (FUN(f) & ((dom(f) = x) & (rg(f) = y)))  ClassElim 3
6. u = (a,(f'a))  Hyp
7. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  TheoremInt
9. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ImpElim 8 7
11. (u = (a,(f'a))) & ((f'a) = (f'a))  AndInt 6 10
13. ∃b.∃w.((u = (b,w)) & ((f'b) = w))  ExistsInt 12
14. (¬(z ϵ dom(f)) -> ((f'z) = U)) & ((z ϵ dom(f)) -> ((f'z) ϵ U))  TheoremInt
17. (a ϵ dom(f)) -> ((f'a) ϵ U)  ForallElim 16
22. a ϵ dom(f)  EqualitySub 18 21
23. (f'a) ϵ U  ImpElim 22 17
24. ∃w.((f'a) ϵ w)  ExistsInt 23
25. Set((f'a))  DefSub 24
26. ∃w.(a ϵ w)  ExistsInt 18
27. Set(a)  DefSub 26
28. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
35. (Set(a) & Set((f'a))) -> Set((a,(f'a)))  ForallElim 34
36. Set(a) & Set((f'a))  AndInt 27 25
37. Set((a,(f'a)))  ImpElim 36 35
39. Set(u)  EqualitySub 37 38
40. Set(u) & ∃b.∃w.((u = (b,w)) & ((f'b) = w))  AndInt 39 13
41. u ϵ {w: ∃b.∃j.((w = (b,j)) & ((f'b) = j))}  ClassInt 40
43. u ϵ f  EqualitySub 41 42
44. (a,(f'a)) ϵ f  EqualitySub 43 6
45. (u = (a,(f'a))) -> ((a,(f'a)) ϵ f)  ImpInt 44
47. ((a,(f'a)) = (a,(f'a))) -> ((a,(f'a)) ϵ f)  ForallElim 46
49. (a,(f'a)) ϵ f  ImpElim 48 47
50. ∃u.((u,(f'a)) ϵ f)  ExistsInt 49
51. Set((f'a)) & ∃u.((u,(f'a)) ϵ f)  AndInt 25 50
52. u = (f'a)  Hyp
54. Set(u) & ∃k.((k,u) ϵ f)  EqualitySub 51 53
55. u ϵ {w: ∃k.((k,w) ϵ f)}  ClassInt 54
58. u ϵ rg(f)  EqualitySub 55 57
59. (f'a) ϵ rg(f)  EqualitySub 58 52
60. (u = (f'a)) -> ((f'a) ϵ rg(f))  ImpInt 59
62. ((f'a) = (f'a)) -> ((f'a) ϵ rg(f))  ForallElim 61
64. (f'a) ϵ rg(f)  ImpElim 63 62
67. (f'a) ϵ y  EqualitySub 64 66
68. ((f ϵ func(x,y)) & (a ϵ x)) -> ((f'a) ϵ y)  ImpInt 67 Qed
```

Used Theorems

```
1. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
2. (¬(z ϵ dom(f)) -> ((f'z) = U)) & ((z ϵ dom(f)) -> ((f'z) ϵ U))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
```

```
Th94. (Sec(r,z,a) & ((f ϵ func(a,z)) & OP(f,r,r))) -> ((x ϵ a) -> ¬(((f'x),x) ϵ r))

0. Sec(r,z,a) & ((f ϵ func(a,z)) & OP(f,r,r))  Hyp
1. u ϵ a  Hyp
2. c = {u: ((u ϵ a) & (((f'u),u) ϵ r))}  Hyp
4. ((a ⊂ z) & WO(r,z)) & ∀u.∀v.((((u ϵ z) & (v ϵ a)) & ((u,v) ϵ r)) -> (u ϵ a))  DefExp 3
7. Connects(r,z) & ∀y.(((y ⊂ z) & ¬(y = 0)) -> ∃x_8.First(r,y,x_8))  DefExp 6
9. ((c ⊂ z) & ¬(c = 0)) -> ∃x_8.First(r,c,x_8)  ForallElim 8
10. ¬(c = 0)  Hyp
```

```
11. x ∈ c  Hyp
12. x ∈ {u: ((u ∈ a) & (((f'u),u) ∈ r))}  EqualitySub 11 2
13. Set(x) & ((x ∈ a) & (((f'x),x) ∈ r))  ClassElim 12
16. (x ∈ c) -> (x ∈ a)  ImpInt 15
17. ∀x.((x ∈ c) -> (x ∈ a))  ForallInt 16
18. c ⊂ a  DefSub 17
20. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)  TheoremInt
24. ((c ⊂ a) & (a ⊂ z)) -> (c ⊂ z)  ForallElim 23
25. (c ⊂ a) & (a ⊂ z)  AndInt 18 19
26. c ⊂ z  ImpElim 25 24
27. (c ⊂ z) & ¬(c = 0)  AndInt 26 10
28. ∃x_8.First(r,c,x_8)  ImpElim 27 9
29. First(r,c,k)  Hyp
30. (k ∈ c) & ∀y.((y ∈ c) -> ¬((y,k) ∈ r))  DefExp 29
32. k ∈ {u: ((u ∈ a) & (((f'u),u) ∈ r))}  EqualitySub 31 2
33. Set(k) & ((k ∈ a) & (((f'k),k) ∈ r))  ClassElim 32
38. (FUN(f) & (WO(r,dom(f)) & WO(r,rg(f)))) & ∀u.∀v.((((u ∈ dom(f)) & (v ∈ dom(f))) & ((u,v) ∈ r)) -> ((
(f'u),(f'v)) ∈ r))  DefExp 37
45. func(a,z) = {f: (FUN(f) & ((dom(f) = a) & (rg(f) = z)))}  ForallElim 44
46. f ∈ {f: (FUN(f) & ((dom(f) = a) & (rg(f) = z)))}  EqualitySub 40 45
47. Set(f) & (FUN(f) & ((dom(f) = a) & (rg(f) = z)))  ClassElim 46
51. ∀z.((z ∈ c) -> (z ∈ a))  DefExp 18
52. (k ∈ c) -> (k ∈ a)  ForallElim 51
53. k ∈ a  ImpElim 31 52
54. ((f ∈ func(x,y)) & (a ∈ x)) -> ((f'a) ∈ y)  TheoremInt
60. ((f ∈ func(a,z)) & (k ∈ a)) -> ((f'k) ∈ z)  ForallElim 59
61. (f ∈ func(a,z)) & (k ∈ a)  AndInt 40 53
62. (f'k) ∈ z  ImpElim 61 60
64. ∀v.(((((f'k) ∈ z) & (v ∈ a)) & (((f'k),v) ∈ r)) -> ((f'k) ∈ a))  ForallElim 63
65. ((((f'k) ∈ z) & (k ∈ a)) & (((f'k),k) ∈ r)) -> ((f'k) ∈ a)  ForallElim 64
66. ((f'k) ∈ z) & (k ∈ a)  AndInt 62 53
67. (((f'k) ∈ z) & (k ∈ a)) & (((f'k),k) ∈ r)  AndInt 66 35
68. (f'k) ∈ a  ImpElim 67 65
70. k ∈ dom(f)  EqualitySub 53 69
71. (f'k) ∈ dom(f)  EqualitySub 68 69
72. ∀v.(((((f'k) ∈ dom(f)) & (v ∈ dom(f))) & (((f'k),v) ∈ r)) -> (((f'(f'k)),(f'v)) ∈ r))  ForallElim 39
73. ((((f'k) ∈ dom(f)) & (k ∈ dom(f))) & (((f'k),k) ∈ r)) -> (((f'(f'k)),(f'k)) ∈ r)  ForallElim 72
74. ((f'k) ∈ dom(f)) & (k ∈ dom(f))  AndInt 71 70
75. (((f'k) ∈ dom(f)) & (k ∈ dom(f))) & (((f'k),k) ∈ r)  AndInt 74 35
76. ((f'(f'k)),(f'k)) ∈ r  ImpElim 75 73
77. u = (f'k)  Hyp
79. ((f'u),u) ∈ r  EqualitySub 76 78
80. u ∈ a  EqualitySub 68 78
81. (u ∈ a) & (((f'u),u) ∈ r)  AndInt 80 79
82. ∃w.((f'k) ∈ w)  ExistsInt 68
83. Set((f'k))  DefSub 82
84. Set(u)  EqualitySub 83 78
85. Set(u) & ((u ∈ a) & (((f'u),u) ∈ r))  AndInt 84 81
86. u ∈ {w: ((w ∈ a) & (((f'w),w) ∈ r))}  ClassInt 85
87. (f'k) ∈ {w: ((w ∈ a) & (((f'w),w) ∈ r))}  EqualitySub 86 77
89. (f'k) ∈ c  EqualitySub 87 88
90. (u = (f'k)) -> ((f'k) ∈ c)  ImpInt 89
92. ((f'k) = (f'k)) -> ((f'k) ∈ c)  ForallElim 91
94. (f'k) ∈ c  ImpElim 93 92
96. ((f'k) ∈ c) -> ¬(((f'k),k) ∈ r)  ForallElim 95
97. ¬(((f'k),k) ∈ r)  ImpElim 94 96
98. _|_  ImpElim 35 97
99. _|_  ExistsElim 28 29 98
100. ¬¬(c = 0)  ImpInt 99
101. D <-> ¬¬D  TheoremInt
104. ¬¬(c = 0) -> (c = 0)  PolySub 103
```

105. c = 0 ImpElim 100 104
106. {u: ((u ϵ a) & (((f'u),u) ϵ r))} = 0 EqualitySub 105 2
107. (c = {u: ((u ϵ a) & (((f'u),u) ϵ r))}) -> ({u: ((u ϵ a) & (((f'u),u) ϵ r))} = 0) ImpInt 106
109. ({u: ((u ϵ a) & (((f'u),u) ϵ r))} = {x_20: ((x_20 ϵ a) & (((f'x_20),x_20) ϵ r))}) -> ({x_20: ((x_20 ϵ a) & (((f'x_20),x_20) ϵ r))} = 0) ForallElim 108
111. {x_20: ((x_20 ϵ a) & (((f'x_20),x_20) ϵ r))} = 0 ImpElim 110 109
112. x ϵ a Hyp
113. ((f'x),x) ϵ r Hyp
114. (x ϵ a) & (((f'x),x) ϵ r) AndInt 112 113
115. ∃w.(x ϵ w) ExistsInt 112
116. Set(x) DefSub 115
117. Set(x) & ((x ϵ a) & (((f'x),x) ϵ r)) AndInt 116 114
118. x ϵ {w: ((w ϵ a) & (((f'w),w) ϵ r))} ClassInt 117
119. x ϵ 0 EqualitySub 118 111
120. ¬(x ϵ 0) TheoremInt
121. _|_ ImpElim 119 120
122. ¬(((f'x),x) ϵ r) ImpInt 121
123. (x ϵ a) -> ¬(((f'x),x) ϵ r) ImpInt 122
124. (Sec(r,z,a) & ((f ϵ func(a,z)) & OP(f,r,r))) -> ((x ϵ a) -> ¬(((f'x),x) ϵ r)) ImpInt 123 Qed

Used Theorems

1. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)
2. ((f ϵ func(x,y)) & (a ϵ x)) -> ((f'a) ϵ y)
3. D <-> ¬¬D
4. ¬(x ϵ 0)

1-to-1. 1-to-1(f) <-> (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))

0. 1-to-1(f) Hyp
1. FUN(f) & FUN((f)$^{-1}$) DefExp 0
2. (x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y)) Hyp
5. Relation((f)$^{-1}$) & ∀x.∀y.∀z.((((x,y) ϵ (f)$^{-1}$) & ((x,z) ϵ (f)$^{-1}$)) -> (y = z)) DefExp 4
7. (f'x) = (f'y) Hyp
8. ∀y.∀z.(((((f'x),y) ϵ (f)$^{-1}$) & (((f'x),z) ϵ (f)$^{-1}$)) -> (y = z)) ForallElim 6
9. ∀z.(((((f'x),x) ϵ (f)$^{-1}$) & (((f'x),z) ϵ (f)$^{-1}$)) -> (x = z)) ForallElim 8
10. ((((f'x),x) ϵ (f)$^{-1}$) & (((f'x),y) ϵ (f)$^{-1}$)) -> (x = y) ForallElim 9
15. (f)$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} ForallElim 14
16. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
17. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 3 16
20. ((x,(f'x)) = (x,(f'x))) & ((f'x) = (f'x)) AndInt 18 19
21. ∃w.((w = (x,(f'x))) & ((f'x) = (f'x))) ExistsInt 20
22. (w = (x,(f'x))) & ((f'x) = (f'x)) Hyp
24. ∃b.∃a.((w = (b,a)) & ((f'b) = a)) ExistsInt 23
27. ∃w.(x ϵ w) ExistsInt 26
28. Set(x) DefSub 27
29. (¬(z ϵ dom(f)) -> ((f'z) = U)) & ((z ϵ dom(f)) -> ((f'z) ϵ U)) TheoremInt
32. (x ϵ dom(f)) -> ((f'x) ϵ U) ForallElim 31
33. (f'x) ϵ U ImpElim 26 32
34. ∃w.((f'x) ϵ w) ExistsInt 33
35. ∃w.((f'x) ϵ w) DefSub 34
36. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
41. (Set(x) & Set((f'x))) -> Set((x,(f'x))) ForallElim 40
42. Set((f'x)) DefSub 34
43. Set(x) & Set((f'x)) AndInt 28 42
44. Set((x,(f'x))) ImpElim 43 41
47. Set(w) EqualitySub 44 46
48. Set(w) & ∃b.∃a.((w = (b,a)) & ((f'b) = a)) AndInt 47 24
49. w ϵ {w: ∃b.∃a.((w = (b,a)) & ((f'b) = a))} ClassInt 48
51. w ϵ f EqualitySub 49 50
52. (x,(f'x)) ϵ f EqualitySub 51 25

53. (x,(f'x)) ϵ f ExistsElim 21 22 52
54. (x,(f'y)) ϵ f EqualitySub 53 7
56. ((x,(f'x)) ϵ f) & (((f'x),x) = ((f'x),x)) AndInt 52 55
57. ∃w.(((x,(f'x)) ϵ f) & (w = ((f'x),x))) ExistsInt 56
58. ((x,(f'x)) ϵ f) & (w = ((f'x),x)) Hyp
59. Set((f'x)) & Set(x) AndInt 42 28
63. ((Set((f'x)) & Set(x)) <-> Set(((f'x),x))) & (¬Set(((f'x),x)) -> (((f'x),x) = U)) ForallElim 62
67. Set(((f'x),x)) ImpElim 59 66
70. Set(w) EqualitySub 67 69
72. ∃x.∃y.(((x,y) ϵ f) & (w = (y,x))) ExistsInt 71
73. Set(w) & ∃x.∃y.(((x,y) ϵ f) & (w = (y,x))) AndInt 70 72
74. w ϵ {w: ∃x.∃y.(((x,y) ϵ f) & (w = (y,x)))} ClassInt 73
76. w ϵ (f)$^{-1}$ EqualitySub 74 75
77. ((f'x),x) ϵ (f)$^{-1}$ EqualitySub 76 68
78. ((f'x),x) ϵ (f)$^{-1}$ ExistsElim 57 58 77
82. ((y,(f'y)) = (y,(f'y))) & ((f'y) = (f'y)) AndInt 80 81
83. ∃w.((w = (y,(f'y))) & ((f'y) = (f'y))) ExistsInt 82
84. (w = (y,(f'y))) & ((f'y) = (f'y)) Hyp
86. ∃b.∃a.((w = (b,a)) & ((f'b) = a)) ExistsInt 85
89. ∃w.(y ϵ w) ExistsInt 88
90. Set(y) DefSub 89
92. (y ϵ dom(f)) -> ((f'y) ϵ U) ForallElim 91
93. (f'y) ϵ U ImpElim 88 92
94. ∃w.((f'y) ϵ w) ExistsInt 93
95. Set((f'y)) DefSub 94
96. Set(y) & Set((f'y)) AndInt 90 95
97. ∀x.((Set((f'x)) & Set(x)) -> Set(((f'x),x))) ForallInt 66
101. ((Set(y) & Set((f'y))) <-> Set((y,(f'y)))) & (¬Set((y,(f'y))) -> ((y,(f'y)) = U)) ForallElim 100
106. Set((y,(f'y))) ImpElim 96 105
109. Set(w) EqualitySub 106 108
110. Set(w) & ∃b.∃a.((w = (b,a)) & ((f'b) = a)) AndInt 109 86
111. w ϵ {w: ∃b.∃a.((w = (b,a)) & ((f'b) = a))} ClassInt 110
113. w ϵ f EqualitySub 111 112
114. (y,(f'y)) ϵ f EqualitySub 113 107
115. (y,(f'y)) ϵ f ExistsElim 83 84 114
117. ((y,(f'y)) ϵ f) & (((f'y),y) = ((f'y),y)) AndInt 115 116
118. ∃w.(((y,(f'y)) ϵ f) & (w = ((f'y),y))) ExistsInt 117
119. ((y,(f'y)) ϵ f) & (w = ((f'y),y)) Hyp
121. ∃a.∃b.(((a,b) ϵ f) & (w = (b,a))) ExistsInt 120
122. Set(y) & Set((f'y)) AndInt 90 95
124. Set((f'y)) & Set(y) AndInt 95 90
129. (Set((f'y)) & Set(y)) -> Set(((f'y),y)) ForallElim 128
130. Set(((f'y),y)) ImpElim 124 129
132. Set(w) EqualitySub 130 131
133. Set(w) & ∃a.∃b.(((a,b) ϵ f) & (w = (b,a))) AndInt 132 121
134. w ϵ {w: ∃a.∃b.(((a,b) ϵ f) & (w = (b,a)))} ClassInt 133
136. w ϵ (f)$^{-1}$ EqualitySub 134 135
137. ((f'y),y) ϵ (f)$^{-1}$ EqualitySub 136 123
139. ((f'y),y) ϵ (f)$^{-1}$ ExistsElim 118 119 137
140. ((f'x),y) ϵ (f)$^{-1}$ EqualitySub 139 138
141. (((f'x),x) ϵ (f)$^{-1}$) & (((f'x),y) ϵ (f)$^{-1}$) AndInt 79 140
142. x = y ImpElim 141 10
143. _|_ ImpElim 142 12
144. ¬((f'x) = (f'y)) ImpInt 143
145. ((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)) ImpInt 144
147. ∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) ForallInt 145
148. ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) ForallInt 147
149. FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) AndInt 146 148
151. FUN(f) & (((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) AndInt 146 145
152. 1-to-1(f) -> (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))) ImpInt 149
153. FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) Hyp

155. ((x,y) ϵ (f)$^{-1}$) & ((x,z) ϵ (f)$^{-1}$) Hyp
158. (x,y) ϵ {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} EqualitySub 156 15
159. (x,z) ϵ {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} EqualitySub 157 15
160. Set((x,y)) & ∃x_17.∃x_18.(((x_17,x_18) ϵ f) & ((x,y) = (x_18,x_17))) ClassElim 158
161. Set((x,z)) & ∃x_20.∃y.(((x_20,y) ϵ f) & ((x,z) = (y,x_20))) ClassElim 159
164. ∃x_18.(((a,x_18) ϵ f) & ((x,y) = (x_18,a))) Hyp
165. ((a,b) ϵ f) & ((x,y) = (b,a)) Hyp
166. ∃y.(((c,y) ϵ f) & ((x,z) = (y,c))) Hyp
167. ((c,d) ϵ f) & ((x,z) = (d,c)) Hyp
168. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
169. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
174. Set((x,z)) -> (Set(x) & Set(z)) ForallElim 173
177. Set(x) & Set(y) ImpElim 175 172
178. Set(x) & Set(z) ImpElim 176 174
180. (Set(x) & Set(y)) & ((x,y) = (b,a)) AndInt 177 179
184. ((Set(x) & Set(y)) & ((x,y) = (b,a))) -> ((x = b) & (y = a)) ForallElim 183
185. (x = b) & (y = a) ImpElim 180 184
192. ((Set(x) & Set(z)) & ((x,z) = (d,c))) -> ((x = d) & (z = c)) ForallElim 191
193. (Set(x) & Set(z)) & ((x,z) = (d,c)) AndInt 178 186
194. (x = d) & (z = c) ImpElim 193 192
200. b = d EqualitySub 199 198
201. (a,d) ϵ f EqualitySub 195 200
202. ∃d.((a,d) ϵ f) ExistsInt 201
205. Set(a) EqualitySub 203 204
206. Set(a) & ∃d.((a,d) ϵ f) AndInt 205 202
207. a ϵ {w: ∃d.((w,d) ϵ f)} ClassInt 206
210. a ϵ dom(f) EqualitySub 207 209
211. ∃d.((c,d) ϵ f) ExistsInt 196
214. Set(c) EqualitySub 212 213
215. Set(c) & ∃d.((c,d) ϵ f) AndInt 214 211
216. c ϵ {w: ∃d.((w,d) ϵ f)} ClassInt 215
217. c ϵ dom(f) EqualitySub 216 209
218. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
220. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 219 218
221. (c,d) ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 196 220
222. Set((c,d)) & ∃x.∃y.(((c,d) = (x,y)) & ((f'x) = y)) ClassElim 221
223. (a,d) ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 201 220
224. Set((a,d)) & ∃x.∃y.(((a,d) = (x,y)) & ((f'x) = y)) ClassElim 223
227. ∃y.(((c,d) = (c1,y)) & ((f'c1) = y)) Hyp
228. ((c,d) = (c1,d1)) & ((f'c1) = d1) Hyp
229. ∃y.(((a,d) = (a1,y)) & ((f'a1) = y)) Hyp
230. ((a,d) = (a1,d2)) & ((f'a1) = d2) Hyp
240. Set((a,d)) -> (Set(a) & Set(d)) ForallElim 239
241. Set(c) & Set(d) ImpElim 231 236
242. Set(a) & Set(d) ImpElim 232 240
260. ((Set(a) & Set(d)) & ((a,d) = (a1,d2))) -> ((a = a1) & (d = d2)) ForallElim 259
261. (Set(c) & Set(d)) & ((c,d) = (c1,d1)) AndInt 241 243
262. (Set(a) & Set(d)) & ((a,d) = (a1,d2)) AndInt 242 244
263. (c = c1) & (d = d1) ImpElim 261 252
264. (a = a1) & (d = d2) ImpElim 262 260
273. (f'c) = d1 EqualitySub 269 271
274. (f'a) = d2 EqualitySub 270 272
275. d2 = d1 EqualitySub 266 268
276. (f'a) = d1 EqualitySub 274 275
278. (f'a) = (f'c) EqualitySub 276 277
281. (f'y) = (f'c) EqualitySub 278 279
282. (f'y) = (f'z) EqualitySub 281 280
283. y ϵ dom(f) EqualitySub 210 279
284. z ϵ dom(f) EqualitySub 217 280
285. ¬(y = z) Hyp
286. ∀x_24.(((y ϵ dom(f)) & ((x_24 ϵ dom(f)) & ¬(y = x_24))) -> ¬((f'y) = (f'x_24))) ForallElim 154

287. ((y ϵ dom(f)) & ((z ϵ dom(f)) & ¬(y = z))) -> ¬((f’y) = (f’z)) ForallElim 286
288. (z ϵ dom(f)) & ¬(y = z) AndInt 284 285
289. (y ϵ dom(f)) & ((z ϵ dom(f)) & ¬(y = z)) AndInt 283 288
290. ¬((f’y) = (f’z)) ImpElim 289 287
291. _|_ ImpElim 282 290
292. ¬¬(y = z) ImpInt 291
293. D <-> ¬¬D TheoremInt
296. ¬¬(y = z) -> (y = z) PolySub 295
297. y = z ImpElim 292 296
298. y = z ExistsElim 229 230 297
306. (((x,y) ϵ (f)$^{-1}$) & ((x,z) ϵ (f)$^{-1}$)) -> (y = z) ImpInt 305
307. ∀z.((((x,y) ϵ (f)$^{-1}$) & ((x,z) ϵ (f)$^{-1}$)) -> (y = z)) ForallInt 306
308. ∀y.∀z.((((x,y) ϵ (f)$^{-1}$) & ((x,z) ϵ (f)$^{-1}$)) -> (y = z)) ForallInt 307
309. ∀x.∀y.∀z.((((x,y) ϵ (f)$^{-1}$) & ((x,z) ϵ (f)$^{-1}$)) -> (y = z)) ForallInt 308
311. Relation(f) & ∀x.∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z)) DefExp 310
313. z ϵ (f)$^{-1}$ Hyp
316. (f)$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} ForallElim 315
317. ∀z.((z ϵ f) -> ∃x.∃y.(z = (x,y))) DefExp 312
318. z ϵ {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} EqualitySub 313 316
319. Set(z) & ∃x.∃y.(((x,y) ϵ f) & (z = (y,x))) ClassElim 318
321. ∃y.(((x,y) ϵ f) & (z = (y,x))) Hyp
322. ((x,y) ϵ f) & (z = (y,x)) Hyp
325. ∃y.∃x.(z = (y,x)) ExistsInt 324
326. ∃y.∃x.(z = (y,x)) ExistsElim 321 322 325
328. (z ϵ (f)$^{-1}$) -> ∃y.∃x.(z = (y,x)) ImpInt 327
329. ∀z.((z ϵ (f)$^{-1}$) -> ∃y.∃x.(z = (y,x))) ForallInt 328
330. Relation((f)$^{-1}$) DefSub 329
331. Relation((f)$^{-1}$) & ∀x.∀y.∀z.((((x,y) ϵ (f)$^{-1}$) & ((x,z) ϵ (f)$^{-1}$)) -> (y = z)) AndInt 330 309
332. FUN((f)$^{-1}$) DefSub 331
333. FUN(f) & FUN((f)$^{-1}$) AndInt 310 332
334. 1-to-1(f) DefSub 333
335. (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f’x) = (f’y)))) -> 1-to-1(f) ImpInt 334
336. (1-to-1(f) -> (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f’x) = (f’y))))) & (
(FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f’x) = (f’y)))) -> 1-to-1(f)) AndInt 152 335
337. 1-to-1(f) <-> (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f’x) = (f’y)))) EquivConst

Used Theorems

1. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f’x) = y))})
2. (¬(z ϵ dom(f)) -> ((f’z) = U)) & ((z ϵ dom(f)) -> ((f’z) ϵ U))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
4. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
6. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f’x) = y))})
8. D <-> ¬¬D

FunctionRange. (FUN(f) & (a ϵ dom(f))) -> ((f’a) ϵ rg(f))

0. FUN(f) & (a ϵ dom(f)) Hyp
4. a ϵ {x: ∃y.((x,y) ϵ f)} EqualitySub 2 3
5. Set(a) & ∃y.((a,y) ϵ f) ClassElim 4
8. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f’x) = y))}) TheoremInt
