Welcome to PyLog 1.0

Natural Deduction Proof Assistant and Proof Checker


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>>> Load("Kelley-Morse")
True
>>> ShowAxioms()
0. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))
1. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y)))
2. (Set(x) & Set(y)) -> Set((x ∪ y))
3. (Function(f) & Set(domain(f))) -> Set(range(f))
4. Set(x) -> Set(∪x)
5. ¬(x = 0) -> ∃y.((y ε x) & ((y ∩ x) = 0))
6. ∃y.((Set(y) & (0 ε y)) & ∀x.((x ε y) -> (suc x ε y)))
7. ∃f.(Choice(f) & (domain(f) = (U ~ {0})))
>>> ShowDefinitions()
Set(x) <-> ∃y.(x ε y)
(x ⊂ y) <-> ∀z.((z ε x) -> (z ε y))
Relation(r) <-> ∀z.((z ε r) -> ∃x.∃y.(z = (x,y)))
Function(f) <-> (Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)))
Trans(r) <-> ∀x.∀y.∀z.((((x,y) ε r) & ((y,z) ε r)) -> ((x,z) ε r))
Connects(r,x) <-> ∀y.∀z.(((y ε x) & (z ε x)) -> ((y = z) v (((y,z) ε r) v ((z,y) ε r))))
Asymmetric(r,x) <-> ∀y.∀z.(((y ε x) & (z ε x)) -> (((y,z) ε r) -> ¬((z,y) ε r)))
First(r,x,z) <-> ((z ε x) & ∀y.((y ε x) -> ¬((y,z) ε r)))
WellOrders(r,x) <-> (Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z)))
Section(r,x,y) <-> (((y ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε y)) & ((u,v) ε
r)) -> (u ε y)))
OrderPreserving(f,r,s) <-> ((Function(f) & (WellOrders(r,domain(f)) &
WellOrders(r,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) ->
(((f'u),(f'v)) ε r)))
1-to-1(f) <-> (Function(f) & Function((f)⁻¹))
Full(x) <-> ∀y.((y ε x) -> (y ⊂ x))
Ordinal(x) <-> (Full(x) & Connects(E,x))
Integer(x) <-> (Ordinal(x) & WellOrders((E)⁻¹,x))
Choice(f) <-> (Function(f) & ∀y.((y ε domain(f)) -> ((f'y) ε y)))
Equi(x,y) <-> ∃f.(1-to-1(f) & ((domain(f) = x) & (range(f) = y)))
Card(x) <-> (Ordinal(x) & ∀y.(((y ε x) & (y ε ord)) -> ¬Equi(y,x)))
TransIn(r,x) <-> ∀u.∀v.∀w.(((u ε x) & ((v ε x) & (w ε x))) -> ((((u,v) ε r) & ((v,w) ε
r)) -> ((u,w) ε r)))
>>> ShowDefEquations()
0. (x ∪ y) = {z: ((z ε x) v (z ε y))}
1. (x ∩ y) = {z: ((z ε x) & (z ε y))}
2. ~x = {y: ¬(y ε x)}
3. (x ~ y) = (x ∩ ~y)
4. 0 = {x: ¬(x = x)}
5. U = {x: (x = x)}
6. ∪x = {z: ∃y.((y ε x) & (z ε y))}
7. ∩x = {z: ∀y.((y ε x) -> (z ε y))}
8. Px = {y: (y ⊂ x)}
9. {x} = {z: ((x ε U) -> (z = x))}
10. {x,y} = ({x} ∪ {y})
11. (x,y) = {{x},{x,y}}
12. proj1(x) = ∩∩x
13. proj2(x) = (∩∪x ∪ (∪∪x ~ ∪∩x))
14. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}
15. (r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}
16. domain(f) = {x: ∃y.((x,y) ε f)}
17. range(f) = {y: ∃x.((x,y) ε f)}
18. (f'x) = ∩{y: ((x,y) ε f)}
19. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}
20. func(x,y) = {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))}
21. E = {z: ∃x.∃y.((z = (x,y)) & (x ε y))}
22. ord = {x: Ordinal(x)}
23. suc x = (x ∪ {x})
24. (f|x) = (f ∩ (x X U))
25. ω = {x: Integer(x)}
>>> Test()
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Th4. ((z ε (x U y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

0. z ε (x U y) Hyp
1. (x U y) = {z: ((z ε x) v (z ε y))} DefEqInt
2. z ε {z: ((z ε x) v (z ε y))} EqualitySub 0 1
3. Set(z) & ((z ε x) v (z ε y)) ClassElim 2
4. (z ε x) v (z ε y) AndElimR 3
5. (z ε (x U y)) -> ((z ε x) v (z ε y)) ImpInt 4
6. (z ε x) v (z ε y) Hyp
7. z ε x Hyp
8. ∃x.(z ε x) ExistsInt 7
9. Set(z) DefSub 8
10. z ε y Hyp
11. ∃y.(z ε y) ExistsInt 10
12. Set(z) DefSub 11
13. Set(z) OrElim 6 7 9 10 12
14. Set(z) & ((z ε x) v (z ε y)) AndInt 13 6
15. z ε {z: ((z ε x) v (z ε y))} ClassInt 14
16. {z: ((z ε x) v (z ε y))} = (x U y) Symmetry 1
17. z ε (x U y) EqualitySub 15 16
18. ((z ε x) v (z ε y)) -> (z ε (x U y)) ImpInt 17
19. ((z ε (x U y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x U y)))
AndInt 5 18
20. (z ε (x U y)) <-> ((z ε x) v (z ε y)) EquivConst 19
21. z ε (x ∩ y) Hyp
22. (x ∩ y) = {z: ((z ε x) & (z ε y))} DefEqInt
23. z ε {z: ((z ε x) & (z ε y))} EqualitySub 21 22
24. Set(z) & ((z ε x) & (z ε y)) ClassElim 23
25. (z ε x) & (z ε y) AndElimR 24
26. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) ImpInt 25
27. (z ε x) & (z ε y) Hyp
28. z ε x AndElimL 27
29. ∃x.(z ε x) ExistsInt 28
30. Set(z) DefSub 29
31. Set(z) & ((z ε x) & (z ε y)) AndInt 30 27
32. z ε {z: ((z ε x) & (z ε y))} ClassInt 31
33. {z: ((z ε x) & (z ε y))} = (x ∩ y) Symmetry 22
34. z ε (x ∩ y) EqualitySub 32 33
35. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) ImpInt 34
36. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
AndInt 26 35
37. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) EquivConst 36
38. ((z ε (x U y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
AndInt 20 37 Qed

Used Theorems

Th5. ((x U x) = x) & ((x ∩ x) = x)

0. z ε (x U x) Hyp
1. ((z ε (x U y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
2. (z ε (x U y)) <-> ((z ε x) v (z ε y)) AndElimL 1
3. ((z ε (x U y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x U y)))
EquivExp 2
4. (z ε (x U y)) -> ((z ε x) v (z ε y)) AndElimL 3
5. ∀y.((z ε (x U y)) -> ((z ε x) v (z ε y))) ForallInt 4
6. (z ε (x U x)) -> ((z ε x) v (z ε x)) ForallElim 5
7. (z ε x) v (z ε x) ImpElim 0 6
8. z ε x Hyp
9. z ε x Hyp
10. z ε x OrElim 7 8 8 9 9
11. (z ε (x U x)) -> (z ε x) ImpInt 10
12. z ε x Hyp
13. (z ε x) v (z ε x) OrIntL 12
14. ((z ε x) v (z ε y)) -> (z ε (x U y)) AndElimR 3
15. ∀y.(((z ε x) v (z ε y)) -> (z ε (x U y))) ForallInt 14
16. ((z ε x) v (z ε x)) -> (z ε (x U x)) ForallElim 15
17. z ε (x U x) ImpElim 13 16

18. (z ε x) -> (z ε (x ∪ x)) ImpInt 17
19. ((z ε (x ∪ x)) -> (z ε x)) & ((z ε x) -> (z ε (x ∪ x))) AndInt 11 18
20. (z ε (x ∪ x)) <-> (z ε x) EquivConst 19
21. ∀z.((z ε (x ∪ x)) <-> (z ε x)) ForallInt 20
22. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
23. ∀y.(((x ∪ x) = y) <-> ∀z.((z ε (x ∪ x)) <-> (z ε y))) ForallElim 22
24. ((x ∪ x) = x) <-> ∀z.((z ε (x ∪ x)) <-> (z ε x)) ForallElim 23
25. (((x ∪ x) = x) -> ∀z.((z ε (x ∪ x)) <-> (z ε x))) & (∀z.((z ε (x ∪ x)) <-> (z ε x)) -
> ((x ∪ x) = x)) EquivExp 24
26. ∀z.((z ε (x ∪ x)) <-> (z ε x)) -> ((x ∪ x) = x) AndElimR 25
27. (x ∪ x) = x ImpElim 21 26
28. z ε (x ∩ x) Hyp
29. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 1
30. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 29
31. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 30
32. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) ForallInt 31
33. (z ε (x ∩ x)) -> ((z ε x) & (z ε x)) ForallElim 32
34. (z ε x) & (z ε x) ImpElim 28 33
35. z ε x AndElimR 34
36. (z ε (x ∩ x)) -> (z ε x) ImpInt 35
37. z ε x Hyp
38. (z ε x) & (z ε x) AndInt 37 37
39. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 30
40. ∀y.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 39
41. ((z ε x) & (z ε x)) -> (z ε (x ∩ x)) ForallElim 40
42. z ε (x ∩ x) ImpElim 38 41
43. (z ε x) -> (z ε (x ∩ x)) ImpInt 42
44. ((z ε (x ∩ x)) -> (z ε x)) & ((z ε x) -> (z ε (x ∩ x))) AndInt 36 43
45. (z ε (x ∩ x)) <-> (z ε x) EquivConst 44
46. ∀y.(((x ∩ x) = y) <-> ∀z.((z ε (x ∩ x)) <-> (z ε y))) ForallElim 22
47. ((x ∩ x) = x) <-> ∀z.((z ε (x ∩ x)) <-> (z ε x)) ForallElim 46
48. (((x ∩ x) = x) -> ∀z.((z ε (x ∩ x)) <-> (z ε x))) & (∀z.((z ε (x ∩ x)) <-> (z ε x)) -
> ((x ∩ x) = x)) EquivExp 47
49. ∀z.((z ε (x ∩ x)) <-> (z ε x)) -> ((x ∩ x) = x) AndElimR 48
50. ∀z.((z ε (x ∩ x)) <-> (z ε x)) ForallInt 45
51. (x ∩ x) = x ImpElim 50 49
52. ((x ∪ x) = x) & ((x ∩ x) = x) AndInt 27 51 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

Th6. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))

0. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
1. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 0
2. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 1
3. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 2
4. z ε (x ∪ y) Hyp
5. (z ε x) v (z ε y) ImpElim 4 3
6. (A v B) -> (B v A) TheoremInt
7. ((z ε x) v B) -> (B v (z ε x)) PolySub 6
8. ((z ε x) v (z ε y)) -> ((z ε y) v (z ε x)) PolySub 7
9. (z ε y) v (z ε x) ImpElim 5 8
10. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 2
11. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 10
12. ((z ε w) v (z ε y)) -> (z ε (w ∪ y)) ForallElim 11
13. ∀y.(((z ε w) v (z ε y)) -> (z ε (w ∪ y))) ForallInt 12
14. ((z ε w) v (z ε x)) -> (z ε (w ∪ x)) ForallElim 13
15. ∀w.(((z ε w) v (z ε x)) -> (z ε (w ∪ x))) ForallInt 14
16. ((z ε y) v (z ε x)) -> (z ε (y ∪ x)) ForallElim 15
17. z ε (y ∪ x) ImpElim 9 16
18. (z ε (x ∪ y)) -> (z ε (y ∪ x)) ImpInt 17
19. ∀x.((z ε (x ∪ y)) -> (z ε (y ∪ x))) ForallInt 18
20. (z ε (w ∪ y)) -> (z ε (y ∪ w)) ForallElim 19
21. ∀y.((z ε (w ∪ y)) -> (z ε (y ∪ w))) ForallInt 20
22. (z ε (w ∪ v)) -> (z ε (v ∪ w)) ForallElim 21
23. ∀w.((z ε (w ∪ v)) -> (z ε (v ∪ w))) ForallInt 22

24. (z ε (y ∪ v)) -> (z ε (v ∪ y)) ForallElim 23
25. ∀v.((z ε (y ∪ v)) -> (z ε (v ∪ y))) ForallInt 24
26. (z ε (y ∪ x)) -> (z ε (x ∪ y)) ForallElim 25
27. ((z ε (x ∪ y)) -> (z ε (y ∪ x))) & ((z ε (y ∪ x)) -> (z ε (x ∪ y))) AndInt 18 26
28. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
29. ∀e.(((x ∪ y) = e) <-> ∀z.((z ε (x ∪ y)) <-> (z ε e))) ForallElim 28
30. ((x ∪ y) = (y ∪ x)) <-> ∀z.((z ε (x ∪ y)) <-> (z ε (y ∪ x))) ForallElim 29
31. (((x ∪ y) = (y ∪ x)) -> ∀z.((z ε (x ∪ y)) <-> (z ε (y ∪ x)))) & (∀z.((z ε (x ∪ y)) <-
> (z ε (y ∪ x))) -> ((x ∪ y) = (y ∪ x))) EquivExp 30
32. ∀z.((z ε (x ∪ y)) <-> (z ε (y ∪ x))) -> ((x ∪ y) = (y ∪ x)) AndElimR 31
33. (z ε (x ∪ y)) <-> (z ε (y ∪ x)) EquivConst 27
34. ∀z.((z ε (x ∪ y)) <-> (z ε (y ∪ x))) ForallInt 33
35. (x ∪ y) = (y ∪ x) ImpElim 34 32
36. z ε (x ∩ y) Hyp
37. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 0
38. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 37
39. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 38
40. (z ε x) & (z ε y) ImpElim 36 39
41. (A & B) -> (B & A) TheoremInt
42. ((z ε x) & B) -> (B & (z ε x)) PolySub 41
43. ((z ε x) & (z ε y)) -> ((z ε y) & (z ε x)) PolySub 42
44. (z ε y) & (z ε x) ImpElim 40 43
45. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 38
46. ∀w.(((z ε w) & (z ε y)) -> (z ε (w ∩ y))) ForallInt 45
47. ∀v.∀w.(((z ε w) & (z ε v)) -> (z ε (w ∩ v))) ForallInt 46
48. ∀w.(((z ε w) & (z ε x)) -> (z ε (w ∩ x))) ForallElim 47
49. ((z ε y) & (z ε x)) -> (z ε (y ∩ x)) ForallElim 48
50. z ε (y ∩ x) ImpElim 44 49
51. (z ε (x ∩ y)) -> (z ε (y ∩ x)) ImpInt 50
52. ∀v.((z ε (v ∩ y)) -> (z ε (y ∩ v))) ForallInt 51
53. ∀w.∀v.((z ε (v ∩ w)) -> (z ε (w ∩ v))) ForallInt 52
54. ∀v.((z ε (v ∩ x)) -> (z ε (x ∩ v))) ForallElim 53
55. (z ε (y ∩ x)) -> (z ε (x ∩ y)) ForallElim 54
56. ((z ε (x ∩ y)) -> (z ε (y ∩ x))) & ((z ε (y ∩ x)) -> (z ε (x ∩ y))) AndInt 51 55
57. ∀g.(((x ∩ y) = g) <-> ∀z.((z ε (x ∩ y)) <-> (z ε g))) ForallElim 28
58. ((x ∩ y) = (y ∩ x)) <-> ∀z.((z ε (x ∩ y)) <-> (z ε (y ∩ x))) ForallElim 57
59. (((x ∩ y) = (y ∩ x)) -> ∀z.((z ε (x ∩ y)) <-> (z ε (y ∩ x)))) & (∀z.((z ε (x ∩ y)) <-
> (z ε (y ∩ x))) -> ((x ∩ y) = (y ∩ x))) EquivExp 58
60. ∀z.((z ε (x ∩ y)) <-> (z ε (y ∩ x))) -> ((x ∩ y) = (y ∩ x)) AndElimR 59
61. (z ε (x ∩ y)) <-> (z ε (y ∩ x)) EquivConst 56
62. ∀z.((z ε (x ∩ y)) <-> (z ε (y ∩ x))) ForallInt 61
63. (x ∩ y) = (y ∩ x) ImpElim 62 60
64. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) AndInt 35 63 Qed

Used Theorems

2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
1. (A v B) -> (B v A)
3. (A & B) -> (B & A)

Th7. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))

0. w ε ((x ∪ y) ∪ z) Hyp
1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
2. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 1
3. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 2
4. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 3
5. ∀z.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 4
6. (w ε (x ∪ y)) -> ((w ε x) v (w ε y)) ForallElim 5
7. ∀x.((w ε (x ∪ y)) -> ((w ε x) v (w ε y))) ForallInt 6
8. (w ε (a ∪ y)) -> ((w ε a) v (w ε y)) ForallElim 7
9. ∀y.((w ε (a ∪ y)) -> ((w ε a) v (w ε y))) ForallInt 8
10. (w ε (a ∪ z)) -> ((w ε a) v (w ε z)) ForallElim 9
11. ∀a.((w ε (a ∪ z)) -> ((w ε a) v (w ε z))) ForallInt 10
12. (w ε ((x ∪ y) ∪ z)) -> ((w ε (x ∪ y)) v (w ε z)) ForallElim 11
13. (w ε (x ∪ y)) v (w ε z) ImpElim 0 12
14. w ε (x ∪ y) Hyp
15. (w ε x) v (w ε y) ImpElim 14 6

16. ((w ε x) v (w ε y)) v (w ε z) OrIntR 15
17. w ε z Hyp
18. ((w ε x) v (w ε y)) v (w ε z) OrIntL 17
19. ((w ε x) v (w ε y)) v (w ε z) OrElim 13 14 16 17 18
20. ((A v B) v C) <-> (A v (B v C)) TheoremInt
21. (((w ε x) v B) v C) <-> ((w ε x) v (B v C)) PolySub 20
22. (((w ε x) v (w ε y)) v C) <-> ((w ε x) v ((w ε y) v C)) PolySub 21
23. (((w ε x) v (w ε y)) v (w ε z)) <-> ((w ε x) v ((w ε y) v (w ε z))) PolySub 22
24. ((((w ε x) v (w ε y)) v (w ε z)) -> ((w ε x) v ((w ε y) v (w ε z)))) & (((w ε x) v ((w ε y) v (w ε z))) -> (((w ε x) v (w ε y)) v (w ε z))) EquivExp 23
25. (((w ε x) v (w ε y)) v (w ε z)) -> ((w ε x) v ((w ε y) v (w ε z))) AndElimL 24
26. (w ε x) v ((w ε y) v (w ε z)) ImpElim 19 25
27. ((z ε x) v (z ε y)) -> (z ε (x U y)) AndElimR 3
28. ∀z.(((z ε x) v (z ε y)) -> (z ε (x U y))) ForallInt 27
29. ((w ε x) v (w ε y)) -> (w ε (x U y)) ForallElim 28
30. ∀x.(((w ε x) v (w ε y)) -> (w ε (x U y))) ForallInt 29
31. ((w ε a) v (w ε y)) -> (w ε (a U y)) ForallElim 30
32. ∀y.(((w ε a) v (w ε y)) -> (w ε (a U y))) ForallInt 31
33. ((w ε a) v (w ε z)) -> (w ε (a U z)) ForallElim 32
34. ∀a.(((w ε a) v (w ε z)) -> (w ε (a U z))) ForallInt 33
35. ((w ε y) v (w ε z)) -> (w ε (y U z)) ForallElim 34
36. (w ε y) v (w ε z) Hyp
37. w ε (y U z) ImpElim 36 35
38. (w ε x) v (w ε (y U z)) OrIntL 37
39. ∀y.(((w ε a) v (w ε y)) -> (w ε (a U y))) ForallInt 31
40. ((w ε a) v (w ε (y U z))) -> (w ε (a U (y U z))) ForallElim 32
41. ∀a.(((w ε a) v (w ε (y U z))) -> (w ε (a U (y U z)))) ForallInt 40
42. ((w ε x) v (w ε (y U z))) -> (w ε (x U (y U z))) ForallElim 41
43. w ε (x U (y U z)) ImpElim 38 42
44. w ε x Hyp
45. (w ε x) v (w ε (y U z)) OrIntR 44
46. ∀y.(((w ε a) v (w ε y)) -> (w ε (a U y))) ForallInt 31
47. ((w ε a) v (w ε (y U z))) -> (w ε (a U (y U z))) ForallElim 32
48. ∀a.(((w ε a) v (w ε (y U z))) -> (w ε (a U (y U z)))) ForallInt 47
49. ((w ε x) v (w ε (y U z))) -> (w ε (x U (y U z))) ForallElim 48
50. w ε (x U (y U z)) ImpElim 45 49
51. w ε (x U (y U z)) OrElim 26 44 50 36 43
52. (w ε ((x U y) U z)) -> (w ε (x U (y U z))) ImpInt 51
53. w ε (x U (y U z)) Hyp
54. ∀y.((w ε (a U y)) -> ((w ε a) v (w ε y))) ForallInt 8
55. (w ε (a U (y U z))) -> ((w ε a) v (w ε (y U z))) ForallElim 9
56. ∀a.((w ε (a U (y U z))) -> ((w ε a) v (w ε (y U z)))) ForallInt 55
57. (w ε (x U (y U z))) -> ((w ε x) v (w ε (y U z))) ForallElim 56
58. (w ε x) v (w ε (y U z)) ImpElim 53 57
59. w ε x Hyp
60. (w ε x) v ((w ε y) v (w ε z)) OrIntR 59
61. w ε (y U z) Hyp
62. ∀a.((w ε (a U z)) -> ((w ε a) v (w ε z))) ForallInt 10
63. (w ε (y U z)) -> ((w ε y) v (w ε z)) ForallElim 11
64. (w ε y) v (w ε z) ImpElim 61 63
65. (w ε x) v ((w ε y) v (w ε z)) OrIntL 64
66. (w ε x) v ((w ε y) v (w ε z)) OrElim 58 59 60 61 65
67. ((w ε x) v ((w ε y) v (w ε z))) -> (((w ε x) v (w ε y)) v (w ε z)) AndElimR 24
68. ((w ε x) v (w ε y)) v (w ε z) ImpElim 66 67
69. (w ε x) v (w ε y) Hyp
70. ∀z.(((z ε x) v (z ε y)) -> (z ε (x U y))) ForallInt 27
71. ((w ε x) v (w ε y)) -> (w ε (x U y)) ForallElim 28
72. w ε (x U y) ImpElim 69 71
73. (w ε (x U y)) v (w ε z) OrIntR 72
74. w ε z Hyp
75. (w ε (x U y)) v (w ε z) OrIntL 74
76. (w ε (x U y)) v (w ε z) OrElim 68 69 73 74 75
77. ∀a.(((w ε a) v (w ε z)) -> (w ε (a U z))) ForallInt 33
78. ((w ε (x U y)) v (w ε z)) -> (w ε ((x U y) U z)) ForallElim 34
79. w ε ((x U y) U z) ImpElim 76 78
80. (w ε (x U (y U z))) -> (w ε ((x U y) U z)) ImpInt 79
81. ((w ε ((x U y) U z)) -> (w ε (x U (y U z)))) & ((w ε (x U (y U z))) -> (w ε ((x U y) U z))) AndInt 52 80
82. (w ε ((x U y) U z)) <-> (w ε (x U (y U z))) EquivConst 81
83. w ε ((x ∩ y) ∩ z) Hyp
84. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 1

85. ∀z.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) ForallInt 84
86. (w ε (x ∩ y)) <-> ((w ε x) & (w ε y)) ForallElim 85
87. ∀x.((w ε (x ∩ y)) <-> ((w ε x) & (w ε y))) ForallInt 86
88. (w ε (a ∩ y)) <-> ((w ε a) & (w ε y)) ForallElim 87
89. ∀y.((w ε (a ∩ y)) <-> ((w ε a) & (w ε y))) ForallInt 88
90. (w ε (a ∩ b)) <-> ((w ε a) & (w ε b)) ForallElim 89
91. ∀a.((w ε (a ∩ b)) <-> ((w ε a) & (w ε b))) ForallInt 90
92. (w ε ((x ∩ y) ∩ b)) <-> ((w ε (x ∩ y)) & (w ε b)) ForallElim 91
93. ∀b.((w ε ((x ∩ y) ∩ b)) <-> ((w ε (x ∩ y)) & (w ε b))) ForallInt 92
94. (w ε ((x ∩ y) ∩ z)) <-> ((w ε (x ∩ y)) & (w ε z)) ForallElim 93
95. ((w ε ((x ∩ y) ∩ z)) -> ((w ε (x ∩ y)) & (w ε z))) & (((w ε (x ∩ y)) & (w ε z)) -> (w ε ((x ∩ y) ∩ z))) EquivExp 94
96. (w ε ((x ∩ y) ∩ z)) -> ((w ε (x ∩ y)) & (w ε z)) AndElimL 95
97. (w ε (x ∩ y)) & (w ε z) ImpElim 83 96
98. w ε (x ∩ y) AndElimL 97
99. ((w ε (x ∩ y)) -> ((w ε x) & (w ε y))) & (((w ε x) & (w ε y)) -> (w ε (x ∩ y))) EquivExp 86
100. (w ε (x ∩ y)) -> ((w ε x) & (w ε y)) AndElimL 99
101. (w ε x) & (w ε y) ImpElim 98 100
102. w ε z AndElimR 97
103. w ε x AndElimL 101
104. w ε y AndElimR 101
105. (w ε y) & (w ε z) AndInt 104 102
106. ((w ε (a ∩ b)) -> ((w ε a) & (w ε b))) & (((w ε a) & (w ε b)) -> (w ε (a ∩ b))) EquivExp 90
107. ((w ε a) & (w ε b)) -> (w ε (a ∩ b)) AndElimR 106
108. ∀a.(((w ε a) & (w ε b)) -> (w ε (a ∩ b))) ForallInt 107
109. ((w ε y) & (w ε b)) -> (w ε (y ∩ b)) ForallElim 108
110. ∀b.(((w ε y) & (w ε b)) -> (w ε (y ∩ b))) ForallInt 109
111. ((w ε y) & (w ε z)) -> (w ε (y ∩ z)) ForallElim 110
112. w ε (y ∩ z) ImpElim 105 111
113. (w ε x) & (w ε (y ∩ z)) AndInt 103 112
114. ∀a.(((w ε a) & (w ε b)) -> (w ε (a ∩ b))) ForallInt 107
115. ((w ε x) & (w ε b)) -> (w ε (x ∩ b)) ForallElim 108
116. ∀b.(((w ε x) & (w ε b)) -> (w ε (x ∩ b))) ForallInt 115
117. ((w ε x) & (w ε (y ∩ z))) -> (w ε (x ∩ (y ∩ z))) ForallElim 116
118. w ε (x ∩ (y ∩ z)) ImpElim 113 117
119. (w ε ((x ∩ y) ∩ z)) -> (w ε (x ∩ (y ∩ z))) ImpInt 118
120. w ε (x ∩ (y ∩ z)) Hyp
121. (w ε (a ∩ b)) -> ((w ε a) & (w ε b)) AndElimL 106
122. ∀a.((w ε (a ∩ b)) -> ((w ε a) & (w ε b))) ForallInt 121
123. (w ε (x ∩ b)) -> ((w ε x) & (w ε b)) ForallElim 122
124. ∀b.((w ε (x ∩ b)) -> ((w ε x) & (w ε b))) ForallInt 123
125. ∀b.((w ε (x ∩ b)) -> ((w ε x) & (w ε b))) ForallInt 123
126. (w ε (x ∩ (y ∩ z))) -> ((w ε x) & (w ε (y ∩ z))) ForallElim 124
127. (w ε x) & (w ε (y ∩ z)) ImpElim 120 126
128. w ε (y ∩ z) AndElimR 127
129. w ε x AndElimL 127
130. ∀a.((w ε (a ∩ b)) -> ((w ε a) & (w ε b))) ForallInt 121
131. (w ε (y ∩ b)) -> ((w ε y) & (w ε b)) ForallElim 122
132. ∀b.((w ε (y ∩ b)) -> ((w ε y) & (w ε b))) ForallInt 131
133. (w ε (y ∩ z)) -> ((w ε y) & (w ε z)) ForallElim 132
134. (w ε y) & (w ε z) ImpElim 128 133
135. w ε y AndElimL 134
136. w ε z AndElimR 134
137. (w ε x) & (w ε y) AndInt 129 135
138. ((w ε x) & (w ε y)) -> (w ε (x ∩ y)) AndElimR 99
139. w ε (x ∩ y) ImpElim 137 138
140. (w ε (x ∩ y)) & (w ε z) AndInt 139 136
141. ∀a.((w ε (a ∩ b)) -> ((w ε a) & (w ε b))) ForallInt 121
142. ∀a.(((w ε a) & (w ε b)) -> (w ε (a ∩ b))) ForallInt 107
143. ((w ε (x ∩ y)) & (w ε b)) -> (w ε ((x ∩ y) ∩ b)) ForallElim 108
144. ∀b.(((w ε (x ∩ y)) & (w ε b)) -> (w ε ((x ∩ y) ∩ b))) ForallInt 143
145. ((w ε (x ∩ y)) & (w ε z)) -> (w ε ((x ∩ y) ∩ z)) ForallElim 144
146. w ε ((x ∩ y) ∩ z) ImpElim 140 145
147. (w ε (x ∩ (y ∩ z))) -> (w ε ((x ∩ y) ∩ z)) ImpInt 146
148. ((w ε ((x ∩ y) ∩ z)) -> (w ε (x ∩ (y ∩ z)))) & ((w ε (x ∩ (y ∩ z))) -> (w ε ((x ∩ y) ∩ z))) AndInt 119 147
149. (w ε ((x ∩ y) ∩ z)) <-> (w ε (x ∩ (y ∩ z))) EquivConst 148
150. ((w ε ((x ∪ y) ∪ z)) <-> (w ε (x ∪ (y ∪ z)))) & ((w ε ((x ∩ y) ∩ z)) <-> (w ε (x ∩ (y ∩ z)))) AndInt 82 149

151. (w ε ((x ∩ y) ∩ z)) <-> (w ε (x ∩ (y ∩ z))) AndElimR 150
152. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
153. ∀h.((((x ∩ y) ∩ z) = h) <-> ∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε h))) ForallElim 152
154. (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) <-> ∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε (x ∩ (y ∩ z)))) ForallElim 153
155. ∀w.((w ε ((x ∩ y) ∩ z)) <-> (w ε (x ∩ (y ∩ z)))) ForallInt 151
156. ((((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) -> ∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε (x ∩ (y ∩ z))))) & (∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε (x ∩ (y ∩ z)))) -> (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))) EquivExp 154
157. ∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε (x ∩ (y ∩ z)))) -> (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) AndElimR 156
158. ((x ∩ y) ∩ z) = (x ∩ (y ∩ z)) ImpElim 155 157
159. ∀j.((((x ∪ y) ∪ z) = j) <-> ∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε j))) ForallElim 152
160. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) <-> ∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε (x ∪ (y ∪ z)))) ForallElim 159
161. ((((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) -> ∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε (x ∪ (y ∪ z))))) & (∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε (x ∪ (y ∪ z)))) -> (((x ∪ y) ∪ z) = (x ∪ (y ∪ z)))) EquivExp 160
162. ∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε (x ∪ (y ∪ z)))) -> (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) AndElimR 161
163. (w ε ((x ∪ y) ∪ z)) <-> (w ε (x ∪ (y ∪ z))) AndElimL 150
164. ∀w.((w ε ((x ∪ y) ∪ z)) <-> (w ε (x ∪ (y ∪ z)))) ForallInt 163
165. ((x ∪ y) ∪ z) = (x ∪ (y ∪ z)) ImpElim 164 162
166. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) AndInt 165 158
Qed

Used Theorems

3. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
1. ((A v B) v C) <-> (A v (B v C))

Th8. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))

0. w ε (x ∩ (y ∪ z)) Hyp
1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
2. ∀z.(((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))) ForallInt 1
3. ((w ε (x ∪ y)) <-> ((w ε x) v (w ε y))) & ((w ε (x ∩ y)) <-> ((w ε x) & (w ε y))) ForallElim 2
4. ∀y.(((w ε (x ∪ y)) <-> ((w ε x) v (w ε y))) & ((w ε (x ∩ y)) <-> ((w ε x) & (w ε y)))) ForallInt 3
5. ((w ε (x ∪ a)) <-> ((w ε x) v (w ε a))) & ((w ε (x ∩ a)) <-> ((w ε x) & (w ε a))) ForallElim 4
6. (w ε (x ∩ a)) <-> ((w ε x) & (w ε a)) AndElimR 5
7. ((w ε (x ∩ a)) -> ((w ε x) & (w ε a))) & (((w ε x) & (w ε a)) -> (w ε (x ∩ a))) EquivExp 6
8. (w ε (x ∩ a)) -> ((w ε x) & (w ε a)) AndElimL 7
9. ∀a.((w ε (x ∩ a)) -> ((w ε x) & (w ε a))) ForallInt 8
10. (w ε (x ∩ (y ∪ z))) -> ((w ε x) & (w ε (y ∪ z))) ForallElim 9
11. (w ε x) & (w ε (y ∪ z)) ImpElim 0 10
12. w ε (y ∪ z) AndElimR 11
13. w ε x AndElimL 11
14. (w ε (x ∪ a)) <-> ((w ε x) v (w ε a)) AndElimL 5
15. ∀x.((w ε (x ∪ a)) <-> ((w ε x) v (w ε a))) ForallInt 14
16. (w ε (b ∪ a)) <-> ((w ε b) v (w ε a)) ForallElim 15
17. ∀b.((w ε (b ∪ a)) <-> ((w ε b) v (w ε a))) ForallInt 16
18. (w ε (y ∪ a)) <-> ((w ε y) v (w ε a)) ForallElim 17
19. ∀a.((w ε (y ∪ a)) <-> ((w ε y) v (w ε a))) ForallInt 18
20. (w ε (y ∪ z)) <-> ((w ε y) v (w ε z)) ForallElim 19
21. ((w ε (y ∪ z)) -> ((w ε y) v (w ε z))) & (((w ε y) v (w ε z)) -> (w ε (y ∪ z))) EquivExp 20
22. (w ε (y ∪ z)) -> ((w ε y) v (w ε z)) AndElimL 21
23. (w ε y) v (w ε z) ImpElim 12 22
24. (w ε x) & ((w ε y) v (w ε z)) AndInt 13 23
25. (A & (B v C)) <-> ((A & B) v (A & C)) TheoremInt
26. ((w ε x) & (B v C)) <-> (((w ε x) & B) v ((w ε x) & C)) PolySub 25
27. ((w ε x) & ((w ε y) v C)) <-> (((w ε x) & (w ε y)) v ((w ε x) & C)) PolySub 26
28. ((w ε x) & ((w ε y) v (w ε z))) <-> (((w ε x) & (w ε y)) v ((w ε x) & (w ε z))) PolySub 27

29. (((w ε x) & ((w ε y) v (w ε z))) -> (((w ε x) & (w ε y)) v ((w ε x) & (w ε z)))) & ((((w ε x) & (w ε y)) v ((w ε x) & (w ε z))) -> ((w ε x) & ((w ε y) v (w ε z)))) EquivExp 28
30. ((w ε x) & ((w ε y) v (w ε z))) -> (((w ε x) & (w ε y)) v ((w ε x) & (w ε z))) AndElimL 29
31. ((w ε x) & (w ε y)) v ((w ε x) & (w ε z)) ImpElim 24 30
32. (w ε x) & (w ε y) Hyp
33. (w ε (x ∩ y)) <-> ((w ε x) & (w ε y)) AndElimR 3
34. ((w ε (x ∩ y)) -> ((w ε x) & (w ε y))) & (((w ε x) & (w ε y)) -> (w ε (x ∩ y))) EquivExp 33
35. ((w ε x) & (w ε y)) -> (w ε (x ∩ y)) AndElimR 34
36. w ε (x ∩ y) ImpElim 32 35
37. (w ε (x ∩ y)) v (w ε (x ∩ z)) OrIntR 36
38. (w ε x) & (w ε z) Hyp
39. ∀y.(((w ε x) & (w ε y)) -> (w ε (x ∩ y))) ForallInt 35
40. ((w ε x) & (w ε z)) -> (w ε (x ∩ z)) ForallElim 39
41. w ε (x ∩ z) ImpElim 38 40
42. (w ε (x ∩ y)) v (w ε (x ∩ z)) OrIntL 41
43. (w ε (x ∩ y)) v (w ε (x ∩ z)) OrElim 31 32 37 38 42
44. ((w ε (b ∪ a)) -> ((w ε b) v (w ε a))) & (((w ε b) v (w ε a)) -> (w ε (b ∪ a))) EquivExp 16
45. ((w ε b) v (w ε a)) -> (w ε (b ∪ a)) AndElimR 44
46. ∀b.(((w ε b) v (w ε a)) -> (w ε (b ∪ a))) ForallInt 45
47. ((w ε (x ∩ y)) v (w ε a)) -> (w ε ((x ∩ y) ∪ a)) ForallElim 46
48. ∀a.(((w ε (x ∩ y)) v (w ε a)) -> (w ε ((x ∩ y) ∪ a))) ForallInt 47
49. ((w ε (x ∩ y)) v (w ε (x ∩ z))) -> (w ε ((x ∩ y) ∪ (x ∩ z))) ForallElim 48
50. w ε ((x ∩ y) ∪ (x ∩ z)) ImpElim 43 49
51. (w ε (x ∩ (y ∪ z))) -> (w ε ((x ∩ y) ∪ (x ∩ z))) ImpInt 50
52. w ε ((x ∩ y) ∪ (x ∩ z)) Hyp
53. (w ε (b ∪ a)) -> ((w ε b) v (w ε a)) AndElimL 44
54. ∀b.((w ε (b ∪ a)) -> ((w ε b) v (w ε a))) ForallInt 53
55. (w ε ((x ∩ y) ∪ a)) -> ((w ε (x ∩ y)) v (w ε a)) ForallElim 54
56. ∀a.((w ε ((x ∩ y) ∪ a)) -> ((w ε (x ∩ y)) v (w ε a))) ForallInt 55
57. (w ε ((x ∩ y) ∪ (x ∩ z))) -> ((w ε (x ∩ y)) v (w ε (x ∩ z))) ForallElim 56
58. (w ε (x ∩ y)) v (w ε (x ∩ z)) ImpElim 52 57
59. ∀a.((w ε (x ∩ a)) -> ((w ε x) & (w ε a))) ForallInt 8
60. (w ε (x ∩ y)) -> ((w ε x) & (w ε y)) ForallElim 9
61. ∀a.((w ε (x ∩ a)) -> ((w ε x) & (w ε a))) ForallInt 8
62. (w ε (x ∩ z)) -> ((w ε x) & (w ε z)) ForallElim 9
63. w ε (x ∩ y) Hyp
64. (w ε x) & (w ε y) ImpElim 63 60
65. w ε y AndElimR 64
66. (w ε y) v (w ε z) OrIntR 65
67. ((w ε b) v (w ε a)) -> (w ε (b ∪ a)) AndElimR 44
68. ∀b.(((w ε b) v (w ε a)) -> (w ε (b ∪ a))) ForallInt 67
69. ((w ε y) v (w ε a)) -> (w ε (y ∪ a)) ForallElim 68
70. ∀a.(((w ε y) v (w ε a)) -> (w ε (y ∪ a))) ForallInt 69
71. ((w ε y) v (w ε z)) -> (w ε (y ∪ z)) ForallElim 70
72. w ε (y ∪ z) ImpElim 66 71
73. w ε x AndElimL 64
74. (w ε x) & (w ε (y ∪ z)) AndInt 73 72
75. ((w ε x) & (w ε a)) -> (w ε (x ∩ a)) AndElimR 7
76. ∀a.(((w ε x) & (w ε a)) -> (w ε (x ∩ a))) ForallInt 75
77. ((w ε x) & (w ε (y ∪ z))) -> (w ε (x ∩ (y ∪ z))) ForallElim 76
78. w ε (x ∩ (y ∪ z)) ImpElim 74 77
79. w ε (x ∩ z) Hyp
80. (w ε x) & (w ε z) ImpElim 79 62
81. w ε x AndElimL 80
82. w ε z AndElimR 80
83. (w ε y) v (w ε z) OrIntL 82
84. w ε (y ∪ z) ImpElim 83 71
85. (w ε x) & (w ε (y ∪ z)) AndInt 81 84
86. w ε (x ∩ (y ∪ z)) ImpElim 85 77
87. w ε (x ∩ (y ∪ z)) OrElim 58 63 78 79 86
88. (w ε ((x ∩ y) ∪ (x ∩ z))) -> (w ε (x ∩ (y ∪ z))) ImpInt 87
89. ((w ε (x ∩ (y ∪ z))) -> (w ε ((x ∩ y) ∪ (x ∩ z)))) & ((w ε ((x ∩ y) ∪ (x ∩ z))) -> (w ε (x ∩ (y ∪ z)))) AndInt 51 88
90. (w ε (x ∩ (y ∪ z))) <-> (w ε ((x ∩ y) ∪ (x ∩ z))) EquivConst 89
91. w ε (x ∪ (y ∩ z)) Hyp
92. ((w ε (b ∪ a)) -> ((w ε b) v (w ε a))) & (((w ε b) v (w ε a)) -> (w ε (b ∪ a))) EquivExp 16

93. ∀b.(((w ε (b U a)) -> ((w ε b) v (w ε a))) & (((w ε b) v (w ε a)) -> (w ε (b U a)))) ForallInt 92

94. ((w ε (x U a)) -> ((w ε x) v (w ε a))) & (((w ε x) v (w ε a)) -> (w ε (x U a))) ForallElim 93

95. ∀a.(((w ε (x U a)) -> ((w ε x) v (w ε a))) & (((w ε x) v (w ε a)) -> (w ε (x U a)))) ForallInt 94

96. ((w ε (x U (y ∩ z))) -> ((w ε x) v (w ε (y ∩ z)))) & (((w ε x) v (w ε (y ∩ z))) -> (w ε (x U (y ∩ z)))) ForallElim 95

97. (w ε (x U (y ∩ z))) -> ((w ε x) v (w ε (y ∩ z))) AndElimL 96

98. (w ε x) v (w ε (y ∩ z)) ImpElim 91 97

99. w ε x Hyp

100. (w ε x) v (w ε y) OrIntR 99

101. ((w ε b) v (w ε a)) -> (w ε (b U a)) AndElimR 92

102. ∀b.(((w ε b) v (w ε a)) -> (w ε (b U a))) ForallInt 101

103. ((w ε x) v (w ε a)) -> (w ε (x U a)) ForallElim 102

104. ∀a.(((w ε x) v (w ε a)) -> (w ε (x U a))) ForallInt 103

105. ((w ε x) v (w ε y)) -> (w ε (x U y)) ForallElim 104

106. w ε (x U y) ImpElim 100 105

107. (w ε x) v (w ε z) OrIntR 99

108. ∀a.(((w ε x) v (w ε a)) -> (w ε (x U a))) ForallInt 103

109. ((w ε x) v (w ε z)) -> (w ε (x U z)) ForallElim 104

110. w ε (x U z) ImpElim 107 109

111. (w ε (x U y)) & (w ε (x U z)) AndInt 106 110

112. ∀x.((w ε (x ∩ a)) <-> ((w ε x) & (w ε a))) ForallInt 6

113. (w ε (b ∩ a)) <-> ((w ε b) & (w ε a)) ForallElim 112

114. ((w ε (b ∩ a)) -> ((w ε b) & (w ε a))) & (((w ε b) & (w ε a)) -> (w ε (b ∩ a))) EquivExp 113

115. ((w ε b) & (w ε a)) -> (w ε (b ∩ a)) AndElimR 114

116. ∀b.(((w ε b) & (w ε a)) -> (w ε (b ∩ a))) ForallInt 115

117. ((w ε (x U y)) & (w ε a)) -> (w ε ((x U y) ∩ a)) ForallElim 116

118. ∀a.(((w ε (x U y)) & (w ε a)) -> (w ε ((x U y) ∩ a))) ForallInt 117

119. ((w ε (x U y)) & (w ε (x U z))) -> (w ε ((x U y) ∩ (x U z))) ForallElim 118

120. w ε ((x U y) ∩ (x U z)) ImpElim 111 119

121. w ε (y ∩ z) Hyp

122. (w ε (b ∩ a)) -> ((w ε b) & (w ε a)) AndElimL 114

123. ∀b.((w ε (b ∩ a)) -> ((w ε b) & (w ε a))) ForallInt 122

124. (w ε (y ∩ a)) -> ((w ε y) & (w ε a)) ForallElim 123

125. ∀a.((w ε (y ∩ a)) -> ((w ε y) & (w ε a))) ForallInt 124

126. (w ε (y ∩ z)) -> ((w ε y) & (w ε z)) ForallElim 125

127. (w ε y) & (w ε z) ImpElim 121 126

128. w ε y AndElimL 127

129. w ε z AndElimR 127

130. (w ε x) v (w ε y) OrIntL 128

131. (w ε x) v (w ε z) OrIntL 129

132. w ε (x U z) ImpElim 131 109

133. (z ε (x U y)) <-> ((z ε x) v (z ε y)) AndElimL 1

134. ((z ε (x U y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x U y))) EquivExp 133

135. ((z ε x) v (z ε y)) -> (z ε (x U y)) AndElimR 134

136. ∀z.(((z ε x) v (z ε y)) -> (z ε (x U y))) ForallInt 135

137. ((w ε x) v (w ε y)) -> (w ε (x U y)) ForallElim 136

138. w ε (x U y) ImpElim 130 137

139. (w ε (x U y)) & (w ε (x U z)) AndInt 138 132

140. w ε ((x U y) ∩ (x U z)) ImpElim 139 119

141. w ε ((x U y) ∩ (x U z)) OrElim 98 99 120 121 140

142. (w ε (x U (y ∩ z))) -> (w ε ((x U y) ∩ (x U z))) ImpInt 141

143. w ε ((x U y) ∩ (x U z)) Hyp

144. (w ε (b ∩ a)) -> ((w ε b) & (w ε a)) AndElimL 114

145. ∀b.(((w ε (b ∩ a)) -> ((w ε b) & (w ε a))) & (((w ε b) & (w ε a)) -> (w ε (b ∩ a)))) ForallInt 114

146. ((w ε ((x U y) ∩ a)) -> ((w ε (x U y)) & (w ε a))) & (((w ε (x U y)) & (w ε a)) -> (w ε ((x U y) ∩ a))) ForallElim 145

147. ∀a.(((w ε ((x U y) ∩ a)) -> ((w ε (x U y)) & (w ε a))) & (((w ε (x U y)) & (w ε a)) -> (w ε ((x U y) ∩ a)))) ForallInt 146

148. ((w ε ((x U y) ∩ (x U z))) -> ((w ε (x U y)) & (w ε (x U z)))) & (((w ε (x U y)) & (w ε (x U z))) -> (w ε ((x U y) ∩ (x U z)))) ForallElim 147

149. (w ε ((x U y) ∩ (x U z))) -> ((w ε (x U y)) & (w ε (x U z))) AndElimL 148

150. (w ε (x U y)) & (w ε (x U z)) ImpElim 143 149

151. w ε (x U y) AndElimL 150

152. w ε (x U z) AndElimR 150

153. (z ε (x U y)) -> ((z ε x) v (z ε y)) AndElimL 134

154. ∀z.((z ε (x U y)) -> ((z ε x) v (z ε y))) ForallInt 153
155. (w ε (x U y)) -> ((w ε x) v (w ε y)) ForallElim 154
156. ∀y.((w ε (x U y)) -> ((w ε x) v (w ε y))) ForallInt 155
157. (w ε (x U z)) -> ((w ε x) v (w ε z)) ForallElim 156
158. (w ε x) v (w ε y) ImpElim 151 155
159. (w ε x) v (w ε z) ImpElim 152 157
160. w ε x Hyp
161. (w ε x) v (w ε (y ∩ z)) OrIntR 160
162. ((w ε (x U a)) -> ((w ε x) v (w ε a))) & (((w ε x) v (w ε a)) -> (w ε (x U a))) EquivExp 14
163. ((w ε x) v (w ε a)) -> (w ε (x U a)) AndElimR 162
164. ∀a.(((w ε x) v (w ε a)) -> (w ε (x U a))) ForallInt 163
165. ((w ε x) v (w ε (y ∩ z))) -> (w ε (x U (y ∩ z))) ForallElim 164
166. w ε (x U (y ∩ z)) ImpElim 161 165
167. (w ε x) -> (w ε (x U (y ∩ z))) ImpInt 166
168. w ε y Hyp
169. w ε x Hyp
170. w ε (x U (y ∩ z)) ImpElim 169 167
171. w ε z Hyp
172. (w ε y) & (w ε z) AndInt 168 171
173. ∀a.(((w ε b) & (w ε a)) -> (w ε (b ∩ a))) ForallInt 115
174. ((w ε y) & (w ε a)) -> (w ε (y ∩ a)) ForallElim 116
175. ∀a.(((w ε y) & (w ε a)) -> (w ε (y ∩ a))) ForallInt 174
176. ((w ε y) & (w ε z)) -> (w ε (y ∩ z)) ForallElim 175
177. w ε (y ∩ z) ImpElim 172 176
178. (w ε x) v (w ε (y ∩ z)) OrIntL 177
179. w ε (x U (y ∩ z)) ImpElim 178 165
180. w ε (x U (y ∩ z)) OrElim 159 169 170 171 179
181. w ε (x U (y ∩ z)) OrElim 158 160 166 168 180
182. (w ε ((x U y) ∩ (x U z))) -> (w ε (x U (y ∩ z))) ImpInt 181
183. ((w ε (x U (y ∩ z))) -> (w ε ((x U y) ∩ (x U z)))) & ((w ε ((x U y) ∩ (x U z))) -> (w ε (x U (y ∩ z)))) AndInt 142 182
184. (w ε (x U (y ∩ z))) <-> (w ε ((x U y) ∩ (x U z))) EquivConst 183
185. ((w ε (x ∩ (y U z))) <-> (w ε ((x ∩ y) U (x ∩ z)))) & ((w ε (x U (y ∩ z))) <-> (w ε ((x U y) ∩ (x U z)))) AndInt 90 184
186. (w ε (x U (y ∩ z))) <-> (w ε ((x U y) ∩ (x U z))) AndElimR 185
187. (w ε (x ∩ (y U z))) <-> (w ε ((x ∩ y) U (x ∩ z))) AndElimL 185
188. ∀w.((w ε (x U (y ∩ z))) <-> (w ε ((x U y) ∩ (x U z)))) ForallInt 186
189. ∀w.((w ε (x ∩ (y U z))) <-> (w ε ((x ∩ y) U (x ∩ z)))) ForallInt 187
190. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
191. ∀j.(((x ∩ (y U z)) = j) <-> ∀k.((k ε (x ∩ (y U z))) <-> (k ε j))) ForallElim 190
192. ((x ∩ (y U z)) = ((x ∩ y) U (x ∩ z))) <-> ∀k.((k ε (x ∩ (y U z))) <-> (k ε ((x ∩ y) U (x ∩ z)))) ForallElim 191
193. (((x ∩ (y U z)) = ((x ∩ y) U (x ∩ z))) -> ∀k.((k ε (x ∩ (y U z))) <-> (k ε ((x ∩ y) U (x ∩ z))))) & (∀k.((k ε (x ∩ (y U z))) <-> (k ε ((x ∩ y) U (x ∩ z)))) -> ((x ∩ (y U z)) = ((x ∩ y) U (x ∩ z)))) EquivExp 192
194. ∀k.((k ε (x ∩ (y U z))) <-> (k ε ((x ∩ y) U (x ∩ z)))) -> ((x ∩ (y U z)) = ((x ∩ y) U (x ∩ z))) AndElimR 193
195. (x ∩ (y U z)) = ((x ∩ y) U (x ∩ z)) ImpElim 189 194
196. ∀l.(((x U (y ∩ z)) = l) <-> ∀m.((m ε (x U (y ∩ z))) <-> (m ε l))) ForallElim 190
197. ((x U (y ∩ z)) = ((x U y) ∩ (x U z))) <-> ∀m.((m ε (x U (y ∩ z))) <-> (m ε ((x U y) ∩ (x U z)))) ForallElim 196
198. (((x U (y ∩ z)) = ((x U y) ∩ (x U z))) -> ∀m.((m ε (x U (y ∩ z))) <-> (m ε ((x U y) ∩ (x U z))))) & (∀m.((m ε (x U (y ∩ z))) <-> (m ε ((x U y) ∩ (x U z)))) -> ((x U (y ∩ z)) = ((x U y) ∩ (x U z)))) EquivExp 197
199. ∀m.((m ε (x U (y ∩ z))) <-> (m ε ((x U y) ∩ (x U z)))) -> ((x U (y ∩ z)) = ((x U y) ∩ (x U z))) AndElimR 198
200. (x U (y ∩ z)) = ((x U y) ∩ (x U z)) ImpElim 188 199
201. ((x ∩ (y U z)) = ((x ∩ y) U (x ∩ z))) & ((x U (y ∩ z)) = ((x U y) ∩ (x U z))) AndInt 195 200 Qed

Used Theorems

1. ((z ε (x U y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
2. (A & (B v C)) <-> ((A & B) v (A & C))

Th11. ~~x = x

0. z ε ~~x Hyp
1. ~x = {y: ¬(y ε x)} DefEqInt
2. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1

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3. ~~x = {y: ¬(y ε ~x)}  ForallElim 2
4. z ε {y: ¬(y ε ~x)}  EqualitySub 0 3
5. Set(z) & ¬(z ε ~x)  ClassElim 4
6. ¬(z ε ~x)  AndElimR 5
7. ¬(z ε x)  Hyp
8. Set(z)  AndElimL 5
9. Set(z) & ¬(z ε x)  AndInt 8 7
10. z ε {y: ¬(y ε x)}  ClassInt 9
11. {y: ¬(y ε x)} = ~x  Symmetry 1
12. z ε ~x  EqualitySub 10 11
13. _|_  ImpElim 12 6
14. ¬¬(z ε x)  ImpInt 13
15. D <-> ¬¬D  TheoremInt
16. (z ε x) <-> ¬¬(z ε x)  PolySub 15
17. ((z ε x) -> ¬¬(z ε x)) & (¬¬(z ε x) -> (z ε x))  EquivExp 16
18. ¬¬(z ε x) -> (z ε x)  AndElimR 17
19. z ε x  ImpElim 14 18
20. (z ε ~~x) -> (z ε x)  ImpInt 19
21. z ε x  Hyp
22. (z ε x) -> ¬¬(z ε x)  AndElimL 17
23. ¬¬(z ε x)  ImpElim 21 22
24. z ε ~x  Hyp
25. z ε {y: ¬(y ε x)}  EqualitySub 24 1
26. Set(z) & ¬(z ε x)  ClassElim 25
27. ¬(z ε x)  AndElimR 26
28. _|_  ImpElim 27 23
29. ¬(z ε ~x)  ImpInt 28
30. ∃y.(z ε y)  ExistsInt 21
31. Set(z)  DefSub 30
32. Set(z) & ¬(z ε ~x)  AndInt 31 29
33. z ε {y: ¬(y ε ~x)}  ClassInt 32
34. {y: ¬(y ε ~x)} = ~~x  Symmetry 3
35. z ε ~~x  EqualitySub 33 34
36. (z ε x) -> (z ε ~~x)  ImpInt 35
37. ((z ε ~~x) -> (z ε x)) & ((z ε x) -> (z ε ~~x))  AndInt 20 36
38. (z ε ~~x) <-> (z ε x)  EquivConst 37
39. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
40. ∀y.((~~x = y) <-> ∀z.((z ε ~~x) <-> (z ε y)))  ForallElim 39
41. (~~x = x) <-> ∀z.((z ε ~~x) <-> (z ε x))  ForallElim 40
42. ((~~x = x) -> ∀z.((z ε ~~x) <-> (z ε x))) & (∀z.((z ε ~~x) <-> (z ε x)) -> (~~x = x))
EquivExp 41
43. ∀z.((z ε ~~x) <-> (z ε x)) -> (~~x = x)  AndElimR 42
44. ∀z.((z ε ~~x) <-> (z ε x))  ForallInt 38
45. ~~x = x  ImpElim 44 43 Qed
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Used Theorems

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1. D <-> ¬¬D
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Th12. (~(x ∪ y) = (~x ∩ ~y)) & (~(x ∩ y) = (~x ∪ ~y))
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```
0. z ε ~(x ∪ y)  Hyp
1. ~x = {y: ¬(y ε x)}  DefEqInt
2. ∀a.(~a = {y: ¬(y ε a)})  ForallInt 1
3. ~(x ∪ y) = {t: ¬(t ε (x ∪ y))}  ForallElim 2
4. z ε {t: ¬(t ε (x ∪ y))}  EqualitySub 0 3
5. Set(z) & ¬(z ε (x ∪ y))  ClassElim 4
6. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
7. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  AndElimL 6
8. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 7
9. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  AndElimR 8
10. (A -> B) -> (¬B -> ¬A)  TheoremInt
11. (((z ε x) v (z ε y)) -> B) -> (¬B -> ¬((z ε x) v (z ε y)))  PolySub 10
12. (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) -> (¬(z ε (x ∪ y)) -> ¬((z ε x) v (z ε y)))
PolySub 11
13. ¬(z ε (x ∪ y)) -> ¬((z ε x) v (z ε y))  ImpElim 9 12
14. ¬(z ε (x ∪ y))  AndElimR 5
15. ¬((z ε x) v (z ε y))  ImpElim 14 13
16. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
```

17. (¬((z ε x) v B) <-> (¬(z ε x) & ¬B)) & (¬((z ε x) & B) <-> (¬(z ε x) v ¬B)) PolySub 16
18. (¬((z ε x) v (z ε y)) <-> (¬(z ε x) & ¬(z ε y))) & (¬((z ε x) & (z ε y)) <-> (¬(z ε x) v ¬(z ε y))) PolySub 17
19. ¬((z ε x) v (z ε y)) <-> (¬(z ε x) & ¬(z ε y)) AndElimL 18
20. (¬((z ε x) v (z ε y)) -> (¬(z ε x) & ¬(z ε y))) & ((¬(z ε x) & ¬(z ε y)) -> ¬((z ε x) v (z ε y))) EquivExp 19
21. ¬((z ε x) v (z ε y)) -> (¬(z ε x) & ¬(z ε y)) AndElimL 20
22. ¬(z ε x) & ¬(z ε y) ImpElim 15 21
23. Set(z) AndElimL 5
24. ¬(z ε x) AndElimL 22
25. ¬(z ε y) AndElimR 22
26. Set(z) & ¬(z ε y) AndInt 23 25
27. z ε {z: ¬(z ε y)} ClassInt 26
28. Set(z) & ¬(z ε x) AndInt 23 24
29. z ε {z: ¬(z ε x)} ClassInt 28
30. ~x = {y: ¬(y ε x)} DefEqInt
31. {y: ¬(y ε x)} = ~x Symmetry 30
32. z ε ~x EqualitySub 29 31
33. ∀w.(~w = {y: ¬(y ε w)}) ForallInt 30
34. ~y = {x_0: ¬(x_0 ε y)} ForallElim 33
35. {x_0: ¬(x_0 ε y)} = ~y Symmetry 34
36. z ε ~y EqualitySub 27 35
37. (z ε ~x) & (z ε ~y) AndInt 32 36
38. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 6
39. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 38
40. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 39
41. ∀x.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 40
42. ((z ε ~x) & (z ε y)) -> (z ε (~x ∩ y)) ForallElim 41
43. ∀y.(((z ε ~x) & (z ε y)) -> (z ε (~x ∩ y))) ForallInt 42
44. ((z ε ~x) & (z ε ~y)) -> (z ε (~x ∩ ~y)) ForallElim 43
45. z ε (~x ∩ ~y) ImpElim 37 44
46. (z ε ~(x ∪ y)) -> (z ε (~x ∩ ~y)) ImpInt 45
47. z ε (~x ∩ ~y) Hyp
48. ∀x.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) ForallInt 38
49. (z ε (~x ∩ y)) <-> ((z ε ~x) & (z ε y)) ForallElim 48
50. ∀y.((z ε (~x ∩ y)) <-> ((z ε ~x) & (z ε y))) ForallInt 49
51. (z ε (~x ∩ ~y)) <-> ((z ε ~x) & (z ε ~y)) ForallElim 50
52. ((z ε (~x ∩ ~y)) -> ((z ε ~x) & (z ε ~y))) & (((z ε ~x) & (z ε ~y)) -> (z ε (~x ∩ ~y))) EquivExp 51
53. (z ε (~x ∩ ~y)) -> ((z ε ~x) & (z ε ~y)) AndElimL 52
54. (z ε ~x) & (z ε ~y) ImpElim 47 53
55. z ε ~y AndElimR 54
56. z ε ~x AndElimL 54
57. z ε {y: ¬(y ε x)} EqualitySub 56 30
58. z ε {x_0: ¬(x_0 ε y)} EqualitySub 55 34
59. Set(z) & ¬(z ε x) ClassElim 57
60. Set(z) & ¬(z ε y) ClassElim 58
61. ¬(z ε x) AndElimR 59
62. ¬(z ε y) AndElimR 60
63. ¬(z ε x) & ¬(z ε y) AndInt 61 62
64. (¬(z ε x) & ¬(z ε y)) -> ¬((z ε x) v (z ε y)) AndElimR 20
65. ¬((z ε x) v (z ε y)) ImpElim 63 64
66. z ε (x ∪ y) Hyp
67. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 8
68. (z ε x) v (z ε y) ImpElim 66 67
69. _|_ ImpElim 68 65
70. ¬(z ε (x ∪ y)) ImpInt 69
71. Set(z) AndElimL 59
72. Set(z) & ¬(z ε (x ∪ y)) AndInt 71 70
73. z ε {w: ¬(w ε (x ∪ y))} ClassInt 72
74. ∀y.({x_0: ¬(x_0 ε y)} = ~y) ForallInt 35
75. {x_0: ¬(x_0 ε (x ∪ y))} = ~(x ∪ y) ForallElim 74
76. z ε ~(x ∪ y) EqualitySub 73 75
77. (z ε (~x ∩ ~y)) -> (z ε ~(x ∪ y)) ImpInt 76
78. ((z ε ~(x ∪ y)) -> (z ε (~x ∩ ~y))) & ((z ε (~x ∩ ~y)) -> (z ε ~(x ∪ y))) AndInt 46 77
79. (z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y)) EquivConst 78
80. z ε ~(x ∩ y) Hyp
81. ∀y.(~y = {x_0: ¬(x_0 ε y)}) ForallInt 34

82. ~(x ∩ y) = {x_0: ¬(x_0 ε (x ∩ y))} ForallElim 81
83. z ε {x_0: ¬(x_0 ε (x ∩ y))} EqualitySub 80 82
84. Set(z) & ¬(z ε (x ∩ y)) ClassElim 83
85. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 39
86. (((z ε x) & (z ε y)) -> B) -> (¬B -> ¬((z ε x) & (z ε y))) PolySub 10
87. (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) -> (¬(z ε (x ∩ y)) -> ¬((z ε x) & (z ε y))) PolySub 86
88. ¬(z ε (x ∩ y)) -> ¬((z ε x) & (z ε y)) ImpElim 85 87
89. ¬(z ε (x ∩ y)) AndElimR 84
90. ¬((z ε x) & (z ε y)) ImpElim 89 88
91. ¬(A & B) <-> (¬A v ¬B) AndElimR 16
92. ¬((z ε x) & B) <-> (¬(z ε x) v ¬B) PolySub 91
93. ¬((z ε x) & (z ε y)) <-> (¬(z ε x) v ¬(z ε y)) PolySub 92
94. (¬((z ε x) & (z ε y)) -> (¬(z ε x) v ¬(z ε y))) & ((¬(z ε x) v ¬(z ε y)) -> ¬((z ε x) & (z ε y))) EquivExp 93
95. ¬((z ε x) & (z ε y)) -> (¬(z ε x) v ¬(z ε y)) AndElimL 94
96. ¬(z ε x) v ¬(z ε y) ImpElim 90 95
97. ¬(z ε x) Hyp
98. Set(z) AndElimL 84
99. Set(z) & ¬(z ε x) AndInt 98 97
100. z ε {w: ¬(w ε x)} ClassInt 99
101. (z ε {w: ¬(w ε x)}) v (z ε {w: ¬(w ε y)}) OrIntR 100
102. {y: ¬(y ε x)} = ~x Symmetry 30
103. ∀x.({y: ¬(y ε x)} = ~x) ForallInt 102
104. {x_1: ¬(x_1 ε y)} = ~y ForallElim 103
105. (z ε ~x) v (z ε {w: ¬(w ε y)}) EqualitySub 101 102
106. (z ε ~x) v (z ε ~y) EqualitySub 105 104
107. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 9
108. ((z ε ~x) v (z ε y)) -> (z ε (~x ∪ y)) ForallElim 107
109. ∀y.(((z ε ~x) v (z ε y)) -> (z ε (~x ∪ y))) ForallInt 108
110. ((z ε ~x) v (z ε ~y)) -> (z ε (~x ∪ ~y)) ForallElim 109
111. z ε (~x ∪ ~y) ImpElim 106 110
112. ¬(z ε y) Hyp
113. Set(z) & ¬(z ε y) AndInt 98 112
114. z ε {z: ¬(z ε y)} ClassInt 113
115. (z ε {z: ¬(z ε x)}) v (z ε {z: ¬(z ε y)}) OrIntL 114
116. (z ε ~x) v (z ε {z: ¬(z ε y)}) EqualitySub 115 102
117. (z ε ~x) v (z ε ~y) EqualitySub 116 104
118. z ε (~x ∪ ~y) ImpElim 117 110
119. z ε (~x ∪ ~y) OrElim 96 97 111 112 118
120. (z ε ~(x ∩ y)) -> (z ε (~x ∪ ~y)) ImpInt 119
121. z ε (~x ∪ ~y) Hyp
122. ∃w.(z ε w) ExistsInt 121
123. Set(z) DefSub 122
124. x = x Identity
125. x = x Identity
126. x = x Identity
127. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 8
128. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 127
129. (z ε (~x ∪ y)) -> ((z ε ~x) v (z ε y)) ForallElim 128
130. ∀y.((z ε (~x ∪ y)) -> ((z ε ~x) v (z ε y))) ForallInt 129
131. (z ε (~x ∪ ~y)) -> ((z ε ~x) v (z ε ~y)) ForallElim 130
132. (z ε ~x) v (z ε ~y) ImpElim 121 131
133. z ε ~x Hyp
134. z ε {y: ¬(y ε x)} EqualitySub 133 30
135. Set(z) & ¬(z ε x) ClassElim 134
136. ¬(z ε x) AndElimR 135
137. z ε ~y Hyp
138. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 30
139. ~y = {x_3: ¬(x_3 ε y)} ForallElim 138
140. z ε {x_3: ¬(x_3 ε y)} EqualitySub 137 139
141. Set(z) & ¬(z ε y) ClassElim 140
142. ¬(z ε y) AndElimR 141
143. ¬(z ε x) v ¬(z ε y) OrIntR 136
144. ¬(z ε x) v ¬(z ε y) OrIntL 142
145. ¬(z ε x) v ¬(z ε y) OrElim 132 133 143 137 144
146. ¬(A & B) <-> (¬A v ¬B) AndElimR 16
147. (¬(A & B) -> (¬A v ¬B)) & ((¬A v ¬B) -> ¬(A & B)) EquivExp 146
148. (¬A v ¬B) -> ¬(A & B) AndElimR 147
149. (¬(z ε x) v ¬B) -> ¬((z ε x) & B) PolySub 148
150. (¬(z ε x) v ¬(z ε y)) -> ¬((z ε x) & (z ε y)) PolySub 149

151. ¬((z ε x) & (z ε y)) ImpElim 145 150
152. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 6
153. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 152
154. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 153
155. ((z ε (x ∩ y)) -> B) -> (¬B -> ¬(z ε (x ∩ y))) PolySub 10
156. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) -> (¬((z ε x) & (z ε y)) -> ¬(z ε (x ∩ y))) PolySub 155
157. ¬((z ε x) & (z ε y)) -> ¬(z ε (x ∩ y)) ImpElim 154 156
158. ¬(z ε (x ∩ y)) ImpElim 151 157
159. Set(z) DefSub 122
160. Set(z) & ¬(z ε (x ∩ y)) AndInt 159 158
161. z ε {w: ¬(w ε (x ∩ y))} ClassInt 160
162. ∀x.({y: ¬(y ε x)} = ~x) ForallInt 31
163. {x_5: ¬(x_5 ε (x ∩ y))} = ~(x ∩ y) ForallElim 162
164. z ε ~(x ∩ y) EqualitySub 161 163
165. (z ε (~x ∪ ~y)) -> (z ε ~(x ∩ y)) ImpInt 164
166. ((z ε ~(x ∩ y)) -> (z ε (~x ∪ ~y))) & ((z ε (~x ∪ ~y)) -> (z ε ~(x ∩ y))) AndInt 120 165
167. (z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y)) EquivConst 166
168. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
169. ∀x_6.((~(x ∪ y) = x_6) <-> ∀z.((z ε ~(x ∪ y)) <-> (z ε x_6))) ForallElim 168
170. (~(x ∪ y) = (~x ∩ ~y)) <-> ∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y))) ForallElim 169
171. ∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y))) ForallInt 79
172. ((~(x ∪ y) = (~x ∩ ~y)) -> ∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y)))) & (∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y))) -> (~(x ∪ y) = (~x ∩ ~y))) EquivExp 170
173. ∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y))) -> (~(x ∪ y) = (~x ∩ ~y)) AndElimR 172
174. ~(x ∪ y) = (~x ∩ ~y) ImpElim 171 173
175. ∀x_7.((~(x ∩ y) = x_7) <-> ∀z.((z ε ~(x ∩ y)) <-> (z ε x_7))) ForallElim 168
176. (~(x ∩ y) = (~x ∪ ~y)) <-> ∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y))) ForallElim 175
177. ((~(x ∩ y) = (~x ∪ ~y)) -> ∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y)))) & (∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y))) -> (~(x ∩ y) = (~x ∪ ~y))) EquivExp 176
178. ∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y))) -> (~(x ∩ y) = (~x ∪ ~y)) AndElimR 177
179. ∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y))) ForallInt 167
180. ~(x ∩ y) = (~x ∪ ~y) ImpElim 179 178
181. (~(x ∪ y) = (~x ∩ ~y)) & (~(x ∩ y) = (~x ∪ ~y)) AndInt 174 180 Qed

Used Theorems

2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
3. (A -> B) -> (¬B -> ¬A)
1. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))

Th14. (x ∩ (y ~ z)) = ((x ∩ y) ∩ ~z)

0. (x ~ y) = (x ∩ ~y) DefEqInt
1. ∀a.((a ~ y) = (a ∩ ~y)) ForallInt 0
2. ∀b.∀a.((a ~ b) = (a ∩ ~b)) ForallInt 1
3. ∀a.((a ~ z) = (a ∩ ~z)) ForallElim 2
4. (y ~ z) = (y ∩ ~z) ForallElim 3
5. (x ∩ (y ~ z)) = (x ∩ (y ~ z)) Identity
6. (x ∩ (y ~ z)) = (x ∩ (y ∩ ~z)) EqualitySub 5 4
7. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) TheoremInt
8. ((x ∩ y) ∩ z) = (x ∩ (y ∩ z)) AndElimR 7
9. (x ∩ (y ∩ z)) = ((x ∩ y) ∩ z) Symmetry 8
10. ∀z.((x ∩ (y ∩ z)) = ((x ∩ y) ∩ z)) ForallInt 9
11. (x ∩ (y ∩ ~z)) = ((x ∩ y) ∩ ~z) ForallElim 10
12. (x ∩ (y ~ z)) = ((x ∩ y) ∩ ~z) EqualitySub 6 11 Qed

Used Theorems

4. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))

Th16. ¬(x ε 0)

0. x ε 0 Hyp
1. 0 = {x: ¬(x = x)} DefEqInt
2. x ε {x: ¬(x = x)} EqualitySub 0 1
3. Set(x) & ¬(x = x) ClassElim 2
4. ¬(x = x) AndElimR 3
5. x = x Identity

```
6. _|_  ImpElim 5 4
7. ¬(x ε 0)  ImpInt 6 Qed
```

Used Theorems

Th17. ((0 ∪ x) = x) & ((0 ∩ x) = 0)

```
0. z ε (0 ∪ x)  Hyp
1. (x ∪ y) = {z: ((z ε x) v (z ε y))}  DefEqInt
2. ∀x.((x ∪ y) = {z: ((z ε x) v (z ε y))})  ForallInt 1
3. (0 ∪ y) = {z: ((z ε 0) v (z ε y))}  ForallElim 2
4. ∀y.((0 ∪ y) = {z: ((z ε 0) v (z ε y))})  ForallInt 3
5. (0 ∪ x) = {z: ((z ε 0) v (z ε x))}  ForallElim 4
6. z ε {z: ((z ε 0) v (z ε x))}  EqualitySub 0 5
7. Set(z) & ((z ε 0) v (z ε x))  ClassElim 6
8. (z ε 0) v (z ε x)  AndElimR 7
9. z ε 0  Hyp
10. ¬(x ε 0)  TheoremInt
11. ∀x.¬(x ε 0)  ForallInt 10
12. ¬(z ε 0)  ForallElim 11
13. _|_  ImpElim 9 12
14. z ε x  AbsI 13
15. z ε x  Hyp
16. z ε x  OrElim 8 9 14 15 15
17. (z ε (0 ∪ x)) -> (z ε x)  ImpInt 16
18. z ε x  Hyp
19. (z ε 0) v (z ε x)  OrIntL 18
20. ∃x.(z ε x)  ExistsInt 18
21. Set(z)  DefSub 20
22. Set(z) & ((z ε 0) v (z ε x))  AndInt 21 19
23. z ε {z: ((z ε 0) v (z ε x))}  ClassInt 22
24. {z: ((z ε 0) v (z ε x))} = (0 ∪ x)  Symmetry 5
25. z ε (0 ∪ x)  EqualitySub 23 24
26. (z ε x) -> (z ε (0 ∪ x))  ImpInt 25
27. ((z ε (0 ∪ x)) -> (z ε x)) & ((z ε x) -> (z ε (0 ∪ x)))  AndInt 17 26
28. (z ε (0 ∪ x)) <-> (z ε x)  EquivConst 27
29. ∀z.((z ε (0 ∪ x)) <-> (z ε x))  ForallInt 28
30. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
31. ∀y.(((0 ∪ x) = y) <-> ∀z.((z ε (0 ∪ x)) <-> (z ε y)))  ForallElim 30
32. ((0 ∪ x) = x) <-> ∀z.((z ε (0 ∪ x)) <-> (z ε x))  ForallElim 31
33. (((0 ∪ x) = x) -> ∀z.((z ε (0 ∪ x)) <-> (z ε x))) & (∀z.((z ε (0 ∪ x)) <-> (z ε x)) -
> ((0 ∪ x) = x))  EquivExp 32
34. ∀z.((z ε (0 ∪ x)) <-> (z ε x)) -> ((0 ∪ x) = x)  AndElimR 33
35. (0 ∪ x) = x  ImpElim 29 34
36. z ε (0 ∩ x)  Hyp
37. (x ∩ y) = {z: ((z ε x) & (z ε y))}  DefEqInt
38. ∀x.((x ∩ y) = {z: ((z ε x) & (z ε y))})  ForallInt 37
39. (0 ∩ y) = {z: ((z ε 0) & (z ε y))}  ForallElim 38
40. ∀y.((0 ∩ y) = {z: ((z ε 0) & (z ε y))})  ForallInt 39
41. (0 ∩ x) = {z: ((z ε 0) & (z ε x))}  ForallElim 40
42. z ε {z: ((z ε 0) & (z ε x))}  EqualitySub 36 41
43. Set(z) & ((z ε 0) & (z ε x))  ClassElim 42
44. (z ε 0) & (z ε x)  AndElimR 43
45. z ε 0  AndElimL 44
46. (z ε (0 ∩ x)) -> (z ε 0)  ImpInt 45
47. z ε 0  Hyp
48. _|_  ImpElim 47 12
49. z ε (0 ∩ x)  AbsI 48
50. (z ε 0) -> (z ε (0 ∩ x))  ImpInt 49
51. ((z ε (0 ∩ x)) -> (z ε 0)) & ((z ε 0) -> (z ε (0 ∩ x)))  AndInt 46 50
52. (z ε (0 ∩ x)) <-> (z ε 0)  EquivConst 51
53. ∀z.((z ε (0 ∩ x)) <-> (z ε 0))  ForallInt 52
54. ∀y.(((0 ∩ x) = y) <-> ∀z.((z ε (0 ∩ x)) <-> (z ε y)))  ForallElim 30
55. ((0 ∩ x) = 0) <-> ∀z.((z ε (0 ∩ x)) <-> (z ε 0))  ForallElim 54
56. (((0 ∩ x) = 0) -> ∀z.((z ε (0 ∩ x)) <-> (z ε 0))) & (∀z.((z ε (0 ∩ x)) <-> (z ε 0)) -
> ((0 ∩ x) = 0))  EquivExp 55
57. ∀z.((z ε (0 ∩ x)) <-> (z ε 0)) -> ((0 ∩ x) = 0)  AndElimR 56
58. (0 ∩ x) = 0  ImpElim 53 57
59. ((0 ∪ x) = x) & ((0 ∩ x) = 0)  AndInt 35 58 Qed
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Used Theorems

```
2. ¬(x ε 0)
```

Th19. (x ε U) <-> Set(x)

```
0. x ε U  Hyp
1. U = {x: (x = x)}  DefEqInt
2. x ε {x: (x = x)}  EqualitySub 0 1
3. Set(x) & (x = x)  ClassElim 2
4. Set(x)  AndElimL 3
5. (x ε U) -> Set(x)  ImpInt 4
6. Set(x)  Hyp
7. x = x  Identity
8. Set(x) & (x = x)  AndInt 6 7
9. x ε {x: (x = x)}  ClassInt 8
10. {x: (x = x)} = U  Symmetry 1
11. x ε U  EqualitySub 9 10
12. Set(x) -> (x ε U)  ImpInt 11
13. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  AndInt 5 12
14. (x ε U) <-> Set(x)  EquivConst 13 Qed
```

Used Theorems

Th20. ((x ∪ U) = U) & ((x ∩ U) = x)

```
0. z ε (x ∪ U)  Hyp
1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
2. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  AndElimL 1
3. ∀y.((z ε (x ∪ y)) <-> ((z ε x) v (z ε y)))  ForallInt 2
4. (z ε (x ∪ U)) <-> ((z ε x) v (z ε U))  ForallElim 3
5. ((z ε (x ∪ U)) -> ((z ε x) v (z ε U))) & (((z ε x) v (z ε U)) -> (z ε (x ∪ U)))
EquivExp 4
6. (z ε (x ∪ U)) -> ((z ε x) v (z ε U))  AndElimL 5
7. (z ε x) v (z ε U)  ImpElim 0 6
8. z ε x  Hyp
9. ∃y.(z ε y)  ExistsInt 8
10. Set(z)  DefSub 9
11. (x ε U) <-> Set(x)  TheoremInt
12. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 11
13. Set(x) -> (x ε U)  AndElimR 12
14. ∀x.(Set(x) -> (x ε U))  ForallInt 13
15. Set(z) -> (z ε U)  ForallElim 14
16. z ε U  ImpElim 10 15
17. z ε U  Hyp
18. z ε U  OrElim 7 8 16 17 17
19. (z ε (x ∪ U)) -> (z ε U)  ImpInt 18
20. z ε U  Hyp
21. (z ε x) v (z ε U)  OrIntL 20
22. ((z ε x) v (z ε U)) -> (z ε (x ∪ U))  AndElimR 5
23. z ε (x ∪ U)  ImpElim 21 22
24. (z ε U) -> (z ε (x ∪ U))  ImpInt 23
25. ((z ε (x ∪ U)) -> (z ε U)) & ((z ε U) -> (z ε (x ∪ U)))  AndInt 19 24
26. (z ε (x ∪ U)) <-> (z ε U)  EquivConst 25
27. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
28. ∀y.(((x ∪ U) = y) <-> ∀z.((z ε (x ∪ U)) <-> (z ε y)))  ForallElim 27
29. ((x ∪ U) = U) <-> ∀z.((z ε (x ∪ U)) <-> (z ε U))  ForallElim 28
30. ∀z.((z ε (x ∪ U)) <-> (z ε U))  ForallInt 26
31. (((x ∪ U) = U) -> ∀z.((z ε (x ∪ U)) <-> (z ε U))) & (∀z.((z ε (x ∪ U)) <-> (z ε U)) -
> ((x ∪ U) = U))  EquivExp 29
32. ∀z.((z ε (x ∪ U)) <-> (z ε U)) -> ((x ∪ U) = U)  AndElimR 31
33. (x ∪ U) = U  ImpElim 30 32
34. z ε (x ∩ U)  Hyp
35. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 1
36. ∀y.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))  ForallInt 35
37. (z ε (x ∩ U)) <-> ((z ε x) & (z ε U))  ForallElim 36
38. ((z ε (x ∩ U)) -> ((z ε x) & (z ε U))) & (((z ε x) & (z ε U)) -> (z ε (x ∩ U)))
EquivExp 37
39. (z ε (x ∩ U)) -> ((z ε x) & (z ε U))  AndElimL 38
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40. (z ε x) & (z ε U) ImpElim 34 39
41. z ε x AndElimL 40
42. (z ε (x ∩ U)) -> (z ε x) ImpInt 41
43. z ε x Hyp
44. ∃y.(z ε y) ExistsInt 43
45. Set(z) DefSub 44
46. z ε U ImpElim 45 15
47. (z ε x) & (z ε U) AndInt 43 46
48. ((z ε x) & (z ε U)) -> (z ε (x ∩ U)) AndElimR 38
49. z ε (x ∩ U) ImpElim 47 48
50. (z ε x) -> (z ε (x ∩ U)) ImpInt 49
51. ((z ε (x ∩ U)) -> (z ε x)) & ((z ε x) -> (z ε (x ∩ U))) AndInt 42 50
52. (z ε (x ∩ U)) <-> (z ε x) EquivConst 51
53. ∀z.((z ε (x ∩ U)) <-> (z ε x)) ForallInt 52
54. ∀y.(((x ∩ U) = y) <-> ∀z.((z ε (x ∩ U)) <-> (z ε y))) ForallElim 27
55. ((x ∩ U) = x) <-> ∀z.((z ε (x ∩ U)) <-> (z ε x)) ForallElim 54
56. (((x ∩ U) = x) -> ∀z.((z ε (x ∩ U)) <-> (z ε x))) & (∀z.((z ε (x ∩ U)) <-> (z ε x)) -> ((x ∩ U) = x)) EquivExp 55
57. ∀z.((z ε (x ∩ U)) <-> (z ε x)) -> ((x ∩ U) = x) AndElimR 56
58. (x ∩ U) = x ImpElim 53 57
59. ((x ∪ U) = U) & ((x ∩ U) = x) AndInt 33 58 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
2. (x ε U) <-> Set(x)

Th21. (~0 = U) & (~U = 0)

0. z ε ~0 Hyp
1. ~x = {y: ¬(y ε x)} DefEqInt
2. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1
3. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1
4. ~0 = {y: ¬(y ε 0)} ForallElim 3
5. z ε {y: ¬(y ε 0)} EqualitySub 0 4
6. Set(z) & ¬(z ε 0) ClassElim 5
7. Set(z) AndElimL 6
8. (x ε U) <-> Set(x) TheoremInt
9. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 8
10. Set(x) -> (x ε U) AndElimR 9
11. ∀x.(Set(x) -> (x ε U)) ForallInt 10
12. Set(z) -> (z ε U) ForallElim 11
13. z ε U ImpElim 7 12
14. (z ε ~0) -> (z ε U) ImpInt 13
15. z ε U Hyp
16. (x ε U) -> Set(x) AndElimL 9
17. ∀x.((x ε U) -> Set(x)) ForallInt 16
18. (z ε U) -> Set(z) ForallElim 17
19. Set(z) ImpElim 15 18
20. ¬(x ε 0) TheoremInt
21. ∀x.¬(x ε 0) ForallInt 20
22. ¬(z ε 0) ForallElim 21
23. Set(z) & ¬(z ε 0) AndInt 19 22
24. z ε {y: ¬(y ε 0)} ClassInt 23
25. {y: ¬(y ε 0)} = ~0 Symmetry 4
26. z ε ~0 EqualitySub 24 25
27. (z ε U) -> (z ε ~0) ImpInt 26
28. ((z ε ~0) -> (z ε U)) & ((z ε U) -> (z ε ~0)) AndInt 14 27
29. (z ε ~0) <-> (z ε U) EquivConst 28
30. ∀z.((z ε ~0) <-> (z ε U)) ForallInt 29
31. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
32. ∀y.((~0 = y) <-> ∀z.((z ε ~0) <-> (z ε y))) ForallElim 31
33. (~0 = U) <-> ∀z.((z ε ~0) <-> (z ε U)) ForallElim 32
34. ((~0 = U) -> ∀z.((z ε ~0) <-> (z ε U))) & (∀z.((z ε ~0) <-> (z ε U)) -> (~0 = U)) EquivExp 33
35. ∀z.((z ε ~0) <-> (z ε U)) -> (~0 = U) AndElimR 34
36. ~0 = U ImpElim 30 35
37. z ε ~U Hyp
38. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1
39. ~U = {y: ¬(y ε U)} ForallElim 38
40. z ε {y: ¬(y ε U)} EqualitySub 37 39

41. Set(z) & ¬(z ε U) ClassElim 40
42. ¬(z ε U) AndElimR 41
43. Set(z) AndElimL 41
44. z ε U ImpElim 43 12
45. _|_ ImpElim 44 42
46. z ε 0 AbsI 45
47. (z ε ~U) -> (z ε 0) ImpInt 46
48. z ε 0 Hyp
49. 0 = {x: ¬(x = x)} DefEqInt
50. z ε {x: ¬(x = x)} EqualitySub 48 49
51. Set(z) & ¬(z = z) ClassElim 50
52. Set(z) AndElimL 51
53. ¬(z = z) AndElimR 51
54. z = z Identity
55. _|_ ImpElim 54 53
56. z ε ~U AbsI 55
57. (z ε 0) -> (z ε ~U) ImpInt 56
58. ((z ε ~U) -> (z ε 0)) & ((z ε 0) -> (z ε ~U)) AndInt 47 57
59. (z ε ~U) <-> (z ε 0) EquivConst 58
60. ∀z.((z ε ~U) <-> (z ε 0)) ForallInt 59
61. ∀y.((~U = y) <-> ∀z.((z ε ~U) <-> (z ε y))) ForallElim 31
62. (~U = 0) <-> ∀z.((z ε ~U) <-> (z ε 0)) ForallElim 61
63. ((~U = 0) -> ∀z.((z ε ~U) <-> (z ε 0))) & (∀z.((z ε ~U) <-> (z ε 0)) -> (~U = 0))
EquivExp 62
64. ∀z.((z ε ~U) <-> (z ε 0)) -> (~U = 0) AndElimR 63
65. ~U = 0 ImpElim 60 64
66. (~0 = U) & (~U = 0) AndInt 36 65 Qed

Used Theorems

1. (x ε U) <-> Set(x)
2. ¬(x ε 0)

Th24. (∩0 = U) & (∪0 = 0)

0. x ε ∩0 Hyp
1. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
2. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 1
3. ∩0 = {z: ∀y.((y ε 0) -> (z ε y))} ForallElim 2
4. x ε {z: ∀y.((y ε 0) -> (z ε y))} EqualitySub 0 3
5. Set(x) & ∀y.((y ε 0) -> (x ε y)) ClassElim 4
6. Set(x) AndElimL 5
7. (x ε U) <-> Set(x) TheoremInt
8. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 7
9. Set(x) -> (x ε U) AndElimR 8
10. x ε U ImpElim 6 9
11. (x ε ∩0) -> (x ε U) ImpInt 10
12. x ε U Hyp
13. y ε 0 Hyp
14. ¬(x ε 0) TheoremInt
15. ∀x.¬(x ε 0) ForallInt 14
16. ¬(y ε 0) ForallElim 15
17. _|_ ImpElim 13 16
18. x ε y AbsI 17
19. (y ε 0) -> (x ε y) ImpInt 18
20. ∀y.((y ε 0) -> (x ε y)) ForallInt 19
21. (x ε U) -> Set(x) AndElimL 8
22. Set(x) ImpElim 12 21
23. Set(x) & ∀y.((y ε 0) -> (x ε y)) AndInt 22 20
24. x ε {z: ∀y.((y ε 0) -> (z ε y))} ClassInt 23
25. {z: ∀y.((y ε 0) -> (z ε y))} = ∩0 Symmetry 3
26. x ε ∩0 EqualitySub 24 25
27. (x ε U) -> (x ε ∩0) ImpInt 26
28. ((x ε ∩0) -> (x ε U)) & ((x ε U) -> (x ε ∩0)) AndInt 11 27
29. (x ε ∩0) <-> (x ε U) EquivConst 28
30. ∀z.((z ε ∩0) <-> (z ε U)) ForallInt 29
31. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
32. ∀y.((∩0 = y) <-> ∀z.((z ε ∩0) <-> (z ε y))) ForallElim 31
33. (∩0 = U) <-> ∀z.((z ε ∩0) <-> (z ε U)) ForallElim 32
34. ((∩0 = U) -> ∀z.((z ε ∩0) <-> (z ε U))) & (∀z.((z ε ∩0) <-> (z ε U)) -> (∩0 = U))
EquivExp 33

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35. ∀z.((z ε ∩0) <-> (z ε U)) -> (∩0 = U)  AndElimR 34
36. ∩0 = U  ImpElim 30 35
37. z ε ∪0  Hyp
38. ∪x = {z: ∃y.((y ε x) & (z ε y))}  DefEqInt
39. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))})  ForallInt 38
40. ∪0 = {z: ∃y.((y ε 0) & (z ε y))}  ForallElim 39
41. z ε {z: ∃y.((y ε 0) & (z ε y))}  EqualitySub 37 40
42. Set(z) & ∃y.((y ε 0) & (z ε y))  ClassElim 41
43. ∃y.((y ε 0) & (z ε y))  AndElimR 42
44. (a ε 0) & (z ε a)  Hyp
45. ∀x.¬(x ε 0)  ForallInt 14
46. ¬(a ε 0)  ForallElim 45
47. a ε 0  AndElimL 44
48. _|_  ImpElim 47 46
49. z ε 0  AbsI 48
50. z ε 0  ExistsElim 43 44 49
51. (z ε ∪0) -> (z ε 0)  ImpInt 50
52. z ε 0  Hyp
53. ∀x.¬(x ε 0)  ForallInt 14
54. ¬(z ε 0)  ForallElim 53
55. _|_  ImpElim 52 54
56. z ε ∪0  AbsI 55
57. (z ε 0) -> (z ε ∪0)  ImpInt 56
58. ((z ε ∪0) -> (z ε 0)) & ((z ε 0) -> (z ε ∪0))  AndInt 51 57
59. (z ε ∪0) <-> (z ε 0)  EquivConst 58
60. ∀z.((z ε ∪0) <-> (z ε 0))  ForallInt 59
61. ∀y.((∪0 = y) <-> ∀z.((z ε ∪0) <-> (z ε y)))  ForallElim 31
62. (∪0 = 0) <-> ∀z.((z ε ∪0) <-> (z ε 0))  ForallElim 61
63. ((∪0 = 0) -> ∀z.((z ε ∪0) <-> (z ε 0))) & (∀z.((z ε ∪0) <-> (z ε 0)) -> (∪0 = 0))
EquivExp 62
64. ∀z.((z ε ∪0) <-> (z ε 0)) -> (∪0 = 0)  AndElimR 63
65. ∪0 = 0  ImpElim 60 64
66. (∩0 = U) & (∪0 = 0)  AndInt 36 65 Qed
```

Used Theorems

```
1. (x ε U) <-> Set(x)
2. ¬(x ε 0)
```

Th26. (0 ⊂ x) & (x ⊂ U)

```
0. z ε 0  Hyp
1. ¬(x ε 0)  TheoremInt
2. ∀x.¬(x ε 0)  ForallInt 1
3. ¬(z ε 0)  ForallElim 2
4. _|_  ImpElim 0 3
5. z ε x  AbsI 4
6. (z ε 0) -> (z ε x)  ImpInt 5
7. ∀z.((z ε 0) -> (z ε x))  ForallInt 6
8. 0 ⊂ x  DefSub 7
9. z ε x  Hyp
10. ∃y.(z ε y)  ExistsInt 9
11. Set(z)  DefSub 10
12. (x ε U) <-> Set(x)  TheoremInt
13. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 12
14. Set(x) -> (x ε U)  AndElimR 13
15. ∀x.(Set(x) -> (x ε U))  ForallInt 14
16. Set(z) -> (z ε U)  ForallElim 15
17. z ε U  ImpElim 11 16
18. (z ε x) -> (z ε U)  ImpInt 17
19. ∀z.((z ε x) -> (z ε U))  ForallInt 18
20. x ⊂ U  DefSub 19
21. (0 ⊂ x) & (x ⊂ U)  AndInt 8 20 Qed
```

Used Theorems

```
1. ¬(x ε 0)
2. (x ε U) <-> Set(x)
```

Th27. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

```
0. a = b  Hyp
1. z ε a  Hyp
2. z ε b  EqualitySub 1 0
3. (z ε a) -> (z ε b)  ImpInt 2
4. ∀z.((z ε a) -> (z ε b))  ForallInt 3
5. a ⊂ b  DefSub 4
6. z ε b  Hyp
7. b = a  Symmetry 0
8. z ε a  EqualitySub 6 7
9. (z ε b) -> (z ε a)  ImpInt 8
10. ∀z.((z ε b) -> (z ε a))  ForallInt 9
11. b ⊂ a  DefSub 10
12. (a ⊂ b) & (b ⊂ a)  AndInt 5 11
13. (a = b) -> ((a ⊂ b) & (b ⊂ a))  ImpInt 12
14. (a ⊂ b) & (b ⊂ a)  Hyp
15. a ⊂ b  AndElimL 14
16. b ⊂ a  AndElimR 14
17. z ε a  Hyp
18. ∀z.((z ε a) -> (z ε b))  DefExp 15
19. (z ε a) -> (z ε b)  ForallElim 18
20. z ε b  ImpElim 17 19
21. (z ε a) -> (z ε b)  ImpInt 20
22. z ε b  Hyp
23. ∀z.((z ε b) -> (z ε a))  DefExp 16
24. (z ε b) -> (z ε a)  ForallElim 23
25. z ε a  ImpElim 22 24
26. (z ε b) -> (z ε a)  ImpInt 25
27. ((z ε a) -> (z ε b)) & ((z ε b) -> (z ε a))  AndInt 21 26
28. (z ε a) <-> (z ε b)  EquivConst 27
29. ∀z.((z ε a) <-> (z ε b))  ForallInt 28
30. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
31. ∀y.((a = y) <-> ∀z.((z ε a) <-> (z ε y)))  ForallElim 30
32. (a = b) <-> ∀z.((z ε a) <-> (z ε b))  ForallElim 31
33. ((a = b) -> ∀z.((z ε a) <-> (z ε b))) & (∀z.((z ε a) <-> (z ε b)) -> (a = b))
EquivExp 32
34. ∀z.((z ε a) <-> (z ε b)) -> (a = b)  AndElimR 33
35. a = b  ImpElim 29 34
36. ((a ⊂ b) & (b ⊂ a)) -> (a = b)  ImpInt 35
37. ((a = b) -> ((a ⊂ b) & (b ⊂ a))) & (((a ⊂ b) & (b ⊂ a)) -> (a = b))  AndInt 13 36
38. (a = b) <-> ((a ⊂ b) & (b ⊂ a))  EquivConst 37
39. ∀a.((a = b) <-> ((a ⊂ b) & (b ⊂ a)))  ForallInt 38
40. (x = b) <-> ((x ⊂ b) & (b ⊂ x))  ForallElim 39
41. ∀b.((x = b) <-> ((x ⊂ b) & (b ⊂ x)))  ForallInt 40
42. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  ForallElim 41 Qed
```

Used Theorems

Th28. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)

```
0. (a ⊂ b) & (b ⊂ c)  Hyp
1. b ⊂ c  AndElimR 0
2. a ⊂ b  AndElimL 0
3. ∀z.((z ε b) -> (z ε c))  DefExp 1
4. ∀z.((z ε a) -> (z ε b))  DefExp 2
5. (z ε b) -> (z ε c)  ForallElim 3
6. (z ε a) -> (z ε b)  ForallElim 4
7. z ε a  Hyp
8. z ε b  ImpElim 7 6
9. z ε c  ImpElim 8 5
10. (z ε a) -> (z ε c)  ImpInt 9
11. ∀z.((z ε a) -> (z ε c))  ForallInt 10
12. a ⊂ c  DefSub 11
13. ((a ⊂ b) & (b ⊂ c)) -> (a ⊂ c)  ImpInt 12
14. ∀a.(((a ⊂ b) & (b ⊂ c)) -> (a ⊂ c))  ForallInt 13
15. ((x ⊂ b) & (b ⊂ c)) -> (x ⊂ c)  ForallElim 14
16. ∀b.(((x ⊂ b) & (b ⊂ c)) -> (x ⊂ c))  ForallInt 15
17. ((x ⊂ y) & (y ⊂ c)) -> (x ⊂ c)  ForallElim 16
18. ∀c.(((x ⊂ y) & (y ⊂ c)) -> (x ⊂ c))  ForallInt 17
19. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)  ForallElim 18 Qed
```

Used Theorems

Th29. (x ⊂ y) <-> ((x ∪ y) = y)

0. a ⊂ b Hyp
1. z ε (a ∪ b) Hyp
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
3. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 2
4. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 3
5. ∀x.(((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))))
ForallInt 4
6. ((z ε (a ∪ y)) -> ((z ε a) v (z ε y))) & (((z ε a) v (z ε y)) -> (z ε (a ∪ y)))
ForallElim 5
7. ∀y.(((z ε (a ∪ y)) -> ((z ε a) v (z ε y))) & (((z ε a) v (z ε y)) -> (z ε (a ∪ y))))
ForallInt 6
8. ((z ε (a ∪ b)) -> ((z ε a) v (z ε b))) & (((z ε a) v (z ε b)) -> (z ε (a ∪ b)))
ForallElim 7
9. (z ε (a ∪ b)) -> ((z ε a) v (z ε b)) AndElimL 8
10. (z ε a) v (z ε b) ImpElim 1 9
11. z ε a Hyp
12. ∀z.((z ε a) -> (z ε b)) DefExp 0
13. (z ε a) -> (z ε b) ForallElim 12
14. z ε b ImpElim 11 13
15. z ε b Hyp
16. z ε b OrElim 10 11 14 15 15
17. (z ε (a ∪ b)) -> (z ε b) ImpInt 16
18. z ε b Hyp
19. (z ε a) v (z ε b) OrIntL 18
20. ((z ε a) v (z ε b)) -> (z ε (a ∪ b)) AndElimR 8
21. z ε (a ∪ b) ImpElim 19 20
22. (z ε b) -> (z ε (a ∪ b)) ImpInt 21
23. ((z ε (a ∪ b)) -> (z ε b)) & ((z ε b) -> (z ε (a ∪ b))) AndInt 17 22
24. (z ε (a ∪ b)) <-> (z ε b) EquivConst 23
25. ∀z.((z ε (a ∪ b)) <-> (z ε b)) ForallInt 24
26. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
27. ∀y.(((a ∪ b) = y) <-> ∀z.((z ε (a ∪ b)) <-> (z ε y))) ForallElim 26
28. ((a ∪ b) = b) <-> ∀z.((z ε (a ∪ b)) <-> (z ε b)) ForallElim 27
29. (((a ∪ b) = b) -> ∀z.((z ε (a ∪ b)) <-> (z ε b))) & (∀z.((z ε (a ∪ b)) <-> (z ε b)) -
> ((a ∪ b) = b)) EquivExp 28
30. ∀z.((z ε (a ∪ b)) <-> (z ε b)) -> ((a ∪ b) = b) AndElimR 29
31. (a ∪ b) = b ImpElim 25 30
32. (a ⊂ b) -> ((a ∪ b) = b) ImpInt 31
33. (a ∪ b) = b Hyp
34. z ε a Hyp
35. (z ε a) v (z ε b) OrIntR 34
36. ((z ε a) v (z ε b)) -> (z ε (a ∪ b)) AndElimR 8
37. z ε (a ∪ b) ImpElim 35 36
38. z ε b EqualitySub 37 33
39. (z ε a) -> (z ε b) ImpInt 38
40. ∀z.((z ε a) -> (z ε b)) ForallInt 39
41. a ⊂ b DefSub 40
42. ((a ∪ b) = b) -> (a ⊂ b) ImpInt 41
43. ((a ⊂ b) -> ((a ∪ b) = b)) & (((a ∪ b) = b) -> (a ⊂ b)) AndInt 32 42
44. (a ⊂ b) <-> ((a ∪ b) = b) EquivConst 43
45. ∀a.((a ⊂ b) <-> ((a ∪ b) = b)) ForallInt 44
46. (x ⊂ b) <-> ((x ∪ b) = b) ForallElim 45
47. ∀b.((x ⊂ b) <-> ((x ∪ b) = b)) ForallInt 46
48. (x ⊂ y) <-> ((x ∪ y) = y) ForallElim 47 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

Th30. (x ⊂ y) <-> ((x ∩ y) = x)

0. a ⊂ b Hyp
1. z ε (a ∩ b) Hyp

2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
3. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 2
4. ∀x.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) ForallInt 3
5. (z ε (a ∩ y)) <-> ((z ε a) & (z ε y)) ForallElim 4
6. ∀y.((z ε (a ∩ y)) <-> ((z ε a) & (z ε y))) ForallInt 5
7. (z ε (a ∩ b)) <-> ((z ε a) & (z ε b)) ForallElim 6
8. ((z ε (a ∩ b)) -> ((z ε a) & (z ε b))) & (((z ε a) & (z ε b)) -> (z ε (a ∩ b))) EquivExp 7
9. (z ε (a ∩ b)) -> ((z ε a) & (z ε b)) AndElimL 8
10. (z ε a) & (z ε b) ImpElim 1 9
11. z ε a AndElimL 10
12. (z ε (a ∩ b)) -> (z ε a) ImpInt 11
13. z ε a Hyp
14. ∀z.((z ε a) -> (z ε b)) DefExp 0
15. (z ε a) -> (z ε b) ForallElim 14
16. z ε b ImpElim 13 15
17. (z ε a) & (z ε b) AndInt 13 16
18. ((z ε a) & (z ε b)) -> (z ε (a ∩ b)) AndElimR 8
19. z ε (a ∩ b) ImpElim 17 18
20. (z ε a) -> (z ε (a ∩ b)) ImpInt 19
21. ((z ε (a ∩ b)) -> (z ε a)) & ((z ε a) -> (z ε (a ∩ b))) AndInt 12 20
22. (z ε (a ∩ b)) <-> (z ε a) EquivConst 21
23. ∀z.((z ε (a ∩ b)) <-> (z ε a)) ForallInt 22
24. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
25. ∀y.(((a ∩ b) = y) <-> ∀z.((z ε (a ∩ b)) <-> (z ε y))) ForallElim 24
26. ((a ∩ b) = a) <-> ∀z.((z ε (a ∩ b)) <-> (z ε a)) ForallElim 25
27. (((a ∩ b) = a) -> ∀z.((z ε (a ∩ b)) <-> (z ε a))) & (∀z.((z ε (a ∩ b)) <-> (z ε a)) -> ((a ∩ b) = a)) EquivExp 26
28. ∀z.((z ε (a ∩ b)) <-> (z ε a)) -> ((a ∩ b) = a) AndElimR 27
29. (a ∩ b) = a ImpElim 23 28
30. (a ⊂ b) -> ((a ∩ b) = a) ImpInt 29
31. (a ∩ b) = a Hyp
32. z ε a Hyp
33. a = (a ∩ b) Symmetry 31
34. z ε (a ∩ b) EqualitySub 32 33
35. (z ε a) & (z ε b) ImpElim 34 9
36. z ε b AndElimR 35
37. (z ε a) -> (z ε b) ImpInt 36
38. ∀z.((z ε a) -> (z ε b)) ForallInt 37
39. a ⊂ b DefSub 38
40. ((a ∩ b) = a) -> (a ⊂ b) ImpInt 39
41. ((a ⊂ b) -> ((a ∩ b) = a)) & (((a ∩ b) = a) -> (a ⊂ b)) AndInt 30 40
42. (a ⊂ b) <-> ((a ∩ b) = a) EquivConst 41
43. ∀a.((a ⊂ b) <-> ((a ∩ b) = a)) ForallInt 42
44. (x ⊂ b) <-> ((x ∩ b) = x) ForallElim 43
45. ∀b.((x ⊂ b) <-> ((x ∩ b) = x)) ForallInt 44
46. (x ⊂ y) <-> ((x ∩ y) = x) ForallElim 45 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

Th31. (x ⊂ y) -> ((∪x ⊂ ∪y) & (∩y ⊂ ∩x))

0. a ⊂ b Hyp
1. z ε ∪a Hyp
2. ∪x = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
3. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 2
4. ∪a = {z: ∃y.((y ε a) & (z ε y))} ForallElim 3
5. z ε {z: ∃y.((y ε a) & (z ε y))} EqualitySub 1 4
6. Set(z) & ∃y.((y ε a) & (z ε y)) ClassElim 5
7. ∃y.((y ε a) & (z ε y)) AndElimR 6
8. (y ε a) & (z ε y) Hyp
9. ∀z.((z ε a) -> (z ε b)) DefExp 0
10. (y ε a) -> (y ε b) ForallElim 9
11. y ε a AndElimL 8
12. y ε b ImpElim 11 10
13. z ε y AndElimR 8
14. (y ε b) & (z ε y) AndInt 12 13
15. ∃y.((y ε b) & (z ε y)) ExistsInt 14

16. Set(z) AndElimL 6
17. Set(z) & ∃y.((y ε b) & (z ε y)) AndInt 16 15
18. z ε {z: ∃y.((y ε b) & (z ε y))} ClassInt 17
19. ∀x.(Ux = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 2
20. Ub = {z: ∃y.((y ε b) & (z ε y))} ForallElim 19
21. {z: ∃y.((y ε b) & (z ε y))} = Ub Symmetry 20
22. z ε Ub EqualitySub 18 21
23. z ε Ub ExistsElim 7 8 22
24. (z ε Ua) -> (z ε Ub) ImpInt 23
25. ∀z.((z ε Ua) -> (z ε Ub)) ForallInt 24
26. Ua ⊂ Ub DefSub 25
27. z ε ∩b Hyp
28. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
29. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 28
30. ∩b = {z: ∀y.((y ε b) -> (z ε y))} ForallElim 29
31. z ε {z: ∀y.((y ε b) -> (z ε y))} EqualitySub 27 30
32. Set(z) & ∀y.((y ε b) -> (z ε y)) ClassElim 31
33. Set(z) AndElimL 32
34. ∀y.((y ε b) -> (z ε y)) AndElimR 32
35. (y ε b) -> (z ε y) ForallElim 34
36. y ε a Hyp
37. y ε b ImpElim 36 10
38. z ε y ImpElim 37 35
39. (y ε a) -> (z ε y) ImpInt 38
40. ∀y.((y ε a) -> (z ε y)) ForallInt 39
41. Set(z) & ∀y.((y ε a) -> (z ε y)) AndInt 33 40
42. z ε {z: ∀y.((y ε a) -> (z ε y))} ClassInt 41
43. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 28
44. ∩a = {z: ∀y.((y ε a) -> (z ε y))} ForallElim 43
45. {z: ∀y.((y ε a) -> (z ε y))} = ∩a Symmetry 44
46. z ε ∩a EqualitySub 42 45
47. (z ε ∩b) -> (z ε ∩a) ImpInt 46
48. ∀z.((z ε ∩b) -> (z ε ∩a)) ForallInt 47
49. ∩b ⊂ ∩a DefSub 48
50. (Ua ⊂ Ub) & (∩b ⊂ ∩a) AndInt 26 49
51. (a ⊂ b) -> ((Ua ⊂ Ub) & (∩b ⊂ ∩a)) ImpInt 50
52. ∀a.((a ⊂ b) -> ((Ua ⊂ Ub) & (∩b ⊂ ∩a))) ForallInt 51
53. (x ⊂ b) -> ((Ux ⊂ Ub) & (∩b ⊂ ∩x)) ForallElim 52
54. ∀b.((x ⊂ b) -> ((Ux ⊂ Ub) & (∩b ⊂ ∩x))) ForallInt 53
55. (x ⊂ y) -> ((Ux ⊂ Uy) & (∩y ⊂ ∩x)) ForallElim 54 Qed

Used Theorems

Th32. (x ε y) -> ((x ⊂ Uy) & (∩y ⊂ x))

0. a ε b Hyp
1. x ε a Hyp
2. (a ε b) & (x ε a) AndInt 0 1
3. ∃y.((y ε b) & (x ε y)) ExistsInt 2
4. ∃y.(x ε y) ExistsInt 1
5. Set(x) DefSub 4
6. Set(x) & ∃y.((y ε b) & (x ε y)) AndInt 5 3
7. x ε {z: ∃y.((y ε b) & (z ε y))} ClassInt 6
8. Ux = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
9. {z: ∃y.((y ε x) & (z ε y))} = Ux Symmetry 8
10. ∀x.({z: ∃y.((y ε x) & (z ε y))} = Ux) ForallInt 9
11. {z: ∃y.((y ε b) & (z ε y))} = Ub ForallElim 10
12. x ε Ub EqualitySub 7 11
13. (x ε a) -> (x ε Ub) ImpInt 12
14. ∀z.((z ε a) -> (z ε Ub)) ForallInt 13
15. a ⊂ Ub DefSub 14
16. x ε ∩b Hyp
17. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
18. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 17
19. ∩b = {z: ∀y.((y ε b) -> (z ε y))} ForallElim 18
20. x ε {z: ∀y.((y ε b) -> (z ε y))} EqualitySub 16 19
21. Set(x) & ∀y.((y ε b) -> (x ε y)) ClassElim 20
22. ∀y.((y ε b) -> (x ε y)) AndElimR 21
23. (a ε b) -> (x ε a) ForallElim 22
24. x ε a ImpElim 0 23

```
25. (x ε ∩b) -> (x ε a)  ImpInt 24
26. ∀z.((z ε ∩b) -> (z ε a))  ForallInt 25
27. ∩b ⊂ a  DefSub 26
28. (a ⊂ ∪b) & (∩b ⊂ a)  AndInt 15 27
29. (a ε b) -> ((a ⊂ ∪b) & (∩b ⊂ a))  ImpInt 28
30. ∀a.((a ε b) -> ((a ⊂ ∪b) & (∩b ⊂ a)))  ForallInt 29
31. (x ε b) -> ((x ⊂ ∪b) & (∩b ⊂ x))  ForallElim 30
32. ∀b.((x ε b) -> ((x ⊂ ∪b) & (∩b ⊂ x)))  ForallInt 31
33. (x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x))  ForallElim 32 Qed
```

Used Theorems

Th33. (Set(x) & (y ⊂ x)) -> Set(y)

```
0. Set(a) & (b ⊂ a)  Hyp
1. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y)))  AxInt
2. ∀x.(Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y))))  ForallInt 1
3. Set(a) -> ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ε y)))  ForallElim 2
4. Set(a)  AndElimL 0
5. ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ε y)))  ImpElim 4 3
6. Set(w) & ∀z.((z ⊂ a) -> (z ε w))  Hyp
7. ∀z.((z ⊂ a) -> (z ε w))  AndElimR 6
8. (b ⊂ a) -> (b ε w)  ForallElim 7
9. b ⊂ a  AndElimR 0
10. b ε w  ImpElim 9 8
11. ∃z.(b ε z)  ExistsInt 10
12. Set(b)  DefSub 11
13. Set(b)  ExistsElim 5 6 12
14. (Set(a) & (b ⊂ a)) -> Set(b)  ImpInt 13
15. ∀a.((Set(a) & (b ⊂ a)) -> Set(b))  ForallInt 14
16. (Set(x) & (b ⊂ x)) -> Set(b)  ForallElim 15
17. ∀b.((Set(x) & (b ⊂ x)) -> Set(b))  ForallInt 16
18. (Set(x) & (y ⊂ x)) -> Set(y)  ForallElim 17 Qed
```

Used Theorems

Th34. (0 = ∩U) & (U = ∪U)

```
0. z ε 0  Hyp
1. 0 = {x: ¬(x = x)}  DefEqInt
2. z ε {x: ¬(x = x)}  EqualitySub 0 1
3. Set(z) & ¬(z = z)  ClassElim 2
4. ¬(z = z)  AndElimR 3
5. z = z  Identity
6. _|_  ImpElim 5 4
7. z ε ∩U  AbsI 6
8. (z ε 0) -> (z ε ∩U)  ImpInt 7
9. z ε ∩U  Hyp
10. U = {x: (x = x)}  DefEqInt
11. ∩x = {z: ∀y.((y ε x) -> (z ε y))}  DefEqInt
12. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))})  ForallInt 11
13. ∩U = {z: ∀y.((y ε U) -> (z ε y))}  ForallElim 12
14. z ε {z: ∀y.((y ε U) -> (z ε y))}  EqualitySub 9 13
15. Set(z) & ∀y.((y ε U) -> (z ε y))  ClassElim 14
16. ∀y.((y ε U) -> (z ε y))  AndElimR 15
17. (0 ε U) -> (z ε 0)  ForallElim 16
18. (0 ⊂ x) & (x ⊂ U)  TheoremInt
19. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
20. 0 ⊂ x  AndElimL 18
21. ∀x.(0 ⊂ x)  ForallInt 20
22. 0 ⊂ z  ForallElim 21
23. ∀x.((Set(x) & (y ⊂ x)) -> Set(y))  ForallInt 19
24. (Set(z) & (y ⊂ z)) -> Set(y)  ForallElim 23
25. ∀y.((Set(z) & (y ⊂ z)) -> Set(y))  ForallInt 24
26. (Set(z) & (0 ⊂ z)) -> Set(0)  ForallElim 25
27. Set(z)  AndElimL 15
28. Set(z) & (0 ⊂ z)  AndInt 27 22
29. Set(0)  ImpElim 28 26
30. (x ε U) <-> Set(x)  TheoremInt
```

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31. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 30
32. Set(x) -> (x ε U)  AndElimR 31
33. ∀x.(Set(x) -> (x ε U))  ForallInt 32
34. Set(0) -> (0 ε U)  ForallElim 33
35. 0 ε U  ImpElim 29 34
36. z ε 0  ImpElim 35 17
37. (z ε ∩U) -> (z ε 0)  ImpInt 36
38. ((z ε 0) -> (z ε ∩U)) & ((z ε ∩U) -> (z ε 0))  AndInt 8 37
39. (z ε 0) <-> (z ε ∩U)  EquivConst 38
40. ∀z.((z ε 0) <-> (z ε ∩U))  ForallInt 39
41. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
42. ∀y.((0 = y) <-> ∀z.((z ε 0) <-> (z ε y)))  ForallElim 41
43. (0 = ∩U) <-> ∀z.((z ε 0) <-> (z ε ∩U))  ForallElim 42
44. ((0 = ∩U) -> ∀z.((z ε 0) <-> (z ε ∩U))) & (∀z.((z ε 0) <-> (z ε ∩U)) -> (0 = ∩U))
EquivExp 43
45. ∀z.((z ε 0) <-> (z ε ∩U)) -> (0 = ∩U)  AndElimR 44
46. 0 = ∩U  ImpElim 40 45
47. z ε U  Hyp
48. ∪x = {z: ∃y.((y ε x) & (z ε y))}  DefEqInt
49. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))})  ForallInt 48
50. ∪U = {z: ∃y.((y ε U) & (z ε y))}  ForallElim 49
51. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y)))  AxInt
52. (x ε U) -> Set(x)  AndElimL 31
53. ∀x.((x ε U) -> Set(x))  ForallInt 52
54. (z ε U) -> Set(z)  ForallElim 53
55. Set(z)  ImpElim 47 54
56. ∀x.(Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y))))  ForallInt 51
57. Set(z) -> ∃y.(Set(y) & ∀i.((i ⊂ z) -> (i ε y)))  ForallElim 56
58. ∃y.(Set(y) & ∀i.((i ⊂ z) -> (i ε y)))  ImpElim 55 57
59. Set(a) & ∀i.((i ⊂ z) -> (i ε a))  Hyp
60. z = z  Identity
61. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
62. ∀x.((x = y) <-> ((x ⊂ y) & (y ⊂ x)))  ForallInt 61
63. (z = y) <-> ((z ⊂ y) & (y ⊂ z))  ForallElim 62
64. ∀y.((z = y) <-> ((z ⊂ y) & (y ⊂ z)))  ForallInt 63
65. (z = z) <-> ((z ⊂ z) & (z ⊂ z))  ForallElim 64
66. ((z = z) -> ((z ⊂ z) & (z ⊂ z))) & (((z ⊂ z) & (z ⊂ z)) -> (z = z))  EquivExp 65
67. (z = z) -> ((z ⊂ z) & (z ⊂ z))  AndElimL 66
68. (z ⊂ z) & (z ⊂ z)  ImpElim 60 67
69. z ⊂ z  AndElimL 68
70. ∀i.((i ⊂ z) -> (i ε a))  AndElimR 59
71. (z ⊂ z) -> (z ε a)  ForallElim 70
72. z ε a  ImpElim 69 71
73. Set(a)  AndElimL 59
74. ∀x.(Set(x) -> (x ε U))  ForallInt 32
75. Set(a) -> (a ε U)  ForallElim 74
76. a ε U  ImpElim 73 75
77. (a ε U) & (z ε a)  AndInt 76 72
78. ∃y.((y ε U) & (z ε y))  ExistsInt 77
79. ∃y.((y ε U) & (z ε y))  ExistsElim 58 59 78
80. Set(z) & ∃y.((y ε U) & (z ε y))  AndInt 55 79
81. z ε {y: ∃j.((j ε U) & (y ε j))}  ClassInt 80
82. {z: ∃y.((y ε U) & (z ε y))} = ∪U  Symmetry 50
83. z ε ∪U  EqualitySub 81 82
84. (z ε U) -> (z ε ∪U)  ImpInt 83
85. z ε ∪U  Hyp
86. ∃y.(z ε y)  ExistsInt 85
87. Set(z)  DefSub 86
88. ∀x.(Set(x) -> (x ε U))  ForallInt 32
89. Set(z) -> (z ε U)  ForallElim 88
90. z ε U  ImpElim 87 89
91. (z ε ∪U) -> (z ε U)  ImpInt 90
92. ((z ε U) -> (z ε ∪U)) & ((z ε ∪U) -> (z ε U))  AndInt 84 91
93. (z ε U) <-> (z ε ∪U)  EquivConst 92
94. ∀z.((z ε U) <-> (z ε ∪U))  ForallInt 93
95. ∀y.((U = y) <-> ∀z.((z ε U) <-> (z ε y)))  ForallElim 41
96. (U = ∪U) <-> ∀z.((z ε U) <-> (z ε ∪U))  ForallElim 95
97. ((U = ∪U) -> ∀z.((z ε U) <-> (z ε ∪U))) & (∀z.((z ε U) <-> (z ε ∪U)) -> (U = ∪U))
EquivExp 96
98. ∀z.((z ε U) <-> (z ε ∪U)) -> (U = ∪U)  AndElimR 97
99. U = ∪U  ImpElim 94 98
```

100. (0 = ∩U) & (U = ∪U) AndInt 46 99 Qed

Used Theorems

1. (0 ⊂ x) & (x ⊂ U)
2. (Set(x) & (y ⊂ x)) -> Set(y)
3. (x ε U) <-> Set(x)
4. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

Th35. ¬(x = 0) -> Set(∩x)

0. ∀z.¬(z ε a) Hyp
1. z ε a Hyp
2. ¬(z ε a) ForallElim 0
3. _|_ ImpElim 1 2
4. z ε 0 AbsI 3
5. (z ε a) -> (z ε 0) ImpInt 4
6. z ε 0 Hyp
7. 0 = {x: ¬(x = x)} DefEqInt
8. z ε {x: ¬(x = x)} EqualitySub 6 7
9. Set(z) & ¬(z = z) ClassElim 8
10. ¬(z = z) AndElimR 9
11. z = z Identity
12. _|_ ImpElim 11 10
13. z ε a AbsI 12
14. (z ε 0) -> (z ε a) ImpInt 13
15. ((z ε a) -> (z ε 0)) & ((z ε 0) -> (z ε a)) AndInt 5 14
16. (z ε a) <-> (z ε 0) EquivConst 15
17. ∀z.((z ε a) <-> (z ε 0)) ForallInt 16
18. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
19. ∀y.((a = y) <-> ∀z.((z ε a) <-> (z ε y))) ForallElim 18
20. (a = 0) <-> ∀z.((z ε a) <-> (z ε 0)) ForallElim 19
21. ((a = 0) -> ∀z.((z ε a) <-> (z ε 0))) & (∀z.((z ε a) <-> (z ε 0)) -> (a = 0))
EquivExp 20
22. ∀z.((z ε a) <-> (z ε 0)) -> (a = 0) AndElimR 21
23. a = 0 ImpElim 17 22
24. ∀z.¬(z ε a) -> (a = 0) ImpInt 23
25. (A -> B) -> (¬B -> ¬A) TheoremInt
26. (∀z.¬(z ε a) -> B) -> (¬B -> ¬∀z.¬(z ε a)) PolySub 25
27. (∀z.¬(z ε a) -> (a = 0)) -> (¬(a = 0) -> ¬∀z.¬(z ε a)) PolySub 26
28. ¬(a = 0) -> ¬∀z.¬(z ε a) ImpElim 24 27
29. ¬∀z.¬(z ε a) Hyp
30. ¬∃z.(z ε a) Hyp
31. z ε a Hyp
32. ∃z.(z ε a) ExistsInt 31
33. _|_ ImpElim 32 30
34. ¬(z ε a) ImpInt 33
35. ∀z.¬(z ε a) ForallInt 34
36. ¬∃z.(z ε a) -> ∀z.¬(z ε a) ImpInt 35
37. (A -> B) -> (¬B -> ¬A) TheoremInt
38. (¬∃z.(z ε a) -> B) -> (¬B -> ¬¬∃z.(z ε a)) PolySub 37
39. (¬∃x_0.(x_0 ε a) -> ∀z.¬(z ε a)) -> (¬∀z.¬(z ε a) -> ¬¬∃x_0.(x_0 ε a)) PolySub 38
40. ¬∀z.¬(z ε a) -> ¬¬∃x_0.(x_0 ε a) ImpElim 36 39
41. D <-> ¬¬D TheoremInt
42. ∃l.(l ε a) <-> ¬¬∃l.(l ε a) PolySub 41
43. (∃l.(l ε a) -> ¬¬∃l.(l ε a)) & (¬¬∃l.(l ε a) -> ∃l.(l ε a)) EquivExp 42
44. ¬¬∃l.(l ε a) -> ∃l.(l ε a) AndElimR 43
45. ¬(a = 0) Hyp
46. ¬∀z.¬(z ε a) ImpElim 45 28
47. ¬¬∃x_0.(x_0 ε a) ImpElim 46 40
48. ∃l.(l ε a) ImpElim 47 44
49. ¬(a = 0) -> ∃l.(l ε a) ImpInt 48
50. ∃l.(l ε a) Hyp
51. b ε a Hyp
52. (x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x)) TheoremInt
53. ∀x.((x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x))) ForallInt 52
54. (b ε y) -> ((b ⊂ ∪y) & (∩y ⊂ b)) ForallElim 53
55. ∀y.((b ε y) -> ((b ⊂ ∪y) & (∩y ⊂ b))) ForallInt 54
56. (b ε a) -> ((b ⊂ ∪a) & (∩a ⊂ b)) ForallElim 55
57. (b ⊂ ∪a) & (∩a ⊂ b) ImpElim 51 56
58. ∩a ⊂ b AndElimR 57

```
59. ∃y.(b ε y)  ExistsInt 51
60. Set(b)  DefSub 59
61. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
62. ∀x.((Set(x) & (y ⊂ x)) -> Set(y))  ForallInt 61
63. (Set(b) & (y ⊂ b)) -> Set(y)  ForallElim 62
64. ∀y.((Set(b) & (y ⊂ b)) -> Set(y))  ForallInt 63
65. (Set(b) & (∩a ⊂ b)) -> Set(∩a)  ForallElim 64
66. Set(b) & (∩a ⊂ b)  AndInt 60 58
67. Set(∩a)  ImpElim 66 65
68. Set(∩a)  ExistsElim 50 51 67
69. ∃l.(l ε a) -> Set(∩a)  ImpInt 68
70. ¬(a = 0)  Hyp
71. ∃l.(l ε a)  ImpElim 70 49
72. Set(∩a)  ImpElim 71 69
73. ¬(a = 0) -> Set(∩a)  ImpInt 72
74. ∀a.(¬(a = 0) -> Set(∩a))  ForallInt 73
75. ¬(x = 0) -> Set(∩x)  ForallElim 74 Qed
```

Used Theorems

```
1. (A -> B) -> (¬B -> ¬A)
1. (A -> B) -> (¬B -> ¬A)
2. D <-> ¬¬D
4. (x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x))
5. (Set(x) & (y ⊂ x)) -> Set(y)
```

Th37. U = PU

```
0. x ε U  Hyp
1. (0 ⊂ x) & (x ⊂ U)  TheoremInt
2. x ⊂ U  AndElimR 1
3. Px = {y: (y ⊂ x)}  DefEqInt
4. ∀x.(Px = {y: (y ⊂ x)})  ForallInt 3
5. PU = {y: (y ⊂ U)}  ForallElim 4
6. ∃y.(x ε y)  ExistsInt 0
7. Set(x)  DefSub 6
8. Set(x) & (x ⊂ U)  AndInt 7 2
9. x ε {y: (y ⊂ U)}  ClassInt 8
10. {y: (y ⊂ U)} = PU  Symmetry 5
11. x ε PU  EqualitySub 9 10
12. (x ε U) -> (x ε PU)  ImpInt 11
13. x ε PU  Hyp
14. ∃y.(x ε y)  ExistsInt 13
15. Set(x)  DefSub 14
16. (x ε U) <-> Set(x)  TheoremInt
17. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 16
18. Set(x) -> (x ε U)  AndElimR 17
19. x ε U  ImpElim 15 18
20. (x ε PU) -> (x ε U)  ImpInt 19
21. ((x ε U) -> (x ε PU)) & ((x ε PU) -> (x ε U))  AndInt 12 20
22. (x ε U) <-> (x ε PU)  EquivConst 21
23. ∀z.((z ε U) <-> (z ε PU))  ForallInt 22
24. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
25. ∀y.((U = y) <-> ∀z.((z ε U) <-> (z ε y)))  ForallElim 24
26. (U = PU) <-> ∀z.((z ε U) <-> (z ε PU))  ForallElim 25
27. ((U = PU) -> ∀z.((z ε U) <-> (z ε PU))) & (∀z.((z ε U) <-> (z ε PU)) -> (U = PU))
EquivExp 26
28. ∀z.((z ε U) <-> (z ε PU)) -> (U = PU)  AndElimR 27
29. U = PU  ImpElim 23 28 Qed
```

Used Theorems

```
1. (0 ⊂ x) & (x ⊂ U)
2. (x ε U) <-> Set(x)
```

Th38. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))

```
0. Set(a)  Hyp
1. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y)))  AxInt
2. ∀x.(Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y))))  ForallInt 1
3. Set(a) -> ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ε y)))  ForallElim 2
```

4. ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ε y))) ImpElim 0 3
5. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
6. ∀y.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 5
7. (Set(x) & (Pa ⊂ x)) -> Set(Pa) ForallElim 6
8. Set(b) & ∀z.((z ⊂ a) -> (z ε b)) Hyp
9. ∀x.((Set(x) & (Pa ⊂ x)) -> Set(Pa)) ForallInt 7
10. (Set(b) & (Pa ⊂ b)) -> Set(Pa) ForallElim 9
11. z ε Pa Hyp
12. Px = {y: (y ⊂ x)} DefEqInt
13. ∀x.(Px = {y: (y ⊂ x)}) ForallInt 12
14. Pa = {y: (y ⊂ a)} ForallElim 13
15. z ε {y: (y ⊂ a)} EqualitySub 11 14
16. Set(z) & (z ⊂ a) ClassElim 15
17. ∀z.((z ⊂ a) -> (z ε b)) AndElimR 8
18. z ⊂ a AndElimR 16
19. (z ⊂ a) -> (z ε b) ForallElim 17
20. z ε b ImpElim 18 19
21. (z ε Pa) -> (z ε b) ImpInt 20
22. ∀z.((z ε Pa) -> (z ε b)) ForallInt 21
23. Pa ⊂ b DefSub 22
24. Set(b) AndElimL 8
25. Set(b) & (Pa ⊂ b) AndInt 24 23
26. Set(Pa) ImpElim 25 10
27. Set(Pa) ExistsElim 4 8 26
28. z ⊂ a Hyp
29. Set(a) & (z ⊂ a) AndInt 0 28
30. ∀x.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 5
31. (Set(a) & (y ⊂ a)) -> Set(y) ForallElim 30
32. ∀y.((Set(a) & (y ⊂ a)) -> Set(y)) ForallInt 31
33. (Set(a) & (z ⊂ a)) -> Set(z) ForallElim 32
34. Set(z) ImpElim 29 33
35. Set(z) & (z ⊂ a) AndInt 34 28
36. z ε {y: (y ⊂ a)} ClassInt 35
37. {y: (y ⊂ a)} = Pa Symmetry 14
38. z ε Pa EqualitySub 36 37
39. (z ⊂ a) -> (z ε Pa) ImpInt 38
40. z ε Pa Hyp
41. z ε {y: (y ⊂ a)} EqualitySub 40 14
42. Set(z) & (z ⊂ a) ClassElim 41
43. z ⊂ a AndElimR 42
44. (z ε Pa) -> (z ⊂ a) ImpInt 43
45. ((z ⊂ a) -> (z ε Pa)) & ((z ε Pa) -> (z ⊂ a)) AndInt 39 44
46. (z ⊂ a) <-> (z ε Pa) EquivConst 45
47. Set(Pa) & ((z ⊂ a) <-> (z ε Pa)) AndInt 27 46
48. Set(a) -> (Set(Pa) & ((z ⊂ a) <-> (z ε Pa))) ImpInt 47
49. ∀a.(Set(a) -> (Set(Pa) & ((z ⊂ a) <-> (z ε Pa)))) ForallInt 48
50. Set(x) -> (Set(Px) & ((z ⊂ x) <-> (z ε Px))) ForallElim 49
51. ∀z.(Set(x) -> (Set(Px) & ((z ⊂ x) <-> (z ε Px)))) ForallInt 50
52. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px))) ForallElim 51 Qed

Used Theorems

1. (Set(x) & (y ⊂ x)) -> Set(y)

Th39. ¬Set(U)

0. rus = {z: ¬(z ε z)} DefEqInt
1. rus ε rus Hyp
2. rus ε {z: ¬(z ε z)} EqualitySub 1 0
3. Set(rus) & ¬(rus ε rus) ClassElim 2
4. ¬(rus ε rus) AndElimR 3
5. _|_ ImpElim 1 4
6. ¬Set(rus) AbsI 5
7. ¬(rus ε rus) Hyp
8. Set(rus) Hyp
9. Set(rus) & ¬(rus ε rus) AndInt 8 7
10. rus ε {z: ¬(z ε z)} ClassInt 9
11. {z: ¬(z ε z)} = rus Symmetry 0
12. rus ε rus EqualitySub 10 11
13. _|_ ImpElim 12 7
14. ¬Set(rus) ImpInt 13

```
15. A v ¬A  TheoremInt
16. (rus ε rus) v ¬(rus ε rus)  PolySub 15
17. ¬Set(rus)  OrElim 16 1 6 7 14
18. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
19. (0 ⊂ x) & (x ⊂ U)  TheoremInt
20. x ⊂ U  AndElimR 19
21. Set(U)  Hyp
22. ∀x.(x ⊂ U)  ForallInt 20
23. rus ⊂ U  ForallElim 22
24. Set(U) & (rus ⊂ U)  AndInt 21 23
25. ∀x.((Set(x) & (y ⊂ x)) -> Set(y))  ForallInt 18
26. (Set(U) & (y ⊂ U)) -> Set(y)  ForallElim 25
27. ∀y.((Set(U) & (y ⊂ U)) -> Set(y))  ForallInt 26
28. (Set(U) & (rus ⊂ U)) -> Set(rus)  ForallElim 27
29. Set(rus)  ImpElim 24 28
30. _|_  ImpElim 29 17
31. ¬Set(U)  ImpInt 30 Qed
```

Used Theorems

```
1. A v ¬A
2. (Set(x) & (y ⊂ x)) -> Set(y)
3. (0 ⊂ x) & (x ⊂ U)
```

```
Th41. Set(x) -> ((y ε {x}) <-> (y = x))
```

```
0. Set(x)  Hyp
1. y ε {x}  Hyp
2. {x} = {z: ((x ε U) -> (z = x))}  DefEqInt
3. y ε {z: ((x ε U) -> (z = x))}  EqualitySub 1 2
4. Set(y) & ((x ε U) -> (y = x))  ClassElim 3
5. (x ε U) <-> Set(x)  TheoremInt
6. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 5
7. Set(x) -> (x ε U)  AndElimR 6
8. x ε U  ImpElim 0 7
9. (x ε U) -> (y = x)  AndElimR 4
10. y = x  ImpElim 8 9
11. (y ε {x}) -> (y = x)  ImpInt 10
12. y = x  Hyp
13. x = y  Symmetry 12
14. Set(y)  EqualitySub 0 13
15. y = x  Hyp
16. x ε U  Hyp
17. (x ε U) -> (y = x)  ImpInt 15
18. (y = x) -> ((x ε U) -> (y = x))  ImpInt 17
19. (x ε U) -> (y = x)  ImpElim 12 18
20. Set(y) & ((x ε U) -> (y = x))  AndInt 14 19
21. y ε {z: ((x ε U) -> (z = x))}  ClassInt 20
22. {z: ((x ε U) -> (z = x))} = {x}  Symmetry 2
23. y ε {x}  EqualitySub 21 22
24. (y = x) -> (y ε {x})  ImpInt 23
25. ((y ε {x}) -> (y = x)) & ((y = x) -> (y ε {x}))  AndInt 11 24
26. (y ε {x}) <-> (y = x)  EquivConst 25
27. Set(x) -> ((y ε {x}) <-> (y = x))  ImpInt 26 Qed
```

Used Theorems