9. f = {w: ∃x.∃y.((w = (x,y)) & ((f’x) = y))} ImpElim 1 8
10. (a,y) ϵ f Hyp
11. (a,y) ϵ {w: ∃x.∃y.((w = (x,y)) & ((f’x) = y))} EqualitySub 10 9
12. Set((a,y)) & ∃x.∃x_0.(((a,y) = (x,x_0)) & ((f’x) = x_0)) ClassElim 11
15. ∃x_0.(((a,y) = (b,x_0)) & ((f’b) = x_0)) Hyp
16. ((a,y) = (b,c)) & ((f’b) = c) Hyp
17. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
22. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 21
23. Set(a) & Set(y) ImpElim 13 22

24. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
30. ((Set(a) & Set(y)) & ((a,y) = (b,c))) -> ((a = b) & (y = c)) ForallElim 29
32. (Set(a) & Set(y)) & ((a,y) = (b,c)) AndInt 23 31
33. (a = b) & (y = c) ImpElim 32 30
39. (f'b) = y EqualitySub 37 38
41. (a,(f'b)) ϵ f EqualitySub 10 40
42. ∃a.((a,(f'b)) ϵ f) ExistsInt 41
44. ∃a.((a,y) ϵ f) ExistsInt 10
45. Set(y) & ∃a.((a,y) ϵ f) AndInt 43 44
46. y ϵ {w: ∃a.((a,w) ϵ f)} ClassInt 45
48. y ϵ rg(f) EqualitySub 46 47
49. (f'b) ϵ rg(f) EqualitySub 48 40
50. (f'b) ϵ rg(f) ExistsElim 15 16 49
52. (f'a) ϵ rg(f) EqualitySub 50 51
53. (f'a) ϵ rg(f) ExistsElim 15 16 52
56. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f)) ImpInt 55 Qed

Used Theorems

4. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
6. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th96. OP(f,r,s) -> (1-to-1(f) & OP($(f)^{-1}$,s,r))

0. OP(f,r,s) Hyp
1. (x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y)) Hyp
2. (FUN(f) & (WO(r,dom(f)) & WO(s,rg(f)))) & ∀u.∀v.((((u ϵ dom(f)) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (((f'u),(f'v)) ϵ s)) DefExp 0
3. (f'x) = (f'y) Hyp
7. Connects(r,dom(f)) & ∀y.(((y ⊂ dom(f)) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 6
9. ∀y.∀z.(((y ϵ dom(f)) & (z ϵ dom(f))) -> ((y = z) v (((y,z) ϵ r) v ((z,y) ϵ r)))) DefExp 8
10. ∀z.(((x ϵ dom(f)) & (z ϵ dom(f))) -> ((x = z) v (((x,z) ϵ r) v ((z,x) ϵ r)))) ForallElim 9
11. ((x ϵ dom(f)) & (y ϵ dom(f))) -> ((x = y) v (((x,y) ϵ r) v ((y,x) ϵ r))) ForallElim 10
15. (x ϵ dom(f)) & (y ϵ dom(f)) AndInt 12 14
16. (x = y) v (((x,y) ϵ r) v ((y,x) ϵ r)) ImpElim 15 11
18. x = y Hyp
19. _|_ ImpElim 18 17
20. ((x,y) ϵ r) v ((y,x) ϵ r) AbsI 19
21. ((x,y) ϵ r) v ((y,x) ϵ r) Hyp
22. ((x,y) ϵ r) v ((y,x) ϵ r) OrElim 16 18 20 21 21
24. ∀v.((((x ϵ dom(f)) & (v ϵ dom(f))) & ((x,v) ϵ r)) -> (((f'x),(f'v)) ϵ s)) ForallElim 23
25. (((x ϵ dom(f)) & (y ϵ dom(f))) & ((x,y) ϵ r)) -> (((f'x),(f'y)) ϵ s) ForallElim 24
28. ((x,y) ϵ r) v ((y,x) ϵ r) AbsI 19
29. ((x,y) ϵ r) v ((y,x) ϵ r) Hyp
30. ((x,y) ϵ r) v ((y,x) ϵ r) OrElim 16 18 28 29 29
31. (x,y) ϵ r Hyp
33. ((x ϵ dom(f)) & (y ϵ dom(f))) & ((x,y) ϵ r) AndInt 15 31
34. ((f'x),(f'y)) ϵ s ImpElim 33 25
36. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x)) TheoremInt
40. WO(s,rg(f)) -> (Asymmetric(s,rg(f)) & TransIn(s,rg(f))) ForallElim 39
41. Asymmetric(s,rg(f)) & TransIn(s,rg(f)) ImpElim 35 40
43. ∀y.∀z.(((y ϵ rg(f)) & (z ϵ rg(f))) -> (((y,z) ϵ s) -> ¬((z,y) ϵ s))) DefExp 42
44. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f)) TheoremInt
46. FUN(f) & (x ϵ dom(f)) AndInt 45 12
48. (FUN(f) & (x ϵ dom(f))) -> ((f'x) ϵ rg(f)) ForallElim 47
49. (f'x) ϵ rg(f) ImpElim 46 48
50. ∀z.((((f'x) ϵ rg(f)) & (z ϵ rg(f))) -> ((((f'x),z) ϵ s) -> ¬((z,(f'x)) ϵ s))) ForallElim 43
51. (((f'x) ϵ rg(f)) & ((f'x) ϵ rg(f))) -> ((((f'x),(f'x)) ϵ s) -> ¬(((f'x),(f'x)) ϵ s)) ForallElim 50
52. ((f'x) ϵ rg(f)) & ((f'x) ϵ rg(f)) AndInt 49 49
53. (((f'x),(f'x)) ϵ s) -> ¬(((f'x),(f'x)) ϵ s) ImpElim 52 51

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55. ((f'x),(f'x)) ϵ s  EqualitySub 34 54
56. ¬(((f'x),(f'x)) ϵ s)  ImpElim 55 53
57. _|_  ImpElim 55 56
58. (y,x) ϵ r  Hyp
59. ∀v.((((y ϵ dom(f)) & (v ϵ dom(f))) & ((y,v) ϵ r)) -> (((f'y),(f'v)) ϵ s))  ForallElim 23
60. (((y ϵ dom(f)) & (x ϵ dom(f))) & ((y,x) ϵ r)) -> (((f'y),(f'x)) ϵ s)  ForallElim 59
61. (y ϵ dom(f)) & (x ϵ dom(f))  AndInt 14 12
62. ((y ϵ dom(f)) & (x ϵ dom(f))) & ((y,x) ϵ r)  AndInt 61 58
63. ((f'y),(f'x)) ϵ s  ImpElim 62 60
64. ((f'x),(f'x)) ϵ s  EqualitySub 63 54
65. ¬(((f'x),(f'x)) ϵ s)  ImpElim 64 53
66. _|_  ImpElim 64 65
67. _|_  OrElim 30 31 57 58 66
68. ¬((f'x) = (f'y))  ImpInt 67
69. ((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))  ImpInt 68
70. ∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))  ForallInt 69
71. ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))  ForallInt 70
72. 1-to-1(f) <-> (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))  TheoremInt
75. FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))  AndInt 45 71
76. 1-to-1(f)  ImpElim 75 74
77. OP(f,r,s) -> 1-to-1(f)  ImpInt 76
78. (x ϵ dom(f)) & (y ϵ dom(f))  Hyp
79. ((f'x),(f'y)) ϵ s  Hyp
80. x = y  Hyp
81. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  TheoremInt
83. WO(s,x) -> (Asymmetric(s,x) & TransIn(s,x))  ForallElim 82
84. (FUN(f) & (WO(r,dom(f)) & WO(s,rg(f)))) & ∀u.∀v.((((u ϵ dom(f)) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> ((
(f'u),(f'v)) ϵ s))  DefExp 0
89. WO(s,rg(f)) -> (Asymmetric(s,rg(f)) & TransIn(s,rg(f)))  ForallElim 88
90. Asymmetric(s,rg(f)) & TransIn(s,rg(f))  ImpElim 87 89
92. ∀y.∀z.(((y ϵ rg(f)) & (z ϵ rg(f))) -> (((y,z) ϵ s) -> ¬((z,y) ϵ s)))  DefExp 91
93. ∀z.((((f'x) ϵ rg(f)) & (z ϵ rg(f))) -> ((((f'x),z) ϵ s) -> ¬((z,(f'x)) ϵ s)))  ForallElim 92
94. (((f'x) ϵ rg(f)) & ((f'y) ϵ rg(f))) -> ((((f'x),(f'y)) ϵ s) -> ¬(((f'y),(f'x)) ϵ s))  ForallElim 93
95. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))  TheoremInt
100. FUN(f) & (x ϵ dom(f))  AndInt 99 96
102. (FUN(f) & (x ϵ dom(f))) -> ((f'x) ϵ rg(f))  ForallElim 101
103. (f'x) ϵ rg(f)  ImpElim 100 102
105. (((f'x) ϵ rg(f)) & ((f'x) ϵ rg(f))) -> ((((f'x),(f'x)) ϵ s) -> ¬(((f'x),(f'x)) ϵ s))  EqualitySub 94 104
106. ((f'x) ϵ rg(f)) & ((f'x) ϵ rg(f))  AndInt 103 103
107. (((f'x),(f'x)) ϵ s) -> ¬(((f'x),(f'x)) ϵ s)  ImpElim 106 105
108. ((f'x),(f'x)) ϵ s  EqualitySub 79 104
109. ¬(((f'x),(f'x)) ϵ s)  ImpElim 108 107
110. _|_  ImpElim 108 109
111. ¬(x = y)  ImpInt 110
113. Connects(r,dom(f)) & ∀y.(((y ⊂ dom(f)) & ¬(y = 0)) -> ∃z.First(r,y,z))  DefExp 112
115. ∀y.∀z.(((y ϵ dom(f)) & (z ϵ dom(f))) -> ((y = z) v (((y,z) ϵ r) v ((z,y) ϵ r))))  DefExp 114
116. ∀z.(((x ϵ dom(f)) & (z ϵ dom(f))) -> ((x = z) v (((x,z) ϵ r) v ((z,x) ϵ r))))  ForallElim 115
117. ((x ϵ dom(f)) & (y ϵ dom(f))) -> ((x = y) v (((x,y) ϵ r) v ((y,x) ϵ r)))  ForallElim 116
118. (x ϵ dom(f)) & (y ϵ dom(f))  AndInt 96 97
119. (x = y) v (((x,y) ϵ r) v ((y,x) ϵ r))  ImpElim 118 117
120. x = y  Hyp
121. _|_  ImpElim 120 111
122. ((x,y) ϵ r) v ((y,x) ϵ r)  AbsI 121
123. ((x,y) ϵ r) v ((y,x) ϵ r)  Hyp
124. ((x,y) ϵ r) v ((y,x) ϵ r)  OrElim 119 120 122 123 123
125. (x,y) ϵ r  Hyp
126. (y,x) ϵ r  Hyp
128. ∀v.((((y ϵ dom(f)) & (v ϵ dom(f))) & ((y,v) ϵ r)) -> (((f'y),(f'v)) ϵ s))  ForallElim 127
129. (((y ϵ dom(f)) & (x ϵ dom(f))) & ((y,x) ϵ r)) -> (((f'y),(f'x)) ϵ s)  ForallElim 128
130. (y ϵ dom(f)) & (x ϵ dom(f))  AndInt 97 96
131. ((y ϵ dom(f)) & (x ϵ dom(f))) & ((y,x) ϵ r)  AndInt 130 126
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132. ((f'y),(f'x)) ϵ s ImpElim 131 129

134. (FUN(f) & (y ϵ dom(f))) -> ((f'y) ϵ rg(f)) ForallElim 133

135. FUN(f) & (y ϵ dom(f)) AndInt 99 97

136. (f'y) ϵ rg(f) ImpElim 135 134

137. ((f'y) ϵ rg(f)) & ((f'x) ϵ rg(f)) AndInt 136 103

138. ∀z.((((f'y) ϵ rg(f)) & (z ϵ rg(f))) -> ((((f'y),z) ϵ s) -> ¬((z,(f'y)) ϵ s))) ForallElim 92

139. (((f'y) ϵ rg(f)) & ((f'x) ϵ rg(f))) -> ((((f'y),(f'x)) ϵ s) -> ¬(((f'x),(f'y)) ϵ s)) ForallElim 138

140. (((f'y),(f'x)) ϵ s) -> ¬(((f'x),(f'y)) ϵ s) ImpElim 137 139

141. ¬(((f'x),(f'y)) ϵ s) ImpElim 132 140

142. _|_ ImpElim 79 141

143. (x,y) ϵ r AbsI 142

144. (x,y) ϵ r OrElim 124 125 125 126 143

145. (((f'x),(f'y)) ϵ s) -> ((x,y) ϵ r) ImpInt 144

146. ((x ϵ dom(f)) & (y ϵ dom(f))) -> ((((f'x),(f'y)) ϵ s) -> ((x,y) ϵ r)) ImpInt 145

147. FUN(f) & FUN((f)$^{-1}$) DefExp 76

149. a ϵ dom((f)$^{-1}$) Hyp

152. dom((f)$^{-1}$) = {x: ∃y.((x,y) ϵ (f)$^{-1}$)} ForallElim 151

153. a ϵ {x: ∃y.((x,y) ϵ (f)$^{-1}$)} EqualitySub 149 152

154. Set(a) & ∃y.((a,y) ϵ (f)$^{-1}$) ClassElim 153

156. (a,b) ϵ (f)$^{-1}$ Hyp

159. (f)$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} ForallElim 158

160. (a,b) ϵ {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} EqualitySub 156 159

161. Set((a,b)) & ∃x.∃y.(((x,y) ϵ f) & ((a,b) = (y,x))) ClassElim 160

163. ∃y.(((x1,y) ϵ f) & ((a,b) = (y,x1))) Hyp

164. ((x1,y1) ϵ f) & ((a,b) = (y1,x1)) Hyp

166. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt

173. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 172

174. Set(a) & Set(b) ImpElim 165 173

175. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt

183. ((Set(a) & Set(b)) & ((a,b) = (u,x1))) -> ((a = u) & (b = x1)) ForallElim 182

185. (Set(a) & Set(b)) & ((a,b) = (y1,x1)) AndInt 174 184

187. ((Set(a) & Set(b)) & ((a,b) = (y1,x1))) -> ((a = y1) & (b = x1)) ForallElim 186

188. (a = y1) & (b = x1) ImpElim 185 187

194. (b,y1) ϵ f EqualitySub 193 192

195. (b,a) ϵ f EqualitySub 194 191

196. ∃b.((b,a) ϵ f) ExistsInt 195

198. Set(a) & ∃b.((b,a) ϵ f) AndInt 197 196

199. a ϵ {w: ∃b.((b,w) ϵ f)} ClassInt 198

202. a ϵ rg(f) EqualitySub 199 201

203. a ϵ rg(f) ExistsElim 163 164 202

206. (a ϵ dom((f)$^{-1}$)) -> (a ϵ rg(f)) ImpInt 205

207. a ϵ rg(f) Hyp

208. a ϵ {y: ∃x.((x,y) ϵ f)} EqualitySub 207 200

209. Set(a) & ∃x.((x,a) ϵ f) ClassElim 208

211. (b,a) ϵ f Hyp

213. ((b,a) ϵ f) & ((a,b) = (a,b)) AndInt 211 212

214. ∃w.(((b,a) ϵ f) & (w = (a,b))) ExistsInt 213

215. ((b,a) ϵ f) & (w = (a,b)) Hyp

217. ∃b.∃a.(((b,a) ϵ f) & (w = (a,b))) ExistsInt 216

219. ∃f.((b,a) ϵ f) ExistsInt 211

220. Set((b,a)) DefSub 219

224. Set((b,a)) -> (Set(b) & Set(a)) ForallElim 223

225. Set(b) & Set(a) ImpElim 220 224

228. Set(a) & Set(b) AndInt 227 226

234. (Set(a) & Set(b)) -> Set((a,b)) ForallElim 233

235. Set((a,b)) ImpElim 228 234

237. Set(w) EqualitySub 235 236

238. Set(w) & ∃b.∃a.(((b,a) ϵ f) & (w = (a,b))) AndInt 237 217

239. w ϵ {w: ∃b.∃a.(((b,a) ϵ f) & (w = (a,b)))} ClassInt 238

240. (a,b) ϵ {w: ∃x_26.∃x_27.(((x_26,x_27) ϵ f) & (w = (x_27,x_26)))} EqualitySub 239 218

242. (a,b) ϵ (f)$^{-1}$ EqualitySub 240 241

243. ∃b.((a,b) ϵ (f)$^{-1}$) ExistsInt 242
244. Set(a) & ∃b.((a,b) ϵ (f)$^{-1}$) AndInt 227 243
245. a ϵ {w: ∃b.((w,b) ϵ (f)$^{-1}$)} ClassInt 244
247. a ϵ dom((f)$^{-1}$) EqualitySub 245 246
248. a ϵ dom((f)$^{-1}$) ExistsElim 214 215 247
250. (a ϵ rg(f)) -> (a ϵ dom((f)$^{-1}$)) ImpInt 249
251. ((a ϵ dom((f)$^{-1}$)) -> (a ϵ rg(f))) & ((a ϵ rg(f)) -> (a ϵ dom((f)$^{-1}$))) AndInt 206 250
253. ∀a.((a ϵ dom((f)$^{-1}$)) <-> (a ϵ rg(f))) ForallInt 252
254. ∀x.∀y.((x = y) <-> ∀z.((z ϵ x) <-> (z ϵ y))) AxInt
255. ∀y.((dom((f)$^{-1}$) = y) <-> ∀z.((z ϵ dom((f)$^{-1}$)) <-> (z ϵ y))) ForallElim 254
256. (dom((f)$^{-1}$) = rg(f)) <-> ∀z.((z ϵ dom((f)$^{-1}$)) <-> (z ϵ rg(f))) ForallElim 255
259. dom((f)$^{-1}$) = rg(f) ImpElim 253 258
261. Relation(f) & ∀x.∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z)) DefExp 260
263. Relation(r) -> (((r)$^{-1}$)$^{-1}$ = r) TheoremInt
265. Relation(f) -> (((f)$^{-1}$)$^{-1}$ = f) ForallElim 264
266. ((f)$^{-1}$)$^{-1}$ = f ImpElim 262 265
268. dom(((f)$^{-1}$)$^{-1}$) = rg((f)$^{-1}$) ForallElim 267
269. dom(f) = rg((f)$^{-1}$) EqualitySub 268 266
274. WO(r,rg((f)$^{-1}$)) EqualitySub 272 269
276. WO(s,dom((f)$^{-1}$)) EqualitySub 273 275
277. WO(s,dom((f)$^{-1}$)) & WO(r,rg((f)$^{-1}$)) AndInt 276 274
278. FUN((f)$^{-1}$) & (WO(s,dom((f)$^{-1}$)) & WO(r,rg((f)$^{-1}$))) AndInt 148 277
279. ((x ϵ dom((f)$^{-1}$)) & (y ϵ dom((f)$^{-1}$))) & ((x,y) ϵ s) Hyp
280. ((x ϵ rg(f)) & (y ϵ rg(f))) & ((x,y) ϵ s) EqualitySub 279 259
281. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
286. x ϵ {y: ∃x.((x,y) ϵ f)} EqualitySub 283 285
287. y ϵ {y: ∃x.((x,y) ϵ f)} EqualitySub 284 285
288. Set(x) & ∃x_29.((x_29,x) ϵ f) ClassElim 286
289. Set(y) & ∃x.((x,y) ϵ f) ClassElim 287
292. (a,x) ϵ f Hyp
293. (b,y) ϵ f Hyp
294. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 260 281
295. (a,x) ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 292 294
296. (b,y) ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 293 294
297. Set((a,x)) & ∃x_30.∃y.(((a,x) = (x_30,y)) & ((f'x_30) = y)) ClassElim 295
298. Set((b,y)) & ∃x.∃x_31.(((b,y) = (x,x_31)) & ((f'x) = x_31)) ClassElim 296
301. ∃y.(((a,x) = (x1,y)) & ((f'x1) = y)) Hyp
302. ((a,x) = (x1,y1)) & ((f'x1) = y1) Hyp
303. ∃x_31.(((b,y) = (x2,x_31)) & ((f'x2) = x_31)) Hyp
304. ((b,y) = (x2,y2)) & ((f'x2) = y2) Hyp
305. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
314. Set((b,y)) -> (Set(b) & Set(y)) ForallElim 313
317. Set(a) & Set(x) ImpElim 315 312
318. Set(b) & Set(y) ImpElim 316 314
319. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
331. ((Set(a) & Set(x)) & ((a,x) = (x1,y1))) -> ((a = x1) & (x = y1)) ForallElim 330
332. (Set(a) & Set(x)) & ((a,x) = (x1,y1)) AndInt 317 320
333. (a = x1) & (x = y1) ImpElim 332 331
339. ((Set(b) & Set(y)) & ((b,y) = (x2,y2))) -> ((b = x2) & (y = y2)) ForallElim 338
340. (Set(b) & Set(y)) & ((b,y) = (x2,y2)) AndInt 318 321
341. (b = x2) & (y = y2) ImpElim 340 339
352. (f'a) = y1 EqualitySub 346 348
353. (f'a) = x EqualitySub 352 350
354. (f'b) = y2 EqualitySub 347 349
355. (f'b) = y EqualitySub 354 351
360. ∃y.((b,y) ϵ f) ExistsInt 293
363. Set(a) & ∃x.((a,x) ϵ f) AndInt 361 359
364. Set(b) & ∃y.((b,y) ϵ f) AndInt 362 360
365. a ϵ {w: ∃x.((w,x) ϵ f)} ClassInt 363
366. b ϵ {w: ∃y.((w,y) ϵ f)} ClassInt 364
369. a ϵ dom(f) EqualitySub 365 368

370. b ϵ dom(f) EqualitySub 366 368
372. ((f'a),y) ϵ s EqualitySub 371 357
373. ((f'a),(f'b)) ϵ s EqualitySub 372 358
374. (a ϵ dom(f)) & (b ϵ dom(f)) AndInt 369 370
378. ((a ϵ dom(f)) & (b ϵ dom(f))) -> ((((f'a),(f'b)) ϵ s) -> ((a,b) ϵ r)) ForallElim 377
379. (((f'a),(f'b)) ϵ s) -> ((a,b) ϵ r) ImpElim 374 378
380. (a,b) ϵ r ImpElim 373 379
381. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
383. FUN((f)$^{-1}$) -> ((f)$^{-1}$ = {w: ∃x.∃y.((w = (x,y)) & (((f)$^{-1}$'x) = y))}) ForallElim 382
384. (f)$^{-1}$ = {w: ∃x.∃y.((w = (x,y)) & (((f)$^{-1}$'x) = y))} ImpElim 148 383
386. ((a,x) ϵ f) & ((x,a) = (x,a)) AndInt 292 385
388. ((b,y) ϵ f) & ((y,b) = (y,b)) AndInt 293 387
390. ∃v.(((b,y) ϵ f) & (v = (y,b))) ExistsInt 388
391. ((a,x) ϵ f) & (u = (x,a)) Hyp
392. ((b,y) ϵ f) & (v = (y,b)) Hyp
396. ∃b.∃y.(((b,y) ϵ f) & (v = (y,b))) ExistsInt 395
405. Set(x) & Set(a) AndInt 402 401
406. Set(y) & Set(b) AndInt 404 403
413. (Set(y) & Set(b)) -> Set((y,b)) ForallElim 412
414. Set((x,a)) ImpElim 405 409
415. Set((y,b)) ImpElim 406 413
416. Set(u) EqualitySub 414 399
417. Set(v) EqualitySub 415 400
418. Set(u) & ∃a.∃x.(((a,x) ϵ f) & (u = (x,a))) AndInt 416 394
419. Set(v) & ∃b.∃y.(((b,y) ϵ f) & (v = (y,b))) AndInt 417 396
420. u ϵ {w: ∃a.∃x.(((a,x) ϵ f) & (w = (x,a)))} ClassInt 418
421. v ϵ {w: ∃b.∃y.(((b,y) ϵ f) & (w = (y,b)))} ClassInt 419
424. (f)$^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} ForallElim 423
426. u ϵ (f)$^{-1}$ EqualitySub 420 425
427. v ϵ (f)$^{-1}$ EqualitySub 421 425
428. (x,a) ϵ (f)$^{-1}$ EqualitySub 426 397
429. (y,b) ϵ (f)$^{-1}$ EqualitySub 427 398
430. ((y,b) ϵ (f)$^{-1}$) & ((x,a) ϵ (f)$^{-1}$) AndInt 429 428
431. ((y,b) ϵ (f)$^{-1}$) & ((x,a) ϵ (f)$^{-1}$) ExistsElim 390 392 430
435. (y,b) ϵ {w: ∃x.∃y.((w = (x,y)) & (((f)$^{-1}$'x) = y))} EqualitySub 433 384
436. (x,a) ϵ {w: ∃x.∃y.((w = (x,y)) & (((f)$^{-1}$'x) = y))} EqualitySub 434 384
437. Set((y,b)) & ∃x.∃x_32.(((y,b) = (x,x_32)) & (((f)$^{-1}$'x) = x_32)) ClassElim 435
438. Set((x,a)) & ∃x_33.∃y.(((x,a) = (x_33,y)) & (((f)$^{-1}$'x_33) = y)) ClassElim 436
441. ∃x_32.(((y,b) = (n1,x_32)) & (((f)$^{-1}$'n1) = x_32)) Hyp
442. ((y,b) = (n1,n2)) & (((f)$^{-1}$'n1) = n2) Hyp
443. ∃y.(((x,a) = (n3,y)) & (((f)$^{-1}$'n3) = y)) Hyp
444. ((x,a) = (n3,n4)) & (((f)$^{-1}$'n3) = n4) Hyp
447. (Set(y) & Set(b)) & ((y,b) = (n1,n2)) AndInt 406 445
448. (Set(x) & Set(a)) & ((x,a) = (n3,n4)) AndInt 405 446
449. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
457. ((Set(y) & Set(b)) & ((y,b) = (n1,n2))) -> ((y = n1) & (b = n2)) ForallElim 456
458. (y = n1) & (b = n2) ImpElim 447 457
464. ((Set(x) & Set(a)) & ((x,a) = (n3,n4))) -> ((x = n3) & (a = n4)) ForallElim 463
465. (x = n3) & (a = n4) ImpElim 448 464
476. ((f)$^{-1}$'y) = n2 EqualitySub 470 472
477. ((f)$^{-1}$'y) = b EqualitySub 476 473
478. ((f)$^{-1}$'x) = n4 EqualitySub 471 474
479. ((f)$^{-1}$'x) = a EqualitySub 478 475
480. (((f)$^{-1}$'y) = b) & (((f)$^{-1}$'x) = a) AndInt 477 479
481. (((f)$^{-1}$'y) = b) & (((f)$^{-1}$'x) = a) ExistsElim 443 444 480
489. (a,((f)$^{-1}$'y)) ϵ r EqualitySub 380 487
490. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ϵ r EqualitySub 489 488
491. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ϵ r ExistsElim 303 304 490
497. (((x ϵ dom((f)$^{-1}$)) & (y ϵ dom((f)$^{-1}$))) & ((x,y) ϵ s)) -> ((((f)$^{-1}$'x),((f)$^{-1}$'y)) ϵ r) ImpInt 496
498. ∀y.((((x ϵ dom((f)$^{-1}$)) & (y ϵ dom((f)$^{-1}$))) & ((x,y) ϵ s)) -> ((((f)$^{-1}$'x),((f)$^{-1}$'y)) ϵ r)) ForallInt 497
499. ∀x.∀y.((((x ϵ dom((f)$^{-1}$)) & (y ϵ dom((f)$^{-1}$))) & ((x,y) ϵ s)) -> ((((f)$^{-1}$'x),((f)$^{-1}$'y)) ϵ r)) ForallInt 498

500. (FUN((f)$^{-1}$) & (WO(s,dom((f)$^{-1}$)) & WO(r,rg((f)$^{-1}$)))) & ∀x.∀y.((((x ϵ dom((f)$^{-1}$)) & (y ϵ dom((f)$^{-1}$))) & ((x,y) ϵ s)) -> ((((f)$^{-1}$'x),((f)$^{-1}$'y)) ϵ r)) AndInt 278 499
501. OP((f)$^{-1}$,s,r) DefSub 500
502. 1-to-1(f) & OP((f)$^{-1}$,s,r) AndInt 76 501
503. OP(f,r,s) -> (1-to-1(f) & OP((f)$^{-1}$,s,r)) ImpInt 502 Qed

Used Theorems

2. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
3. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))
4. 1-to-1(f) <-> (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
6. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
7. Relation(r) -> (((r)$^{-1}$)$^{-1}$ = r)
8. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

FunctionApp2. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)

0. FUN(f) & ((a,b) ϵ f) Hyp
1. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
3. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 2 1
5. (a,b) ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 4 3
6. Set((a,b)) & ∃x.∃y.(((a,b) = (x,y)) & ((f'x) = y)) ClassElim 5
9. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
16. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 15
17. Set(a) & Set(b) ImpElim 7 16
19. ∃y.(((a,b) = (u,y)) & ((f'u) = y)) Hyp
20. ((a,b) = (u,v)) & ((f'u) = v) Hyp
22. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
26. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)) ForallElim 25
27. (Set(a) & Set(b)) & ((a,b) = (u,v)) AndInt 17 21
28. (a = u) & (b = v) ImpElim 27 26
34. (f'a) = v EqualitySub 33 31
35. (f'a) = b EqualitySub 34 32
36. (f'a) = b ExistsElim 19 20 35
38. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b) ImpInt 37 Qed

Used Theorems

1. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

FunctionInvApp. (FUN(f) & (FUN((f)$^{-1}$) & (a ϵ dom(f)))) -> (((f'a) ϵ dom((f)$^{-1}$)) & (((f)$^{-1}$'(f'a)) = a))

0. FUN(f) & (FUN((f)$^{-1}$) & (a ϵ dom(f))) Hyp
2. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
3. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 1 2
4. s = (a,(f'a)) Hyp
6. (s = (a,(f'a))) & ((f'a) = (f'a)) AndInt 4 5
8. ∃v.∃u.((s = (v,u)) & ((f'v) = u)) ExistsInt 7
11. ∃w.(a ϵ w) ExistsInt 10
12. Set(a) DefSub 11
13. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f)) TheoremInt
14. FUN(f) & (a ϵ dom(f)) AndInt 1 10
15. (f'a) ϵ rg(f) ImpElim 14 13
16. ∃w.((f'a) ϵ w) ExistsInt 15
17. Set((f'a)) DefSub 16
18. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
25. (Set(a) & Set((f'a))) -> Set((a,(f'a))) ForallElim 24
26. Set(a) & Set((f'a)) AndInt 12 17

27. Set((a,(f'a))) ImpElim 26 25
29. Set(s) EqualitySub 27 28
30. Set(s) & ∃v.∃u.((s = (v,u)) & ((f'v) = u)) AndInt 29 8
31. s ϵ {w: ∃v.∃u.((w = (v,u)) & ((f'v) = u))} ClassInt 30
33. s ϵ f EqualitySub 31 32
34. (a,(f'a)) ϵ f EqualitySub 33 4
35. (s = (a,(f'a))) -> ((a,(f'a)) ϵ f) ImpInt 34
37. ((a,(f'a)) = (a,(f'a))) -> ((a,(f'a)) ϵ f) ForallElim 36
39. (a,(f'a)) ϵ f ImpElim 38 37
42. $(f)^{-1}$ = {z: ∃x.∃y.(((x,y) ϵ f) & (z = (y,x)))} ForallElim 41
44. ((a,(f'a)) ϵ f) & (((f'a),a) = ((f'a),a)) AndInt 39 43
45. ∃t.(((a,(f'a)) ϵ f) & (t = ((f'a),a))) ExistsInt 44
46. ((a,(f'a)) ϵ f) & (t = ((f'a),a)) Hyp
48. ∃v.∃u.(((v,u) ϵ f) & (t = (u,v))) ExistsInt 47
50. Set((f'a)) & Set(a) AndInt 17 12
54. (Set((f'a)) & Set(a)) -> Set(((f'a),a)) ForallElim 53
55. Set(((f'a),a)) ImpElim 50 54
57. Set(t) EqualitySub 55 56
58. Set(t) & ∃v.∃u.(((v,u) ϵ f) & (t = (u,v))) AndInt 57 48
59. t ϵ {w: ∃v.∃u.(((v,u) ϵ f) & (w = (u,v)))} ClassInt 58
61. t ϵ $(f)^{-1}$ EqualitySub 59 60
62. ((f'a),a) ϵ $(f)^{-1}$ EqualitySub 61 49
63. ((f'a),a) ϵ $(f)^{-1}$ ExistsElim 45 46 62
64. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b) TheoremInt
72. (FUN($(f)^{-1}$) & (((f'a),a) ϵ $(f)^{-1}$)) -> ((($(f)^{-1}$'(f'a)) = a) ForallElim 71
74. FUN($(f)^{-1}$) & (((f'a),a) ϵ $(f)^{-1}$) AndInt 73 63
75. (($(f)^{-1}$'(f'a)) = a ImpElim 74 72
76. (FUN(f) & (FUN($(f)^{-1}$) & (a ϵ dom(f)))) -> ((($(f)^{-1}$'(f'a)) = a) ImpInt 75
77. ∃w.(((f'a),w) ϵ $(f)^{-1}$) ExistsInt 63
78. x = (f'a) Hyp
80. Set(x) EqualitySub 17 79
81. ∃w.((x,w) ϵ $(f)^{-1}$) EqualitySub 77 79
82. Set(x) & ∃w.((x,w) ϵ $(f)^{-1}$) AndInt 80 81
83. x ϵ {w: ∃x_2.((w,x_2) ϵ $(f)^{-1}$)} ClassInt 82
87. {x: ∃y.((x,y) ϵ $(f)^{-1}$)} = dom($(f)^{-1}$) ForallElim 86
88. x ϵ dom($(f)^{-1}$) EqualitySub 83 87
89. (f'a) ϵ dom($(f)^{-1}$) EqualitySub 88 78
90. (x = (f'a)) -> ((f'a) ϵ dom($(f)^{-1}$)) ImpInt 89
92. ((f'a) = (f'a)) -> ((f'a) ϵ dom($(f)^{-1}$)) ForallElim 91
94. (f'a) ϵ dom($(f)^{-1}$) ImpElim 93 92
95. ((f'a) ϵ dom($(f)^{-1}$)) & ((($(f)^{-1}$'(f'a)) = a) AndInt 94 75
96. (FUN(f) & (FUN($(f)^{-1}$) & (a ϵ dom(f)))) -> (((f'a) ϵ dom($(f)^{-1}$)) & ((($(f)^{-1}$'(f'a)) = a)) ImpInt 95 Qed

Used Theorems

1. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
2. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
4. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)

FunctionDomRange. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))

0. (a,b) ϵ f Hyp
1. ∃w.((a,w) ϵ f) ExistsInt 0
4. ∃w.((w,b) ϵ f) ExistsInt 0
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
12. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 11
13. ∃w.((a,b) ϵ w) ExistsInt 0
14. Set((a,b)) DefSub 13
15. Set(a) & Set(b) ImpElim 14 12
18. Set(a) & ∃w.((a,w) ϵ f) AndInt 16 1

```
19. a ϵ {w: ∃h.((w,h) ϵ f)}  ClassInt 18
21. a ϵ dom(f)  EqualitySub 19 20
22. Set(b) & ∃w.((w,b) ϵ f)  AndInt 17 4
23. b ϵ {w: ∃i.((i,w) ϵ f)}  ClassInt 22
25. b ϵ rg(f)  EqualitySub 23 24
26. (a ϵ dom(f)) & (b ϵ rg(f))  AndInt 21 25
27. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))  ImpInt 26 Qed
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Used Theorems

```
1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
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FunctionPair. (FUN(f) & (x ϵ dom(f))) -> ((x,(f'x)) ϵ f)
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```
0. FUN(f) & (x ϵ dom(f))  Hyp
1. z = (x,(f'x))  Hyp
3. (z = (x,(f'x))) & ((f'x) = (f'x))  AndInt 1 2
5. ∃a.∃b.((z = (a,b)) & (b = (f'a)))  ExistsInt 4
7. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))  TheoremInt
9. (FUN(f) & (x ϵ dom(f))) -> ((f'x) ϵ rg(f))  ForallElim 8
10. (f'x) ϵ rg(f)  ImpElim 0 9
12. ∃w.((f'x) ϵ w)  ExistsInt 10
13. Set(x)  DefSub 11
14. Set((f'x))  DefSub 12
15. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
20. (Set(x) & Set((f'x))) -> Set((x,(f'x)))  ForallElim 19
21. Set(x) & Set((f'x))  AndInt 13 14
22. Set((x,(f'x)))  ImpElim 21 20
24. Set(z)  EqualitySub 22 23
25. Set(z) & ∃a.∃b.((z = (a,b)) & (b = (f'a)))  AndInt 24 5
26. z ϵ {w: ∃a.∃b.((w = (a,b)) & (b = (f'a)))}  ClassInt 25
27. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  TheoremInt
29. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ImpElim 28 27
32. ∃a.∃b.((z = (a,b)) & ((f'a) = b))  ExistsInt 31
33. Set(z) & ∃a.∃b.((z = (a,b)) & ((f'a) = b))  AndInt 24 32
34. z ϵ {w: ∃a.∃b.((w = (a,b)) & ((f'a) = b))}  ClassInt 33
35. z ϵ f  EqualitySub 34 30
36. (x,(f'x)) ϵ f  EqualitySub 35 1
37. (z = (x,(f'x))) -> ((x,(f'x)) ϵ f)  ImpInt 36
39. ((x,(f'x)) = (x,(f'x))) -> ((x,(f'x)) ϵ f)  ForallElim 38
41. (x,(f'x)) ϵ f  ImpElim 40 39
42. (FUN(f) & (x ϵ dom(f))) -> ((x,(f'x)) ϵ f)  ImpInt 41 Qed
```

Used Theorems

```
1. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
```

```
Th97. (OP(f,r,s) & (OP(g,r,s) & (Sec(r,x,dom(f)) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(f)) & Sec(s,y,rg(g))))))
) -> ((f ⊂ g) v (g ⊂ f))
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```
0. OP(f,r,s) & (OP(g,r,s) & (Sec(r,x,dom(f)) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(f)) & Sec(s,y,rg(g))))))
Hyp
1. (Sec(r,z,a) & Sec(r,z,b)) -> ((a ⊂ b) v (b ⊂ a))  TheoremInt
7. (Sec(r,x,dom(f)) & Sec(r,x,dom(g))) -> ((dom(f) ⊂ dom(g)) v (dom(g) ⊂ dom(f)))  ForallElim 6
13. Sec(r,x,dom(f)) & Sec(r,x,dom(g))  AndInt 10 12
14. (dom(f) ⊂ dom(g)) v (dom(g) ⊂ dom(f))  ImpElim 13 7
15. dom(f) ⊂ dom(g)  Hyp
16. class = {z: ((z ϵ dom(f)) & ((z ϵ dom(g)) & ¬((g'z) = (f'z))))}  Hyp
21. ((dom(f) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (u ϵ dom(f)))  DefExp 20
```

```
24. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  DefExp 23
27. ((class ⊂ x) & ¬(class = 0)) -> ∃z.First(r,class,z)  ForallElim 26
28. a ϵ class  Hyp
29. a ϵ {z: ((z ϵ dom(f)) & ((z ϵ dom(g)) & ¬((g'z) = (f'z))))}  EqualitySub 28 16
30. Set(a) & ((a ϵ dom(f)) & ((a ϵ dom(g)) & ¬((g'a) = (f'a))))  ClassElim 29
33. ∀z.((z ϵ dom(f)) -> (z ϵ x))  DefExp 25
34. (a ϵ dom(f)) -> (a ϵ x)  ForallElim 33
35. a ϵ x  ImpElim 32 34
36. (a ϵ class) -> (a ϵ x)  ImpInt 35
37. ∀a.((a ϵ class) -> (a ϵ x))  ForallInt 36
38. class ⊂ x  DefSub 37
39. ¬(class = 0)  Hyp
40. (class ⊂ x) & ¬(class = 0)  AndInt 38 39
41. ∃z.First(r,class,z)  ImpElim 40 27
42. First(r,class,u)  Hyp
43. (u ϵ class) & ∀y.((y ϵ class) -> ¬((y,u) ϵ r))  DefExp 42
45. u ϵ {z: ((z ϵ dom(f)) & ((z ϵ dom(g)) & ¬((g'z) = (f'z))))}  EqualitySub 44 16
46. Set(u) & ((u ϵ dom(f)) & ((u ϵ dom(g)) & ¬((g'u) = (f'u))))  ClassElim 45
55. ((rg(f) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(f))) & ((u,v) ϵ s)) -> (u ϵ rg(f)))  DefExp 54
58. Connects(s,y) & ∀x_34.(((x_34 ⊂ y) & ¬(x_34 = 0)) -> ∃z.First(s,x_34,z))  DefExp 57
60. ∀x_38.∀z.(((x_38 ϵ y) & (z ϵ y)) -> ((x_38 = z) v (((x_38,z) ϵ s) v ((z,x_38) ϵ s))))  DefExp 59
61. ∀z.((((g'u) ϵ y) & (z ϵ y)) -> (((g'u) = z) v ((((g'u),z) ϵ s) v ((z,(g'u)) ϵ s))))  ForallElim 60
62. (((g'u) ϵ y) & ((f'u) ϵ y)) -> (((g'u) = (f'u)) v ((((g'u),(f'u)) ϵ s) v (((f'u),(g'u)) ϵ s)))  ForallElim 61
64. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))  TheoremInt
65. (FUN(f) & (WO(r,dom(f)) & WO(s,rg(f)))) & ∀u.∀v.((((u ϵ dom(f)) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> ((
(f'u),(f'v)) ϵ s))  DefExp 17
69. (FUN(f) & (u ϵ dom(f))) -> ((f'u) ϵ rg(f))  ForallElim 68
72. FUN(f) & (u ϵ dom(f))  AndInt 67 71
73. (f'u) ϵ rg(f)  ImpElim 72 69
75. (FUN(g) & (u ϵ dom(g))) -> ((g'u) ϵ rg(g))  ForallElim 74
77. (FUN(g) & (WO(r,dom(g)) & WO(s,rg(g)))) & ∀u.∀v.((((u ϵ dom(g)) & (v ϵ dom(g))) & ((u,v) ϵ r)) -> ((
(g'u),(g'v)) ϵ s))  DefExp 76
80. FUN(g) & (u ϵ dom(g))  AndInt 79 70
81. (g'u) ϵ rg(g)  ImpElim 80 75
83. ((rg(g) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(g))) & ((u,v) ϵ s)) -> (u ϵ rg(g)))  DefExp 82
86. ∀z.((z ϵ rg(f)) -> (z ϵ y))  DefExp 63
87. ∀z.((z ϵ rg(g)) -> (z ϵ y))  DefExp 85
88. ((f'u) ϵ rg(f)) -> ((f'u) ϵ y)  ForallElim 86
89. ((g'u) ϵ rg(g)) -> ((g'u) ϵ y)  ForallElim 87
90. (f'u) ϵ y  ImpElim 73 88
91. (g'u) ϵ y  ImpElim 81 89
92. ((g'u) ϵ y) & ((f'u) ϵ y)  AndInt 91 90
93. ((g'u) = (f'u)) v ((((g'u),(f'u)) ϵ s) v (((f'u),(g'u)) ϵ s))  ImpElim 92 62
94. (g'u) = (f'u)  Hyp
95. _|_  ImpElim 94 49
96. (((g'u),(f'u)) ϵ s) v (((f'u),(g'u)) ϵ s)  AbsI 95
97. (((g'u),(f'u)) ϵ s) v (((f'u),(g'u)) ϵ s)  Hyp
98. (((g'u),(f'u)) ϵ s) v (((f'u),(g'u)) ϵ s)  OrElim 93 94 96 97 97
99. ((f'u),(g'u)) ϵ s  Hyp
103. ((rg(g) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(g))) & ((u,v) ϵ s)) -> (u ϵ rg(g)))  DefExp 102
105. ∀v.(((((f'u) ϵ y) & (v ϵ rg(g))) & (((f'u),v) ϵ s)) -> ((f'u) ϵ rg(g)))  ForallElim 104
106. ((((f'u) ϵ y) & ((g'u) ϵ rg(g))) & (((f'u),(g'u)) ϵ s)) -> ((f'u) ϵ rg(g))  ForallElim 105
107. ((f'u) ϵ y) & ((g'u) ϵ rg(g))  AndInt 90 81
108. (((f'u) ϵ y) & ((g'u) ϵ rg(g))) & (((f'u),(g'u)) ϵ s)  AndInt 107 99
109. (f'u) ϵ rg(g)  ImpElim 108 106
112. rg(g) = {y: ∃x.((x,y) ϵ g)}  ForallElim 111
113. (f'u) ϵ {y: ∃x.((x,y) ϵ g)}  EqualitySub 109 112
114. Set((f'u)) & ∃x.((x,(f'u)) ϵ g)  ClassElim 113
116. (v,(f'u)) ϵ g  Hyp
117. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)  TheoremInt
123. (FUN(g) & ((v,(f'u)) ϵ g)) -> ((g'v) = (f'u))  ForallElim 122
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124. FUN(g) & ((v,(f'u)) ϵ g) AndInt 79 116
125. (g'v) = (f'u) ImpElim 124 123
127. ((g'v),(g'u)) ϵ s EqualitySub 99 126
128. OP(f,r,s) -> (1-to-1(f) & OP((f)$^{-1}$,s,r)) TheoremInt
130. OP(g,r,s) -> (1-to-1(g) & OP((g)$^{-1}$,s,r)) ForallElim 129
133. 1-to-1(g) & OP((g)$^{-1}$,s,r) ImpElim 132 130
135. (FUN((g)$^{-1}$) & (WO(s,dom((g)$^{-1}$)) & WO(r,rg((g)$^{-1}$)))) & ∀u.∀v.((((u ϵ dom((g)$^{-1}$)) & (v ϵ dom((g)$^{-1}$)))
& ((u,v) ϵ s)) -> ((((g)$^{-1}$'u),((g)$^{-1}$'v)) ϵ r)) DefExp 134
136. (FUN(f) & (FUN((f)$^{-1}$) & (a ϵ dom(f)))) -> (((f'a) ϵ dom((f)$^{-1}$)) & (((f)$^{-1}$'(f'a)) = a)) TheoremInt
140. (FUN(g) & (FUN((g)$^{-1}$) & (u ϵ dom(g)))) -> (((g'u) ϵ dom((g)$^{-1}$)) & (((g)$^{-1}$'(g'u)) = u)) ForallElim 139
145. FUN((g)$^{-1}$) & (u ϵ dom(g)) AndInt 144 141
146. FUN(g) & (FUN((g)$^{-1}$) & (u ϵ dom(g))) AndInt 142 145
147. ((g'u) ϵ dom((g)$^{-1}$)) & (((g)$^{-1}$'(g'u)) = u) ImpElim 146 140
150. ∃w.((v,(f'u)) ϵ w) ExistsInt 116
151. Set((v,(f'u))) DefSub 150
152. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
159. Set((v,(f'u))) -> (Set(v) & Set((f'u))) ForallElim 158
160. Set(v) & Set((f'u)) ImpElim 151 159
162. Set(v) & ∃w.((v,w) ϵ g) AndInt 161 149
163. v ϵ {w: ∃x_59.((w,x_59) ϵ g)} ClassInt 162
166. dom(g) = {x: ∃y.((x,y) ϵ g)} ForallElim 165
168. v ϵ dom(g) EqualitySub 163 167
170. (FUN(g) & (FUN((g)$^{-1}$) & (v ϵ dom(g)))) -> (((g'v) ϵ dom((g)$^{-1}$)) & (((g)$^{-1}$'(g'v)) = v)) ForallElim 169
171. FUN((g)$^{-1}$) & (v ϵ dom(g)) AndInt 144 168
172. FUN(g) & (FUN((g)$^{-1}$) & (v ϵ dom(g))) AndInt 142 171
173. ((g'v) ϵ dom((g)$^{-1}$)) & (((g)$^{-1}$'(g'v)) = v) ImpElim 172 170
175. ((g'u) ϵ dom((g)$^{-1}$)) & ((g'v) ϵ dom((g)$^{-1}$)) AndInt 148 174
177. ∀x_60.(((((g'v) ϵ dom((g)$^{-1}$)) & (x_60 ϵ dom((g)$^{-1}$))) & (((g'v),x_60) ϵ s)) -> (((((g)$^{-1}$'(g'v)),((g)$^{-1}$'x_60))
178. ((((g'v) ϵ dom((g)$^{-1}$)) & ((g'u) ϵ dom((g)$^{-1}$))) & (((g'v),(g'u)) ϵ s)) -> (((((g)$^{-1}$'(g'v)),((g)$^{-1}$'(g'u
))) ϵ r) ForallElim 177
179. ((g'v) ϵ dom((g)$^{-1}$)) & ((g'u) ϵ dom((g)$^{-1}$)) AndInt 174 148
180. (((g'v) ϵ dom((g)$^{-1}$)) & ((g'u) ϵ dom((g)$^{-1}$))) & (((g'v),(g'u)) ϵ s) AndInt 179 127
181. ((((g)$^{-1}$'(g'v)),((g)$^{-1}$'(g'u))) ϵ r ImpElim 180 178
184. (v,((g)$^{-1}$'(g'u))) ϵ r EqualitySub 181 182
185. (v,u) ϵ r EqualitySub 184 183
186. (u ϵ class) & ∀y.((y ϵ class) -> ¬((y,u) ϵ r)) DefExp 42
188. (v ϵ class) -> ¬((v,u) ϵ r) ForallElim 187
189. (A -> B) -> (¬B -> ¬A) TheoremInt
190. ((v ϵ class) -> B) -> (¬B -> ¬(v ϵ class)) PolySub 189
191. ((v ϵ class) -> ¬((v,u) ϵ r)) -> (¬¬((v,u) ϵ r) -> ¬(v ϵ class)) PolySub 190
192. ¬¬((v,u) ϵ r) -> ¬(v ϵ class) ImpElim 188 191
193. D <-> ¬¬D TheoremInt
196. ((v,u) ϵ r) -> ¬¬((v,u) ϵ r) PolySub 195
197. (v,u) ϵ r Hyp
198. ¬¬((v,u) ϵ r) ImpElim 197 196
199. ¬(v ϵ class) ImpElim 198 192
200. ((v,u) ϵ r) -> ¬(v ϵ class) ImpInt 199
201. ¬(v ϵ class) ImpElim 185 200
203. ((dom(f) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (u ϵ dom(f))) DefExp 202
205. ∀x_67.((((v ϵ x) & (x_67 ϵ dom(f))) & ((v,x_67) ϵ r)) -> (v ϵ dom(f))) ForallElim 204
206. (((v ϵ x) & (u ϵ dom(f))) & ((v,u) ϵ r)) -> (v ϵ dom(f)) ForallElim 205
209. ((dom(g) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(g))) & ((u,v) ϵ r)) -> (u ϵ dom(g))) DefExp 208
212. ∀z.((z ϵ dom(g)) -> (z ϵ x)) DefExp 211
213. (v ϵ dom(g)) -> (v ϵ x) ForallElim 212
214. v ϵ x ImpElim 168 213
215. (v ϵ x) & (u ϵ dom(f)) AndInt 214 207
216. ((v ϵ x) & (u ϵ dom(f))) & ((v,u) ϵ r) AndInt 215 197
217. v ϵ dom(f) ImpElim 216 206
218. ¬((g'v) = (f'v)) Hyp
219. (v ϵ dom(g)) & ¬((g'v) = (f'v)) AndInt 168 218
220. (v ϵ dom(f)) & ((v ϵ dom(g)) & ¬((g'v) = (f'v))) AndInt 217 219

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221. Set(v) & ((v ϵ dom(f)) & ((v ϵ dom(g)) & ¬((g'v) = (f'v))))  AndInt 161 220
222. v ϵ {w: ((w ϵ dom(f)) & ((w ϵ dom(g)) & ¬((g'w) = (f'w))))}  ClassInt 221
224. v ϵ class  EqualitySub 222 223
225. _|_  ImpElim 224 201
226. ¬¬((g'v) = (f'v))  ImpInt 225
227. D <-> ¬¬D  TheoremInt
230. ¬¬((g'v) = (f'v)) -> ((g'v) = (f'v))  PolySub 229
231. (g'v) = (f'v)  ImpElim 226 230
232. (f'u) = (f'v)  EqualitySub 231 125
233. (FUN(f) & (WO(r,dom(f)) & WO(s,rg(f)))) & ∀u.∀v.((((u ϵ dom(f)) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> ((
(f'u),(f'v)) ϵ s))  DefExp 17
235. ∀x_76.((((v ϵ dom(f)) & (x_76 ϵ dom(f))) & ((v,x_76) ϵ r)) -> (((f'v),(f'x_76)) ϵ s))  ForallElim 234