```
1. (x ε U) <-> Set(x)
```

```
Th42. Set(x) -> Set({x})
```

```
0. Set(x)  Hyp
1. z ε {x}  Hyp
2. {x} = {z: ((x ε U) -> (z = x))}  DefEqInt
3. z ε {z: ((x ε U) -> (z = x))}  EqualitySub 1 2
4. Set(z) & ((x ε U) -> (z = x))  ClassElim 3
5. (x ε U) -> (z = x)  AndElimR 4
6. (x ε U) <-> Set(x)  TheoremInt
7. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 6
8. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 6
9. Set(x) -> (x ε U)  AndElimR 8
```

10. x ε U ImpElim 0 9
11. z = x ImpElim 10 5
12. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
13. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y)) EquivExp 12
14. (x = y) -> ((x ⊂ y) & (y ⊂ x)) AndElimL 13
15. ∀x.((x = y) -> ((x ⊂ y) & (y ⊂ x))) ForallInt 14
16. (z = y) -> ((z ⊂ y) & (y ⊂ z)) ForallElim 15
17. ∀y.((z = y) -> ((z ⊂ y) & (y ⊂ z))) ForallInt 16
18. (z = x) -> ((z ⊂ x) & (x ⊂ z)) ForallElim 17
19. (z ⊂ x) & (x ⊂ z) ImpElim 11 18
20. z ⊂ x AndElimL 19
21. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px))) TheoremInt
22. Set(Px) & ((y ⊂ x) <-> (y ε Px)) ImpElim 0 21
23. (y ⊂ x) <-> (y ε Px) AndElimR 22
24. ((y ⊂ x) -> (y ε Px)) & ((y ε Px) -> (y ⊂ x)) EquivExp 23
25. (y ⊂ x) -> (y ε Px) AndElimL 24
26. ∀y.((y ⊂ x) -> (y ε Px)) ForallInt 25
27. (z ⊂ x) -> (z ε Px) ForallElim 26
28. z ε Px ImpElim 20 27
29. (z ε {x}) -> (z ε Px) ImpInt 28
30. ∀z.((z ε {x}) -> (z ε Px)) ForallInt 29
31. {x} ⊂ Px DefSub 30
32. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
33. ∀x.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 32
34. (Set(Px) & (y ⊂ Px)) -> Set(y) ForallElim 33
35. ∀y.((Set(Px) & (y ⊂ Px)) -> Set(y)) ForallInt 34
36. (Set(Px) & ({x} ⊂ Px)) -> Set({x}) ForallElim 35
37. Set(Px) AndElimL 22
38. Set(Px) & ({x} ⊂ Px) AndInt 37 31
39. Set({x}) ImpElim 38 36
40. Set(x) -> Set({x}) ImpInt 39 Qed

Used Theorems

3. (x ε U) <-> Set(x)
2. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
1. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))
4. (Set(x) & (y ⊂ x)) -> Set(y)

Th43. ({x} = U) <-> ¬Set(x)

0. Set(x) Hyp
1. Set(x) -> Set({x}) TheoremInt
2. Set({x}) ImpElim 0 1
3. ¬Set(U) TheoremInt
4. {x} = U Hyp
5. Set(U) EqualitySub 2 4
6. _|_ ImpElim 5 3
7. ¬({x} = U) ImpInt 6
8. ¬Set(x) Hyp
9. x ε U Hyp
10. ∃y.(x ε y) ExistsInt 9
11. Set(x) DefSub 10
12. _|_ ImpElim 11 8
13. ¬(x ε U) ImpInt 12
14. x ε U Hyp
15. _|_ ImpElim 14 13
16. y = x AbsI 15
17. (x ε U) -> (y = x) ImpInt 16
18. y ε U Hyp
19. (x ε U) <-> Set(x) TheoremInt
20. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 19
21. (x ε U) -> Set(x) AndElimL 20
22. ∀x.((x ε U) -> Set(x)) ForallInt 21
23. (y ε U) -> Set(y) ForallElim 22
24. Set(y) ImpElim 18 23
25. Set(y) & ((x ε U) -> (y = x)) AndInt 24 17
26. y ε {z: ((x ε U) -> (z = x))} ClassInt 25
27. {x} = {z: ((x ε U) -> (z = x))} DefEqInt
28. {z: ((x ε U) -> (z = x))} = {x} Symmetry 27
29. y ε {x} EqualitySub 26 28

30. (y ε U) -> (y ε {x}) ImpInt 29
31. ∀z.((z ε U) -> (z ε {x})) ForallInt 30
32. U ⊂ {x} DefSub 31
33. (0 ⊂ x) & (x ⊂ U) TheoremInt
34. ∀x.((0 ⊂ x) & (x ⊂ U)) ForallInt 33
35. (0 ⊂ {x}) & ({x} ⊂ U) ForallElim 34
36. {x} ⊂ U AndElimR 35
37. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
38. ∀x.((x = y) <-> ((x ⊂ y) & (y ⊂ x))) ForallInt 37
39. ({x} = y) <-> (({x} ⊂ y) & (y ⊂ {x})) ForallElim 38
40. ∀y.(({x} = y) <-> (({x} ⊂ y) & (y ⊂ {x}))) ForallInt 39
41. ({x} = U) <-> (({x} ⊂ U) & (U ⊂ {x})) ForallElim 40
42. (({x} = U) -> (({x} ⊂ U) & (U ⊂ {x}))) & ((({x} ⊂ U) & (U ⊂ {x})) -> ({x} = U)) EquivExp 41
43. (({x} = U) -> (({x} ⊂ U) & (U ⊂ {x}))) & ((({x} ⊂ U) & (U ⊂ {x})) -> ({x} = U)) EquivExp 41
44. (({x} ⊂ U) & (U ⊂ {x})) -> ({x} = U) AndElimR 43
45. ({x} ⊂ U) & (U ⊂ {x}) AndInt 36 32
46. {x} = U ImpElim 45 44
47. ¬Set(x) -> ({x} = U) ImpInt 46
48. Set(x) -> ¬({x} = U) ImpInt 7
49. (A -> B) -> (¬B -> ¬A) TheoremInt
50. (Set(x) -> B) -> (¬B -> ¬Set(x)) PolySub 49
51. (Set(x) -> ¬({x} = U)) -> (¬¬({x} = U) -> ¬Set(x)) PolySub 50
52. ¬¬({x} = U) -> ¬Set(x) ImpElim 48 51
53. D <-> ¬¬D TheoremInt
54. (D -> ¬¬D) & (¬¬D -> D) EquivExp 53
55. D -> ¬¬D AndElimL 54
56. ({x} = U) -> ¬¬({x} = U) PolySub 55
57. {x} = U Hyp
58. ¬¬({x} = U) ImpElim 57 56
59. ¬Set(x) ImpElim 58 52
60. ({x} = U) -> ¬Set(x) ImpInt 59
61. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U)) AndInt 60 47
62. ({x} = U) <-> ¬Set(x) EquivConst 61 Qed

Used Theorems

1. Set(x) -> Set({x})
2. ¬Set(U)
3. (x ε U) <-> Set(x)
4. (0 ⊂ x) & (x ⊂ U)
6. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
10. (A -> B) -> (¬B -> ¬A)
9. D <-> ¬¬D

Th44. (Set(x) -> ((∩{x} = x) & (**U**{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (**U**{x} = U)))

0. z ε ∩{x} Hyp
1. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
2. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 1
3. ∩{x} = {z: ∀y.((y ε {x}) -> (z ε y))} ForallElim 2
4. z ε {z: ∀y.((y ε {x}) -> (z ε y))} EqualitySub 0 3
5. Set(z) & ∀y.((y ε {x}) -> (z ε y)) ClassElim 4
6. ∀y.((y ε {x}) -> (z ε y)) AndElimR 5
7. Set(x) Hyp
8. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
9. (y ε {x}) <-> (y = x) ImpElim 7 8
10. ((y ε {x}) -> (y = x)) & ((y = x) -> (y ε {x})) EquivExp 9
11. (y = x) -> (y ε {x}) AndElimR 10
12. ∀y.((y = x) -> (y ε {x})) ForallInt 11
13. (x = x) -> (x ε {x}) ForallElim 12
14. x = x Identity
15. x ε {x} ImpElim 14 13
16. (x ε {x}) -> (z ε x) ForallElim 6
17. z ε x ImpElim 15 16
18. (z ε ∩{x}) -> (z ε x) ImpInt 17
19. z ε x Hyp
20. y ε {x} Hyp
21. (y ε {x}) -> (y = x) AndElimL 10
22. y = x ImpElim 20 21

23. x = y Symmetry 22
24. z ε y EqualitySub 19 23
25. (y ε {x}) -> (z ε y) ImpInt 24
26. ∀y.((y ε {x}) -> (z ε y)) ForallInt 25
27. ∃x.(z ε x) ExistsInt 19
28. Set(z) DefSub 27
29. Set(z) & ∀y.((y ε {x}) -> (z ε y)) AndInt 28 26
30. z ε {z: ∀y.((y ε {x}) -> (z ε y))} ClassInt 29
31. {z: ∀y.((y ε {x}) -> (z ε y))} = ∩{x} Symmetry 3
32. z ε ∩{x} EqualitySub 30 31
33. (z ε x) -> (z ε ∩{x}) ImpInt 32
34. ((z ε ∩{x}) -> (z ε x)) & ((z ε x) -> (z ε ∩{x})) AndInt 18 33
35. (z ε ∩{x}) <-> (z ε x) EquivConst 34
36. ∀z.((z ε ∩{x}) <-> (z ε x)) ForallInt 35
37. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
38. ∀y.((∩{x} = y) <-> ∀z.((z ε ∩{x}) <-> (z ε y))) ForallElim 37
39. (∩{x} = x) <-> ∀z.((z ε ∩{x}) <-> (z ε x)) ForallElim 38
40. ((∩{x} = x) -> ∀z.((z ε ∩{x}) <-> (z ε x))) & (∀z.((z ε ∩{x}) <-> (z ε x)) -> (∩{x} = x)) EquivExp 39
41. ∀z.((z ε ∩{x}) <-> (z ε x)) -> (∩{x} = x) AndElimR 40
42. ∩{x} = x ImpElim 36 41
43. z ε ∪{x} Hyp
44. ∪x = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
45. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 44
46. ∪{x} = {z: ∃y.((y ε {x}) & (z ε y))} ForallElim 45
47. z ε {z: ∃y.((y ε {x}) & (z ε y))} EqualitySub 43 46
48. Set(z) & ∃y.((y ε {x}) & (z ε y)) ClassElim 47
49. ∃y.((y ε {x}) & (z ε y)) AndElimR 48
50. (a ε {x}) & (z ε a) Hyp
51. ∀y.((y ε {x}) -> (y = x)) ForallInt 21
52. (a ε {x}) -> (a = x) ForallElim 51
53. a ε {x} AndElimL 50
54. a = x ImpElim 53 52
55. z ε a AndElimR 50
56. z ε x EqualitySub 55 54
57. z ε x ExistsElim 49 50 56
58. (z ε ∪{x}) -> (z ε x) ImpInt 57
59. z ε x Hyp
60. (y = x) -> (y ε {x}) AndElimR 10
61. ∀y.((y = x) -> (y ε {x})) ForallInt 60
62. (x = x) -> (x ε {x}) ForallElim 61
63. x ε {x} ImpElim 14 62
64. (x ε {x}) & (z ε x) AndInt 63 59
65. ∃y.((y ε {x}) & (z ε y)) ExistsInt 64
66. ∃y.(z ε y) ExistsInt 59
67. Set(z) DefSub 66
68. Set(z) & ∃y.((y ε {x}) & (z ε y)) AndInt 67 65
69. z ε {z: ∃y.((y ε {x}) & (z ε y))} ClassInt 68
70. {z: ∃y.((y ε {x}) & (z ε y))} = ∪{x} Symmetry 46
71. z ε ∪{x} EqualitySub 69 70
72. (z ε x) -> (z ε ∪{x}) ImpInt 71
73. ((z ε ∪{x}) -> (z ε x)) & ((z ε x) -> (z ε ∪{x})) AndInt 58 72
74. (z ε ∪{x}) <-> (z ε x) EquivConst 73
75. ∀z.((z ε ∪{x}) <-> (z ε x)) ForallInt 74
76. ∀y.((∪{x} = y) <-> ∀z.((z ε ∪{x}) <-> (z ε y))) ForallElim 37
77. (∪{x} = x) <-> ∀z.((z ε ∪{x}) <-> (z ε x)) ForallElim 76
78. ((∪{x} = x) -> ∀z.((z ε ∪{x}) <-> (z ε x))) & (∀z.((z ε ∪{x}) <-> (z ε x)) -> (∪{x} = x)) EquivExp 77
79. ∀z.((z ε ∪{x}) <-> (z ε x)) -> (∪{x} = x) AndElimR 78
80. ∪{x} = x ImpElim 75 79
81. (∩{x} = x) & (∪{x} = x) AndInt 42 80
82. Set(x) -> ((∩{x} = x) & (∪{x} = x)) ImpInt 81
83. ¬Set(x) Hyp
84. ({x} = U) <-> ¬Set(x) TheoremInt
85. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U)) EquivExp 84
86. ¬Set(x) -> ({x} = U) AndElimR 85
87. {x} = U ImpElim 83 86
88. (0 = ∩U) & (U = ∪U) TheoremInt
89. U = {x} Symmetry 87
90. (0 = ∩{x}) & (U = ∪{x}) EqualitySub 88 89
91. 0 = ∩{x} AndElimL 90

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92. U = U{x}  AndElimR 90
93. ∩{x} = 0  Symmetry 91
94. U{x} = U  Symmetry 92
95. (∩{x} = 0) & (U{x} = U)  AndInt 93 94
96. ¬Set(x) -> ((∩{x} = 0) & (U{x} = U))  ImpInt 95
97. (Set(x) -> ((∩{x} = x) & (U{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (U{x} = U)))
AndInt 82 96 Qed
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Used Theorems

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1. Set(x) -> ((y ε {x}) <-> (y = x))
2. ({x} = U) <-> ¬Set(x)
3. (0 = ∩U) & (U = UU)
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Th46. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))

```
0. Set(x) & Set(y)  Hyp
1. Set(x) -> Set({x})  TheoremInt
2. Set(x)  AndElimL 0
3. Set(y)  AndElimR 0
4. Set({x})  ImpElim 2 1
5. ∀x.(Set(x) -> Set({x}))  ForallInt 1
6. Set(y) -> Set({y})  ForallElim 5
7. Set({y})  ImpElim 3 6
8. (Set(x) & Set(y)) -> Set((x U y))  AxInt
9. ∀x.((Set(x) & Set(y)) -> Set((x U y)))  ForallInt 8
10. (Set({x}) & Set(y)) -> Set(({x} U y))  ForallElim 9
11. ∀y.((Set({x}) & Set(y)) -> Set(({x} U y)))  ForallInt 10
12. (Set({x}) & Set({y})) -> Set(({x} U {y}))  ForallElim 11
13. Set({x}) & Set({y})  AndInt 4 7
14. Set(({x} U {y}))  ImpElim 13 12
15. {x,y} = ({x} U {y})  DefEqInt
16. ({x} U {y}) = {x,y}  Symmetry 15
17. Set({x,y})  EqualitySub 14 16
18. ((z ε (x U y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
19. (z ε (x U y)) <-> ((z ε x) v (z ε y))  AndElimL 18
20. z ε {x,y}  Hyp
21. z ε ({x} U {y})  EqualitySub 20 15
22. ((z ε (x U y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x U y)))
EquivExp 19
23. (z ε (x U y)) -> ((z ε x) v (z ε y))  AndElimL 22
24. ∀x.((z ε (x U y)) -> ((z ε x) v (z ε y)))  ForallInt 23
25. (z ε ({x} U y)) -> ((z ε {x}) v (z ε y))  ForallElim 24
26. ∀y.((z ε ({x} U y)) -> ((z ε {x}) v (z ε y)))  ForallInt 25
27. (z ε ({x} U {y})) -> ((z ε {x}) v (z ε {y}))  ForallElim 26
28. (z ε {x}) v (z ε {y})  ImpElim 21 27
29. z ε {x}  Hyp
30. Set(x) -> ((y ε {x}) <-> (y = x))  TheoremInt
31. ∀y.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 30
32. Set(x) -> ((z ε {x}) <-> (z = x))  ForallElim 31
33. ∀x.(Set(x) -> ((z ε {x}) <-> (z = x)))  ForallInt 32
34. Set(y) -> ((z ε {y}) <-> (z = y))  ForallElim 33
35. (z ε {x}) <-> (z = x)  ImpElim 2 32
36. ((z ε {x}) -> (z = x)) & ((z = x) -> (z ε {x}))  EquivExp 35
37. (z ε {x}) -> (z = x)  AndElimL 36
38. z = x  ImpElim 29 37
39. (z = x) v (z = y)  OrIntR 38
40. z ε {y}  Hyp
41. (z ε {y}) <-> (z = y)  ImpElim 3 34
42. ((z ε {y}) -> (z = y)) & ((z = y) -> (z ε {y}))  EquivExp 41
43. (z ε {y}) -> (z = y)  AndElimL 42
44. z = y  ImpElim 40 43
45. (z = x) v (z = y)  OrIntL 44
46. (z = x) v (z = y)  OrElim 28 29 39 40 45
47. (z ε {x,y}) -> ((z = x) v (z = y))  ImpInt 46
48. (z = x) v (z = y)  Hyp
49. z = x  Hyp
50. (z = x) -> (z ε {x})  AndElimR 36
51. z ε {x}  ImpElim 49 50
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52. (z ε {x}) v (z ε {y})  OrIntR 51
53. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  AndElimR 22
54. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 53
55. ((z ε {x}) v (z ε y)) -> (z ε ({x} ∪ y))  ForallElim 54
56. ∀y.(((z ε {x}) v (z ε y)) -> (z ε ({x} ∪ y)))  ForallInt 55
57. ((z ε {x}) v (z ε {y})) -> (z ε ({x} ∪ {y}))  ForallElim 56
58. z ε ({x} ∪ {y})  ImpElim 52 57
59. z = y  Hyp
60. (z = y) -> (z ε {y})  AndElimR 42
61. z ε {y}  ImpElim 59 60
62. (z ε {x}) v (z ε {y})  OrIntL 61
63. z ε ({x} ∪ {y})  ImpElim 62 57
64. z ε ({x} ∪ {y})  OrElim 48 49 58 59 63
65. ((z = x) v (z = y)) -> (z ε ({x} ∪ {y}))  ImpInt 64
66. ((z = x) v (z = y)) -> (z ε {x,y})  EqualitySub 65 16
67. ((z ε {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ε {x,y}))  AndInt 47 66
68. (z ε {x,y}) <-> ((z = x) v (z = y))  EquivConst 67
69. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))  AndInt 17 68
70. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))  ImpInt 69
71. {x,y} = U  Hyp
72. ({x} ∪ {y}) = U  EqualitySub 71 15
73. ¬Set(U)  TheoremInt
74. U = ({x} ∪ {y})  Symmetry 72
75. ¬Set(({x} ∪ {y}))  EqualitySub 73 74
76. (Set(x) & Set(y)) -> Set((x ∪ y))  AxInt
77. (A -> B) -> (¬B -> ¬A)  TheoremInt
78. ((Set(x) & Set(y)) -> B) -> (¬B -> ¬(Set(x) & Set(y)))  PolySub 77
79. ((Set(x) & Set(y)) -> Set((x ∪ y))) -> (¬Set((x ∪ y)) -> ¬(Set(x) & Set(y)))  PolySub 78
80. ¬Set((x ∪ y)) -> ¬(Set(x) & Set(y))  ImpElim 76 79
81. ∀x.(¬Set((x ∪ y)) -> ¬(Set(x) & Set(y)))  ForallInt 80
82. ¬Set(({x} ∪ y)) -> ¬(Set({x}) & Set(y))  ForallElim 81
83. ∀y.(¬Set(({x} ∪ y)) -> ¬(Set({x}) & Set(y)))  ForallInt 82
84. ¬Set(({x} ∪ {y})) -> ¬(Set({x}) & Set({y}))  ForallElim 83
85. ¬(Set({x}) & Set({y}))  ImpElim 75 84
86. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
87. ¬(A & B) <-> (¬A v ¬B)  AndElimR 86
88. ¬(Set({x}) & B) <-> (¬Set({x}) v ¬B)  PolySub 87
89. ¬(Set({x}) & Set({y})) <-> (¬Set({x}) v ¬Set({y}))  PolySub 88
90. (¬(Set({x}) & Set({y})) -> (¬Set({x}) v ¬Set({y}))) & ((¬Set({x}) v ¬Set({y})) -> ¬(Set({x}) & Set({y})))  EquivExp 89
91. ¬(Set({x}) & Set({y})) -> (¬Set({x}) v ¬Set({y}))  AndElimL 90
92. ¬Set({x}) v ¬Set({y})  ImpElim 85 91
93. ¬Set({x})  Hyp
94. Set(x) -> Set({x})  TheoremInt
95. (Set(x) -> B) -> (¬B -> ¬Set(x))  PolySub 77
96. (Set(x) -> Set({x})) -> (¬Set({x}) -> ¬Set(x))  PolySub 95
97. ¬Set({x}) -> ¬Set(x)  ImpElim 94 96
98. ¬Set(x)  ImpElim 93 97
99. ¬Set({x}) -> ¬Set(x)  ImpInt 98
100. ∀a.(¬Set({a}) -> ¬Set(a))  ForallInt 99
101. ¬Set({y})  Hyp
102. ¬Set({y}) -> ¬Set(y)  ForallElim 100
103. ¬Set(y)  ImpElim 101 102
104. ¬Set(x) v ¬Set(y)  OrIntR 98
105. ¬Set(x) v ¬Set(y)  OrIntL 103
106. ¬Set(x) v ¬Set(y)  OrElim 92 93 104 101 105
107. ({x,y} = U) -> (¬Set(x) v ¬Set(y))  ImpInt 106
108. ¬Set(x) v ¬Set(y)  Hyp
109. ¬Set(x)  Hyp
110. ({x} = U) <-> ¬Set(x)  TheoremInt
111. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U))  EquivExp 110
112. ¬Set(x) -> ({x} = U)  AndElimR 111
113. {x} = U  ImpElim 109 112
114. ((x ∪ U) = U) & ((x ∩ U) = x)  TheoremInt
115. (x ∪ U) = U  AndElimL 114
116. ∀x.((x ∪ U) = U)  ForallInt 115
117. ({y} ∪ U) = U  ForallElim 116
118. U = {x}  Symmetry 113
119. ({y} ∪ {x}) = U  EqualitySub 117 118
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120. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
121. (x ∪ y) = (y ∪ x) AndElimL 120
122. ∀x.((x ∪ y) = (y ∪ x)) ForallInt 121
123. ({x} ∪ y) = (y ∪ {x}) ForallElim 122
124. ∀y.(({x} ∪ y) = (y ∪ {x})) ForallInt 123
125. ({x} ∪ {y}) = ({y} ∪ {x}) ForallElim 124
126. ({y} ∪ {x}) = ({x} ∪ {y}) Symmetry 125
127. ({x} ∪ {y}) = U EqualitySub 119 126
128. {x,y} = U EqualitySub 127 16
129. ¬Set(x) -> ({x,y} = U) ImpInt 128
130. ∀a.(¬Set(a) -> ({a,y} = U)) ForallInt 129
131. ∀b.∀a.(¬Set(a) -> ({a,b} = U)) ForallInt 130
132. ¬Set(y) Hyp
133. ∀a.(¬Set(a) -> ({a,z} = U)) ForallElim 131
134. ¬Set(y) -> ({y,z} = U) ForallElim 133
135. ∀z.(¬Set(y) -> ({y,z} = U)) ForallInt 134
136. ¬Set(y) -> ({y,x} = U) ForallElim 135
137. ∀x.({x,y} = ({x} ∪ {y})) ForallInt 15
138. {a,y} = ({a} ∪ {y}) ForallElim 137
139. ∀y.({a,y} = ({a} ∪ {y})) ForallInt 138
140. {a,b} = ({a} ∪ {b}) ForallElim 139
141. ∀a.({a,b} = ({a} ∪ {b})) ForallInt 140
142. {y,b} = ({y} ∪ {b}) ForallElim 141
143. ∀b.({y,b} = ({y} ∪ {b})) ForallInt 142
144. {y,x} = ({y} ∪ {x}) ForallElim 143
145. {y,x} = ({x} ∪ {y}) EqualitySub 144 126
146. {y,x} = {x,y} EqualitySub 145 16
147. ¬Set(y) -> ({x,y} = U) EqualitySub 136 146
148. {x,y} = U ImpElim 132 147
149. {x,y} = U OrElim 108 109 128 132 148
150. (¬Set(x) v ¬Set(y)) -> ({x,y} = U) ImpInt 149
151. (({x,y} = U) -> (¬Set(x) v ¬Set(y))) & ((¬Set(x) v ¬Set(y)) -> ({x,y} = U)) AndInt 107 150
152. ({x,y} = U) <-> (¬Set(x) v ¬Set(y)) EquivConst 151
153. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) AndInt 70 152 Qed

Used Theorems

1. Set(x) -> Set({x})
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
3. Set(x) -> ((y ε {x}) <-> (y = x))
4. ¬Set(U)
5. (A -> B) -> (¬B -> ¬A)
6. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
1. Set(x) -> Set({x})
7. ({x} = U) <-> ¬Set(x)
8. ((x ∪ U) = U) & ((x ∩ U) = x)
10. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))

Th47. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})))

0. Set(x) & Set(y) Hyp
1. z ε ∩{x,y} Hyp
2. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
3. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 2
4. ∩{x,y} = {z: ∀x_0.((x_0 ε {x,y}) -> (z ε x_0))} ForallElim 3
5. z ε {z: ∀x_0.((x_0 ε {x,y}) -> (z ε x_0))} EqualitySub 1 4
6. Set(z) & ∀x_0.((x_0 ε {x,y}) -> (z ε x_0)) ClassElim 5
7. ∀x_0.((x_0 ε {x,y}) -> (z ε x_0)) AndElimR 6
8. (x ε {x,y}) -> (z ε x) ForallElim 7
9. (y ε {x,y}) -> (z ε y) ForallElim 7
10. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
11. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))) AndElimL 10
12. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))) ImpElim 0 11
13. (z ε {x,y}) <-> ((z = x) v (z = y)) AndElimR 12
14. ((z ε {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ε {x,y})) EquivExp 13

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15. ((z = x) v (z = y)) -> (z ε {x,y})  AndElimR 14
16. ∀z.(((z = x) v (z = y)) -> (z ε {x,y}))  ForallInt 15
17. ((x = x) v (x = y)) -> (x ε {x,y})  ForallElim 16
18. ∀z.(((z = x) v (z = y)) -> (z ε {x,y}))  ForallInt 15
19. ((y = x) v (y = y)) -> (y ε {x,y})  ForallElim 18
20. x = x  Identity
21. y = y  Identity
22. (x = x) v (x = y)  OrIntR 20
23. x ε {x,y}  ImpElim 22 17
24. z ε x  ImpElim 23 8
25. (y = x) v (y = y)  OrIntL 21
26. y ε {x,y}  ImpElim 25 19
27. z ε y  ImpElim 26 9
28. (z ε x) & (z ε y)  AndInt 24 27
29. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
30. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 29
31. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 30
32. ((z ε x) & (z ε y)) -> (z ε (x ∩ y))  AndElimR 31
33. z ε (x ∩ y)  ImpElim 28 32
34. (z ε ∩{x,y}) -> (z ε (x ∩ y))  ImpInt 33
35. z ε (x ∩ y)  Hyp
36. (z ε (x ∩ y)) -> ((z ε x) & (z ε y))  AndElimL 31
37. (z ε x) & (z ε y)  ImpElim 35 36
38. c ε {x,y}  Hyp
39. (z ε {x,y}) -> ((z = x) v (z = y))  AndElimL 14
40. ∀z.((z ε {x,y}) -> ((z = x) v (z = y)))  ForallInt 39
41. (c ε {x,y}) -> ((c = x) v (c = y))  ForallElim 40
42. (c = x) v (c = y)  ImpElim 38 41
43. c = x  Hyp
44. z ε x  AndElimL 37
45. x = c  Symmetry 43
46. z ε c  EqualitySub 44 45
47. c = y  Hyp
48. z ε y  AndElimR 37
49. y = c  Symmetry 47
50. z ε c  EqualitySub 48 49
51. z ε c  OrElim 42 43 46 47 50
52. (c ε {x,y}) -> (z ε c)  ImpInt 51
53. ∀c.((c ε {x,y}) -> (z ε c))  ForallInt 52
54. ∃c.(z ε c)  ExistsInt 35
55. Set(z)  DefSub 54
56. Set(z) & ∀c.((c ε {x,y}) -> (z ε c))  AndInt 55 53
57. z ε {c: ∀x_4.((x_4 ε {x,y}) -> (c ε x_4))}  ClassInt 56
58. {z: ∀x_0.((x_0 ε {x,y}) -> (z ε x_0))} = ∩{x,y}  Symmetry 4
59. z ε ∩{x,y}  EqualitySub 57 58
60. (z ε (x ∩ y)) -> (z ε ∩{x,y})  ImpInt 59
61. ((z ε ∩{x,y}) -> (z ε (x ∩ y))) & ((z ε (x ∩ y)) -> (z ε ∩{x,y}))  AndInt 34 60
62. (z ε ∩{x,y}) <-> (z ε (x ∩ y))  EquivConst 61
63. ∀z.((z ε ∩{x,y}) <-> (z ε (x ∩ y)))  ForallInt 62
64. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
65. ∀x_6.((∩{x,y} = x_6) <-> ∀z.((z ε ∩{x,y}) <-> (z ε x_6)))  ForallElim 64
66. (∩{x,y} = (x ∩ y)) <-> ∀z.((z ε ∩{x,y}) <-> (z ε (x ∩ y)))  ForallElim 65
67. ((∩{x,y} = (x ∩ y)) -> ∀z.((z ε ∩{x,y}) <-> (z ε (x ∩ y)))) & (∀z.((z ε ∩{x,y}) <->
(z ε (x ∩ y))) -> (∩{x,y} = (x ∩ y)))  EquivExp 66
68. ∀z.((z ε ∩{x,y}) <-> (z ε (x ∩ y))) -> (∩{x,y} = (x ∩ y))  AndElimR 67
69. ∩{x,y} = (x ∩ y)  ImpElim 63 68
70. z ε ∪{x,y}  Hyp
71. ∪x = {z: ∃y.((y ε x) & (z ε y))}  DefEqInt
72. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))})  ForallInt 71
73. ∪{x,y} = {z: ∃x_8.((x_8 ε {x,y}) & (z ε x_8))}  ForallElim 72
74. z ε {z: ∃x_8.((x_8 ε {x,y}) & (z ε x_8))}  EqualitySub 70 73
75. Set(z) & ∃x_8.((x_8 ε {x,y}) & (z ε x_8))  ClassElim 74
76. ∃x_8.((x_8 ε {x,y}) & (z ε x_8))  AndElimR 75
77. (u ε {x,y}) & (z ε u)  Hyp
78. u ε {x,y}  AndElimL 77
79. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))  TheoremInt
80. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))  AndElimL
79
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81. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))  ImpElim 0 80
82. (z ε {x,y}) <-> ((z = x) v (z = y))  AndElimR 81
83. ((z ε {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ε {x,y}))  EquivExp 82
84. (z ε {x,y}) -> ((z = x) v (z = y))  AndElimL 83
85. ∀z.((z ε {x,y}) -> ((z = x) v (z = y)))  ForallInt 84
86. (u ε {x,y}) -> ((u = x) v (u = y))  ForallElim 85
87. (u = x) v (u = y)  ImpElim 78 86
88. u = x  Hyp
89. z ε u  AndElimR 77
90. z ε x  EqualitySub 89 88
91. (z ε x) v (z ε y)  OrIntR 90
92. u = y  Hyp
93. z ε y  EqualitySub 89 92
94. (z ε x) v (z ε y)  OrIntL 93
95. (z ε x) v (z ε y)  OrElim 87 88 91 92 94
96. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
97. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  AndElimL 96
98. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 97
99. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  AndElimR 98
100. z ε (x ∪ y)  ImpElim 95 99
101. z ε (x ∪ y)  ExistsElim 76 77 100
102. (z ε ∪{x,y}) -> (z ε (x ∪ y))  ImpInt 101
103. z ε (x ∪ y)  Hyp
104. (z ε (x ∪ y)) -> ((z ε x) v (z ε y))  AndElimL 98
105. (z ε x) v (z ε y)  ImpElim 103 104
106. z ε x  Hyp
107. ((z ε {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ε {x,y})) EquivExp 82
108. ((z = x) v (z = y)) -> (z ε {x,y})  AndElimR 107
109. ∀z.(((z = x) v (z = y)) -> (z ε {x,y}))  ForallInt 108
110. ((x = x) v (x = y)) -> (x ε {x,y})  ForallElim 109
111. x = x  Identity
112. (x = x) v (x = y)  OrIntR 111
113. x ε {x,y}  ImpElim 112 110
114. (x ε {x,y}) & (z ε x)  AndInt 113 106
115. ∃a.((a ε {x,y}) & (z ε a))  ExistsInt 114
116. ∃y.(z ε y)  ExistsInt 106
117. Set(z)  DefSub 116
118. Set(z) & ∃a.((a ε {x,y}) & (z ε a))  AndInt 117 115
119. z ε {b: ∃a.((a ε {x,y}) & (b ε a))}  ClassInt 118
120. {z: ∃x_8.((x_8 ε {x,y}) & (z ε x_8))} = ∪{x,y}  Symmetry 73
121. z ε ∪{x,y}  EqualitySub 119 120
122. z ε y  Hyp
123. y = y  Identity
124. ∀z.(((z = x) v (z = y)) -> (z ε {x,y}))  ForallInt 108
125. ((y = x) v (y = y)) -> (y ε {x,y})  ForallElim 124
126. (y = x) v (y = y)  OrIntL 123
127. y ε {x,y}  ImpElim 126 125
128. (y ε {x,y}) & (z ε y)  AndInt 127 122
129. ∃a.((a ε {x,y}) & (z ε a))  ExistsInt 128
130. ∃y.(z ε y)  ExistsInt 122
131. Set(z)  DefSub 130
132. Set(z) & ∃a.((a ε {x,y}) & (z ε a))  AndInt 131 129
133. z ε {b: ∃a.((a ε {x,y}) & (b ε a))}  ClassInt 132
134. z ε ∪{x,y}  EqualitySub 133 120
135. z ε ∪{x,y}  OrElim 105 106 121 122 134
136. (z ε (x ∪ y)) -> (z ε ∪{x,y})  ImpInt 135
137. ((z ε ∪{x,y}) -> (z ε (x ∪ y))) & ((z ε (x ∪ y)) -> (z ε ∪{x,y}))  AndInt 102 136
138. (z ε ∪{x,y}) <-> (z ε (x ∪ y))  EquivConst 137
139. ∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y)))  ForallInt 138
140. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
141. ∀x_14.((∪{x,y} = x_14) <-> ∀z.((z ε ∪{x,y}) <-> (z ε x_14)))  ForallElim 140
142. (∪{x,y} = (x ∪ y)) <-> ∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y)))  ForallElim 141
143. ((∪{x,y} = (x ∪ y)) -> ∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y)))) & (∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y))) -> (∪{x,y} = (x ∪ y)))  EquivExp 142
144. ∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y))) -> (∪{x,y} = (x ∪ y))  AndElimR 143
145. ∪{x,y} = (x ∪ y)  ImpElim 139 144
146. (∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y))  AndInt 69 145
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147. (Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y))) ImpInt 146
148. ¬Set(x) v ¬Set(y) Hyp
149. ({x} = U) <-> ¬Set(x) TheoremInt
150. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U)) EquivExp 149
151. ¬Set(x) -> ({x} = U) AndElimR 150
152. ¬Set(x) Hyp
153. {x} = U ImpElim 152 151
154. {x,y} = ({x} ∪ {y}) DefEqInt
155. {x,y} = (U ∪ {y}) EqualitySub 154 153
156. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
157. (x ∪ U) = U AndElimL 156
158. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
159. (x ∪ y) = (y ∪ x) AndElimL 158
160. ∀y.((x ∪ y) = (y ∪ x)) ForallInt 159
161. (x ∪ U) = (U ∪ x) ForallElim 160
162. (U ∪ x) = U EqualitySub 157 161
163. ∀x.((U ∪ x) = U) ForallInt 162
164. (U ∪ {y}) = U ForallElim 163
165. {x,y} = U EqualitySub 155 164
166. (0 = ∩U) & (U = ∪U) TheoremInt
167. U = {x,y} Symmetry 165
168. (0 = ∩{x,y}) & (U = ∪{x,y}) EqualitySub 166 167
169. ¬Set(y) Hyp
170. ∀x.(¬Set(x) -> ({x} = U)) ForallInt 151
171. ¬Set(y) -> ({y} = U) ForallElim 170
172. {y} = U ImpElim 169 171
173. {x,y} = ({x} ∪ U) EqualitySub 154 172
174. ∀x.((x ∪ U) = U) ForallInt 157
175. ({x} ∪ U) = U ForallElim 174
176. {x,y} = U EqualitySub 173 175
177. U = {x,y} Symmetry 176
178. (0 = ∩{x,y}) & (U = ∪{x,y}) EqualitySub 166 177
179. (0 = ∩{x,y}) & (U = ∪{x,y}) OrElim 148 152 168 169 178
180. (¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})) ImpInt 179
181. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y}))) AndInt 147 180 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
1. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
3. ({x} = U) <-> ¬Set(x)
4. ((x ∪ U) = U) & ((x ∩ U) = x)
5. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
6. (0 = ∩U) & (U = ∪U)

Th49. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

0. Set(x) & Set(y) Hyp
1. Set(x) AndElimL 0
2. Set(x) -> Set({x}) TheoremInt
3. Set({x}) ImpElim 1 2
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
5. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))) AndElimL 4
6. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))) ImpElim 0 5
7. Set({x,y}) AndElimL 6
8. ∀x.((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) ForallInt 5
9. (Set({x}) & Set(y)) -> (Set({{x},y}) & ((z ε {{x},y}) <-> ((z = {x}) v (z = y)))) ForallElim 8
10. ∀y.((Set({x}) & Set(y)) -> (Set({{x},y}) & ((z ε {{x},y}) <-> ((z = {x}) v (z = y))))) ForallInt 9
11. (Set({x}) & Set({x,y})) -> (Set({{x},{x,y}}) & ((z ε {{x},{x,y}}) <-> ((z = {x}) v (z = {x,y})))) ForallElim 10
12. Set({x}) & Set({x,y}) AndInt 3 7
13. Set({{x},{x,y}}) & ((z ε {{x},{x,y}}) <-> ((z = {x}) v (z = {x,y}))) ImpElim 12 11

14. Set({{x},{x,y}}) AndElimL 13
15. (x,y) = {{x},{x,y}} DefEqInt
16. {{x},{x,y}} = (x,y) Symmetry 15
17. Set((x,y)) EqualitySub 14 16
18. (Set(x) & Set(y)) -> Set((x,y)) ImpInt 17
19. ¬Set(x) v ¬Set(y) Hyp
20. ¬Set(x) Hyp
21. ({x} = U) <-> ¬Set(x) TheoremInt
22. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U)) EquivExp 21
23. ¬Set(x) -> ({x} = U) AndElimR 22
24. {x} = U ImpElim 20 23
25. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
26. ({x,y} = U) <-> (¬Set(x) v ¬Set(y)) AndElimR 25
27. (({x,y} = U) -> (¬Set(x) v ¬Set(y))) & ((¬Set(x) v ¬Set(y)) -> ({x,y} = U)) EquivExp 26
28. (¬Set(x) v ¬Set(y)) -> ({x,y} = U) AndElimR 27
29. ¬Set(x) v ¬Set(y) OrIntR 20
30. {x,y} = U ImpElim 29 28
31. ¬Set(U) TheoremInt
32. U = {x} Symmetry 24
33. ¬Set({x}) EqualitySub 31 32
34. ∀x.(¬Set(x) -> ({x} = U)) ForallInt 23
35. ¬Set({x}) -> ({{x}} = U) ForallElim 34
36. {{x}} = U ImpElim 33 35
37. {x,y} = ({x} ∪ {y}) DefEqInt
38. ∀x.({x,y} = ({x} ∪ {y})) ForallInt 37
39. {{x},y} = ({{x}} ∪ {y}) ForallElim 38
40. ∀y.({{x},y} = ({{x}} ∪ {y})) ForallInt 39
41. {{x},{x,y}} = ({{x}} ∪ {{x,y}}) ForallElim 40
42. U = {x,y} Symmetry 30
43. ¬Set({x,y}) EqualitySub 31 42
44. ∀x.(¬Set(x) -> ({x} = U)) ForallInt 23
45. ¬Set({x,y}) -> ({{x,y}} = U) ForallElim 44
46. {{x,y}} = U ImpElim 43 45
47. {{x},{x,y}} = ({{x}} ∪ U) EqualitySub 41 46
48. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
49. (x ∪ U) = U AndElimL 48
50. ∀x.((x ∪ U) = U) ForallInt 49
51. ({{x}} ∪ U) = U ForallElim 50
52. {{x},{x,y}} = U EqualitySub 47 51
53. (x,y) = U EqualitySub 15 52
54. U = (x,y) Symmetry 53
55. ¬Set((x,y)) EqualitySub 31 54
56. ¬Set(y) Hyp
57. ¬Set(x) v ¬Set(y) OrIntL 56
58. {x,y} = U ImpElim 57 28
59. U = {x,y} Symmetry 58
60. ¬Set({x,y}) EqualitySub 31 59
61. {{x,y}} = U ImpElim 60 45
62. {{x},{x,y}} = ({{x}} ∪ U) EqualitySub 41 61
63. {{x},{x,y}} = U EqualitySub 62 51
64. (x,y) = U EqualitySub 15 63
65. U = (x,y) Symmetry 64
66. ¬Set((x,y)) EqualitySub 31 65
67. ¬Set((x,y)) OrElim 19 20 55 56 66
68. (¬Set(x) v ¬Set(y)) -> ¬Set((x,y)) ImpInt 67
69. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B)) TheoremInt
70. ¬(A & B) <-> (¬A v ¬B) AndElimR 69
71. (¬(A & B) -> (¬A v ¬B)) & ((¬A v ¬B) -> ¬(A & B)) EquivExp 70
72. ¬(A & B) -> (¬A v ¬B) AndElimL 71
73. ¬(Set(x) & B) -> (¬Set(x) v ¬B) PolySub 72
74. ¬(Set(x) & Set(y)) -> (¬Set(x) v ¬Set(y)) PolySub 73
75. ¬(Set(x) & Set(y)) Hyp
76. ¬Set(x) v ¬Set(y) ImpElim 75 74
77. ¬Set((x,y)) ImpElim 76 68
78. ¬(Set(x) & Set(y)) -> ¬Set((x,y)) ImpInt 77
79. (A -> B) -> (¬B -> ¬A) TheoremInt
80. (¬(Set(x) & Set(y)) -> B) -> (¬B -> ¬¬(Set(x) & Set(y))) PolySub 79
81. (¬(Set(x) & Set(y)) -> ¬Set((x,y))) -> (¬¬Set((x,y)) -> ¬¬(Set(x) & Set(y))) PolySub 80

82. ¬¬Set((x,y)) -> ¬¬(Set(x) & Set(y)) ImpElim 78 81
83. D <-> ¬¬D TheoremInt
84. (D -> ¬¬D) & (¬¬D -> D) EquivExp 83
85. D -> ¬¬D AndElimL 84
86. (D -> ¬¬D) & (¬¬D -> D) EquivExp 83
87. ¬¬D -> D AndElimR 86
88. Set((x,y)) -> ¬¬Set((x,y)) PolySub 85
89. ¬¬(Set(x) & Set(y)) -> (Set(x) & Set(y)) PolySub 87
90. Set((x,y)) Hyp
91. ¬¬Set((x,y)) ImpElim 90 88
92. ¬¬(Set(x) & Set(y)) ImpElim 91 82
93. Set(x) & Set(y) ImpElim 92 89
94. Set((x,y)) -> (Set(x) & Set(y)) ImpInt 93
95. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) AndInt 18 94
96. (Set(x) & Set(y)) <-> Set((x,y)) EquivConst 95
97. ¬Set((x,y)) Hyp
98. ((Set(x) & Set(y)) -> B) -> (¬B -> ¬(Set(x) & Set(y))) PolySub 79
99. ((Set(x) & Set(y)) -> Set((x,y))) -> (¬Set((x,y)) -> ¬(Set(x) & Set(y))) PolySub 98
100. ¬Set((x,y)) -> ¬(Set(x) & Set(y)) ImpElim 18 99
101. ¬(Set(x) & Set(y)) ImpElim 97 100
102. ¬Set(x) v ¬Set(y) ImpElim 101 74
103. ¬Set(x) Hyp
104. {x} = U ImpElim 103 23
105. U = {x} Symmetry 104
106. ¬Set({x}) EqualitySub 31 105
107. {{x}} = U ImpElim 106 35
108. {{x},{x,y}} = (U ∪ {{x,y}}) EqualitySub 41 107
109. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
110. (x ∪ y) = (y ∪ x) AndElimL 109
111. ∀x.((x ∪ y) = (y ∪ x)) ForallInt 110
112. (U ∪ y) = (y ∪ U) ForallElim 111
113. ∀y.((U ∪ y) = (y ∪ U)) ForallInt 112
114. (U ∪ {{x,y}}) = ({{x,y}} ∪ U) ForallElim 113
115. {{x},{x,y}} = ({{x,y}} ∪ U) EqualitySub 108 114
116. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
117. (x ∪ U) = U AndElimL 116
118. ∀x.((x ∪ U) = U) ForallInt 117
119. ({{x,y}} ∪ U) = U ForallElim 118
120. (U ∪ {{x,y}}) = U EqualitySub 114 119
121. {{x},{x,y}} = U EqualitySub 108 120
122. (x,y) = U EqualitySub 15 121
123. ¬Set(y) Hyp
124. ({x,y} = U) <-> (¬Set(x) v ¬Set(y)) AndElimR 25
125. (({x,y} = U) -> (¬Set(x) v ¬Set(y))) & ((¬Set(x) v ¬Set(y)) -> ({x,y} = U)) EquivExp 124
126. (¬Set(x) v ¬Set(y)) -> ({x,y} = U) AndElimR 125
127. ¬Set(x) v ¬Set(y) OrIntL 123
128. {x,y} = U ImpElim 127 126
129. U = {x,y} Symmetry 128
130. ¬Set({x,y}) EqualitySub 31 129
131. {{x,y}} = U ImpElim 130 45
132. {{x},{x,y}} = ({{x}} ∪ U) EqualitySub 41 131
133. ∀x.((x ∪ U) = U) ForallInt 117
134. ({{x}} ∪ U) = U ForallElim 133
135. {{x},{x,y}} = U EqualitySub 132 134
136. (x,y) = U EqualitySub 15 135
137. (x,y) = U OrElim 102 103 122 123 136
138. ¬Set((x,y)) -> ((x,y) = U) ImpInt 137
139. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) AndInt 96 138 Qed

Used Theorems

1. Set(x) -> Set({x})
2. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))
3. ({x} = U) <-> ¬Set(x)
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))
5. ¬Set(U)
6. ((x ∪ U) = U) & ((x ∩ U) = x)
9. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))

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7. (A -> B) -> (¬B -> ¬A)
8. D <-> ¬¬D
10. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
6. ((x ∪ U) = U) & ((x ∩ U) = x)

Th50. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) &
(∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) ->
(((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))

0. Set(x) & Set(y)  Hyp
1. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v
¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})))  TheoremInt
2. (Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))  AndElimL 1
3. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))  TheoremInt
4. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))  AndElimL 3
5. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))  ImpElim 0 4
6. Set({x,y})  AndElimL 5
7. Set(x) -> Set({x})  TheoremInt
8. Set(x)  AndElimL 0
9. Set({x})  ImpElim 8 7
10. ∀x.(((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v
¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y}))))  ForallInt 1
11. ((Set({x}) & Set(y)) -> ((∩{{x},y} = ({x} ∩ y)) & (∪{{x},y} = ({x} ∪ y)))) &
((¬Set({x}) v ¬Set(y)) -> ((0 = ∩{{x},y}) & (U = ∪{{x},y})))  ForallElim 10
12. ∀y.(((Set({x}) & Set(y)) -> ((∩{{x},y} = ({x} ∩ y)) & (∪{{x},y} = ({x} ∪ y)))) &
((¬Set({x}) v ¬Set(y)) -> ((0 = ∩{{x},y}) & (U = ∪{{x},y}))))  ForallInt 11
13. ((Set({x}) & Set({x,y})) -> ((∩{{x},{x,y}} = ({x} ∩ {x,y})) & (∪{{x},{x,y}} = ({x} ∪
{x,y})))) & ((¬Set({x}) v ¬Set({x,y})) -> ((0 = ∩{{x},{x,y}}) & (U = ∪{{x},{x,y}})))
ForallElim 12
14. Set({x}) & Set({x,y})  AndInt 9 6
15. (Set({x}) & Set({x,y})) -> ((∩{{x},{x,y}} = ({x} ∩ {x,y})) & (∪{{x},{x,y}} = ({x} ∪
{x,y})))  AndElimL 13
16. (∩{{x},{x,y}} = ({x} ∩ {x,y})) & (∪{{x},{x,y}} = ({x} ∪ {x,y}))  ImpElim 14 15
17. {x,y} = ({x} ∪ {y})  DefEqInt
18. (∩{{x},{x,y}} = ({x} ∩ ({x} ∪ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))
EqualitySub 16 17
19. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))
TheoremInt
20. ∀x.(((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))))
ForallInt 19
21. (({x} ∩ (y ∪ z)) = (({x} ∩ y) ∪ ({x} ∩ z))) & (({x} ∪ (y ∩ z)) = (({x} ∪ y) ∩ ({x} ∪
z)))  ForallElim 20
22. ∀y.((({x} ∩ (y ∪ z)) = (({x} ∩ y) ∪ ({x} ∩ z))) & (({x} ∪ (y ∩ z)) = (({x} ∪ y) ∩
({x} ∪ z))))  ForallInt 21
23. (({x} ∩ ({x} ∪ z)) = (({x} ∩ {x}) ∪ ({x} ∩ z))) & (({x} ∪ ({x} ∩ z)) = (({x} ∪ {x}) ∩
({x} ∪ z)))  ForallElim 22
24. ∀z.((({x} ∩ ({x} ∪ z)) = (({x} ∩ {x}) ∪ ({x} ∩ z))) & (({x} ∪ ({x} ∩ z)) = (({x} ∪
{x}) ∩ ({x} ∪ z))))  ForallInt 23
25. (({x} ∩ ({x} ∪ {y})) = (({x} ∩ {x}) ∪ ({x} ∩ {y}))) & (({x} ∪ ({x} ∩ {y})) = (({x} ∪
{x}) ∩ ({x} ∪ {y})))  ForallElim 24
26. ((x ∪ x) = x) & ((x ∩ x) = x)  TheoremInt
27. ∀x.(((x ∪ x) = x) & ((x ∩ x) = x))  ForallInt 26
28. (({x} ∪ {x}) = {x}) & (({x} ∩ {x}) = {x})  ForallElim 27
29. ({x} ∪ {x}) = {x}  AndElimL 28
30. ({x} ∩ {x}) = {x}  AndElimR 28
31. ({x} ∩ ({x} ∪ {y})) = (({x} ∩ {x}) ∪ ({x} ∩ {y}))  AndElimL 25
32. ({x} ∪ ({x} ∩ {y})) = (({x} ∪ {x}) ∩ ({x} ∪ {y}))  AndElimR 25
33. (∩{{x},{x,y}} = (({x} ∩ {x}) ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))
EqualitySub 18 31
34. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))
EqualitySub 33 30
35. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))  TheoremInt
36. ((x ∪ y) ∪ z) = (x ∪ (y ∪ z))  AndElimL 35
37. ∀x.(((x ∪ y) ∪ z) = (x ∪ (y ∪ z)))  ForallInt 36
38. (({x} ∪ y) ∪ z) = ({x} ∪ (y ∪ z))  ForallElim 37
39. ∀y.((({x} ∪ y) ∪ z) = ({x} ∪ (y ∪ z)))  ForallInt 38
40. (({x} ∪ {x}) ∪ z) = ({x} ∪ ({x} ∪ z))  ForallElim 39
41. ∀z.((({x} ∪ {x}) ∪ z) = ({x} ∪ ({x} ∪ z)))  ForallInt 40
42. (({x} ∪ {x}) ∪ {y}) = ({x} ∪ ({x} ∪ {y}))  ForallElim 41
43. ({x} ∪ ({x} ∪ {y})) = (({x} ∪ {x}) ∪ {y})  Symmetry 42
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44. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = (({x} ∪ {x}) ∪ {y})) EqualitySub 34 43
45. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ {y})) EqualitySub 44 29
46. z ε ({x} ∩ {y}) Hyp
47. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
48. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 47
49. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 48
50. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 49
51. ∀x.((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) ForallInt 50
52. (z ε ({x} ∩ y)) -> ((z ε {x}) & (z ε y)) ForallElim 51
53. ∀y.((z ε ({x} ∩ y)) -> ((z ε {x}) & (z ε y))) ForallInt 52
54. (z ε ({x} ∩ {y})) -> ((z ε {x}) & (z ε {y})) ForallElim 53
55. (z ε {x}) & (z ε {y}) ImpElim 46 54
56. z ε {x} AndElimL 55
57. (z ε ({x} ∩ {y})) -> (z ε {x}) ImpInt 56
58. ∀z.((z ε ({x} ∩ {y})) -> (z ε {x})) ForallInt 57
59. ∀x.∀z.((z ε ({x} ∩ {y})) -> (z ε {x})) ForallInt 58
60. ∀z.((z ε ({a} ∩ {y})) -> (z ε {a})) ForallElim 59
61. ∀y.∀z.((z ε ({a} ∩ {y})) -> (z ε {a})) ForallInt 60
62. ∀z.((z ε ({a} ∩ {b})) -> (z ε {a})) ForallElim 61
63. ({a} ∩ {b}) ⊂ {a} DefSub 62
64. (x ⊂ y) <-> ((x ∪ y) = y) TheoremInt
65. ∀x.((x ⊂ y) <-> ((x ∪ y) = y)) ForallInt 64
66. (({a} ∩ {b}) ⊂ y) <-> ((({a} ∩ {b}) ∪ y) = y) ForallElim 65
67. ∀y.((({a} ∩ {b}) ⊂ y) <-> ((({a} ∩ {b}) ∪ y) = y)) ForallInt 66
68. (({a} ∩ {b}) ⊂ {a}) <-> ((({a} ∩ {b}) ∪ {a}) = {a}) ForallElim 67
69. ((({a} ∩ {b}) ⊂ {a}) -> ((({a} ∩ {b}) ∪ {a}) = {a})) & (((({a} ∩ {b}) ∪ {a}) = {a}) -> (({a} ∩ {b}) ⊂ {a})) EquivExp 68
70. (({a} ∩ {b}) ⊂ {a}) -> ((({a} ∩ {b}) ∪ {a}) = {a}) AndElimL 69
71. (({a} ∩ {b}) ∪ {a}) = {a} ImpElim 63 70
72. ∀a.((({a} ∩ {b}) ∪ {a}) = {a}) ForallInt 71
73. (({x} ∩ {b}) ∪ {x}) = {x} ForallElim 72
74. ∀b.((({x} ∩ {b}) ∪ {x}) = {x}) ForallInt 73
75. (({x} ∩ {y}) ∪ {x}) = {x} ForallElim 74
76. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
77. (x ∪ y) = (y ∪ x) AndElimL 76
78. ∀x.((x ∪ y) = (y ∪ x)) ForallInt 77
79. (({x} ∩ {a}) ∪ y) = (y ∪ ({x} ∩ {a})) ForallElim 78
80. ∀y.((({x} ∩ {a}) ∪ y) = (y ∪ ({x} ∩ {a}))) ForallInt 79
81. (({x} ∩ {a}) ∪ {x}) = ({x} ∪ ({x} ∩ {a})) ForallElim 80
82. ∀a.((({x} ∩ {a}) ∪ {x}) = ({x} ∪ ({x} ∩ {a}))) ForallInt 81
83. (({x} ∩ {y}) ∪ {x}) = ({x} ∪ ({x} ∩ {y})) ForallElim 82
84. ({x} ∪ ({x} ∩ {y})) = {x} EqualitySub 75 83
85. (∩{{x},{x,y}} = {x}) & (∪{{x},{x,y}} = ({x} ∪ {y})) EqualitySub 45 84
86. ({x} ∪ {y}) = {x,y} Symmetry 17
87. (∩{{x},{x,y}} = {x}) & (∪{{x},{x,y}} = {x,y}) EqualitySub 85 86
88. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U))) TheoremInt
89. Set(x) -> ((∩{x} = x) & (∪{x} = x)) AndElimL 88
90. (∩{x} = x) & (∪{x} = x) ImpElim 8 89
91. (x,y) = {{x},{x,y}} DefEqInt
92. {{x},{x,y}} = (x,y) Symmetry 91
93. (∩(x,y) = {x}) & (∪(x,y) = {x,y}) EqualitySub 87 92
94. ∩(x,y) = {x} AndElimL 93
95. ∪(x,y) = {x,y} AndElimR 93
96. {x} = ∩(x,y) Symmetry 94
97. {x,y} = ∪(x,y) Symmetry 95
98. ∩{x} = x AndElimL 90
99. ∩∩(x,y) = x EqualitySub 98 96
100. ∪{x} = x AndElimR 90
101. ∪∩(x,y) = x EqualitySub 100 96
102. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y}))) TheoremInt
103. (Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y))) AndElimL 102
104. (∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)) ImpElim 0 103
105. ∩{x,y} = (x ∩ y) AndElimL 104
106. ∪{x,y} = (x ∪ y) AndElimR 104
107. ∩∪(x,y) = (x ∩ y) EqualitySub 105 97

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108. ∪∪(x,y) = (x ∪ y)  EqualitySub 106 97
109. (¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y}))  AndElimR 102
110. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
111. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 110
112. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 111
113. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 112
114. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
115. ¬(A & B) <-> (¬A v ¬B)  AndElimR 114
116. (¬(A & B) -> (¬A v ¬B)) & ((¬A v ¬B) -> ¬(A & B))  EquivExp 115
117. (¬A v ¬B) -> ¬(A & B)  AndElimR 116
118. (¬Set(x) v ¬B) -> ¬(Set(x) & B)  PolySub 117
119. (¬Set(x) v ¬Set(y)) -> ¬(Set(x) & Set(y))  PolySub 118
120. (A -> B) -> (¬B -> ¬A)  TheoremInt
121. (Set((x,y)) -> B) -> (¬B -> ¬Set((x,y)))  PolySub 120
122. (Set((x,y)) -> (Set(x) & Set(y))) -> (¬(Set(x) & Set(y)) -> ¬Set((x,y)))  PolySub
121
123. ¬(Set(x) & Set(y)) -> ¬Set((x,y))  ImpElim 113 122
124. ¬Set((x,y)) -> ((x,y) = U)  AndElimR 110
125. ¬Set(x) v ¬Set(y)  Hyp
126. ¬(Set(x) & Set(y))  ImpElim 125 119
127. ¬Set((x,y))  ImpElim 126 123
128. (x,y) = U  ImpElim 127 124
129. U = (x,y)  Symmetry 128
130. (0 = ∩U) & (U = ∪U)  TheoremInt
131. (0 = ∩(x,y)) & (U = ∪(x,y))  EqualitySub 130 129
132. U = ∪(x,y)  AndElimR 131
133. 0 = ∩(x,y)  AndElimL 131
134. (∩0 = U) & (∪0 = 0)  TheoremInt
135. (0 = ∩∪(x,y)) & (U = ∪∪(x,y))  EqualitySub 130 132
136. (∩∩(x,y) = U) & (∪∩(x,y) = 0)  EqualitySub 134 133
137. 0 = ∩∪(x,y)  AndElimL 135
138. U = ∪∪(x,y)  AndElimR 135
139. ∩∪(x,y) = 0  Symmetry 137
140. ∪∪(x,y) = U  Symmetry 138
141. (∪∪(x,y) = U) & (∩∪(x,y) = 0)  AndInt 140 139
142. ∩∩(x,y) = U  AndElimL 136
143. ∪∩(x,y) = 0  AndElimR 136
144. (∪∩(x,y) = 0) & (∩∩(x,y) = U)  AndInt 143 142
145. ((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))  AndInt 144 141
146. (¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y)
= 0)))  ImpInt 145
147. (∪(x,y) = {x,y}) & (∩(x,y) = {x})  AndInt 95 94
148. (∪∩(x,y) = x) & (∩∩(x,y) = x)  AndInt 101 99
149. (∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))  AndInt 108 107
150. ((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))  AndInt 147
148
151. (((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y)
= (x ∪ y)) & (∩∪(x,y) = (x ∩ y)))  AndInt 150 149
152. (Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) &
(∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))  ImpInt 151
153. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) &
(∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) ->
(((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))  AndInt 152 146 Qed
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Used Theorems

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1. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v
¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})))
2. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))
3. Set(x) -> Set({x})
4. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))
5. ((x ∪ x) = x) & ((x ∩ x) = x)
6. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))
7. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
9. (x ⊂ y) <-> ((x ∪ y) = y)
10. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
11. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))
1. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v
¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})))
12. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
```

13. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
14. (A -> B) -> (¬B -> ¬A)
15. (0 = ∩U) & (U = ∪U)
16. (∩0 = U) & (∪0 = 0)

Th53. proj2(U) = U

0. proj2(x) = (∩∪x ∪ (∪∪x ~ ∪∩x)) DefEqInt
1. ∀x.(proj2(x) = (∩∪x ∪ (∪∪x ~ ∪∩x))) ForallInt 0
2. proj2(U) = (∩∪U ∪ (∪∪U ~ ∪∩U)) ForallElim 1
3. (0 = ∩U) & (U = ∪U) TheoremInt
4. (∩0 = U) & (∪0 = 0) TheoremInt
5. 0 = ∩U AndElimL 3
6. U = ∪U AndElimR 3
7. ∩0 = U AndElimL 4
8. ∪0 = 0 AndElimR 4
9. ∩U = 0 Symmetry 5
10. ∪U = U Symmetry 6
11. proj2(U) = (∩U ∪ (∪U ~ ∪∩U)) EqualitySub 2 10
12. proj2(U) = (0 ∪ (∪U ~ ∪0)) EqualitySub 11 9
13. proj2(U) = (0 ∪ (U ~ ∪0)) EqualitySub 12 10
14. proj2(U) = (0 ∪ (U ~ 0)) EqualitySub 13 8
15. ((0 ∪ x) = x) & ((0 ∩ x) = 0) TheoremInt
16. (0 ∪ x) = x AndElimL 15
17. ∀x.((0 ∪ x) = x) ForallInt 16
18. (0 ∪ (U ~ 0)) = (U ~ 0) ForallElim 17
19. proj2(U) = (U ~ 0) EqualitySub 14 18
20. (x ~ y) = (x ∩ ~y) DefEqInt
21. ∀x.((x ~ y) = (x ∩ ~y)) ForallInt 20
22. (U ~ y) = (U ∩ ~y) ForallElim 21
23. ∀y.((U ~ y) = (U ∩ ~y)) ForallInt 22
24. (U ~ 0) = (U ∩ ~0) ForallElim 23
25. (~0 = U) & (~U = 0) TheoremInt
26. ~0 = U AndElimL 25
27. (U ~ 0) = (U ∩ U) EqualitySub 24 26
28. ((x ∪ x) = x) & ((x ∩ x) = x) TheoremInt
29. (x ∩ x) = x AndElimR 28
30. ∀x.((x ∩ x) = x) ForallInt 29
31. (U ∩ U) = U ForallElim 30
32. (U ~ 0) = U EqualitySub 27 31
33. proj2(U) = U EqualitySub 19 32 Qed

Used Theorems

1. (0 = ∩U) & (U = ∪U)
2. (∩0 = U) & (∪0 = 0)
3. ((0 ∪ x) = x) & ((0 ∩ x) = 0)
5. (~0 = U) & (~U = 0)
6. ((x ∪ x) = x) & ((x ∩ x) = x)

Th54. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)))

0. Set(x) & Set(y) Hyp
1. proj1(x) = ∩∩x DefEqInt
2. proj2(x) = (∩∪x ∪ (∪∪x ~ ∪∩x)) DefEqInt
3. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0)))) TheoremInt
4. (Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y)))) AndElimL 3
5. (((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))) ImpElim 0 4
6. ((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x)) AndElimL 5
7. (∪∩(x,y) = x) & (∩∩(x,y) = x) AndElimR 6
8. ∩∩(x,y) = x AndElimR 7
9. ∀x.(proj1(x) = ∩∩x) ForallInt 1
10. ∀x.(proj1(x) = ∩∩x) ForallInt 1
11. proj1((x,y)) = ∩∩(x,y) ForallElim 10
12. proj1((x,y)) = x EqualitySub 11 8
13. ∀x.(proj2(x) = (∩∪x ∪ (∪∪x ~ ∪∩x))) ForallInt 2

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14. proj2((x,y)) = (∩U(x,y) U (UU(x,y) ~ U∩(x,y)))  ForallElim 13
15. U∩(x,y) = x  AndElimL 7
16. (UU(x,y) = (x U y)) & (∩U(x,y) = (x ∩ y))  AndElimR 5
17. UU(x,y) = (x U y)  AndElimL 16
18. ∩U(x,y) = (x ∩ y)  AndElimR 16
19. proj2((x,y)) = (∩U(x,y) U ((x U y) ~ U∩(x,y)))  EqualitySub 14 17
20. proj2((x,y)) = ((x ∩ y) U ((x U y) ~ U∩(x,y)))  EqualitySub 19 18
21. proj2((x,y)) = ((x ∩ y) U ((x U y) ~ x))  EqualitySub 20 15
22. z ε ((x U y) ~ x)  Hyp
23. (x ~ y) = (x ∩ ~y)  DefEqInt
24. ∀x.((x ~ y) = (x ∩ ~y))  ForallInt 23
25. (a ~ y) = (a ∩ ~y)  ForallElim 24
26. ∀y.((a ~ y) = (a ∩ ~y))  ForallInt 25
27. (a ~ b) = (a ∩ ~b)  ForallElim 26
28. ∀a.((a ~ b) = (a ∩ ~b))  ForallInt 27
29. ((x U y) ~ b) = ((x U y) ∩ ~b)  ForallElim 28
30. ∀b.(((x U y) ~ b) = ((x U y) ∩ ~b))  ForallInt 29
31. ((x U y) ~ x) = ((x U y) ∩ ~x)  ForallElim 30
32. z ε ((x U y) ∩ ~x)  EqualitySub 22 31
33. ((z ε (x U y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
34. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 33
35. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 34
36. (z ε (x ∩ y)) -> ((z ε x) & (z ε y))  AndElimL 35
37. ∀x.((z ε (x ∩ y)) -> ((z ε x) & (z ε y)))  ForallInt 36
38. (z ε (a ∩ y)) -> ((z ε a) & (z ε y))  ForallElim 37
39. ∀y.((z ε (a ∩ y)) -> ((z ε a) & (z ε y)))  ForallInt 38
40. (z ε (a ∩ b)) -> ((z ε a) & (z ε b))  ForallElim 39
41. ∀a.((z ε (a ∩ b)) -> ((z ε a) & (z ε b)))  ForallInt 40
42. (z ε ((x U y) ∩ b)) -> ((z ε (x U y)) & (z ε b))  ForallElim 41
43. ∀b.((z ε ((x U y) ∩ b)) -> ((z ε (x U y)) & (z ε b)))  ForallInt 42
44. (z ε ((x U y) ∩ ~x)) -> ((z ε (x U y)) & (z ε ~x))  ForallElim 43
45. (z ε (x U y)) & (z ε ~x)  ImpElim 32 44
46. z ε (x U y)  AndElimL 45
47. (z ε (x U y)) <-> ((z ε x) v (z ε y))  AndElimL 33
48. ((z ε (x U y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x U y)))
EquivExp 47
49. (z ε (x U y)) -> ((z ε x) v (z ε y))  AndElimL 48
50. (z ε x) v (z ε y)  ImpElim 46 49
51. z ε ~x  AndElimR 45
52. ~x = {y: ¬(y ε x)}  DefEqInt
53. z ε {y: ¬(y ε x)}  EqualitySub 51 52
54. Set(z) & ¬(z ε x)  ClassElim 53
55. ¬(z ε x)  AndElimR 54
56. z ε x  Hyp
57. _|_  ImpElim 56 55
58. z ε (y ∩ ~x)  AbsI 57
59. z ε y  Hyp
60. (z ε y) & (z ε ~x)  AndInt 59 51
61. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 34
62. ((z ε x) & (z ε y)) -> (z ε (x ∩ y))  AndElimR 61
63. ∀y.(((z ε x) & (z ε y)) -> (z ε (x ∩ y)))  ForallInt 62
64. ((z ε x) & (z ε a)) -> (z ε (x ∩ a))  ForallElim 63
65. ∀x.(((z ε x) & (z ε a)) -> (z ε (x ∩ a)))  ForallInt 64
66. ((z ε y) & (z ε a)) -> (z ε (y ∩ a))  ForallElim 65
67. ∀a.(((z ε y) & (z ε a)) -> (z ε (y ∩ a)))  ForallInt 66
68. ∀a.(((z ε y) & (z ε a)) -> (z ε (y ∩ a)))  ForallInt 66
69. ((z ε y) & (z ε ~x)) -> (z ε (y ∩ ~x))  ForallElim 68
70. z ε (y ∩ ~x)  ImpElim 60 69
71. z ε (y ∩ ~x)  OrElim 50 56 58 59 70
72. (z ε ((x U y) ~ x)) -> (z ε (y ∩ ~x))  ImpInt 71
73. z ε (y ∩ ~x)  Hyp
74. (z ε (x ∩ y)) -> ((z ε x) & (z ε y))  AndElimL 61
75. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y)))  ForallInt 74
76. (z ε (x ∩ a)) -> ((z ε x) & (z ε a))  ForallElim 75
77. ∀x.((z ε (x ∩ a)) -> ((z ε x) & (z ε a)))  ForallInt 76
78. (z ε (y ∩ a)) -> ((z ε y) & (z ε a))  ForallElim 77
79. ∀a.((z ε (y ∩ a)) -> ((z ε y) & (z ε a)))  ForallInt 78
80. (z ε (y ∩ ~x)) -> ((z ε y) & (z ε ~x))  ForallElim 79
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81. (z ε y) & (z ε ~x) ImpElim 73 80
82. z ε y AndElimL 81
83. (z ε x) v (z ε y) OrIntL 82
84. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 48
85. z ε (x ∪ y) ImpElim 83 84
86. z ε ~x AndElimR 81
87. (z ε (x ∪ y)) & (z ε ~x) AndInt 85 86
88. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 35
89. ∀y.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 88
90. ((z ε x) & (z ε a)) -> (z ε (x ∩ a)) ForallElim 89
91. ∀x.(((z ε x) & (z ε a)) -> (z ε (x ∩ a))) ForallInt 90
92. ((z ε (x ∪ y)) & (z ε a)) -> (z ε ((x ∪ y) ∩ a)) ForallElim 91
93. ∀a.(((z ε (x ∪ y)) & (z ε a)) -> (z ε ((x ∪ y) ∩ a))) ForallInt 92
94. ((z ε (x ∪ y)) & (z ε ~x)) -> (z ε ((x ∪ y) ∩ ~x)) ForallElim 93
95. z ε ((x ∪ y) ∩ ~x) ImpElim 87 94
96. ((x ∪ y) ∩ ~x) = ((x ∪ y) ~ x) Symmetry 31
97. z ε ((x ∪ y) ~ x) EqualitySub 95 96
98. (z ε (y ∩ ~x)) -> (z ε ((x ∪ y) ~ x)) ImpInt 97
99. ((z ε ((x ∪ y) ~ x)) -> (z ε (y ∩ ~x))) & ((z ε (y ∩ ~x)) -> (z ε ((x ∪ y) ~ x)))
AndInt 72 98
100. (z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x)) EquivConst 99
101. ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x))) ForallInt 100
102. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
103. ∀o.((((x ∪ y) ~ x) = o) <-> ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε o))) ForallElim 102
104. (((x ∪ y) ~ x) = (y ∩ ~x)) <-> ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x)))
ForallElim 103
105. ((((x ∪ y) ~ x) = (y ∩ ~x)) -> ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x)))) & (∀z.
((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x))) -> (((x ∪ y) ~ x) = (y ∩ ~x))) EquivExp 104
106. ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x))) -> (((x ∪ y) ~ x) = (y ∩ ~x)) AndElimR
105
107. ((x ∪ y) ~ x) = (y ∩ ~x) ImpElim 101 106
108. proj2((x,y)) = ((x ∩ y) ∪ (y ∩ ~x)) EqualitySub 21 107
109. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
110. (x ∩ y) = (y ∩ x) AndElimR 109
111. proj2((x,y)) = ((y ∩ x) ∪ (y ∩ ~x)) EqualitySub 108 110
112. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))
TheoremInt
113. (x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z)) AndElimL 112
114. ((x ∩ y) ∪ (x ∩ z)) = (x ∩ (y ∪ z)) Symmetry 113
115. ∀x.(((x ∩ y) ∪ (x ∩ z)) = (x ∩ (y ∪ z))) ForallInt 114
116. ((a ∩ y) ∪ (a ∩ z)) = (a ∩ (y ∪ z)) ForallElim 115
117. ∀y.(((a ∩ y) ∪ (a ∩ z)) = (a ∩ (y ∪ z))) ForallInt 116
118. ((a ∩ b) ∪ (a ∩ z)) = (a ∩ (b ∪ z)) ForallElim 117
119. ∀a.(((a ∩ b) ∪ (a ∩ z)) = (a ∩ (b ∪ z))) ForallInt 118
120. ((y ∩ b) ∪ (y ∩ z)) = (y ∩ (b ∪ z)) ForallElim 119
121. ∀b.(((y ∩ b) ∪ (y ∩ z)) = (y ∩ (b ∪ z))) ForallInt 120
122. ((y ∩ x) ∪ (y ∩ z)) = (y ∩ (x ∪ z)) ForallElim 121
123. ∀z.(((y ∩ x) ∪ (y ∩ z)) = (y ∩ (x ∪ z))) ForallInt 122
124. ((y ∩ x) ∪ (y ∩ ~x)) = (y ∩ (x ∪ ~x)) ForallElim 123
125. proj2((x,y)) = (y ∩ (x ∪ ~x)) EqualitySub 111 124
126. z ε U Hyp
127. A v ¬A TheoremInt
128. (z ε x) v ¬(z ε x) PolySub 127
129. z ε x Hyp
130. (z ε x) v (z ε ~x) OrIntR 129
131. ∀y.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 84
132. ((z ε x) v (z ε ~x)) -> (z ε (x ∪ ~x)) ForallElim 131
133. z ε (x ∪ ~x) ImpElim 130 132
134. ¬(z ε x) Hyp
135. ∃y.(z ε y) ExistsInt 126
136. Set(z) DefSub 135
137. ¬(z ε x) & Set(z) AndInt 134 136
138. z ε {z: ¬(z ε x)} ClassInt 137
139. {y: ¬(y ε x)} = ~x Symmetry 52
140. z ε ~x EqualitySub 138 139
141. (z ε x) v (z ε ~x) OrIntL 140
142. z ε (x ∪ ~x) ImpElim 141 132
143. z ε (x ∪ ~x) OrElim 128 129 133 134 142
144. (z ε U) -> (z ε (x ∪ ~x)) ImpInt 143
145. ∀z.((z ε U) -> (z ε (x ∪ ~x))) ForallInt 144
146. U ⊂ (x ∪ ~x) DefSub 145

147. (0 ⊂ x) & (x ⊂ U) TheoremInt
148. x ⊂ U AndElimR 147
149. ∀x.(x ⊂ U) ForallInt 148
150. (x ∪ ~x) ⊂ U ForallElim 149
151. (U ⊂ (x ∪ ~x)) & ((x ∪ ~x) ⊂ U) AndInt 146 150
152. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
153. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y)) EquivExp 152
154. ((x ⊂ y) & (y ⊂ x)) -> (x = y) AndElimR 153
155. ∀x.(((x ⊂ y) & (y ⊂ x)) -> (x = y)) ForallInt 154
156. ((U ⊂ y) & (y ⊂ U)) -> (U = y) ForallElim 155
157. ∀y.(((U ⊂ y) & (y ⊂ U)) -> (U = y)) ForallInt 156
158. ((U ⊂ (x ∪ ~x)) & ((x ∪ ~x) ⊂ U)) -> (U = (x ∪ ~x)) ForallElim 157
159. U = (x ∪ ~x) ImpElim 151 158
160. (x ∪ ~x) = U Symmetry 159
161. proj2((x,y)) = (y ∩ U) EqualitySub 125 160
162. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
163. (x ∩ U) = x AndElimR 162
164. ∀x.((x ∩ U) = x) ForallInt 163
165. (y ∩ U) = y ForallElim 164
166. proj2((x,y)) = y EqualitySub 161 165
167. (proj1((x,y)) = x) & (proj2((x,y)) = y) AndInt 12 166
168. (Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y)) ImpInt 167
169. ¬Set(x) v ¬Set(y) Hyp
170. (¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))) AndElimR 3
171. ((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0)) ImpElim 169 170
172. (∪∩(x,y) = 0) & (∩∩(x,y) = U) AndElimL 171
173. ∩∩(x,y) = U AndElimR 172
174. proj1((x,y)) = U EqualitySub 11 173
175. (∪∪(x,y) = U) & (∩∪(x,y) = 0) AndElimR 171
176. ∩∪(x,y) = 0 AndElimR 175
177. ∪∪(x,y) = U AndElimL 175
178. ∪∩(x,y) = 0 AndElimL 172
179. proj2((x,y)) = (∩∪(x,y) ∪ (U ~ ∪∩(x,y))) EqualitySub 14 177
180. proj2((x,y)) = (∩∪(x,y) ∪ (U ~ 0)) EqualitySub 179 178
181. proj2((x,y)) = (0 ∪ (U ~ 0)) EqualitySub 180 176
182. ((0 ∪ x) = x) & ((0 ∩ x) = 0) TheoremInt
183. (0 ∪ x) = x AndElimL 182
184. ∀x.((0 ∪ x) = x) ForallInt 183
185. (0 ∪ (U ~ 0)) = (U ~ 0) ForallElim 184
186. proj2((x,y)) = (U ~ 0) EqualitySub 181 185
187. ∀x.((x ~ y) = (x ∩ ~y)) ForallInt 23
188. (U ~ y) = (U ∩ ~y) ForallElim 187
189. ∀y.((U ~ y) = (U ∩ ~y)) ForallInt 188
190. (U ~ 0) = (U ∩ ~0) ForallElim 189
191. proj2((x,y)) = (U ∩ ~0) EqualitySub 186 190
192. (~0 = U) & (~U = 0) TheoremInt
193. ~0 = U AndElimL 192
194. proj2((x,y)) = (U ∩ U) EqualitySub 191 193
195. ((x ∪ x) = x) & ((x ∩ x) = x) TheoremInt
196. (x ∩ x) = x AndElimR 195
197. ∀x.((x ∩ x) = x) ForallInt 196
198. (U ∩ U) = U ForallElim 197
199. proj2((x,y)) = U EqualitySub 194 198
200. (proj1((x,y)) = U) & (proj2((x,y)) = U) AndInt 174 199
201. (¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)) ImpInt 200
202. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U))) AndInt 168 201 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
3. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
4. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))
0. A v ¬A
5. (0 ⊂ x) & (x ⊂ U)
6. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
8. ((x ∪ U) = U) & ((x ∩ U) = x)

7. ((0 ∪ x) = x) & ((0 ∩ x) = 0)
9. (~0 = U) & (~U = 0)
10. ((x ∪ x) = x) & ((x ∩ x) = x)

Th55. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

0. (Set(x) & Set(y)) & ((x,y) = (u,v)) Hyp
1. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U))) TheoremInt
2. (Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y)) AndElimL 1
3. Set(x) & Set(y) AndElimL 0
4. (proj1((x,y)) = x) & (proj2((x,y)) = y) ImpElim 3 2
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
6. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 5
7. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 6
8. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 7
9. Set((x,y)) ImpElim 3 8
10. (x,y) = (u,v) AndElimR 0
11. Set((u,v)) EqualitySub 9 10
12. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 6
13. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 12
14. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 13
15. Set((u,y)) -> (Set(u) & Set(y)) ForallElim 14
16. ∀y.(Set((u,y)) -> (Set(u) & Set(y))) ForallInt 15
17. Set((u,v)) -> (Set(u) & Set(v)) ForallElim 16
18. Set(u) & Set(v) ImpElim 11 17
19. ∀x.((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) ForallInt 2
20. (Set(u) & Set(y)) -> ((proj1((u,y)) = u) & (proj2((u,y)) = y)) ForallElim 19
21. ∀y.((Set(u) & Set(y)) -> ((proj1((u,y)) = u) & (proj2((u,y)) = y))) ForallInt 20
22. (Set(u) & Set(v)) -> ((proj1((u,v)) = u) & (proj2((u,v)) = v)) ForallElim 21
23. (proj1((u,v)) = u) & (proj2((u,v)) = v) ImpElim 18 22
24. proj1((x,y)) = x AndElimL 4
25. proj2((x,y)) = y AndElimR 4
26. proj1((u,v)) = u AndElimL 23
27. proj2((u,v)) = v AndElimR 23
28. proj1((u,v)) = x EqualitySub 24 10
29. u = x EqualitySub 28 26
30. proj2((u,v)) = y EqualitySub 25 10
31. v = y EqualitySub 30 27
32. x = u Symmetry 29
33. y = v Symmetry 31
34. (x = u) & (y = v) AndInt 32 33
35. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) ImpInt 34 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th58. ((r∘s)∘t) = (r∘(s∘t))

0. z ε ((r∘s)∘t) Hyp
1. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))} DefEqInt
2. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
3. ((r∘s)∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (r∘s))) & (w = (x,z)))} ForallElim 2
4. ∀b.(((r∘s)∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (r∘s))) & (w = (x,z)))}) ForallInt 3
5. ((r∘s)∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε (r∘s))) & (w = (x,z)))} ForallElim 4
6. z ε {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε (r∘s))) & (w = (x,z)))} EqualitySub 0 5
7. Set(z) & ∃x.∃y.∃x_1.((((x,y) ε t) & ((y,x_1) ε (r∘s))) & (z = (x,x_1))) ClassElim 6
8. ∃x.∃y.∃x_1.((((x,y) ε t) & ((y,x_1) ε (r∘s))) & (z = (x,x_1))) AndElimR 7
9. ∃y.∃x_1.((((x,y) ε t) & ((y,x_1) ε (r∘s))) & (z = (x,x_1))) Hyp
10. ∃x_1.((((x,y) ε t) & ((y,x_1) ε (r∘s))) & (z = (x,x_1))) Hyp
11. (((x,y) ε t) & ((y,c) ε (r∘s))) & (z = (x,c)) Hyp
12. ((x,y) ε t) & ((y,c) ε (r∘s)) AndElimL 11
13. (y,c) ε (r∘s) AndElimR 12
14. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
15. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 14

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16. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 15
17. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 16
18. (y,c) ε {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  EqualitySub 13 17
19. Set((y,c)) & ∃x.∃x_2.∃z.((((x,x_2) ε s) & ((x_2,z) ε r)) & ((y,c) = (x,z)))
ClassElim 18
20. ∃x.∃x_2.∃z.((((x,x_2) ε s) & ((x_2,z) ε r)) & ((y,c) = (x,z)))  AndElimR 19
21. ∃x_2.∃z.((((a,x_2) ε s) & ((x_2,z) ε r)) & ((y,c) = (a,z)))  Hyp
22. ∃z.((((a,b) ε s) & ((b,z) ε r)) & ((y,c) = (a,z)))  Hyp
23. (((a,b) ε s) & ((b,d) ε r)) & ((y,c) = (a,d))  Hyp
24. ((a,b) ε s) & ((b,d) ε r)  AndElimL 23
25. (x,y) ε t  AndElimL 12
26. (a,b) ε s  AndElimL 24
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
28. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 27
29. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 28
30. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 29
31. ∀y.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 30
32. Set((x,c)) -> (Set(x) & Set(c))  ForallElim 31
33. ∀x.(Set((x,c)) -> (Set(x) & Set(c)))  ForallInt 32
34. Set((y,c)) -> (Set(y) & Set(c))  ForallElim 33
35. Set((y,c))  AndElimL 19
36. Set(y) & Set(c)  ImpElim 35 34
37. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
38. ∀y.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 37
39. ((Set(x) & Set(c)) & ((x,c) = (u,v))) -> ((x = u) & (c = v))  ForallElim 38
40. ∀x.(((Set(x) & Set(c)) & ((x,c) = (u,v))) -> ((x = u) & (c = v)))  ForallInt 39
41. ((Set(y) & Set(c)) & ((y,c) = (u,v))) -> ((y = u) & (c = v))  ForallElim 40
42. ∀u.(((Set(y) & Set(c)) & ((y,c) = (u,v))) -> ((y = u) & (c = v)))  ForallInt 41
43. ((Set(y) & Set(c)) & ((y,c) = (a,v))) -> ((y = a) & (c = v))  ForallElim 42
44. ∀v.(((Set(y) & Set(c)) & ((y,c) = (a,v))) -> ((y = a) & (c = v)))  ForallInt 43
45. ((Set(y) & Set(c)) & ((y,c) = (a,d))) -> ((y = a) & (c = d))  ForallElim 44
46. (y,c) = (a,d)  AndElimR 23
47. (Set(y) & Set(c)) & ((y,c) = (a,d))  AndInt 36 46
48. (y = a) & (c = d)  ImpElim 47 45
49. y = a  AndElimL 48
50. c = d  AndElimR 48
51. (x,a) ε t  EqualitySub 25 49
52. ((x,a) ε t) & ((a,b) ε s)  AndInt 51 26
53. (b,d) ε r  AndElimR 24
54. g = (x,b)  Hyp
55. (((x,a) ε t) & ((a,b) ε s)) & (g = (x,b))  AndInt 52 54
56. ∃b.((((x,a) ε t) & ((a,b) ε s)) & (g = (x,b)))  ExistsInt 55
57. ∃a.∃b.((((x,a) ε t) & ((a,b) ε s)) & (g = (x,b)))  ExistsInt 56
58. ∃x.∃a.∃b.((((x,a) ε t) & ((a,b) ε s)) & (g = (x,b)))  ExistsInt 57
59. ∃r.((b,d) ε r)  ExistsInt 53
60. Set((b,d))  DefSub 59
61. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 30
62. Set((b,y)) -> (Set(b) & Set(y))  ForallElim 61
63. ∀y.(Set((b,y)) -> (Set(b) & Set(y)))  ForallInt 62
64. Set((b,d)) -> (Set(b) & Set(d))  ForallElim 63
65. Set(b) & Set(d)  ImpElim 60 64
66. Set(b)  AndElimL 65
67. ∃t.((x,a) ε t)  ExistsInt 51
68. Set((x,a))  DefSub 67
69. ∀y.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 30
70. Set((x,a)) -> (Set(x) & Set(a))  ForallElim 69
71. Set(x) & Set(a)  ImpElim 68 70
72. Set(x)  AndElimL 71
73. Set(x) & Set(b)  AndInt 72 66
74. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 28
75. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 74
76. ∀y.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 75
77. (Set(x) & Set(b)) -> Set((x,b))  ForallElim 76
78. Set((x,b))  ImpElim 73 77
79. (x,b) = g  Symmetry 54
80. Set(g)  EqualitySub 78 79
81. Set(g) & ∃x.∃a.∃b.((((x,a) ε t) & ((a,b) ε s)) & (g = (x,b)))  AndInt 80 58
82. g ε {w: ∃x.∃a.∃b.((((x,a) ε t) & ((a,b) ε s)) & (w = (x,b)))}  ClassInt 81
83. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
84. (s∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε s)) & (w = (x,z)))}  ForallElim 83
85. ∀b.((s∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε s)) & (w = (x,z)))})  ForallInt 84
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86. (s∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε s)) & (w = (x,z)))}  ForallElim 85
87. {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε s)) & (w = (x,z)))} = (s∘t)  Symmetry 86
88. g ε (s∘t)  EqualitySub 82 87
89. (x,b) ε (s∘t)  EqualitySub 88 54
90. (g = (x,b)) -> ((x,b) ε (s∘t))  ImpInt 89
91. ∀g.((g = (x,b)) -> ((x,b) ε (s∘t)))  ForallInt 90
92. ((x,b) = (x,b)) -> ((x,b) ε (s∘t))  ForallElim 91
93. (x,b) = (x,b)  Identity
94. (x,b) ε (s∘t)  ImpElim 93 92
95. ((b,d) ε r) & ((x,b) ε (s∘t))  AndInt 53 94
96. d = c  Symmetry 50
97. z = (x,c)  AndElimR 11
98. ((x,b) ε (s∘t)) & ((b,d) ε r)  AndInt 94 53
99. (((x,b) ε (s∘t)) & ((b,d) ε r)) & (z = (x,c))  AndInt 98 97
100. (((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c))  EqualitySub 99 96
101. ∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c)))  ExistsInt 100
102. ∃b.∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c)))  ExistsInt 101
103. ∃x.∃b.∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c)))  ExistsInt 102
104. Set(z)  AndElimL 7
105. Set(z) & ∃x.∃b.∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c)))  AndInt 104 103
106. z ε {w: ∃x.∃b.∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (w = (x,c)))}  ClassInt 105
107. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
108. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 107
109. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 108
110. (r∘(s∘t)) = {w: ∃x.∃y.∃z.((((x,y) ε (s∘t)) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 109
111. {w: ∃x.∃y.∃z.((((x,y) ε (s∘t)) & ((y,z) ε r)) & (w = (x,z)))} = (r∘(s∘t))  Symmetry 110
112. z ε (r∘(s∘t))  EqualitySub 106 111
113. z ε (r∘(s∘t))  ExistsElim 22 23 112
114. z ε (r∘(s∘t))  ExistsElim 21 22 113
115. z ε (r∘(s∘t))  ExistsElim 20 21 114
116. z ε (r∘(s∘t))  ExistsElim 10 11 115
117. z ε (r∘(s∘t))  ExistsElim 9 10 116
118. z ε (r∘(s∘t))  ExistsElim 8 9 117
119. (z ε ((r∘s)∘t)) -> (z ε (r∘(s∘t)))  ImpInt 118
120. z ε (r∘(s∘t))  Hyp
121. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
122. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 121
123. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 122
124. (r∘(s∘t)) = {w: ∃x.∃y.∃z.((((x,y) ε (s∘t)) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 123
125. z ε {w: ∃x.∃y.∃z.((((x,y) ε (s∘t)) & ((y,z) ε r)) & (w = (x,z)))}  EqualitySub 120 124
126. Set(z) & ∃x.∃y.∃x_7.((((x,y) ε (s∘t)) & ((y,x_7) ε r)) & (z = (x,x_7)))  ClassElim 125
127. ∃x.∃y.∃x_7.((((x,y) ε (s∘t)) & ((y,x_7) ε r)) & (z = (x,x_7)))  AndElimR 126
128. ∃y.∃x_7.((((x,y) ε (s∘t)) & ((y,x_7) ε r)) & (z = (x,x_7)))  Hyp
129. ∃x_7.((((x,y) ε (s∘t)) & ((y,x_7) ε r)) & (z = (x,x_7)))  Hyp
130. (((x,y) ε (s∘t)) & ((y,c) ε r)) & (z = (x,c))  Hyp
131. z = (x,c)  AndElimR 130
132. ((x,y) ε (s∘t)) & ((y,c) ε r)  AndElimL 130
133. (x,y) ε (s∘t)  AndElimL 132
134. (y,c) ε r  AndElimR 132
135. (x,y) ε {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε s)) & (w = (x,z)))}  EqualitySub 133 86
136. Set((x,y)) & ∃x_8.∃x_9.∃z.((((x_8,x_9) ε t) & ((x_9,z) ε s)) & ((x,y) = (x_8,z))) ClassElim 135
137. Set((x,y))  AndElimL 136
138. ∃x_8.∃x_9.∃z.((((x_8,x_9) ε t) & ((x_9,z) ε s)) & ((x,y) = (x_8,z)))  AndElimR 136
139. ∃x_9.∃z.((((a,x_9) ε t) & ((x_9,z) ε s)) & ((x,y) = (a,z)))  Hyp
140. ∃z.((((a,b) ε t) & ((b,z) ε s)) & ((x,y) = (a,z)))  Hyp
141. (((a,b) ε t) & ((b,d) ε s)) & ((x,y) = (a,d))  Hyp
142. (x,y) = (a,d)  AndElimR 141
143. Set((a,d))  EqualitySub 137 142
144. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 74
145. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 144
146. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 145
147. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 146
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148. Set((a,d)) -> (Set(a) & Set(d)) ForallElim 147
149. Set(a) & Set(d) ImpElim 143 148
150. Set(a) AndElimL 149
151. Set(d) AndElimR 149
152. ((a,b) ε t) & ((b,d) ε s) AndElimL 141
153. (b,d) ε s AndElimR 152
154. ((b,d) ε s) & ((y,c) ε r) AndInt 153 134
155. Set(x) & Set(y) ImpElim 137 144
156. (Set(x) & Set(y)) & ((x,y) = (a,d)) AndInt 155 142
157. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
158. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 157
159. ((Set(x) & Set(y)) & ((x,y) = (a,v))) -> ((x = a) & (y = v)) ForallElim 158
160. ∀v.(((Set(x) & Set(y)) & ((x,y) = (a,v))) -> ((x = a) & (y = v))) ForallInt 159
161. ((Set(x) & Set(y)) & ((x,y) = (a,d))) -> ((x = a) & (y = d)) ForallElim 160
162. (x = a) & (y = d) ImpElim 156 161
163. y = d AndElimR 162
164. d = y Symmetry 163
165. ((b,y) ε s) & ((y,c) ε r) EqualitySub 154 164
166. h = (b,c) Hyp
167. ∃w.((b,d) ε w) ExistsInt 153
168. ∃w.((y,c) ε w) ExistsInt 134
169. Set((b,d)) DefSub 167
170. Set((y,c)) DefSub 168
171. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 144
172. Set((b,y)) -> (Set(b) & Set(y)) ForallElim 171
173. ∀y.(Set((b,y)) -> (Set(b) & Set(y))) ForallInt 172
174. Set((b,d)) -> (Set(b) & Set(d)) ForallElim 173
175. ∀y.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 144
176. Set((x,c)) -> (Set(x) & Set(c)) ForallElim 175
177. ∀x.(Set((x,c)) -> (Set(x) & Set(c))) ForallInt 176
178. Set((y,c)) -> (Set(y) & Set(c)) ForallElim 177
179. Set(b) & Set(d) ImpElim 169 174
180. Set(y) & Set(c) ImpElim 170 178
181. Set(b) AndElimL 179
182. Set(c) AndElimR 180
183. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 74
184. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 183
185. (Set(b) & Set(y)) -> Set((b,y)) ForallElim 184
186. ∀y.((Set(b) & Set(y)) -> Set((b,y))) ForallInt 185
187. (Set(b) & Set(c)) -> Set((b,c)) ForallElim 186
188. Set(b) & Set(c) AndInt 181 182
189. Set((b,c)) ImpElim 188 187
190. (b,c) = h Symmetry 166
191. Set(h) EqualitySub 189 190
192. (((b,y) ε s) & ((y,c) ε r)) & (h = (b,c)) AndInt 165 166
193. ∃c.((((b,y) ε s) & ((y,c) ε r)) & (h = (b,c))) ExistsInt 192
194. ∃y.∃c.((((b,y) ε s) & ((y,c) ε r)) & (h = (b,c))) ExistsInt 193
195. ∃b.∃y.∃c.((((b,y) ε s) & ((y,c) ε r)) & (h = (b,c))) ExistsInt 194
196. Set(h) & ∃b.∃y.∃c.((((b,y) ε s) & ((y,c) ε r)) & (h = (b,c))) AndInt 191 195
197. h ε {w: ∃b.∃y.∃c.((((b,y) ε s) & ((y,c) ε r)) & (w = (b,c)))} ClassInt 196
198. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
199. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 198
200. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}) ForallInt 199
201. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 200
202. {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} = (r∘s) Symmetry 201
203. h ε (r∘s) EqualitySub 197 202
204. (b,c) ε (r∘s) EqualitySub 203 166
205. (h = (b,c)) -> ((b,c) ε (r∘s)) ImpInt 204
206. ∀h.((h = (b,c)) -> ((b,c) ε (r∘s))) ForallInt 205
207. ((b,c) = (b,c)) -> ((b,c) ε (r∘s)) ForallElim 206
208. (b,c) = (b,c) Identity
209. (b,c) ε (r∘s) ImpElim 208 207
210. (a,b) ε t AndElimL 152
211. x = a AndElimL 162
212. a = x Symmetry 211
213. (x,b) ε t EqualitySub 210 212
214. ((x,b) ε t) & ((b,c) ε (r∘s)) AndInt 213 209
215. (((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c)) AndInt 214 131
216. ∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c))) ExistsInt 215
217. ∃b.∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c))) ExistsInt 216

218. ∃x.∃b.∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c))) ExistsInt 217
219. Set(z) AndElimL 126
220. Set(z) & ∃x.∃b.∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c))) AndInt 219 218
221. z ε {w: ∃x.∃b.∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (w = (x,c)))} ClassInt 220
222. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
223. ((r∘s)∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (r∘s))) & (w = (x,z)))} ForallElim 222
224. ∀b.(((r∘s)∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (r∘s))) & (w = (x,z)))}) ForallInt 223
225. ((r∘s)∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε (r∘s))) & (w = (x,z)))} ForallElim 224
226. {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε (r∘s))) & (w = (x,z)))} = ((r∘s)∘t) Symmetry 225
227. z ε ((r∘s)∘t) EqualitySub 221 226
228. z ε ((r∘s)∘t) ExistsElim 140 141 227
229. z ε ((r∘s)∘t) ExistsElim 139 140 228
230. z ε ((r∘s)∘t) ExistsElim 138 139 229
231. z ε ((r∘s)∘t) ExistsElim 129 130 230
232. z ε ((r∘s)∘t) ExistsElim 128 129 231
233. z ε ((r∘s)∘t) ExistsElim 127 128 232
234. (z ε (r∘(s∘t))) -> (z ε ((r∘s)∘t)) ImpInt 233
235. ((z ε ((r∘s)∘t)) -> (z ε (r∘(s∘t)))) & ((z ε (r∘(s∘t))) -> (z ε ((r∘s)∘t))) AndInt 119 234
236. (z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t))) EquivConst 235
237. ∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t)))) ForallInt 236
238. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
239. ∀y.((((r∘s)∘t) = y) <-> ∀z.((z ε ((r∘s)∘t)) <-> (z ε y))) ForallElim 238
240. (((r∘s)∘t) = (r∘(s∘t))) <-> ∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t)))) ForallElim 239
241. ((((r∘s)∘t) = (r∘(s∘t))) -> ∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t))))) & (∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t)))) -> (((r∘s)∘t) = (r∘(s∘t)))) EquivExp 240
242. ∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t)))) -> (((r∘s)∘t) = (r∘(s∘t))) AndElimR 241
243. ((r∘s)∘t) = (r∘(s∘t)) ImpElim 237 242 Qed

Used Theorems

2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
1. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
1. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th59. ((r∘(s ∪ t)) = ((r∘s) ∪ (r∘t))) & ((r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t)))

0. z ε (r∘(s ∪ t)) Hyp
1. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))} DefEqInt
2. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
3. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 2
4. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}) ForallInt 3
5. (r∘(s ∪ t)) = {w: ∃x.∃y.∃z.((((x,y) ε (s ∪ t)) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 4
6. z ε {w: ∃x.∃y.∃z.((((x,y) ε (s ∪ t)) & ((y,z) ε r)) & (w = (x,z)))} EqualitySub 0 5
7. Set(z) & ∃x.∃y.∃x_1.((((x,y) ε (s ∪ t)) & ((y,x_1) ε r)) & (z = (x,x_1))) ClassElim 6
8. ∃x.∃y.∃x_1.((((x,y) ε (s ∪ t)) & ((y,x_1) ε r)) & (z = (x,x_1))) AndElimR 7
9. ∃y.∃x_1.((((x,y) ε (s ∪ t)) & ((y,x_1) ε r)) & (z = (x,x_1))) Hyp
10. ∃x_1.((((x,y) ε (s ∪ t)) & ((y,x_1) ε r)) & (z = (x,x_1))) Hyp
11. (((x,y) ε (s ∪ t)) & ((y,c) ε r)) & (z = (x,c)) Hyp
12. ((x,y) ε (s ∪ t)) & ((y,c) ε r) AndElimL 11
13. (x,y) ε (s ∪ t) AndElimL 12
14. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
15. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 14
16. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 15
17. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 16
18. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 17
19. (z ε (s ∪ y)) -> ((z ε s) v (z ε y)) ForallElim 18
20. ∀y.((z ε (s ∪ y)) -> ((z ε s) v (z ε y))) ForallInt 19
21. (z ε (s ∪ t)) -> ((z ε s) v (z ε t)) ForallElim 20
22. ∀z.((z ε (s ∪ t)) -> ((z ε s) v (z ε t))) ForallInt 21
23. ((x,y) ε (s ∪ t)) -> (((x,y) ε s) v ((x,y) ε t)) ForallElim 22
24. ((x,y) ε s) v ((x,y) ε t) ImpElim 13 23
25. (x,y) ε s Hyp
26. (y,c) ε r AndElimR 12

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27. ((x,y) ε s) & ((y,c) ε r)  AndInt 25 26
28. z = (x,c)  AndElimR 11
29. (((x,y) ε s) & ((y,c) ε r)) & (z = (x,c))  AndInt 27 28
30. ∃c.((((x,y) ε s) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 29
31. ∃y.∃c.((((x,y) ε s) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 30
32. ∃x.∃y.∃c.((((x,y) ε s) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 31
33. Set(z)  AndElimL 7
34. Set(z) & ∃x.∃y.∃c.((((x,y) ε s) & ((y,c) ε r)) & (z = (x,c)))  AndInt 33 32
35. z ε {w: ∃x.∃y.∃c.((((x,y) ε s) & ((y,c) ε r)) & (w = (x,c)))}  ClassInt 34
36. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
37. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 36
38. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 37
39. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 38
40. {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} = (r∘s)  Symmetry 39
41. z ε (r∘s)  EqualitySub 35 40
42. (z ε (r∘s)) v (z ε (r∘t))  OrIntR 41
43. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  AndElimR 16
44. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 43
45. ((z ε (r∘s)) v (z ε y)) -> (z ε ((r∘s) ∪ y))  ForallElim 44
46. ∀y.(((z ε (r∘s)) v (z ε y)) -> (z ε ((r∘s) ∪ y)))  ForallInt 45
47. ((z ε (r∘s)) v (z ε (r∘t))) -> (z ε ((r∘s) ∪ (r∘t)))  ForallElim 46
48. z ε ((r∘s) ∪ (r∘t))  ImpElim 42 47
49. (x,y) ε t  Hyp
50. ((x,y) ε t) & ((y,c) ε r)  AndInt 49 26
51. (((x,y) ε t) & ((y,c) ε r)) & (z = (x,c))  AndInt 50 28
52. ∃c.((((x,y) ε t) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 51
53. ∃y.∃c.((((x,y) ε t) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 52
54. ∃x.∃y.∃c.((((x,y) ε t) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 53
55. Set(z) & ∃x.∃y.∃c.((((x,y) ε t) & ((y,c) ε r)) & (z = (x,c)))  AndInt 33 54
56. z ε {w: ∃x.∃y.∃c.((((x,y) ε t) & ((y,c) ε r)) & (w = (x,c)))}  ClassInt 55
57. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
58. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 57
59. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 58
60. (r∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 59
61. {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε r)) & (w = (x,z)))} = (r∘t)  Symmetry 60
62. z ε (r∘t)  EqualitySub 56 61
63. (z ε (r∘s)) v (z ε (r∘t))  OrIntL 62
64. z ε ((r∘s) ∪ (r∘t))  ImpElim 63 47
65. z ε ((r∘s) ∪ (r∘t))  OrElim 24 25 48 49 64
66. z ε ((r∘s) ∪ (r∘t))  ExistsElim 10 11 65
67. z ε ((r∘s) ∪ (r∘t))  ExistsElim 9 10 66
68. z ε ((r∘s) ∪ (r∘t))  ExistsElim 8 9 67
69. (z ε (r∘(s ∪ t))) -> (z ε ((r∘s) ∪ (r∘t)))  ImpInt 68
70. z ε ((r∘s) ∪ (r∘t))  Hyp
71. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y)))  ForallInt 17
72. (z ε ((r∘s) ∪ y)) -> ((z ε (r∘s)) v (z ε y))  ForallElim 71
73. ∀y.((z ε ((r∘s) ∪ y)) -> ((z ε (r∘s)) v (z ε y)))  ForallInt 72
74. (z ε ((r∘s) ∪ (r∘t))) -> ((z ε (r∘s)) v (z ε (r∘t)))  ForallElim 73
75. (z ε (r∘s)) v (z ε (r∘t))  ImpElim 70 74
76. z ε (r∘s)  Hyp
77. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
78. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 77
79. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 78
80. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 79
81. z ε {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  EqualitySub 76 80
82. Set(z) & ∃x.∃y.∃x_2.((((x,y) ε s) & ((y,x_2) ε r)) & (z = (x,x_2)))  ClassElim 81
83. ∃x.∃y.∃x_2.((((x,y) ε s) & ((y,x_2) ε r)) & (z = (x,x_2)))  AndElimR 82
84. ∃y.∃x_2.((((x,y) ε s) & ((y,x_2) ε r)) & (z = (x,x_2)))  Hyp
85. ∃x_2.((((x,y) ε s) & ((y,x_2) ε r)) & (z = (x,x_2)))  Hyp
86. (((x,y) ε s) & ((y,m) ε r)) & (z = (x,m))  Hyp
87. ((x,y) ε s) & ((y,m) ε r)  AndElimL 86
88. (x,y) ε s  AndElimL 87
89. ((x,y) ε s) v ((x,y) ε t)  OrIntR 88
90. (y,m) ε r  AndElimR 87
91. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 15
92. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  AndElimR 91
93. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 92
94. ((z ε s) v (z ε y)) -> (z ε (s ∪ y))  ForallElim 93
95. ∀y.(((z ε s) v (z ε y)) -> (z ε (s ∪ y)))  ForallInt 94
96. ((z ε s) v (z ε t)) -> (z ε (s ∪ t))  ForallElim 95
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97. ∀z.(((z ε s) v (z ε t)) -> (z ε (s U t)))  ForallInt 96
98. (((x,y) ε s) v ((x,y) ε t)) -> ((x,y) ε (s U t))  ForallElim 97
99. (x,y) ε (s U t)  ImpElim 89 98
100. ((x,y) ε (s U t)) & ((y,m) ε r)  AndInt 99 90
101. z = (x,m)  AndElimR 86
102. (((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m))  AndInt 100 101
103. ∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m)))  ExistsInt 102
104. ∃y.∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m)))  ExistsInt 103
105. ∃x.∃y.∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m)))  ExistsInt 104
106. Set(z)  AndElimL 82
107. Set(z) & ∃x.∃y.∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m)))  AndInt 106 105
108. z ε {w: ∃x.∃y.∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (w = (x,m)))}  ClassInt 107
109. {w: ∃x.∃y.∃z.((((x,y) ε (s U t)) & ((y,z) ε r)) & (w = (x,z)))} = (r∘(s U t)) Symmetry 5
110. z ε (r∘(s U t))  EqualitySub 108 109
111. z ε (r∘(s U t))  ExistsElim 85 86 110
112. z ε (r∘(s U t))  ExistsElim 84 85 111
113. z ε (r∘(s U t))  ExistsElim 83 84 112
114. z ε (r∘t)  Hyp
115. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 78
116. (r∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 115
117. z ε {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε r)) & (w = (x,z)))}  EqualitySub 114 116
118. Set(z) & ∃x.∃y.∃x_4.((((x,y) ε t) & ((y,x_4) ε r)) & (z = (x,x_4)))  ClassElim 117
119. ∃x.∃y.∃x_4.((((x,y) ε t) & ((y,x_4) ε r)) & (z = (x,x_4)))  AndElimR 118
120. ∃y.∃x_4.((((x,y) ε t) & ((y,x_4) ε r)) & (z = (x,x_4)))  Hyp
121. ∃x_4.((((x,y) ε t) & ((y,x_4) ε r)) & (z = (x,x_4)))  Hyp
122. (((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))  Hyp
123. ((x,y) ε t) & ((y,e) ε r)  AndElimL 122
124. (x,y) ε t  AndElimL 123
125. ((x,y) ε s) v ((x,y) ε t)  OrIntL 124
126. (x,y) ε (s U t)  ImpElim 125 98
127. (y,e) ε r  AndElimR 123
128. ((x,y) ε (s U t)) & ((y,e) ε r)  AndInt 126 127
129. z = (x,e)  AndElimR 122
130. (((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e))  AndInt 128 129
131. ∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e)))  ExistsInt 130
132. ∃y.∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e)))  ExistsInt 131
133. ∃x.∃y.∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e)))  ExistsInt 132
134. Set(z)  AndElimL 118
135. Set(z) & ∃x.∃y.∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e)))  AndInt 134 133
136. z ε {w: ∃x.∃y.∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (w = (x,e)))}  ClassInt 135
137. z ε (r∘(s U t))  EqualitySub 136 109
138. z ε (r∘(s U t))  ExistsElim 121 122 137
139. z ε (r∘(s U t))  ExistsElim 120 121 138
140. z ε (r∘(s U t))  ExistsElim 119 120 139
141. z ε (r∘(s U t))  OrElim 75 76 113 114 140
142. (z ε ((r∘s) U (r∘t))) -> (z ε (r∘(s U t)))  ImpInt 141
143. ((z ε (r∘(s U t))) -> (z ε ((r∘s) U (r∘t)))) & ((z ε ((r∘s) U (r∘t))) -> (z ε (r∘(s U t))))  AndInt 69 142
144. (z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t)))  EquivConst 143
145. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
146. ∀y.(((r∘(s U t)) = y) <-> ∀z.((z ε (r∘(s U t))) <-> (z ε y)))  ForallElim 145
147. ((r∘(s U t)) = ((r∘s) U (r∘t))) <-> ∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t)))) ForallElim 146
148. (((r∘(s U t)) = ((r∘s) U (r∘t))) -> ∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t))))) & (∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t)))) -> ((r∘(s U t)) = ((r∘s) U (r∘t))))  EquivExp 147
149. ∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t)))) -> ((r∘(s U t)) = ((r∘s) U (r∘t))) AndElimR 148
150. ∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t))))  ForallInt 144
151. (r∘(s U t)) = ((r∘s) U (r∘t))  ImpElim 150 149
152. z ε (r∘(s ∩ t))  Hyp
153. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
154. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 153
155. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 154
156. (r∘(s ∩ t)) = {w: ∃x.∃y.∃z.((((x,y) ε (s ∩ t)) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 155
157. z ε {w: ∃x.∃y.∃z.((((x,y) ε (s ∩ t)) & ((y,z) ε r)) & (w = (x,z)))}  EqualitySub 152 156
```

158. Set(z) & ∃x.∃y.∃x_5.((((x,y) ε (s ∩ t)) & ((y,x_5) ε r)) & (z = (x,x_5))) ClassElim 157
159. ∃x.∃y.∃x_5.((((x,y) ε (s ∩ t)) & ((y,x_5) ε r)) & (z = (x,x_5))) AndElimR 158
160. ∃y.∃x_5.((((x,y) ε (s ∩ t)) & ((y,x_5) ε r)) & (z = (x,x_5))) Hyp
161. ∃x_5.((((x,y) ε (s ∩ t)) & ((y,x_5) ε r)) & (z = (x,x_5))) Hyp
162. (((x,y) ε (s ∩ t)) & ((y,e) ε r)) & (z = (x,e)) Hyp
163. ((x,y) ε (s ∩ t)) & ((y,e) ε r) AndElimL 162
164. (x,y) ε (s ∩ t) AndElimL 163
165. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 14
166. ∀x.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) ForallInt 165
167. (z ε (s ∩ y)) <-> ((z ε s) & (z ε y)) ForallElim 166
168. ∀y.((z ε (s ∩ y)) <-> ((z ε s) & (z ε y))) ForallInt 167
169. (z ε (s ∩ t)) <-> ((z ε s) & (z ε t)) ForallElim 168
170. ∀z.((z ε (s ∩ t)) <-> ((z ε s) & (z ε t))) ForallInt 169
171. ((x,y) ε (s ∩ t)) <-> (((x,y) ε s) & ((x,y) ε t)) ForallElim 170
172. (((x,y) ε (s ∩ t)) -> (((x,y) ε s) & ((x,y) ε t))) & ((((x,y) ε s) & ((x,y) ε t)) -> ((x,y) ε (s ∩ t))) EquivExp 171
173. ((x,y) ε (s ∩ t)) -> (((x,y) ε s) & ((x,y) ε t)) AndElimL 172
174. ((x,y) ε s) & ((x,y) ε t) ImpElim 164 173
175. (x,y) ε s AndElimL 174
176. (y,e) ε r AndElimR 163
177. ((x,y) ε s) & ((y,e) ε r) AndInt 175 176
178. z = (x,e) AndElimR 162
179. (((x,y) ε s) & ((y,e) ε r)) & (z = (x,e)) AndInt 177 178
180. ∃e.((((x,y) ε s) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 179
181. ∃y.∃e.((((x,y) ε s) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 180
182. ∃x.∃y.∃e.((((x,y) ε s) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 181
183. Set(z) AndElimL 158
184. Set(z) & ∃x.∃y.∃e.((((x,y) ε s) & ((y,e) ε r)) & (z = (x,e))) AndInt 183 182
185. z ε {w: ∃x.∃y.∃e.((((x,y) ε s) & ((y,e) ε r)) & (w = (x,e)))} ClassInt 184
186. z ε (r∘s) EqualitySub 185 40
187. (x,y) ε t AndElimR 174
188. ((x,y) ε t) & ((y,e) ε r) AndInt 187 176
189. (((x,y) ε t) & ((y,e) ε r)) & (z = (x,e)) AndInt 188 178
190. ∃e.((((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 189
191. ∃y.∃e.((((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 190
192. ∃x.∃y.∃e.((((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 191
193. Set(z) & ∃x.∃y.∃e.((((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))) AndInt 183 192
194. z ε {w: ∃x.∃y.∃e.((((x,y) ε t) & ((y,e) ε r)) & (w = (x,e)))} ClassInt 193
195. z ε (r∘t) EqualitySub 194 61
196. (z ε (r∘s)) & (z ε (r∘t)) AndInt 186 195
197. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 165
198. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 197
199. ∀x.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 198
200. ((z ε (r∘s)) & (z ε y)) -> (z ε ((r∘s) ∩ y)) ForallElim 199
201. ∀y.(((z ε (r∘s)) & (z ε y)) -> (z ε ((r∘s) ∩ y))) ForallInt 200
202. ((z ε (r∘s)) & (z ε (r∘t))) -> (z ε ((r∘s) ∩ (r∘t))) ForallElim 201
203. z ε ((r∘s) ∩ (r∘t)) ImpElim 196 202
204. z ε ((r∘s) ∩ (r∘t)) ExistsElim 161 162 203
205. z ε ((r∘s) ∩ (r∘t)) ExistsElim 160 161 204
206. z ε ((r∘s) ∩ (r∘t)) ExistsElim 159 160 205
207. (z ε (r∘(s ∩ t))) -> (z ε ((r∘s) ∩ (r∘t))) ImpInt 206
208. ∀z.((z ε (r∘(s ∩ t))) -> (z ε ((r∘s) ∩ (r∘t)))) ForallInt 207
209. (r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t)) DefSub 208
210. ((r∘(s ∪ t)) = ((r∘s) ∪ (r∘t))) & ((r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t))) AndInt 151 209
Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

Th61. Relation(r) -> (((r)$^{-1}$)$^{-1}$ = r)

0. z ε ((r)$^{-1}$)$^{-1}$ Hyp
1. (r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} DefEqInt
2. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 1
3. ((r)$^{-1}$)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x)))} ForallElim 2
4. z ε {z: ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x)))} EqualitySub 0 3
5. Set(z) & ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x))) ClassElim 4
6. ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x))) AndElimR 5

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7. ∃y.(((x,y) ε (r)⁻¹) & (z = (y,x)))  Hyp
8. ((x,y) ε (r)⁻¹) & (z = (y,x))  Hyp
9. (x,y) ε (r)⁻¹  AndElimL 8
10. (x,y) ε {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}  EqualitySub 9 1
11. Set((x,y)) & ∃x_0.∃x_2.(((x_0,x_2) ε r) & ((x,y) = (x_2,x_0)))  ClassElim 10
12. ∃x_0.∃x_2.(((x_0,x_2) ε r) & ((x,y) = (x_2,x_0)))  AndElimR 11
13. ∃x_2.(((c,x_2) ε r) & ((x,y) = (x_2,c)))  Hyp
14. ((c,d) ε r) & ((x,y) = (d,c))  Hyp
15. z = (y,x)  AndElimR 8
16. Set(z)  AndElimL 5
17. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
18. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
19. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 18
20. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 19
21. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 20
22. Set((y,x))  EqualitySub 16 15
23. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 21
24. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 23
25. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 24
26. Set((a,x)) -> (Set(a) & Set(x))  ForallElim 25
27. ∀a.(Set((a,x)) -> (Set(a) & Set(x)))  ForallInt 26
28. Set((y,x)) -> (Set(y) & Set(x))  ForallElim 27
29. Set(y) & Set(x)  ImpElim 22 28
30. Set(y)  AndElimL 29
31. Set(x)  AndElimR 29
32. Set(x) & Set(y)  AndInt 31 30
33. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 17
34. ((Set(x) & Set(y)) & ((x,y) = (d,v))) -> ((x = d) & (y = v))  ForallElim 33
35. ∀v.(((Set(x) & Set(y)) & ((x,y) = (d,v))) -> ((x = d) & (y = v)))  ForallInt 34
36. ((Set(x) & Set(y)) & ((x,y) = (d,c))) -> ((x = d) & (y = c))  ForallElim 35
37. (x,y) = (d,c)  AndElimR 14
38. (Set(x) & Set(y)) & ((x,y) = (d,c))  AndInt 32 37
39. (x = d) & (y = c)  ImpElim 38 36
40. x = d  AndElimL 39
41. y = c  AndElimR 39
42. (c,d) ε r  AndElimL 14
43. d = x  Symmetry 40
44. c = y  Symmetry 41
45. (c,x) ε r  EqualitySub 42 43
46. (y,x) ε r  EqualitySub 45 44
47. (y,x) ε r  ExistsElim 13 14 46
48. (y,x) ε r  ExistsElim 12 13 47
49. (y,x) = z  Symmetry 15
50. z ε r  EqualitySub 48 49
51. z ε r  ExistsElim 7 8 50
52. z ε r  ExistsElim 6 7 51
53. (z ε ((r)⁻¹)⁻¹) -> (z ε r)  ImpInt 52
54. Relation(r)  Hyp
55. z ε r  Hyp
56. ∀z.((z ε r) -> ∃x.∃y.(z = (x,y)))  DefExp 54
57. (z ε r) -> ∃x.∃y.(z = (x,y))  ForallElim 56
58. ∃x.∃y.(z = (x,y))  ImpElim 55 57
59. ∃y.(z = (x,y))  Hyp
60. z = (x,y)  Hyp
61. f = (y,x)  Hyp
62. (x,y) ε r  EqualitySub 55 60
63. ((x,y) ε r) & (f = (y,x))  AndInt 62 61
64. Set((y,x))  EqualitySub 16 15
65. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
66. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 65
67. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 66
68. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 67
69. ∃w.(z ε w)  ExistsInt 55
70. Set(z)  DefSub 69
71. Set((x,y))  EqualitySub 70 60
72. Set(x) & Set(y)  ImpElim 71 68
73. Set(x)  AndElimL 72
74. Set(y)  AndElimR 72
75. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 66
76. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 75
77. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 76
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78. (Set(a) & Set(y)) -> Set((a,y)) ForallElim 77
79. ∀y.((Set(a) & Set(y)) -> Set((a,y))) ForallInt 78
80. (Set(a) & Set(x)) -> Set((a,x)) ForallElim 79
81. ∀a.((Set(a) & Set(x)) -> Set((a,x))) ForallInt 80
82. (Set(y) & Set(x)) -> Set((y,x)) ForallElim 81
83. Set(y) & Set(x) AndInt 74 73
84. Set((y,x)) ImpElim 83 82
85. (y,x) = f Symmetry 61
86. Set(f) EqualitySub 84 85
87. ∃y.(((x,y) ε r) & (f = (y,x))) ExistsInt 63
88. ∃x.∃y.(((x,y) ε r) & (f = (y,x))) ExistsInt 87
89. Set(f) & ∃x.∃y.(((x,y) ε r) & (f = (y,x))) AndInt 86 88
90. f ε {w: ∃x.∃y.(((x,y) ε r) & (w = (y,x)))} ClassInt 89
91. {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} = (r)$^{-1}$ Symmetry 1
92. f ε (r)$^{-1}$ EqualitySub 90 91
93. (y,x) ε (r)$^{-1}$ EqualitySub 92 61
94. (f = (y,x)) -> ((y,x) ε (r)$^{-1}$) ImpInt 93
95. ∀f.((f = (y,x)) -> ((y,x) ε (r)$^{-1}$)) ForallInt 94
96. ((y,x) = (y,x)) -> ((y,x) ε (r)$^{-1}$) ForallElim 95
97. (y,x) = (y,x) Identity
98. (y,x) ε (r)$^{-1}$ ImpElim 97 96
99. ((y,x) ε (r)$^{-1}$) & (z = (x,y)) AndInt 98 60
100. ∃x.(((y,x) ε (r)$^{-1}$) & (z = (x,y))) ExistsInt 99
101. ∃y.∃x.(((y,x) ε (r)$^{-1}$) & (z = (x,y))) ExistsInt 100
102. Set(z) & ∃y.∃x.(((y,x) ε (r)$^{-1}$) & (z = (x,y))) AndInt 70 101
103. z ε {w: ∃y.∃x.(((y,x) ε (r)$^{-1}$) & (w = (x,y)))} ClassInt 102
104. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 1
105. ((r)$^{-1}$)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x)))} ForallElim 104
106. {z: ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x)))} = ((r)$^{-1}$)$^{-1}$ Symmetry 105
107. z ε ((r)$^{-1}$)$^{-1}$ EqualitySub 103 106
108. z ε ((r)$^{-1}$)$^{-1}$ ExistsElim 59 60 107
109. z ε ((r)$^{-1}$)$^{-1}$ ExistsElim 58 59 108
110. (z ε r) -> (z ε ((r)$^{-1}$)$^{-1}$) ImpInt 109
111. ((z ε ((r)$^{-1}$)$^{-1}$) -> (z ε r)) & ((z ε r) -> (z ε ((r)$^{-1}$)$^{-1}$)) AndInt 53 110
112. (z ε ((r)$^{-1}$)$^{-1}$) <-> (z ε r) EquivConst 111
113. ∀z.((z ε ((r)$^{-1}$)$^{-1}$) <-> (z ε r)) ForallInt 112
114. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
115. ∀y.((((r)$^{-1}$)$^{-1}$ = y) <-> ∀z.((z ε ((r)$^{-1}$)$^{-1}$) <-> (z ε y))) ForallElim 114
116. (((r)$^{-1}$)$^{-1}$ = r) <-> ∀z.((z ε ((r)$^{-1}$)$^{-1}$) <-> (z ε r)) ForallElim 115
117. ((((r)$^{-1}$)$^{-1}$ = r) -> ∀z.((z ε ((r)$^{-1}$)$^{-1}$) <-> (z ε r))) & (∀z.((z ε ((r)$^{-1}$)$^{-1}$) <-> (z ε r)) -> (((r)$^{-1}$)$^{-1}$ = r)) EquivExp 116
118. ∀z.((z ε ((r)$^{-1}$)$^{-1}$) <-> (z ε r)) -> (((r)$^{-1}$)$^{-1}$ = r) AndElimR 117
119. ((r)$^{-1}$)$^{-1}$ = r ImpElim 113 118
120. Relation(r) -> (((r)$^{-1}$)$^{-1}$ = r) ImpInt 119 Qed

Used Theorems

1. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th62. ((r∘s))$^{-1}$ = ((s)$^{-1}$∘(r)$^{-1}$)

0. z ε ((r∘s))$^{-1}$ Hyp
1. (r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} DefEqInt
2. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 1
3. ((r∘s))$^{-1}$ = {z: ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))} ForallElim 2
4. z ε {z: ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))} EqualitySub 0 3
5. Set(z) & ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x))) ClassElim 4
6. ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x))) AndElimR 5
7. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))} DefEqInt
8. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 7
9. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 8
10. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}) ForallInt 9
11. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 10
12. ∃y.(((x,y) ε (r∘s)) & (z = (y,x))) Hyp
13. ((x,y) ε (r∘s)) & (z = (y,x)) Hyp
14. (x,y) ε (r∘s) AndElimL 13
15. (x,y) ε {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} EqualitySub 14 11
16. Set((x,y)) & ∃x_0.∃x_1.∃z.((((x_0,x_1) ε s) & ((x_1,z) ε r)) & ((x,y) = (x_0,z))) ClassElim 15

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17. ∃x_0.∃x_1.∃z.((((x_0,x_1) ε s) & ((x_1,z) ε r)) & ((x,y) = (x_0,z)))  AndElimR 16
18. ∃x_1.∃z.((((c,x_1) ε s) & ((x_1,z) ε r)) & ((x,y) = (c,z)))  Hyp
19. ∃z.((((c,d) ε s) & ((d,z) ε r)) & ((x,y) = (c,z)))  Hyp
20. (((c,d) ε s) & ((d,b) ε r)) & ((x,y) = (c,b))  Hyp
21. ∃w.((x,y) ε w)  ExistsInt 14
22. Set((x,y))  DefSub 21
23. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
24. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 23
25. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 24
26. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 25
27. Set(x) & Set(y)  ImpElim 22 26
28. (x,y) = (c,b)  AndElimR 20
29. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
30. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 29
31. ((Set(x) & Set(y)) & ((x,y) = (c,v))) -> ((x = c) & (y = v))  ForallElim 30
32. ∀v.(((Set(x) & Set(y)) & ((x,y) = (c,v))) -> ((x = c) & (y = v)))  ForallInt 31
33. ((Set(x) & Set(y)) & ((x,y) = (c,b))) -> ((x = c) & (y = b))  ForallElim 32
34. (Set(x) & Set(y)) & ((x,y) = (c,b))  AndInt 27 28
35. (x = c) & (y = b)  ImpElim 34 33
36. x = c  AndElimL 35
37. y = b  AndElimR 35
38. c = x  Symmetry 36
39. b = y  Symmetry 37
40. (((x,d) ε s) & ((d,b) ε r)) & ((x,y) = (x,b))  EqualitySub 20 38
41. (((x,d) ε s) & ((d,y) ε r)) & ((x,y) = (x,y))  EqualitySub 40 39
42. ((x,d) ε s) & ((d,y) ε r)  AndElimL 41
43. h = (d,x)  Hyp
44. (x,d) ε s  AndElimL 42
45. ((x,d) ε s) & (h = (d,x))  AndInt 44 43
46. ∃d.(((x,d) ε s) & (h = (d,x)))  ExistsInt 45
47. ∃x.∃d.(((x,d) ε s) & (h = (d,x)))  ExistsInt 46
48. (x,d) ε s  AndElimL 45
49. ∃w.((x,d) ε w)  ExistsInt 48
50. Set((x,d))  DefSub 49
51. ∀y.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 26
52. Set((x,d)) -> (Set(x) & Set(d))  ForallElim 51
53. Set(x) & Set(d)  ImpElim 50 52
54. Set(d)  AndElimR 53
55. Set(x)  AndElimL 53
56. Set(x) & Set(d)  AndInt 55 54
57. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 25
58. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 57
59. (Set(d) & Set(y)) -> Set((d,y))  ForallElim 58
60. ∀y.((Set(d) & Set(y)) -> Set((d,y)))  ForallInt 59
61. (Set(d) & Set(x)) -> Set((d,x))  ForallElim 60
62. Set(d) & Set(x)  AndInt 54 55
63. Set((d,x))  ImpElim 62 61
64. (d,x) = h  Symmetry 43
65. Set(h)  EqualitySub 63 64
66. Set(h) & ∃x.∃d.(((x,d) ε s) & (h = (d,x)))  AndInt 65 47
67. h ε {w: ∃x.∃d.(((x,d) ε s) & (w = (d,x)))}  ClassInt 66
68. ∀r.((r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))})  ForallInt 1
69. (s)⁻¹ = {z: ∃x.∃y.(((x,y) ε s) & (z = (y,x)))}  ForallElim 68
70. {z: ∃x.∃y.(((x,y) ε s) & (z = (y,x)))} = (s)⁻¹  Symmetry 69
71. h ε (s)⁻¹  EqualitySub 67 70
72. (d,x) ε (s)⁻¹  EqualitySub 71 43
73. (h = (d,x)) -> ((d,x) ε (s)⁻¹)  ImpInt 72
74. ∀h.((h = (d,x)) -> ((d,x) ε (s)⁻¹))  ForallInt 73
75. ((d,x) = (d,x)) -> ((d,x) ε (s)⁻¹)  ForallElim 74
76. (d,x) = (d,x)  Identity
77. (d,x) ε (s)⁻¹  ImpElim 76 75
78. f = (y,d)  Hyp
79. (d,y) ε r  AndElimR 42
80. ((d,y) ε r) & (f = (y,d))  AndInt 79 78
81. ∃y.(((d,y) ε r) & (f = (y,d)))  ExistsInt 80
82. ∃d.∃y.(((d,y) ε r) & (f = (y,d)))  ExistsInt 81
83. Set(y)  AndElimR 27
84. Set(y) & Set(d)  AndInt 83 54
85. ∀y.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 57
86. (Set(x) & Set(d)) -> Set((x,d))  ForallElim 85
87. ∀x.((Set(x) & Set(d)) -> Set((x,d)))  ForallInt 86
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88. (Set(y) & Set(d)) -> Set((y,d))  ForallElim 87
89. Set((y,d))  ImpElim 84 88
90. (y,d) = f  Symmetry 78
91. Set(f)  EqualitySub 89 90
92. Set(f) & ∃d.∃y.(((d,y) ε r) & (f = (y,d)))  AndInt 91 82
93. f ε {w: ∃d.∃y.(((d,y) ε r) & (w = (y,d)))}  ClassInt 92
94. {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} = (r)⁻¹  Symmetry 1
95. f ε (r)⁻¹  EqualitySub 93 94
96. (y,d) ε (r)⁻¹  EqualitySub 95 78
97. (f = (y,d)) -> ((y,d) ε (r)⁻¹)  ImpInt 96
98. ∀f.((f = (y,d)) -> ((y,d) ε (r)⁻¹))  ForallInt 97
99. ((y,d) = (y,d)) -> ((y,d) ε (r)⁻¹)  ForallElim 98
100. (y,d) = (y,d)  Identity
101. (y,d) ε (r)⁻¹  ImpElim 100 99
102. ((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)  AndInt 101 77
103. z = (y,x)  AndElimR 13
104. (((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x))  AndInt 102 103
105. ∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x)))  ExistsInt 104
106. ∃d.∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x)))  ExistsInt 105
107. ∃y.∃d.∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x)))  ExistsInt 106
108. Set(z)  AndElimL 5
109. Set(z) & ∃y.∃d.∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x)))  AndInt 108
107
110. z ε {w: ∃y.∃d.∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (w = (y,x)))}  ClassInt 109
111. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 7
112. ((s)⁻¹∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))}
ForallElim 111
113. ∀b.(((s)⁻¹∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))})
ForallInt 112
114. ((s)⁻¹∘(r)⁻¹) = {w: ∃x.∃y.∃z.((((x,y) ε (r)⁻¹) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))}
ForallElim 113
115. {w: ∃x.∃y.∃z.((((x,y) ε (r)⁻¹) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))} = ((s)⁻¹∘(r)⁻¹)
Symmetry 114
116. z ε ((s)⁻¹∘(r)⁻¹)  EqualitySub 110 115
117. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 19 20 116
118. (h = (d,x)) -> (z ε ((s)⁻¹∘(r)⁻¹))  ImpInt 117
119. ∀h.((h = (d,x)) -> (z ε ((s)⁻¹∘(r)⁻¹)))  ForallInt 118
120. ((d,x) = (d,x)) -> (z ε ((s)⁻¹∘(r)⁻¹))  ForallElim 119
121. (d,x) = (d,x)  Identity
122. z ε ((s)⁻¹∘(r)⁻¹)  ImpElim 121 120
123. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 18 19 122
124. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 17 18 123
125. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 12 13 124
126. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 6 12 125
127. (z ε ((r∘s))⁻¹) -> (z ε ((s)⁻¹∘(r)⁻¹))  ImpInt 126
128. z ε ((s)⁻¹∘(r)⁻¹)  Hyp
129. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 7
130. ((s)⁻¹∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))}
ForallElim 129
131. ∀b.(((s)⁻¹∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))})
ForallInt 130
132. ((s)⁻¹∘(r)⁻¹) = {w: ∃x.∃y.∃z.((((x,y) ε (r)⁻¹) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))}
ForallElim 131
133. z ε {w: ∃x.∃y.∃z.((((x,y) ε (r)⁻¹) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))}  EqualitySub
128 132
134. Set(z) & ∃x.∃y.∃x_9.((((x,y) ε (r)⁻¹) & ((y,x_9) ε (s)⁻¹)) & (z = (x,x_9)))
ClassElim 133
135. Set(z)  AndElimL 134
136. ∃x.∃y.∃x_9.((((x,y) ε (r)⁻¹) & ((y,x_9) ε (s)⁻¹)) & (z = (x,x_9)))  AndElimR 134
137. ∃y.∃x_9.((((x,y) ε (r)⁻¹) & ((y,x_9) ε (s)⁻¹)) & (z = (x,x_9)))  Hyp
138. ∃x_9.((((x,y) ε (r)⁻¹) & ((y,x_9) ε (s)⁻¹)) & (z = (x,x_9)))  Hyp
139. (((x,y) ε (r)⁻¹) & ((y,a) ε (s)⁻¹)) & (z = (x,a))  Hyp
140. z = (x,a)  AndElimR 139
141. ((x,y) ε (r)⁻¹) & ((y,a) ε (s)⁻¹)  AndElimL 139
142. (x,y) ε (r)⁻¹  AndElimL 141
143. (y,a) ε (s)⁻¹  AndElimR 141
144. ∀r.((r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))})  ForallInt 1
145. (s)⁻¹ = {z: ∃x.∃y.(((x,y) ε s) & (z = (y,x)))}  ForallElim 144
146. (x,y) ε {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}  EqualitySub 142 1
147. (y,a) ε {z: ∃x.∃y.(((x,y) ε s) & (z = (y,x)))}  EqualitySub 143 145
148. Set((x,y)) & ∃x_10.∃x_11.(((x_10,x_11) ε r) & ((x,y) = (x_11,x_10)))  ClassElim 146
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149. Set((y,a)) & ∃x.∃x_12.(((x,x_12) ε s) & ((y,a) = (x_12,x)))  ClassElim 147
150. Set((x,y))  AndElimL 148
151. ∃x_10.∃x_11.(((x_10,x_11) ε r) & ((x,y) = (x_11,x_10)))  AndElimR 148
152. Set((y,a))  AndElimL 149
153. ∃x.∃x_12.(((x,x_12) ε s) & ((y,a) = (x_12,x)))  AndElimR 149
154. ∃x_11.(((b,x_11) ε r) & ((x,y) = (x_11,b)))  Hyp
155. ((b,c) ε r) & ((x,y) = (c,b))  Hyp
156. ∃x_12.(((d,x_12) ε s) & ((y,a) = (x_12,d)))  Hyp
157. ((d,e) ε s) & ((y,a) = (e,d))  Hyp
158. (b,c) ε r  AndElimL 155
159. (d,e) ε s  AndElimL 157
160. (x,y) = (c,b)  AndElimR 155
161. (y,a) = (e,d)  AndElimR 157
162. Set(x) & Set(y)  ImpElim 150 26
163. (Set(x) & Set(y)) & ((x,y) = (c,b))  AndInt 162 160
164. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 29
165. ((Set(x) & Set(y)) & ((x,y) = (c,v))) -> ((x = c) & (y = v))  ForallElim 164
166. ∀v.(((Set(x) & Set(y)) & ((x,y) = (c,v))) -> ((x = c) & (y = v)))  ForallInt 165
167. ((Set(x) & Set(y)) & ((x,y) = (c,b))) -> ((x = c) & (y = b))  ForallElim 166
168. (x = c) & (y = b)  ImpElim 163 167
169. x = c  AndElimL 168
170. y = b  AndElimR 168
171. c = x  Symmetry 169
172. b = y  Symmetry 170
173. ∀y.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 26
174. Set((x,a)) -> (Set(x) & Set(a))  ForallElim 173
175. ∀x.(Set((x,a)) -> (Set(x) & Set(a)))  ForallInt 174
176. Set((y,a)) -> (Set(y) & Set(a))  ForallElim 175
177. Set(y) & Set(a)  ImpElim 152 176
178. ((d,e) ε s) & ((b,c) ε r)  AndInt 159 158
179. ((d,e) ε s) & ((b,x) ε r)  EqualitySub 178 171
180. (Set(y) & Set(a)) & ((y,a) = (e,d))  AndInt 177 161
181. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 29
182. ((Set(x) & Set(y)) & ((x,y) = (e,v))) -> ((x = e) & (y = v))  ForallElim 181
183. ∀y.(((Set(x) & Set(y)) & ((x,y) = (e,v))) -> ((x = e) & (y = v)))  ForallInt 182
184. ((Set(x) & Set(a)) & ((x,a) = (e,v))) -> ((x = e) & (a = v))  ForallElim 183
185. ∀x.(((Set(x) & Set(a)) & ((x,a) = (e,v))) -> ((x = e) & (a = v)))  ForallInt 184
186. ((Set(y) & Set(a)) & ((y,a) = (e,v))) -> ((y = e) & (a = v))  ForallElim 185
187. ∀v.(((Set(y) & Set(a)) & ((y,a) = (e,v))) -> ((y = e) & (a = v)))  ForallInt 186
188. ((Set(y) & Set(a)) & ((y,a) = (e,d))) -> ((y = e) & (a = d))  ForallElim 187
189. (y = e) & (a = d)  ImpElim 180 188
190. y = e  AndElimL 189
191. a = d  AndElimR 189
192. e = y  Symmetry 190
193. ((d,y) ε s) & ((b,x) ε r)  EqualitySub 179 192
194. ((d,y) ε s) & ((y,x) ε r)  EqualitySub 193 172
195. d = a  Symmetry 191
196. ((a,y) ε s) & ((y,x) ε r)  EqualitySub 194 195
197. h = (a,x)  Hyp
198. Set(a)  AndElimR 177
199. Set(x)  AndElimL 162
200. Set(a) & Set(x)  AndInt 198 199
201. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 57
202. (Set(a) & Set(y)) -> Set((a,y))  ForallElim 201
203. ∀y.((Set(a) & Set(y)) -> Set((a,y)))  ForallInt 202
204. (Set(a) & Set(x)) -> Set((a,x))  ForallElim 203
205. Set((a,x))  ImpElim 200 204
206. (a,x) = h  Symmetry 197
207. Set(h)  EqualitySub 205 206
208. (((a,y) ε s) & ((y,x) ε r)) & (h = (a,x))  AndInt 196 197
209. ∃x.((((a,y) ε s) & ((y,x) ε r)) & (h = (a,x)))  ExistsInt 208
210. ∃y.∃x.((((a,y) ε s) & ((y,x) ε r)) & (h = (a,x)))  ExistsInt 209
211. ∃a.∃y.∃x.((((a,y) ε s) & ((y,x) ε r)) & (h = (a,x)))  ExistsInt 210
212. Set(h) & ∃a.∃y.∃x.((((a,y) ε s) & ((y,x) ε r)) & (h = (a,x)))  AndInt 207 211
213. h ε {w: ∃a.∃y.∃x.((((a,y) ε s) & ((y,x) ε r)) & (w = (a,x)))}  ClassInt 212
214. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 7
215. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 214
216. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt
215
217. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 216
218. {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} = (r∘s)  Symmetry 217
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219. h ε (r∘s)  EqualitySub 213 218
220. (a,x) ε (r∘s)  EqualitySub 219 197
221. (h = (a,x)) -> ((a,x) ε (r∘s))  ImpInt 220
222. ∀h.((h = (a,x)) -> ((a,x) ε (r∘s)))  ForallInt 221
223. ((a,x) = (a,x)) -> ((a,x) ε (r∘s))  ForallElim 222
224. (a,x) = (a,x)  Identity
225. (a,x) ε (r∘s)  ImpElim 224 223
226. f = (x,a)  Hyp
227. (x,a) = f  Symmetry 226
228. Set((x,a))  EqualitySub 135 140
229. Set(f)  EqualitySub 228 227
230. ((a,x) ε (r∘s)) & (f = (x,a))  AndInt 220 226
231. ∃x.(((a,x) ε (r∘s)) & (f = (x,a)))  ExistsInt 230
232. ∃a.∃x.(((a,x) ε (r∘s)) & (f = (x,a)))  ExistsInt 231
233. Set(f) & ∃a.∃x.(((a,x) ε (r∘s)) & (f = (x,a)))  AndInt 229 232
234. ∀r.((r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))})  ForallInt 1
235. ∀r.((r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))})  ForallInt 1
236. ((r∘s))⁻¹ = {z: ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))}  ForallElim 235
237. {z: ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))} = ((r∘s))⁻¹  Symmetry 236
238. f ε {w: ∃a.∃x.(((a,x) ε (r∘s)) & (w = (x,a)))}  ClassInt 233
239. f ε ((r∘s))⁻¹  EqualitySub 238 237
240. (x,a) ε ((r∘s))⁻¹  EqualitySub 239 226
241. (f = (x,a)) -> ((x,a) ε ((r∘s))⁻¹)  ImpInt 240
242. ∀f.((f = (x,a)) -> ((x,a) ε ((r∘s))⁻¹))  ForallInt 241
243. ((x,a) = (x,a)) -> ((x,a) ε ((r∘s))⁻¹)  ForallElim 242
244. (x,a) = (x,a)  Identity
245. (x,a) ε ((r∘s))⁻¹  ImpElim 244 243
246. f ε ((r∘s))⁻¹  EqualitySub 245 227
247. f ε ((r∘s))⁻¹  ExistsElim 156 157 246
248. f ε ((r∘s))⁻¹  ExistsElim 153 156 247
249. f ε ((r∘s))⁻¹  ExistsElim 154 155 248
250. f ε ((r∘s))⁻¹  ExistsElim 151 154 249
251. f ε ((r∘s))⁻¹  ExistsElim 154 155 250
252. (h = (a,x)) -> (f ε ((r∘s))⁻¹)  ImpInt 251
253. ∀h.((h = (a,x)) -> (f ε ((r∘s))⁻¹))  ForallInt 252
254. ∀h.((h = (a,x)) -> (f ε ((r∘s))⁻¹))  ForallInt 252
255. ((a,x) = (a,x)) -> (f ε ((r∘s))⁻¹)  ForallElim 254
256. (a,x) = (a,x)  Identity
257. f ε ((r∘s))⁻¹  ImpElim 256 255
258. (x,a) ε ((r∘s))⁻¹  EqualitySub 257 226
259. (f = (x,a)) -> ((x,a) ε ((r∘s))⁻¹)  ImpInt 258
260. ∀f.((f = (x,a)) -> ((x,a) ε ((r∘s))⁻¹))  ForallInt 259
261. ((x,a) = (x,a)) -> ((x,a) ε ((r∘s))⁻¹)  ForallElim 260
262. (x,a) = (x,a)  Identity
263. (x,a) ε ((r∘s))⁻¹  ImpElim 262 261
264. (x,a) = z  Symmetry 140
265. z ε ((r∘s))⁻¹  EqualitySub 263 264
266. z ε ((r∘s))⁻¹  ExistsElim 151 154 265
267. z ε ((r∘s))⁻¹  ExistsElim 138 139 266
268. z ε ((r∘s))⁻¹  ExistsElim 137 138 267
269. z ε ((r∘s))⁻¹  ExistsElim 136 137 268
270. (z ε ((s)⁻¹∘(r)⁻¹)) -> (z ε ((r∘s))⁻¹)  ImpInt 269
271. ((z ε ((r∘s))⁻¹) -> (z ε ((s)⁻¹∘(r)⁻¹))) & ((z ε ((s)⁻¹∘(r)⁻¹)) -> (z ε ((r∘s))⁻¹))
AndInt 127 270
272. (z ε ((r∘s))⁻¹) <-> (z ε ((s)⁻¹∘(r)⁻¹))  EquivConst 271
273. ∀z.((z ε ((r∘s))⁻¹) <-> (z ε ((s)⁻¹∘(r)⁻¹)))  ForallInt 272
274. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
275. ∀y.((((r∘s))⁻¹ = y) <-> ∀z.((z ε ((r∘s))⁻¹) <-> (z ε y)))  ForallElim 274
276. (((r∘s))⁻¹ = ((s)⁻¹∘(r)⁻¹)) <-> ∀z.((z ε ((r∘s))⁻¹) <-> (z ε ((s)⁻¹∘(r)⁻¹)))
ForallElim 275
277. ((((r∘s))⁻¹ = ((s)⁻¹∘(r)⁻¹)) -> ∀z.((z ε ((r∘s))⁻¹) <-> (z ε ((s)⁻¹∘(r)⁻¹)))) & (∀z.
((z ε ((r∘s))⁻¹) <-> (z ε ((s)⁻¹∘(r)⁻¹))) -> (((r∘s))⁻¹ = ((s)⁻¹∘(r)⁻¹)))  EquivExp 276
278. ∀z.((z ε ((r∘s))⁻¹) <-> (z ε ((s)⁻¹∘(r)⁻¹))) -> (((r∘s))⁻¹ = ((s)⁻¹∘(r)⁻¹))
AndElimR 277
279. ((r∘s))⁻¹ = ((s)⁻¹∘(r)⁻¹)  ImpElim 273 278 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
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Th64. (Function(f) & Function(g)) -> Function((f∘g))