236. (((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)) -> (((f'v),(f'u)) ϵ s)  ForallElim 235
237. (v ϵ dom(f)) & (u ϵ dom(f))  AndInt 217 207
238. ((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)  AndInt 237 185
239. ((f'v),(f'u)) ϵ s  ImpElim 238 236
240. ((f'v),(f'v)) ϵ s  EqualitySub 239 232
241. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  TheoremInt
245. WO(s,y) -> (Asymmetric(s,y) & TransIn(s,y))  ForallElim 244
249. ((rg(f) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(f))) & ((u,v) ϵ s)) -> (u ϵ rg(f)))  DefExp 248
252. Asymmetric(s,y) & TransIn(s,y)  ImpElim 251 245
254. ∀x_82.∀z.(((x_82 ϵ y) & (z ϵ y)) -> (((x_82,z) ϵ s) -> ¬((z,x_82) ϵ s)))  DefExp 253
255. ∀z.((((f'v) ϵ y) & (z ϵ y)) -> ((((f'v),z) ϵ s) -> ¬((z,(f'v)) ϵ s)))  ForallElim 254
256. (((f'v) ϵ y) & ((f'v) ϵ y)) -> ((((f'v),(f'v)) ϵ s) -> ¬(((f'v),(f'v)) ϵ s))  ForallElim 255
258. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))  TheoremInt
260. (FUN(f) & (v ϵ dom(f))) -> ((f'v) ϵ rg(f))  ForallElim 259
261. FUN(f) & (v ϵ dom(f))  AndInt 67 217
262. (f'v) ϵ rg(f)  ImpElim 261 260
263. ∀z.((z ϵ rg(f)) -> (z ϵ y))  DefExp 257
264. ((f'v) ϵ rg(f)) -> ((f'v) ϵ y)  ForallElim 263
265. (f'v) ϵ y  ImpElim 262 264
266. ((f'v) ϵ y) & ((f'v) ϵ y)  AndInt 265 265
267. (((f'v),(f'v)) ϵ s) -> ¬(((f'v),(f'v)) ϵ s)  ImpElim 266 256
268. ¬(((f'v),(f'v)) ϵ s)  ImpElim 240 267
269. _|_  ImpElim 240 268
270. ¬((v,u) ϵ r)  ImpInt 269
271. _|_  ImpElim 185 270
272. _|_  ExistsElim 115 116 271
273. ((g'u),(f'u)) ϵ s  Hyp
275. ((rg(f) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(f))) & ((u,v) ϵ s)) -> (u ϵ rg(f)))  DefExp 274
277. ∀v.(((((g'u) ϵ y) & (v ϵ rg(f))) & (((g'u),v) ϵ s)) -> ((g'u) ϵ rg(f)))  ForallElim 276
278. ((((g'u) ϵ y) & ((f'u) ϵ rg(f))) & (((g'u),(f'u)) ϵ s)) -> ((g'u) ϵ rg(f))  ForallElim 277
281. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))  TheoremInt
283. (FUN(f) & (u ϵ dom(f))) -> ((f'u) ϵ rg(f))  ForallElim 282
284. FUN(f) & (u ϵ dom(f))  AndInt 67 279
285. (f'u) ϵ rg(f)  ImpElim 284 283
287. (FUN(g) & (u ϵ dom(g))) -> ((g'u) ϵ rg(g))  ForallElim 286
288. FUN(g) & (u ϵ dom(g))  AndInt 79 280
289. (g'u) ϵ rg(g)  ImpElim 288 287
290. ∀z.((z ϵ rg(g)) -> (z ϵ y))  DefExp 85
291. ((g'u) ϵ rg(g)) -> ((g'u) ϵ y)  ForallElim 290
292. (g'u) ϵ y  ImpElim 289 291
293. ((g'u) ϵ y) & ((f'u) ϵ rg(f))  AndInt 292 285
294. (((g'u) ϵ y) & ((f'u) ϵ rg(f))) & (((g'u),(f'u)) ϵ s)  AndInt 293 273
295. (g'u) ϵ rg(f)  ImpElim 294 278
297. (g'u) ϵ {y: ∃x.((x,y) ϵ f)}  EqualitySub 295 296
298. Set((g'u)) & ∃x.((x,(g'u)) ϵ f)  ClassElim 297
300. (v,(g'u)) ϵ f  Hyp
301. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)  TheoremInt
305. (FUN(f) & ((v,(g'u)) ϵ f)) -> ((f'v) = (g'u))  ForallElim 304
306. FUN(f) & ((v,(g'u)) ϵ f)  AndInt 67 300
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307. (f'v) = (g'u) ImpElim 306 305
309. ((f'v),(f'u)) ϵ s EqualitySub 273 308
311. OP(f,r,s) -> (1-to-1(f) & OP((f)$^{-1}$,s,r)) TheoremInt
312. 1-to-1(f) & OP((f)$^{-1}$,s,r) ImpElim 310 311
314. (FUN((f)$^{-1}$) & (WO(s,dom((f)$^{-1}$)) & WO(r,rg((f)$^{-1}$)))) & ∀u.∀v.((((u ϵ dom((f)$^{-1}$)) & (v ϵ dom((f)$^{-1}$))) & ((u,v) ϵ s)) -> ((((f)$^{-1}$'u),((f)$^{-1}$'v)) ϵ r)) DefExp 313
316. ∀x_93.(((((f'v) ϵ dom((f)$^{-1}$)) & (x_93 ϵ dom((f)$^{-1}$))) & (((f'v),x_93) ϵ s)) -> ((((f)$^{-1}$'(f'v)),((f)$^{-1}$'x_93))
317. ((((f'v) ϵ dom((f)$^{-1}$)) & ((f'u) ϵ dom((f)$^{-1}$))) & (((f'v),(f'u)) ϵ s)) -> ((((f)$^{-1}$'(f'v)),((f)$^{-1}$'(f'u))) ϵ r) ForallElim 316
319. ∃w.((v,(g'u)) ϵ w) ExistsInt 300
320. Set((v,(g'u))) DefSub 319
321. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
328. Set((v,(g'u))) -> (Set(v) & Set((g'u))) ForallElim 327
329. Set(v) & Set((g'u)) ImpElim 320 328
331. Set(v) & ∃w.((v,w) ϵ f) AndInt 330 318
332. v ϵ {w: ∃x_95.((w,x_95) ϵ f)} ClassInt 331
335. v ϵ dom(f) EqualitySub 332 334
337. (FUN(f) & (v ϵ dom(f))) -> ((f'v) ϵ rg(f)) ForallElim 336
338. FUN(f) & (v ϵ dom(f)) AndInt 67 335
339. (f'v) ϵ rg(f) ImpElim 338 337
340. ((f'u) ϵ rg(f)) & ((f'v) ϵ rg(f)) AndInt 285 339
341. (FUN(f) & (FUN((f)$^{-1}$) & (a ϵ dom(f)))) -> (((f'a) ϵ dom((f)$^{-1}$)) & (((f)$^{-1}$'(f'a)) = a)) TheoremInt
343. (FUN((f)$^{-1}$) & (WO(s,dom((f)$^{-1}$)) & WO(r,rg((f)$^{-1}$)))) & ∀u.∀v.((((u ϵ dom((f)$^{-1}$)) & (v ϵ dom((f)$^{-1}$))) & ((u,v) ϵ s)) -> ((((f)$^{-1}$'u),((f)$^{-1}$'v)) ϵ r)) DefExp 342
347. (FUN(f) & (FUN((f)$^{-1}$) & (v ϵ dom(f)))) -> (((f'v) ϵ dom((f)$^{-1}$)) & (((f)$^{-1}$'(f'v)) = v)) ForallElim 346
348. FUN((f)$^{-1}$) & (v ϵ dom(f)) AndInt 345 335
349. FUN(f) & (FUN((f)$^{-1}$) & (v ϵ dom(f))) AndInt 67 348
350. ((f'v) ϵ dom((f)$^{-1}$)) & (((f)$^{-1}$'(f'v)) = v) ImpElim 349 347
351. FUN((f)$^{-1}$) & (u ϵ dom(f)) AndInt 345 279
352. FUN(f) & (FUN((f)$^{-1}$) & (u ϵ dom(f))) AndInt 67 351
354. (FUN(f) & (FUN((f)$^{-1}$) & (u ϵ dom(f)))) -> (((f'u) ϵ dom((f)$^{-1}$)) & (((f)$^{-1}$'(f'u)) = u)) ForallElim 353
355. ((f'u) ϵ dom((f)$^{-1}$)) & (((f)$^{-1}$'(f'u)) = u) ImpElim 352 354
358. ((f'v) ϵ dom((f)$^{-1}$)) & ((f'u) ϵ dom((f)$^{-1}$)) AndInt 356 357
359. (((f'v) ϵ dom((f)$^{-1}$)) & ((f'u) ϵ dom((f)$^{-1}$))) & (((f'v),(f'u)) ϵ s) AndInt 358 309
360. (((f)$^{-1}$'(f'v)),((f)$^{-1}$'(f'u))) ϵ r ImpElim 359 317
363. (v,((f)$^{-1}$'(f'u))) ϵ r EqualitySub 360 361
364. (v,u) ϵ r EqualitySub 363 362
365. ¬(v ϵ class) ImpElim 364 200
366. ¬((g'v) = (f'v)) Hyp
367. (u ϵ dom(g)) & (v ϵ dom(f)) AndInt 280 335
372. ((dom(g) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(g))) & ((u,v) ϵ r)) -> (u ϵ dom(g))) DefExp 371
374. ∀x_102.((((v ϵ x) & (x_102 ϵ dom(g))) & ((v,x_102) ϵ r)) -> (v ϵ dom(g))) ForallElim 373
375. (((v ϵ x) & (u ϵ dom(g))) & ((v,u) ϵ r)) -> (v ϵ dom(g)) ForallElim 374
377. ((dom(f) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (u ϵ dom(f))) DefExp 376
380. ∀z.((z ϵ dom(f)) -> (z ϵ x)) DefExp 379
381. (v ϵ dom(f)) -> (v ϵ x) ForallElim 380
383. v ϵ x ImpElim 382 381
385. (v ϵ x) & (u ϵ dom(g)) AndInt 383 384
386. ((v ϵ x) & (u ϵ dom(g))) & ((v,u) ϵ r) AndInt 385 364
387. v ϵ dom(g) ImpElim 386 375
388. (v ϵ dom(g)) & ¬((g'v) = (f'v)) AndInt 387 366
389. (v ϵ dom(f)) & ((v ϵ dom(g)) & ¬((g'v) = (f'v))) AndInt 382 388
390. ∃w.(v ϵ w) ExistsInt 383
391. Set(v) DefSub 390
392. Set(v) & ((v ϵ dom(f)) & ((v ϵ dom(g)) & ¬((g'v) = (f'v)))) AndInt 391 389
393. v ϵ {w: ((w ϵ dom(f)) & ((w ϵ dom(g)) & ¬((g'w) = (f'w))))} ClassInt 392
395. v ϵ class EqualitySub 393 394
396. _|_ ImpElim 395 365
397. ¬¬((g'v) = (f'v)) ImpInt 396
398. ¬¬((g'v) = (f'v)) -> ((g'v) = (f'v)) PolySub 229
399. (g'v) = (f'v) ImpElim 397 398

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401. (g'u) = (g'v)  EqualitySub 308 400
402. 1-to-1(f) <-> (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))  TheoremInt
406. 1-to-1(g) -> (FUN(g) & ∀x.∀y.(((x ϵ dom(g)) & ((y ϵ dom(g)) & ¬(x = y))) -> ¬((g'x) = (g'y))))  ForallElim 4
407. OP(f,r,s) -> (1-to-1(f) & OP((f)^-1,s,r))  TheoremInt
409. OP(g,r,s) -> (1-to-1(g) & OP((g)^-1,s,r))  ForallElim 408
411. 1-to-1(g) & OP((g)^-1,s,r)  ImpElim 410 409
413. FUN(g) & ∀x.∀y.(((x ϵ dom(g)) & ((y ϵ dom(g)) & ¬(x = y))) -> ¬((g'x) = (g'y)))  ImpElim 412 406
415. ∀y.(((u ϵ dom(g)) & ((y ϵ dom(g)) & ¬(u = y))) -> ¬((g'u) = (g'y)))  ForallElim 414
416. ((u ϵ dom(g)) & ((v ϵ dom(g)) & ¬(u = v))) -> ¬((g'u) = (g'v))  ForallElim 415
417. (u ϵ dom(f)) & (u ϵ dom(g))  AndInt 279 280
418. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  TheoremInt
419. Asymmetric(r,x) & TransIn(r,x)  ImpElim 23 418
421. ∀y.∀z.(((y ϵ x) & (z ϵ x)) -> (((y,z) ϵ r) -> ¬((z,y) ϵ r)))  DefExp 420
422. ∀z.(((v ϵ x) & (z ϵ x)) -> (((v,z) ϵ r) -> ¬((z,v) ϵ r)))  ForallElim 421
423. ((v ϵ x) & (u ϵ x)) -> (((v,u) ϵ r) -> ¬((u,v) ϵ r))  ForallElim 422
424. (u ϵ dom(f)) -> (u ϵ x)  ForallElim 380
426. u ϵ x  ImpElim 425 424
427. (v ϵ x) & (u ϵ x)  AndInt 383 426
428. ((v,u) ϵ r) -> ¬((u,v) ϵ r)  ImpElim 427 423
429. ¬((u,v) ϵ r)  ImpElim 364 428
430. u = v  Hyp
431. (v,v) ϵ r  EqualitySub 364 430
432. ¬((v,v) ϵ r)  EqualitySub 429 430
433. _|_  ImpElim 431 432
434. ¬(u = v)  ImpInt 433
436. (v ϵ dom(g)) & ¬(u = v)  AndInt 387 434
437. (u ϵ dom(g)) & ((v ϵ dom(g)) & ¬(u = v))  AndInt 384 436
438. ¬((g'u) = (g'v))  ImpElim 437 416
439. _|_  ImpElim 401 438
440. _|_  ExistsElim 299 300 439
441. _|_  OrElim 98 273 440 99 272
442. _|_  ExistsElim 41 42 441
443. ¬¬(class = 0)  ImpInt 442
444. ¬¬(class = 0) -> (class = 0)  PolySub 229
445. class = 0  ImpElim 443 444
446. {z: ((z ϵ dom(f)) & ((z ϵ dom(g)) & ¬((g'z) = (f'z))))} = 0  EqualitySub 445 16
447. (class = {z: ((z ϵ dom(f)) & ((z ϵ dom(g)) & ¬((g'z) = (f'z))))}) -> ({z: ((z ϵ dom(f)) & ((z ϵ dom(
g)) & ¬((g'z) = (f'z))))} = 0)  ImpInt 446
449. ({z: ((z ϵ dom(f)) & ((z ϵ dom(g)) & ¬((g'z) = (f'z))))} = {x_111: ((x_111 ϵ dom(f)) & ((x_111 ϵ dom
(g)) & ¬((g'x_111) = (f'x_111))))}) -> ({x_111: ((x_111 ϵ dom(f)) & ((x_111 ϵ dom(g)) & ¬((g'x_111) =
(f'x_111))))} = 0)  ForallElim 448
451. {x_111: ((x_111 ϵ dom(f)) & ((x_111 ϵ dom(g)) & ¬((g'x_111) = (f'x_111))))} = 0  ImpElim 450 449
452. z ϵ f  Hyp
453. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  TheoremInt
454. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ImpElim 67 453
455. z ϵ {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  EqualitySub 452 454
456. Set(z) & ∃x.∃y.((z = (x,y)) & ((f'x) = y))  ClassElim 455
458. ∃y.((z = (a,y)) & ((f'a) = y))  Hyp
459. (z = (a,b)) & ((f'a) = b)  Hyp
460. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))  TheoremInt
462. (a,b) ϵ f  EqualitySub 452 461
463. (a ϵ dom(f)) & (b ϵ rg(f))  ImpElim 462 460
465. ∀z.((z ϵ dom(f)) -> (z ϵ dom(g)))  DefExp 15
466. (a ϵ dom(f)) -> (a ϵ dom(g))  ForallElim 465
467. a ϵ dom(g)  ImpElim 464 466
468. ¬((g'a) = (f'a))  Hyp
469. (a ϵ dom(g)) & ¬((g'a) = (f'a))  AndInt 467 468
470. (a ϵ dom(f)) & ((a ϵ dom(g)) & ¬((g'a) = (f'a)))  AndInt 464 469
471. ∃w.(a ϵ w)  ExistsInt 464
472. Set(a)  DefSub 471
473. Set(a) & ((a ϵ dom(f)) & ((a ϵ dom(g)) & ¬((g'a) = (f'a))))  AndInt 472 470
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474. a ∈ {w: ((w ∈ dom(f)) & ((w ∈ dom(g)) & ¬((g'w) = (f'w))))}  ClassInt 473
475. a ∈ 0  EqualitySub 474 451
477. a ∈ {x: ¬(x = x)}  EqualitySub 475 476
478. Set(a) & ¬(a = a)  ClassElim 477
481. _|_  ImpElim 480 479
482. ¬¬((g'a) = (f'a))  ImpInt 481
483. ¬¬((g'a) = (f'a)) -> ((g'a) = (f'a))  PolySub 229
484. (g'a) = (f'a)  ImpElim 482 483
488. b = (g'a)  EqualitySub 486 487
489. z = (a,(g'a))  EqualitySub 461 488
490. (FUN(f) & (x ∈ dom(f))) -> ((x,(f'x)) ∈ f)  TheoremInt
494. (FUN(g) & (a ∈ dom(g))) -> ((a,(g'a)) ∈ g)  ForallElim 493
495. FUN(g) & (a ∈ dom(g))  AndInt 79 467
496. (a,(g'a)) ∈ g  ImpElim 495 494
498. z ∈ g  EqualitySub 496 497
499. z ∈ g  ExistsElim 458 459 498
501. (z ∈ f) -> (z ∈ g)  ImpInt 500
502. ∀z.((z ∈ f) -> (z ∈ g))  ForallInt 501
503. f ⊂ g  DefSub 502
504. dom(g) ⊂ dom(f)  Hyp
505. z ∈ g  Hyp
507. FUN(g) -> (g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))})  ForallElim 506
508. g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  ImpElim 79 507
509. z ∈ {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  EqualitySub 505 508
510. Set(z) & ∃x.∃y.((z = (x,y)) & ((g'x) = y))  ClassElim 509
512. ∃y.((z = (a,y)) & ((g'a) = y))  Hyp
513. (z = (a,b)) & ((g'a) = b)  Hyp
515. (a,b) ∈ g  EqualitySub 505 514
517. ((a,b) ∈ g) -> ((a ∈ dom(g)) & (b ∈ rg(g)))  ForallElim 516
518. (a ∈ dom(g)) & (b ∈ rg(g))  ImpElim 515 517
519. ∀z.((z ∈ dom(g)) -> (z ∈ dom(f)))  DefExp 504
520. (a ∈ dom(g)) -> (a ∈ dom(f))  ForallElim 519
522. a ∈ dom(f)  ImpElim 521 520
523. ¬((g'a) = (f'a))  Hyp
524. (a ∈ dom(g)) & ¬((g'a) = (f'a))  AndInt 521 523
525. (a ∈ dom(f)) & ((a ∈ dom(g)) & ¬((g'a) = (f'a)))  AndInt 522 524
526. ∃w.(a ∈ w)  ExistsInt 521
527. Set(a)  DefSub 526
528. Set(a) & ((a ∈ dom(f)) & ((a ∈ dom(g)) & ¬((g'a) = (f'a))))  AndInt 527 525
529. a ∈ {w: ((w ∈ dom(f)) & ((w ∈ dom(g)) & ¬((g'w) = (f'w))))}  ClassInt 528
530. a ∈ 0  EqualitySub 529 451
531. a ∈ {x: ¬(x = x)}  EqualitySub 530 476
532. Set(a) & ¬(a = a)  ClassElim 531
535. _|_  ImpElim 534 533
536. ¬¬((g'a) = (f'a))  ImpInt 535
537. (g'a) = (f'a)  ImpElim 536 483
540. b = (f'a)  EqualitySub 539 537
541. z = (a,(f'a))  EqualitySub 514 540
542. (FUN(f) & (x ∈ dom(f))) -> ((x,(f'x)) ∈ f)  TheoremInt
544. (FUN(f) & (a ∈ dom(f))) -> ((a,(f'a)) ∈ f)  ForallElim 543
545. FUN(f) & (a ∈ dom(f))  AndInt 67 522
546. (a,(f'a)) ∈ f  ImpElim 545 544
548. z ∈ f  EqualitySub 546 547
549. z ∈ f  ExistsElim 512 513 548
551. (z ∈ g) -> (z ∈ f)  ImpInt 550
552. ∀z.((z ∈ g) -> (z ∈ f))  ForallInt 551
553. g ⊂ f  DefSub 552
554. (f ⊂ g) v (g ⊂ f)  OrIntR 503
555. (f ⊂ g) v (g ⊂ f)  OrIntL 553
556. (f ⊂ g) v (g ⊂ f)  OrElim 14 15 554 504 555
557. (OP(f,r,s) & (OP(g,r,s) & (Sec(r,x,dom(f)) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(f)) & Sec(s,y,rg(g))))))
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) -> ((f ⊂ g) v (g ⊂ f)) ImpInt 556 Qed

Used Theorems

1. (Sec(r,z,a) & Sec(r,z,b)) -> ((a ⊂ b) v (b ⊂ a))
2. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))
3. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)
4. OP(f,r,s) -> (1-to-1(f) & OP((f)$^{-1}$,s,r))
5. (FUN(f) & (FUN((f)$^{-1}$) & (a ϵ dom(f)))) -> (((f'a) ϵ dom((f)$^{-1}$)) & (((f)$^{-1}$'(f'a)) = a))
6. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
8. (A -> B) -> (¬B -> ¬A)
9. D <-> ¬¬D
10. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
11. (FUN(f) & (a ϵ dom(f))) -> ((f'a) ϵ rg(f))
12. 1-to-1(f) <-> (FUN(f) & ∀x.∀y.(((x ϵ dom(f)) & ((y ϵ dom(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))
13. FUN(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
14. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))
15. (FUN(f) & (x ϵ dom(f))) -> ((x,(f'x)) ϵ f)

PairEquals. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))

0. Set((a,b)) & ((a,b) = (x,y)) Hyp
1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
9. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 8
10. Set(a) & Set(b) ImpElim 5 9
11. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
19. ((Set(a) & Set(b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) ForallElim 18
21. (Set(a) & Set(b)) & ((a,b) = (x,y)) AndInt 10 20
22. (a = x) & (b = y) ImpElim 21 19
23. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) ImpInt 22 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

WellOrdersSubset. (WO(r,a) & (b ⊂ a)) -> WO(r,b)

0. WO(r,a) & (b ⊂ a) Hyp
1. (x ϵ b) & (y ϵ b) Hyp
3. ∀z.((z ϵ b) -> (z ϵ a)) DefExp 2
4. (x ϵ b) -> (x ϵ a) ForallElim 3
5. (y ϵ b) -> (y ϵ a) ForallElim 3
8. x ϵ a ImpElim 6 4
9. y ϵ a ImpElim 7 5
11. Connects(r,a) & ∀y.(((y ⊂ a) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 10
13. ∀y.∀z.(((y ϵ a) & (z ϵ a)) -> ((y = z) v (((y,z) ϵ r) v ((z,y) ϵ r)))) DefExp 12
14. ∀z.(((x ϵ a) & (z ϵ a)) -> ((x = z) v (((x,z) ϵ r) v ((z,x) ϵ r)))) ForallElim 13
15. ((x ϵ a) & (y ϵ a)) -> ((x = y) v (((x,y) ϵ r) v ((y,x) ϵ r))) ForallElim 14
16. (x ϵ a) & (y ϵ a) AndInt 8 9
17. (x = y) v (((x,y) ϵ r) v ((y,x) ϵ r)) ImpElim 16 15
18. ((x ϵ b) & (y ϵ b)) -> ((x = y) v (((x,y) ϵ r) v ((y,x) ϵ r))) ImpInt 17
19. ∀y.(((x ϵ b) & (y ϵ b)) -> ((x = y) v (((x,y) ϵ r) v ((y,x) ϵ r)))) ForallInt 18
20. ∀x.∀y.(((x ϵ b) & (y ϵ b)) -> ((x = y) v (((x,y) ϵ r) v ((y,x) ϵ r)))) ForallInt 19
21. Connects(r,b) DefSub 20
22. (y ⊂ b) & ¬(y = 0) Hyp
23. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z) TheoremInt
25. ((y ⊂ a) & ¬(y = 0)) -> ∃z.First(r,y,z) ForallElim 24
32. ((y ⊂ b) & (b ⊂ a)) -> (y ⊂ a) ForallElim 31
33. (y ⊂ b) & (b ⊂ a) AndInt 26 2
34. y ⊂ a ImpElim 33 32

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36. (y ⊂ a) & ¬(y = 0)  AndInt 34 35
37. ∃z.First(r,y,z)  ImpElim 36 25
38. ((y ⊂ b) & ¬(y = 0)) -> ∃z.First(r,y,z)  ImpInt 37
39. ∀y.(((y ⊂ b) & ¬(y = 0)) -> ∃z.First(r,y,z))  ForallInt 38
40. Connects(r,b) & ∀y.(((y ⊂ b) & ¬(y = 0)) -> ∃z.First(r,y,z))  AndInt 21 39
41. WO(r,b)  DefSub 40
42. (WO(r,a) & (b ⊂ a)) -> WO(r,b)  ImpInt 41 Qed

Used Theorems

1. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)

ContCompl. ((y ⊂ x) & ((x ~ y) = 0)) -> (x = y)

0. (y ⊂ x) & ((x ~ y) = 0)  Hyp
1. a ϵ x  Hyp
2. ¬(a ϵ y)  Hyp
3. ∃x.(a ϵ x)  ExistsInt 1
4. Set(a)  DefSub 3
5. Set(a) & ¬(a ϵ y)  AndInt 4 2
6. a ϵ {w: ¬(w ϵ y)}  ClassInt 5
9. ~y = {i: ¬(i ϵ y)}  ForallElim 8
11. a ϵ ~y  EqualitySub 6 10
12. (a ϵ x) & (a ϵ ~y)  AndInt 1 11
13. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
20. ((a ϵ x) & (a ϵ ~y)) -> (a ϵ (x ∩ ~y))  ForallElim 19
21. a ϵ (x ∩ ~y)  ImpElim 12 20
24. a ϵ (x ~ y)  EqualitySub 21 23
26. a ϵ 0  EqualitySub 24 25
28. a ϵ {x: ¬(x = x)}  EqualitySub 26 27
29. Set(a) & ¬(a = a)  ClassElim 28