0. Function(f) & Function(g)  Hyp
1. Function(f)  AndElimL 0
2. Function(g)  AndElimR 0
3. (a,b) ε (f∘g)  Hyp
4. (a,c) ε (f∘g)  Hyp
5. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}  DefEqInt
6. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 5
7. (f∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε f)) & (w = (x,z)))}  ForallElim 6
8. ∀b.((f∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε f)) & (w = (x,z)))})  ForallInt 7
9. (f∘g) = {w: ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & (w = (x,z)))}  ForallElim 8
10. (a,b) ε {w: ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & (w = (x,z)))}  EqualitySub 3 9
11. (a,c) ε {w: ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & (w = (x,z)))}  EqualitySub 4 9
12. Set((a,b)) & ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z)))  ClassElim 10
13. Set((a,c)) & ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,c) = (x,z)))  ClassElim 11
14. ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z)))  AndElimR 12
15. ∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z)))  Hyp
16. ∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z)))  Hyp
17. (((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z))  Hyp
18. ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,c) = (x,z)))  AndElimR 13
19. ∃y.∃z.((((u,y) ε g) & ((y,z) ε f)) & ((a,c) = (u,z)))  Hyp
20. ∃z.((((u,v) ε g) & ((v,z) ε f)) & ((a,c) = (u,z)))  Hyp
21. (((u,v) ε g) & ((v,w) ε f)) & ((a,c) = (u,w))  Hyp
22. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
23. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 22
24. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 23
25. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 24
26. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 25
27. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 26
28. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 27
29. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 28
30. Set((a,b))  AndElimL 12
31. Set(a) & Set(b)  ImpElim 30 29
32. Set(a)  AndElimL 31
33. Set(b)  AndElimR 31
34. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 25
35. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 34
36. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 35
37. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 36
38. Set((a,c))  AndElimL 13
39. Set(a) & Set(c)  ImpElim 38 37
40. Set(c)  AndElimR 39
41. (a,b) = (x,z)  AndElimR 17
42. (Set(a) & Set(b)) & ((a,b) = (x,z))  AndInt 31 41
43. (a,c) = (u,w)  AndElimR 21
44. (Set(a) & Set(c)) & ((a,c) = (u,w))  AndInt 39 43
45. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
46. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 45
47. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))  ForallElim 46
48. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 47
49. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))  ForallElim 48
50. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)))  ForallInt 49
51. ((Set(a) & Set(b)) & ((a,b) = (x,v))) -> ((a = x) & (b = v))  ForallElim 50
52. ∀v.(((Set(a) & Set(b)) & ((a,b) = (x,v))) -> ((a = x) & (b = v)))  ForallInt 51
53. ((Set(a) & Set(b)) & ((a,b) = (x,z))) -> ((a = x) & (b = z))  ForallElim 52
54. (a = x) & (b = z)  ImpElim 42 53
55. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 47
56. ((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v))  ForallElim 55
57. ∀v.(((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v)))  ForallInt 56
58. ((Set(a) & Set(c)) & ((a,c) = (u,w))) -> ((a = u) & (c = w))  ForallElim 57
59. (a = u) & (c = w)  ImpElim 44 58
60. a = x  AndElimL 54
61. b = z  AndElimR 54
62. a = u  AndElimL 59
63. c = w  AndElimR 59
64. ((x,y) ε g) & ((y,z) ε f)  AndElimL 17
65. ((u,v) ε g) & ((v,w) ε f)  AndElimL 21
66. (y,z) ε f  AndElimR 64
67. (v,w) ε f  AndElimR 65
68. (x,y) ε g  AndElimL 64
```

69. (u,v) ε g AndElimL 65
70. x = u EqualitySub 62 60
71. (u,y) ε g EqualitySub 68 70
72. Relation(g) & ∀x.∀y.∀z.((((x,y) ε g) & ((x,z) ε g)) -> (y = z)) DefExp 2
73. ∀x.∀y.∀z.((((x,y) ε g) & ((x,z) ε g)) -> (y = z)) AndElimR 72
74. ∀y.∀z.((((u,y) ε g) & ((u,z) ε g)) -> (y = z)) ForallElim 73
75. ∀z.((((u,y) ε g) & ((u,z) ε g)) -> (y = z)) ForallElim 74
76. (((u,y) ε g) & ((u,v) ε g)) -> (y = v) ForallElim 75
77. ((u,y) ε g) & ((u,v) ε g) AndInt 71 69
78. y = v ImpElim 77 76
79. (v,z) ε f EqualitySub 66 78
80. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 1
81. ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) AndElimR 80
82. ∀y.∀z.((((v,y) ε f) & ((v,z) ε f)) -> (y = z)) ForallElim 81
83. ∀x_0.((((v,z) ε f) & ((v,x_0) ε f)) -> (z = x_0)) ForallElim 82
84. (((v,z) ε f) & ((v,w) ε f)) -> (z = w) ForallElim 83
85. ((v,z) ε f) & ((v,w) ε f) AndInt 79 67
86. z = w ImpElim 85 84
87. b = w EqualitySub 61 86
88. w = c Symmetry 63
89. b = c EqualitySub 87 88
90. b = c ExistsElim 20 21 89
91. b = c ExistsElim 19 20 90
92. b = c ExistsElim 18 19 91
93. b = c ExistsElim 16 17 92
94. b = c ExistsElim 15 16 93
95. b = c ExistsElim 14 15 94
96. ((a,c) ε (f∘g)) -> (b = c) ImpInt 95
97. ((a,b) ε (f∘g)) -> (((a,c) ε (f∘g)) -> (b = c)) ImpInt 96
98. A -> (B -> C) Hyp
99. A & B Hyp
100. A AndElimL 99
101. B -> C ImpElim 100 98
102. B AndElimR 99
103. C ImpElim 102 101
104. (A & B) -> C ImpInt 103
105. (A -> (B -> C)) -> ((A & B) -> C) ImpInt 104
106. (((a,b) ε (f∘g)) -> (B -> C)) -> ((((a,b) ε (f∘g)) & B) -> C) PolySub 105
107. (((a,b) ε (f∘g)) -> (((a,c) ε (f∘g)) -> C)) -> ((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> C) PolySub 106
108. (((a,b) ε (f∘g)) -> (((a,c) ε (f∘g)) -> (b = c))) -> ((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) PolySub 107
109. (((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c) ImpElim 97 108
110. Relation(g) AndElimL 72
111. Relation(f) AndElimL 80
112. z ε (f∘g) Hyp
113. z ε {w: ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & (w = (x,z)))} EqualitySub 112 9
114. Set(z) & ∃x.∃y.∃x_2.((((x,y) ε g) & ((y,x_2) ε f)) & (z = (x,x_2))) ClassElim 113
115. ∃x.∃y.∃x_2.((((x,y) ε g) & ((y,x_2) ε f)) & (z = (x,x_2))) AndElimR 114
116. ∃y.∃x_2.((((x,y) ε g) & ((y,x_2) ε f)) & (z = (x,x_2))) Hyp
117. ∃x_2.((((x,y) ε g) & ((y,x_2) ε f)) & (z = (x,x_2))) Hyp
118. (((x,y) ε g) & ((y,l) ε f)) & (z = (x,l)) Hyp
119. z = (x,l) AndElimR 118
120. ∃l.(z = (x,l)) ExistsInt 119
121. ∃x.∃l.(z = (x,l)) ExistsInt 120
122. ∃x.∃l.(z = (x,l)) ExistsElim 117 118 121
123. ∃x.∃l.(z = (x,l)) ExistsElim 116 117 122
124. ∃x.∃l.(z = (x,l)) ExistsElim 115 116 123
125. (z ε (f∘g)) -> ∃x.∃l.(z = (x,l)) ImpInt 124
126. ∀z.((z ε (f∘g)) -> ∃x.∃l.(z = (x,l))) ForallInt 125
127. Relation((f∘g)) DefSub 126
128. ∀c.((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) ForallInt 109
129. ∀b.∀c.((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) ForallInt 128
130. ∀a.∀b.∀c.((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) ForallInt 129
131. Relation((f∘g)) & ∀a.∀b.∀c.((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) AndInt 127 130
132. Function((f∘g)) DefSub 131
133. (Function(f) & Function(g)) -> Function((f∘g)) ImpInt 132 Qed

Used Theorems

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1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th67. (domain(U) = U) & (range(U) = U)

0. z ε domain(U)  Hyp
1. ∃w.(z ε w)  ExistsInt 0
2. Set(z)  DefSub 1
3. (x ε U) <-> Set(x)  TheoremInt
4. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 3
5. Set(x) -> (x ε U)  AndElimR 4
6. ∀x.(Set(x) -> (x ε U))  ForallInt 5
7. Set(z) -> (z ε U)  ForallElim 6
8. z ε U  ImpElim 2 7
9. (z ε domain(U)) -> (z ε U)  ImpInt 8
10. z ε U  Hyp
11. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 4
12. (x ε U) -> Set(x)  AndElimL 11
13. ∀x.((x ε U) -> Set(x))  ForallInt 12
14. (z ε U) -> Set(z)  ForallElim 13
15. Set(z)  ImpElim 10 14
16. (0 ⊂ x) & (x ⊂ U)  TheoremInt
17. 0 ⊂ x  AndElimL 16
18. ∀x.(0 ⊂ x)  ForallInt 17
19. 0 ⊂ z  ForallElim 18
20. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
21. ∀x.((Set(x) & (y ⊂ x)) -> Set(y))  ForallInt 20
22. (Set(z) & (y ⊂ z)) -> Set(y)  ForallElim 21
23. ∀y.((Set(z) & (y ⊂ z)) -> Set(y))  ForallInt 22
24. (Set(z) & (0 ⊂ z)) -> Set(0)  ForallElim 23
25. Set(z) & (0 ⊂ z)  AndInt 15 19
26. Set(0)  ImpElim 25 24
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
28. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 27
29. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 28
30. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 29
31. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 30
32. (Set(z) & Set(y)) -> Set((z,y))  ForallElim 31
33. ∀y.((Set(z) & Set(y)) -> Set((z,y)))  ForallInt 32
34. (Set(z) & Set(0)) -> Set((z,0))  ForallElim 33
35. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
36. Set(z) & Set(0)  AndInt 15 26
37. Set((z,0))  ImpElim 36 34
38. Set(x) -> (x ε U)  AndElimR 11
39. ∀x.(Set(x) -> (x ε U))  ForallInt 38
40. Set((z,0)) -> ((z,0) ε U)  ForallElim 39
41. (z,0) ε U  ImpElim 37 40
42. ∃w.((z,w) ε U)  ExistsInt 41
43. Set(z) & ∃w.((z,w) ε U)  AndInt 15 42
44. z ε {w: ∃i.((w,i) ε U)}  ClassInt 43
45. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 35
46. ∀f.({x: ∃y.((x,y) ε f)} = domain(f))  ForallInt 45
47. {x: ∃y.((x,y) ε U)} = domain(U)  ForallElim 46
48. z ε domain(U)  EqualitySub 44 47
49. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
50. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 30
51. (Set(0) & Set(y)) -> Set((0,y))  ForallElim 50
52. ∀y.((Set(0) & Set(y)) -> Set((0,y)))  ForallInt 51
53. (Set(0) & Set(z)) -> Set((0,z))  ForallElim 52
54. Set(0) & Set(z)  AndInt 26 15
55. Set((0,z))  ImpElim 54 53
56. ∀x.(Set(x) -> (x ε U))  ForallInt 38
57. Set((0,z)) -> ((0,z) ε U)  ForallElim 56
58. (0,z) ε U  ImpElim 55 57
59. ∃w.((w,z) ε U)  ExistsInt 58
60. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
61. {y: ∃x.((x,y) ε f)} = range(f)  Symmetry 60
62. ∀f.({y: ∃x.((x,y) ε f)} = range(f))  ForallInt 61
63. {y: ∃x.((x,y) ε U)} = range(U)  ForallElim 62
64. Set(z) & ∃w.((w,z) ε U)  AndInt 15 59
65. z ε {w: ∃j.((j,w) ε U)}  ClassInt 64
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66. z ε range(U)  EqualitySub 65 63
67. (z ε U) -> (z ε domain(U))  ImpInt 48
68. (z ε U) -> (z ε range(U))  ImpInt 66
69. z ε range(U)  Hyp
70. ∃w.(z ε w)  ExistsInt 69
71. Set(z)  DefSub 70
72. z ε U  ImpElim 71 7
73. (z ε range(U)) -> (z ε U)  ImpInt 72
74. ((z ε domain(U)) -> (z ε U)) & ((z ε U) -> (z ε domain(U)))  AndInt 9 67
75. (z ε domain(U)) <-> (z ε U)  EquivConst 74
76. ∀z.((z ε domain(U)) <-> (z ε U))  ForallInt 75
77. ((z ε range(U)) -> (z ε U)) & ((z ε U) -> (z ε range(U)))  AndInt 73 68
78. (z ε range(U)) <-> (z ε U)  EquivConst 77
79. ∀z.((z ε range(U)) <-> (z ε U))  ForallInt 78
80. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
81. ∀y.((domain(U) = y) <-> ∀z.((z ε domain(U)) <-> (z ε y)))  ForallElim 80
82. (domain(U) = U) <-> ∀z.((z ε domain(U)) <-> (z ε U))  ForallElim 81
83. ((domain(U) = U) -> ∀z.((z ε domain(U)) <-> (z ε U))) & (∀z.((z ε domain(U)) <-> (z ε
U)) -> (domain(U) = U))  EquivExp 82
84. ∀z.((z ε domain(U)) <-> (z ε U)) -> (domain(U) = U)  AndElimR 83
85. domain(U) = U  ImpElim 76 84
86. ∀y.((range(U) = y) <-> ∀z.((z ε range(U)) <-> (z ε y)))  ForallElim 80
87. (range(U) = U) <-> ∀z.((z ε range(U)) <-> (z ε U))  ForallElim 86
88. ((range(U) = U) -> ∀z.((z ε range(U)) <-> (z ε U))) & (∀z.((z ε range(U)) <-> (z ε
U)) -> (range(U) = U))  EquivExp 87
89. ∀z.((z ε range(U)) <-> (z ε U)) -> (range(U) = U)  AndElimR 88
90. range(U) = U  ImpElim 79 89
91. (domain(U) = U) & (range(U) = U)  AndInt 85 90 Qed
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Used Theorems

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1. (x ε U) <-> Set(x)
2. (0 ⊂ x) & (x ⊂ U)
3. (Set(x) & (y ⊂ x)) -> Set(y)
4. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
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Th69. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U))

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0. ¬(z ε domain(f))  Hyp
1. a ε {y: ((z,y) ε f)}  Hyp
2. Set(a) & ((z,a) ε f)  ClassElim 1
3. (z,a) ε f  AndElimR 2
4. ∃w.((z,w) ε f)  ExistsInt 3
5. ∃v.((z,a) ε v)  ExistsInt 3
6. Set((z,a))  DefSub 5
7. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
8. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 7
9. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 8
10. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 9
11. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 10
12. Set((z,y)) -> (Set(z) & Set(y))  ForallElim 11
13. ∀y.(Set((z,y)) -> (Set(z) & Set(y)))  ForallInt 12
14. Set((z,a)) -> (Set(z) & Set(a))  ForallElim 13
15. Set(z) & Set(a)  ImpElim 6 14
16. Set(z)  AndElimL 15
17. Set(z) & ∃w.((z,w) ε f)  AndInt 16 4
18. z ε {w: ∃x_1.((w,x_1) ε f)}  ClassInt 17
19. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
20. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 19
21. z ε domain(f)  EqualitySub 18 20
22. _|_  ImpElim 21 0
23. ¬(a ε {y: ((z,y) ε f)})  ImpInt 22
24. ∀a.¬(a ε {y: ((z,y) ε f)})  ForallInt 23
25. b ε 0  Hyp
26. 0 = {x: ¬(x = x)}  DefEqInt
27. b ε {x: ¬(x = x)}  EqualitySub 25 26
28. Set(b) & ¬(b = b)  ClassElim 27
29. ¬(b = b)  AndElimR 28
30. b = b  Identity
31. _|_  ImpElim 30 29
32. b ε {y: ((z,y) ε f)}  AbsI 31
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33. (b ε 0) -> (b ε {y: ((z,y) ε f)})  ImpInt 32
34. b ε {y: ((z,y) ε f)}  Hyp
35. ¬(b ε {y: ((z,y) ε f)})  ForallElim 24
36. _|_  ImpElim 34 35
37. b ε 0  AbsI 36
38. (b ε {y: ((z,y) ε f)}) -> (b ε 0)  ImpInt 37
39. ((b ε {y: ((z,y) ε f)}) -> (b ε 0)) & ((b ε 0) -> (b ε {y: ((z,y) ε f)}))  AndInt 38 33
40. (b ε {y: ((z,y) ε f)}) <-> (b ε 0)  EquivConst 39
41. ∀b.((b ε {y: ((z,y) ε f)}) <-> (b ε 0))  ForallInt 40
42. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
43. ∀x_2.(({y: ((z,y) ε f)} = x_2) <-> ∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε x_2))) ForallElim 42
44. ({y: ((z,y) ε f)} = 0) <-> ∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε 0))  ForallElim 43
45. (({y: ((z,y) ε f)} = 0) -> ∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε 0))) & (∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε 0)) -> ({y: ((z,y) ε f)} = 0))  EquivExp 44
46. ∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε 0)) -> ({y: ((z,y) ε f)} = 0)  AndElimR 45
47. {y: ((z,y) ε f)} = 0  ImpElim 41 46
48. (∩0 = U) & (∪0 = 0)  TheoremInt
49. ∩0 = U  AndElimL 48
50. 0 = {y: ((z,y) ε f)}  Symmetry 47
51. ∩{y: ((z,y) ε f)} = U  EqualitySub 49 50
52. (f'x) = ∩{y: ((x,y) ε f)}  DefEqInt
53. ∀x.((f'x) = ∩{y: ((x,y) ε f)})  ForallInt 52
54. (f'z) = ∩{y: ((z,y) ε f)}  ForallElim 53
55. ∩{y: ((z,y) ε f)} = (f'z)  Symmetry 54
56. (f'z) = U  EqualitySub 51 55
57. ¬(z ε domain(f)) -> ((f'z) = U)  ImpInt 56
58. z ε domain(f)  Hyp
59. z ε {x: ∃y.((x,y) ε f)}  EqualitySub 58 19
60. Set(z) & ∃y.((z,y) ε f)  ClassElim 59
61. Set(z)  AndElimL 60
62. ∃y.((z,y) ε f)  AndElimR 60
63. {a: ((z,a) ε f)} = 0  Hyp
64. (z,y) ε f  Hyp
65. ∃v.((z,y) ε v)  ExistsInt 64
66. Set((z,y))  DefSub 65
67. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
68. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 67
69. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 68
70. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 69
71. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 70
72. Set((z,y)) -> (Set(z) & Set(y))  ForallElim 71
73. Set(z) & Set(y)  ImpElim 66 72
74. Set(y)  AndElimR 73
75. Set(y) & ((z,y) ε f)  AndInt 74 64
76. y ε {w: ((z,w) ε f)}  ClassInt 75
77. y ε 0  EqualitySub 76 63
78. 0 = {x: ¬(x = x)}  DefEqInt
79. y ε {x: ¬(x = x)}  EqualitySub 77 78
80. Set(y) & ¬(y = y)  ClassElim 79
81. ¬(y = y)  AndElimR 80
82. y = y  Identity
83. _|_  ImpElim 82 81
84. ¬({a: ((z,a) ε f)} = 0)  ImpInt 83
85. ¬(x = 0) -> Set(∩x)  TheoremInt
86. ∀x.(¬(x = 0) -> Set(∩x))  ForallInt 85
87. ¬({a: ((z,a) ε f)} = 0) -> Set(∩{a: ((z,a) ε f)})  ForallElim 86
88. Set(∩{a: ((z,a) ε f)})  ImpElim 84 87
89. (f'x) = ∩{y: ((x,y) ε f)}  DefEqInt
90. ∀x.((f'x) = ∩{y: ((x,y) ε f)})  ForallInt 89
91. (f'z) = ∩{y: ((z,y) ε f)}  ForallElim 90
92. ∩{y: ((z,y) ε f)} = (f'z)  Symmetry 91
93. Set((f'z))  EqualitySub 88 92
94. (x ε U) <-> Set(x)  TheoremInt
95. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 94
96. Set(x) -> (x ε U)  AndElimR 95
97. ∀x.(Set(x) -> (x ε U))  ForallInt 96
98. Set((f'z)) -> ((f'z) ε U)  ForallElim 97
99. (f'z) ε U  ImpElim 93 98
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100. (f'z) ε U ExistsElim 62 64 99
101. (z ε domain(f)) -> ((f'z) ε U) ImpInt 100
102. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U)) AndInt 57 101
Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. (∩0 = U) & (∪0 = 0)
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
4. ¬(x = 0) -> Set(∩x)
5. (x ε U) <-> Set(x)

Th70. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

0. Function(f) Hyp
1. z ε f Hyp
2. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 0
3. Relation(f) AndElimL 2
4. ∀z.((z ε f) -> ∃x.∃y.(z = (x,y))) DefExp 3
5. (z ε f) -> ∃x.∃y.(z = (x,y)) ForallElim 4
6. ∃x.∃y.(z = (x,y)) ImpElim 1 5
7. ∃y.(z = (x,y)) Hyp
8. z = (x,y) Hyp
9. ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) AndElimR 2
10. (f'x) = ∩{y: ((x,y) ε f)} DefEqInt
11. a ε {y: ((x,y) ε f)} Hyp
12. Set(a) & ((x,a) ε f) ClassElim 11
13. (x,a) ε f AndElimR 12
14. ∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) ForallElim 9
15. ∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) ForallElim 14
16. (((x,y) ε f) & ((x,a) ε f)) -> (y = a) ForallElim 15
17. (x,y) ε f EqualitySub 1 8
18. ((x,y) ε f) & ((x,a) ε f) AndInt 17 13
19. y = a ImpElim 18 16
20. {x} = {z: ((x ε U) -> (z = x))} DefEqInt
21. ∀x.({x} = {z: ((x ε U) -> (z = x))}) ForallInt 20
22. {y} = {z: ((y ε U) -> (z = y))} ForallElim 21
23. (a ε {y: ((x,y) ε f)}) -> (y = a) ImpInt 19
24. ∃w.(z ε w) ExistsInt 1
25. Set(z) DefSub 24
26. Set((x,y)) EqualitySub 25 8
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
28. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 27
29. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 28
30. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 29
31. Set(x) & Set(y) ImpElim 26 30
32. Set(y) AndElimR 31
33. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
34. ∀y.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 33
35. Set(x) -> ((a ε {x}) <-> (a = x)) ForallElim 34
36. ∀x.(Set(x) -> ((a ε {x}) <-> (a = x))) ForallInt 35
37. Set(y) -> ((a ε {y}) <-> (a = y)) ForallElim 36
38. (a ε {y}) <-> (a = y) ImpElim 32 37
39. ((a ε {y}) -> (a = y)) & ((a = y) -> (a ε {y})) EquivExp 38
40. (a = y) -> (a ε {y}) AndElimR 39
41. a = y Symmetry 19
42. a ε {y} ImpElim 41 40
43. (a ε {y: ((x,y) ε f)}) -> (a ε {y}) ImpInt 42
44. a ε {y} Hyp
45. ((a ε {y}) -> (a = y)) & ((a = y) -> (a ε {y})) EquivExp 38
46. (a ε {y}) -> (a = y) AndElimL 45
47. a = y ImpElim 44 46
48. y = a Symmetry 47
49. (x,y) ε f EqualitySub 1 8
50. (x,a) ε f EqualitySub 49 48
51. Set(a) EqualitySub 32 48
52. Set(a) & ((x,a) ε f) AndInt 51 50
53. a ε {y: ((x,y) ε f)} ClassInt 52
54. (a ε {y}) -> (a ε {y: ((x,y) ε f)}) ImpInt 53

55. ((a ε {y: ((x,y) ε f)}) -> (a ε {y})) & ((a ε {y}) -> (a ε {y: ((x,y) ε f)})) AndInt 43 54
56. (a ε {y: ((x,y) ε f)}) <-> (a ε {y}) EquivConst 55
57. ∀a.((a ε {y: ((x,y) ε f)}) <-> (a ε {y})) ForallInt 56
58. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
59. ∀x_3.(({y: ((x,y) ε f)} = x_3) <-> ∀z.((z ε {y: ((x,y) ε f)}) <-> (z ε x_3))) ForallElim 58
60. ({x_4: ((x,x_4) ε f)} = {y}) <-> ∀z.((z ε {x_4: ((x,x_4) ε f)}) <-> (z ε {y})) ForallElim 59
61. (({x_4: ((x,x_4) ε f)} = {y}) -> ∀z.((z ε {x_4: ((x,x_4) ε f)}) <-> (z ε {y}))) & (∀z.((z ε {x_4: ((x,x_4) ε f)}) <-> (z ε {y})) -> ({x_4: ((x,x_4) ε f)} = {y})) EquivExp 60
62. ∀z.((z ε {x_4: ((x,x_4) ε f)}) <-> (z ε {y})) -> ({x_4: ((x,x_4) ε f)} = {y}) AndElimR 61
63. {x_4: ((x,x_4) ε f)} = {y} ImpElim 57 62
64. (f'x) = ∩{y} EqualitySub 10 63
65. (Set(x) -> ((∩{x} = x) & (U{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (U{x} = U))) TheoremInt
66. Set(x) -> ((∩{x} = x) & (U{x} = x)) AndElimL 65
67. ∀x.(Set(x) -> ((∩{x} = x) & (U{x} = x))) ForallInt 66
68. Set(y) -> ((∩{y} = y) & (U{y} = y)) ForallElim 67
69. (∩{y} = y) & (U{y} = y) ImpElim 32 68
70. ∩{y} = y AndElimL 69
71. (f'x) = y EqualitySub 64 70
72. (z = (x,y)) & ((f'x) = y) AndInt 8 71
73. ∃y.((z = (x,y)) & ((f'x) = y)) ExistsInt 72
74. ∃x.∃y.((z = (x,y)) & ((f'x) = y)) ExistsInt 73
75. Set(z) & ∃x.∃y.((z = (x,y)) & ((f'x) = y)) AndInt 25 74
76. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ClassInt 75
77. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ExistsElim 7 8 76
78. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ExistsElim 6 7 77
79. (z ε f) -> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) ImpInt 78
80. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} Hyp
81. Set(z) & ∃x.∃y.((z = (x,y)) & ((f'x) = y)) ClassElim 80
82. Set(z) AndElimL 81
83. ∃x.∃y.((z = (x,y)) & ((f'x) = y)) AndElimR 81
84. ∃y.((z = (x,y)) & ((f'x) = y)) Hyp
85. (z = (x,y)) & ((f'x) = y) Hyp
86. z = (x,y) AndElimL 85
87. (f'x) = y AndElimR 85
88. ∩{y: ((x,y) ε f)} = y EqualitySub 87 10
89. Set((x,y)) EqualitySub 82 86
90. Set(x) & Set(y) ImpElim 89 30
91. Set(y) AndElimR 90
92. y = (f'x) Symmetry 87
93. Set((f'x)) EqualitySub 91 92
94. (f'x) = U Hyp
95. ¬Set(U) TheoremInt
96. Set(U) EqualitySub 93 94
97. _|_ ImpElim 96 95
98. ¬((f'x) = U) ImpInt 97
99. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U)) TheoremInt
100. ¬(z ε domain(f)) -> ((f'z) = U) AndElimL 99
101. (A -> B) -> (¬B -> ¬A) TheoremInt
102. (¬(z ε domain(f)) -> B) -> (¬B -> ¬¬(z ε domain(f))) PolySub 101
103. (¬(z ε domain(f)) -> ((f'z) = U)) -> (¬((f'z) = U) -> ¬¬(z ε domain(f))) PolySub 102
104. ¬((f'z) = U) -> ¬¬(z ε domain(f)) ImpElim 100 103
105. D <-> ¬¬D TheoremInt
106. (D -> ¬¬D) & (¬¬D -> D) EquivExp 105
107. ¬¬D -> D AndElimR 106
108. ¬¬(z ε domain(f)) -> (z ε domain(f)) PolySub 107
109. ¬((f'z) = U) Hyp
110. ¬¬(z ε domain(f)) ImpElim 109 104
111. z ε domain(f) ImpElim 110 108
112. ¬((f'z) = U) -> (z ε domain(f)) ImpInt 111
113. ∀z.(¬((f'z) = U) -> (z ε domain(f))) ForallInt 112
114. ¬((f'x) = U) -> (x ε domain(f)) ForallElim 113
115. x ε domain(f) ImpElim 98 114
116. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
117. x ε {x: ∃y.((x,y) ε f)} EqualitySub 115 116

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118. Set(x) & ∃y.((x,y) ε f)  ClassElim 117
119. ∃y.((x,y) ε f)  AndElimR 118
120. (x,b) ε f  Hyp
121. e ε {b}  Hyp
122. ∃w.((x,b) ε w)  ExistsInt 120
123. Set((x,b))  DefSub 122
124. ∀y.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 30
125. Set((x,b)) -> (Set(x) & Set(b))  ForallElim 124
126. Set(x) & Set(b)  ImpElim 123 125
127. Set(b)  AndElimR 126
128. Set(x) -> ((y ε {x}) <-> (y = x))  TheoremInt
129. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 128
130. Set(b) -> ((y ε {b}) <-> (y = b))  ForallElim 129
131. (y ε {b}) <-> (y = b)  ImpElim 127 130
132. ∀y.((y ε {b}) <-> (y = b))  ForallInt 131
133. (e ε {b}) <-> (e = b)  ForallElim 132
134. ((e ε {b}) -> (e = b)) & ((e = b) -> (e ε {b}))  EquivExp 133
135. (e ε {b}) -> (e = b)  AndElimL 134
136. e = b  ImpElim 121 135
137. b = e  Symmetry 136
138. (x,e) ε f  EqualitySub 120 137
139. Set(e)  EqualitySub 127 137
140. Set(e) & ((x,e) ε f)  AndInt 139 138
141. e ε {y: ((x,y) ε f)}  ClassInt 140
142. e ε {y: ((x,y) ε f)}  Hyp
143. Set(e) & ((x,e) ε f)  ClassElim 142
144. (x,e) ε f  AndElimR 143
145. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  DefExp 0
146. ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  AndElimR 145
147. (e ε {b}) -> (e ε {y: ((x,y) ε f)})  ImpInt 141
148. ((x,b) ε f) & ((x,e) ε f)  AndInt 120 144
149. ∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  ForallElim 146
150. ∀z.((((x,b) ε f) & ((x,z) ε f)) -> (b = z))  ForallElim 149
151. (((x,b) ε f) & ((x,e) ε f)) -> (b = e)  ForallElim 150
152. b = e  ImpElim 148 151
153. ((y ε {b}) -> (y = b)) & ((y = b) -> (y ε {b}))  EquivExp 131
154. ((e ε {b}) -> (e = b)) & ((e = b) -> (e ε {b}))  EquivExp 133
155. (e = b) -> (e ε {b})  AndElimR 154
156. e = b  Symmetry 152
157. e ε {b}  ImpElim 156 155
158. (e ε {y: ((x,y) ε f)}) -> (e ε {b})  ImpInt 157
159. ((e ε {b}) -> (e ε {y: ((x,y) ε f)})) & ((e ε {y: ((x,y) ε f)}) -> (e ε {b}))
AndInt 147 158
160. (e ε {b}) <-> (e ε {y: ((x,y) ε f)})  EquivConst 159
161. ∀e.((e ε {b}) <-> (e ε {y: ((x,y) ε f)}))  ForallInt 160
162. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
163. ∀y.(({b} = y) <-> ∀z.((z ε {b}) <-> (z ε y)))  ForallElim 162
164. ({b} = {y: ((x,y) ε f)}) <-> ∀z.((z ε {b}) <-> (z ε {y: ((x,y) ε f)}))  ForallElim
163
165. (({b} = {y: ((x,y) ε f)}) -> ∀z.((z ε {b}) <-> (z ε {y: ((x,y) ε f)}))) & (∀z.((z ε
{b}) <-> (z ε {y: ((x,y) ε f)})) -> ({b} = {y: ((x,y) ε f)}))  EquivExp 164
166. ∀z.((z ε {b}) <-> (z ε {y: ((x,y) ε f)})) -> ({b} = {y: ((x,y) ε f)})  AndElimR 165
167. {b} = {y: ((x,y) ε f)}  ImpElim 161 166
168. {y: ((x,y) ε f)} = {b}  Symmetry 167
169. ∩{b} = y  EqualitySub 88 168
170. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))
TheoremInt
171. Set(x) -> ((∩{x} = x) & (∪{x} = x))  AndElimL 170
172. ∀x.(Set(x) -> ((∩{x} = x) & (∪{x} = x)))  ForallInt 171
173. Set(b) -> ((∩{b} = b) & (∪{b} = b))  ForallElim 172
174. (∩{b} = b) & (∪{b} = b)  ImpElim 127 173
175. ∩{b} = b  AndElimL 174
176. b = y  EqualitySub 169 175
177. (x,y) ε f  EqualitySub 120 176
178. (x,y) ε f  EqualitySub 120 176
179. (x,y) = z  Symmetry 86
180. z ε f  EqualitySub 178 179
181. x = x  Identity
182. z ε f  ExistsElim 119 120 180
183. z ε f  ExistsElim 84 85 182
184. z ε f  ExistsElim 83 84 183
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185. (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) -> (z ε f) ImpInt 184
186. ((z ε f) -> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) & ((z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) -> (z ε f)) AndInt 79 185
187. (z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) EquivConst 186
188. ∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) ForallInt 187
189. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
190. ∀y.((f = y) <-> ∀z.((z ε f) <-> (z ε y))) ForallElim 189
191. (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) <-> ∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) ForallElim 190
192. ((f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) -> ∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}))) & (∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) EquivExp 191
193. ∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) AndElimR 192
194. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 188 193
195. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) ImpInt 194 Qed

Used Theorems

2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. Set(x) -> ((y ε {x}) <-> (y = x))
4. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))
5. ¬Set(U)
6. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U))
7. (A -> B) -> (¬B -> ¬A)
8. D <-> ¬¬D
3. Set(x) -> ((y ε {x}) <-> (y = x))
4. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))

Th71. (Function(f) & Function(g)) -> ((f = g) <-> ∀z.((f'z) = (g'z)))

0. Function(f) & Function(g) Hyp
1. ∀z.((f'z) = (g'z)) Hyp
2. e ε f Hyp
3. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
4. Function(f) AndElimL 0
5. Function(g) AndElimR 0
6. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 4 3
7. e ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 2 6
8. Set(e) & ∃x.∃y.((e = (x,y)) & ((f'x) = y)) ClassElim 7
9. Set(e) AndElimL 8
10. ∃x.∃y.((e = (x,y)) & ((f'x) = y)) AndElimR 8
11. ∃y.((e = (x,y)) & ((f'x) = y)) Hyp
12. (e = (x,y)) & ((f'x) = y) Hyp
13. (f'x) = (g'x) ForallElim 1
14. (e = (x,y)) & ((g'x) = y) EqualitySub 12 13
15. ∃y.((e = (x,y)) & ((g'x) = y)) ExistsInt 14
16. ∃x.∃y.((e = (x,y)) & ((g'x) = y)) ExistsInt 15
17. Set(e) & ∃x.∃y.((e = (x,y)) & ((g'x) = y)) AndInt 9 16
18. e ε {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))} ClassInt 17
19. ∀f.(Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) ForallInt 3
20. Function(g) -> (g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}) ForallElim 19
21. g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))} ImpElim 5 20
22. {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))} = g Symmetry 21
23. e ε g EqualitySub 18 22
24. e ε g ExistsElim 11 12 23
25. e ε g ExistsElim 10 11 24
26. (e ε f) -> (e ε g) ImpInt 25
27. e ε g Hyp
28. e ε {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))} EqualitySub 27 21
29. Set(e) & ∃x.∃y.((e = (x,y)) & ((g'x) = y)) ClassElim 28
30. Set(e) AndElimL 29
31. ∃x.∃y.((e = (x,y)) & ((g'x) = y)) AndElimR 29
32. ∃y.((e = (x,y)) & ((g'x) = y)) Hyp
33. (e = (x,y)) & ((g'x) = y) Hyp
34. (g'x) = (f'x) Symmetry 13
35. (e = (x,y)) & ((f'x) = y) EqualitySub 33 34
36. ∃y.((e = (x,y)) & ((f'x) = y)) ExistsInt 35
37. ∃x.∃y.((e = (x,y)) & ((f'x) = y)) ExistsInt 36
38. Set(e) & ∃x.∃y.((e = (x,y)) & ((f'x) = y)) AndInt 30 37
39. e ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ClassInt 38

40. {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} = f Symmetry 6
41. e ε f EqualitySub 39 40
42. e ε f ExistsElim 32 33 41
43. e ε f ExistsElim 31 32 42
44. (e ε g) -> (e ε f) ImpInt 43
45. ((e ε f) -> (e ε g)) & ((e ε g) -> (e ε f)) AndInt 26 44
46. (e ε f) <-> (e ε g) EquivConst 45
47. ∀e.((e ε f) <-> (e ε g)) ForallInt 46
48. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
49. ∀y.((f = y) <-> ∀z.((z ε f) <-> (z ε y))) ForallElim 48
50. (f = g) <-> ∀z.((z ε f) <-> (z ε g)) ForallElim 49
51. ((f = g) -> ∀z.((z ε f) <-> (z ε g))) & (∀z.((z ε f) <-> (z ε g)) -> (f = g)) EquivExp 50
52. ∀z.((z ε f) <-> (z ε g)) -> (f = g) AndElimR 51
53. f = g ImpElim 47 52
54. ∀z.((f'z) = (g'z)) -> (f = g) ImpInt 53
55. f = g Hyp
56. (f'z) = (f'z) Identity
57. (f'z) = (g'z) EqualitySub 56 55
58. ∀z.((f'z) = (g'z)) ForallInt 57
59. (f = g) -> ∀z.((f'z) = (g'z)) ImpInt 58
60. ((f = g) -> ∀z.((f'z) = (g'z))) & (∀z.((f'z) = (g'z)) -> (f = g)) AndInt 59 54
61. (f = g) <-> ∀z.((f'z) = (g'z)) EquivConst 60
62. (Function(f) & Function(g)) -> ((f = g) <-> ∀z.((f'z) = (g'z))) ImpInt 61 Qed

Used Theorems

1. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

Th73. (Set(u) & Set(y)) -> Set(({u} X y))

0. Set(u) & Set(y) Hyp
1. f = {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} Hyp
2. x ε domain(f) Hyp
3. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
4. x ε {x: ∃y.((x,y) ε f)} EqualitySub 2 3
5. Set(x) & ∃y.((x,y) ε f) ClassElim 4
6. Set(x) & ∃x_0.((x,x_0) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}) EqualitySub 5 1
7. Set(x) AndElimL 6
8. ∃x_0.((x,x_0) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}) AndElimR 6
9. (x,c) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} Hyp
10. Set((x,c)) & ∃w.∃z.(((x,c) = (w,z)) & ((w ε y) & (z = (u,w)))) ClassElim 9
11. Set((x,c)) AndElimL 10
12. ∃w.∃z.(((x,c) = (w,z)) & ((w ε y) & (z = (u,w)))) AndElimR 10
13. ∃z.(((x,c) = (w,z)) & ((w ε y) & (z = (u,w)))) Hyp
14. ((x,c) = (w,z)) & ((w ε y) & (z = (u,w))) Hyp
15. (x,c) = (w,z) AndElimL 14
16. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
17. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 16
18. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 17
19. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 18
20. ∀y.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 19
21. Set((x,c)) -> (Set(x) & Set(c)) ForallElim 20
22. Set(x) & Set(c) ImpElim 11 21
23. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
24. ∀y.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 23
25. ((Set(x) & Set(c)) & ((x,c) = (u,v))) -> ((x = u) & (c = v)) ForallElim 24
26. ∀u.(((Set(x) & Set(c)) & ((x,c) = (u,v))) -> ((x = u) & (c = v))) ForallInt 25
27. ((Set(x) & Set(c)) & ((x,c) = (w,v))) -> ((x = w) & (c = v)) ForallElim 26
28. ∀v.(((Set(x) & Set(c)) & ((x,c) = (w,v))) -> ((x = w) & (c = v))) ForallInt 27
29. ((Set(x) & Set(c)) & ((x,c) = (w,z))) -> ((x = w) & (c = z)) ForallElim 28
30. (Set(x) & Set(c)) & ((x,c) = (w,z)) AndInt 22 15
31. (x = w) & (c = z) ImpElim 30 29
32. x = w AndElimL 31
33. (w ε y) & (z = (u,w)) AndElimR 14
34. w ε y AndElimL 33
35. w = x Symmetry 32
36. x ε y EqualitySub 34 35
37. x ε y ExistsElim 13 14 36
38. x ε y ExistsElim 12 13 37

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39. x ε y  ExistsElim 8 9 38
40. (x ε domain(f)) -> (x ε y)  ImpInt 39
41. x ε y  Hyp
42. z = (u,x)  Hyp
43. a = (x,z)  Hyp
44. (a = (x,z)) & (z = (u,x))  AndInt 43 42
45. ∃z.((a = (x,z)) & (z = (u,x)))  ExistsInt 44
46. ∃x.∃z.((a = (x,z)) & (z = (u,x)))  ExistsInt 45
47. ∃y.(x ε y)  ExistsInt 41
48. Set(x)  DefSub 47
49. Set(u)  AndElimL 0
50. Set(u) & Set(x)  AndInt 49 48
51. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 17
52. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 51
53. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 52
54. (Set(u) & Set(y)) -> Set((u,y))  ForallElim 53
55. ∀y.((Set(u) & Set(y)) -> Set((u,y)))  ForallInt 54
56. (Set(u) & Set(x)) -> Set((u,x))  ForallElim 55
57. Set((u,x))  ImpElim 50 56
58. (u,x) = z  Symmetry 42
59. Set(z)  EqualitySub 57 58
60. Set(x) & Set(z)  AndInt 48 59
61. ∀y.(((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))))  ForallInt
51
62. ∀y.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 52
63. (Set(x) & Set(z)) -> Set((x,z))  ForallElim 62
64. Set((x,z))  ImpElim 60 63
65. (x,z) = a  Symmetry 43
66. Set(a)  EqualitySub 64 65
67. Set(a) & ∃x.∃z.((a = (x,z)) & (z = (u,x)))  AndInt 66 46
68. {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} = f  Symmetry 1
69. a ε {a: ∃x.∃z.((a = (x,z)) & (z = (u,x)))}  ClassInt 67
70. (x ε y) & (z = (u,x))  AndInt 41 42
71. (a = (x,z)) & ((x ε y) & (z = (u,x)))  AndInt 43 70
72. ∃z.((a = (x,z)) & ((x ε y) & (z = (u,x))))  ExistsInt 71
73. ∃x.∃z.((a = (x,z)) & ((x ε y) & (z = (u,x))))  ExistsInt 72
74. Set(a) & ∃x.∃z.((a = (x,z)) & ((x ε y) & (z = (u,x))))  AndInt 66 73
75. a ε {a: ∃x.∃z.((a = (x,z)) & ((x ε y) & (z = (u,x))))}  ClassInt 74
76. a ε f  EqualitySub 75 68
77. (x,z) ε f  EqualitySub 76 43
78. ∃z.((x,z) ε f)  ExistsInt 77
79. Set(x) & ∃z.((x,z) ε f)  AndInt 48 78
80. x ε {w: ∃z.((w,z) ε f)}  ClassInt 79
81. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 3
82. x ε domain(f)  EqualitySub 80 81
83. (a = (x,z)) -> (x ε domain(f))  ImpInt 82
84. ∀a.((a = (x,z)) -> (x ε domain(f)))  ForallInt 83
85. ((x,z) = (x,z)) -> (x ε domain(f))  ForallElim 84
86. (x,z) = (x,z)  Identity
87. x ε domain(f)  ImpElim 86 85
88. (z = (u,x)) -> (x ε domain(f))  ImpInt 87
89. ∀z.((z = (u,x)) -> (x ε domain(f)))  ForallInt 88
90. ((u,x) = (u,x)) -> (x ε domain(f))  ForallElim 89
91. (u,x) = (u,x)  Identity
92. x ε domain(f)  ImpElim 91 90
93. (x ε y) -> (x ε domain(f))  ImpInt 92
94. ((x ε domain(f)) -> (x ε y)) & ((x ε y) -> (x ε domain(f)))  AndInt 40 93
95. (x ε domain(f)) <-> (x ε y)  EquivConst 94
96. ∀x.((x ε domain(f)) <-> (x ε y))  ForallInt 95
97. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
98. ∀y.((domain(f) = y) <-> ∀z.((z ε domain(f)) <-> (z ε y)))  ForallElim 97
99. (domain(f) = y) <-> ∀z.((z ε domain(f)) <-> (z ε y))  ForallElim 98
100. ((domain(f) = y) -> ∀z.((z ε domain(f)) <-> (z ε y))) & (∀z.((z ε domain(f)) <-> (z
ε y)) -> (domain(f) = y))  EquivExp 99
101. ∀z.((z ε domain(f)) <-> (z ε y)) -> (domain(f) = y)  AndElimR 100
102. domain(f) = y  ImpElim 96 101
103. x ε range(f)  Hyp
104. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
105. x ε {y: ∃x.((x,y) ε f)}  EqualitySub 103 104
106. Set(x) & ∃x_4.((x_4,x) ε f)  ClassElim 105
107. ∃x_4.((x_4,x) ε f)  AndElimR 106
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108. ∃x_4.((x_4,x) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}) EqualitySub 107 1
109. (c,x) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} Hyp
110. Set((c,x)) & ∃w.∃z.(((c,x) = (w,z)) & ((w ε y) & (z = (u,w)))) ClassElim 109
111. ∃w.∃z.(((c,x) = (w,z)) & ((w ε y) & (z = (u,w)))) AndElimR 110
112. ∃z.(((c,x) = (w,z)) & ((w ε y) & (z = (u,w)))) Hyp
113. ((c,x) = (w,z)) & ((w ε y) & (z = (u,w))) Hyp
114. Set((c,x)) AndElimL 110
115. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 19
116. Set((c,y)) -> (Set(c) & Set(y)) ForallElim 115
117. ∀y.(Set((c,y)) -> (Set(c) & Set(y))) ForallInt 116
118. Set((c,x)) -> (Set(c) & Set(x)) ForallElim 117
119. Set(c) & Set(x) ImpElim 114 118
120. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 23
121. ((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v)) ForallElim 120
122. ∀y.(((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v))) ForallInt 121
123. ((Set(c) & Set(x)) & ((c,x) = (u,v))) -> ((c = u) & (x = v)) ForallElim 122
124. ∀u.(((Set(c) & Set(x)) & ((c,x) = (u,v))) -> ((c = u) & (x = v))) ForallInt 123
125. ((Set(c) & Set(x)) & ((c,x) = (w,v))) -> ((c = w) & (x = v)) ForallElim 124
126. ∀v.(((Set(c) & Set(x)) & ((c,x) = (w,v))) -> ((c = w) & (x = v))) ForallInt 125
127. ((Set(c) & Set(x)) & ((c,x) = (w,z))) -> ((c = w) & (x = z)) ForallElim 126
128. (c,x) = (w,z) AndElimL 113
129. (Set(c) & Set(x)) & ((c,x) = (w,z)) AndInt 119 128
130. (c = w) & (x = z) ImpElim 129 127
131. (w ε y) & (z = (u,w)) AndElimR 113
132. w ε y AndElimL 131
133. z = (u,w) AndElimR 131
134. x = z AndElimR 130
135. z = x Symmetry 134
136. x = (u,w) EqualitySub 133 135
137. Set(c) AndElimL 119
138. c = w AndElimL 130
139. Set(w) EqualitySub 137 138
140. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
141. Set(u) AndElimL 0
142. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 140
143. Set(u) -> ((y ε {u}) <-> (y = u)) ForallElim 142
144. ∀y.(Set(u) -> ((y ε {u}) <-> (y = u))) ForallInt 143
145. Set(u) -> ((u ε {u}) <-> (u = u)) ForallElim 144
146. (u ε {u}) <-> (u = u) ImpElim 141 145
147. ((u ε {u}) -> (u = u)) & ((u = u) -> (u ε {u})) EquivExp 146
148. (u = u) -> (u ε {u}) AndElimR 147
149. u = u Identity
150. u ε {u} ImpElim 149 148
151. (u ε {u}) & (w ε y) AndInt 150 132
152. (x = (u,w)) & ((u ε {u}) & (w ε y)) AndInt 136 151
153. Set(x) AndElimR 119
154. ∃w.((x = (u,w)) & ((u ε {u}) & (w ε y))) ExistsInt 152
155. ∃b.∃w.((x = (b,w)) & ((b ε {u}) & (w ε y))) ExistsInt 154
156. Set(x) & ∃b.∃w.((x = (b,w)) & ((b ε {u}) & (w ε y))) AndInt 153 155
157. x ε {e: ∃b.∃w.((e = (b,w)) & ((b ε {u}) & (w ε y)))} ClassInt 156
158. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} DefEqInt
159. ∀x.((x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}) ForallInt 158
160. ({u} X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))} ForallElim 159
161. {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))} = ({u} X y) Symmetry 160
162. x ε ({u} X y) EqualitySub 157 161
163. x ε ({u} X y) ExistsElim 112 113 162
164. x ε ({u} X y) Hyp
165. x ε {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))} EqualitySub 164 160
166. Set(x) & ∃a.∃b.((x = (a,b)) & ((a ε {u}) & (b ε y))) ClassElim 165
167. ∃a.∃b.((x = (a,b)) & ((a ε {u}) & (b ε y))) AndElimR 166
168. x ε ({u} X y) ExistsElim 111 112 163
169. x ε ({u} X y) ExistsElim 108 109 168
170. (x ε range(f)) -> (x ε ({u} X y)) ImpInt 169
171. ∃b.((x = (a,b)) & ((a ε {u}) & (b ε y))) Hyp
172. (x = (a,b)) & ((a ε {u}) & (b ε y)) Hyp
173. x = (a,b) AndElimL 172
174. (a ε {u}) & (b ε y) AndElimR 172
175. a ε {u} AndElimL 174
176. b ε y AndElimR 174
177. ∀y.(Set(u) -> ((y ε {u}) <-> (y = u))) ForallInt 143

178. Set(u) -> ((a ε {u}) <-> (a = u)) ForallElim 177
179. Set(u) AndElimL 0
180. (a ε {u}) <-> (a = u) ImpElim 179 178
181. ((a ε {u}) -> (a = u)) & ((a = u) -> (a ε {u})) EquivExp 180
182. (a ε {u}) -> (a = u) AndElimL 181
183. a = u ImpElim 175 182
184. x = (u,b) EqualitySub 173 183
185. c = (b,x) Hyp
186. (b ε y) & (x = (u,b)) AndInt 176 184
187. (c = (b,x)) & ((b ε y) & (x = (u,b))) AndInt 185 186
188. ∃x.((c = (b,x)) & ((b ε y) & (x = (u,b)))) ExistsInt 187
189. ∃b.∃x.((c = (b,x)) & ((b ε y) & (x = (u,b)))) ExistsInt 188
190. Set(x) AndElimL 166
191. ∃y.(b ε y) ExistsInt 176
192. Set(b) DefSub 191
193. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 52
194. (Set(b) & Set(y)) -> Set((b,y)) ForallElim 193
195. ∀y.((Set(b) & Set(y)) -> Set((b,y))) ForallInt 194
196. (Set(b) & Set(x)) -> Set((b,x)) ForallElim 195
197. Set(b) & Set(x) AndInt 192 190
198. Set((b,x)) ImpElim 197 196
199. (b,x) = c Symmetry 185
200. Set(c) EqualitySub 198 199
201. Set(c) & ∃b.∃x.((c = (b,x)) & ((b ε y) & (x = (u,b)))) AndInt 200 189
202. c ε {w: ∃b.∃x.((w = (b,x)) & ((b ε y) & (x = (u,b))))} ClassInt 201
203. {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} = f Symmetry 1
204. c ε f EqualitySub 202 203
205. (b,x) ε f EqualitySub 204 185
206. ∃b.((b,x) ε f) ExistsInt 205
207. Set(x) & ∃b.((b,x) ε f) AndInt 190 206
208. x ε {w: ∃b.((b,w) ε f)} ClassInt 207
209. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 104
210. x ε range(f) EqualitySub 208 209
211. (c = (b,x)) -> (x ε range(f)) ImpInt 210
212. ∀c.((c = (b,x)) -> (x ε range(f))) ForallInt 211
213. ((b,x) = (b,x)) -> (x ε range(f)) ForallElim 212
214. (b,x) = (b,x) Identity
215. x ε range(f) ImpElim 214 213
216. x ε range(f) ExistsElim 171 172 215
217. x ε range(f) ExistsElim 167 171 216
218. (x ε ({u} X y)) -> (x ε range(f)) ImpInt 217
219. ((x ε range(f)) -> (x ε ({u} X y))) & ((x ε ({u} X y)) -> (x ε range(f))) AndInt 170 218
220. (x ε range(f)) <-> (x ε ({u} X y)) EquivConst 219
221. ∀x.((x ε range(f)) <-> (x ε ({u} X y))) ForallInt 220
222. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
223. ∀y.((range(f) = y) <-> ∀z.((z ε range(f)) <-> (z ε y))) ForallElim 222
224. (range(f) = ({u} X y)) <-> ∀z.((z ε range(f)) <-> (z ε ({u} X y))) ForallElim 223
225. ((range(f) = ({u} X y)) -> ∀z.((z ε range(f)) <-> (z ε ({u} X y)))) & (∀z.((z ε range(f)) <-> (z ε ({u} X y))) -> (range(f) = ({u} X y))) EquivExp 224
226. ∀z.((z ε range(f)) <-> (z ε ({u} X y))) -> (range(f) = ({u} X y)) AndElimR 225
227. range(f) = ({u} X y) ImpElim 221 226
228. (Function(f) & Set(domain(f))) -> Set(range(f)) AxInt
229. Set(y) AndElimR 0
230. y = domain(f) Symmetry 102
231. Set(domain(f)) EqualitySub 229 230
232. x ε f Hyp
233. x ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} EqualitySub 232 1
234. Set(x) & ∃w.∃z.((x = (w,z)) & ((w ε y) & (z = (u,w)))) ClassElim 233
235. ∃w.∃z.((x = (w,z)) & ((w ε y) & (z = (u,w)))) AndElimR 234
236. ∃z.((x = (w,z)) & ((w ε y) & (z = (u,w)))) Hyp
237. (x = (w,z)) & ((w ε y) & (z = (u,w))) Hyp
238. x = (w,z) AndElimL 237
239. ∃z.(x = (w,z)) ExistsInt 238
240. ∃w.∃z.(x = (w,z)) ExistsInt 239
241. ∃w.∃z.(x = (w,z)) ExistsElim 236 237 240
242. ∃w.∃z.(x = (w,z)) ExistsElim 235 236 241
243. (x ε f) -> ∃w.∃z.(x = (w,z)) ImpInt 242
244. ∀x.((x ε f) -> ∃w.∃z.(x = (w,z))) ForallInt 243
245. Relation(f) DefSub 244
246. (a,b) ε f Hyp

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247. (a,c) ε f  Hyp
248. (a,b) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}  EqualitySub 246 1
249. (a,c) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}  EqualitySub 247 1
250. Set((a,b)) & ∃w.∃z.(((a,b) = (w,z)) & ((w ε y) & (z = (u,w))))  ClassElim 248
251. Set((a,c)) & ∃w.∃z.(((a,c) = (w,z)) & ((w ε y) & (z = (u,w))))  ClassElim 249
252. ∃w.∃z.(((a,b) = (w,z)) & ((w ε y) & (z = (u,w))))  AndElimR 250
253. ∃w.∃z.(((a,c) = (w,z)) & ((w ε y) & (z = (u,w))))  AndElimR 251
254. ∃z.(((a,b) = (x1,z)) & ((x1 ε y) & (z = (u,x1))))  Hyp
255. ((a,b) = (x1,y1)) & ((x1 ε y) & (y1 = (u,x1)))  Hyp
256. ∃z.(((a,c) = (x2,z)) & ((x2 ε y) & (z = (u,x2))))  Hyp
257. ((a,c) = (x2,y2)) & ((x2 ε y) & (y2 = (u,x2)))  Hyp
258. (a,b) = (x1,y1)  AndElimL 255
259. (a,c) = (x2,y2)  AndElimL 257
260. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
261. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 260
262. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 261
263. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 262
264. Set((a,b))  AndElimL 250
265. Set((a,c))  AndElimL 251
266. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 263
267. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 266
268. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 267
269. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 268
270. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 267
271. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 270
272. Set(a) & Set(b)  ImpElim 264 269
273. Set(a) & Set(c)  ImpElim 265 271
274. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
275. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 274
276. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))  ForallElim 275
277. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 276
278. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))  ForallElim 277
279. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)))  ForallInt 278
280. ((Set(a) & Set(b)) & ((a,b) = (x1,v))) -> ((a = x1) & (b = v))  ForallElim 279
281. ∀v.(((Set(a) & Set(b)) & ((a,b) = (x1,v))) -> ((a = x1) & (b = v)))  ForallInt 280
282. ((Set(a) & Set(b)) & ((a,b) = (x1,y1))) -> ((a = x1) & (b = y1))  ForallElim 281
283. (Set(a) & Set(b)) & ((a,b) = (x1,y1))  AndInt 272 258
284. (a = x1) & (b = y1)  ImpElim 283 282
285. (Set(a) & Set(c)) & ((a,c) = (x2,y2))  AndInt 273 259
286. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 276
287. ((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v))  ForallElim 286
288. ∀u.(((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v)))  ForallInt 287
289. ((Set(a) & Set(c)) & ((a,c) = (x2,v))) -> ((a = x2) & (c = v))  ForallElim 288
290. ∀v.(((Set(a) & Set(c)) & ((a,c) = (x2,v))) -> ((a = x2) & (c = v)))  ForallInt 289
291. ((Set(a) & Set(c)) & ((a,c) = (x2,y2))) -> ((a = x2) & (c = y2))  ForallElim 290
292. (a = x2) & (c = y2)  ImpElim 285 291
293. (x1 ε y) & (y1 = (u,x1))  AndElimR 255
294. (x2 ε y) & (y2 = (u,x2))  AndElimR 257
295. a = x1  AndElimL 284
296. a = x2  AndElimL 292
297. x1 = x2  EqualitySub 296 295
298. y1 = (u,x1)  AndElimR 293
299. y2 = (u,x2)  AndElimR 294
300. x2 = x1  Symmetry 297
301. y2 = (u,x1)  EqualitySub 299 300
302. (u,x1) = y2  Symmetry 301
303. y1 = y2  EqualitySub 298 302
304. (a,b) = (x2,y1)  EqualitySub 258 297
305. (a,b) = (x2,y2)  EqualitySub 304 303
306. (x2,y2) = (a,c)  Symmetry 259
307. (a,b) = (a,c)  EqualitySub 305 306
308. (Set(a) & Set(b)) & ((a,b) = (a,c))  AndInt 272 307
309. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)))  ForallInt 278
310. ((Set(a) & Set(b)) & ((a,b) = (a,v))) -> ((a = a) & (b = v))  ForallElim 309
311. ∀v.(((Set(a) & Set(b)) & ((a,b) = (a,v))) -> ((a = a) & (b = v)))  ForallInt 310
312. ((Set(a) & Set(b)) & ((a,b) = (a,c))) -> ((a = a) & (b = c))  ForallElim 311
313. (a = a) & (b = c)  ImpElim 308 312
314. b = c  AndElimR 313
315. b = c  ExistsElim 256 257 314
316. b = c  ExistsElim 253 256 315
317. b = c  ExistsElim 254 255 316
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318. b = c ExistsElim 252 254 317
319. ((a,c) ε f) -> (b = c) ImpInt 318
320. ((a,b) ε f) -> (((a,c) ε f) -> (b = c)) ImpInt 319
321. A -> (B -> C) Hyp
322. A & B Hyp
323. A AndElimL 322
324. B -> C ImpElim 323 321
325. B AndElimR 322
326. C ImpElim 325 324
327. (A & B) -> C ImpInt 326
328. (A -> (B -> C)) -> ((A & B) -> C) ImpInt 327
329. (((a,b) ε f) -> (B -> C)) -> ((((a,b) ε f) & B) -> C) PolySub 328
330. (((a,b) ε f) -> (((a,c) ε f) -> C)) -> ((((a,b) ε f) & ((a,c) ε f)) -> C) PolySub 329
331. (((a,b) ε f) -> (((a,c) ε f) -> (b = c))) -> ((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) PolySub 330
332. (((a,b) ε f) & ((a,c) ε f)) -> (b = c) ImpElim 320 331
333. ∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 332
334. ∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 333
335. ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 334
336. Relation(f) & ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) AndInt 245 335
337. Function(f) DefSub 336
338. Function(f) & Set(domain(f)) AndInt 337 231
339. (Function(f) & Set(domain(f))) -> Set(range(f)) AxInt
340. Set(range(f)) ImpElim 338 339
341. Set(({u} X y)) EqualitySub 340 227
342. (f = {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}) -> Set(({u} X y)) ImpInt 341
343. ∀f.((f = {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}) -> Set(({u} X y))) ForallInt 342
344. ({a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} = {x_8: ∃x_9.∃x_10.((x_8 = (x_9,x_10)) & ((x_9 ε y) & (x_10 = (u,x_9))))}) -> Set(({u} X y)) ForallElim 343
345. {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} = {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} Identity
346. Set(({u} X y)) ImpElim 345 344
347. (Set(u) & Set(y)) -> Set(({u} X y)) ImpInt 346 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
3. Set(x) -> ((y ε {x}) <-> (y = x))
1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th74. (Set(x) & Set(y)) -> Set((x X y))

0. f = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} Hyp
1. c ε f Hyp
2. c ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} EqualitySub 1 0
3. Set(c) & ∃u.∃z.((c = (u,z)) & ((u ε x) & (z = ({u} X y)))) ClassElim 2
4. ∃u.∃z.((c = (u,z)) & ((u ε x) & (z = ({u} X y)))) AndElimR 3
5. ∃z.((c = (u,z)) & ((u ε x) & (z = ({u} X y)))) Hyp
6. (c = (u,z)) & ((u ε x) & (z = ({u} X y))) Hyp
7. c = (u,z) AndElimL 6
8. ∃z.(c = (u,z)) ExistsInt 7
9. ∃u.∃z.(c = (u,z)) ExistsInt 8
10. ∃u.∃z.(c = (u,z)) ExistsElim 5 6 9
11. ∃u.∃z.(c = (u,z)) ExistsElim 4 5 10
12. (c ε f) -> ∃u.∃z.(c = (u,z)) ImpInt 11
13. ∀c.((c ε f) -> ∃u.∃z.(c = (u,z))) ForallInt 12
14. Relation(f) DefSub 13
15. ((a,b) ε f) & ((a,c) ε f) Hyp
16. (a,b) ε f AndElimL 15
17. (a,c) ε f AndElimR 15
18. (a,b) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} EqualitySub 16 0
19. (a,c) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} EqualitySub 17 0
20. Set((a,b)) & ∃u.∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y)))) ClassElim 18
21. Set((a,c)) & ∃u.∃z.(((a,c) = (u,z)) & ((u ε x) & (z = ({u} X y)))) ClassElim 19
22. ∃u.∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y)))) AndElimR 20
23. ∃u.∃z.(((a,c) = (u,z)) & ((u ε x) & (z = ({u} X y)))) AndElimR 21

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24. ∃z.(((a,b) = (x1,z)) & ((x1 ε x) & (z = ({x1} X y))))  Hyp
25. ((a,b) = (x1,y1)) & ((x1 ε x) & (y1 = ({x1} X y)))  Hyp
26. ∃z.(((a,c) = (x2,z)) & ((x2 ε x) & (z = ({x2} X y))))  Hyp
27. ((a,c) = (x2,y2)) & ((x2 ε x) & (y2 = ({x2} X y)))  Hyp
28. Set((a,b))  AndElimL 20
29. Set((a,c))  AndElimL 21
30. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
31. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 30
32. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 31
33. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 32
34. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 33
35. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 34
36. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 35
37. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 36
38. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 35
39. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 38
40. Set(a) & Set(b)  ImpElim 28 37
41. Set(a) & Set(c)  ImpElim 29 39
42. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
43. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 42
44. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))  ForallElim 43
45. ∀x.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 44
46. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 44
47. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))  ForallElim 46
48. (a,b) = (x1,y1)  AndElimL 25
49. (a,c) = (x2,y2)  AndElimL 27
50. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)))  ForallInt 47
51. ((Set(a) & Set(b)) & ((a,b) = (x1,v))) -> ((a = x1) & (b = v))  ForallElim 50
52. ∀v.(((Set(a) & Set(b)) & ((a,b) = (x1,v))) -> ((a = x1) & (b = v)))  ForallInt 51
53. ((Set(a) & Set(b)) & ((a,b) = (x1,y1))) -> ((a = x1) & (b = y1))  ForallElim 52
54. (Set(a) & Set(b)) & ((a,b) = (x1,y1))  AndInt 40 48
55. (a = x1) & (b = y1)  ImpElim 54 53
56. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 44
57. ((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v))  ForallElim 56
58. ∀u.(((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v)))  ForallInt 57
59. ((Set(a) & Set(c)) & ((a,c) = (x2,v))) -> ((a = x2) & (c = v))  ForallElim 58
60. ∀v.(((Set(a) & Set(c)) & ((a,c) = (x2,v))) -> ((a = x2) & (c = v)))  ForallInt 59
61. ((Set(a) & Set(c)) & ((a,c) = (x2,y2))) -> ((a = x2) & (c = y2))  ForallElim 60
62. (Set(a) & Set(c)) & ((a,c) = (x2,y2))  AndInt 41 49
63. (a = x2) & (c = y2)  ImpElim 62 61
64. a = x1  AndElimL 55
65. a = x2  AndElimL 63
66. x2 = x1  EqualitySub 64 65
67. (x1 ε x) & (y1 = ({x1} X y))  AndElimR 25
68. (x2 ε x) & (y2 = ({x2} X y))  AndElimR 27
69. y1 = ({x1} X y)  AndElimR 67
70. y2 = ({x2} X y)  AndElimR 68
71. y2 = ({x1} X y)  EqualitySub 70 66
72. ({x1} X y) = y2  Symmetry 71
73. y1 = y2  EqualitySub 69 72
74. b = y1  AndElimR 55
75. c = y2  AndElimR 63
76. b = y2  EqualitySub 74 73
77. y2 = b  Symmetry 76
78. c = b  EqualitySub 75 77
79. c = b  ExistsElim 26 27 78
80. c = b  ExistsElim 23 26 79
81. c = b  ExistsElim 24 25 80
82. c = b  ExistsElim 22 24 81
83. b = c  Symmetry 82
84. (((a,b) ε f) & ((a,c) ε f)) -> (b = c)  ImpInt 83
85. ∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c))  ForallInt 84
86. ∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c))  ForallInt 85
87. ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c))  ForallInt 86
88. Relation(f) & ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c))  AndInt 14 87
89. Function(f)  DefSub 88
90. a ε x  Hyp
91. b = ({a} X y)  Hyp
92. (a ε x) & (b = ({a} X y))  AndInt 90 91
93. c = (a,b)  Hyp
94. (c = (a,b)) & ((a ε x) & (b = ({a} X y)))  AndInt 93 92
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95. ∃b.((c = (a,b)) & ((a ε x) & (b = ({a} X y))))  ExistsInt 94
96. ∃a.∃b.((c = (a,b)) & ((a ε x) & (b = ({a} X y))))  ExistsInt 95
97. Set(x) & Set(y)  Hyp
98. ∃w.(a ε w)  ExistsInt 90
99. Set(a)  DefSub 98
100. Set(x) -> Set({x})  TheoremInt
101. ∀x.(Set(x) -> Set({x}))  ForallInt 100
102. Set(a) -> Set({a})  ForallElim 101
103. Set({a})  ImpElim 99 102
104. Set(y)  AndElimR 97
105. (Set(u) & Set(y)) -> Set(({u} X y))  TheoremInt
106. ∀u.((Set(u) & Set(y)) -> Set(({u} X y)))  ForallInt 105
107. (Set(a) & Set(y)) -> Set(({a} X y))  ForallElim 106
108. Set(a) & Set(y)  AndInt 99 104
109. Set(({a} X y))  ImpElim 108 107
110. ({a} X y) = b  Symmetry 91
111. Set(b)  EqualitySub 109 110
112. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
113. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 112
114. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 113
115. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 114
116. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 115
117. (Set(a) & Set(y)) -> Set((a,y))  ForallElim 116
118. ∀y.((Set(a) & Set(y)) -> Set((a,y)))  ForallInt 117
119. (Set(a) & Set(b)) -> Set((a,b))  ForallElim 118
120. Set(a) & Set(b)  AndInt 99 111
121. Set((a,b))  ImpElim 120 119
122. (a,b) = c  Symmetry 93
123. Set(c)  EqualitySub 121 122
124. Set(c) & ∃a.∃b.((c = (a,b)) & ((a ε x) & (b = ({a} X y))))  AndInt 123 96
125. c ε {w: ∃a.∃b.((w = (a,b)) & ((a ε x) & (b = ({a} X y))))}  ClassInt 124
126. (a,b) ε {w: ∃x_6.∃x_8.((w = (x_6,x_8)) & ((x_6 ε x) & (x_8 = ({x_6} X y))))}
EqualitySub 125 93
127. {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = f  Symmetry 0
128. (a,b) ε f  EqualitySub 126 127
129. ∃b.((a,b) ε f)  ExistsInt 128
130. Set(a) & ∃b.((a,b) ε f)  AndInt 99 129
131. a ε {w: ∃b.((w,b) ε f)}  ClassInt 130
132. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
133. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 132
134. a ε domain(f)  EqualitySub 131 133
135. (c = (a,b)) -> (a ε domain(f))  ImpInt 134
136. ∀c.((c = (a,b)) -> (a ε domain(f)))  ForallInt 135
137. ((a,b) = (a,b)) -> (a ε domain(f))  ForallElim 136
138. (a,b) = (a,b)  Identity
139. a ε domain(f)  ImpElim 138 137
140. (b = ({a} X y)) -> (a ε domain(f))  ImpInt 139
141. ∀b.((b = ({a} X y)) -> (a ε domain(f)))  ForallInt 140
142. (({a} X y) = ({a} X y)) -> (a ε domain(f))  ForallElim 141
143. ({a} X y) = ({a} X y)  Identity
144. a ε domain(f)  ImpElim 143 142
145. (a ε x) -> (a ε domain(f))  ImpInt 144
146. a ε domain(f)  Hyp
147. a ε {x: ∃y.((x,y) ε f)}  EqualitySub 146 132
148. Set(a) & ∃y.((a,y) ε f)  ClassElim 147
149. ∃y.((a,y) ε f)  AndElimR 148
150. (a,b) ε f  Hyp
151. (a,b) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}  EqualitySub 150 0
152. Set((a,b)) & ∃u.∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y))))  ClassElim 151
153. Set((a,b))  AndElimL 152
154. ∃u.∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y))))  AndElimR 152
155. ∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y))))  Hyp
156. ((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y)))  Hyp
157. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
158. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 157
159. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 158
160. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 159
161. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 160
162. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 161
163. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 162
164. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 163
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165. Set(a) & Set(b) ImpElim 153 164
166. (a,b) = (u,z) AndElimL 156
167. (Set(a) & Set(b)) & ((a,b) = (u,z)) AndInt 165 166
168. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
169. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 168
170. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)) ForallElim 169
171. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 170
172. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)) ForallElim 171
173. ∀v.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))) ForallInt 172
174. ((Set(a) & Set(b)) & ((a,b) = (u,z))) -> ((a = u) & (b = z)) ForallElim 173
175. (a = u) & (b = z) ImpElim 167 174
176. a = u AndElimL 175
177. (u ε x) & (z = ({u} X y)) AndElimR 156
178. u ε x AndElimL 177
179. u = a Symmetry 176
180. a ε x EqualitySub 178 179
181. a ε x ExistsElim 155 156 180
182. a ε x ExistsElim 154 155 181
183. a ε x ExistsElim 149 150 182
184. (a ε domain(f)) -> (a ε x) ImpInt 183
185. ((a ε x) -> (a ε domain(f))) & ((a ε domain(f)) -> (a ε x)) AndInt 145 184
186. (a ε x) <-> (a ε domain(f)) EquivConst 185
187. ∀a.((a ε x) <-> (a ε domain(f))) ForallInt 186
188. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
189. ∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) ForallElim 188
190. (x = domain(f)) <-> ∀z.((z ε x) <-> (z ε domain(f))) ForallElim 189
191. ((x = domain(f)) -> ∀z.((z ε x) <-> (z ε domain(f)))) & (∀z.((z ε x) <-> (z ε domain(f))) -> (x = domain(f))) EquivExp 190
192. ∀z.((z ε x) <-> (z ε domain(f))) -> (x = domain(f)) AndElimR 191
193. x = domain(f) ImpElim 187 192
194. Function(f) & (x = domain(f)) AndInt 89 193
195. (f = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) -> (Function(f) & (x = domain(f))) ImpInt 194
196. ({a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) -> (Function(f) & (x = domain(f))) EqualitySub 195 0
197. {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} Identity
198. Function(f) & (x = domain(f)) ImpElim 197 196
199. x = domain(f) AndElimR 198
200. Set(x) AndElimL 97
201. Set(domain(f)) EqualitySub 200 199
202. Function(f) AndElimL 198
203. Function(f) & Set(domain(f)) AndInt 202 201
204. (Function(f) & Set(domain(f))) -> Set(range(f)) AxInt
205. Set(range(f)) ImpElim 203 204
206. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
207. range(f) = {x_10: ∃x_11.((x_11,x_10) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))})} EqualitySub 206 0
208. e ε range(f) Hyp
209. e ε {x_10: ∃x_11.((x_11,x_10) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))})} EqualitySub 208 207
210. Set(e) & ∃x_11.((x_11,e) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) ClassElim 209
211. ∃x_11.((x_11,e) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) AndElimR 210
212. (c,e) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} Hyp
213. Set((c,e)) & ∃u.∃z.(((c,e) = (u,z)) & ((u ε x) & (z = ({u} X y)))) ClassElim 212
214. ∃u.∃z.(((c,e) = (u,z)) & ((u ε x) & (z = ({u} X y)))) AndElimR 213
215. ∃z.(((c,e) = (u,z)) & ((u ε x) & (z = ({u} X y)))) Hyp
216. ((c,e) = (u,z)) & ((u ε x) & (z = ({u} X y))) Hyp
217. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
218. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 217
219. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 218
220. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 219
221. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 220
222. Set((c,y)) -> (Set(c) & Set(y)) ForallElim 221
223. ∀y.(Set((c,y)) -> (Set(c) & Set(y))) ForallInt 222
224. Set((c,e)) -> (Set(c) & Set(e)) ForallElim 223
225. Set((c,e)) AndElimL 213
226. Set(c) & Set(e) ImpElim 225 224
227. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt

228. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 227
229. ((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v)) ForallElim 228
230. ∀y.(((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v))) ForallInt 229
231. ((Set(c) & Set(e)) & ((c,e) = (u,v))) -> ((c = u) & (e = v)) ForallElim 230
232. (c,e) = (u,z) AndElimL 216
233. (Set(c) & Set(e)) & ((c,e) = (u,z)) AndInt 226 232
234. ∀v.(((Set(c) & Set(e)) & ((c,e) = (u,v))) -> ((c = u) & (e = v))) ForallInt 231
235. ((Set(c) & Set(e)) & ((c,e) = (u,z))) -> ((c = u) & (e = z)) ForallElim 234
236. (c = u) & (e = z) ImpElim 233 235
237. (u ε x) & (z = ({u} X y)) AndElimR 216
238. z = ({u} X y) AndElimR 237
239. e = z AndElimR 236
240. z = e Symmetry 239
241. e = ({u} X y) EqualitySub 238 240
242. u ε x AndElimL 237
243. (u ε x) & (e = ({u} X y)) AndInt 242 241
244. ∃u.((u ε x) & (e = ({u} X y))) ExistsInt 243
245. Set(e) AndElimR 226
246. Set(e) & ∃u.((u ε x) & (e = ({u} X y))) AndInt 245 244
247. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} ClassInt 246
248. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} ExistsElim 215 216 247
249. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} ExistsElim 214 215 248
250. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} ExistsElim 211 212 249
251. (e ε range(f)) -> (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) ImpInt 250
252. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} Hyp
253. Set(e) & ∃u.((u ε x) & (e = ({u} X y))) ClassElim 252
254. Set(e) AndElimL 253
255. ∃u.((u ε x) & (e = ({u} X y))) AndElimR 253
256. (u ε x) & (e = ({u} X y)) Hyp
257. (u,e) = (u,e) Identity
258. ((u,e) = (u,e)) & ((u ε x) & (e = ({u} X y))) AndInt 257 256
259. ∃b.(((u,e) = (u,b)) & ((u ε x) & (b = ({u} X y)))) ExistsInt 258
260. ∃v.∃b.(((u,e) = (v,b)) & ((v ε x) & (b = ({v} X y)))) ExistsInt 259
261. u ε x AndElimL 256
262. ∃w.(u ε w) ExistsInt 261
263. Set(u) DefSub 262
264. Set(u) & Set(e) AndInt 263 254
265. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 219
266. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 265
267. (Set(u) & Set(y)) -> Set((u,y)) ForallElim 266
268. ∀y.((Set(u) & Set(y)) -> Set((u,y))) ForallInt 267
269. (Set(u) & Set(e)) -> Set((u,e)) ForallElim 268
270. Set((u,e)) ImpElim 264 269
271. Set((u,e)) & ∃v.∃b.(((u,e) = (v,b)) & ((v ε x) & (b = ({v} X y)))) AndInt 270 260
272. c = (u,e) Hyp
273. (u,e) = c Symmetry 272
274. Set(c) & ∃v.∃b.((c = (v,b)) & ((v ε x) & (b = ({v} X y)))) EqualitySub 271 273
275. c ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))} ClassInt 274
276. (u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))} EqualitySub 275 272
277. (c = (u,e)) -> ((u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))}) ImpInt 276
278. ∀c.((c = (u,e)) -> ((u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))})) ForallInt 277
279. ((u,e) = (u,e)) -> ((u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))}) ForallElim 278
280. (u,e) = (u,e) Identity
281. (u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))} ImpElim 280 279
282. {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = f Symmetry 0
283. (u,e) ε f EqualitySub 281 282
284. ∃u.((u,e) ε f) ExistsInt 283
285. ∃u.((u,e) ε f) ExistsElim 255 256 284
286. Set(e) & ∃u.((u,e) ε f) AndInt 254 285
287. e ε {w: ∃u.((u,w) ε f)} ClassInt 286
288. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
289. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 288
290. e ε range(f) EqualitySub 287 289
291. (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) -> (e ε range(f)) ImpInt 290
292. ((e ε range(f)) -> (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))})) & ((e ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) -> (e ε range(f))) AndInt 251 291
293. (e ε range(f)) <-> (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) EquivConst 292
294. ∀e.((e ε range(f)) <-> (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))})) ForallInt 293

295. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
296. ∀y.((range(f) = y) <-> ∀z.((z ε range(f)) <-> (z ε y))) ForallElim 295
297. (range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))}) <-> ∀z.((z ε range(f)) <-> (z ε {w: ∃u.((u ε x) & (w = ({u} X y)))})) ForallElim 296
298. ((range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))}) -> ∀z.((z ε range(f)) <-> (z ε {w: ∃u.((u ε x) & (w = ({u} X y)))}))) & (∀z.((z ε range(f)) <-> (z ε {w: ∃u.((u ε x) & (w = ({u} X y)))})) -> (range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))})) EquivExp 297
299. ∀z.((z ε range(f)) <-> (z ε {w: ∃u.((u ε x) & (w = ({u} X y)))})) -> (range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))}) AndElimR 298
300. range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))} ImpElim 294 299
301. e ε ∪range(f) Hyp
302. e ε ∪{w: ∃u.((u ε x) & (w = ({u} X y)))} EqualitySub 301 300
303. ∪x = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
304. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 303
305. ∪range(f) = {z: ∃y.((y ε range(f)) & (z ε y))} ForallElim 304
306. ∪range(f) = {z: ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (z ε x_13))} EqualitySub 305 300
307. e ε {z: ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (z ε x_13))} EqualitySub 301 306
308. Set(e) & ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (e ε x_13)) ClassElim 307
309. ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (e ε x_13)) AndElimR 308
310. (x_5 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (e ε x_5) Hyp
311. e ε x_5 AndElimR 310
312. x_5 ε {w: ∃u.((u ε x) & (w = ({u} X y)))} AndElimL 310
313. Set(x_5) & ∃u.((u ε x) & (x_5 = ({u} X y))) ClassElim 312
314. Set(x_5) AndElimL 313
315. ∃u.((u ε x) & (x_5 = ({u} X y))) AndElimR 313
316. (u ε x) & (x_5 = ({u} X y)) Hyp
317. x_5 = ({u} X y) AndElimR 316
318. e ε ({u} X y) EqualitySub 311 317
319. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} DefEqInt
320. ∀x.((x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}) ForallInt 319
321. ({u} X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))} ForallElim 320
322. e ε {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))} EqualitySub 318 321
323. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ε {u}) & (b ε y))) ClassElim 322
324. ∃a.∃b.((e = (a,b)) & ((a ε {u}) & (b ε y))) AndElimR 323
325. ∃b.((e = (a,b)) & ((a ε {u}) & (b ε y))) Hyp
326. (e = (a,b)) & ((a ε {u}) & (b ε y)) Hyp
327. (a ε {u}) & (b ε y) AndElimR 326
328. a ε {u} AndElimL 327
329. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
330. u ε x AndElimL 316
331. ∃w.(u ε w) ExistsInt 330
332. Set(u) DefSub 331
333. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 329
334. Set(u) -> ((y ε {u}) <-> (y = u)) ForallElim 333
335. ∀y.(Set(u) -> ((y ε {u}) <-> (y = u))) ForallInt 334
336. Set(u) -> ((a ε {u}) <-> (a = u)) ForallElim 335
337. (a ε {u}) <-> (a = u) ImpElim 332 336
338. ((a ε {u}) -> (a = u)) & ((a = u) -> (a ε {u})) EquivExp 337
339. (a ε {u}) -> (a = u) AndElimL 338
340. a = u ImpElim 328 339
341. u = a Symmetry 340
342. a ε x EqualitySub 330 341
343. b ε y AndElimR 327
344. (a ε x) & (b ε y) AndInt 342 343
345. e = (a,b) AndElimL 326
346. (e = (a,b)) & ((a ε x) & (b ε y)) AndInt 345 344
347. ∃b.((e = (a,b)) & ((a ε x) & (b ε y))) ExistsInt 346
348. ∃a.∃b.((e = (a,b)) & ((a ε x) & (b ε y))) ExistsInt 347
349. Set(e) AndElimL 323
350. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ε x) & (b ε y))) AndInt 349 348
351. e ε {w: ∃a.∃b.((w = (a,b)) & ((a ε x) & (b ε y)))} ClassInt 350
352. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} DefEqInt
353. {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} = (x X y) Symmetry 352
354. e ε (x X y) EqualitySub 351 353
355. e ε (x X y) ExistsElim 325 326 354
356. e ε (x X y) ExistsElim 324 325 355
357. e ε (x X y) ExistsElim 315 316 356
358. e ε (x X y) ExistsElim 309 310 357

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359. (e ε Urange(f)) -> (e ε (x X y))  ImpInt 358
360. e ε (x X y)  Hyp
361. e ε {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}  EqualitySub 360 352
362. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  ClassElim 361
363. Set(e)  AndElimL 362
364. ∃a.∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  AndElimR 362
365. ∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  Hyp
366. (e = (a,b)) & ((a ε x) & (b ε y))  Hyp
367. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 218
368. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 367
369. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 368
370. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 369
371. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 370
372. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 371
373. e = (a,b)  AndElimL 366
374. Set((a,b))  EqualitySub 363 373
375. Set(a) & Set(b)  ImpElim 374 372
376. Set(a)  AndElimL 375
377. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 329
378. Set(a) -> ((y ε {a}) <-> (y = a))  ForallElim 377
379. ∀y.(Set(a) -> ((y ε {a}) <-> (y = a)))  ForallInt 378
380. Set(a) -> ((a ε {a}) <-> (a = a))  ForallElim 379
381. (a ε {a}) <-> (a = a)  ImpElim 376 380
382. ((a ε {a}) -> (a = a)) & ((a = a) -> (a ε {a}))  EquivExp 381
383. (a = a) -> (a ε {a})  AndElimR 382
384. a = a  Identity
385. a ε {a}  ImpElim 384 383
386. e = (a,b)  AndElimL 366
387. (a ε x) & (b ε y)  AndElimR 366
388. a ε x  AndElimL 387
389. b ε y  AndElimR 387
390. (a ε {a}) & (b ε y)  AndInt 385 389
391. (e = (a,b)) & ((a ε {a}) & (b ε y))  AndInt 386 390
392. ∃u.((e = (a,u)) & ((a ε {a}) & (u ε y)))  ExistsInt 391
393. ∃v.∃u.((e = (v,u)) & ((v ε {a}) & (u ε y)))  ExistsInt 392
394. Set(e) & ∃v.∃u.((e = (v,u)) & ((v ε {a}) & (u ε y)))  AndInt 363 393
395. e ε {w: ∃v.∃u.((w = (v,u)) & ((v ε {a}) & (u ε y)))}  ClassInt 394
396. ∀x.((x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))})  ForallInt 319
397. ({a} X y) = {z: ∃x_15.∃b.((z = (x_15,b)) & ((x_15 ε {a}) & (b ε y)))}  ForallElim
396
398. {z: ∃x_15.∃b.((z = (x_15,b)) & ((x_15 ε {a}) & (b ε y)))} = ({a} X y)  Symmetry 397
399. e ε ({a} X y)  EqualitySub 395 398
400. g = ({a} X y)  Hyp
401. ({a} X y) = g  Symmetry 400
402. (a ε x) & (g = ({a} X y))  AndInt 388 400
403. ∃a.((a ε x) & (g = ({a} X y)))  ExistsInt 402
404. (Set(u) & Set(y)) -> Set(({u} X y))  TheoremInt
405. ∀u.((Set(u) & Set(y)) -> Set(({u} X y)))  ForallInt 404
406. (Set(a) & Set(y)) -> Set(({a} X y))  ForallElim 405
407. Set(y)  AndElimR 97
408. Set(a) & Set(y)  AndInt 376 407
409. Set(({a} X y))  ImpElim 408 406
410. Set(g)  EqualitySub 409 401
411. Set(g) & ∃a.((a ε x) & (g = ({a} X y)))  AndInt 410 403
412. g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}  ClassInt 411
413. e ε g  EqualitySub 399 401
414. (g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}) & (e ε g)  AndInt 412 413
415. ∃g.((g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}) & (e ε g))  ExistsInt 414
416. Set(e) & ∃g.((g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}) & (e ε g))  AndInt 363 415
417. e ε {d: ∃g.((g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}) & (d ε g))}  ClassInt 416
418. {z: ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (z ε x_13))} = Urange(f)
Symmetry 306
419. e ε Urange(f)  EqualitySub 417 418
420. (g = ({a} X y)) -> (e ε Urange(f))  ImpInt 419
421. ∀g.((g = ({a} X y)) -> (e ε Urange(f)))  ForallInt 420
422. (({a} X y) = ({a} X y)) -> (e ε Urange(f))  ForallElim 421
423. ({a} X y) = ({a} X y)  Identity
424. e ε Urange(f)  ImpElim 423 422
425. e ε Urange(f)  ExistsElim 365 366 424
426. e ε Urange(f)  ExistsElim 364 365 425
427. (e ε (x X y)) -> (e ε Urange(f))  ImpInt 426
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428. ((e ε Urange(f)) -> (e ε (x X y))) & ((e ε (x X y)) -> (e ε Urange(f))) AndInt 359 427
429. (e ε Urange(f)) <-> (e ε (x X y)) EquivConst 428
430. ∀e.((e ε Urange(f)) <-> (e ε (x X y))) ForallInt 429
431. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
432. ∀y.((Urange(f) = y) <-> ∀z.((z ε Urange(f)) <-> (z ε y))) ForallElim 431
433. (Urange(f) = (x X y)) <-> ∀z.((z ε Urange(f)) <-> (z ε (x X y))) ForallElim 432
434. ((Urange(f) = (x X y)) -> ∀z.((z ε Urange(f)) <-> (z ε (x X y)))) & (∀z.((z ε Urange(f)) <-> (z ε (x X y))) -> (Urange(f) = (x X y))) EquivExp 433
435. ∀z.((z ε Urange(f)) <-> (z ε (x X y))) -> (Urange(f) = (x X y)) AndElimR 434
436. Urange(f) = (x X y) ImpElim 430 435
437. Set(x) -> Set(Ux) AxInt
438. ∀x.(Set(x) -> Set(Ux)) ForallInt 437
439. Set(range(f)) -> Set(Urange(f)) ForallElim 438
440. Set(Urange(f)) ImpElim 205 439
441. Set((x X y)) EqualitySub 440 436
442. (Set(x) & Set(y)) -> Set((x X y)) ImpInt 441
443. (f = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) -> ((Set(x) & Set(y)) -> Set((x X y))) ImpInt 442
444. ∀f.((f = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) -> ((Set(x) & Set(y)) -> Set((x X y)))) ForallInt 443
445. ({a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = {x_16: ∃x_17.∃x_18.((x_16 = (x_17,x_18)) & ((x_17 ε x) & (x_18 = ({x_17} X y))))}) -> ((Set(x) & Set(y)) -> Set((x X y))) ForallElim 444
446. {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} Identity
447. (Set(x) & Set(y)) -> Set((x X y)) ImpElim 446 445 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
3. Set(x) -> Set({x})
4. (Set(u) & Set(y)) -> Set(({u} X y))
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
6. Set(x) -> ((y ε {x}) <-> (y = x))
7. (Set(u) & Set(y)) -> Set(({u} X y))

Th75. (Function(f) & Set(domain(f))) -> (f ⊂ (domain(f) X range(f)))

0. Function(f) & Set(domain(f)) Hyp
1. z ε f Hyp
2. Function(f) AndElimL 0
3. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 2
4. Relation(f) AndElimL 3
5. ∀z.((z ε f) -> ∃x.∃y.(z = (x,y))) DefExp 4
6. (z ε f) -> ∃x.∃y.(z = (x,y)) ForallElim 5
7. ∃x.∃y.(z = (x,y)) ImpElim 1 6
8. ∃y.(z = (x,y)) Hyp
9. z = (x,y) Hyp
10. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
11. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
12. ∃y.(z = (x,y)) ExistsInt 9
13. ∃f.(z ε f) ExistsInt 1
14. Set(z) DefSub 13
15. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
16. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 15
17. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 16
18. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 17
19. Set((x,y)) EqualitySub 14 9
20. Set(x) & Set(y) ImpElim 19 18
21. Set(x) AndElimL 20
22. (x,y) ε f EqualitySub 1 9
23. ∃y.((x,y) ε f) ExistsInt 22
24. Set(x) & ∃y.((x,y) ε f) AndInt 21 23
25. x ε {w: ∃y.((w,y) ε f)} ClassInt 24
26. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 10

27. x ε domain(f) EqualitySub 25 26
28. ∃x.((x,y) ε f) ExistsInt 22
29. Set(y) AndElimR 20
30. Set(y) & ∃x.((x,y) ε f) AndInt 29 28
31. y ε {w: ∃x.((x,w) ε f)} ClassInt 30
32. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 11
33. y ε range(f) EqualitySub 31 32
34. (x ε domain(f)) & (y ε range(f)) AndInt 27 33
35. (z = (x,y)) & ((x ε domain(f)) & (y ε range(f))) AndInt 9 34
36. ∃y.((z = (x,y)) & ((x ε domain(f)) & (y ε range(f)))) ExistsInt 35
37. ∃x.∃y.((z = (x,y)) & ((x ε domain(f)) & (y ε range(f)))) ExistsInt 36
38. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} DefEqInt
39. ∀x.((x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}) ForallInt 38
40. (domain(f) X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε domain(f)) & (b ε y)))} ForallElim 39
41. ∀y.((domain(f) X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε domain(f)) & (b ε y)))}) ForallInt 40
42. (domain(f) X range(f)) = {z: ∃a.∃b.((z = (a,b)) & ((a ε domain(f)) & (b ε range(f))))} ForallElim 41
43. Set(z) & ∃x.∃y.((z = (x,y)) & ((x ε domain(f)) & (y ε range(f)))) AndInt 14 37
44. z ε {w: ∃x.∃y.((w = (x,y)) & ((x ε domain(f)) & (y ε range(f))))} ClassInt 43
45. {z: ∃a.∃b.((z = (a,b)) & ((a ε domain(f)) & (b ε range(f))))} = (domain(f) X range(f)) Symmetry 42
46. z ε (domain(f) X range(f)) EqualitySub 44 45
47. z ε (domain(f) X range(f)) ExistsElim 8 9 46
48. z ε (domain(f) X range(f)) ExistsElim 7 8 47
49. (z ε f) -> (z ε (domain(f) X range(f))) ImpInt 48
50. ∀z.((z ε f) -> (z ε (domain(f) X range(f)))) ForallInt 49
51. f ⊂ (domain(f) X range(f)) DefSub 50
52. (Function(f) & Set(domain(f))) -> (f ⊂ (domain(f) X range(f))) ImpInt 51 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th77. (Set(x) & Set(y)) -> Set(func(x,y))

0. Set(x) & Set(y) Hyp
1. f ε func(x,y) Hyp
2. func(x,y) = {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))} DefEqInt
3. f ε {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))} EqualitySub 1 2
4. Set(f) & (Function(f) & ((domain(f) = x) & (range(f) = y))) ClassElim 3
5. Set(f) AndElimL 4
6. Function(f) & ((domain(f) = x) & (range(f) = y)) AndElimR 4
7. Function(f) AndElimL 6
8. (domain(f) = x) & (range(f) = y) AndElimR 6
9. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 7
10. Relation(f) AndElimL 9
11. ∀z.((z ε f) -> ∃x.∃y.(z = (x,y))) DefExp 10
12. z ε f Hyp
13. (z ε f) -> ∃x.∃y.(z = (x,y)) ForallElim 11
14. ∃x.∃y.(z = (x,y)) ImpElim 12 13
15. ∃y.(z = (a,y)) Hyp
16. z = (a,b) Hyp
17. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} DefEqInt
18. (a,b) ε f EqualitySub 12 16
19. ∃w.((a,w) ε f) ExistsInt 18
20. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
21. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
22. ∃w.((a,b) ε w) ExistsInt 18
23. Set((a,b)) DefSub 22
24. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
25. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 24
26. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 25
27. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 26
28. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 27
29. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 28
30. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 29
31. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 30
32. Set(a) & Set(b) ImpElim 23 31
33. Set(a) AndElimL 32

34. Set(a) & ∃w.((a,w) ε f) AndInt 33 19
35. a ε {w: ∃x_1.((w,x_1) ε f)} ClassInt 34
36. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 20
37. a ε domain(f) EqualitySub 35 36
38. domain(f) = x AndElimL 8
39. a ε x EqualitySub 37 38
40. ∃w.((w,b) ε f) ExistsInt 18
41. Set(b) AndElimR 32
42. Set(b) & ∃w.((w,b) ε f) AndInt 41 40
43. b ε {w: ∃x_3.((x_3,w) ε f)} ClassInt 42
44. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 21
45. b ε range(f) EqualitySub 43 44
46. range(f) = y AndElimR 8
47. b ε y EqualitySub 45 46
48. (a ε x) & (b ε y) AndInt 39 47
49. (z = (a,b)) & ((a ε x) & (b ε y)) AndInt 16 48
50. (a,b) = z Symmetry 16
51. Set(z) EqualitySub 23 50
52. ∃b.((z = (a,b)) & ((a ε x) & (b ε y))) ExistsInt 49
53. ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y))) ExistsInt 52
54. Set(z) & ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y))) AndInt 51 53
55. z ε {w: ∃a.∃b.((w = (a,b)) & ((a ε x) & (b ε y)))} ClassInt 54
56. {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} = (x X y) Symmetry 17
57. z ε (x X y) EqualitySub 55 56
58. z ε (x X y) ExistsElim 15 16 57
59. z ε (x X y) ExistsElim 14 15 58
60. (z ε f) -> (z ε (x X y)) ImpInt 59
61. ∀z.((z ε f) -> (z ε (x X y))) ForallInt 60
62. f ⊂ (x X y) DefSub 61
63. (Set(x) & Set(y)) -> Set((x X y)) TheoremInt
64. Set((x X y)) ImpElim 0 63
65. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px))) TheoremInt
66. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
67. ∀y.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 66
68. (Set(x) & (c ⊂ x)) -> Set(c) ForallElim 67
69. ∀x.((Set(x) & (c ⊂ x)) -> Set(c)) ForallInt 68
70. (Set((x X y)) & (c ⊂ (x X y))) -> Set(c) ForallElim 69
71. ∀c.((Set((x X y)) & (c ⊂ (x X y))) -> Set(c)) ForallInt 70
72. (Set((x X y)) & (f ⊂ (x X y))) -> Set(f) ForallElim 71
73. Set((x X y)) & (f ⊂ (x X y)) AndInt 64 62
74. Set(f) ImpElim 73 72
75. ∀y.(Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))) ForallInt 65
76. Set(x) -> (Set(Px) & ((f ⊂ x) <-> (f ε Px))) ForallElim 75
77. ∀x.(Set(x) -> (Set(Px) & ((f ⊂ x) <-> (f ε Px)))) ForallInt 76
78. Set((x X y)) -> (Set(P(x X y)) & ((f ⊂ (x X y)) <-> (f ε P(x X y)))) ForallElim 77
79. Set(P(x X y)) & ((f ⊂ (x X y)) <-> (f ε P(x X y))) ImpElim 64 78
80. Set(P(x X y)) AndElimL 79
81. (f ⊂ (x X y)) <-> (f ε P(x X y)) AndElimR 79
82. ((f ⊂ (x X y)) -> (f ε P(x X y))) & ((f ε P(x X y)) -> (f ⊂ (x X y))) EquivExp 81
83. (f ⊂ (x X y)) -> (f ε P(x X y)) AndElimL 82
84. f ε P(x X y) ImpElim 62 83
85. (f ε func(x,y)) -> (f ε P(x X y)) ImpInt 84
86. ∀f.((f ε func(x,y)) -> (f ε P(x X y))) ForallInt 85
87. func(x,y) ⊂ P(x X y) DefSub 86
88. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
89. ∀y.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 88
90. (Set(x) & (c ⊂ x)) -> Set(c) ForallElim 89
91. ∀x.((Set(x) & (c ⊂ x)) -> Set(c)) ForallInt 90
92. (Set(P(x X y)) & (c ⊂ P(x X y))) -> Set(c) ForallElim 91
93. ∀c.((Set(P(x X y)) & (c ⊂ P(x X y))) -> Set(c)) ForallInt 92
94. (Set(P(x X y)) & (func(x,y) ⊂ P(x X y))) -> Set(func(x,y)) ForallElim 93
95. Set(P(x X y)) & (func(x,y) ⊂ P(x X y)) AndInt 80 87
96. Set(func(x,y)) ImpElim 95 94
97. (Set(x) & Set(y)) -> Set(func(x,y)) ImpInt 96 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. (Set(x) & Set(y)) -> Set((x X y))
3. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))
4. (Set(x) & (y ⊂ x)) -> Set(y)

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4. (Set(x) & (y ⊂ x)) -> Set(y)

Th88. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))

0. WellOrders(r,x)  Hyp
1. (u ε x) & ((v ε x) & (w ε x))  Hyp
2. ((u,v) ε r) & ((v,w) ε r)  Hyp
3. z ε {u,v}  Hyp
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))  TheoremInt
5. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))  AndElimL 4
6. ∀x.((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))))
ForallInt 5
7. (Set(c) & Set(y)) -> (Set({c,y}) & ((z ε {c,y}) <-> ((z = c) v (z = y))))  ForallElim
6
8. ∀y.((Set(c) & Set(y)) -> (Set({c,y}) & ((z ε {c,y}) <-> ((z = c) v (z = y)))))
ForallInt 7
9. (Set(c) & Set(d)) -> (Set({c,d}) & ((z ε {c,d}) <-> ((z = c) v (z = d))))  ForallElim
8
10. ∀z.((Set(c) & Set(d)) -> (Set({c,d}) & ((z ε {c,d}) <-> ((z = c) v (z = d)))))
ForallInt 9
11. (Set(c) & Set(d)) -> (Set({c,d}) & ((e ε {c,d}) <-> ((e = c) v (e = d))))  ForallElim
10
12. u ε x  AndElimL 1
13. (v ε x) & (w ε x)  AndElimR 1
14. v ε x  AndElimL 13
15. ∃x.(u ε x)  ExistsInt 12
16. Set(u)  DefSub 15
17. ∃x.(v ε x)  ExistsInt 14
18. Set(v)  DefSub 17
19. ∀c.((Set(c) & Set(d)) -> (Set({c,d}) & ((e ε {c,d}) <-> ((e = c) v (e = d)))))
ForallInt 11
20. (Set(u) & Set(d)) -> (Set({u,d}) & ((e ε {u,d}) <-> ((e = u) v (e = d))))  ForallElim
19
21. ∀d.((Set(u) & Set(d)) -> (Set({u,d}) & ((e ε {u,d}) <-> ((e = u) v (e = d)))))
ForallInt 20
22. (Set(u) & Set(v)) -> (Set({u,v}) & ((e ε {u,v}) <-> ((e = u) v (e = v))))  ForallElim
21
23. Set(u) & Set(v)  AndInt 16 18
24. Set({u,v}) & ((e ε {u,v}) <-> ((e = u) v (e = v)))  ImpElim 23 22
25. (e ε {u,v}) <-> ((e = u) v (e = v))  AndElimR 24
26. ∀e.((e ε {u,v}) <-> ((e = u) v (e = v)))  ForallInt 25
27. (z ε {u,v}) <-> ((z = u) v (z = v))  ForallElim 26
28. ((z ε {u,v}) -> ((z = u) v (z = v))) & (((z = u) v (z = v)) -> (z ε {u,v}))  EquivExp
27
29. (z ε {u,v}) -> ((z = u) v (z = v))  AndElimL 28
30. (z = u) v (z = v)  ImpElim 3 29
31. z = u  Hyp
32. u ε x  AndElimL 1
33. u = z  Symmetry 31
34. z ε x  EqualitySub 32 33
35. z = v  Hyp
36. (v ε x) & (w ε x)  AndElimR 1
37. v ε x  AndElimL 36
38. v = z  Symmetry 35
39. z ε x  EqualitySub 37 38
40. z ε x  OrElim 30 31 34 35 39
41. (z ε {u,v}) -> (z ε x)  ImpInt 40
42. ∀z.((z ε {u,v}) -> (z ε x))  ForallInt 41
43. {u,v} ⊂ x  DefSub 42
44. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  DefExp 0
45. ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  AndElimR 44
46. (({u,v} ⊂ x) & ¬({u,v} = 0)) -> ∃z.First(r,{u,v},z)  ForallElim 45
47. u = u  Identity
48. (u = u) v (v = v)  OrIntR 47
49. ((e ε {u,v}) -> ((e = u) v (e = v))) & (((e = u) v (e = v)) -> (e ε {u,v}))  EquivExp
25
50. ((e = u) v (e = v)) -> (e ε {u,v})  AndElimR 49
51. ∀e.(((e = u) v (e = v)) -> (e ε {u,v}))  ForallInt 50
52. ((u = u) v (u = v)) -> (u ε {u,v})  ForallElim 51
53. (u = u) v (u = v)  OrIntR 47
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54. u ε {u,v} ImpElim 53 52
55. {u,v} = 0 Hyp
56. u ε 0 EqualitySub 54 55
57. ¬(x ε 0) TheoremInt
58. ∀x.¬(x ε 0) ForallInt 57
59. ¬(u ε 0) ForallElim 58
60. _|_ ImpElim 56 59
61. ¬({u,v} = 0) ImpInt 60
62. ({u,v} ⊂ x) & ¬({u,v} = 0) AndInt 43 61
63. ∃z.First(r,{u,v},z) ImpElim 62 46
64. First(r,{u,v},f) Hyp
65. (f ε {u,v}) & ∀y.((y ε {u,v}) -> ¬((y,f) ε r)) DefExp 64
66. f ε {u,v} AndElimL 65
67. ((e ε {u,v}) -> ((e = u) v (e = v))) & (((e = u) v (e = v)) -> (e ε {u,v})) EquivExp 25
68. (e ε {u,v}) -> ((e = u) v (e = v)) AndElimL 67
69. ∀e.((e ε {u,v}) -> ((e = u) v (e = v))) ForallInt 68
70. (f ε {u,v}) -> ((f = u) v (f = v)) ForallElim 69
71. (f = u) v (f = v) ImpElim 66 70
72. ∀y.((y ε {u,v}) -> ¬((y,f) ε r)) AndElimR 65
73. (u ε {u,v}) -> ¬((u,f) ε r) ForallElim 72
74. (v ε {u,v}) -> ¬((v,f) ε r) ForallElim 72
75. f = u Hyp
76. ∀e.(((e = u) v (e = v)) -> (e ε {u,v})) ForallInt 50
77. ((v = u) v (v = v)) -> (v ε {u,v}) ForallElim 76
78. v = v Identity
79. (v = u) v (v = v) OrIntL 78
80. v ε {u,v} ImpElim 79 77
81. ¬((v,f) ε r) ImpElim 80 74
82. ¬((v,u) ε r) EqualitySub 81 75
83. ¬((v,u) ε r) v ¬((u,v) ε r) OrIntR 82
84. f = v Hyp
85. ∀e.(((e = u) v (e = v)) -> (e ε {u,v})) ForallInt 50
86. ((u = u) v (u = v)) -> (u ε {u,v}) ForallElim 85
87. u = u Identity
88. (u = u) v (u = v) OrIntR 87
89. u ε {u,v} ImpElim 88 86
90. (u ε {u,v}) -> ¬((u,f) ε r) ForallElim 72
91. ¬((u,f) ε r) ImpElim 89 90
92. ¬((u,v) ε r) EqualitySub 91 84
93. ¬((v,u) ε r) v ¬((u,v) ε r) OrIntL 92
94. ¬((v,u) ε r) v ¬((u,v) ε r) OrElim 71 75 83 84 93
95. ¬((v,u) ε r) v ¬((u,v) ε r) ExistsElim 63 64 94
96. (B v ¬A) -> (A -> B) TheoremInt
97. (¬((v,u) ε r) v ¬A) -> (A -> ¬((v,u) ε r)) PolySub 96
98. (¬((v,u) ε r) v ¬((u,v) ε r)) -> (((u,v) ε r) -> ¬((v,u) ε r)) PolySub 97
99. ((u,v) ε r) -> ¬((v,u) ε r) ImpElim 95 98
100. ((u ε x) & ((v ε x) & (w ε x))) -> (((u,v) ε r) -> ¬((v,u) ε r)) ImpInt 99
101. ∀w.(((u ε x) & ((v ε x) & (w ε x))) -> (((u,v) ε r) -> ¬((v,u) ε r))) ForallInt 100
102. ((u ε x) & ((v ε x) & (v ε x))) -> (((u,v) ε r) -> ¬((v,u) ε r)) ForallElim 101
103. (u ε x) & (v ε x) Hyp
104. (u,v) ε r Hyp
105. u ε x AndElimL 103
106. v ε x AndElimR 103
107. (v ε x) & (v ε x) AndInt 106 106
108. (u ε x) & ((v ε x) & (v ε x)) AndInt 105 107
109. ((u,v) ε r) -> ¬((v,u) ε r) ImpElim 108 102
110. ¬((v,u) ε r) ImpElim 104 109
111. ((u,v) ε r) -> ¬((v,u) ε r) ImpInt 110
112. ((u ε x) & (v ε x)) -> (((u,v) ε r) -> ¬((v,u) ε r)) ImpInt 111
113. ∀z.(((u ε x) & (z ε x)) -> (((u,z) ε r) -> ¬((z,u) ε r))) ForallInt 112
114. ∀y.∀z.(((y ε x) & (z ε x)) -> (((y,z) ε r) -> ¬((z,y) ε r))) ForallInt 113
115. Asymmetric(r,x) DefSub 114
116. ¬TransIn(r,x) Hyp
117. ¬∀u.∀v.∀w.(((u ε x) & ((v ε x) & (w ε x))) -> ((((u,v) ε r) & ((v,w) ε r)) -> ((u,w) ε r))) DefExp 116
118. ¬∀i.P(i) -> ∃c.¬P(c) TheoremInt
119. ¬∀i.∀v.∀w.(((i ε x) & ((v ε x) & (w ε x))) -> ((((i,v) ε r) & ((v,w) ε r)) -> ((i,w) ε r))) -> ∃c.¬∀v.∀w.(((c ε x) & ((v ε x) & (w ε x))) -> ((((c,v) ε r) & ((v,w) ε r)) -> ((c,w) ε r))) PredSub 118

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120. ∃c.¬∀v.∀w.(((c ε x) & ((v ε x) & (w ε x))) -> ((((c,v) ε r) & ((v,w) ε r)) -> ((c,w) ε r)))  ImpElim 117 119
121. ¬∀v.∀w.(((k ε x) & ((v ε x) & (w ε x))) -> ((((k,v) ε r) & ((v,w) ε r)) -> ((k,w) ε r)))  Hyp
122. ¬∀i.∀w.(((k ε x) & ((i ε x) & (w ε x))) -> ((((k,i) ε r) & ((i,w) ε r)) -> ((k,w) ε r))) -> ∃c.¬∀w.(((k ε x) & ((c ε x) & (w ε x))) -> ((((k,c) ε r) & ((c,w) ε r)) -> ((k,w) ε r)))  PredSub 118
123. ∃c.¬∀w.(((k ε x) & ((c ε x) & (w ε x))) -> ((((k,c) ε r) & ((c,w) ε r)) -> ((k,w) ε r)))  ImpElim 121 122
124. ¬∀w.(((k ε x) & ((p ε x) & (w ε x))) -> ((((k,p) ε r) & ((p,w) ε r)) -> ((k,w) ε r)))  Hyp
125. ¬∀i.(((k ε x) & ((p ε x) & (i ε x))) -> ((((k,p) ε r) & ((p,i) ε r)) -> ((k,i) ε r))) -> ∃c.¬(((k ε x) & ((p ε x) & (c ε x))) -> ((((k,p) ε r) & ((p,c) ε r)) -> ((k,c) ε r)))  PredSub 118
126. ∃c.¬(((k ε x) & ((p ε x) & (c ε x))) -> ((((k,p) ε r) & ((p,c) ε r)) -> ((k,c) ε r)))  ImpElim 124 125
127. ¬(((k ε x) & ((p ε x) & (q ε x))) -> ((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)))  Hyp
128. (A -> B) -> (¬B -> ¬A)  TheoremInt
129. (A -> C) -> (¬C -> ¬A)  PolySub 128
130. ((B v ¬A) -> C) -> (¬C -> ¬(B v ¬A))  PolySub 129
131. ((B v ¬A) -> (A -> B)) -> (¬(A -> B) -> ¬(B v ¬A))  PolySub 130
132. (B v ¬A) -> (A -> B)  TheoremInt
133. ¬(A -> B) -> ¬(B v ¬A)  ImpElim 132 131
134. ¬(((k ε x) & ((p ε x) & (q ε x))) -> B) -> ¬(B v ¬((k ε x) & ((p ε x) & (q ε x))))  PolySub 133
135. ¬(((k ε x) & ((p ε x) & (q ε x))) -> ((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r))) -> ¬(((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) v ¬((k ε x) & ((p ε x) & (q ε x))))  PolySub 134
136. ¬(((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) v ¬((k ε x) & ((p ε x) & (q ε x))))  ImpElim 127 135
137. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
138. ¬(A v B) <-> (¬A & ¬B)  AndElimL 137
139. ¬(A v C) <-> (¬A & ¬C)  PolySub 138
140. ¬(B v C) <-> (¬B & ¬C)  PolySub 139
141. ¬(B v ¬A) <-> (¬B & ¬¬A)  PolySub 140
142. (¬(B v ¬A) -> (¬B & ¬¬A)) & ((¬B & ¬¬A) -> ¬(B v ¬A))  EquivExp 141
143. ¬(B v ¬A) -> (¬B & ¬¬A)  AndElimL 142
144. D <-> ¬¬D  TheoremInt
145. (D -> ¬¬D) & (¬¬D -> D)  EquivExp 144
146. ¬¬D -> D  AndElimR 145
147. ¬¬A -> A  PolySub 146
148. ¬(B v ¬A)  Hyp
149. ¬B & ¬¬A  ImpElim 148 143
150. ¬B  AndElimL 149
151. ¬¬A  AndElimR 149
152. A  ImpElim 151 147
153. ¬B & A  AndInt 150 152
154. ¬(B v ¬A) -> (¬B & A)  ImpInt 153
155. ¬(A -> B)  Hyp
156. ¬(B v ¬A)  ImpElim 155 133
157. ¬B & A  ImpElim 156 154
158. ¬(A -> B) -> (¬B & A)  ImpInt 157
159. ¬(((k ε x) & ((p ε x) & (q ε x))) -> B) -> (¬B & ((k ε x) & ((p ε x) & (q ε x))))  PolySub 158
160. ¬(((k ε x) & ((p ε x) & (q ε x))) -> ((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r))) -> (¬((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) & ((k ε x) & ((p ε x) & (q ε x))))  PolySub 159
161. ¬((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) & ((k ε x) & ((p ε x) & (q ε x)))  ImpElim 127 160
162. ¬((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r))  AndElimL 161
163. (k ε x) & ((p ε x) & (q ε x))  AndElimR 161
164. ¬((((k,p) ε r) & ((p,q) ε r)) -> B) -> (¬B & (((k,p) ε r) & ((p,q) ε r)))  PolySub 158
165. ¬((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) -> (¬((k,q) ε r) & (((k,p) ε r) & ((p,q) ε r)))  PolySub 164
166. ¬((k,q) ε r) & (((k,p) ε r) & ((p,q) ε r))  ImpElim 162 165
167. ¬((k,q) ε r)  AndElimL 166
168. k ε x  AndElimL 163
169. (p ε x) & (q ε x)  AndElimR 163
170. q ε x  AndElimR 169
```