32. _|_  ImpElim 31 30
33. ¬¬(a ϵ y)  ImpInt 32
34. D <-> ¬¬D  TheoremInt
37. ¬¬(a ϵ y) -> (a ϵ y)  PolySub 36
38. a ϵ y  ImpElim 33 37
39. (a ϵ x) -> (a ϵ y)  ImpInt 38
40. ∀a.((a ϵ x) -> (a ϵ y))  ForallInt 39
41. x ⊂ y  DefSub 40
43. (x ⊂ y) & (y ⊂ x)  AndInt 41 42
44. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
47. x = y  ImpElim 43 46
48. ((y ⊂ x) & ((x ~ y) = 0)) -> (x = y)  ImpInt 47 Qed

Used Theorems

1. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
2. D <-> ¬¬D
3. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

Th99. (WO(r,x) & WO(s,y)) -> ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = r
g(f))))

0. WO(r,x) & WO(s,y)  Hyp
1. f = {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ d
om(g)) & ((u,v) ϵ g)))))))}  Hyp
2. a ϵ f  Hyp
3. a ϵ {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ d
om(g)) & ((u,v) ϵ g)))))))}  EqualitySub 2 1
4. Set(a) & ∃u.∃v.((a = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ
dom(g)) & ((u,v) ϵ g)))))))  ClassElim 3
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6. ∃v.((a = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ dom(g)) & ((u,v) ϵ g))
7. (a = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ dom(g)) & ((u,v)
ϵ g)))))) Hyp
10. ∃u.∃v.(a = (u,v)) ExistsInt 9
11. ∃u.∃v.(a = (u,v)) ExistsElim 6 7 10
13. (a ϵ f) -> ∃u.∃v.(a = (u,v)) ImpInt 12
14. ∀a.((a ϵ f) -> ∃u.∃v.(a = (u,v))) ForallInt 13
15. Relation(f) DefSub 14
16. ((a,b) ϵ f) & ((a,c) ϵ f) Hyp
19. (a,b) ϵ {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u
ϵ dom(g)) & ((u,v) ϵ g)))))))} EqualitySub 17 1
20. (a,c) ϵ {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u
ϵ dom(g)) & ((u,v) ϵ g)))))))} EqualitySub 18 1
21. Set((a,b)) & ∃u.∃v.(((a,b) = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g))
& ((u ϵ dom(g)) & ((u,v) ϵ g))))))) ClassElim 19
22. Set((a,c)) & ∃u.∃v.(((a,c) = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g))
& ((u ϵ dom(g)) & ((u,v) ϵ g))))))) ClassElim 20
25. ∃v.(((a,b) = (u1,v)) & ((u1 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u1 ϵ dom(g)) & ((u1
26. ((a,b) = (u1,v1)) & ((u1 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u1 ϵ dom(g))
& ((u1,v1) ϵ g)))))) Hyp
27. ∃v.(((a,c) = (u2,v)) & ((u2 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u2 ϵ dom(g)) & ((u2
28. ((a,c) = (u2,v2)) & ((u2 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u2 ϵ dom(g))
& ((u2,v2) ϵ g)))))) Hyp
33. OP(g1,r,s) & (Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((u1 ϵ dom(g1)) & ((u1,v1) ϵ g1)))) Hyp
34. OP(g2,r,s) & (Sec(r,x,dom(g2)) & (Sec(s,y,rg(g2)) & ((u2 ϵ dom(g2)) & ((u2,v2) ϵ g2)))) Hyp
35. (OP(f,r,s) & (OP(g,r,s) & (Sec(r,x,dom(f)) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(f)) & Sec(s,y,rg(g)))))
) -> ((f ⊂ g) v (g ⊂ f)) TheoremInt
39. (OP(g1,r,s) & (OP(g2,r,s) & (Sec(r,x,dom(g1)) & (Sec(r,x,dom(g2)) & (Sec(s,y,rg(g1)) & Sec(s,y,rg(g2
))))))) -> ((g1 ⊂ g2) v (g2 ⊂ g1)) ForallElim 38
52. Sec(s,y,rg(g1)) & Sec(s,y,rg(g2)) AndInt 44 50
53. Sec(r,x,dom(g2)) & (Sec(s,y,rg(g1)) & Sec(s,y,rg(g2))) AndInt 48 52
54. Sec(r,x,dom(g1)) & (Sec(r,x,dom(g2)) & (Sec(s,y,rg(g1)) & Sec(s,y,rg(g2)))) AndInt 42 53
55. OP(g2,r,s) & (Sec(r,x,dom(g1)) & (Sec(r,x,dom(g2)) & (Sec(s,y,rg(g1)) & Sec(s,y,rg(g2))))) AndInt 46 54
56. OP(g1,r,s) & (OP(g2,r,s) & (Sec(r,x,dom(g1)) & (Sec(r,x,dom(g2)) & (Sec(s,y,rg(g1)) & Sec(s,y,rg(g2)
))))) AndInt 40 55
57. (g1 ⊂ g2) v (g2 ⊂ g1) ImpElim 56 39
58. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
67. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 66
68. Set(a) & Set(b) ImpElim 62 67
70. Set((a,c)) -> (Set(a) & Set(c)) ForallElim 69
71. Set(a) & Set(c) ImpElim 63 70
72. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
75. (Set(a) & Set(b)) & ((a,b) = (u1,v1)) AndInt 68 73
76. (Set(a) & Set(c)) & ((a,c) = (u2,v2)) AndInt 71 74
84. ((Set(a) & Set(b)) & ((a,b) = (u1,v1))) -> ((a = u1) & (b = v1)) ForallElim 83
85. (a = u1) & (b = v1) ImpElim 75 84
91. ((Set(a) & Set(c)) & ((a,c) = (u2,v2))) -> ((a = u2) & (c = v2)) ForallElim 90
92. (a = u2) & (c = v2) ImpElim 76 91
103. (a,v1) ϵ g1 EqualitySub 93 99
104. (a,b) ϵ g1 EqualitySub 103 100
105. (a,v2) ϵ g2 EqualitySub 94 101
106. (a,c) ϵ g2 EqualitySub 105 102
107. g1 ⊂ g2 Hyp
108. ∀z.((z ϵ g1) -> (z ϵ g2)) DefExp 107
109. ((a,b) ϵ g1) -> ((a,b) ϵ g2) ForallElim 108
110. (a,b) ϵ g2 ImpElim 104 109
112. (FUN(g2) & (WO(r,dom(g2)) & WO(s,rg(g2)))) & ∀u.∀v.((((u ϵ dom(g2)) & (v ϵ dom(g2))) & ((u,v) ϵ r))
-> (((g2'u),(g2'v)) ϵ s)) DefExp 111
115. Relation(g2) & ∀x.∀y.∀z.((((x,y) ϵ g2) & ((x,z) ϵ g2)) -> (y = z)) DefExp 114
117. ∀y.∀z.((((a,y) ϵ g2) & ((a,z) ϵ g2)) -> (y = z)) ForallElim 116
118. ∀z.((((a,b) ϵ g2) & ((a,z) ϵ g2)) -> (b = z)) ForallElim 117
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119. (((a,b) ϵ g2) & ((a,c) ϵ g2)) -> (b = c)  ForallElim 118
120. ((a,b) ϵ g2) & ((a,c) ϵ g2)  AndInt 110 106
121. b = c  ImpElim 120 119
122. g2 ⊂ g1  Hyp
123. ∀z.((z ϵ g2) -> (z ϵ g1))  DefExp 122
124. ((a,c) ϵ g2) -> ((a,c) ϵ g1)  ForallElim 123
125. (a,c) ϵ g1  ImpElim 106 124
127. (FUN(g1) & (WO(r,dom(g1)) & WO(s,rg(g1)))) & ∀u.∀v.((((u ϵ dom(g1)) & (v ϵ dom(g1))) & ((u,v) ϵ r))
-> (((g1'u),(g1'v)) ϵ s))  DefExp 126
130. Relation(g1) & ∀x.∀y.∀z.((((x,y) ϵ g1) & ((x,z) ϵ g1)) -> (y = z))  DefExp 129
132. ∀y.∀z.((((a,y) ϵ g1) & ((a,z) ϵ g1)) -> (y = z))  ForallElim 131
133. ∀z.((((a,b) ϵ g1) & ((a,z) ϵ g1)) -> (b = z))  ForallElim 132
134. (((a,b) ϵ g1) & ((a,c) ϵ g1)) -> (b = c)  ForallElim 133
135. ((a,b) ϵ g1) & ((a,c) ϵ g1)  AndInt 104 125
136. b = c  ImpElim 135 134
137. b = c  OrElim 57 107 121 122 136
138. b = c  ExistsElim 32 34 137
144. (((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c)  ImpInt 143
145. ∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  ForallInt 144
146. ∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  ForallInt 145
147. ∀a.∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  ForallInt 146
148. Relation(f) & ∀a.∀b.∀c.((((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c))  AndInt 15 147
149. FUN(f)  DefSub 148
150. ((a ϵ x) & (b ϵ dom(f))) & ((a,b) ϵ r)  Hyp
154. b ϵ {x: ∃y.((x,y) ϵ f)}  EqualitySub 152 153
155. Set(b) & ∃y.((b,y) ϵ f)  ClassElim 154
157. (b,j) ϵ f  Hyp
158. (b,j) ϵ {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u
ϵ dom(g)) & ((u,v) ϵ g)))))))}  EqualitySub 157 1
159. Set((b,j)) & ∃u.∃v.(((b,j) = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g))
& ((u ϵ dom(g)) & ((u,v) ϵ g)))))))  ClassElim 158
161. ∃v.(((b,j) = (u1,v)) & ((u1 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u1 ϵ dom(g)) & ((u
162. ((b,j) = (u1,v1)) & ((u1 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u1 ϵ dom(g))
& ((u1,v1) ϵ g))))))  Hyp
165. OP(g1,r,s) & (Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((u1 ϵ dom(g1)) & ((u1,v1) ϵ g1))))  Hyp
173. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
181. (Set((b,j)) & ((b,j) = (u1,v1))) -> ((b = u1) & (j = v1))  ForallElim 180
182. Set((b,j)) & ((b,j) = (u1,v1))  AndInt 172 171
183. (b = u1) & (j = v1)  ImpElim 182 181
188. b ϵ dom(g1)  EqualitySub 170 186
190. (b,v1) ϵ g1  EqualitySub 189 186
191. (b,j) ϵ g1  EqualitySub 190 187
195. ((dom(g1) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(g1))) & ((u,v) ϵ r)) -> (u ϵ dom(g1)))  DefExp 167
197. ∀v.((((a ϵ x) & (v ϵ dom(g1))) & ((a,v) ϵ r)) -> (a ϵ dom(g1)))  ForallElim 196
198. (((a ϵ x) & (b ϵ dom(g1))) & ((a,b) ϵ r)) -> (a ϵ dom(g1))  ForallElim 197
199. (a ϵ x) & (b ϵ dom(g1))  AndInt 194 188
200. ((a ϵ x) & (b ϵ dom(g1))) & ((a,b) ϵ r)  AndInt 199 192
201. a ϵ dom(g1)  ImpElim 200 198
204. dom(g1) = {x: ∃y.((x,y) ϵ g1)}  ForallElim 203
205. a ϵ {x: ∃y.((x,y) ϵ g1)}  EqualitySub 201 204
206. Set(a) & ∃y.((a,y) ϵ g1)  ClassElim 205
208. (a,d) ϵ g1  Hyp
209. w = (a,d)  Hyp
210. (a ϵ dom(g1)) & ((a,d) ϵ g1)  AndInt 201 208
212. Sec(s,y,rg(g1)) & ((a ϵ dom(g1)) & ((a,d) ϵ g1))  AndInt 211 210
213. Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((a ϵ dom(g1)) & ((a,d) ϵ g1)))  AndInt 167 212
215. OP(g1,r,s) & (Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((a ϵ dom(g1)) & ((a,d) ϵ g1))))  AndInt 214 213
216. ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((a ϵ dom(g)) & ((a,d) ϵ g)))))  ExistsInt 215
217. (a ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((a ϵ dom(g)) & ((a,d) ϵ g)))))  AndInt 194 21
218. (w = (a,d)) & ((a ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((a ϵ dom(g)) & ((a,d)
ϵ g))))))  AndInt 209 217
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221. ∃w.((a,d) ϵ w)  ExistsInt 208
222. Set((a,d))  DefSub 221
224. Set(w)  EqualitySub 222 223
225. Set(w) & ∃a.∃d.((w = (a,d)) & ((a ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((a ϵ
dom(g)) & ((a,d) ϵ g)))))))  AndInt 224 220
226. w ϵ {w: ∃a.∃d.((w = (a,d)) & ((a ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((a ϵ d
om(g)) & ((a,d) ϵ g)))))))}  ClassInt 225
228. w ϵ f  EqualitySub 226 227
229. (a,d) ϵ f  EqualitySub 228 209
230. (w = (a,d)) -> ((a,d) ϵ f)  ImpInt 229
232. ((a,d) = (a,d)) -> ((a,d) ϵ f)  ForallElim 231
234. (a,d) ϵ f  ImpElim 233 232
235. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))  TheoremInt
237. ((a,d) ϵ f) -> ((a ϵ dom(f)) & (d ϵ rg(f)))  ForallElim 236
238. (a ϵ dom(f)) & (d ϵ rg(f))  ImpElim 234 237
240. a ϵ dom(f)  ExistsElim 207 208 239
245. (((a ϵ x) & (b ϵ dom(f))) & ((a,b) ϵ r)) -> (a ϵ dom(f))  ImpInt 244
247. w ϵ dom(f)  Hyp
248. w ϵ {x: ∃y.((x,y) ϵ f)}  EqualitySub 247 202
249. Set(w) & ∃y.((w,y) ϵ f)  ClassElim 248
251. (w,y1) ϵ f  Hyp
252. (w,y1) ϵ {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((
u ϵ dom(g)) & ((u,v) ϵ g)))))))}  EqualitySub 251 1
253. Set((w,y1)) & ∃u.∃v.(((w,y1) = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)
) & ((u ϵ dom(g)) & ((u,v) ϵ g)))))))  ClassElim 252
255. ∃v.(((w,y1) = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ dom(g)) & ((u,v
256. ((w,y1) = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ dom(g)) & (
(u,v) ϵ g))))))  Hyp
261. Set((w,y1)) & ((w,y1) = (u,v))  AndInt 260 259
262. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
272. (Set((w,y1)) & ((w,y1) = (u,v))) -> ((w = u) & (y1 = v))  ForallElim 271
273. (w = u) & (y1 = v)  ImpElim 261 272
276. w ϵ x  EqualitySub 258 275
277. w ϵ x  ExistsElim 255 256 276
280. (w ϵ dom(f)) -> (w ϵ x)  ImpInt 279
281. ∀w.((w ϵ dom(f)) -> (w ϵ x))  ForallInt 280
282. dom(f) ⊂ x  DefSub 281
283. (dom(f) ⊂ x) & WO(r,x)  AndInt 282 246
284. ∀b.((((a ϵ x) & (b ϵ dom(f))) & ((a,b) ϵ r)) -> (a ϵ dom(f)))  ForallInt 245
285. ∀a.∀b.((((a ϵ x) & (b ϵ dom(f))) & ((a,b) ϵ r)) -> (a ϵ dom(f)))  ForallInt 284
286. ((dom(f) ⊂ x) & WO(r,x)) & ∀a.∀b.((((a ϵ x) & (b ϵ dom(f))) & ((a,b) ϵ r)) -> (a ϵ dom(f)))  AndInt 283 285
287. Sec(r,x,dom(f))  DefSub 286
288. ((a ϵ y) & (b ϵ rg(f))) & ((a,b) ϵ s)  Hyp
292. b ϵ {y: ∃x.((x,y) ϵ f)}  EqualitySub 290 291
293. Set(b) & ∃x.((x,b) ϵ f)  ClassElim 292
295. (i,b) ϵ f  Hyp
296. (i,b) ϵ {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u
ϵ dom(g)) & ((u,v) ϵ g)))))))}  EqualitySub 295 1
297. Set((i,b)) & ∃u.∃v.(((i,b) = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g))
& ((u ϵ dom(g)) & ((u,v) ϵ g)))))))  ClassElim 296
299. ∃v.(((i,b) = (u1,v)) & ((u1 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u1 ϵ dom(g)) & ((u
300. ((i,b) = (u1,v1)) & ((u1 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u1 ϵ dom(g))
& ((u1,v1) ϵ g))))))  Hyp
303. OP(g1,r,s) & (Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((u1 ϵ dom(g1)) & ((u1,v1) ϵ g1))))  Hyp
307. ((rg(g1) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(g1))) & ((u,v) ϵ s)) -> (u ϵ rg(g1)))  DefExp 306
311. Set((i,b)) & ((i,b) = (u1,v1))  AndInt 310 309
312. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
318. (Set((i,b)) & ((i,b) = (u1,v1))) -> ((i = u1) & (b = v1))  ForallElim 317
319. (i = u1) & (b = v1)  ImpElim 311 318
324. ∀v.((((a ϵ y) & (v ϵ rg(g1))) & ((a,v) ϵ s)) -> (a ϵ rg(g1)))  ForallElim 308
325. (((a ϵ y) & (b ϵ rg(g1))) & ((a,b) ϵ s)) -> (a ϵ rg(g1))  ForallElim 324
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329. (u1,b) ϵ g1 EqualitySub 328 322
330. (i,b) ϵ g1 EqualitySub 329 323
331. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f))) TheoremInt
335. ((i,b) ϵ g1) -> ((i ϵ dom(g1)) & (b ϵ rg(g1))) ForallElim 334
336. (i ϵ dom(g1)) & (b ϵ rg(g1)) ImpElim 330 335
341. (a ϵ y) & (b ϵ rg(g1)) AndInt 340 337
342. ((a ϵ y) & (b ϵ rg(g1))) & ((a,b) ϵ s) AndInt 341 339
343. a ϵ rg(g1) ImpElim 342 325
346. rg(g1) = {y: ∃x.((x,y) ϵ g1)} ForallElim 345
347. a ϵ {y: ∃x.((x,y) ϵ g1)} EqualitySub 343 346
348. Set(a) & ∃x.((x,a) ϵ g1) ClassElim 347
350. (k,a) ϵ g1 Hyp
351. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f))) TheoremInt
357. ((k,a) ϵ g1) -> ((k ϵ dom(g1)) & (a ϵ rg(g1))) ForallElim 356
358. (k ϵ dom(g1)) & (a ϵ rg(g1)) ImpElim 350 357
360. (k ϵ dom(g1)) & ((k,a) ϵ g1) AndInt 359 350
364. Sec(s,y,rg(g1)) & ((k ϵ dom(g1)) & ((k,a) ϵ g1)) AndInt 361 360
365. Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((k ϵ dom(g1)) & ((k,a) ϵ g1))) AndInt 363 364
366. OP(g1,r,s) & (Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((k ϵ dom(g1)) & ((k,a) ϵ g1)))) AndInt 362 365
367. ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((k ϵ dom(g)) & ((k,a) ϵ g))))) ExistsInt 366
368. ((dom(g1) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(g1))) & ((u,v) ϵ r)) -> (u ϵ dom(g1))) DefExp 363
371. ∀z.((z ϵ dom(g1)) -> (z ϵ x)) DefExp 370
372. (k ϵ dom(g1)) -> (k ϵ x) ForallElim 371
373. k ϵ x ImpElim 359 372
374. (k ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((k ϵ dom(g)) & ((k,a) ϵ g))))) AndInt 373 36
375. v = (k,a) Hyp
376. (v = (k,a)) & ((k ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((k ϵ dom(g)) & ((k,a)
ϵ g)))))) AndInt 375 374
379. ∃w.((k,a) ϵ w) ExistsInt 350
380. Set((k,a)) DefSub 379
382. Set(v) EqualitySub 380 381
383. Set(v) & ∃k.∃a.((v = (k,a)) & ((k ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((k ϵ
dom(g)) & ((k,a) ϵ g))))))) AndInt 382 378
384. v ϵ {w: ∃k.∃a.((w = (k,a)) & ((k ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((k ϵ d
om(g)) & ((k,a) ϵ g)))))))} ClassInt 383
386. v ϵ f EqualitySub 384 385
387. (k,a) ϵ f EqualitySub 386 375
388. (v = (k,a)) -> ((k,a) ϵ f) ImpInt 387
390. ((k,a) = (k,a)) -> ((k,a) ϵ f) ForallElim 389
392. (k,a) ϵ f ImpElim 391 390
393. ∃w.((w,a) ϵ f) ExistsInt 392
394. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
401. Set((k,a)) -> (Set(k) & Set(a)) ForallElim 400
402. Set(k) & Set(a) ImpElim 380 401
404. Set(a) & ∃w.((w,a) ϵ f) AndInt 403 393
406. a ϵ {w: ∃x_66.((x_66,w) ϵ f)} ClassInt 404
408. a ϵ rg(f) EqualitySub 406 407
409. a ϵ rg(f) ExistsElim 349 350 408
414. (((a ϵ y) & (b ϵ rg(f))) & ((a,b) ϵ s)) -> (a ϵ rg(f)) ImpInt 413
415. j ϵ rg(f) Hyp
416. j ϵ {y: ∃x.((x,y) ϵ f)} EqualitySub 415 405
417. Set(j) & ∃x.((x,j) ϵ f) ClassElim 416
419. (k,j) ϵ f Hyp
420. (k,j) ϵ {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u
ϵ dom(g)) & ((u,v) ϵ g)))))))} EqualitySub 419 1
421. Set((k,j)) & ∃u.∃v.(((k,j) = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g))
& ((u ϵ dom(g)) & ((u,v) ϵ g))))))) ClassElim 420
423. ∃v.(((k,j) = (u1,v)) & ((u1 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u1 ϵ dom(g)) & ((u
424. ((k,j) = (u1,v1)) & ((u1 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u1 ϵ dom(g))
& ((u1,v1) ϵ g)))))) Hyp
427. OP(g1,r,s) & (Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((u1 ϵ dom(g1)) & ((u1,v1) ϵ g1)))) Hyp

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433. ((rg(g1) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(g1))) & ((u,v) ϵ s)) -> (u ϵ rg(g1)))  DefExp 430
434. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))  TheoremInt
440. ((u1,v1) ϵ g1) -> ((u1 ϵ dom(g1)) & (v1 ϵ rg(g1)))  ForallElim 439
441. (u1 ϵ dom(g1)) & (v1 ϵ rg(g1))  ImpElim 432 440
444. ∀z.((z ϵ rg(g1)) -> (z ϵ y)) & WO(s,y)  DefExp 443
446. (v1 ϵ rg(g1)) -> (v1 ϵ y)  ForallElim 445
447. v1 ϵ y  ImpElim 442 446
450. Set((k,j)) & ((k,j) = (u1,v1))  AndInt 449 448
451. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
459. (Set((k,j)) & ((k,j) = (u1,v1))) -> ((k = u1) & (j = v1))  ForallElim 458
460. (k = u1) & (j = v1)  ImpElim 450 459
464. j ϵ y  EqualitySub 447 463
465. j ϵ y  ExistsElim 426 427 464
469. (j ϵ rg(f)) -> (j ϵ y)  ImpInt 468
470. ∀j.((j ϵ rg(f)) -> (j ϵ y))  ForallInt 469
471. rg(f) ⊂ y  DefSub 470
472. ∀b.((((a ϵ y) & (b ϵ rg(f))) & ((a,b) ϵ s)) -> (a ϵ rg(f)))  ForallInt 414
473. ∀a.∀b.((((a ϵ y) & (b ϵ rg(f))) & ((a,b) ϵ s)) -> (a ϵ rg(f)))  ForallInt 472
475. (rg(f) ⊂ y) & WO(s,y)  AndInt 471 474