171. Connects(r,x) AndElimL 44
172. ∀y.∀z.(((y ε x) & (z ε x)) -> ((y = z) v (((y,z) ε r) v ((z,y) ε r)))) DefExp 171
173. ∀z.(((k ε x) & (z ε x)) -> ((k = z) v (((k,z) ε r) v ((z,k) ε r)))) ForallElim 172
174. ((k ε x) & (q ε x)) -> ((k = q) v (((k,q) ε r) v ((q,k) ε r))) ForallElim 173
175. (k ε x) & (q ε x) AndInt 168 170
176. (k = q) v (((k,q) ε r) v ((q,k) ε r)) ImpElim 175 174
177. k = q Hyp
178. ((k,p) ε r) & ((p,q) ε r) AndElimR 166
179. ((q,p) ε r) & ((p,q) ε r) EqualitySub 178 177
180. ∀z.(((q ε x) & (z ε x)) -> (((q,z) ε r) -> ¬((z,q) ε r))) ForallElim 114
181. ((q ε x) & (p ε x)) -> (((q,p) ε r) -> ¬((p,q) ε r)) ForallElim 180
182. p ε x AndElimL 169
183. (q ε x) & (p ε x) AndInt 170 182
184. ((q,p) ε r) -> ¬((p,q) ε r) ImpElim 183 181
185. (q,p) ε r AndElimL 179
186. ¬((p,q) ε r) ImpElim 185 184
187. (p,q) ε r AndElimR 178
188. _|_ ImpElim 187 186
189. (q,k) ε r AbsI 188
190. ((k,q) ε r) v ((q,k) ε r) Hyp
191. (k,q) ε r Hyp
192. _|_ ImpElim 191 167
193. (q,k) ε r AbsI 192
194. (q,k) ε r Hyp
195. (q,k) ε r OrElim 190 191 193 194 194
196. (q,k) ε r OrElim 176 177 189 190 195
197. ((q,k) ε r) & (((k,p) ε r) & ((p,q) ε r)) AndInt 196 178
198. cyc = {p,{q,k}} Hyp
199. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) &
(({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
200. k ε x AndElimL 163
201. ∃w.(k ε w) ExistsInt 200
202. Set(k) DefSub 201
203. (p ε x) & (q ε x) AndElimR 163
204. q ε x AndElimR 203
205. ∃w.(q ε w) ExistsInt 204
206. Set(q) DefSub 205
207. p ε x AndElimL 203
208. ∃w.(p ε w) ExistsInt 207
209. Set(p) DefSub 208
210. triad = ({p} ∪ ({q} ∪ {k})) Hyp
211. z ε triad Hyp
212. Set(x) -> Set({x}) TheoremInt
213. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
214. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 213
215. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 214
216. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 215
217. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 216
218. (z ε ({p} ∪ y)) -> ((z ε {p}) v (z ε y)) ForallElim 217
219. ∀y.((z ε ({p} ∪ y)) -> ((z ε {p}) v (z ε y))) ForallInt 218
220. (z ε ({p} ∪ ({q} ∪ {k}))) -> ((z ε {p}) v (z ε ({q} ∪ {k}))) ForallElim 219
221. z ε ({p} ∪ ({q} ∪ {k})) EqualitySub 211 210
222. (z ε {p}) v (z ε ({q} ∪ {k})) ImpElim 221 220
223. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
224. z ε {p} Hyp
225. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 223
226. Set(p) -> ((y ε {p}) <-> (y = p)) ForallElim 225
227. (y ε {p}) <-> (y = p) ImpElim 209 226
228. ((y ε {p}) -> (y = p)) & ((y = p) -> (y ε {p})) EquivExp 227
229. (y ε {p}) -> (y = p) AndElimL 228
230. ∀y.((y ε {p}) -> (y = p)) ForallInt 229
231. (z ε {p}) -> (z = p) ForallElim 230
232. z = p ImpElim 224 231
233. p = z Symmetry 232
234. z ε x EqualitySub 207 233
235. z ε ({q} ∪ {k}) Hyp
236. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 216
237. (z ε ({q} ∪ y)) -> ((z ε {q}) v (z ε y)) ForallElim 236
238. ∀y.((z ε ({q} ∪ y)) -> ((z ε {q}) v (z ε y))) ForallInt 237

239. (z ε ({q} ∪ {k})) -> ((z ε {q}) v (z ε {k})) ForallElim 238
240. (z ε {q}) v (z ε {k}) ImpElim 235 239
241. z ε {q} Hyp
242. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 223
243. Set(q) -> ((y ε {q}) <-> (y = q)) ForallElim 242
244. (y ε {q}) <-> (y = q) ImpElim 206 243
245. ((y ε {q}) -> (y = q)) & ((y = q) -> (y ε {q})) EquivExp 244
246. (y ε {q}) -> (y = q) AndElimL 245
247. ∀y.((y ε {q}) -> (y = q)) ForallInt 246
248. (z ε {q}) -> (z = q) ForallElim 247
249. z = q ImpElim 241 248
250. q = z Symmetry 249
251. z ε x EqualitySub 204 250
252. z ε {k} Hyp
253. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 223
254. Set(k) -> ((y ε {k}) <-> (y = k)) ForallElim 253
255. (y ε {k}) <-> (y = k) ImpElim 202 254
256. ((y ε {k}) -> (y = k)) & ((y = k) -> (y ε {k})) EquivExp 255
257. (y ε {k}) -> (y = k) AndElimL 256
258. ∀y.((y ε {k}) -> (y = k)) ForallInt 257
259. (z ε {k}) -> (z = k) ForallElim 258
260. z = k ImpElim 252 259
261. k = z Symmetry 260
262. z ε x EqualitySub 200 261
263. z ε x OrElim 240 241 251 252 262
264. z ε x OrElim 222 224 234 235 263
265. (z ε triad) -> (z ε x) ImpInt 264
266. ∀z.((z ε triad) -> (z ε x)) ForallInt 265
267. triad ⊂ x DefSub 266
268. ((triad ⊂ x) & ¬(triad = 0)) -> ∃z.First(r,triad,z) ForallElim 45
269. ∀y.((y ε {p}) <-> (y = p)) ForallInt 227
270. (p ε {p}) <-> (p = p) ForallElim 269
271. ((p ε {p}) -> (p = p)) & ((p = p) -> (p ε {p})) EquivExp 270
272. (p = p) -> (p ε {p}) AndElimR 271
273. p = p Identity
274. p ε {p} ImpElim 273 272
275. (p ε {p}) v (p ε ({q} ∪ {k})) OrIntR 274
276. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 214
277. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 276
278. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 277
279. ((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y)) ForallElim 278
280. ∀y.(((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y))) ForallInt 279
281. ((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k}))) ForallElim 280
282. ∀z.(((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k})))) ForallInt 281
283. ((p ε {p}) v (p ε ({q} ∪ {k}))) -> (p ε ({p} ∪ ({q} ∪ {k}))) ForallElim 282
284. p ε ({p} ∪ ({q} ∪ {k})) ImpElim 275 283
285. ({p} ∪ ({q} ∪ {k})) = triad Symmetry 210
286. p ε triad EqualitySub 284 285
287. ¬(x ε 0) TheoremInt
288. triad = 0 Hyp
289. 0 = triad Symmetry 288
290. p ε 0 EqualitySub 286 288
291. ∀x.¬(x ε 0) ForallInt 287
292. ¬(p ε 0) ForallElim 291
293. _|_ ImpElim 290 292
294. ¬(triad = 0) ImpInt 293
295. (triad ⊂ x) & ¬(triad = 0) AndInt 267 294
296. ∃z.First(r,triad,z) ImpElim 295 268
297. First(r,triad,l) Hyp
298. (l ε triad) & ∀y.((y ε triad) -> ¬((y,l) ε r)) DefExp 297
299. l ε triad AndElimL 298
300. l ε ({p} ∪ ({q} ∪ {k})) EqualitySub 299 210
301. ∀z.((z ε ({p} ∪ ({q} ∪ {k}))) -> ((z ε {p}) v (z ε ({q} ∪ {k})))) ForallInt 220
302. (l ε ({p} ∪ ({q} ∪ {k}))) -> ((l ε {p}) v (l ε ({q} ∪ {k}))) ForallElim 301
303. (l ε {p}) v (l ε ({q} ∪ {k})) ImpElim 300 302
304. l ε {p} Hyp
305. ∀y.((y ε {p}) -> (y = p)) ForallInt 229
306. (l ε {p}) -> (l = p) ForallElim 305
307. l = p ImpElim 304 306
308. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt

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309. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 308
310. Set(k) -> ((y ε {k}) <-> (y = k))  ForallElim 309
311. k = k  Identity
312. (y ε {k}) <-> (y = k)  ImpElim 202 310
313. ∀y.((y ε {k}) <-> (y = k))  ForallInt 312
314. (k ε {k}) <-> (k = k)  ForallElim 313
315. ((k ε {k}) -> (k = k)) & ((k = k) -> (k ε {k}))  EquivExp 314
316. (k = k) -> (k ε {k})  AndElimR 315
317. k ε {k}  ImpElim 311 316
318. (k ε {q}) v (k ε {k})  OrIntL 317
319. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 277
320. ((z ε {q}) v (z ε y)) -> (z ε ({q} ∪ y))  ForallElim 319
321. ∀y.(((z ε {q}) v (z ε y)) -> (z ε ({q} ∪ y)))  ForallInt 320
322. ((z ε {q}) v (z ε {k})) -> (z ε ({q} ∪ {k}))  ForallElim 321
323. ∀z.(((z ε {q}) v (z ε {k})) -> (z ε ({q} ∪ {k})))  ForallInt 322
324. ((k ε {q}) v (k ε {k})) -> (k ε ({q} ∪ {k}))  ForallElim 323
325. k ε ({q} ∪ {k})  ImpElim 318 324
326. (k ε {p}) v (k ε ({q} ∪ {k}))  OrIntL 325
327. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 277
328. ((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y))  ForallElim 327
329. ∀y.(((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y)))  ForallInt 328
330. ((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k})))  ForallElim 329
331. ∀z.(((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k}))))  ForallInt 330
332. ((k ε {p}) v (k ε ({q} ∪ {k}))) -> (k ε ({p} ∪ ({q} ∪ {k})))  ForallElim 331
333. k ε ({p} ∪ ({q} ∪ {k}))  ImpElim 326 332
334. ({p} ∪ ({q} ∪ {k})) = triad  Symmetry 210
335. k ε triad  EqualitySub 333 334
336. ∀y.((y ε triad) -> ¬((y,l) ε r))  AndElimR 298
337. ∀y.((y ε triad) -> ¬((y,p) ε r))  EqualitySub 336 307
338. (k ε triad) -> ¬((k,p) ε r)  ForallElim 337
339. ¬((k,p) ε r)  ImpElim 335 338
340. ((k,p) ε r) & ((p,q) ε r)  AndElimR 197
341. (k,p) ε r  AndElimL 340
342. _|_  ImpElim 341 339
343. l ε ({q} ∪ {k})  Hyp
344. (z ε (x ∪ y)) -> ((z ε x) v (z ε y))  AndElimL 276
345. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y)))  ForallInt 344
346. (z ε ({q} ∪ y)) -> ((z ε {q}) v (z ε y))  ForallElim 345
347. ∀y.((z ε ({q} ∪ y)) -> ((z ε {q}) v (z ε y)))  ForallInt 346
348. (z ε ({q} ∪ {k})) -> ((z ε {q}) v (z ε {k}))  ForallElim 347
349. ∀z.((z ε ({q} ∪ {k})) -> ((z ε {q}) v (z ε {k})))  ForallInt 348
350. (l ε ({q} ∪ {k})) -> ((l ε {q}) v (l ε {k}))  ForallElim 349
351. (l ε {q}) v (l ε {k})  ImpElim 343 350
352. l ε {q}  Hyp
353. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 308
354. Set(q) -> ((y ε {q}) <-> (y = q))  ForallElim 353
355. ∀y.(Set(q) -> ((y ε {q}) <-> (y = q)))  ForallInt 354
356. Set(q) -> ((l ε {q}) <-> (l = q))  ForallElim 355
357. (l ε {q}) <-> (l = q)  ImpElim 206 356
358. ((l ε {q}) -> (l = q)) & ((l = q) -> (l ε {q}))  EquivExp 357
359. (l ε {q}) -> (l = q)  AndElimL 358
360. l = q  ImpElim 352 359
361. ∀y.((y ε triad) -> ¬((y,l) ε r))  AndElimR 298
362. ∀y.((y ε triad) -> ¬((y,q) ε r))  EqualitySub 361 360
363. (p ε triad) -> ¬((p,q) ε r)  ForallElim 362
364. ¬((p,q) ε r)  ImpElim 286 363
365. (p,q) ε r  AndElimR 340
366. _|_  ImpElim 365 364
367. l ε {k}  Hyp
368. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 308
369. Set(k) -> ((y ε {k}) <-> (y = k))  ForallElim 368
370. (y ε {k}) <-> (y = k)  ImpElim 202 369
371. ∀y.((y ε {k}) <-> (y = k))  ForallInt 370
372. (l ε {k}) <-> (l = k)  ForallElim 371
373. ((l ε {k}) -> (l = k)) & ((l = k) -> (l ε {k}))  EquivExp 372
374. (l ε {k}) -> (l = k)  AndElimL 373
375. l = k  ImpElim 367 374
376. ∀y.((y ε triad) -> ¬((y,k) ε r))  EqualitySub 361 375
377. (q ε triad) -> ¬((q,k) ε r)  ForallElim 376
378. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 308
379. Set(q) -> ((y ε {q}) <-> (y = q))  ForallElim 378
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380. (y ε {q}) <-> (y = q)  ImpElim 206 379
381. ∀y.((y ε {q}) <-> (y = q))  ForallInt 380
382. (q ε {q}) <-> (q = q)  ForallElim 381
383. q = q  Identity
384. ((q ε {q}) -> (q = q)) & ((q = q) -> (q ε {q}))  EquivExp 382
385. (q = q) -> (q ε {q})  AndElimR 384
386. q ε {q}  ImpElim 383 385
387. (q ε {q}) v (q ε {k})  OrIntR 386
388. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 277
389. ((z ε {q}) v (z ε y)) -> (z ε ({q} ∪ y))  ForallElim 388
390. ∀y.(((z ε {q}) v (z ε y)) -> (z ε ({q} ∪ y)))  ForallInt 389
391. ((z ε {q}) v (z ε {k})) -> (z ε ({q} ∪ {k}))  ForallElim 390
392. ∀z.(((z ε {q}) v (z ε {k})) -> (z ε ({q} ∪ {k})))  ForallInt 391
393. ((q ε {q}) v (q ε {k})) -> (q ε ({q} ∪ {k}))  ForallElim 392
394. q ε ({q} ∪ {k})  ImpElim 387 393
395. (q ε {p}) v (q ε ({q} ∪ {k}))  OrIntL 394
396. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 277
397. ((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y))  ForallElim 396
398. ∀y.(((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y)))  ForallInt 397
399. ((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k})))  ForallElim 398
400. ∀z.(((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k}))))  ForallInt 399
401. ((q ε {p}) v (q ε ({q} ∪ {k}))) -> (q ε ({p} ∪ ({q} ∪ {k})))  ForallElim 400
402. q ε ({p} ∪ ({q} ∪ {k}))  ImpElim 395 401
403. ({p} ∪ ({q} ∪ {k})) = triad  Symmetry 210
404. q ε triad  EqualitySub 402 403
405. ∀y.((y ε triad) -> ¬((y,k) ε r))  EqualitySub 361 375
406. (q ε triad) -> ¬((q,k) ε r)  ForallElim 405
407. ¬((q,k) ε r)  ImpElim 404 406
408. (q,k) ε r  AndElimL 197
409. _|_  ImpElim 408 407
410. _|_  OrElim 351 352 366 367 409
411. _|_  OrElim 303 304 342 343 410
412. _|_  ExistsElim 296 297 411
413. ¬(triad = ({p} ∪ ({q} ∪ {k})))  ImpInt 412
414. ∀triad.¬(triad = ({p} ∪ ({q} ∪ {k})))  ForallInt 413
415. ¬(({p} ∪ ({q} ∪ {k})) = ({p} ∪ ({q} ∪ {k})))  ForallElim 414
416. ({p} ∪ ({q} ∪ {k})) = ({p} ∪ ({q} ∪ {k}))  Identity
417. _|_  ImpElim 416 415
418. _|_  ExistsElim 126 127 417
419. _|_  ExistsElim 123 124 418
420. _|_  ExistsElim 120 121 419
421. ¬¬TransIn(r,x)  ImpInt 420
422. D <-> ¬¬D  TheoremInt
423. (D -> ¬¬D) & (¬¬D -> D)  EquivExp 422
424. ¬¬D -> D  AndElimR 423
425. ¬¬TransIn(r,x) -> TransIn(r,x)  PolySub 424
426. TransIn(r,x)  ImpElim 421 425
427. Asymmetric(r,x) & TransIn(r,x)  AndInt 115 426
428. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  ImpInt 427 Qed
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Used Theorems

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1. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))
2. ¬(x ε 0)
3. (B v ¬A) -> (A -> B)
5. ¬∀i.P(i) -> ∃c.¬P(c)
7. (A -> B) -> (¬B -> ¬A)
6. (B v ¬A) -> (A -> B)
8. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
9. D <-> ¬¬D
10. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))
11. Set(x) -> Set({x})
12. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
13. Set(x) -> ((y ε {x}) <-> (y = x))
14. ¬(x ε 0)
13. Set(x) -> ((y ε {x}) <-> (y = x))
9. D <-> ¬¬D
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Succesfully checked 52 theorems with a total of 5579 lines in 23 seconds.