476. ((rg(f) ⊂ y) & WO(s,y)) & ∀a.∀b.((((a ϵ y) & (b ϵ rg(f))) & ((a,b) ϵ s)) -> (a ϵ rg(f)))  AndInt 475 473
477. Sec(s,y,rg(f))  DefSub 476
478. ((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)  Hyp
482. u ϵ {x: ∃y.((x,y) ϵ f)}  EqualitySub 480 481
483. Set(u) & ∃y.((u,y) ϵ f)  ClassElim 482
485. (u,v1) ϵ f  Hyp
486. (u,v1) ϵ {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((
u ϵ dom(g)) & ((u,v) ϵ g)))))))}  EqualitySub 485 1
487. Set((u,v1)) & ∃x_87.∃v.(((u,v1) = (x_87,v)) & ((x_87 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(
s,y,rg(g)) & ((x_87 ϵ dom(g)) & ((x_87,v) ϵ g)))))))  ClassElim 486
489. ∃v.(((u,v1) = (u2,v)) & ((u2 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u2 ϵ dom(g)) & ((
490. ((u,v1) = (u2,v2)) & ((u2 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u2 ϵ dom(g))
& ((u2,v2) ϵ g))))))  Hyp
493. OP(g1,r,s) & (Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((u2 ϵ dom(g1)) & ((u2,v2) ϵ g1))))  Hyp
495. (FUN(g1) & (WO(r,dom(g1)) & WO(s,rg(g1)))) & ∀u.∀v.((((u ϵ dom(g1)) & (v ϵ dom(g1))) & ((u,v) ϵ r))
-> (((g1'u),(g1'v)) ϵ s))  DefExp 494
498. ((dom(g1) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(g1))) & ((u,v) ϵ r)) -> (u ϵ dom(g1)))  DefExp 497
506. Set((u,v1)) & ((u,v1) = (u2,v2))  AndInt 504 505
507. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
515. (Set((u,v1)) & ((u,v1) = (u2,v2))) -> ((u = u2) & (v1 = v2))  ForallElim 514
516. (u = u2) & (v1 = v2)  ImpElim 506 515
519. u ϵ dom(g1)  EqualitySub 503 518
520. ∀x_98.((((v ϵ x) & (x_98 ϵ dom(g1))) & ((v,x_98) ϵ r)) -> (v ϵ dom(g1)))  ForallElim 499
521. (((v ϵ x) & (u ϵ dom(g1))) & ((v,u) ϵ r)) -> (v ϵ dom(g1))  ForallElim 520
522. ((dom(f) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (u ϵ dom(f)))  DefExp 287
526. ∀z.((z ϵ dom(f)) -> (z ϵ x))  DefExp 525
527. (v ϵ dom(f)) -> (v ϵ x)  ForallElim 526
528. v ϵ x  ImpElim 524 527
529. (v ϵ x) & (u ϵ dom(g1))  AndInt 528 519
530. ((v ϵ x) & (u ϵ dom(g1))) & ((v,u) ϵ r)  AndInt 529 500
531. v ϵ dom(g1)  ImpElim 530 521
533. ∀x_104.((((v ϵ dom(g1)) & (x_104 ϵ dom(g1))) & ((v,x_104) ϵ r)) -> (((g1'v),(g1'x_104)) ϵ s))  ForallElim 532
534. (((v ϵ dom(g1)) & (u ϵ dom(g1))) & ((v,u) ϵ r)) -> (((g1'v),(g1'u)) ϵ s)  ForallElim 533
535. (v ϵ dom(g1)) & (u ϵ dom(g1))  AndInt 531 519
536. ((v ϵ dom(g1)) & (u ϵ dom(g1))) & ((v,u) ϵ r)  AndInt 535 500
537. ((g1'v),(g1'u)) ϵ s  ImpElim 536 534
541. (u,v2) ϵ g1  EqualitySub 540 518
544. (u,v1) ϵ g1  EqualitySub 541 543
547. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)  TheoremInt
553. (FUN(g1) & ((u,v1) ϵ g1)) -> ((g1'u) = v1)  ForallElim 552
554. FUN(g1) & ((u,v1) ϵ g1)  AndInt 546 544
555. (g1'u) = v1  ImpElim 554 553
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556. FUN(f) & ((u,v1) ϵ f) AndInt 149 485
560. (FUN(f) & ((u,v1) ϵ f)) -> ((f'u) = v1) ForallElim 559
561. (f'u) = v1 ImpElim 556 560
564. dom(g1) = {x: ∃y.((x,y) ϵ g1)} ForallElim 563
565. v ϵ {x: ∃y.((x,y) ϵ g1)} EqualitySub 531 564
566. Set(v) & ∃y.((v,y) ϵ g1) ClassElim 565
568. (v,j) ϵ g1 Hyp
569. (v ϵ dom(g1)) & ((v,j) ϵ g1) AndInt 531 568
571. Sec(s,y,rg(g1)) & ((v ϵ dom(g1)) & ((v,j) ϵ g1)) AndInt 570 569
572. Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((v ϵ dom(g1)) & ((v,j) ϵ g1))) AndInt 497 571
573. OP(g1,r,s) & (Sec(r,x,dom(g1)) & (Sec(s,y,rg(g1)) & ((v ϵ dom(g1)) & ((v,j) ϵ g1)))) AndInt 494 572
574. ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((v ϵ dom(g)) & ((v,j) ϵ g))))) ExistsInt 573
576. ((dom(g1) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(g1))) & ((u,v) ϵ r)) -> (u ϵ dom(g1))) DefExp 575
578. ∀z.((z ϵ dom(g1)) -> (z ϵ x)) & WO(r,x) DefExp 577
580. (v ϵ dom(g1)) -> (v ϵ x) ForallElim 579
582. v ϵ x ImpElim 581 580
583. (v ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((v ϵ dom(g)) & ((v,j) ϵ g))))) AndInt 582 5
584. w = (v,j) Hyp
585. (w = (v,j)) & ((v ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((v ϵ dom(g)) & ((v,j) ϵ g)))))) AndInt 584 583
588. ∃w.((v,j) ϵ w) ExistsInt 568
589. Set((v,j)) DefSub 588
590. Set((v,j)) & ∃v.∃j.((w = (v,j)) & ((v ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((v ϵ dom(g)) & ((v,j) ϵ g))))))) AndInt 589 587
592. Set(w) & ∃v.∃j.((w = (v,j)) & ((v ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((v ϵ dom(g)) & ((v,j) ϵ g))))))) EqualitySub 590 591
593. w ϵ {w: ∃v.∃j.((w = (v,j)) & ((v ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((v ϵ dom(g)) & ((v,j) ϵ g)))))))} ClassInt 592
595. w ϵ f EqualitySub 593 594
596. (v,j) ϵ f EqualitySub 595 584
597. FUN(f) & ((v,j) ϵ f) AndInt 149 596
598. FUN(g1) & ((v,j) ϵ g1) AndInt 546 568
599. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b) TheoremInt
603. (FUN(f) & ((v,j) ϵ f)) -> ((f'v) = j) ForallElim 602
604. (f'v) = j ImpElim 597 603
610. (FUN(g1) & ((v,j) ϵ g1)) -> ((g1'v) = j) ForallElim 609
611. (g1'v) = j ImpElim 598 610
613. (g1'v) = (f'v) EqualitySub 611 612
615. (g1'u) = (f'u) EqualitySub 555 614
616. ((f'v),(g1'u)) ϵ s EqualitySub 537 613
617. ((f'v),(f'u)) ϵ s EqualitySub 616 615
618. (w = (v,j)) -> (((f'v),(f'u)) ϵ s) ImpInt 617
620. ((v,j) = (v,j)) -> (((f'v),(f'u)) ϵ s) ForallElim 619
622. ((f'v),(f'u)) ϵ s ImpElim 621 620
623. ((f'v),(f'u)) ϵ s ExistsElim 567 568 622
628. (((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)) -> (((f'v),(f'u)) ϵ s) ImpInt 627
629. ∀v.((((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)) -> (((f'v),(f'u)) ϵ s)) ForallInt 628
630. ∀u.∀v.((((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)) -> (((f'v),(f'u)) ϵ s)) ForallInt 629
631. (WO(r,a) & (b ⊂ a)) -> WO(r,b) TheoremInt
633. ((dom(f) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (u ϵ dom(f))) DefExp 287
639. (WO(r,x) & (dom(f) ⊂ x)) -> WO(r,dom(f)) ForallElim 638
640. WO(r,x) & (dom(f) ⊂ x) AndInt 632 635
641. WO(r,dom(f)) ImpElim 640 639
643. ((rg(f) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(f))) & ((u,v) ϵ s)) -> (u ϵ rg(f))) DefExp 477
651. (WO(s,y) & (rg(f) ⊂ y)) -> WO(s,rg(f)) ForallElim 650
652. WO(s,y) & (rg(f) ⊂ y) AndInt 642 645
653. WO(s,rg(f)) ImpElim 652 651
654. WO(r,dom(f)) & WO(s,rg(f)) AndInt 641 653
655. FUN(f) & (WO(r,dom(f)) & WO(s,rg(f))) AndInt 149 654
656. ∀u.((((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)) -> (((f'v),(f'u)) ϵ s)) ForallInt 628
657. ∀v.∀u.((((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)) -> (((f'v),(f'u)) ϵ s)) ForallInt 656

658. (FUN(f) & (WO(r,dom(f)) & WO(s,rg(f)))) & ∀v.∀u.((((v ϵ dom(f)) & (u ϵ dom(f))) & ((v,u) ϵ r)) -> (((f'v),(f'u)) ϵ s)) AndInt 655 657
659. OP(f,r,s) DefSub 658
660. Sec(r,x,dom(f)) & Sec(s,y,rg(f)) AndInt 287 477
661. OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f))) AndInt 659 660
662. ¬((x ~ dom(f)) = 0) & ¬((y ~ rg(f)) = 0) Hyp
663. z ϵ (x ~ dom(f)) Hyp
666. (x ~ dom(f)) = (x ∩ ~dom(f)) ForallElim 665
667. z ϵ (x ∩ ~dom(f)) EqualitySub 663 666
668. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
673. (z ϵ (x ∩ ~dom(f))) -> ((z ϵ x) & (z ϵ ~dom(f))) ForallElim 672
674. (z ϵ x) & (z ϵ ~dom(f)) ImpElim 667 673
676. (z ϵ (x ~ dom(f))) -> (z ϵ x) ImpInt 675
677. ∀z.((z ϵ (x ~ dom(f))) -> (z ϵ x)) ForallInt 676
678. (x ~ dom(f)) ⊂ x DefSub 677
679. z ϵ (y ~ rg(f)) Hyp
683. (y ~ rg(f)) = (y ∩ ~rg(f)) ForallElim 682
684. z ϵ (y ∩ ~rg(f)) EqualitySub 679 683
688. (z ϵ (y ∩ ~rg(f))) -> ((z ϵ y) & (z ϵ ~rg(f))) ForallElim 687
689. (z ϵ y) & (z ϵ ~rg(f)) ImpElim 684 688
691. (z ϵ (y ~ rg(f))) -> (z ϵ y) ImpInt 690
692. ∀z.((z ϵ (y ~ rg(f))) -> (z ϵ y)) ForallInt 691
693. (y ~ rg(f)) ⊂ y DefSub 692
695. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 694
697. (((x ~ dom(f)) ⊂ x) & ¬((x ~ dom(f)) = 0)) -> ∃z.First(r,(x ~ dom(f)),z) ForallElim 696
699. ((x ~ dom(f)) ⊂ x) & ¬((x ~ dom(f)) = 0) AndInt 678 698
700. ∃z.First(r,(x ~ dom(f)),z) ImpElim 699 697
702. Connects(s,y) & ∀x_128.(((x_128 ⊂ y) & ¬(x_128 = 0)) -> ∃z.First(s,x_128,z)) DefExp 701
704. (((y ~ rg(f)) ⊂ y) & ¬((y ~ rg(f)) = 0)) -> ∃z.First(s,(y ~ rg(f)),z) ForallElim 703
706. ((y ~ rg(f)) ⊂ y) & ¬((y ~ rg(f)) = 0) AndInt 693 705
707. ∃z.First(s,(y ~ rg(f)),z) ImpElim 706 704
708. First(r,(x ~ dom(f)),m) Hyp
709. First(s,(y ~ rg(f)),n) Hyp
710. (a ϵ dom(f)) & ((m,a) ϵ r) Hyp
712. ((dom(f) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (u ϵ dom(f))) DefExp 711
714. ∀v.((((m ϵ x) & (v ϵ dom(f))) & ((m,v) ϵ r)) -> (m ϵ dom(f))) ForallElim 713
715. (((m ϵ x) & (a ϵ dom(f))) & ((m,a) ϵ r)) -> (m ϵ dom(f)) ForallElim 714
716. (m ϵ (x ~ dom(f))) & ∀y.((y ϵ (x ~ dom(f))) -> ¬((y,m) ϵ r)) DefExp 708
718. ∀z.((z ϵ (x ~ dom(f))) -> (z ϵ x)) DefExp 678
719. (m ϵ (x ~ dom(f))) -> (m ϵ x) ForallElim 718
720. m ϵ x ImpElim 717 719
721. (m ϵ x) & (m ϵ (x ~ dom(f))) AndInt 720 717
724. (m ϵ x) & (a ϵ dom(f)) AndInt 720 723
726. ((m ϵ x) & (a ϵ dom(f))) & ((m,a) ϵ r) AndInt 724 725
727. m ϵ dom(f) ImpElim 726 715
728. (m ϵ (x ~ dom(f))) & ∀y.((y ϵ (x ~ dom(f))) -> ¬((y,m) ϵ r)) DefExp 708
732. (x ~ dom(f)) = (x ∩ ~dom(f)) ForallElim 731
733. m ϵ (x ∩ ~dom(f)) EqualitySub 729 732
734. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
741. (m ϵ (x ∩ ~dom(f))) -> ((m ϵ x) & (m ϵ ~dom(f))) ForallElim 740
742. (m ϵ x) & (m ϵ ~dom(f)) ImpElim 733 741
746. ~dom(f) = {y: ¬(y ϵ dom(f))} ForallElim 745
747. m ϵ {y: ¬(y ϵ dom(f))} EqualitySub 743 746
748. Set(m) & ¬(m ϵ dom(f)) ClassElim 747
750. _|_ ImpElim 727 749
751. ¬((a ϵ dom(f)) & ((m,a) ϵ r)) ImpInt 750
752. (a ϵ rg(f)) & ((n,a) ϵ s) Hyp
754. ((rg(f) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(f))) & ((u,v) ϵ s)) -> (u ϵ rg(f))) DefExp 753
756. ∀v.((((n ϵ y) & (v ϵ rg(f))) & ((n,v) ϵ s)) -> (n ϵ rg(f))) ForallElim 755
757. (((n ϵ y) & (a ϵ rg(f))) & ((n,a) ϵ s)) -> (n ϵ rg(f)) ForallElim 756
758. ∀z.((z ϵ (y ~ rg(f))) -> (z ϵ y)) DefExp 693

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759. (n ϵ (y ~ rg(f))) -> (n ϵ y)  ForallElim 758
760. (n ϵ (y ~ rg(f))) & ∀x_148.((x_148 ϵ (y ~ rg(f))) -> ¬((x_148,n) ϵ s))  DefExp 709
762. n ϵ y  ImpElim 761 759
764. (n ϵ y) & (a ϵ rg(f))  AndInt 762 763
766. ((n ϵ y) & (a ϵ rg(f))) & ((n,a) ϵ s)  AndInt 764 765
767. n ϵ rg(f)  ImpElim 766 757
771. (y ~ rg(f)) = (y ∩ ~rg(f))  ForallElim 770
772. n ϵ (y ∩ ~rg(f))  EqualitySub 761 771
778. (n ϵ (y ∩ ~rg(f))) -> ((n ϵ y) & (n ϵ ~rg(f)))  ForallElim 777
779. (n ϵ y) & (n ϵ ~rg(f))  ImpElim 772 778
782. ~rg(f) = {y: ¬(y ϵ rg(f))}  ForallElim 781
783. n ϵ {y: ¬(y ϵ rg(f))}  EqualitySub 780 782
784. Set(n) & ¬(n ϵ rg(f))  ClassElim 783
786. _|_  ImpElim 767 785
787. ¬((a ϵ rg(f)) & ((n,a) ϵ s))  ImpInt 786
788. ¬((a ϵ dom(f)) & ((m,a) ϵ r)) & ¬((a ϵ rg(f)) & ((n,a) ϵ s))  AndInt 751 787
789. g = (f ∪ {(m,n)})  Hyp
790. z ϵ g  Hyp
791. z ϵ (f ∪ {(m,n)})  EqualitySub 790 789
798. (z ϵ (f ∪ {(m,n)})) -> ((z ϵ f) v (z ϵ {(m,n)}))  ForallElim 797
799. (z ϵ f) v (z ϵ {(m,n)})  ImpElim 791 798
800. z ϵ f  Hyp
801. Relation(f) & ∀x.∀y.∀z.((((x,y) ϵ f) & ((x,z) ϵ f)) -> (y = z))  DefExp 149
803. ∀z.((z ϵ f) -> ∃x.∃y.(z = (x,y)))  DefExp 802
804. (z ϵ f) -> ∃x.∃y.(z = (x,y))  ForallElim 803
805. ∃x.∃y.(z = (x,y))  ImpElim 800 804
806. z ϵ {(m,n)}  Hyp
807. ∃w.(m ϵ w)  ExistsInt 720
808. Set(m)  DefSub 807
809. ∃w.(n ϵ w)  ExistsInt 762
810. Set(n)  DefSub 809
811. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
818. (Set(m) & Set(n)) -> Set((m,n))  ForallElim 817
819. Set(m) & Set(n)  AndInt 808 810
820. Set((m,n))  ImpElim 819 818
821. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
825. Set((m,n)) -> ((z ϵ {(m,n)}) <-> (z = (m,n)))  ForallElim 824
826. (z ϵ {(m,n)}) <-> (z = (m,n))  ImpElim 820 825
829. z = (m,n)  ImpElim 806 828
831. ∃x.∃y.(z = (x,y))  ExistsInt 830
832. ∃x.∃y.(z = (x,y))  OrElim 799 800 805 806 831
833. (z ϵ g) -> ∃x.∃y.(z = (x,y))  ImpInt 832
834. ∀z.((z ϵ g) -> ∃x.∃y.(z = (x,y)))  ForallInt 833
835. Relation(g)  DefSub 834
836. ((a,b) ϵ g) & ((a,c) ϵ g)  Hyp
838. (a,b) ϵ (f ∪ {(m,n)})  EqualitySub 837 789
840. ((a,b) ϵ (f ∪ {(m,n)})) -> (((a,b) ϵ f) v ((a,b) ϵ {(m,n)}))  ForallElim 839
841. ((a,b) ϵ f) v ((a,b) ϵ {(m,n)})  ImpElim 838 840
842. (a,b) ϵ f  Hyp
845. ((a,c) ϵ (f ∪ {(m,n)})) -> (((a,c) ϵ f) v ((a,c) ϵ {(m,n)}))  ForallElim 844
846. (a,c) ϵ (f ∪ {(m,n)})  EqualitySub 843 789
847. ((a,c) ϵ f) v ((a,c) ϵ {(m,n)})  ImpElim 846 845
848. (a,c) ϵ f  Hyp
850. ∀y.∀z.((((a,y) ϵ f) & ((a,z) ϵ f)) -> (y = z))  ForallElim 849
851. ∀z.((((a,b) ϵ f) & ((a,z) ϵ f)) -> (b = z))  ForallElim 850
852. (((a,b) ϵ f) & ((a,c) ϵ f)) -> (b = c)  ForallElim 851
853. ((a,b) ϵ f) & ((a,c) ϵ f)  AndInt 842 848
854. b = c  ImpElim 853 852
855. (a,c) ϵ {(m,n)}  Hyp
856. ∀z.((z ϵ {(m,n)}) -> (z = (m,n)))  ForallInt 828
858. ((a,c) ϵ {(m,n)}) -> ((a,c) = (m,n))  ForallElim 857
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859. (a,c) = (m,n)  ImpElim 855 858
860. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
862. Set((m,n)) & ((m,n) = (a,c))  AndInt 820 861
870. (Set((m,n)) & ((m,n) = (a,c))) -> ((m = a) & (n = c))  ForallElim 869
871. (m = a) & (n = c)  ImpElim 862 870
873. ∃w.((a,c) ϵ w)  ExistsInt 848
874. Set((a,c))  DefSub 873
875. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
882. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 881
883. Set(a) & Set(c)  ImpElim 874 882
885. Set(a) & ∃w.((a,w) ϵ f)  AndInt 884 872
886. a ϵ {w: ∃x_155.((w,x_155) ϵ f)}  ClassInt 885
889. a ϵ dom(f)  EqualitySub 886 888
892. m ϵ dom(f)  EqualitySub 889 891
893. _|_  ImpElim 892 749
894. b = c  AbsI 893
895. b = c  OrElim 847 848 854 855 894
896. (a,b) ϵ {(m,n)}  Hyp
897. (a,c) ϵ f  Hyp
898. ((a,b) ϵ {(m,n)}) -> ((a,b) = (m,n))  ForallElim 857
899. (a,b) = (m,n)  ImpElim 896 898
902. (Set((m,n)) & ((m,n) = (a,b))) -> ((m = a) & (n = b))  ForallElim 901
903. Set((m,n)) & ((m,n) = (a,b))  AndInt 820 900
904. (m = a) & (n = b)  ImpElim 903 902
906. ∃w.((a,c) ϵ w)  ExistsInt 897
907. Set((a,c))  DefSub 906
908. Set(a) & Set(c)  ImpElim 907 882
910. ∃w.((a,w) ϵ f)  ExistsInt 897
911. Set(a) & ∃w.((a,w) ϵ f)  AndInt 909 910
912. a ϵ {w: ∃x_157.((w,x_157) ϵ f)}  ClassInt 911
913. a ϵ dom(f)  EqualitySub 912 888
915. m ϵ dom(f)  EqualitySub 913 914
916. _|_  ImpElim 915 749
917. b = c  AbsI 916
918. (a,c) ϵ {(m,n)}  Hyp
919. (a,c) = (m,n)  ImpElim 918 858
921. Set((m,n)) & ((m,n) = (a,c))  AndInt 820 920
922. (m = a) & (n = c)  ImpElim 921 870
926. b = c  EqualitySub 925 924
927. b = c  OrElim 847 897 917 918 926
929. (((a,b) ϵ g) & ((a,c) ϵ g)) -> (b = c)  ImpInt 928
930. ∀c.((((a,b) ϵ g) & ((a,c) ϵ g)) -> (b = c))  ForallInt 929
931. ∀b.∀c.((((a,b) ϵ g) & ((a,c) ϵ g)) -> (b = c))  ForallInt 930
932. ∀a.∀b.∀c.((((a,b) ϵ g) & ((a,c) ϵ g)) -> (b = c))  ForallInt 931
933. Relation(g) & ∀a.∀b.∀c.((((a,b) ϵ g) & ((a,c) ϵ g)) -> (b = c))  AndInt 835 932
934. FUN(g)  DefSub 933
935. (a ϵ dom(g)) & ((b ϵ dom(g)) & ((a,b) ϵ r))  Hyp
937. ∀g.(dom(f) = {x: ∃y.((x,y) ϵ f)})  ForallInt 936
939. dom(g) = {x: ∃y.((x,y) ϵ g)}  ForallElim 938
940. (a ϵ {x: ∃y.((x,y) ϵ g)}) & ((b ϵ {x: ∃y.((x,y) ϵ g)}) & ((a,b) ϵ r))  EqualitySub 935 939
944. Set(a) & ∃y.((a,y) ϵ g)  ClassElim 941
945. Set(b) & ∃y.((b,y) ϵ g)  ClassElim 943
948. (a,p) ϵ g  Hyp
949. (b,q) ϵ g  Hyp
950. (a,p) ϵ (f ∪ {(m,n)})  EqualitySub 948 789
951. (b,q) ϵ (f ∪ {(m,n)})  EqualitySub 949 789
952. ((a,p) ϵ (f ∪ {(m,n)})) -> (((a,p) ϵ f) v ((a,p) ϵ {(m,n)}))  ForallElim 844
953. ((a,p) ϵ f) v ((a,p) ϵ {(m,n)})  ImpElim 950 952
954. (a,p) ϵ f  Hyp
955. ((b,q) ϵ (f ∪ {(m,n)})) -> (((b,q) ϵ f) v ((b,q) ϵ {(m,n)}))  ForallElim 844
956. ((b,q) ϵ f) v ((b,q) ϵ {(m,n)})  ImpElim 951 955
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957. (b,q) ϵ f  Hyp
958. ∃w.((a,p) ϵ w)  ExistsInt 954
959. Set((a,p))  DefSub 958
963. Set((a,p)) -> (Set(a) & Set(p))  ForallElim 962
964. Set(a) & Set(p)  ImpElim 959 963
966. ∃w.((a,w) ϵ f)  ExistsInt 954
967. Set(a) & ∃w.((a,w) ϵ f)  AndInt 965 966
968. a ϵ {w: ∃x_160.((w,x_160) ϵ f)}  ClassInt 967
971. a ϵ dom(f)  EqualitySub 968 970
972. ∃w.((b,q) ϵ w)  ExistsInt 957
973. Set((b,q))  DefSub 972
977. Set((b,q)) -> (Set(b) & Set(q))  ForallElim 976
978. Set(b) & Set(q)  ImpElim 973 977
980. ∃w.((b,w) ϵ f)  ExistsInt 957
981. Set(b) & ∃w.((b,w) ϵ f)  AndInt 979 980
982. b ϵ {w: ∃x_162.((w,x_162) ϵ f)}  ClassInt 981
983. b ϵ dom(f)  EqualitySub 982 970
984. (FUN(f) & (WO(r,dom(f)) & WO(s,rg(f)))) & ∀u.∀v.((((u ϵ dom(f)) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> ((
(f'u),(f'v)) ϵ s))  DefExp 659
986. ∀v.((((a ϵ dom(f)) & (v ϵ dom(f))) & ((a,v) ϵ r)) -> (((f'a),(f'v)) ϵ s))  ForallElim 985
987. (((a ϵ dom(f)) & (b ϵ dom(f))) & ((a,b) ϵ r)) -> (((f'a),(f'b)) ϵ s)  ForallElim 986
988. (a ϵ dom(f)) & (b ϵ dom(f))  AndInt 971 983
991. ((a ϵ dom(f)) & (b ϵ dom(f))) & ((a,b) ϵ r)  AndInt 988 990
992. ((f'a),(f'b)) ϵ s  ImpElim 991 987
993. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)  TheoremInt
998. (FUN(g) & ((a,p) ϵ g)) -> ((g'a) = p)  ForallElim 997
999. FUN(g) & ((a,p) ϵ g)  AndInt 934 948
1000. (g'a) = p  ImpElim 999 998
1002. FUN(f) & ((a,p) ϵ f)  AndInt 1001 954
1003. (f'a) = p  ImpElim 1002 996
1007. (FUN(f) & ((b,q) ϵ f)) -> ((f'b) = q)  ForallElim 1006
1008. FUN(f) & ((b,q) ϵ f)  AndInt 1001 957
1009. (f'b) = q  ImpElim 1008 1007
1011. (FUN(g) & ((b,q) ϵ g)) -> ((g'b) = q)  ForallElim 1010
1012. FUN(g) & ((b,q) ϵ g)  AndInt 934 949
1013. (g'b) = q  ImpElim 1012 1011
1016. (f'a) = (g'a)  EqualitySub 1003 1014
1017. (f'b) = (g'b)  EqualitySub 1009 1015
1018. ((g'a),(f'b)) ϵ s  EqualitySub 992 1016
1019. ((g'a),(g'b)) ϵ s  EqualitySub 1018 1017
1020. (b,q) ϵ {(m,n)}  Hyp
1021. Set((m,n)) & ((b,q) ϵ {(m,n)})  AndInt 820 1020
1022. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
1026. Set((m,n)) -> (((b,q) ϵ {(m,n)}) <-> ((b,q) = (m,n)))  ForallElim 1025
1027. ((b,q) ϵ {(m,n)}) <-> ((b,q) = (m,n))  ImpElim 820 1026
1030. (b,q) = (m,n)  ImpElim 1020 1029
1032. Set((m,n)) & ((m,n) = (b,q))  AndInt 820 1031
1033. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
1041. (Set((m,n)) & ((m,n) = (b,q))) -> ((m = b) & (n = q))  ForallElim 1040
1042. (m = b) & (n = q)  ImpElim 1032 1041
1047. (m,q) ϵ g  EqualitySub 949 1045
1048. (m,n) ϵ g  EqualitySub 1047 1046
1049. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)  TheoremInt
1055. (FUN(g) & ((m,n) ϵ g)) -> ((g'm) = n)  ForallElim 1054
1056. FUN(g) & ((m,n) ϵ g)  AndInt 934 1048
1057. (g'm) = n  ImpElim 1056 1055
1058. (g'b) = n  EqualitySub 1057 1043
1059. ∃w.((w,p) ϵ f)  ExistsInt 954
1061. Set(p) & ∃w.((w,p) ϵ f)  AndInt 1060 1059
1062. p ϵ {w: ∃x_166.((x_166,w) ϵ f)}  ClassInt 1061
1065. p ϵ rg(f)  EqualitySub 1062 1064
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1067. ¬((p ϵ rg(f)) & ((n,p) ϵ s)) ForallElim 1066
1068. (n,p) ϵ s Hyp
1069. (p ϵ rg(f)) & ((n,p) ϵ s) AndInt 1065 1068
1070. _|_ ImpElim 1069 1067
1071. ¬((n,p) ϵ s) ImpInt 1070
1072. n = p Hyp
1074. n ϵ rg(f) EqualitySub 1065 1073
1075. _|_ ImpElim 1074 785
1076. ¬(n = p) ImpInt 1075
1078. Connects(s,y) & ∀x_169.(((x_169 ⊂ y) & ¬(x_169 = 0)) -> ∃z.First(s,x_169,z)) DefExp 1077
1080. ∀x_172.∀z.(((x_172 ϵ y) & (z ϵ y)) -> ((x_172 = z) v (((x_172,z) ϵ s) v ((z,x_172) ϵ s)))) DefExp 1079
1081. ∀z.(((n ϵ y) & (z ϵ y)) -> ((n = z) v (((n,z) ϵ s) v ((z,n) ϵ s)))) ForallElim 1080
1082. ((n ϵ y) & (p ϵ y)) -> ((n = p) v (((n,p) ϵ s) v ((p,n) ϵ s))) ForallElim 1081
1083. (p ϵ rg(f)) -> (p ϵ y) ForallElim 470
1084. p ϵ y ImpElim 1065 1083
1085. (n ϵ y) & (p ϵ y) AndInt 762 1084
1086. (n = p) v (((n,p) ϵ s) v ((p,n) ϵ s)) ImpElim 1085 1082
1087. n = p Hyp
1088. _|_ ImpElim 1087 1076
1089. (p,n) ϵ s AbsI 1088
1090. ((n,p) ϵ s) v ((p,n) ϵ s) Hyp
1091. (n,p) ϵ s Hyp
1092. _|_ ImpElim 1091 1071
1093. (p,n) ϵ s AbsI 1092
1094. (p,n) ϵ s Hyp
1095. (p,n) ϵ s OrElim 1090 1091 1093 1094 1094
1098. (p,(g'b)) ϵ s EqualitySub 1096 1097
1100. ((g'a),(g'b)) ϵ s EqualitySub 1098 1099
1101. ((g'a),(g'b)) ϵ s OrElim 956 957 1019 1020 1100
1102. (a,p) ϵ {(m,n)} Hyp
1103. Set((m,n)) -> (((a,p) ϵ {(m,n)}) <-> ((a,p) = (m,n))) ForallElim 1025
1104. ((a,p) ϵ {(m,n)}) <-> ((a,p) = (m,n)) ImpElim 820 1103
1107. (a,p) = (m,n) ImpElim 1102 1106
1109. Set((m,n)) & ((m,n) = (a,p)) AndInt 820 1108
1113. (Set((m,n)) & ((m,n) = (a,p))) -> ((m = a) & (n = p)) ForallElim 1112
1114. (m = a) & (n = p) ImpElim 1109 1113
1122. ¬((b ϵ dom(f)) & ((m,b) ϵ r)) ForallElim 1121
1123. (b,q) ϵ f Hyp
1124. ∃w.((b,q) ϵ w) ExistsInt 1123
1125. Set((b,q)) DefSub 1124
1126. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
1133. Set((b,q)) -> (Set(b) & Set(q)) ForallElim 1132
1134. Set(b) & Set(q) ImpElim 1125 1133
1136. ∃w.((b,w) ϵ f) ExistsInt 1123
1137. Set(b) & ∃w.((b,w) ϵ f) AndInt 1135 1136
1138. b ϵ {w: ∃x_174.((w,x_174) ϵ f)} ClassInt 1137
1141. b ϵ dom(f) EqualitySub 1138 1140
1142. (m,b) ϵ r EqualitySub 1119 1116
1143. (b ϵ dom(f)) & ((m,b) ϵ r) AndInt 1141 1142
1144. _|_ ImpElim 1143 1122
1145. ((g'a),(g'b)) ϵ s AbsI 1144
1146. (b,q) ϵ {(m,n)} Hyp
1147. (b,q) = (m,n) ImpElim 1146 1029
1149. Set((m,n)) & ((m,n) = (b,q)) AndInt 820 1148
1150. (m = b) & (n = q) ImpElim 1149 1041
1152. (m,b) ϵ r EqualitySub 1119 1116
1154. (m,m) ϵ r EqualitySub 1152 1153
1155. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x)) TheoremInt
1157. Asymmetric(r,x) & TransIn(r,x) ImpElim 1156 1155
1159. ∀y.∀z.(((y ϵ x) & (z ϵ x)) -> (((y,z) ϵ r) -> ¬((z,y) ϵ r))) DefExp 1158
1160. ∀z.(((m ϵ x) & (z ϵ x)) -> (((m,z) ϵ r) -> ¬((z,m) ϵ r))) ForallElim 1159

1161. ((m ϵ x) & (m ϵ x)) -> (((m,m) ϵ r) -> ¬((m,m) ϵ r)) ForallElim 1160
1163. (m ϵ x) & (m ϵ x) AndInt 1162 1162
1164. ((m,m) ϵ r) -> ¬((m,m) ϵ r) ImpElim 1163 1161
1165. ¬((m,m) ϵ r) ImpElim 1154 1164
1166. _|_ ImpElim 1154 1165
1167. ((g'a),(g'b)) ϵ s AbsI 1166
1168. ((g'a),(g'b)) ϵ s OrElim 956 1123 1145 1146 1167
1170. ((g'a),(g'b)) ϵ s ExistsElim 947 949 1169
1172. ((a ϵ dom(g)) & ((b ϵ dom(g)) & ((a,b) ϵ r))) -> (((g'a),(g'b)) ϵ s) ImpInt 1171
1173. ∀b.(((a ϵ dom(g)) & ((b ϵ dom(g)) & ((a,b) ϵ r))) -> (((g'a),(g'b)) ϵ s)) ForallInt 1172
1174. ∀a.∀b.(((a ϵ dom(g)) & ((b ϵ dom(g)) & ((a,b) ϵ r))) -> (((g'a),(g'b)) ϵ s)) ForallInt 1173
1175. a ϵ dom(g) Hyp
1178. dom(g) = {x: ∃y.((x,y) ϵ g)} ForallElim 1177
1179. a ϵ {x: ∃y.((x,y) ϵ g)} EqualitySub 1175 1178
1180. Set(a) & ∃y.((a,y) ϵ g) ClassElim 1179
1182. (a,b) ϵ g Hyp
1183. (a,b) ϵ (f ∪ {(m,n)}) EqualitySub 1182 789
1184. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
1193. ((a,b) ϵ (f ∪ {(m,n)})) -> (((a,b) ϵ f) v ((a,b) ϵ {(m,n)})) ForallElim 1192
1194. ((a,b) ϵ f) v ((a,b) ϵ {(m,n)}) ImpElim 1183 1193
1195. (a,b) ϵ f Hyp
1196. ∃b.((a,b) ϵ f) ExistsInt 1195
1198. Set(a) & ∃b.((a,b) ϵ f) AndInt 1197 1196
1199. a ϵ {w: ∃b.((w,b) ϵ f)} ClassInt 1198
1201. a ϵ dom(f) EqualitySub 1199 1200
1202. (a ϵ dom(f)) v (a ϵ {m}) OrIntR 1201
1210. ((a ϵ dom(f)) v (a ϵ {m})) -> (a ϵ (dom(f) ∪ {m})) ForallElim 1209
1211. a ϵ (dom(f) ∪ {m}) ImpElim 1202 1210
1212. (a,b) ϵ {(m,n)} Hyp
1213. Set((m,n)) & ((a,b) ϵ {(m,n)}) AndInt 820 1212
1214. Set(x) -> ((y ϵ {x}) <-> (y = x)) TheoremInt
1218. Set((m,n)) -> (((a,b) ϵ {(m,n)}) <-> ((a,b) = (m,n))) ForallElim 1217
1220. ((a,b) ϵ {(m,n)}) <-> ((a,b) = (m,n)) ImpElim 1219 1218
1223. (a,b) = (m,n) ImpElim 1212 1222
1225. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
1233. (Set((m,n)) & ((m,n) = (a,b))) -> ((m = a) & (n = b)) ForallElim 1232
1234. Set((m,n)) & ((m,n) = (a,b)) AndInt 820 1224
1235. (m = a) & (n = b) ImpElim 1234 1233
1237. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
1244. Set((m,n)) -> (Set(m) & Set(n)) ForallElim 1243
1245. Set(m) & Set(n) ImpElim 1219 1244
1247. Set(x) -> ((y ϵ {x}) <-> (y = x)) TheoremInt
1251. Set(m) -> ((a ϵ {m}) <-> (a = m)) ForallElim 1250
1252. (a ϵ {m}) <-> (a = m) ImpElim 1246 1251
1256. a ϵ {m} ImpElim 1255 1254
1257. (a ϵ dom(f)) v (a ϵ {m}) OrIntL 1256
1258. a ϵ (dom(f) ∪ {m}) ImpElim 1257 1210
1259. a ϵ (dom(f) ∪ {m}) OrElim 1194 1195 1211 1212 1258
1260. a ϵ (dom(f) ∪ {m}) ExistsElim 1181 1182 1259
1261. (a ϵ dom(g)) -> (a ϵ (dom(f) ∪ {m})) ImpInt 1260
1262. ∀a.((a ϵ dom(g)) -> (a ϵ (dom(f) ∪ {m}))) ForallInt 1261
1263. dom(g) ⊂ (dom(f) ∪ {m}) DefSub 1262
1264. a ϵ (dom(f) ∪ {m}) Hyp
1265. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
1274. (a ϵ (dom(f) ∪ {m})) -> ((a ϵ dom(f)) v (a ϵ {m})) ForallElim 1273
1275. (a ϵ dom(f)) v (a ϵ {m}) ImpElim 1264 1274
1276. a ϵ dom(f) Hyp
1278. a ϵ {x: ∃y.((x,y) ϵ f)} EqualitySub 1276 1277
1279. Set(a) & ∃y.((a,y) ϵ f) ClassElim 1278
1281. (a,b) ϵ f Hyp
1282. ((a,b) ϵ f) v ((a,b) ϵ {(m,n)}) OrIntR 1281

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1289. (((a,b) ϵ f) v ((a,b) ϵ {(m,n)})) -> ((a,b) ϵ (f ∪ {(m,n)}))  ForallElim 1288
1290. (a,b) ϵ (f ∪ {(m,n)})  ImpElim 1282 1289
1292. (a,b) ϵ g  EqualitySub 1290 1291
1293. ∃b.((a,b) ϵ g)  ExistsInt 1292
1295. Set(a) & ∃b.((a,b) ϵ g)  AndInt 1294 1293
1296. a ϵ {w: ∃b.((w,b) ϵ g)}  ClassInt 1295
1298. dom(g) = {x: ∃y.((x,y) ϵ g)}  ForallElim 1297
1300. a ϵ dom(g)  EqualitySub 1296 1299
1301. a ϵ dom(g)  ExistsElim 1280 1281 1300
1302. a ϵ {m}  Hyp
1303. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
1307. Set(m) -> ((a ϵ {m}) <-> (a = m))  ForallElim 1306
1308. (a ϵ {m}) <-> (a = m)  ImpElim 808 1307
1311. a = m  ImpElim 1302 1310
1315. Set((m,n)) -> (((m,n) ϵ {(m,n)}) <-> ((m,n) = (m,n)))  ForallElim 1314
1316. ((m,n) ϵ {(m,n)}) <-> ((m,n) = (m,n))  ImpElim 820 1315
1320. (m,n) ϵ {(m,n)}  ImpElim 1319 1318
1321. ((m,n) ϵ f) v ((m,n) ϵ {(m,n)})  OrIntL 1320
1327. (((m,n) ϵ f) v ((m,n) ϵ {(m,n)})) -> ((m,n) ϵ (f ∪ {(m,n)}))  ForallElim 1326
1328. (m,n) ϵ (f ∪ {(m,n)})  ImpElim 1321 1327
1329. (m,n) ϵ g  EqualitySub 1328 1291
1330. ∃n.((m,n) ϵ g)  ExistsInt 1329
1331. Set(m) & ∃n.((m,n) ϵ g)  AndInt 808 1330
1332. m ϵ {w: ∃n.((w,n) ϵ g)}  ClassInt 1331
1333. m ϵ dom(g)  EqualitySub 1332 1299
1335. a ϵ dom(g)  EqualitySub 1333 1334
1336. a ϵ dom(g)  OrElim 1275 1276 1301 1302 1335
1337. (a ϵ (dom(f) ∪ {m})) -> (a ϵ dom(g))  ImpInt 1336
1338. ∀a.((a ϵ (dom(f) ∪ {m})) -> (a ϵ dom(g)))  ForallInt 1337
1339. (dom(f) ∪ {m}) ⊂ dom(g)  DefSub 1338
1340. (dom(g) ⊂ (dom(f) ∪ {m})) & ((dom(f) ∪ {m}) ⊂ dom(g))  AndInt 1263 1339
1341. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
1347. ((dom(g) ⊂ (dom(f) ∪ {m})) & ((dom(f) ∪ {m}) ⊂ dom(g))) -> (dom(g) = (dom(f) ∪ {m}))  ForallElim 1346
1348. dom(g) = (dom(f) ∪ {m})  ImpElim 1340 1347
1349. a ϵ rg(g)  Hyp
1352. rg(g) = {y: ∃x.((x,y) ϵ g)}  ForallElim 1351
1353. a ϵ {y: ∃x.((x,y) ϵ g)}  EqualitySub 1349 1352
1354. Set(a) & ∃x.((x,a) ϵ g)  ClassElim 1353
1356. (b,a) ϵ g  Hyp
1357. (b,a) ϵ (f ∪ {(m,n)})  EqualitySub 1356 789
1359. ((b,a) ϵ (f ∪ {(m,n)})) -> (((b,a) ϵ f) v ((b,a) ϵ {(m,n)}))  ForallElim 1358
1360. ((b,a) ϵ f) v ((b,a) ϵ {(m,n)})  ImpElim 1357 1359
1361. (b,a) ϵ f  Hyp
1362. ∃b.((b,a) ϵ f)  ExistsInt 1361
1364. Set(a) & ∃b.((b,a) ϵ f)  AndInt 1363 1362
1365. a ϵ {w: ∃b.((b,w) ϵ f)}  ClassInt 1364
1368. a ϵ rg(f)  EqualitySub 1365 1367
1369. (a ϵ rg(f)) v (a ϵ {n})  OrIntR 1368
1370. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
1379. ((a ϵ rg(f)) v (a ϵ {n})) -> (a ϵ (rg(f) ∪ {n}))  ForallElim 1378
1380. a ϵ (rg(f) ∪ {n})  ImpElim 1369 1379
1381. (b,a) ϵ {(m,n)}  Hyp
1382. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
1386. Set((m,n)) -> (((b,a) ϵ {(m,n)}) <-> ((b,a) = (m,n)))  ForallElim 1385
1387. ((b,a) ϵ {(m,n)}) <-> ((b,a) = (m,n))  ImpElim 820 1386
1390. (b,a) = (m,n)  ImpElim 1381 1389
1392. Set((m,n)) & ((m,n) = (b,a))  AndInt 820 1391
1393. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
1401. (Set((m,n)) & ((m,n) = (b,a))) -> ((m = b) & (n = a))  ForallElim 1400
1402. (m = b) & (n = a)  ImpElim 1392 1401
1405. Set(m) & Set(n)  ImpElim 820 1244
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1410. Set(n) -> ((a ϵ {n}) <-> (a = n)) ForallElim 1409
1412. (a ϵ {n}) <-> (a = n) ImpElim 1411 1410
1415. a ϵ {n} ImpElim 1404 1414
1416. (a ϵ rg(f)) v (a ϵ {n}) OrIntL 1415
1417. a ϵ (rg(f) ∪ {n}) ImpElim 1416 1379
1418. a ϵ (rg(f) ∪ {n}) OrElim 1360 1361 1380 1381 1417
1419. a ϵ (rg(f) ∪ {n}) ExistsElim 1355 1356 1418
1420. (a ϵ rg(g)) -> (a ϵ (rg(f) ∪ {n})) ImpInt 1419
1421. ∀a.((a ϵ rg(g)) -> (a ϵ (rg(f) ∪ {n}))) ForallInt 1420
1422. rg(g) ⊂ (rg(f) ∪ {n}) DefSub 1421
1423. a ϵ dom(g) Hyp
1424. a ϵ (dom(f) ∪ {m}) EqualitySub 1423 1348
1425. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
1434. (a ϵ (dom(f) ∪ {m})) -> ((a ϵ dom(f)) v (a ϵ {m})) ForallElim 1433
1435. (a ϵ dom(f)) v (a ϵ {m}) ImpElim 1424 1434
1436. a ϵ dom(f) Hyp
1437. (a ϵ dom(f)) -> (a ϵ x) ForallElim 281
1438. a ϵ x ImpElim 1436 1437
1439. a ϵ {m} Hyp
1440. Set(x) -> ((y ϵ {x}) <-> (y = x)) TheoremInt
1444. Set(m) -> ((a ϵ {m}) <-> (a = m)) ForallElim 1443
1445. (a ϵ {m}) <-> (a = m) ImpElim 1406 1444
1448. a = m ImpElim 1439 1447
1450. a ϵ x EqualitySub 720 1449
1451. a ϵ x OrElim 1435 1436 1438 1439 1450
1452. (a ϵ dom(g)) -> (a ϵ x) ImpInt 1451
1453. ∀a.((a ϵ dom(g)) -> (a ϵ x)) ForallInt 1452
1454. dom(g) ⊂ x DefSub 1453
1455. a ϵ rg(g) Hyp
1456. (a ϵ rg(g)) -> (a ϵ (rg(f) ∪ {n})) ForallElim 1421
1457. a ϵ (rg(f) ∪ {n}) ImpElim 1455 1456
1461. (a ϵ (rg(f) ∪ {n})) -> ((a ϵ rg(f)) v (a ϵ {n})) ForallElim 1460
1462. (a ϵ rg(f)) v (a ϵ {n}) ImpElim 1457 1461
1463. a ϵ rg(f) Hyp
1464. (a ϵ rg(f)) -> (a ϵ y) ForallElim 470
1465. a ϵ y ImpElim 1463 1464
1466. a ϵ {n} Hyp
1468. Set(n) -> ((y ϵ {n}) <-> (y = n)) ForallElim 1467
1471. Set(n) -> ((a ϵ {n}) <-> (a = n)) ForallElim 1470
1472. (a ϵ {n}) <-> (a = n) ImpElim 1469 1471
1475. a = n ImpElim 1466 1474
1477. a ϵ y EqualitySub 762 1476
1478. a ϵ y OrElim 1462 1463 1465 1466 1477
1479. (a ϵ rg(g)) -> (a ϵ y) ImpInt 1478
1480. ∀a.((a ϵ rg(g)) -> (a ϵ y)) ForallInt 1479
1481. rg(g) ⊂ y DefSub 1480
1482. (WO(r,a) & (b ⊂ a)) -> WO(r,b) TheoremInt
1486. (WO(r,x) & (dom(g) ⊂ x)) -> WO(r,dom(g)) ForallElim 1485
1488. WO(r,x) & (dom(g) ⊂ x) AndInt 1487 1454
1489. WO(r,dom(g)) ImpElim 1488 1486
1496. (WO(s,y) & (rg(g) ⊂ y)) -> WO(s,rg(g)) ForallElim 1495
1497. WO(s,y) & (rg(g) ⊂ y) AndInt 1490 1481
1498. WO(s,rg(g)) ImpElim 1497 1496
1499. WO(r,dom(g)) & WO(s,rg(g)) AndInt 1489 1498
1500. FUN(g) & (WO(r,dom(g)) & WO(s,rg(g))) AndInt 934 1499
1501. ((a ϵ dom(g)) & (b ϵ dom(g))) & ((a,b) ϵ r) Hyp
1506. (b ϵ dom(g)) & ((a,b) ϵ r) AndInt 1505 1503
1507. (a ϵ dom(g)) & ((b ϵ dom(g)) & ((a,b) ϵ r)) AndInt 1504 1506
1508. ∀b.(((a ϵ dom(g)) & ((b ϵ dom(g)) & ((a,b) ϵ r))) -> (((g'a),(g'b)) ϵ s)) ForallElim 1174
1509. ((a ϵ dom(g)) & ((b ϵ dom(g)) & ((a,b) ϵ r))) -> (((g'a),(g'b)) ϵ s) ForallElim 1508
1510. ((g'a),(g'b)) ϵ s ImpElim 1507 1509

1511. (((a ϵ dom(g)) & (b ϵ dom(g))) & ((a,b) ϵ r)) -> (((g'a),(g'b)) ϵ s) ImpInt 1510
1512. ∀b.((((a ϵ dom(g)) & (b ϵ dom(g))) & ((a,b) ϵ r)) -> (((g'a),(g'b)) ϵ s)) ForallInt 1511
1513. ∀a.∀b.((((a ϵ dom(g)) & (b ϵ dom(g))) & ((a,b) ϵ r)) -> (((g'a),(g'b)) ϵ s)) ForallInt 1512
1514. (FUN(g) & (WO(r,dom(g)) & WO(s,rg(g)))) & ∀a.∀b.((((a ϵ dom(g)) & (b ϵ dom(g))) & ((a,b) ϵ r)) -> ((
(g'a),(g'b)) ϵ s)) AndInt 1500 1513
1515. OP(g,r,s) DefSub 1514
1516. ((a ϵ x) & (b ϵ dom(g))) & ((a,b) ϵ r) Hyp
1519. (b ϵ dom(g)) -> (b ϵ (dom(f) ∪ {m})) ForallElim 1262
1520. b ϵ (dom(f) ∪ {m}) ImpElim 1518 1519
1526. (b ϵ (dom(f) ∪ {m})) -> ((b ϵ dom(f)) v (b ϵ {m})) ForallElim 1525
1527. (b ϵ dom(f)) v (b ϵ {m}) ImpElim 1520 1526
1528. b ϵ dom(f) Hyp
1529. ((dom(f) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (u ϵ dom(f))) DefExp 287
1531. ∀v.((((a ϵ x) & (v ϵ dom(f))) & ((a,v) ϵ r)) -> (a ϵ dom(f))) ForallElim 1530
1532. (((a ϵ x) & (b ϵ dom(f))) & ((a,b) ϵ r)) -> (a ϵ dom(f)) ForallElim 1531
1534. (a ϵ x) & (b ϵ dom(f)) AndInt 1533 1528
1536. ((a ϵ x) & (b ϵ dom(f))) & ((a,b) ϵ r) AndInt 1534 1535
1537. a ϵ dom(f) ImpElim 1536 1532
1538. (a ϵ dom(f)) v (a ϵ {m}) OrIntR 1537
1545. ((a ϵ dom(f)) v (a ϵ {m})) -> (a ϵ (dom(f) ∪ {m})) ForallElim 1544
1546. a ϵ (dom(f) ∪ {m}) ImpElim 1538 1545
1547. b ϵ {m} Hyp
1548. Set(x) -> ((y ϵ {x}) <-> (y = x)) TheoremInt
1552. Set(m) -> ((b ϵ {m}) <-> (b = m)) ForallElim 1551
1553. (b ϵ {m}) <-> (b = m) ImpElim 1406 1552
1556. b = m ImpElim 1547 1555
1558. (a,m) ϵ r EqualitySub 1557 1556
1559. (m ϵ (x ~ dom(f))) & ∀y.((y ϵ (x ~ dom(f))) -> ¬((y,m) ϵ r)) DefExp 708
1561. (a ϵ (x ~ dom(f))) -> ¬((a,m) ϵ r) ForallElim 1560
1562. ¬(a ϵ dom(f)) Hyp
1563. ∃w.(a ϵ w) ExistsInt 1533
1564. Set(a) DefSub 1563
1565. Set(a) & ¬(a ϵ dom(f)) AndInt 1564 1562
1566. a ϵ {w: ¬(w ϵ dom(f))} ClassInt 1565
1569. ~dom(f) = {y: ¬(y ϵ dom(f))} ForallElim 1568
1571. a ϵ ~dom(f) EqualitySub 1566 1570
1572. (a ϵ x) & (a ϵ ~dom(f)) AndInt 1533 1571
1573. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y))) TheoremInt
1580. ((a ϵ x) & (a ϵ ~dom(f))) -> (a ϵ (x ∩ ~dom(f))) ForallElim 1579
1581. a ϵ (x ∩ ~dom(f)) ImpElim 1572 1580
1584. (x ~ dom(f)) = (x ∩ ~dom(f)) ForallElim 1583
1586. a ϵ (x ~ dom(f)) EqualitySub 1581 1585
1587. ¬((a,m) ϵ r) ImpElim 1586 1561
1588. _|_ ImpElim 1558 1587
1589. ¬¬(a ϵ dom(f)) ImpInt 1588
1590. D <-> ¬¬D TheoremInt
1593. ¬¬(a ϵ dom(f)) -> (a ϵ dom(f)) PolySub 1592
1594. a ϵ dom(f) ImpElim 1589 1593
1595. (a ϵ dom(f)) v (a ϵ {m}) OrIntR 1594
1596. a ϵ (dom(f) ∪ {m}) ImpElim 1595 1545
1597. a ϵ (dom(f) ∪ {m}) OrElim 1527 1528 1546 1547 1596
1599. a ϵ dom(g) EqualitySub 1597 1598
1600. (((a ϵ x) & (b ϵ dom(g))) & ((a,b) ϵ r)) -> (a ϵ dom(g)) ImpInt 1599
1601. ∀b.((((a ϵ x) & (b ϵ dom(g))) & ((a,b) ϵ r)) -> (a ϵ dom(g))) ForallInt 1600
1602. ∀a.∀b.((((a ϵ x) & (b ϵ dom(g))) & ((a,b) ϵ r)) -> (a ϵ dom(g))) ForallInt 1601
1604. (dom(g) ⊂ x) & WO(r,x) AndInt 1454 1603
1605. ((dom(g) ⊂ x) & WO(r,x)) & ∀a.∀b.((((a ϵ x) & (b ϵ dom(g))) & ((a,b) ϵ r)) -> (a ϵ dom(g))) AndInt 1604 16
1606. Sec(r,x,dom(g)) DefSub 1605
1607. ((a ϵ y) & (b ϵ rg(g))) & ((a,b) ϵ s) Hyp
1610. (b ϵ rg(g)) -> (b ϵ (rg(f) ∪ {n})) ForallElim 1421
1611. b ϵ (rg(f) ∪ {n}) ImpElim 1609 1610

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1612. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
1621. (b ϵ (rg(f) ∪ {n})) -> ((b ϵ rg(f)) v (b ϵ {n}))  ForallElim 1620
1622. (b ϵ rg(f)) v (b ϵ {n})  ImpElim 1611 1621
1623. b ϵ rg(f)  Hyp
1624. ((rg(f) ⊂ y) & WO(s,y)) & ∀u.∀v.((((u ϵ y) & (v ϵ rg(f))) & ((u,v) ϵ s)) -> (u ϵ rg(f)))  DefExp 477
1626. ∀v.((((a ϵ y) & (v ϵ rg(f))) & ((a,v) ϵ s)) -> (a ϵ rg(f)))  ForallElim 1625
1627. (((a ϵ y) & (b ϵ rg(f))) & ((a,b) ϵ s)) -> (a ϵ rg(f))  ForallElim 1626
1629. (a ϵ y) & (b ϵ rg(f))  AndInt 1628 1623
1631. ((a ϵ y) & (b ϵ rg(f))) & ((a,b) ϵ s)  AndInt 1629 1630
1632. a ϵ rg(f)  ImpElim 1631 1627
1633. b ϵ {n}  Hyp
1634. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
1639. Set(n) -> ((b ϵ {n}) <-> (b = n))  ForallElim 1638
1640. (b ϵ {n}) <-> (b = n)  ImpElim 1635 1639
1643. b = n  ImpElim 1633 1642
1645. (n ϵ (y ~ rg(f))) & ∀x_206.((x_206 ϵ (y ~ rg(f))) -> ¬((x_206,n) ϵ s))  DefExp 709
1647. (a ϵ (y ~ rg(f))) -> ¬((a,n) ϵ s)  ForallElim 1646
1648. (a,n) ϵ s  EqualitySub 1630 1643
1649. ¬(a ϵ rg(f))  Hyp
1650. ∃w.(a ϵ w)  ExistsInt 1628
1651. Set(a)  DefSub 1650
1652. Set(a) & ¬(a ϵ rg(f))  AndInt 1651 1649
1653. a ϵ {w: ¬(w ϵ rg(f))}  ClassInt 1652
1656. ~rg(f) = {y: ¬(y ϵ rg(f))}  ForallElim 1655
1658. a ϵ ~rg(f)  EqualitySub 1653 1657
1659. (a ϵ y) & (a ϵ ~rg(f))  AndInt 1628 1658
1668. ((a ϵ y) & (a ϵ ~rg(f))) -> (a ϵ (y ∩ ~rg(f)))  ForallElim 1667
1669. a ϵ (y ∩ ~rg(f))  ImpElim 1659 1668
1674. (y ~ rg(f)) = (y ∩ ~rg(f))  ForallElim 1673
1676. a ϵ (y ~ rg(f))  EqualitySub 1669 1675
1677. ¬((a,n) ϵ s)  ImpElim 1676 1647
1678. _|_  ImpElim 1648 1677
1679. ¬¬(a ϵ rg(f))  ImpInt 1678
1680. ¬¬(a ϵ rg(f)) -> (a ϵ rg(f))  PolySub 1592
1681. a ϵ rg(f)  ImpElim 1679 1680
1682. a ϵ rg(f)  OrElim 1622 1623 1632 1633 1681
1684. a ϵ {y: ∃x.((x,y) ϵ f)}  EqualitySub 1682 1683
1685. Set(a) & ∃x.((x,a) ϵ f)  ClassElim 1684
1687. (b,a) ϵ f  Hyp
1688. ((b,a) ϵ f) v ((b,a) ϵ {(m,n)})  OrIntR 1687
1695. (((b,a) ϵ f) v ((b,a) ϵ {(m,n)})) -> ((b,a) ϵ (f ∪ {(m,n)}))  ForallElim 1694
1696. (b,a) ϵ (f ∪ {(m,n)})  ImpElim 1688 1695
1698. (b,a) ϵ g  EqualitySub 1696 1697
1699. ∃b.((b,a) ϵ g)  ExistsInt 1698
1700. ∃b.((b,a) ϵ g)  ExistsElim 1686 1687 1699
1702. Set(a) & ∃b.((b,a) ϵ g)  AndInt 1701 1700
1703. a ϵ {w: ∃b.((b,w) ϵ g)}  ClassInt 1702
1707. {y: ∃x.((x,y) ϵ g)} = rg(g)  ForallElim 1706
1708. a ϵ rg(g)  EqualitySub 1703 1707
1709. (((a ϵ y) & (b ϵ rg(g))) & ((a,b) ϵ s)) -> (a ϵ rg(g))  ImpInt 1708
1710. ∀b.((((a ϵ y) & (b ϵ rg(g))) & ((a,b) ϵ s)) -> (a ϵ rg(g)))  ForallInt 1709
1711. ∀a.∀b.((((a ϵ y) & (b ϵ rg(g))) & ((a,b) ϵ s)) -> (a ϵ rg(g)))  ForallInt 1710
1713. WO(s,y) & (rg(g) ⊂ y)  AndInt 1712 1481
1714. (rg(g) ⊂ y) & WO(s,y)  AndInt 1481 1712
1715. ((rg(g) ⊂ y) & WO(s,y)) & ∀a.∀b.((((a ϵ y) & (b ϵ rg(g))) & ((a,b) ϵ s)) -> (a ϵ rg(g)))  AndInt 1714 1711
1716. Sec(s,y,rg(g))  DefSub 1715
1717. Set(x) -> ((y ϵ {x}) <-> (y = x))  TheoremInt
1721. Set((m,n)) -> (((m,n) ϵ {(m,n)}) <-> ((m,n) = (m,n)))  ForallElim 1720
1723. ((m,n) ϵ {(m,n)}) <-> ((m,n) = (m,n))  ImpElim 820 1721
1727. (m,n) ϵ {(m,n)}  ImpElim 1726 1725
1728. ((m,n) ϵ f) v ((m,n) ϵ {(m,n)})  OrIntL 1727
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1734. (((m,n) ϵ f) v ((m,n) ϵ {(m,n)})) -> ((m,n) ϵ (f ∪ {(m,n)}))  ForallElim 1733
1735. (m,n) ϵ (f ∪ {(m,n)})  ImpElim 1728 1734
1737. (m,n) ϵ g  EqualitySub 1735 1736
1738. ∃n.((m,n) ϵ g)  ExistsInt 1737
1739. Set(m) & ∃n.((m,n) ϵ g)  AndInt 808 1738
1740. m ϵ {w: ∃n.((w,n) ϵ g)}  ClassInt 1739
1743. dom(g) = {x: ∃y.((x,y) ϵ g)}  ForallElim 1742
1745. m ϵ dom(g)  EqualitySub 1740 1744
1746. (m ϵ dom(g)) & ((m,n) ϵ g)  AndInt 1745 1737
1747. Sec(s,y,rg(g)) & ((m ϵ dom(g)) & ((m,n) ϵ g))  AndInt 1716 1746
1748. Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((m ϵ dom(g)) & ((m,n) ϵ g)))  AndInt 1606 1747
1749. OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((m ϵ dom(g)) & ((m,n) ϵ g))))  AndInt 1515 1748
1750. ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((m ϵ dom(g)) & ((m,n) ϵ g)))))  ExistsInt 1749
1751. (m ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((m ϵ dom(g)) & ((m,n) ϵ g)))))  AndInt 1162
1752. w = (m,n)  Hyp
1753. (w = (m,n)) & ((m ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((m ϵ dom(g)) & ((m,n)
ϵ g))))))  AndInt 1752 1751
1755. ∃m.∃n.((w = (m,n)) & ((m ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((m ϵ dom(g)) & ((m,n)
1757. Set(w)  EqualitySub 820 1756
1758. Set(w) & ∃m.∃n.((w = (m,n)) & ((m ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((m ϵ
dom(g)) & ((m,n) ϵ g)))))))  AndInt 1757 1755
1759. w ϵ {w: ∃m.∃n.((w = (m,n)) & ((m ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((m ϵ d
om(g)) & ((m,n) ϵ g)))))))}  ClassInt 1758
1760. (m,n) ϵ {w: ∃x_211.∃x_212.((w = (x_211,x_212)) & ((x_211 ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (
Sec(s,y,rg(g)) & ((x_211 ϵ dom(g)) & ((x_211,x_212) ϵ g)))))))}  EqualitySub 1759 1752
1762. (m,n) ϵ f  EqualitySub 1760 1761
1763. (w = (m,n)) -> ((m,n) ϵ f)  ImpInt 1762
1765. ((m,n) = (m,n)) -> ((m,n) ϵ f)  ForallElim 1764
1767. (m,n) ϵ f  ImpElim 1766 1765
1768. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))  TheoremInt
1772. ((m,n) ϵ f) -> ((m ϵ dom(f)) & (n ϵ rg(f)))  ForallElim 1771
1773. (m ϵ dom(f)) & (n ϵ rg(f))  ImpElim 1767 1772
1775. (g = (f ∪ {(m,n)})) -> (m ϵ dom(f))  ImpInt 1774
1777. ((f ∪ {(m,n)}) = (f ∪ {(m,n)})) -> (m ϵ dom(f))  ForallElim 1776
1779. m ϵ dom(f)  ImpElim 1778 1777
1780. m ϵ dom(f)  ExistsElim 707 709 1779
1781. (m ϵ (x ~ dom(f))) & ∀y.((y ϵ (x ~ dom(f))) -> ¬((y,m) ϵ r))  DefExp 708
1785. (x ~ dom(f)) = (x ∩ ~dom(f))  ForallElim 1784
1786. m ϵ (x ∩ ~dom(f))  EqualitySub 1782 1785
1787. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))  TheoremInt
1794. (m ϵ (x ∩ ~dom(f))) -> ((m ϵ x) & (m ϵ ~dom(f)))  ForallElim 1793
1795. (m ϵ x) & (m ϵ ~dom(f))  ImpElim 1786 1794
1799. ~dom(f) = {y: ¬(y ϵ dom(f))}  ForallElim 1798
1800. m ϵ {y: ¬(y ϵ dom(f))}  EqualitySub 1796 1799
1801. Set(m) & ¬(m ϵ dom(f))  ClassElim 1800
1803. _|_  ImpElim 1780 1802
1804. _|_  ExistsElim 700 708 1803
1805. ¬(¬((x ~ dom(f)) = 0) & ¬((y ~ rg(f)) = 0))  ImpInt 1804
1806. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
1808. ¬(¬((x ~ dom(f)) = 0) & B) <-> (¬¬((x ~ dom(f)) = 0) v ¬B)  PolySub 1807
1809. ¬(¬((x ~ dom(f)) = 0) & ¬((y ~ rg(f)) = 0)) <-> (¬¬((x ~ dom(f)) = 0) v ¬¬((y ~ rg(f)) = 0))  PolySub 180
1812. ¬¬((x ~ dom(f)) = 0) v ¬¬((y ~ rg(f)) = 0)  ImpElim 1805 1811
1813. ¬¬((y ~ rg(f)) = 0)  Hyp
1814. ¬¬((y ~ rg(f)) = 0) -> ((y ~ rg(f)) = 0)  PolySub 1592
1815. (y ~ rg(f)) = 0  ImpElim 1813 1814
1816. ((x ~ dom(f)) = 0) v ((y ~ rg(f)) = 0)  OrIntL 1815
1817. ¬¬((x ~ dom(f)) = 0)  Hyp
1818. ¬¬((x ~ dom(f)) = 0) -> ((x ~ dom(f)) = 0)  PolySub 1592
1819. (x ~ dom(f)) = 0  ImpElim 1817 1818
1820. ((x ~ dom(f)) = 0) v ((y ~ rg(f)) = 0)  OrIntR 1819
1821. ((x ~ dom(f)) = 0) v ((y ~ rg(f)) = 0)  OrElim 1812 1817 1820 1813 1816
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1822. ((y ⊂ x) & ((x ˜ y) = 0)) -> (x = y)  TheoremInt
1828. ((rg(f) ⊂ y) & ((y ˜ rg(f)) = 0)) -> (y = rg(f))  ForallElim 1827
1829. (dom(f) ⊂ x) & (rg(f) ⊂ y)  AndInt 282 471
1830. (x ˜ dom(f)) = 0  Hyp
1832. (dom(f) ⊂ x) & ((x ˜ dom(f)) = 0)  AndInt 1831 1830
1833. x = dom(f)  ImpElim 1832 1824
1834. (x = dom(f)) v (y = rg(f))  OrIntR 1833
1835. (y ˜ rg(f)) = 0  Hyp
1837. (rg(f) ⊂ y) & ((y ˜ rg(f)) = 0)  AndInt 1836 1835
1838. y = rg(f)  ImpElim 1837 1828
1839. (x = dom(f)) v (y = rg(f))  OrIntL 1838
1840. (x = dom(f)) v (y = rg(f))  OrElim 1821 1830 1834 1835 1839
1841. (OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = rg(f)))  AndInt 661 1840
1842. ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = rg(f))))  ExistsInt 1841
1843. (f = {w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ
dom(g)) & ((u,v) ϵ g)))))))}) -> ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f))
v (y = rg(f))))  ImpInt 1842
1845. ({w: ∃u.∃v.((w = (u,v)) & ((u ϵ x) & ∃g.(OP(g,r,s) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(g)) & ((u ϵ dom(
g)) & ((u,v) ϵ g)))))))} = {x_217: ∃x_218.∃x_219.((x_217 = (x_218,x_219)) & ((x_218 ϵ x) & ∃x_220.(OP
x_220,r,s) & (Sec(r,x,dom(x_220)) & (Sec(s,y,rg(x_220)) & ((x_218 ϵ dom(x_220)) & ((x_218,x_219) ϵ x_220)))))))})
1847. ∃x_216.((OP(x_216,r,s) & (Sec(r,x,dom(x_216)) & Sec(s,y,rg(x_216)))) & ((x = dom(x_216)) v (y = rg(x_216)))
1848. (OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = rg(f)))  Hyp
1849. ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = rg(f))))  ExistsInt 1848
1850. ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = rg(f))))  ExistsElim 1847 1848
1851. (WO(r,x) & WO(s,y)) -> ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = r
g(f))))  ImpInt 1850 Qed
```

Used Theorems

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1. (OP(f,r,s) & (OP(g,r,s) & (Sec(r,x,dom(f)) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(f)) & Sec(s,y,rg(g))))))
) -> ((f ⊂ g) v (g ⊂ f))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
4. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))
5. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))
6. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)
7. (WO(r,a) & (b ⊂ a)) -> WO(r,b)
8. ((z ϵ (x ∪ y)) <-> ((z ϵ x) v (z ϵ y))) & ((z ϵ (x ∩ y)) <-> ((z ϵ x) & (z ϵ y)))
9. Set(x) -> ((y ϵ {x}) <-> (y = x))
10. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))
11. (FUN(f) & ((a,b) ϵ f)) -> ((f'a) = b)
12. WO(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
13. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
14. D <-> ¬¬D
15. ((a,b) ϵ f) -> ((a ϵ dom(f)) & (b ϵ rg(f)))
16. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
17. ((y ⊂ x) & ((x ˜ y) = 0)) -> (x = y)
```

Th100aux. (FUN(f) & (FUN(g) & ((dom(f) = dom(g)) & (f ⊂ g)))) -> (f = g)

```
0. FUN(f) & (FUN(g) & ((dom(f) = dom(g)) & (f ⊂ g)))  Hyp
1. x ϵ g  Hyp
4. Relation(g) & ∀x.∀y.∀z.((((x,y) ϵ g) & ((x,z) ϵ g)) -> (y = z))  DefExp 3
6. ∀z.((z ϵ g) -> ∃x.∃y.(z = (x,y)))  DefExp 5
7. (x ϵ g) -> ∃x_3.∃y.(x = (x_3,y))  ForallElim 6
8. ∃x_3.∃y.(x = (x_3,y))  ImpElim 1 7
9. ∃y.(x = (n,y))  Hyp
10. x = (n,y)  Hyp
11. (n,y) ϵ g  EqualitySub 1 10
13. ∃c.((n,y) ϵ c)  ExistsInt 11
14. Set((n,y))  DefSub 13
```

```
15. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
20. Set((n,y)) -> (Set(n) & Set(y))  ForallElim 19
21. Set(n) & Set(y)  ImpElim 14 20
23. Set(n) & ∃b.((n,b) ϵ g)  AndInt 22 12
24. n ϵ {m: ∃b.((m,b) ϵ g)}  ClassInt 23
28. {x: ∃y.((x,y) ϵ g)} = dom(g)  ForallElim 27
29. n ϵ dom(g)  EqualitySub 24 28
33. n ϵ dom(f)  EqualitySub 29 32
34. n ϵ {x: ∃y.((x,y) ϵ f)}  EqualitySub 33 25
35. Set(n) & ∃y.((n,y) ϵ f)  ClassElim 34
37. (n,z) ϵ f  Hyp
40. ∀z.((z ϵ f) -> (z ϵ g))  DefExp 39
41. ((n,z) ϵ f) -> ((n,z) ϵ g)  ForallElim 40
42. (n,z) ϵ g  ImpElim 37 41
44. ∀y.∀z.((((n,y) ϵ g) & ((n,z) ϵ g)) -> (y = z))  ForallElim 43
45. ∀z.((((n,y) ϵ g) & ((n,z) ϵ g)) -> (y = z))  ForallElim 44
46. (((n,y) ϵ g) & ((n,z) ϵ g)) -> (y = z)  ForallElim 45
47. ((n,y) ϵ g) & ((n,z) ϵ g)  AndInt 11 42
48. y = z  ImpElim 47 46
49. x = (n,z)  EqualitySub 10 48
51. x ϵ f  EqualitySub 37 50
52. x ϵ f  ExistsElim 9 10 51
57. (x ϵ g) -> (x ϵ f)  ImpInt 56
58. ∀x.((x ϵ g) -> (x ϵ f))  ForallInt 57
59. g ⊂ f  DefSub 58
60. (f ⊂ g) & (g ⊂ f)  AndInt 39 59
61. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
67. ((f ⊂ g) & (g ⊂ f)) -> (f = g)  ForallElim 66
68. f = g  ImpElim 60 67
69. (FUN(f) & (FUN(g) & ((dom(f) = dom(g)) & (f ⊂ g)))) -> (f = g)  ImpInt 68 Qed
```

Used Theorems

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1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
```

```
Th100. ((WO(r,x) & (WO(s,y) & (Set(x) & ¬Set(y)))) -> ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f))))
& (x = dom(f)))) & ((((OP(g,r,s) & (Sec(r,x,dom(g)) & Sec(s,y,rg(g)))) & (x = dom(g))) & ((OP(h,r,s)
(Sec(r,x,dom(h)) & Sec(s,y,rg(h)))) & (x = dom(h)))) -> (g = h))
```

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0. WO(r,x) & (WO(s,y) & (Set(x) & ¬Set(y)))  Hyp
4. WO(r,x) & WO(s,y)  AndInt 1 3
5. (WO(r,x) & WO(s,y)) -> ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = r
g(f))))  TheoremInt
6. ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = rg(f))))  ImpElim 4 5
7. (OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = rg(f)))  Hyp
10. (FUN(f) & (WO(r,dom(f)) & WO(s,rg(f)))) & ∀u.∀v.((((u ϵ dom(f)) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> ((
(f'u),(f'v)) ϵ s))  DefExp 9
13. (FUN(f) & Set(dom(f))) -> Set(rg(f))  AxInt
18. ((dom(f) ⊂ x) & WO(r,x)) & ∀u.∀v.((((u ϵ x) & (v ϵ dom(f))) & ((u,v) ϵ r)) -> (u ϵ dom(f)))  DefExp 17
21. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
26. (Set(x) & (dom(f) ⊂ x)) -> Set(dom(f))  ForallElim 25
27. (FUN(f) & Set(dom(f))) -> Set(rg(f))  AxInt
28. Set(x) & (dom(f) ⊂ x)  AndInt 24 20
29. Set(dom(f))  ImpElim 28 26
30. FUN(f) & Set(dom(f))  AndInt 12 29
31. Set(rg(f))  ImpElim 30 27
32. x = dom(f)  Hyp
33. y = rg(f)  Hyp
35. Set(y)  EqualitySub 31 34
37. _|_  ImpElim 35 36
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38. x = dom(f)  AbsI 37
39. x = dom(f)  OrElim 14 32 32 33 38
40. (OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & (x = dom(f))  AndInt 8 39
41. ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & (x = dom(f)))  ExistsInt 40
42. ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & (x = dom(f)))  ExistsElim 6 7 41
43. ((OP(g,r,s) & (Sec(r,x,dom(g)) & Sec(s,y,rg(g)))) & (x = dom(g))) & ((OP(h,r,s) & (Sec(r,x,dom(h)) &
Sec(s,y,rg(h)))) & (x = dom(h)))  Hyp
56. Sec(s,y,rg(g)) & Sec(s,y,rg(h))  AndInt 49 55
57. Sec(r,x,dom(h)) & (Sec(s,y,rg(g)) & Sec(s,y,rg(h)))  AndInt 54 56
58. Sec(r,x,dom(g)) & (Sec(r,x,dom(h)) & (Sec(s,y,rg(g)) & Sec(s,y,rg(h))))  AndInt 50 57
59. OP(h,r,s) & (Sec(r,x,dom(g)) & (Sec(r,x,dom(h)) & (Sec(s,y,rg(g)) & Sec(s,y,rg(h)))))  AndInt 52 58
60. OP(g,r,s) & (OP(h,r,s) & (Sec(r,x,dom(g)) & (Sec(r,x,dom(h)) & (Sec(s,y,rg(g)) & Sec(s,y,rg(h))))))
AndInt 47 59
61. (OP(f,r,s) & (OP(g,r,s) & (Sec(r,x,dom(f)) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(f)) & Sec(s,y,rg(g))))))
) -> ((f ⊂ g) v (g ⊂ f))  TheoremInt
65. (OP(g,r,s) & (OP(h,r,s) & (Sec(r,x,dom(g)) & (Sec(r,x,dom(h)) & (Sec(s,y,rg(g)) & Sec(s,y,rg(h))))))
) -> ((g ⊂ h) v (h ⊂ g))  ForallElim 64
66. (g ⊂ h) v (h ⊂ g)  ImpElim 60 65
70. dom(g) = dom(h)  EqualitySub 69 68
71. (FUN(g) & (WO(r,dom(g)) & WO(s,rg(g)))) & ∀u.∀v.((((u ϵ dom(g)) & (v ϵ dom(g))) & ((u,v) ϵ r)) -> ((
(g'u),(g'v)) ϵ s))  DefExp 47
72. (FUN(h) & (WO(r,dom(h)) & WO(s,rg(h)))) & ∀u.∀v.((((u ϵ dom(h)) & (v ϵ dom(h))) & ((u,v) ϵ r)) -> ((
(h'u),(h'v)) ϵ s))  DefExp 52
77. (FUN(f) & (FUN(g) & ((dom(f) = dom(g)) & (f ⊂ g)))) -> (f = g)  TheoremInt
81. (FUN(g) & (FUN(h) & ((dom(g) = dom(h)) & (g ⊂ h)))) -> (g = h)  ForallElim 80
82. g ⊂ h  Hyp
83. (dom(g) = dom(h)) & (g ⊂ h)  AndInt 70 82
84. FUN(h) & ((dom(g) = dom(h)) & (g ⊂ h))  AndInt 76 83
85. FUN(g) & (FUN(h) & ((dom(g) = dom(h)) & (g ⊂ h)))  AndInt 74 84
86. g = h  ImpElim 85 81
87. h ⊂ g  Hyp
89. (FUN(h) & (FUN(g) & ((dom(h) = dom(g)) & (h ⊂ g)))) -> (h = g)  ForallElim 88
91. (dom(h) = dom(g)) & (h ⊂ g)  AndInt 90 87
92. FUN(g) & ((dom(h) = dom(g)) & (h ⊂ g))  AndInt 74 91
93. FUN(h) & (FUN(g) & ((dom(h) = dom(g)) & (h ⊂ g)))  AndInt 76 92
94. h = g  ImpElim 93 89
96. g = h  OrElim 66 82 86 87 95
97. (((OP(g,r,s) & (Sec(r,x,dom(g)) & Sec(s,y,rg(g)))) & (x = dom(g))) & ((OP(h,r,s) & (Sec(r,x,dom(h))
& Sec(s,y,rg(h)))) & (x = dom(h)))) -> (g = h)  ImpInt 96
98. (WO(r,x) & (WO(s,y) & (Set(x) & ¬Set(y)))) -> ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) &
(x = dom(f)))  ImpInt 42
99. ((WO(r,x) & (WO(s,y) & (Set(x) & ¬Set(y)))) -> ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f))))
& (x = dom(f)))) & ((((OP(g,r,s) & (Sec(r,x,dom(g)) & Sec(s,y,rg(g)))) & (x = dom(g))) & ((OP(h,r,s)
(Sec(r,x,dom(h)) & Sec(s,y,rg(h)))) & (x = dom(h)))) -> (g = h))  AndInt 98 97 Qed

Used Theorems

1. (WO(r,x) & WO(s,y)) -> ∃f.((OP(f,r,s) & (Sec(r,x,dom(f)) & Sec(s,y,rg(f)))) & ((x = dom(f)) v (y = r
g(f))))
3. (Set(x) & (y ⊂ x)) -> Set(y)
2. (OP(f,r,s) & (OP(g,r,s) & (Sec(r,x,dom(f)) & (Sec(r,x,dom(g)) & (Sec(s,y,rg(f)) & Sec(s,y,rg(g))))))
) -> ((f ⊂ g) v (g ⊂ f))
4. (FUN(f) & (FUN(g) & ((dom(f) = dom(g)) & (f ⊂ g)))) -> (f = g)
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