Th4. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

0. Hyp("Elem(z, union(x,y))") 1. DefEqInt(0) 2. EqualitySub(0,1,[0]) 3. ClassElim(2) 4. AndElimR(3) 5. ImpInt(4,0) 6. Hyp("(Elem(z,x) v Elem(z,y))") 7. Hyp("Elem(z,x)") 8. ExistsInt(7,"x","x",[0]) 9. DefSub(8,"Set",["z"],[0]) 10. Hyp("Elem(z,y)") 11. ExistsInt(10,"y","y",[0]) 12. DefSub(11,"Set",["z"],[0]) 13. OrElim(6,7,9,10,12) 14. AndInt(13,6) 15. ClassInt(14,"z") 16. Symmetry(1) 17. EqualitySub(15,16,[0]) 18. ImpInt(17,6) 19. AndInt(5,18) 20. EquivConst(19) 21. Hyp("Elem(z, intersection(x,y))") 22. DefEqInt(1) 23. EqualitySub(21,22,[0]) 24. ClassElim(23) 25. AndElimR(24) 26. ImpInt(25,21) 27. Hyp("(Elem(z,x) & Elem(z,y))") 28. AndElimL(27) 29. ExistsInt(28,"x","x",[0]) 30. DefSub(29,"Set", ["z"],[0]) 31. AndInt(30,27) 32. ClassInt(31,"z") 33. Symmetry(22) 34. EqualitySub(32,33,[0]) 35. ImpInt(34,27) 36. AndInt(26,35) 37. EquivConst(36) 38. AndInt(20,37)

Th5. ((x ∪ x) = x) & ((x ∩ x) = x)

0. Hyp("Elem(z,union(x,x))") 1. TheoremInt(1) 2. AndElimL(1) 3. EquivExp(2) 4. AndElimL(3) 5. ForallInt(4,"y","y") 6. ForallElim(5,"x") 7. ImpElim(0,6) 8. Hyp("Elem(z,x)") 9. Hyp("Elem(z,x)") 10. OrElim(7,8,8,9,9) 11. ImpInt(10,0) 12. Hyp("Elem(z,x)") 13. OrIntL(12,"Elem(z,x)") 14. AndElimR(3) 15. ForallInt(14,"y","y") 16. ForallElim(15,"x") 17. ImpElim(13,16) 18. ImpInt(17,12) 19. AndInt(11,18) 20. EquivConst(19) 21. ForallInt(20,"z","z") 22. AxInt(0) 23. ForallElim(22,"union(x,x)") 24. ForallElim(23,"x") 25. EquivExp(24) 26. AndElimR(25) 27. ImpElim(21,26) 28. Hyp("Elem(z, intersection(x,x))") 29. AndElimR(1) 30. EquivExp(29) 31. AndElimL(30) 32. ForallInt(31,"y","y") 33. ForallElim(32,"x") 34. ImpElim(28,33) 35. AndElimR(34) 36. ImpInt(35,28) 37. Hyp("Elem(z,x)") 38. AndInt(37,37) 39. AndElimR(30) 40. ForallInt(39,"y","y") 41. ForallElim(40,"x") 42. ImpElim(38,41) 43. ImpInt(42,37) 44. AndInt(36,43) 45. EquivConst(44) 46. ForallElim(22,"intersection(x,x)") 47. ForallElim(46,"x") 48. EquivExp(47) 49. AndElimR(48) 50. ForallInt(45,"z","z") 51. ImpElim(50,49) 52. AndInt(27,51)

Th6. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))

0. TheoremInt(2) 1. AndElimL(0) 2. EquivExp(1) 3. AndElimL(2) 4. Hyp("Elem(z,union(x,y))") 5. ImpElim(4,3) 6. TheoremInt(1) 7. PolySub(6,"A","Elem(z,x)") 8. PolySub(7,"B","Elem(z,y)") 9. ImpElim(5,8) 10. AndElimR(2) 11. ForallInt(10,"x","x") 12. ForallElim(11,"w") 13. ForallInt(12,"y","y") 14. ForallElim(13,"x") 15. ForallInt(14,"w","w") 16. ForallElim(15,"y") 17. ImpElim(9,16) 18. ImpInt(17,4) 19. ForallInt(18,"x","x") 20. ForallElim(19,"w") 21. ForallInt(20,"y","y") 22. ForallElim(21,"v") 23. ForallInt(22,"w","w") 24. ForallElim(23,"y") 25. ForallInt(24,"v","v") 26. ForallElim(25,"x") 27. AndInt(18,26) 28. AxInt(0) 29. ForallElim(28,"union(x,y)") 30. ForallElim(29,"union(y,x)") 31. EquivExp(30) 32. AndElimR(31) 33. EquivConst(27) 34. ForallInt(33,"z","z") 35. ImpElim(34,32) 36. Hyp("Elem(z, intersection(x,y))") 37. AndElimR(0) 38. EquivExp(37) 39. AndElimL(38) 40. ImpElim(36,39) 41. TheoremInt(3) 42. PolySub(41,"A","Elem(z,x)") 43. PolySub(42,"B","Elem(z,y)") 44. ImpElim(40,43) 45. AndElimR(38) 46. ForallInt(45,"x","w") 47. ForallInt(46,"y","v") 48. ForallElim(47,"x") 49. ForallElim(48,"y") 50. ImpElim(44,49) 51. ImpInt(50,36) 52. ForallInt(51,"x","v") 53. ForallInt(52,"y","w") 54. ForallElim(53,"x") 55. ForallElim(54,"y") 56. AndInt(51,55) 57. ForallElim(28,"intersection(x,y)") 58. ForallElim(57,"intersection(y,x)") 59. EquivExp(58) 60. AndElimR(59) 61. EquivConst(56) 62. ForallInt(61,"z","z") 63. ImpElim(62,60) 64. AndInt(35,63)

Th7. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))

0. Hyp("Elem(w, union(union(x,y),z))") 1. TheoremInt(3) 2. AndElimL(1) 3. EquivExp(2) 4. AndElimL(3) 5. ForallInt(4,"z","z") 6. ForallElim(5,"w") 7. ForallInt(6,"x","x") 8. ForallElim(7,"a") 9. ForallInt(8,"y","y") 10. ForallElim(9,"z") 11. ForallInt(10,"a","a") 12. ForallElim(11,"union(x,y)") 13. ImpElim(0,12) 14. Hyp("Elem(w,union(x,y))") 15. ImpElim(14,6) 16. OrIntR(15,"Elem(w,z)") 17. Hyp("Elem(w,z)") 18. OrIntL(17,"(Elem(w,x) v Elem(w,y))") 19. OrElim(13,14,16,17,18) 20. TheoremInt(1) 21. PolySub(20,"A","Elem(w,x)") 22. PolySub(21,"B","Elem(w,y)") 23. PolySub(22,"C","Elem(w,z)") 24. EquivExp(23) 25. AndElimL(24) 26. ImpElim(19,25)

27. AndElimR(3) 28. ForallInt(27,"z","z") 29. ForallElim(28,"w") 30. ForallInt(29,"x","x") 31. ForallElim(30,"a") 32. ForallInt(31,"y","y") 33. ForallElim(32,"z") 34. ForallInt(33,"a","a") 35. ForallElim(34,"y") 36. Hyp("(Elem(w,y) v Elem(w,z))") 37. ImpElim(36,35) 38. OrIntL(37,"Elem(w,x)") 39. ForallInt(31,"y","y") 40. ForallElim(32,"union(y,z)") 41. ForallInt(40,"a","a") 42. ForallElim(41,"x") 43. ImpElim(38,42) 44. Hyp("Elem(w,x)") 45. OrIntR(44,"Elem(w,union(y,z))") 46. ForallInt(31,"y","y") 47. ForallElim(32,"union(y,z)") 48. ForallInt(47,"a","a") 49. ForallElim(48,"x") 50. ImpElim(45,49) 51. OrElim(26,44,50,36,43) 52. ImpInt(51,0) 53. Hyp("Elem(w,union(x, union(y,z)))") 54. ForallInt(8,"y","y") 55. ForallElim(9,"union(y,z)") 56. ForallInt(55,"a","a") 57. ForallElim(56,"x") 58. ImpElim(53,57) 59. Hyp("Elem(w,x)") 60. OrIntR(59,"(Elem(w,y) v Elem(w,z))") 61. Hyp("Elem(w, union(y,z))") 62. ForallInt(10,"a","a") 63. ForallElim(11,"y") 64. ImpElim(61,63) 65. OrIntL(64,"Elem(w,x)") 66. OrElim(58,59,60,61,65) 67. AndElimR(24) 68. ImpElim(66,67) 69. Hyp("(Elem(w,x) v Elem(w,y))") 70. ForallInt(27,"z","z") 71. ForallElim(28,"w") 72. ImpElim(69,71) 73. OrIntR(72,"Elem(w,z)") 74. Hyp("Elem(w,z)") 75. OrIntL(74,"Elem(w,union(x,y))") 76. OrElim(68,69,73,74,75) 77. ForallInt(33,"a","a") 78. ForallElim(34,"union(x,y)") 79. ImpElim(76,78) 80. ImpInt(79,53) 81. AndInt(52,80) 82. EquivConst(81) 83. Hyp("Elem(w, intersection(intersection(x,y), z))") 84. AndElimR(1) 85. ForallInt(84,"z","z") 86. ForallElim(85,"w") 87. ForallInt(86,"x","x") 88. ForallElim(87,"a") 89. ForallInt(88,"y","y") 90. ForallElim(89,"b") 91. ForallInt(90,"a","a") 92. ForallElim(91,"intersection(x,y)") 93. ForallInt(92,"b","b") 94. ForallElim(93,"z") 95. EquivExp(94) 96. AndElimL(95) 97. ImpElim(83,96) 98. AndElimL(97) 99. EquivExp(86) 100. AndElimL(99) 101. ImpElim(98,100) 102. AndElimR(97) 103. AndElimL(101) 104. AndElimR(101) 105. AndInt(104,102) 106. EquivExp(90) 107. AndElimR(106) 108. ForallInt(107,"a","a") 109. ForallElim(108,"y") 110. ForallInt(109,"b","b") 111. ForallElim(110,"z") 112. ImpElim(105,111) 113. AndInt(103,112) 114. ForallInt(107,"a","a") 115. ForallElim(108,"x") 116. ForallInt(115,"b","b") 117. ForallElim(116,"intersection(y,z)") 118. ImpElim(113,117) 119. ImpInt(118,83) 120. Hyp("Elem(w,intersection(x,intersection(y,z)))") 121. AndElimL(106) 122. ForallInt(121,"a","a") 123. ForallElim(122,"x") 124. ForallInt(123,"b","b") 125. ForallInt(123,"b","b") 126. ForallElim(124,"intersection(y,z)") 127. ImpElim(120,126) 128. AndElimR(127) 129. AndElimL(127) 130. ForallInt(121,"a","a") 131. ForallElim(122,"y") 132. ForallInt(131,"b","b") 133. ForallElim(132,"z") 134. ImpElim(128,133) 135. AndElimL(134) 136. AndElimR(134) 137. AndInt(129,135) 138. AndElimR(99) 139. ImpElim(137,138) 140. AndInt(139,136) 141. ForallInt(121,"a","a") 142. ForallInt(107,"a","a") 143. ForallElim(108,"intersection(x,y)") 144. ForallInt(143,"b","b") 145. ForallElim(144,"z") 146. ImpElim(140,145) 147. ImpInt(146,120) 148. AndInt(119,147) 149. EquivConst(148) 150. AndInt(82,149) 151. AndElimR(150) 152. AxInt(0) 153. ForallElim(152,"intersection(intersection(x,y),z)") 154. ForallElim(153,"intersection(x, intersection(y,z))") 155. ForallInt(151,"w","w") 156. EquivExp(154) 157. AndElimR(156) 158. ImpElim(155,157) 159. ForallElim(152,"union(union(x,y),z)") 160. ForallElim(159,"union(x, union(y,z))") 161. EquivExp(160) 162. AndElimR(161) 163. AndElimL(150) 164. ForallInt(163,"w","w") 165. ImpElim(164,162) 166. AndInt(165,158)

Th8. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))

0. Hyp("Elem(w, intersection(x, union(y,z)))") 1. TheoremInt(1) 2. ForallInt(1,"z","z") 3. ForallElim(2,"w") 4. ForallInt(3,"y","y") 5. ForallElim(4,"a") 6. AndElimR(5) 7. EquivExp(6) 8. AndElimL(7) 9. ForallInt(8,"a","a") 10. ForallElim(9,"union(y,z)") 11. ImpElim(0,10) 12. AndElimR(11) 13. AndElimL(11) 14. AndElimL(5) 15. ForallInt(14,"x","x") 16. ForallElim(15,"b") 17. ForallInt(16,"b","b") 18. ForallElim(17,"y") 19. ForallInt(18,"a","a") 20. ForallElim(19,"z") 21. EquivExp(20) 22. AndElimL(21) 23. ImpElim(12,22) 24. AndInt(13,23) 25. TheoremInt(2) 26. PolySub(25,"A","Elem(w,x)") 27. PolySub(26,"B","Elem(w,y)") 28. PolySub(27,"C","Elem(w,z)") 29. EquivExp(28) 30. AndElimL(29) 31. ImpElim(24,30) 32. Hyp("(Elem(w,x) & Elem(w,y))") 33. AndElimR(3) 34. EquivExp(33) 35. AndElimR(34) 36. ImpElim(32,35) 37. OrIntR(36,"Elem(w, intersection(x,z))") 38. Hyp("(Elem(w,x) & Elem(w,z))") 39. ForallInt(35,"y","y") 40. ForallElim(39,"z") 41. ImpElim(38,40) 42. OrIntL(41,"Elem(w,intersection(x,y))") 43. OrElim(31,32,37,38,42) 44. EquivExp(16) 45. AndElimR(44) 46. ForallInt(45,"b","b") 47. ForallElim(46,"intersection(x,y)") 48. ForallInt(47,"a","a") 49. ForallElim(48,"intersection(x,z)") 50. ImpElim(43,49) 51. ImpInt(50,0) 52. Hyp("Elem(w,union(intersection(x,y), intersection(x,z)))") 53. AndElimL(44) 54. ForallInt(53,"b","b") 55. ForallElim(54,"intersection(x,y)")

56. ForallInt(55,"a","a") 57. ForallElim(56,"intersection(x,z)") 58. ImpElim(52,57) 59. ForallInt(8,"a","a") 60. ForallElim(9,"y") 61. ForallInt(8,"a","a") 62. ForallElim(9,"z") 63. Hyp("Elem(w,intersection(x,y))") 64. ImpElim(63,60) 65. AndElimR(64) 66. OrIntR(65,"Elem(w,z)") 67. AndElimR(44) 68. ForallInt(67,"b","b") 69. ForallElim(68,"y") 70. ForallInt(69,"a","a") 71. ForallElim(70,"z") 72. ImpElim(66,71) 73. AndElimL(64) 74. AndInt(73,72) 75. AndElimR(7) 76. ForallInt(75,"a","a") 77. ForallElim(76,"union(y,z)") 78. ImpElim(74,77) 79. Hyp("Elem(w,intersection(x,z))") 80. ImpElim(79,62) 81. AndElimL(80) 82. AndElimR(80) 83. OrIntL(82,"Elem(w,y)") 84. ImpElim(83,71) 85. AndInt(81,84) 86. ImpElim(85,77) 87. OrElim(58,63,78,79,86) 88. ImpInt(87,52) 89. AndInt(51,88) 90. EquivConst(89) 91. Hyp("Elem(w, union(x, intersection(y,z)))") 92. EquivExp(16) 93. ForallInt(92,"b","b") 94. ForallElim(93,"x") 95. ForallInt(94,"a","a") 96. ForallElim(95,"intersection(y,z)") 97. AndElimL(96) 98. ImpElim(91,97) 99. Hyp("Elem(w,x)") 100. OrIntR(99,"Elem(w,y)") 101. AndElimR(92) 102. ForallInt(101,"b","b") 103. ForallElim(102,"x") 104. ForallInt(103,"a","a") 105. ForallElim(104,"y") 106. ImpElim(100,105) 107. OrIntR(99,"Elem(w,z)") 108. ForallInt(103,"a","a") 109. ForallElim(104,"z") 110. ImpElim(107,109) 111. AndInt(106,110) 112. ForallInt(6,"x","x") 113. ForallElim(112,"b") 114. EquivExp(113) 115. AndElimR(114) 116. ForallInt(115,"b","b") 117. ForallElim(116,"union(x,y)") 118. ForallInt(117,"a","a") 119. ForallElim(118,"union(x,z)") 120. ImpElim(111,119) 121. Hyp("Elem(w,intersection(y,z))") 122. AndElimL(114) 123. ForallInt(122,"b","b") 124. ForallElim(123,"y") 125. ForallInt(124,"a","a") 126. ForallElim(125,"z") 127. ImpElim(121,126) 128. AndElimL(127) 129. AndElimR(127) 130. OrIntL(128,"Elem(w,x)") 131. OrIntL(129,"Elem(w,x)") 132. ImpElim(131,109) 133. AndElimL(1) 134. EquivExp(133) 135. AndElimR(134) 136. ForallInt(135,"z","z") 137. ForallElim(136,"w") 138. ImpElim(130,137) 139. AndInt(138,132) 140. ImpElim(139,119) 141. OrElim(98,99,120,121,140) 142. ImpInt(141,91) 143. Hyp("Elem(w, intersection(union(x,y),union(x,z)))") 144. AndElimL(114) 145. ForallInt(114,"b","b") 146. ForallElim(145,"union(x,y)") 147. ForallInt(146,"a","a") 148. ForallElim(147,"union(x,z)") 149. AndElimL(148) 150. ImpElim(143,149) 151. AndElimL(150) 152. AndElimR(150) 153. AndElimL(134) 154. ForallInt(153,"z","z") 155. ForallElim(154,"w") 156. ForallInt(155,"y","y") 157. ForallElim(156,"z") 158. ImpElim(151,155) 159. ImpElim(152,157) 160. Hyp("Elem(w,x)") 161. OrIntR(160,"Elem(w,intersection(y,z))") 162. EquivExp(14) 163. AndElimR(162) 164. ForallInt(163,"a","a") 165. ForallElim(164,"intersection(y,z)") 166. ImpElim(161,165) 167. ImpInt(166,160) 168. Hyp("Elem(w,y)") 169. Hyp("Elem(w,x)") 170. ImpElim(169,167) 171. Hyp("Elem(w,z)") 172. AndInt(168,171) 173. ForallInt(115,"a","a") 174. ForallElim(116,"y") 175. ForallInt(174,"a","a") 176. ForallElim(175,"z") 177. ImpElim(172,176) 178. OrIntL(177,"Elem(w,x)") 179. ImpElim(178,165) 180. OrElim(159,169,170,171,179) 181. OrElim(158,160,166,168,180) 182. ImpInt(181,143) 183. AndInt(142,182) 184. EquivConst(183) 185. AndInt(90,184) 186. AndElimR(185) 187. AndElimL(185) 188. ForallInt(186,"w","w") 189. ForallInt(187,"w","w") 190. AxInt(0) 191. ForallElim(190,"intersection(x,union(y,z))") 192. ForallElim(191,"union(intersection(x,y),intersection(x,z))") 193. EquivExp(192) 194. AndElimR(193) 195. ImpElim(189,194) 196. ForallElim(190,"union(x,intersection(y,z))") 197. ForallElim(196,"intersection(union(x,y),union(x,z))") 198. EquivExp(197) 199. AndElimR(198) 200. ImpElim(188,199) 201. AndInt(195,200)

Th11. ~~x = x

0. Hyp("Elem(z, complement1(complement1(x)))") 1. DefEqInt(2) 2. ForallInt(1,"x","x") 3. ForallElim(2,"complement1(x)") 4. EqualitySub(0,3,[0]) 5. ClassElim(4) 6. AndElimR(5) 7. Hyp("neg Elem(z,x)") 8. AndElimL(5) 9. AndInt(8,7) 10. ClassInt(9,"y") 11. Symmetry(1) 12. EqualitySub(10,11,[0]) 13. ImpElim(12,6) 14. ImpInt(13,7) 15. TheoremInt(1) 16. PolySub(15,"D","Elem(z,x)") 17. EquivExp(16) 18. AndElimR(17) 19. ImpElim(14,18) 20. ImpInt(19,0) 21. Hyp("Elem(z,x)") 22. AndElimL(17) 23. ImpElim(21,22) 24. Hyp("Elem(z,complement1(x))") 25. EqualitySub(24,1,[0]) 26. ClassElim(25) 27. AndElimR(26) 28. ImpElim(27,23) 29. ImpInt(28,24) 30. ExistsInt(21,"x","y",[0]) 31. DefSub(30,"Set",["z"],[0]) 32. AndInt(31,29) 33. ClassInt(32,"y") 34. Symmetry(3) 35. EqualitySub(33,34,[0]) 36. ImpInt(35,21) 37. AndInt(20,36) 38. EquivConst(37) 39. AxInt(0) 40. ForallElim(39,"complement1(complement1(x))") 41. ForallElim(40,"x") 42. EquivExp(41) 43. AndElimR(42) 44. ForallInt(38,"z","z") 45. ImpElim(44,43)

Th12. (~(x ∪ y) = (~x ∩ ~y)) & (~(x ∩ y) = (~x ∪ ~y))

0. Hyp("Elem(z, complement1(union(x,y)))") 1. DefEqInt(2) 2. ForallInt(1,"x","a") 3. ForallElim(2,"union(x,y)") 4. EqualitySub(0,3,[0]) 5. ClassElim(4) 6. TheoremInt(2) 7. AndElimL(6) 8. EquivExp(7) 9. AndElimR(8) 10. TheoremInt(3) 11. PolySub(10,"A","(Elem(z,x) v Elem(z,y))") 12. PolySub(11,"B","Elem(z,union(x,y))") 13. ImpElim(9,12) 14. AndElimR(5) 15. ImpElim(14,13) 16. TheoremInt(1) 17. PolySub(16,"A","Elem(z,x)") 18. PolySub(17,"B","Elem(z,y)") 19. AndElimL(18) 20. EquivExp(19) 21. AndElimL(20) 22. ImpElim(15,21) 23. AndElimL(5) 24. AndElimL(22) 25. AndElimR(22) 26. AndInt(23,25) 27. ClassInt(26,"z") 28. AndInt(23,24) 29. ClassInt(28,"z") 30. DefEqInt(2) 31. Symmetry(30) 32. EqualitySub(29,31,[0]) 33. ForallInt(30,"x","w") 34. ForallElim(33,"y") 35. Symmetry(34) 36. EqualitySub(27,35,[0]) 37. AndInt(32,36) 38. AndElimR(6) 39. EquivExp(38) 40. AndElimR(39) 41. ForallInt(40,"x","x") 42. ForallElim(41,"complement1(x)") 43. ForallInt(42,"y","y") 44. ForallElim(43,"complement1(y)") 45. ImpElim(37,44) 46. ImpInt(45,0) 47. Hyp("Elem(z, intersection(complement1(x), complement1(y)))") 48. ForallInt(38,"x","x") 49. ForallElim(48,"complement1(x)") 50. ForallInt(49,"y","y") 51. ForallElim(50,"complement1(y)") 52. EquivExp(51) 53. AndElimL(52) 54. ImpElim(47,53) 55. AndElimR(54) 56. AndElimL(54) 57. EqualitySub(56,30,[0]) 58. EqualitySub(55,34,[0]) 59. ClassElim(57) 60. ClassElim(58) 61. AndElimR(59) 62. AndElimR(60) 63. AndInt(61,62) 64. AndElimR(20) 65. ImpElim(63,64) 66. Hyp("Elem(z, union(x,y))") 67. AndElimL(8) 68. ImpElim(66,67) 69. ImpElim(68,65) 70. ImpInt(69,66) 71. AndElimL(59) 72. AndInt(71,70) 73. ClassInt(72,"w") 74. ForallInt(35,"y","y") 75. ForallElim(74,"union(x,y)") 76. EqualitySub(73,75,[0]) 77. ImpInt(76,47) 78. AndInt(46,77) 79. EquivConst(78) 80. Hyp("Elem(z,complement1(intersection(x,y)))") 81. ForallInt(34,"y","y") 82. ForallElim(81,"intersection(x,y)") 83. EqualitySub(80,82,[0]) 84. ClassElim(83) 85. AndElimR(39) 86. PolySub(10,"A","(Elem(z,x) & Elem(z,y))") 87. PolySub(86,"B","Elem(z,intersection(x,y))") 88. ImpElim(85,87) 89. AndElimR(84) 90. ImpElim(89,88) 91. AndElimR(16) 92. PolySub(91,"A","Elem(z,x)") 93. PolySub(92,"B","Elem(z,y)") 94. EquivExp(93) 95. AndElimL(94) 96. ImpElim(90,95) 97. Hyp("neg Elem(z,x)") 98. AndElimL(84) 99. AndInt(98,97) 100. ClassInt(99,"w") 101. OrIntR(100,"Elem(z, extension w. neg Elem(w,y))") 102. Symmetry(30) 103. ForallInt(102,"x","x") 104. ForallElim(103,"y") 105. EqualitySub(101,102,[0]) 106. EqualitySub(105,104,[0]) 107. ForallInt(9,"x","x") 108. ForallElim(107,"complement1(x)") 109. ForallInt(108,"y","y") 110. ForallElim(109,"complement1(y)") 111. ImpElim(106,110) 112. Hyp("neg Elem(z,y)") 113. AndInt(98,112) 114. ClassInt(113,"z") 115. OrIntL(114,"Elem(z,extension z. neg Elem(z,x))") 116. EqualitySub(115,102,[0]) 117. EqualitySub(116,104,[0]) 118. ImpElim(117,110) 119. OrElim(96,97,111,112,118) 120. ImpInt(119,80) 121. Hyp("Elem(z, union(complement1(x), complement1(y)))") 122. ExistsInt(121,"union(complement1(x),complement1(y))","w",[0]) 123. DefSub(122,"Set", ["z"],[0]) 124. Identity("x") 125. Identity("x") 126. Identity("x") 127. AndElimL(8) 128. ForallInt(127,"x","x") 129. ForallElim(128,"complement1(x)") 130. ForallInt(129,"y","y") 131. ForallElim(130,"complement1(y)") 132. ImpElim(121,131) 133. Hyp("Elem(z,complement1(x))") 134. EqualitySub(133,30,[0]) 135. ClassElim(134) 136. AndElimR(135) 137. Hyp("Elem(z,complement1(y))") 138. ForallInt(30,"x","x") 139. ForallElim(138,"y") 140. EqualitySub(137,139,[0]) 141. ClassElim(140) 142. AndElimR(141) 143. OrIntR(136,"neg Elem(z,y)") 144. OrIntL(142,"neg Elem(z,x)") 145. OrElim(132,133,143,137,144) 146. AndElimR(16) 147. EquivExp(146) 148. AndElimR(147) 149. PolySub(148,"A","Elem(z,x)") 150. PolySub(149,"B","Elem(z,y)") 151. ImpElim(145,150) 152. AndElimR(6) 153. EquivExp(152) 154. AndElimL(153) 155. PolySub(10,"A","Elem(z, intersection(x,y))") 156. PolySub(155,"B","(Elem(z,x) & Elem(z,y))") 157. ImpElim(154,156) 158. ImpElim(151,157) 159. DefSub(122,"Set",["z"],[0]) 160. AndInt(159,158) 161. ClassInt(160,"w") 162. ForallInt(31,"x","x") 163. ForallElim(162,"intersection(x,y)") 164. EqualitySub(161,163,[0]) 165. ImpInt(164,121) 166. AndInt(120,165) 167. EquivConst(166) 168. AxInt(0) 169. ForallElim(168,"complement1(union(x,y))") 170. ForallElim(169,"intersection(complement1(x),complement1(y))") 171. ForallInt(79,"z","z") 172. EquivExp(170) 173. AndElimR(172) 174. ImpElim(171,173) 175. ForallElim(168,"complement1(intersection(x,y))") 176. ForallElim(175,"union(complement1(x),complement1(y))") 177. EquivExp(176) 178. AndElimR(177) 179. ForallInt(167,"z","z") 180. ImpElim(179,178) 181. AndInt(174,180)

Th14. (x ∩ (y ~ z)) = ((x ∩ y) ∩ ~z)

0. DefEqInt(3) 1. ForallInt(0,"x","a") 2. ForallInt(1,"y","b") 3. ForallElim(2,"z") 4. ForallElim(3,"y") 5.

Identity("intersection(x,complement2(y,z))") 6. EqualitySub(5,4,[1]) 7. TheoremInt(4) 8. AndElimR(7) 9. Symmetry(8) 10. ForallInt(9,"z","z") 11. ForallElim(10,"complement1(z)") 12. EqualitySub(6,11,[0])

Th16. ¬(x ε 0)

0. Hyp("Elem(x,0)") 1. DefEqInt(4) 2. EqualitySub(0,1,[0]) 3. ClassElim(2) 4. AndElimR(3) 5. Identity("x") 6. ImpElim(5,4) 7. ImpInt(6,0)

Th17. ((0 U x) = x) & ((0 ∩ x) = 0)

0. Hyp("Elem(z,union(0,x))") 1. DefEqInt(0) 2. ForallInt(1,"x","x") 3. ForallElim(2,"0") 4. ForallInt(3,"y","y") 5. ForallElim(4,"x") 6. EqualitySub(0,5,[0]) 7. ClassElim(6) 8. AndElimR(7) 9. Hyp("Elem(z,0)") 10. TheoremInt(2) 11. ForallInt(10,"x","x") 12. ForallElim(11,"z") 13. ImpElim(9,12) 14. AbsI(13,"Elem(z,x)") 15. Hyp("Elem(z,x)") 16. OrElim(8,9,14,15,15) 17. ImpInt(16,0) 18. Hyp("Elem(z,x)") 19. OrIntL(18,"Elem(z,0)") 20. ExistsInt(18,"x","x",[0]) 21. DefSub(20,"Set",["z"],[0]) 22. AndInt(21,19) 23. ClassInt(22,"z") 24. Symmetry(5) 25. EqualitySub(23,24,[0]) 26. ImpInt(25,18) 27. AndInt(17,26) 28. EquivConst(27) 29. ForallInt(28,"z","z") 30. AxInt(0) 31. ForallElim(30,"union(0,x)") 32. ForallElim(31,"x") 33. EquivExp(32) 34. AndElimR(33) 35. ImpElim(29,34) 36. Hyp("Elem(z,intersection(0,x))") 37. DefEqInt(1) 38. ForallInt(37,"x","x") 39. ForallElim(38,"0") 40. ForallInt(39,"y","y") 41. ForallElim(40,"x") 42. EqualitySub(36,41,[0]) 43. ClassElim(42) 44. AndElimR(43) 45. AndElimL(44) 46. ImpInt(45,36) 47. Hyp("Elem(z,0)") 48. ImpElim(47,12) 49. AbsI(48,"Elem(z,intersection(0,x))") 50. ImpInt(49,47) 51. AndInt(46,50) 52. EquivConst(51) 53. ForallInt(52,"z","z") 54. ForallElim(30,"intersection(0,x)") 55. ForallElim(54,"0") 56. EquivExp(55) 57. AndElimR(56) 58. ImpElim(53,57) 59. AndInt(35,58)

Th19. (x ε U) <-> Set(x)

0. Hyp("Elem(x,U)") 1. DefEqInt(5) 2. EqualitySub(0,1,[0]) 3. ClassElim(2) 4. AndElimL(3) 5. ImpInt(4,0) 6. Hyp("Set(x)") 7. Identity("x") 8. AndInt(6,7) 9. ClassInt(8,"x") 10. Symmetry(1) 11. EqualitySub(9,10,[0]) 12. ImpInt(11,6) 13. AndInt(5,12) 14. EquivConst(13)

Th20. ((x U U) = U) & ((x ∩ U) = x)

0. Hyp("Elem(z,union(x,U))") 1. TheoremInt(1) 2. AndElimL(1) 3. ForallInt(2,"y","y") 4. ForallElim(3,"U") 5. EquivExp(4) 6. AndElimL(5) 7. ImpElim(0,6) 8. Hyp("Elem(z,x)") 9. ExistsInt(8,"x","y",[0]) 10. DefSub(9,"Set",["z"],[0]) 11. TheoremInt(2) 12. EquivExp(11) 13. AndElimR(12) 14. ForallInt(13,"x","x") 15. ForallElim(14,"z") 16. ImpElim(10,15) 17. Hyp("Elem(z,U)") 18. OrElim(7,8,16,17,17) 19. ImpInt(18,0) 20. Hyp("Elem(z,U)") 21. OrIntL(20,"Elem(z,x)") 22. AndElimR(5) 23. ImpElim(21,22) 24. ImpInt(23,20) 25. AndInt(19,24) 26. EquivConst(25) 27. AxInt(0) 28. ForallElim(27,"union(x,U)") 29. ForallElim(28,"U") 30. ForallInt(26,"z","z") 31. EquivExp(29) 32. AndElimR(31) 33. ImpElim(30,32) 34. Hyp("Elem(z,intersection(x,U))") 35. AndElimR(1) 36. ForallInt(35,"y","y") 37. ForallElim(36,"U") 38. EquivExp(37) 39. AndElimL(38) 40. ImpElim(34,39) 41. AndElimL(40) 42. ImpInt(41,34) 43. Hyp("Elem(z,x)") 44. ExistsInt(43,"x","y",[0]) 45. DefSub(44,"Set",["z"],[0]) 46. ImpElim(45,15) 47. AndInt(43,46) 48. AndElimR(38) 49. ImpElim(47,48) 50. ImpInt(49,43) 51. AndInt(42,50) 52. EquivConst(51) 53. ForallInt(52,"z","z") 54. ForallElim(27,"intersection(x,U)") 55. ForallElim(54,"x") 56. EquivExp(55) 57. AndElimR(56) 58. ImpElim(53,57) 59. AndInt(33,58)

Th21. (~0 = U) & (~U = 0)

0. Hyp("Elem(z,complement1(0))") 1. DefEqInt(2) 2. ForallInt(1,"x","x") 3. ForallInt(1,"x","x") 4. ForallElim(3,"0") 5. EqualitySub(0,4,[0]) 6. ClassElim(5) 7. AndElimL(6) 8. TheoremInt(1) 9. EquivExp(8) 10. AndElimR(9) 11. ForallInt(10,"x","x") 12. ForallElim(11,"z") 13. ImpElim(7,12) 14. ImpInt(13,0) 15. Hyp("Elem(z,U)") 16. AndElimL(9) 17. ForallInt(16,"x","x") 18. ForallElim(17,"z") 19. ImpElim(15,18) 20. TheoremInt(2) 21.

ForallInt(20,"x","x") 22. ForallElim(21,"z") 23. AndInt(19,22) 24. ClassInt(23,"y") 25. Symmetry(4) 26. EqualitySub(24,25,[0]) 27. ImpInt(26,15) 28. AndInt(14,27) 29. EquivConst(28) 30. ForallInt(29,"z","z") 31. AxInt(0) 32. ForallElim(31,"complement1(0)") 33. ForallElim(32,"U") 34. EquivExp(33) 35. AndElimR(34) 36. ImpElim(30,35) 37. Hyp("Elem(z,complement1(U))") 38. ForallInt(1,"x","x") 39. ForallElim(38,"U") 40. EqualitySub(37,39,[0]) 41. ClassElim(40) 42. AndElimR(41) 43. AndElimL(41) 44. ImpElim(43,12) 45. ImpElim(44,42) 46. AbsI(45,"Elem(z,0)") 47. ImpInt(46,37) 48. Hyp("Elem(z,0)") 49. DefEqInt(4) 50. EqualitySub(48,49,[0]) 51. ClassElim(50) 52. AndElimL(51) 53. AndElimR(51) 54. Identity("z") 55. ImpElim(54,53) 56. AbsI(55,"Elem(z,complement1(U))") 57. ImpInt(56,48) 58. AndInt(47,57) 59. EquivConst(58) 60. ForallInt(59,"z","z") 61. ForallElim(31,"complement1(U)") 62. ForallElim(61,"0") 63. EquivExp(62) 64. AndElimR(63) 65. ImpElim(60,64) 66. AndInt(36,65)

Th24. (∩0 = U) & (∪0 = 0)

0. Hyp("Elem(x,bigintersection(0))") 1. DefEqInt(7) 2. ForallInt(1,"x","x") 3. ForallElim(2,"0") 4. EqualitySub(0,3,[0]) 5. ClassElim(4) 6. AndElimL(5) 7. TheoremInt(1) 8. EquivExp(7) 9. AndElimR(8) 10. ImpElim(6,9) 11. ImpInt(10,0) 12. Hyp("Elem(x,U)") 13. Hyp("Elem(y,0)") 14. TheoremInt(2) 15. ForallInt(14,"x","x") 16. ForallElim(15,"y") 17. ImpElim(13,16) 18. AbsI(17,"Elem(x,y)") 19. ImpInt(18,13) 20. ForallInt(19,"y","y") 21. AndElimL(8) 22. ImpElim(12,21) 23. AndInt(22,20) 24. ClassInt(23,"z") 25. Symmetry(3) 26. EqualitySub(24,25,[0]) 27. ImpInt(26,12) 28. AndInt(11,27) 29. EquivConst(28) 30. ForallInt(29,"x","z") 31. AxInt(0) 32. ForallElim(31,"bigintersection(0)") 33. ForallElim(32,"U") 34. EquivExp(33) 35. AndElimR(34) 36. ImpElim(30,35) 37. Hyp("Elem(z, bigunion(0))") 38. DefEqInt(6) 39. ForallInt(38,"x","x") 40. ForallElim(39,"0") 41. EqualitySub(37,40,[0]) 42. ClassElim(41) 43. AndElimR(42) 44. ExistsInst(43,"a") 45. ForallInt(14,"x","x") 46. ForallElim(45,"a") 47. AndElimL(44) 48. ImpElim(47,46) 49. AbsI(48,"Elem(z,0)") 50. ExistsElim(43,44,49,"a") 51. ImpInt(50,37) 52. Hyp("Elem(z,0)") 53. ForallInt(14,"x","x") 54. ForallElim(53,"z") 55. ImpElim(52,54) 56. AbsI(55,"Elem(z,bigunion(0))") 57. ImpInt(56,52) 58. AndInt(51,57) 59. EquivConst(58) 60. ForallInt(59,"z","z") 61. ForallElim(31,"bigunion(0)") 62. ForallElim(61,"0") 63. EquivExp(62) 64. AndElimR(63) 65. ImpElim(60,64) 66. AndInt(36,65)

Th26. (0 ⊂ x) & (x ⊂ U)

0. Hyp("Elem(z,0)") 1. TheoremInt(1) 2. ForallInt(1,"x","x") 3. ForallElim(2,"z") 4. ImpElim(0,3) 5. AbsI(4,"Elem(z,x)") 6. ImpInt(5,0) 7. ForallInt(6,"z","z") 8. DefSub(7,"Contains",["0","x"],[0]) 9. Hyp("Elem(z,x)") 10. ExistsInt(9,"x","y",[0]) 11. DefSub(10,"Set",["z"],[0]) 12. TheoremInt(2) 13. EquivExp(12) 14. AndElimR(13) 15. ForallInt(14,"x","x") 16. ForallElim(15,"z") 17. ImpElim(11,16) 18. ImpInt(17,9) 19. ForallInt(18,"z","z") 20. DefSub(19,"Contains",["x","U"],[0]) 21. AndInt(8,20)

Th27. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

0. Hyp("(a = b)") 1. Hyp("Elem(z,a)") 2. EqualitySub(1,0,[0]) 3. ImpInt(2,1) 4. ForallInt(3,"z","z") 5. DefSub(4,"Contains",["a","b"],[0]) 6. Hyp("Elem(z,b)") 7. Symmetry(0) 8. EqualitySub(6,7,[0]) 9. ImpInt(8,6) 10. ForallInt(9,"z","z") 11. DefSub(10,"Contains",["b","a"],[0]) 12. AndInt(5,11) 13. ImpInt(12,0) 14. Hyp("(Contains(a,b) & Contains(b,a))") 15. AndElimL(14) 16. AndElimR(14) 17. Hyp("Elem(z,a)") 18. DefExp(15,"Contains",[0]) 19. ForallElim(18,"z") 20. ImpElim(17,19) 21. ImpInt(20,17) 22. Hyp("Elem(z,b)") 23. DefExp(16,"Contains",[0]) 24. ForallElim(23,"z") 25. ImpElim(22,24) 26. ImpInt(25,22) 27. AndInt(21,26) 28. EquivConst(27) 29. ForallInt(28,"z","z") 30. AxInt(0) 31. ForallElim(30,"a") 32. ForallElim(31,"b") 33. EquivExp(32) 34. AndElimR(33) 35. ImpElim(29,34) 36. ImpInt(35,14) 37. AndInt(13,36) 38. EquivConst(37) 39. ForallInt(38,"a","a") 40. ForallElim(39,"x") 41. ForallInt(40,"b","b") 42. ForallElim(41,"y")

Th28. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)

0. Hyp("(Contains(a,b) & Contains(b,c))") 1. AndElimR(0) 2. AndElimL(0) 3. DefExp(1,"Contains",[0]) 4. DefExp(2,"Contains",[0]) 5. ForallElim(3,"z") 6. ForallElim(4,"z") 7. Hyp("Elem(z,a)") 8. ImpElim(7,6) 9. ImpElim(8,5) 10. ImpInt(9,7) 11. ForallInt(10,"z","z") 12. DefSub(11,"Contains",["a","c"],[0]) 13. ImpInt(12,0) 14. ForallInt(13,"a","a") 15. ForallElim(14,"x") 16. ForallInt(15,"b","b") 17. ForallElim(16,"y") 18. ForallInt(17,"c","c") 19. ForallElim(18,"z")

Th29. (x ⊂ y) <-> ((x ∪ y) = y)

0. Hyp("Contains(a,b)") 1. Hyp("Elem(z,union(a,b))") 2. TheoremInt(1) 3. AndElimL(2) 4. EquivExp(3) 5. ForallInt(4,"x","x") 6. ForallElim(5,"a") 7. ForallInt(6,"y","y") 8. ForallElim(7,"b") 9. AndElimL(8) 10. ImpElim(1,9) 11. Hyp("Elem(z,a)") 12. DefExp(0,"Contains",[0]) 13. ForallElim(12,"z") 14. ImpElim(11,13) 15. Hyp("Elem(z,b)") 16. OrElim(10,11,14,15,15) 17. ImpInt(16,1) 18. Hyp("Elem(z,b)") 19. OrIntL(18,"Elem(z,a)") 20. AndElimR(8) 21. ImpElim(19,20) 22. ImpInt(21,18) 23. AndInt(17,22) 24. EquivConst(23) 25. ForallInt(24,"z","z") 26. AxInt(0) 27. ForallElim(26,"union(a,b)") 28. ForallElim(27,"b") 29. EquivExp(28) 30. AndElimR(29) 31. ImpElim(25,30) 32. ImpInt(31,0) 33. Hyp("(union(a,b) = b)") 34. Hyp("Elem(z,a)") 35. OrIntR(34,"Elem(z,b)") 36. AndElimR(8) 37. ImpElim(35,36) 38. EqualitySub(37,33,[0]) 39. ImpInt(38,34) 40. ForallInt(39,"z","z") 41. DefSub(40,"Contains",["a","b"],[0]) 42. ImpInt(41,33) 43. AndInt(32,42) 44. EquivConst(43) 45. ForallInt(44,"a","a") 46. ForallElim(45,"x") 47. ForallInt(46,"b","b") 48. ForallElim(47,"y")

Th30. (x ⊂ y) <-> ((x ∩ y) = x)

0. Hyp("Contains(a,b)") 1. Hyp("Elem(z,intersection(a,b))") 2. TheoremInt(1) 3. AndElimR(2) 4. ForallInt(3,"x","x") 5. ForallElim(4,"a") 6. ForallInt(5,"y","y") 7. ForallElim(6,"b") 8. EquivExp(7) 9. AndElimL(8) 10. ImpElim(1,9) 11. AndElimL(10) 12. ImpInt(11,1) 13. Hyp("Elem(z,a)") 14. DefExp(0,"Contains",[0]) 15. ForallElim(14,"z") 16. ImpElim(13,15) 17. AndInt(13,16) 18. AndElimR(8) 19. ImpElim(17,18) 20. ImpInt(19,13) 21. AndInt(12,20) 22. EquivConst(21) 23. ForallInt(22,"z","z") 24. AxInt(0) 25. ForallElim(24,"intersection(a,b)") 26. ForallElim(25,"a") 27. EquivExp(26) 28. AndElimR(27) 29. ImpElim(23,28) 30. ImpInt(29,0) 31. Hyp("(intersection(a,b) = a)") 32. Hyp("Elem(z,a)") 33. Symmetry(31) 34. EqualitySub(32,33,[0]) 35. ImpElim(34,9) 36. AndElimR(35) 37. ImpInt(36,32) 38. ForallInt(37,"z","z") 39. DefSub(38,"Contains",["a","b"],[0]) 40. ImpInt(39,31) 41. AndInt(30,40) 42. EquivConst(41) 43. ForallInt(42,"a","a") 44. ForallElim(43,"x") 45. ForallInt(44,"b","b") 46. ForallElim(45,"y")

Th31. (x ⊂ y) -> ((∪x ⊂ ∪y) & (∩y ⊂ ∩x))

0. Hyp("Contains(a,b)") 1. Hyp("Elem(z,bigunion(a))") 2. DefEqInt(6) 3. ForallInt(2,"x","x") 4. ForallElim(3,"a") 5. EqualitySub(1,4,[0]) 6. ClassElim(5) 7. AndElimR(6) 8. Hyp("(Elem(y,a) & Elem(z,y))") 9. DefExp(0,"Contains",[0]) 10. ForallElim(9,"y") 11. AndElimL(8) 12. ImpElim(11,10) 13. AndElimR(8) 14. AndInt(12,13) 15. ExistsInt(14,"y","y",[0,1]) 16. AndElimL(6) 17. AndInt(16,15) 18. ClassInt(17,"z") 19. ForallInt(2,"x","x") 20. ForallElim(19,"b") 21. Symmetry(20) 22. EqualitySub(18,21,[0]) 23. ExistsElim(7,8,22,"y") 24. ImpInt(23,1) 25. ForallInt(24,"z","z") 26. DefSub(25,"Contains",["bigunion(a)","bigunion(b)"],[0]) 27. Hyp("Elem(z,bigintersection(b))") 28. DefEqInt(7) 29. ForallInt(28,"x","x") 30. ForallElim(29,"b") 31. EqualitySub(27,30,[0]) 32. ClassElim(31) 33. AndElimL(32) 34. AndElimR(32) 35. ForallElim(34,"y") 36. Hyp("Elem(y,a)") 37. ImpElim(36,10) 38. ImpElim(37,35) 39. ImpInt(38,36) 40. ForallInt(39,"y","y") 41. AndInt(33,40) 42. ClassInt(41,"z") 43. ForallInt(28,"x","x") 44. ForallElim(43,"a") 45. Symmetry(44) 46. EqualitySub(42,45,[0]) 47. ImpInt(46,27) 48. ForallInt(47,"z","z") 49. DefSub(48,"Contains",["bigintersection(b)","bigintersection(a)"],[0]) 50. AndInt(26,49) 51. ImpInt(50,0) 52. ForallInt(51,"a","a") 53. ForallElim(52,"x") 54. ForallInt(53,"b","b") 55. ForallElim(54,"y")

Th32. (x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x))

0. Hyp("Elem(a,b)") 1. Hyp("Elem(x,a)") 2. AndInt(0,1) 3. ExistsInt(2,"a","y",[0,1]) 4. ExistsInt(1,"a","y",[0]) 5. DefSub(4,"Set",["x"],[0]) 6. AndInt(5,3) 7. ClassInt(6,"z") 8. DefEqInt(6) 9. Symmetry(8) 10. ForallInt(9,"x","x") 11. ForallElim(10,"b") 12. EqualitySub(7,11,[0]) 13. ImpInt(12,1) 14. ForallInt(13,"x","z") 15. DefSub(14,"Contains",["a","bigunion(b)"],[0]) 16. Hyp("Elem(x, bigintersection(b))") 17. DefEqInt(7) 18. ForallInt(17,"x","x") 19. ForallElim(18,"b") 20. EqualitySub(16,19,[0]) 21. ClassElim(20) 22. AndElimR(21) 23. ForallElim(22,"a") 24. ImpElim(0,23) 25. ImpInt(24,16) 26. ForallInt(25,"x","z") 27. DefSub(26,"Contains",["bigintersection(b)","a"],[0]) 28. AndInt(15,27) 29. ImpInt(28,0) 30. ForallInt(29,"a","a") 31. ForallElim(30,"x") 32. ForallInt(31,"b","b") 33. ForallElim(32,"y")

Th33. (Set(x) & (y ⊂ x)) -> Set(y)

0. Hyp("(Set(a) & Contains(b,a))") 1. AxInt(1) 2. ForallInt(1,"x","x") 3. ForallElim(2,"a") 4. AndElimL(0) 5. ImpElim(4,3) 6. ExistsInst(5,"w") 7. AndElimR(6) 8. ForallElim(7,"b") 9. AndElimR(0) 10. ImpElim(9,8) 11. ExistsInt(10,"w","z",[0]) 12. DefSub(11,"Set",["b"],[0]) 13. ExistsElim(5,6,12,"w") 14. ImpInt(13,0) 15. ForallInt(14,"a","a") 16. ForallElim(15,"x") 17. ForallInt(16,"b","b") 18. ForallElim(17,"y")

Th34. (0 = ∩U) & (U = ∪U)

0. Hyp("Elem(z,0)") 1. DefEqInt(4) 2. EqualitySub(0,1,[0]) 3. ClassElim(2) 4. AndElimR(3) 5. Identity("z") 6. ImpElim(5,4) 7. AbsI(6,"Elem(z,bigintersection(U))") 8. ImpInt(7,0) 9. Hyp("Elem(z, bigintersection(U))") 10. DefEqInt(5) 11. DefEqInt(7) 12. ForallInt(11,"x","x") 13. ForallElim(12,"U") 14. EqualitySub(9,13,[0]) 15. ClassElim(14) 16. AndElimR(15) 17. ForallElim(16,"0") 18. TheoremInt(1) 19. TheoremInt(2) 20. AndElimL(18) 21. ForallInt(20,"x","x") 22. ForallElim(21,"z") 23. ForallInt(19,"x","x") 24. ForallElim(23,"z") 25. ForallInt(24,"y","y") 26. ForallElim(25,"0") 27. AndElimL(15) 28. AndInt(27,22) 29. ImpElim(28,26) 30. TheoremInt(3) 31. EquivExp(30) 32. AndElimR(31) 33. ForallInt(32,"x","x") 34. ForallElim(33,"0") 35. ImpElim(29,34) 36. ImpElim(35,17) 37. ImpInt(36,9) 38. AndInt(8,37) 39. EquivConst(38) 40. ForallInt(39,"z","z") 41. AxInt(0) 42. ForallElim(41,"0") 43. ForallElim(42,"bigintersection(U)") 44. EquivExp(43) 45. AndElimR(44) 46. ImpElim(40,45) 47. Hyp("Elem(z,U)") 48. DefEqInt(6) 49. ForallInt(48,"x","x") 50. ForallElim(49,"U") 51. AxInt(1) 52. AndElimL(31) 53. ForallInt(52,"x","x") 54. ForallElim(53,"z") 55. ImpElim(47,54) 56. ForallInt(51,"x","x") 57. ForallElim(56,"z") 58. ImpElim(55,57) 59. ExistsInst(58,"a") 60. Identity("z") 61. TheoremInt(4) 62. ForallInt(61,"x","x") 63. ForallElim(62,"z") 64. ForallInt(63,"y","y") 65. ForallElim(64,"z") 66. EquivExp(65) 67. AndElimL(66) 68. ImpElim(60,67) 69. AndElimL(68) 70. AndElimR(59) 71. ForallElim(70,"z") 72. ImpElim(69,71) 73. AndElimL(59) 74. ForallInt(32,"x","x") 75. ForallElim(74,"a") 76. ImpElim(73,75) 77. AndInt(76,72) 78. ExistsInt(77,"a","y",[0,1]) 79. ExistsElim(58,59,78,"a") 80. AndInt(55,79) 81. ClassInt(80,"y") 82. Symmetry(50) 83. EqualitySub(81,82,[0]) 84. ImpInt(83,47) 85. Hyp("Elem(z,bigunion(U))") 86. ExistsInt(85,"bigunion(U)","y",[0]) 87. DefSub(86,"Set",["z"],[0]) 88. ForallInt(32,"x","x") 89. ForallElim(88,"z") 90. ImpElim(87,89) 91. ImpInt(90,85) 92. AndInt(84,91) 93. EquivConst(92) 94. ForallInt(93,"z","z") 95. ForallElim(41,"U") 96. ForallElim(95,"bigunion(U)") 97. EquivExp(96) 98. AndElimR(97) 99. ImpElim(94,98) 100. AndInt(46,99)

Th35. ¬(x = 0) -> Set(∩x)

0. Hyp("forall z. neg Elem(z,a)") 1. Hyp("Elem(z,a)") 2. ForallElim(0,"z") 3. ImpElim(1,2) 4. AbsI(3,"Elem(z,0)") 5. ImpInt(4,1) 6. Hyp("Elem(z,0)") 7. DefEqInt(4) 8. EqualitySub(6,7,[0]) 9. ClassElim(8) 10. AndElimR(9) 11. Identity("z") 12. ImpElim(11,10) 13. AbsI(12,"Elem(z,a)") 14. ImpInt(13,6) 15. AndInt(5,14) 16. EquivConst(15) 17. ForallInt(16,"z","z") 18. AxInt(0) 19. ForallElim(18,"a") 20. ForallElim(19,"0") 21. EquivExp(20) 22. AndElimR(21) 23. ImpElim(17,22) 24. ImpInt(23,0) 25. TheoremInt(1) 26. PolySub(25,"A","forall z. neg Elem(z,a)") 27. PolySub(26,"B","(a = 0)") 28. ImpElim(24,27) 29. Hyp("neg forall z.neg Elem(z,a)") 30. Hyp("neg exists z. Elem(z,a)") 31. Hyp("Elem(z,a)") 32. ExistsInt(31,"z","z",[0]) 33. ImpElim(32,30) 34. ImpInt(33,31) 35. ForallInt(34,"z","z") 36. ImpInt(35,30) 37. TheoremInt(1) 38. PolySub(37,"A","neg exists z. Elem(z,a)") 39. PolySub(38,"B","forall z. neg

Elem(z,a)") 40. ImpElim(36,39) 41. TheoremInt(2) 42. PolySub(41,"D","exists l.Elem(l,a)") 43. EquivExp(42) 44. AndElimR(43) 45. Hyp("neg (a = 0)") 46. ImpElim(45,28) 47. ImpElim(46,40) 48. ImpElim(47,44) 49. ImpInt(48,45) 50. Hyp("exists l. Elem(l,a)") 51. Hyp("Elem(b,a)") 52. TheoremInt(4) 53. ForallInt(52,"x","x") 54. ForallElim(53,"b") 55. ForallInt(54,"y","y") 56. ForallElim(55,"a") 57. ImpElim(51,56) 58. AndElimR(57) 59. ExistsInt(51,"a","y",[0]) 60. DefSub(59,"Set",["b"],[0]) 61. TheoremInt(5) 62. ForallInt(61,"x","x") 63. ForallElim(62,"b") 64. ForallInt(63,"y","y") 65. ForallElim(64,"bigintersection(a)") 66. AndInt(60,58) 67. ImpElim(66,65) 68. ExistsElim(50,51,67,"b") 69. ImpInt(68,50) 70. Hyp("neg (a = 0)") 71. ImpElim(70,49) 72. ImpElim(71,69) 73. ImpInt(72,70) 74. ForallInt(73,"a","a") 75. ForallElim(74,"x")

Th37. U = PU

0. Hyp("Elem(x,U)") 1. TheoremInt(1) 2. AndElimR(1) 3. DefEqInt(8) 4. ForallInt(3,"x","x") 5. ForallElim(4,"U") 6. ExistsInt(0,"U","y",[0]) 7. DefSub(6,"Set",["x"],[0]) 8. AndInt(7,2) 9. ClassInt(8,"y") 10. Symmetry(5) 11. EqualitySub(9,10,[0]) 12. ImpInt(11,0) 13. Hyp("Elem(x,parts(U))") 14. ExistsInt(13,"parts(U)","y",[0]) 15. DefSub(14,"Set",["x"],[0]) 16. TheoremInt(2) 17. EquivExp(16) 18. AndElimR(17) 19. ImpElim(15,18) 20. ImpInt(19,13) 21. AndInt(12,20) 22. EquivConst(21) 23. ForallInt(22,"x","z") 24. AxInt(0) 25. ForallElim(24,"U") 26. ForallElim(25,"parts(U)") 27. EquivExp(26) 28. AndElimR(27) 29. ImpElim(23,28)

Th38. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))

0. Hyp("Set(a)") 1. AxInt(1) 2. ForallInt(1,"x","x") 3. ForallElim(2,"a") 4. ImpElim(0,3) 5. TheoremInt(1) 6. ForallInt(5,"y","y") 7. ForallElim(6,"parts(a)") 8. ExistsInst(4,"b") 9. ForallInt(7,"x","x") 10. ForallElim(9,"b") 11. Hyp("Elem(z,parts(a))") 12. DefEqInt(8) 13. ForallInt(12,"x","x") 14. ForallElim(13,"a") 15. EqualitySub(11,14,[0]) 16. ClassElim(15) 17. AndElimR(8) 18. AndElimR(16) 19. ForallElim(17,"z") 20. ImpElim(18,19) 21. ImpInt(20,11) 22. ForallInt(21,"z","z") 23. DefSub(22,"Contains",["parts(a)","b"],[0]) 24. AndElimL(8) 25. AndInt(24,23) 26. ImpElim(25,10) 27. ExistsElim(4,8,26,"b") 28. Hyp("Contains(z,a)") 29. AndInt(0,28) 30. ForallInt(5,"x","x") 31. ForallElim(30,"a") 32. ForallInt(31,"y","y") 33. ForallElim(32,"z") 34. ImpElim(29,33) 35. AndInt(34,28) 36. ClassInt(35,"y") 37. Symmetry(14) 38. EqualitySub(36,37,[0]) 39. ImpInt(38,28) 40. Hyp("Elem(z,parts(a))") 41. EqualitySub(40,14,[0]) 42. ClassElim(41) 43. AndElimR(42) 44. ImpInt(43,40) 45. AndInt(39,44) 46. EquivConst(45) 47. AndInt(27,46) 48. ImpInt(47,0) 49. ForallInt(48,"a","a") 50. ForallElim(49,"x") 51. ForallInt(50,"z","z") 52. ForallElim(51,"y")

Th39. ¬Set(U)

0. DefEqInt(26) 1. Hyp("Elem(rus,rus)") 2. EqualitySub(1,0,[1]) 3. ClassElim(2) 4. AndElimR(3) 5. ImpElim(1,4) 6. AbsI(5,"neg Set(rus)") 7. Hyp("neg Elem(rus,rus)") 8. Hyp("Set(rus)") 9. AndInt(8,7) 10. ClassInt(9,"z") 11. Symmetry(0) 12. EqualitySub(10,11,[0]) 13. ImpElim(12,7) 14. ImpInt(13,8) 15. TheoremInt(1) 16. PolySub(15,"A","Elem(rus,rus)") 17. OrElim(16,1,6,7,14) 18. TheoremInt(2) 19. TheoremInt(3) 20. AndElimR(19) 21. Hyp("Set(U)") 22. ForallInt(20,"x","x") 23. ForallElim(22,"rus") 24. AndInt(21,23) 25. ForallInt(18,"x","x") 26. ForallElim(25,"U") 27. ForallInt(26,"y","y") 28. ForallElim(27,"rus") 29. ImpElim(24,28) 30. ImpElim(29,17) 31. ImpInt(30,21)

Th41. Set(x) -> ((y ε {x}) <-> (y = x))

0. Hyp("Set(x)") 1. Hyp("Elem(y,singleton(x))") 2. DefEqInt(9) 3. EqualitySub(1,2,[0]) 4. ClassElim(3) 5. TheoremInt(1) 6. EquivExp(5) 7. AndElimR(6) 8. ImpElim(0,7) 9. AndElimR(4) 10. ImpElim(8,9) 11. ImpInt(10,1) 12. Hyp("(y = x)") 13. Symmetry(12) 14. EqualitySub(0,13,[0]) 15. Hyp("(y = x)") 16. Hyp("Elem(x,U)") 17. ImpInt(15,16) 18. ImpInt(17,15) 19. ImpElim(12,18) 20. AndInt(14,19) 21. ClassInt(20,"z") 22. Symmetry(2) 23. EqualitySub(21,22,[0]) 24. ImpInt(23,12) 25. AndInt(11,24) 26. EquivConst(25) 27. ImpInt(26,0)

Th42. Set(x) -> Set({x})

0. Hyp("Set(x)") 1. Hyp("Elem(z,singleton(x))") 2. DefEqInt(9) 3. EqualitySub(1,2,[0]) 4. ClassElim(3) 5. AndElimR(4) 6. TheoremInt(3) 7. EquivExp(6) 8. EquivExp(6) 9. AndElimR(8) 10. ImpElim(0,9) 11. ImpElim(10,5) 12. TheoremInt(2) 13. EquivExp(12) 14. AndElimL(13) 15. ForallInt(14,"x","x") 16. ForallElim(15,"z") 17. ForallInt(16,"y","y") 18. ForallElim(17,"x") 19. ImpElim(11,18) 20. AndElimL(19) 21. TheoremInt(1) 22. ImpElim(0,21) 23. AndElimR(22) 24. EquivExp(23) 25. AndElimL(24) 26. ForallInt(25,"y","y") 27. ForallElim(26,"z") 28. ImpElim(20,27) 29. ImpInt(28,1) 30. ForallInt(29,"z","z") 31. DefSub(30,"Contains",["singleton(x)","parts(x)"],[0]) 32. TheoremInt(4) 33. ForallInt(32,"x","x") 34. ForallElim(33,"parts(x)") 35. ForallInt(34,"y","y") 36. ForallElim(35,"singleton(x)") 37. AndElimL(22) 38. AndInt(37,31) 39. ImpElim(38,36) 40. ImpInt(39,0)

Th43. ({x} = U) <-> ¬Set(x)

0. Hyp("Set(x)") 1. TheoremInt(1) 2. ImpElim(0,1) 3. TheoremInt(2) 4. Hyp("(singleton(x) =U)") 5. EqualitySub(2,4,[0]) 6. ImpElim(5,3) 7. ImpInt(6,4) 8. Hyp("neg Set(x)") 9. Hyp("Elem(x,U)") 10. ExistsInt(9,"U","y",[0]) 11. DefSub(10,"Set",["x"],[0]) 12. ImpElim(11,8) 13. ImpInt(12,9) 14. Hyp("Elem(x,U)") 15. ImpElim(14,13) 16. AbsI(15,"(y = x)") 17. ImpInt(16,14) 18. Hyp("Elem(y,U)") 19. TheoremInt(3) 20. EquivExp(19) 21. AndElimL(20) 22. ForallInt(21,"x","x") 23. ForallElim(22,"y") 24. ImpElim(18,23) 25. AndInt(24,17) 26. ClassInt(25,"z") 27. DefEqInt(9) 28. Symmetry(27) 29. EqualitySub(26,28,[0]) 30. ImpInt(29,18) 31. ForallInt(30,"y","z") 32. DefSub(31,"Contains",["U","singleton(x)"],[0]) 33. TheoremInt(4) 34. ForallInt(33,"x","x") 35. ForallElim(34,"singleton(x)") 36. AndElimR(35) 37. TheoremInt(6) 38. ForallInt(37,"x","x") 39. ForallElim(38,"singleton(x)") 40. ForallInt(39,"y","y") 41. ForallElim(40,"U") 42. EquivExp(41) 43. EquivExp(41) 44. AndElimR(43) 45. AndInt(36,32) 46. ImpElim(45,44) 47. ImpInt(46,8) 48. ImpInt(7,0) 49. TheoremInt(10) 50. PolySub(49,"A","Set(x)") 51. PolySub(50,"B","neg (singleton(x) = U)") 52. ImpElim(48,51) 53. TheoremInt(9) 54. EquivExp(53) 55. AndElimL(54) 56. PolySub(55,"D","(singleton(x) =U)") 57. Hyp("(singleton(x) = U)") 58. ImpElim(57,56) 59. ImpElim(58,52) 60. ImpInt(59,57) 61. AndInt(60,47) 62. EquivConst(61)

Th44. (Set(x) -> ((∩{x} = x) & (U{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (U{x} = U)))

0. Hyp("Elem(z,bigintersection(singleton(x)))") 1. DefEqInt(7) 2. ForallInt(1,"x","x") 3. ForallElim(2,"singleton(x)") 4. EqualitySub(0,3,[0]) 5. ClassElim(4) 6. AndElimR(5) 7. Hyp("Set(x)") 8. TheoremInt(1) 9. ImpElim(7,8) 10. EquivExp(9) 11. AndElimR(10) 12. ForallInt(11,"y","y") 13. ForallElim(12,"x") 14. Identity("x") 15. ImpElim(14,13) 16. ForallElim(6,"x") 17. ImpElim(15,16) 18. ImpInt(17,0) 19. Hyp("Elem(z,x)") 20. Hyp("Elem(y,singleton(x))") 21. AndElimL(10) 22. ImpElim(20,21) 23. Symmetry(22) 24. EqualitySub(19,23,[0]) 25. ImpInt(24,20) 26. ForallInt(25,"y","y") 27. ExistsInt(19,"x","x",[0]) 28. DefSub(27,"Set",["z"],[0]) 29. AndInt(28,26) 30. ClassInt(29,"z") 31. Symmetry(3) 32. EqualitySub(30,31,[0]) 33. ImpInt(32,19) 34. AndInt(18,33) 35. EquivConst(34) 36. ForallInt(35,"z","z") 37. AxInt(0) 38. ForallElim(37,"bigintersection(singleton(x))") 39. ForallElim(38,"x") 40. EquivExp(39) 41. AndElimR(40) 42. ImpElim(36,41) 43. Hyp("Elem(z,bigunion(singleton(x)))") 44. DefEqInt(6) 45. ForallInt(44,"x","x") 46. ForallElim(45,"singleton(x)") 47. EqualitySub(43,46,[0]) 48. ClassElim(47) 49. AndElimR(48) 50. ExistsInst(49,"a") 51. ForallInt(21,"y","y") 52. ForallElim(51,"a") 53. AndElimL(50) 54. ImpElim(53,52) 55. AndElimR(50) 56. EqualitySub(55,54,[0]) 57. ExistsElim(49,50,56,"a") 58. ImpInt(57,43) 59. Hyp("Elem(z,x)") 60. AndElimR(10) 61. ForallInt(60,"y","y") 62. ForallElim(61,"x") 63. ImpElim(14,62) 64. AndInt(63,59) 65. ExistsInt(64,"x","y",[0,2]) 66. ExistsInt(59,"x","y",[0]) 67. DefSub(66,"Set", ["z"],[0]) 68. AndInt(67,65) 69. ClassInt(68,"z") 70. Symmetry(46) 71. EqualitySub(69,70,[0]) 72. ImpInt(71,59) 73. AndInt(58,72) 74. EquivConst(73) 75. ForallInt(74,"z","z") 76. ForallElim(37,"bigunion(singleton(x))") 77. ForallElim(76,"x") 78. EquivExp(77) 79. AndElimR(78) 80. ImpElim(75,79) 81. AndInt(42,80) 82. ImpInt(81,7) 83. Hyp("neg Set(x)") 84. TheoremInt(2) 85. EquivExp(84) 86. AndElimR(85) 87. ImpElim(83,86) 88. TheoremInt(3) 89. Symmetry(87) 90. EqualitySub(88,89,[0,2]) 91. AndElimL(90) 92. AndElimR(90) 93. Symmetry(91) 94. Symmetry(92) 95. AndInt(93,94) 96. ImpInt(95,83) 97. AndInt(82,96)

Th46. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))

0. Hyp("(Set(x) & Set(y))") 1. TheoremInt(1) 2. AndElimL(0) 3. AndElimR(0) 4. ImpElim(2,1) 5. ForallInt(1,"x","x") 6. ForallElim(5,"y") 7. ImpElim(3,6) 8. AxInt(2) 9. ForallInt(8,"x","x") 10. ForallElim(9,"singleton(x)") 11. ForallInt(10,"y","y") 12. ForallElim(11,"singleton(y)") 13. AndInt(4,7) 14. ImpElim(13,12) 15. DefEqInt(10) 16. Symmetry(15) 17. EqualitySub(14,16,[0]) 18. TheoremInt(2) 19. AndElimL(18) 20. Hyp("Elem(z,pair(x,y))") 21. EqualitySub(20,15,[0]) 22. EquivExp(19) 23. AndElimL(22) 24. ForallInt(23,"x","x") 25. ForallElim(24,"singleton(x)") 26. ForallInt(25,"y","y") 27. ForallElim(26,"singleton(y)") 28. ImpElim(21,27) 29. Hyp("Elem(z,singleton(x))") 30. TheoremInt(3) 31. ForallInt(30,"y","y") 32. ForallElim(31,"z") 33. ForallInt(32,"x","x") 34. ForallElim(33,"y") 35. ImpElim(2,32) 36. EquivExp(35) 37. AndElimL(36) 38. ImpElim(29,37) 39. OrIntR(38,"(z = y)") 40. Hyp("Elem(z,singleton(y))") 41. ImpElim(3,34) 42. EquivExp(41) 43. AndElimL(42) 44. ImpElim(40,43) 45. OrIntL(44,"(z = x)") 46. OrElim(28,29,39,40,45) 47. ImpInt(46,20) 48. Hyp("((z = x) v (z = y))") 49. Hyp("(z =x)") 50. AndElimR(36) 51. ImpElim(49,50) 52. OrIntR(51,"Elem(z,singleton(y))") 53. AndElimR(22) 54. ForallInt(53,"x","x") 55. ForallElim(54,"singleton(x)") 56. ForallInt(55,"y","y") 57. ForallElim(56,"singleton(y)") 58. ImpElim(52,57) 59. Hyp("(z = y)") 60. AndElimR(42) 61. ImpElim(59,60) 62. OrIntL(61,"Elem(z,singleton(x))") 63. ImpElim(62,57) 64. OrElim(48,49,58,59,63) 65. ImpInt(64,48) 66. EqualitySub(65,16,[0]) 67. AndInt(47,66) 68. EquivConst(67) 69. AndInt(17,68) 70. ImpInt(69,0) 71. Hyp("(pair(x,y) = U)") 72. EqualitySub(71,15,[0]) 73. TheoremInt(4) 74. Symmetry(72) 75. EqualitySub(73,74,[0]) 76. AxInt(2) 77. TheoremInt(5) 78. PolySub(77,"A","(Set(x) & Set(y))") 79. PolySub(78,"B","Set(union(x,y))") 80. ImpElim(76,79) 81. ForallInt(80,"x","x") 82. ForallElim(81,"singleton(x)") 83. ForallInt(82,"y","y") 84. ForallElim(83,"singleton(y)") 85. ImpElim(75,84) 86. TheoremInt(6) 87. AndElimR(86) 88. PolySub(87,"A","Set(singleton(x))") 89. PolySub(88,"B","Set(singleton(y))") 90. EquivExp(89) 91. AndElimL(90) 92. ImpElim(85,91) 93. Hyp("neg Set(singleton(x))") 94. TheoremInt(1) 95. PolySub(77,"A","Set(x)") 96. PolySub(95,"B","Set(singleton(x))") 97. ImpElim(94,96) 98. ImpElim(93,97) 99. ImpInt(98,93) 100. ForallInt(99,"x","a") 101. Hyp("neg Set(singleton(y))") 102. ForallElim(100,"y") 103. ImpElim(101,102) 104. OrIntR(98,"neg Set(y)") 105. OrIntL(103,"neg Set(x)") 106. OrElim(92,93,104,101,105) 107. ImpInt(106,71) 108. Hyp("(neg Set(x) v neg Set(y))") 109. Hyp("neg Set(x)") 110. TheoremInt(7) 111. EquivExp(110) 112. AndElimR(111) 113. ImpElim(109,112) 114. TheoremInt(8) 115. AndElimL(114) 116. ForallInt(115,"x","x") 117. ForallElim(116,"singleton(y)") 118. Symmetry(113) 119. EqualitySub(117,118,[0]) 120. TheoremInt(10) 121. AndElimL(120) 122. ForallInt(121,"x","x") 123. ForallElim(122,"singleton(x)") 124. ForallInt(123,"y","y") 125. ForallElim(124,"singleton(y)") 126. Symmetry(125) 127. EqualitySub(119,126,[0]) 128. EqualitySub(127,16,[0]) 129. ImpInt(128,109) 130. ForallInt(129,"x","a") 131. ForallInt(130,"y","b") 132. Hyp("neg Set(y)") 133. ForallElim(131,"z") 134. ForallElim(133,"y") 135. ForallInt(134,"z","z") 136. ForallElim(135,"x") 137. ForallInt(15,"x","x") 138. ForallElim(137,"a") 139. ForallInt(138,"y","y") 140. ForallElim(139,"b") 141. ForallInt(140,"a","a") 142. ForallElim(141,"y") 143. ForallInt(142,"b","b") 144. ForallElim(143,"x") 145. EqualitySub(144,126,[0]) 146. EqualitySub(145,16,[0]) 147. EqualitySub(136,146,[0]) 148. ImpElim(132,147) 149. OrElim(108,109,128,132,148) 150. ImpInt(149,108) 151. AndInt(107,150) 152. EquivConst(151) 153. AndInt(70,152)

Th47. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (**U**{x,y} = (x **U** y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = **U**{x,y})))

0. Hyp("(Set(x) & Set(y))") 1. Hyp("Elem(z,bigintersection(pair(x,y)))") 2. DefEqInt(7) 3. ForallInt(2,"x","x") 4. ForallElim(3,"pair(x,y)") 5. EqualitySub(1,4,[0]) 6. ClassElim(5) 7. AndElimR(6) 8. ForallElim(7,"x") 9. ForallElim(7,"y") 10. TheoremInt(1) 11. AndElimL(10) 12. ImpElim(0,11) 13. AndElimR(12) 14. EquivExp(13) 15. AndElimR(14) 16. ForallInt(15,"z","z") 17. ForallElim(16,"x") 18. ForallInt(15,"z","z") 19. ForallElim(18,"y") 20. Identity("x") 21. Identity("y") 22. OrIntR(20,"(x = y)") 23. ImpElim(22,17) 24. ImpElim(23,8) 25. OrIntL(21,"(y = x)") 26. ImpElim(25,19) 27. ImpElim(26,9) 28. AndInt(24,27) 29. TheoremInt(2) 30. AndElimR(29) 31. EquivExp(30) 32.

AndElimR(31) 33. ImpElim(28,32) 34. ImpInt(33,1) 35. Hyp("Elem(z,intersection(x,y))") 36. AndElimL(31) 37. ImpElim(35,36) 38. Hyp("Elem(c,pair(x,y))") 39. AndElimL(14) 40. ForallInt(39,"z","z") 41. ForallElim(40,"c") 42. ImpElim(38,41) 43. Hyp("(c = x)") 44. AndElimL(37) 45. Symmetry(43) 46. EqualitySub(44,45,[0]) 47. Hyp("(c = y)") 48. AndElimR(37) 49. Symmetry(47) 50. EqualitySub(48,49,[0]) 51. OrElim(42,43,46,47,50) 52. ImpInt(51,38) 53. ForallInt(52,"c","c") 54. ExistsInt(35,"intersection(x,y)","c",[0]) 55. DefSub(54,"Set",["z"],[0]) 56. AndInt(55,53) 57. ClassInt(56,"c") 58. Symmetry(4) 59. EqualitySub(57,58,[0]) 60. ImpInt(59,35) 61. AndInt(34,60) 62. EquivConst(61) 63. ForallInt(62,"z","z") 64. AxInt(0) 65. ForallElim(64,"bigintersection(pair(x,y))") 66. ForallElim(65,"intersection(x,y)") 67. EquivExp(66) 68. AndElimR(67) 69. ImpElim(63,68) 70. Hyp("Elem(z,bigunion(pair(x,y)))") 71. DefEqInt(6) 72. ForallInt(71,"x","x") 73. ForallElim(72,"pair(x,y)") 74. EqualitySub(70,73,[0]) 75. ClassElim(74) 76. AndElimR(75) 77. ExistsInst(76,"u") 78. AndElimL(77) 79. TheoremInt(1) 80. AndElimL(79) 81. ImpElim(0,80) 82. AndElimR(81) 83. EquivExp(82) 84. AndElimL(83) 85. ForallInt(84,"z","z") 86. ForallElim(85,"u") 87. ImpElim(78,86) 88. Hyp("(u = x)") 89. AndElimR(77) 90. EqualitySub(89,88,[0]) 91. OrIntR(90,"Elem(z,y)") 92. Hyp("(u = y)") 93. EqualitySub(89,92,[0]) 94. OrIntL(93,"Elem(z,x)") 95. OrElim(87,88,91,92,94) 96. TheoremInt(2) 97. AndElimL(96) 98. EquivExp(97) 99. AndElimR(98) 100. ImpElim(95,99) 101. ExistsElim(76,77,100,"u") 102. ImpInt(101,70) 103. Hyp("Elem(z,union(x,y))") 104. AndElimL(98) 105. ImpElim(103,104) 106. Hyp("Elem(z,x)") 107. EquivExp(82) 108. AndElimR(107) 109. ForallInt(108,"z","z") 110. ForallElim(109,"x") 111. Identity("x") 112. OrIntR(111,"(x = y)") 113. ImpElim(112,110) 114. AndInt(113,106) 115. ExistsInt(114,"x","a",[0,2]) 116. ExistsInt(106,"x","y",[0]) 117. DefSub(116,"Set",["z"],[0]) 118. AndInt(117,115) 119. ClassInt(118,"b") 120. Symmetry(73) 121. EqualitySub(119,120,[0]) 122. Hyp("Elem(z,y)") 123. Identity("y") 124. ForallInt(108,"z","z") 125. ForallElim(124,"y") 126. OrIntL(123,"(y = x)") 127. ImpElim(126,125) 128. AndInt(127,122) 129. ExistsInt(128,"y","a",[0,2]) 130. ExistsInt(122,"y","y",[0]) 131. DefSub(130,"Set",["z"],[0]) 132. AndInt(131,129) 133. ClassInt(132,"b") 134. EqualitySub(133,120,[0]) 135. OrElim(105,106,121,122,134) 136. ImpInt(135,103) 137. AndInt(102,136) 138. EquivConst(137) 139. ForallInt(138,"z","z") 140. AxInt(0) 141. ForallElim(140,"bigunion(pair(x,y))") 142. ForallElim(141,"union(x,y)") 143. EquivExp(142) 144. AndElimR(143) 145. ImpElim(139,144) 146. AndInt(69,145) 147. ImpInt(146,0) 148. Hyp("(neg Set(x) v neg Set(y))") 149. TheoremInt(3) 150. EquivExp(149) 151. AndElimR(150) 152. Hyp("neg Set(x)") 153. ImpElim(152,151) 154. DefEqInt(10) 155. EqualitySub(154,153,[0]) 156. TheoremInt(4) 157. AndElimL(156) 158. TheoremInt(5) 159. AndElimL(158) 160. ForallInt(159,"y","y") 161. ForallElim(160,"U") 162. EqualitySub(157,161,[0]) 163. ForallInt(162,"x","x") 164. ForallElim(163,"singleton(y)") 165. EqualitySub(155,164,[0]) 166. TheoremInt(6) 167. Symmetry(165) 168. EqualitySub(166,167,[0,2]) 169. Hyp("neg Set(y)") 170. ForallInt(151,"x","x") 171. ForallElim(170,"y") 172. ImpElim(169,171) 173. EqualitySub(154,172,[0]) 174. ForallInt(157,"x","x") 175. ForallElim(174,"singleton(x)") 176. EqualitySub(173,175,[0]) 177. Symmetry(176) 178. EqualitySub(166,177,[0,2]) 179. OrElim(148,152,168,169,178) 180. ImpInt(179,148) 181. AndInt(147,180)

Th49. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

0. Hyp("(Set(x) & Set(y))") 1. AndElimL(0) 2. TheoremInt(1) 3. ImpElim(1,2) 4. TheoremInt(2) 5. AndElimL(4) 6. ImpElim(0,5) 7. AndElimL(6) 8. ForallInt(5,"x","x") 9. ForallElim(8,"singleton(x)") 10. ForallInt(9,"y","y") 11. ForallElim(10,"pair(x,y)") 12. AndInt(3,7) 13. ImpElim(12,11) 14. AndElimL(13) 15. DefEqInt(11) 16. Symmetry(15) 17. EqualitySub(14,16,[0]) 18. ImpInt(17,0) 19. Hyp("(neg Set(x) v neg Set(y))") 20. Hyp("neg Set(x)") 21. TheoremInt(3) 22. EquivExp(21) 23. AndElimR(22) 24. ImpElim(20,23) 25. TheoremInt(4) 26. AndElimR(25) 27. EquivExp(26) 28. AndElimR(27) 29. OrIntR(20,"neg Set(y)") 30. ImpElim(29,28) 31. TheoremInt(5) 32. Symmetry(24) 33. EqualitySub(31,32,[0]) 34. ForallInt(23,"x","x") 35. ForallElim(34,"singleton(x)") 36. ImpElim(33,35) 37. DefEqInt(10) 38. ForallInt(37,"x","x") 39. ForallElim(38,"singleton(x)") 40. ForallInt(39,"y","y") 41. ForallElim(40,"pair(x,y)") 42. Symmetry(30) 43. EqualitySub(31,42,[0]) 44. ForallInt(23,"x","x") 45. ForallElim(44,"pair(x,y)") 46. ImpElim(43,45) 47. EqualitySub(41,46,[0]) 48. TheoremInt(6) 49. AndElimL(48) 50. ForallInt(49,"x","x") 51. ForallElim(50,"singleton(singleton(x))") 52. EqualitySub(47,51,[0]) 53. EqualitySub(15,52,[0]) 54. Symmetry(53) 55. EqualitySub(31,54,[0]) 56. Hyp("neg Set(y)") 57. OrIntL(56,"neg Set(x)") 58. ImpElim(57,28) 59. Symmetry(58) 60. EqualitySub(31,59,[0]) 61. ImpElim(60,45)

62. EqualitySub(41,61,[0])   63. EqualitySub(62,51,[0])   64. EqualitySub(15,63,[0])
65. Symmetry(64)   66. EqualitySub(31,65,[0])   67. OrElim(19,20,55,56,66)   68.
ImpInt(67,19)   69. TheoremInt(9)   70. AndElimR(69)   71. EquivExp(70)   72.
AndElimL(71)   73. PolySub(72,"A","Set(x)")   74. PolySub(73,"B","Set(y)")   75.
Hyp("neg (Set(x) & Set(y))")   76. ImpElim(75,74)   77. ImpElim(76,68)   78.
ImpInt(77,75)   79. TheoremInt(7)   80. PolySub(79,"A","neg (Set(x) & Set(y))")   81.
PolySub(80,"B","neg Set(orderedpair(x,y))")   82. ImpElim(78,81)   83. TheoremInt(8)
84. EquivExp(83)   85. AndElimL(84)   86. EquivExp(83)   87. AndElimR(86)   88.
PolySub(85,"D","Set(orderedpair(x,y))")   89. PolySub(87,"D","(Set(x) & Set(y))")   90.
Hyp("Set(orderedpair(x,y))")   91. ImpElim(90,88)   92. ImpElim(91,82)   93.
ImpElim(92,89)   94. ImpInt(93,90)   95. AndInt(18,94)   96. EquivConst(95)   97.
Hyp("neg Set(orderedpair(x,y))")   98. PolySub(79,"A","(Set(x) & Set(y))")   99.
PolySub(98,"B","Set(orderedpair(x,y))")   100. ImpElim(18,99)   101. ImpElim(97,100)
102. ImpElim(101,74)   103. Hyp("neg Set(x)")   104. ImpElim(103,23)   105.
Symmetry(104)   106. EqualitySub(31,105,[0])   107. ImpElim(106,35)   108.
EqualitySub(41,107,[0])   109. TheoremInt(10)   110. AndElimL(109)   111.
ForallInt(110,"x","x")   112. ForallElim(111,"U")   113. ForallInt(112,"y","y")   114.
ForallElim(113,"singleton(pair(x,y))")   115. EqualitySub(108,114,[0])   116.
TheoremInt(6)   117. AndElimL(116)   118. ForallInt(117,"x","x")   119.
ForallElim(118,"singleton(pair(x,y))")   120. EqualitySub(114,119,[0])   121.
EqualitySub(108,120,[0])   122. EqualitySub(15,121,[0])   123. Hyp("neg Set(y)")
124. AndElimR(25)   125. EquivExp(124)   126. AndElimR(125)   127. OrIntL(123,"neg
Set(x)")   128. ImpElim(127,126)   129. Symmetry(128)   130. EqualitySub(31,129,[0])
131. ImpElim(130,45)   132. EqualitySub(41,131,[0])   133. ForallInt(117,"x","x")
134. ForallElim(133,"singleton(singleton(x))")   135. EqualitySub(132,134,[0])   136.
EqualitySub(15,135,[0])   137. OrElim(102,103,122,123,136)   138. ImpInt(137,97)
139. AndInt(96,138)

Th50. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))

0. Hyp("(Set(x) & Set(y))")   1. TheoremInt(1)   2. AndElimL(1)   3. TheoremInt(2)
4. AndElimL(3)   5. ImpElim(0,4)   6. AndElimL(5)   7. TheoremInt(3)   8. AndElimL(0)
9. ImpElim(8,7)   10. ForallInt(1,"x","x")   11. ForallElim(10,"singleton(x)")   12.
ForallInt(11,"y","y")   13. ForallElim(12,"pair(x,y)")   14. AndInt(9,6)   15.
AndElimL(13)   16. ImpElim(14,15)   17. DefEqInt(10)   18. EqualitySub(16,17,[1,3])
19. TheoremInt(4)   20. ForallInt(19,"x","x")   21. ForallElim(20,"singleton(x)")
22. ForallInt(21,"y","y")   23. ForallElim(22,"singleton(x)")   24.
ForallInt(23,"z","z")   25. ForallElim(24,"singleton(y)")   26. TheoremInt(5)   27.
ForallInt(26,"x","x")   28. ForallElim(27,"singleton(x)")   29. AndElimL(28)   30.
AndElimR(28)   31. AndElimL(25)   32. AndElimR(25)   33. EqualitySub(18,31,[0])   34.
EqualitySub(33,30,[0])   35. TheoremInt(6)   36. AndElimL(35)   37.
ForallInt(36,"x","x")   38. ForallElim(37,"singleton(x)")   39. ForallInt(38,"y","y")
40. ForallElim(39,"singleton(x)")   41. ForallInt(40,"z","z")   42.
ForallElim(41,"singleton(y)")   43. Symmetry(42)   44. EqualitySub(34,43,[0])   45.
EqualitySub(44,29,[0])   46. Hyp("Elem(z, intersection(singleton(x),singleton(y)))")
47. TheoremInt(7)   48. AndElimR(47)   49. EquivExp(48)   50. AndElimL(49)   51.
ForallInt(50,"x","x")   52. ForallElim(51,"singleton(x)")   53. ForallInt(52,"y","y")
54. ForallElim(53,"singleton(y)")   55. ImpElim(46,54)   56. AndElimL(55)   57.
ImpInt(56,46)   58. ForallInt(57,"z","z")   59. ForallInt(58,"x","x")   60.
ForallElim(59,"a")   61. ForallInt(60,"y","y")   62. ForallElim(61,"b")   63.
DefSub(62,"Contains",["intersection(singleton(a),singleton(b))","singleton(a)"],[0])
64. TheoremInt(9)   65. ForallInt(64,"x","x")   66.
ForallElim(65,"intersection(singleton(a),singleton(b))")   67. ForallInt(66,"y","y")
68. ForallElim(67,"singleton(a)")   69. EquivExp(68)   70. AndElimL(69)   71.
ImpElim(63,70)   72. ForallInt(71,"a","a")   73. ForallElim(72,"x")   74.
ForallInt(73,"b","b")   75. ForallElim(74,"y")   76. TheoremInt(10)   77. AndElimL(76)
78. ForallInt(77,"x","x")   79. ForallElim(78,"intersection(singleton(x),singleton(a))")
80. ForallInt(79,"y","y")   81. ForallElim(80,"singleton(x)")   82.
ForallInt(81,"a","a")   83. ForallElim(82,"y")   84. EqualitySub(75,83,[0])   85.
EqualitySub(45,84,[0])   86. Symmetry(17)   87. EqualitySub(85,86,[0])   88.
TheoremInt(11)   89. AndElimL(88)   90. ImpElim(8,89)   91. DefEqInt(11)   92.
Symmetry(91)   93. EqualitySub(87,92,[0,1])   94. AndElimL(93)   95. AndElimR(93)
96. Symmetry(94)   97. Symmetry(95)   98. AndElimL(90)   99. EqualitySub(98,96,[0])
100. AndElimR(90)   101. EqualitySub(100,96,[0])   102. TheoremInt(1)   103.
AndElimL(102)   104. ImpElim(0,103)   105. AndElimL(104)   106. AndElimR(104)   107.
EqualitySub(105,97,[0])   108. EqualitySub(106,97,[0])   109. AndElimR(102)   110.
TheoremInt(12)   111. AndElimL(110)   112. EquivExp(111)   113. AndElimR(112)   114.
TheoremInt(13)   115. AndElimR(114)   116. EquivExp(115)   117. AndElimR(116)   118.

PolySub(117,"A","Set(x)") 119. PolySub(118,"B","Set(y)") 120. TheoremInt(14) 121. PolySub(120,"A","Set(orderedpair(x,y))") 122. PolySub(121,"B","(Set(x) & Set(y))") 123. ImpElim(113,122) 124. AndElimR(110) 125. Hyp("(neg Set(x) v neg Set(y))") 126. ImpElim(125,119) 127. ImpElim(126,123) 128. ImpElim(127,124) 129. Symmetry(128) 130. TheoremInt(15) 131. EqualitySub(130,129,[0,2]) 132. AndElimR(131) 133. AndElimL(131) 134. TheoremInt(16) 135. EqualitySub(130,132,[0,2]) 136. EqualitySub(134,133,[0,1]) 137. AndElimL(135) 138. AndElimR(135) 139. Symmetry(137) 140. Symmetry(138) 141. AndInt(140,139) 142. AndElimL(136) 143. AndElimR(136) 144. AndInt(143,142) 145. AndInt(144,141) 146. ImpInt(145,125) 147. AndInt(95,94) 148. AndInt(101,99) 149. AndInt(108,107) 150. AndInt(147,148) 151. AndInt(150,149) 152. ImpInt(151,0) 153. AndInt(152,146)

Th53. proj2(U) = U

0. DefEqInt(13) 1. ForallInt(0,"x","x") 2. ForallElim(1,"U") 3. TheoremInt(1) 4. TheoremInt(2) 5. AndElimL(3) 6. AndElimR(3) 7. AndElimL(4) 8. AndElimR(4) 9. Symmetry(5) 10. Symmetry(6) 11. EqualitySub(2,10,[0,1]) 12. EqualitySub(11,9,[0,1]) 13. EqualitySub(12,10,[0]) 14. EqualitySub(13,8,[0]) 15. TheoremInt(3) 16. AndElimL(15) 17. ForallInt(16,"x","x") 18. ForallElim(17,"complement2(U,0)") 19. EqualitySub(14,18,[0]) 20. DefEqInt(3) 21. ForallInt(20,"x","x") 22. ForallElim(21,"U") 23. ForallInt(22,"y","y") 24. ForallElim(23,"0") 25. TheoremInt(5) 26. AndElimL(25) 27. EqualitySub(24,26,[0]) 28. TheoremInt(6) 29. AndElimR(28) 30. ForallInt(29,"x","x") 31. ForallElim(30,"U") 32. EqualitySub(27,31,[0]) 33. EqualitySub(19,32,[0])

Th54. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)))

0. Hyp("(Set(x) & Set(y))") 1. DefEqInt(12) 2. DefEqInt(13) 3. TheoremInt(1) 4. AndElimL(3) 5. ImpElim(0,4) 6. AndElimL(5) 7. AndElimR(6) 8. AndElimR(7) 9. ForallInt(1,"x","x") 10. ForallInt(1,"x","x") 11. ForallElim(10,"orderedpair(x,y)") 12. EqualitySub(11,8,[0]) 13. ForallInt(2,"x","x") 14. ForallElim(13,"orderedpair(x,y)") 15. AndElimL(7) 16. AndElimR(5) 17. AndElimL(16) 18. AndElimR(16) 19. EqualitySub(14,17,[0]) 20. EqualitySub(19,18,[0]) 21. EqualitySub(20,15,[0]) 22. Hyp("Elem(z, complement2(union(x,y),x))") 23. DefEqInt(3) 24. ForallInt(23,"x","x") 25. ForallElim(24,"a") 26. ForallInt(25,"y","y") 27. ForallElim(26,"b") 28. ForallInt(27,"a","a") 29. ForallElim(28,"union(x,y)") 30. ForallInt(29,"b","b") 31. ForallElim(30,"x") 32. EqualitySub(22,31,[0]) 33. TheoremInt(2) 34. AndElimR(33) 35. EquivExp(34) 36. AndElimL(35) 37. ForallInt(36,"x","x") 38. ForallElim(37,"a") 39. ForallInt(38,"y","y") 40. ForallElim(39,"b") 41. ForallInt(40,"a","a") 42. ForallElim(41,"union(x,y)") 43. ForallInt(42,"b","b") 44. ForallElim(43,"complement1(x)") 45. ImpElim(32,44) 46. AndElimL(45) 47. AndElimL(33) 48. EquivExp(47) 49. AndElimL(48) 50. ImpElim(46,49) 51. AndElimR(45) 52. DefEqInt(2) 53. EqualitySub(51,52,[0]) 54. ClassElim(53) 55. AndElimR(54) 56. Hyp("Elem(z,x)") 57. ImpElim(56,55) 58. AbsI(57,"Elem(z,intersection(y,complement1(x)))") 59. Hyp("Elem(z,y)") 60. AndInt(59,51) 61. EquivExp(34) 62. AndElimR(61) 63. ForallInt(62,"y","y") 64. ForallElim(63,"a") 65. ForallInt(64,"x","x") 66. ForallElim(65,"y") 67. ForallInt(66,"a","a") 68. ForallInt(66,"a","a") 69. ForallElim(68,"complement1(x)") 70. ImpElim(60,69) 71. OrElim(50,56,58,59,70) 72. ImpInt(71,22) 73. Hyp("Elem(z,intersection(y,complement1(x)))") 74. AndElimL(61) 75. ForallInt(74,"y","y") 76. ForallElim(75,"a") 77. ForallInt(76,"x","x") 78. ForallElim(77,"y") 79. ForallInt(78,"a","a") 80. ForallElim(79,"complement1(x)") 81. ImpElim(73,80) 82. AndElimL(81) 83. OrIntL(82,"Elem(z,x)") 84. AndElimR(48) 85. ImpElim(83,84) 86. AndElimR(81) 87. AndInt(85,86) 88. AndElimR(35) 89. ForallInt(88,"y","y") 90. ForallElim(89,"a") 91. ForallInt(90,"x","x") 92. ForallElim(91,"union(x,y)") 93. ForallInt(92,"a","a") 94. ForallElim(93,"complement1(x)") 95. ImpElim(87,94) 96. Symmetry(31) 97. EqualitySub(95,96,[0]) 98. ImpInt(97,73) 99. AndInt(72,98) 100. EquivConst(99) 101. ForallInt(100,"z","z") 102. AxInt(0) 103. ForallElim(102,"complement2(union(x,y),x)") 104. ForallElim(103,"intersection(y,complement1(x))") 105. EquivExp(104) 106. AndElimR(105) 107. ImpElim(101,106) 108. EqualitySub(21,107,[0]) 109. TheoremInt(3) 110. AndElimR(109) 111. EqualitySub(108,110,[0]) 112. TheoremInt(4) 113. AndElimL(112) 114. Symmetry(113) 115. ForallInt(114,"x","x") 116. ForallElim(115,"a") 117. ForallInt(116,"y","y") 118. ForallElim(117,"b") 119. ForallInt(118,"a","a") 120. ForallElim(119,"y") 121. ForallInt(120,"b","b")

122. ForallElim(121,"x")   123. ForallInt(122,"z","z")   124. ForallElim(123,"complement1(x)")   125. EqualitySub(111,124,[0])   126. Hyp("Elem(z,U)")   127. TheoremInt(0)   128. PolySub(127,"A","Elem(z,x)")   129. Hyp("Elem(z,x)")   130. OrIntR(129,"Elem(z,complement1(x))")   131. ForallInt(84,"y","y")   132. ForallElim(131,"complement1(x)")   133. ImpElim(130,132)   134. Hyp("neg Elem(z,x)")   135. ExistsInt(126,"U","y",[0])   136. DefSub(135,"Set",["z"],[0])   137. AndInt(134,136)   138. ClassInt(137,"z")   139. Symmetry(52)   140. EqualitySub(138,139,[0])   141. OrIntL(140,"Elem(z,x)")   142. ImpElim(141,132)   143. OrElim(128,129,133,134,142)   144. ImpInt(143,126)   145. ForallInt(144,"z","z")   146. DefSub(145,"Contains",["U","union(x,complement1(x))"],[0])   147. TheoremInt(5)   148. AndElimR(147)   149. ForallInt(148,"x","x")   150. ForallElim(149,"union(x,complement1(x))")   151. AndInt(146,150)   152. TheoremInt(6)   153. EquivExp(152)   154. AndElimR(153)   155. ForallInt(154,"x","x")   156. ForallElim(155,"U")   157. ForallInt(156,"y","y")   158. ForallElim(157,"union(x,complement1(x))")   159. ImpElim(151,158)   160. Symmetry(159)   161. EqualitySub(125,160,[0])   162. TheoremInt(8)   163. AndElimR(162)   164. ForallInt(163,"x","x")   165. ForallElim(164,"y")   166. EqualitySub(161,165,[0])   167. AndInt(12,166)   168. ImpInt(167,0)   169. Hyp("(neg Set(x) v neg Set(y))")   170. AndElimR(3)   171. ImpElim(169,170)   172. AndElimL(171)   173. AndElimR(172)   174. EqualitySub(11,173,[0])   175. AndElimR(171)   176. AndElimR(175)   177. AndElimL(175)   178. AndElimL(172)   179. EqualitySub(14,177,[0])   180. EqualitySub(179,178,[0])   181. EqualitySub(180,176,[0])   182. TheoremInt(7)   183. AndElimL(182)   184. ForallInt(183,"x","x")   185. ForallElim(184,"complement2(U,0)")   186. EqualitySub(181,185,[0])   187. ForallInt(23,"x","x")   188. ForallElim(187,"U")   189. ForallInt(188,"y","y")   190. ForallElim(189,"0")   191. EqualitySub(186,190,[0])   192. TheoremInt(9)   193. AndElimL(192)   194. EqualitySub(191,193,[0])   195. TheoremInt(10)   196. AndElimR(195)   197. ForallInt(196,"x","x")   198. ForallElim(197,"U")   199. EqualitySub(194,198,[0])   200. AndInt(174,199)   201. ImpInt(200,169)   202. AndInt(168,201)

Th55. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

0. Hyp("((Set(x) & Set(y)) & (orderedpair(x,y) = orderedpair(u,v)))")   1. TheoremInt(1)   2. AndElimL(1)   3. AndElimL(0)   4. ImpElim(3,2)   5. TheoremInt(2)   6. AndElimL(5)   7. EquivExp(6)   8. AndElimL(7)   9. ImpElim(3,8)   10. AndElimR(0)   11. EqualitySub(9,10,[0])   12. EquivExp(6)   13. AndElimR(12)   14. ForallInt(13,"x","x")   15. ForallElim(14,"u")   16. ForallInt(15,"y","y")   17. ForallElim(16,"v")   18. ImpElim(11,17)   19. ForallInt(2,"x","x")   20. ForallElim(19,"u")   21. ForallInt(20,"y","y")   22. ForallElim(21,"v")   23. ImpElim(18,22)   24. AndElimL(4)   25. AndElimR(4)   26. AndElimL(23)   27. AndElimR(23)   28. EqualitySub(24,10,[0])   29. EqualitySub(28,26,[0])   30. EqualitySub(25,10,[0])   31. EqualitySub(30,27,[0])   32. Symmetry(29)   33. Symmetry(31)   34. AndInt(32,33)   35. ImpInt(34,0)

Th58. ((r∘s)∘t) = (r∘(s∘t))

0. Hyp("Elem(z, comp(comp(r,s),t))")   1. DefEqInt(14)   2. ForallInt(1,"a","a")   3. ForallElim(2,"comp(r,s)")   4. ForallInt(3,"b","b")   5. ForallElim(4,"t")   6. EqualitySub(0,5,[0])   7. ClassElim(6)   8. AndElimR(7)   9. ExistsInst(8,"x")   10. ExistsInst(9,"y")   11. ExistsInst(10,"c")   12. AndElimL(11)   13. AndElimR(12)   14. ForallInt(1,"a","a")   15. ForallElim(14,"r")   16. ForallInt(15,"b","b")   17. ForallElim(16,"s")   18. EqualitySub(13,17,[0])   19. ClassElim(18)   20. AndElimR(19)   21. ExistsInst(20,"a")   22. ExistsInst(21,"b")   23. ExistsInst(22,"d")   24. AndElimL(23)   25. AndElimL(12)   26. AndElimL(24)   27. TheoremInt(2)   28. AndElimL(27)   29. EquivExp(28)   30. AndElimR(29)   31. ForallInt(30,"y","y")   32. ForallElim(31,"c")   33. ForallInt(32,"x","x")   34. ForallElim(33,"y")   35. AndElimL(19)   36. ImpElim(35,34)   37. TheoremInt(1)   38. ForallInt(37,"y","y")   39. ForallElim(38,"c")   40. ForallInt(39,"x","x")   41. ForallElim(40,"y")   42. ForallInt(41,"u","u")   43. ForallElim(42,"a")   44. ForallInt(43,"v","v")   45. ForallElim(44,"d")   46. AndElimR(23)   47. AndInt(36,46)   48. ImpElim(47,45)   49. AndElimL(48)   50. AndElimR(48)   51. EqualitySub(25,49,[0])   52. AndInt(51,26)   53. AndElimR(24)   54. Hyp("(g = orderedpair(x,b))")   55. AndInt(52,54)   56. ExistsInt(55,"b","b",[0,1])   57. ExistsInt(56,"a","a",[0,1])   58. ExistsInt(57,"x","x",[0,1])   59. ExistsInt(53,"r","r",[0])   60. DefSub(59,"Set",["orderedpair(b,d)"],[0])   61. ForallInt(30,"x","x")   62. ForallElim(61,"b")   63. ForallInt(62,"y","y")   64. ForallElim(63,"d")   65. ImpElim(60,64)   66. AndElimL(65)   67. ExistsInt(51,"t","t",[0])   68. DefSub(67,"Set",["orderedpair(x,a)"],[0])   69. ForallInt(30,"y","y")   70. ForallElim(69,"a")   71. ImpElim(68,70)   72. AndElimL(71)   73. AndInt(72,66)   74. EquivExp(28)   75. AndElimL(74)   76. ForallInt(75,"y","y")   77. ForallElim(76,"b")   78. ImpElim(73,77)   79. Symmetry(54)   80.

EqualitySub(78,79,[0]) 81. AndInt(80,58) 82. ClassInt(81,"w") 83. ForallInt(1,"a","a") 84. ForallElim(83,"s") 85. ForallInt(84,"b","b") 86. ForallElim(85,"t") 87. Symmetry(86) 88. EqualitySub(82,87,[0]) 89. EqualitySub(88,54,[0]) 90. ImpInt(89,54) 91. ForallInt(90,"g","g") 92. ForallElim(91,"orderedpair(x,b)") 93. Identity("orderedpair(x,b)") 94. ImpElim(93,92) 95. AndInt(53,94) 96. Symmetry(50) 97. AndElimR(11) 98. AndInt(94,53) 99. AndInt(98,97) 100. EqualitySub(99,96,[0]) 101. ExistsInt(100,"c","c",[0,1]) 102. ExistsInt(101,"b","b",[0,1]) 103. ExistsInt(102,"x","x",[0,1]) 104. AndElimL(7) 105. AndInt(104,103) 106. ClassInt(105,"w") 107. ForallInt(1,"a","a") 108. ForallElim(107,"r") 109. ForallInt(108,"b","b") 110. ForallElim(109,"comp(s,t)") 111. Symmetry(110) 112. EqualitySub(106,111,[0]) 113. ExistsElim(22,23,112,"d") 114. ExistsElim(21,22,113,"b") 115. ExistsElim(20,21,114,"a") 116. ExistsElim(10,11,115,"c") 117. ExistsElim(9,10,116,"y") 118. ExistsElim(8,9,117,"x") 119. ImpInt(118,0) 120. Hyp("Elem(z,comp(r,comp(s,t)))") 121. ForallInt(1,"a","a") 122. ForallElim(121,"r") 123. ForallInt(122,"b","b") 124. ForallElim(123,"comp(s,t)") 125. EqualitySub(120,124,[0]) 126. ClassElim(125) 127. AndElimR(126) 128. ExistsInst(127,"x") 129. ExistsInst(128,"y") 130. ExistsInst(129,"c") 131. AndElimR(130) 132. AndElimL(130) 133. AndElimL(132) 134. AndElimR(132) 135. EqualitySub(133,86,[0]) 136. ClassElim(135) 137. AndElimL(136) 138. AndElimR(136) 139. ExistsInst(138,"a") 140. ExistsInst(139,"b") 141. ExistsInst(140,"d") 142. AndElimR(141) 143. EqualitySub(137,142,[0]) 144. AndElimR(74) 145. ForallInt(144,"x","x") 146. ForallElim(145,"a") 147. ForallInt(146,"y","y") 148. ForallElim(147,"d") 149. ImpElim(143,148) 150. AndElimL(149) 151. AndElimR(149) 152. AndElimL(141) 153. AndElimR(152) 154. AndInt(153,134) 155. ImpElim(137,144) 156. AndInt(155,142) 157. TheoremInt(1) 158. ForallInt(157,"u","u") 159. ForallElim(158,"a") 160. ForallInt(159,"v","v") 161. ForallElim(160,"d") 162. ImpElim(156,161) 163. AndElimR(162) 164. Symmetry(163) 165. EqualitySub(154,164,[0]) 166. Hyp("(h = orderedpair(b,c))") 167. ExistsInt(153,"s","w",[0]) 168. ExistsInt(134,"r","w",[0]) 169. DefSub(167,"Set",["orderedpair(b,d)"],[0]) 170. DefSub(168,"Set",["orderedpair(y,c)"],[0]) 171. ForallInt(144,"x","x") 172. ForallElim(171,"b") 173. ForallInt(172,"y","y") 174. ForallElim(173,"d") 175. ForallInt(144,"y","y") 176. ForallElim(175,"c") 177. ForallInt(176,"x","x") 178. ForallElim(177,"y") 179. ImpElim(169,174) 180. ImpElim(170,178) 181. AndElimL(179) 182. AndElimR(180) 183. AndElimL(74) 184. ForallInt(183,"x","x") 185. ForallElim(184,"b") 186. ForallInt(185,"y","y") 187. ForallElim(186,"c") 188. AndInt(181,182) 189. ImpElim(188,187) 190. Symmetry(166) 191. EqualitySub(189,190,[0]) 192. AndInt(165,166) 193. ExistsInt(192,"c","c",[0,1]) 194. ExistsInt(193,"y","y",[0,1]) 195. ExistsInt(194,"b","b",[0,1]) 196. AndInt(191,195) 197. ClassInt(196,"w") 198. ForallInt(1,"a","a") 199. ForallElim(198,"r") 200. ForallInt(199,"b","b") 201. ForallElim(200,"s") 202. Symmetry(201) 203. EqualitySub(197,202,[0]) 204. EqualitySub(203,166,[0]) 205. ImpInt(204,166) 206. ForallInt(205,"h","h") 207. ForallElim(206,"orderedpair(b,c)") 208. Identity("orderedpair(b,c)") 209. ImpElim(208,207) 210. AndElimL(152) 211. AndElimL(162) 212. Symmetry(211) 213. EqualitySub(210,212,[0]) 214. AndInt(213,209) 215. AndInt(214,131) 216. ExistsInt(215,"c","c",[0,1]) 217. ExistsInt(216,"b","b",[0,1]) 218. ExistsInt(217,"x","x",[0,1]) 219. AndElimL(126) 220. AndInt(219,218) 221. ClassInt(220,"w") 222. ForallInt(1,"a","a") 223. ForallElim(222,"comp(r,s)") 224. ForallInt(223,"b","b") 225. ForallElim(224,"t") 226. Symmetry(225) 227. EqualitySub(221,226,[0]) 228. ExistsElim(140,141,227,"d") 229. ExistsElim(139,140,228,"b") 230. ExistsElim(138,139,229,"a") 231. ExistsElim(129,130,230,"c") 232. ExistsElim(128,129,231,"y") 233. ExistsElim(127,128,232,"x") 234. ImpInt(233,120) 235. AndInt(119,234) 236. EquivConst(235) 237. ForallInt(236,"z","z") 238. AxInt(0) 239. ForallElim(238,"comp(comp(r,s),t)") 240. ForallElim(239,"comp(r,comp(s,t))") 241. EquivExp(240) 242. AndElimR(241) 243. ImpElim(237,242)

Th59. $((r\circ(s \cup t)) = ((r\circ s) \cup (r\circ t))) \ \&\ ((r\circ(s \cap t)) \subset ((r\circ s) \cap (r\circ t)))$

0. Hyp("Elem(z,comp(r,union(s,t)))") 1. DefEqInt(14) 2. ForallInt(1,"a","a") 3. ForallElim(2,"r") 4. ForallInt(3,"b","b") 5. ForallElim(4,"union(s,t)") 6. EqualitySub(0,5,[0]) 7. ClassElim(6) 8. AndElimR(7) 9. ExistsInst(8,"x") 10. ExistsInst(9,"y") 11. ExistsInst(10,"c") 12. AndElimL(11) 13. AndElimL(12) 14. TheoremInt(1) 15. AndElimL(14) 16. EquivExp(15) 17. AndElimL(16) 18. ForallInt(17,"x","x") 19. ForallElim(18,"s") 20. ForallInt(19,"y","y") 21. ForallElim(20,"t") 22. ForallInt(21,"z","z") 23. ForallElim(22,"orderedpair(x,y)") 24. ImpElim(13,23) 25. Hyp("Elem(orderedpair(x,y),s)") 26. AndElimR(12) 27. AndInt(25,26) 28. AndElimR(11) 29. AndInt(27,28) 30. ExistsInt(29,"c","c",[0,1]) 31. ExistsInt(30,"y","y",[0,1]) 32. ExistsInt(31,"x","x",[0,1]) 33. AndElimL(7)

34. AndInt(33,32)   35. ClassInt(34,"w")   36. ForallInt(1,"a","a")   37. ForallElim(36,"r")   38. ForallInt(37,"b","b")   39. ForallElim(38,"s")   40. Symmetry(39)   41. EqualitySub(35,40,[0])   42. OrIntR(41,"Elem(z,comp(r,t))")   43. AndElimR(16)   44. ForallInt(43,"x","x")   45. ForallElim(44,"comp(r,s)")   46. ForallInt(45,"y","y")   47. ForallElim(46,"comp(r,t)")   48. ImpElim(42,47)   49. Hyp("Elem(orderedpair(x,y), t)")   50. AndInt(49,26)   51. AndInt(50,28)   52. ExistsInt(51,"c","c",[0,1])   53. ExistsInt(52,"y","y",[0,1])   54. ExistsInt(53,"x","x",[0,1])   55. AndInt(33,54)   56. ClassInt(55,"w")   57. ForallInt(1,"a","a")   58. ForallElim(57,"r")   59. ForallInt(58,"b","b")   60. ForallElim(59,"t")   61. Symmetry(60)   62. EqualitySub(56,61,[0])   63. OrIntL(62,"Elem(z,comp(r,s))")   64. ImpElim(63,47)   65. OrElim(24,25,48,49,64)   66. ExistsElim(10,11,65,"c")   67. ExistsElim(9,10,66,"y")   68. ExistsElim(8,9,67,"x")   69. ImpInt(68,0)   70. Hyp("Elem(z,union(comp(r,s), comp(r,t)))")   71. ForallInt(17,"x","x")   72. ForallElim(71,"comp(r,s)")   73. ForallInt(72,"y","y")   74. ForallElim(73,"comp(r,t)")   75. ImpElim(70,74)   76. Hyp("Elem(z,comp(r,s))")   77. ForallInt(1,"a","a")   78. ForallElim(77,"r")   79. ForallInt(78,"b","b")   80. ForallElim(79,"s")   81. EqualitySub(76,80,[0])   82. ClassElim(81)   83. AndElimR(82)   84. ExistsInst(83,"x")   85. ExistsInst(84,"y")   86. ExistsInst(85,"m")   87. AndElimL(86)   88. AndElimL(87)   89. OrIntR(88,"Elem(orderedpair(x,y),t)")   90. AndElimR(87)   91. EquivExp(15)   92. AndElimR(91)   93. ForallInt(92,"x","x")   94. ForallElim(93,"s")   95. ForallInt(94,"y","y")   96. ForallElim(95,"t")   97. ForallInt(96,"z","z")   98. ForallElim(97,"orderedpair(x,y)")   99. ImpElim(89,98)   100. AndInt(99,90)   101. AndElimR(86)   102. AndInt(100,101)   103. ExistsInt(102,"m","m",[0,1])   104. ExistsInt(103,"y","y",[0,1])   105. ExistsInt(104,"x","x",[0,1])   106. AndElimL(82)   107. AndInt(106,105)   108. ClassInt(107,"w")   109. Symmetry(5)   110. EqualitySub(108,109,[0])   111. ExistsElim(85,86,110,"m")   112. ExistsElim(84,85,111,"y")   113. ExistsElim(83,84,112,"x")   114. Hyp("Elem(z, comp(r,t))")   115. ForallInt(78,"b","b")   116. ForallElim(115,"t")   117. EqualitySub(114,116,[0])   118. ClassElim(117)   119. AndElimR(118)   120. ExistsInst(119,"x")   121. ExistsInst(120,"y")   122. ExistsInst(121,"e")   123. AndElimL(122)   124. AndElimL(123)   125. OrIntL(124,"Elem(orderedpair(x,y),s)")   126. ImpElim(125,98)   127. AndElimR(123)   128. AndInt(126,127)   129. AndElimR(122)   130. AndInt(128,129)   131. ExistsInt(130,"e","e",[0,1])   132. ExistsInt(131,"y","y",[0,1])   133. ExistsInt(132,"x","x",[0,1])   134. AndElimL(118)   135. AndInt(134,133)   136. ClassInt(135,"w")   137. EqualitySub(136,109,[0])   138. ExistsElim(121,122,137,"e")   139. ExistsElim(120,121,138,"y")   140. ExistsElim(119,120,139,"x")   141. OrElim(75,76,113,114,140)   142. ImpInt(141,70)   143. AndInt(69,142)   144. EquivConst(143)   145. AxInt(0)   146. ForallElim(145,"comp(r,union(s,t))")   147. ForallElim(146,"union(comp(r,s),comp(r,t))")   148. EquivExp(147)   149. AndElimR(148)   150. ForallInt(144,"z","z")   151. ImpElim(150,149)   152. Hyp("Elem(z,comp(r,intersection(s,t)))")   153. ForallInt(1,"a","a")   154. ForallElim(153,"r")   155. ForallInt(154,"b","b")   156. ForallElim(155,"intersection(s,t)")   157. EqualitySub(152,156,[0])   158. ClassElim(157)   159. AndElimR(158)   160. ExistsInst(159,"x")   161. ExistsInst(160,"y")   162. ExistsInst(161,"e")   163. AndElimL(162)   164. AndElimL(163)   165. AndElimR(14)   166. ForallInt(165,"x","x")   167. ForallElim(166,"s")   168. ForallInt(167,"y","y")   169. ForallElim(168,"t")   170. ForallInt(169,"z","z")   171. ForallElim(170,"orderedpair(x,y)")   172. EquivExp(171)   173. AndElimL(172)   174. ImpElim(164,173)   175. AndElimL(174)   176. AndElimR(163)   177. AndInt(175,176)   178. AndElimR(162)   179. AndInt(177,178)   180. ExistsInt(179,"e","e",[0,1])   181. ExistsInt(180,"y","y",[0,1])   182. ExistsInt(181,"x","x",[0,1])   183. AndElimL(158)   184. AndInt(183,182)   185. ClassInt(184,"w")   186. EqualitySub(185,40,[0])   187. AndElimR(174)   188. AndInt(187,176)   189. AndInt(188,178)   190. ExistsInt(189,"e","e",[0,1])   191. ExistsInt(190,"y","y",[0,1])   192. ExistsInt(191,"x","x",[0,1])   193. AndInt(183,192)   194. ClassInt(193,"w")   195. EqualitySub(194,61,[0])   196. AndInt(186,195)   197. EquivExp(165)   198. AndElimR(197)   199. ForallInt(198,"x","x")   200. ForallElim(199,"comp(r,s)")   201. ForallInt(200,"y","y")   202. ForallElim(201,"comp(r,t)")   203. ImpElim(196,202)   204. ExistsElim(161,162,203,"e")   205. ExistsElim(160,161,204,"y")   206. ExistsElim(159,160,205,"x")   207. ImpInt(206,152)   208. ForallInt(207,"z","z")   209. DefSub(208,"Contains", ["comp(r,intersection(s,t))","intersection(comp(r,s),comp(r,t))"],[0])   210. AndInt(151,209)

Th61. Relation(r) -> $(((r)^{-1})^{-1} = r)$

0. Hyp("Elem(z,inv(inv(r)))")   1. DefEqInt(15)   2. ForallInt(1,"r","r")   3. ForallElim(2,"inv(r)")   4. EqualitySub(0,3,[0])   5. ClassElim(4)   6. AndElimR(5)   7. ExistsInst(6,"x")   8. ExistsInst(7,"y")   9. AndElimL(8)   10. EqualitySub(9,1,

[0]) 11. ClassElim(10) 12. AndElimR(11) 13. ExistsInst(12,"c") 14. ExistsInst(13,"d") 15. AndElimR(8) 16. AndElimL(5) 17. TheoremInt(1) 18. TheoremInt(2) 19. AndElimL(18) 20. EquivExp(19) 21. AndElimR(20) 22. EqualitySub(16,15,[0]) 23. ForallInt(21,"x","x") 24. ForallElim(23,"a") 25. ForallInt(24,"y","y") 26. ForallElim(25,"x") 27. ForallInt(26,"a","a") 28. ForallElim(27,"y") 29. ImpElim(22,28) 30. AndElimL(29) 31. AndElimR(29) 32. AndInt(31,30) 33. ForallInt(17,"u","u") 34. ForallElim(33,"d") 35. ForallInt(34,"v","v") 36. ForallElim(35,"c") 37. AndElimR(14) 38. AndInt(32,37) 39. ImpElim(38,36) 40. AndElimL(39) 41. AndElimR(39) 42. AndElimL(14) 43. Symmetry(40) 44. Symmetry(41) 45. EqualitySub(42,43,[0]) 46. EqualitySub(45,44,[0]) 47. ExistsElim(13,14,46,"d") 48. ExistsElim(12,13,47,"c") 49. Symmetry(15) 50. EqualitySub(48,49,[0]) 51. ExistsElim(7,8,50,"y") 52. ExistsElim(6,7,51,"x") 53. ImpInt(52,0) 54. Hyp("Relation(r)") 55. Hyp("Elem(z,r)") 56. DefExp(54,"Relation",[0]) 57. ForallElim(56,"z") 58. ImpElim(55,57) 59. ExistsInst(58,"x") 60. ExistsInst(59,"y") 61. Hyp("(f = orderedpair(y,x))") 62. EqualitySub(55,60,[0]) 63. AndInt(62,61) 64. EqualitySub(16,15,[0]) 65. TheoremInt(3) 66. AndElimL(65) 67. EquivExp(66) 68. AndElimR(67) 69. ExistsInt(55,"r","w",[0]) 70. DefSub(69,"Set",["z"],[0]) 71. EqualitySub(70,60,[0]) 72. ImpElim(71,68) 73. AndElimL(72) 74. AndElimR(72) 75. EquivExp(66) 76. AndElimL(75) 77. ForallInt(76,"x","x") 78. ForallElim(77,"a") 79. ForallInt(78,"y","y") 80. ForallElim(79,"x") 81. ForallInt(80,"a","a") 82. ForallElim(81,"y") 83. AndInt(74,73) 84. ImpElim(83,82) 85. Symmetry(61) 86. EqualitySub(84,85,[0]) 87. ExistsInt(63,"y","y",[0,1]) 88. ExistsInt(87,"x","x",[0,1]) 89. AndInt(86,88) 90. ClassInt(89,"w") 91. Symmetry(1) 92. EqualitySub(90,91,[0]) 93. EqualitySub(92,61,[0]) 94. ImpInt(93,61) 95. ForallInt(94,"f","f") 96. ForallElim(95,"orderedpair(y,x)") 97. Identity("orderedpair(y,x)") 98. ImpElim(97,96) 99. AndInt(98,60) 100. ExistsInt(99,"x","x",[0,1]) 101. ExistsInt(100,"y","y",[0,1]) 102. AndInt(70,101) 103. ClassInt(102,"w") 104. ForallInt(1,"r","r") 105. ForallElim(104,"inv(r)") 106. Symmetry(105) 107. EqualitySub(103,106,[0]) 108. ExistsElim(59,60,107,"y") 109. ExistsElim(58,59,108,"x") 110. ImpInt(109,55) 111. AndInt(53,110) 112. EquivConst(111) 113. ForallInt(112,"z","z") 114. AxInt(0) 115. ForallElim(114,"inv(inv(r))") 116. ForallElim(115,"r") 117. EquivExp(116) 118. AndElimR(117) 119. ImpElim(113,118) 120. ImpInt(119,54)

Th62. $((r \circ s))^{-1} = ((s)^{-1} \circ (r)^{-1})$

0. Hyp("Elem(z,inv(comp(r,s)))") 1. DefEqInt(15) 2. ForallInt(1,"r","r") 3. ForallElim(2,"comp(r,s)") 4. EqualitySub(0,3,[0]) 5. ClassElim(4) 6. AndElimR(5) 7. DefEqInt(14) 8. ForallInt(7,"a","a") 9. ForallElim(8,"r") 10. ForallInt(9,"b","b") 11. ForallElim(10,"s") 12. ExistsInst(6,"x") 13. ExistsInst(12,"y") 14. AndElimL(13) 15. EqualitySub(14,11,[0]) 16. ClassElim(15) 17. AndElimR(16) 18. ExistsInst(17,"c") 19. ExistsInst(18,"d") 20. ExistsInst(19,"b") 21. ExistsInt(14,"comp(r,s)","w",[0]) 22. DefSub(21,"Set",["orderedpair(x,y)"],[0]) 23. TheoremInt(1) 24. AndElimL(23) 25. EquivExp(24) 26. AndElimR(25) 27. ImpElim(22,26) 28. AndElimR(20) 29. TheoremInt(2) 30. ForallInt(29,"u","u") 31. ForallElim(30,"c") 32. ForallInt(31,"v","v") 33. ForallElim(32,"b") 34. AndInt(27,28) 35. ImpElim(34,33) 36. AndElimL(35) 37. AndElimR(35) 38. Symmetry(36) 39. Symmetry(37) 40. EqualitySub(20,38,[0,1]) 41. EqualitySub(40,39,[0,1]) 42. AndElimL(41) 43. Hyp("(h = orderedpair(d,x))") 44. AndElimL(42) 45. AndInt(44,43) 46. ExistsInt(45,"d","d",[0,1]) 47. ExistsInt(46,"x","x",[0,1]) 48. AndElimL(45) 49. ExistsInt(48,"s","w",[0]) 50. DefSub(49,"Set",["orderedpair(x,d)"],[0]) 51. ForallInt(26,"y","y") 52. ForallElim(51,"d") 53. ImpElim(50,52) 54. AndElimR(53) 55. AndElimL(53) 56. AndInt(55,54) 57. AndElimL(25) 58. ForallInt(57,"x","x") 59. ForallElim(58,"d") 60. ForallInt(59,"y","y") 61. ForallElim(60,"x") 62. AndInt(54,55) 63. ImpElim(62,61) 64. Symmetry(43) 65. EqualitySub(63,64,[0]) 66. AndInt(65,47) 67. ClassInt(66,"w") 68. ForallInt(1,"r","r") 69. ForallElim(68,"s") 70. Symmetry(69) 71. EqualitySub(67,70,[0]) 72. EqualitySub(71,43,[0]) 73. ImpInt(72,43) 74. ForallInt(73,"h","h") 75. ForallElim(74,"orderedpair(d,x)") 76. Identity("orderedpair(d,x)") 77. ImpElim(76,75) 78. Hyp("(f = orderedpair(y,d))") 79. AndElimR(42) 80. AndInt(79,78) 81. ExistsInt(80,"y","y",[0,1]) 82. ExistsInt(81,"d","d",[0,1]) 83. AndElimR(27) 84. AndInt(83,54) 85. ForallInt(57,"y","y") 86. ForallElim(85,"d") 87. ForallInt(86,"x","x") 88. ForallElim(87,"y") 89. ImpElim(84,88) 90. Symmetry(78) 91. EqualitySub(89,90,[0]) 92. AndInt(91,82) 93. ClassInt(92,"w") 94. Symmetry(1) 95. EqualitySub(93,94,[0]) 96. EqualitySub(95,78,[0]) 97. ImpInt(96,78) 98. ForallInt(97,"f","f") 99. ForallElim(98,"orderedpair(y,d)") 100. Identity("orderedpair(y,d)") 101. ImpElim(100,99) 102. AndInt(101,77) 103. AndElimR(13) 104. AndInt(102,103) 105. ExistsInt(104,"x","x",[0,1]) 106.

ExistsInt(105,"d","d",[0,1]) 107. ExistsInt(106,"y","y",[0,1]) 108. AndElimL(5) 109. AndInt(108,107) 110. ClassInt(109,"w") 111. ForallInt(7,"a","a") 112. ForallElim(111,"inv(s)") 113. ForallInt(112,"b","b") 114. ForallElim(113,"inv(r)") 115. Symmetry(114) 116. EqualitySub(110,115,[0]) 117. ExistsElim(19,20,116,"b") 118. ImpInt(117,43) 119. ForallInt(118,"h","h") 120. ForallElim(119,"orderedpair(d,x)") 121. Identity("orderedpair(d,x)") 122. ImpElim(121,120) 123. ExistsElim(18,19,122,"d") 124. ExistsElim(17,18,123,"c") 125. ExistsElim(12,13,124,"y") 126. ExistsElim(6,12,125,"x") 127. ImpInt(126,0) 128. Hyp("Elem(z, comp(inv(s),inv(r)))") 129. ForallInt(7,"a","a") 130. ForallElim(129,"inv(s)") 131. ForallInt(130,"b","b") 132. ForallElim(131,"inv(r)") 133. EqualitySub(128,132,[0]) 134. ClassElim(133) 135. AndElimL(134) 136. AndElimR(134) 137. ExistsInst(136,"x") 138. ExistsInst(137,"y") 139. ExistsInst(138,"a") 140. AndElimR(139) 141. AndElimL(139) 142. AndElimL(141) 143. AndElimR(141) 144. ForallInt(1,"r","r") 145. ForallElim(144,"s") 146. EqualitySub(142,1,[0]) 147. EqualitySub(143,145,[0]) 148. ClassElim(146) 149. ClassElim(147) 150. AndElimL(148) 151. AndElimR(148) 152. AndElimL(149) 153. AndElimR(149) 154. ExistsInst(151,"b") 155. ExistsInst(154,"c") 156. ExistsInst(153,"d") 157. ExistsInst(156,"e") 158. AndElimL(155) 159. AndElimL(157) 160. AndElimR(155) 161. AndElimR(157) 162. ImpElim(150,26) 163. AndInt(162,160) 164. ForallInt(29,"u","u") 165. ForallElim(164,"c") 166. ForallInt(165,"v","v") 167. ForallElim(166,"b") 168. ImpElim(163,167) 169. AndElimL(168) 170. AndElimR(168) 171. Symmetry(169) 172. Symmetry(170) 173. ForallInt(26,"y","y") 174. ForallElim(173,"a") 175. ForallInt(174,"x","x") 176. ForallElim(175,"y") 177. ImpElim(152,176) 178. AndInt(159,158) 179. EqualitySub(178,171,[0,1]) 180. AndInt(177,161) 181. ForallInt(29,"u","u") 182. ForallElim(181,"e") 183. ForallInt(182,"y","y") 184. ForallElim(183,"a") 185. ForallInt(184,"x","x") 186. ForallElim(185,"y") 187. ForallInt(186,"v","v") 188. ForallElim(187,"d") 189. ImpElim(180,188) 190. AndElimL(189) 191. AndElimR(189) 192. Symmetry(190) 193. EqualitySub(179,192,[0,1]) 194. EqualitySub(193,172,[0,1]) 195. Symmetry(191) 196. EqualitySub(194,195,[0,1]) 197. Hyp("(h = orderedpair(a,x))") 198. AndElimR(177) 199. AndElimL(162) 200. AndInt(198,199) 201. ForallInt(57,"x","x") 202. ForallElim(201,"a") 203. ForallInt(202,"y","y") 204. ForallElim(203,"x") 205. ImpElim(200,204) 206. Symmetry(197) 207. EqualitySub(205,206,[0]) 208. AndInt(196,197) 209. ExistsInt(208,"x","x",[0,1]) 210. ExistsInt(209,"y","y",[0,1]) 211. ExistsInt(210,"a","a",[0,1]) 212. AndInt(207,211) 213. ClassInt(212,"w") 214. ForallInt(7,"a","a") 215. ForallElim(214,"r") 216. ForallInt(215,"b","b") 217. ForallElim(216,"s") 218. Symmetry(217) 219. EqualitySub(213,218,[0,1]) 220. EqualitySub(219,197,[0]) 221. ImpInt(220,197) 222. ForallInt(221,"h","h") 223. ForallElim(222,"orderedpair(a,x)") 224. Identity("orderedpair(a,x)") 225. ImpElim(224,223) 226. Hyp("(f = orderedpair(x,a))") 227. Symmetry(226) 228. EqualitySub(135,140,[0]) 229. EqualitySub(228,227,[0]) 230. AndInt(220,226) 231. ExistsInt(230,"x","x",[0,1]) 232. ExistsInt(231,"a","a",[0,1]) 233. AndInt(229,232) 234. ForallInt(1,"r","r") 235. ForallInt(1,"r","r") 236. ForallElim(235,"comp(r,s)") 237. Symmetry(236) 238. ClassInt(233,"w") 239. EqualitySub(238,237,[0]) 240. EqualitySub(239,226,[0]) 241. ImpInt(240,226) 242. ForallInt(241,"f","f") 243. ForallElim(242,"orderedpair(x,a)") 244. Identity("orderedpair(x,a)") 245. ImpElim(244,243) 246. EqualitySub(245,227,[0]) 247. ExistsElim(156,157,246,"e") 248. ExistsElim(153,156,247,"d") 249. ExistsElim(154,155,248,"c") 250. ExistsElim(151,154,249,"b") 251. ExistsElim(154,155,250,"c") 252. ImpInt(251,197) 253. ForallInt(252,"h","h") 254. ForallInt(252,"h","h") 255. ForallElim(254,"orderedpair(a,x)") 256. Identity("orderedpair(a,x)") 257. ImpElim(256,255) 258. EqualitySub(257,226,[0]) 259. ImpInt(258,226) 260. ForallInt(259,"f","f") 261. ForallElim(260,"orderedpair(x,a)") 262. Identity("orderedpair(x,a)") 263. ImpElim(262,261) 264. Symmetry(140) 265. EqualitySub(263,264,[0]) 266. ExistsElim(151,154,265,"b") 267. ExistsElim(138,139,266,"a") 268. ExistsElim(137,138,267,"y") 269. ExistsElim(136,137,268,"x") 270. ImpInt(269,128) 271. AndInt(127,270) 272. EquivConst(271) 273. ForallInt(272,"z","z") 274. AxInt(0) 275. ForallElim(274,"inv(comp(r,s))") 276. ForallElim(275,"comp(inv(s),inv(r))") 277. EquivExp(276) 278. AndElimR(277) 279. ImpElim(273,278)

Th64. (Function(f) & Function(g)) -> Function((f∘g))

0. Hyp("(Function(f) & Function(g))") 1. AndElimL(0) 2. AndElimR(0) 3. Hyp("Elem(orderedpair(a,b), comp(f,g))") 4. Hyp("Elem(orderedpair(a,c), comp(f,g))") 5. DefEqInt(14) 6. ForallInt(5,"a","a") 7. ForallElim(6,"f") 8. ForallInt(7,"b","b") 9. ForallElim(8,"g") 10. EqualitySub(3,9,[0]) 11. EqualitySub(4,9,[0]) 12. ClassElim(10) 13. ClassElim(11) 14. AndElimR(12) 15. ExistsInst(14,"x") 16. ExistsInst(15,"y") 17. ExistsInst(16,"z") 18.

AndElimR(13) 19. ExistsInst(18,"u") 20. ExistsInst(19,"v") 21. ExistsInst(20,"w") 22. TheoremInt(1) 23. AndElimL(22) 24. EquivExp(23) 25. AndElimR(24) 26. ForallInt(25,"x","x") 27. ForallElim(26,"a") 28. ForallInt(27,"y","y") 29. ForallElim(28,"b") 30. AndElimL(12) 31. ImpElim(30,29) 32. AndElimL(31) 33. AndElimR(31) 34. ForallInt(25,"x","x") 35. ForallElim(34,"a") 36. ForallInt(35,"y","y") 37. ForallElim(36,"c") 38. AndElimL(13) 39. ImpElim(38,37) 40. AndElimR(39) 41. AndElimR(17) 42. AndInt(31,41) 43. AndElimR(21) 44. AndInt(39,43) 45. TheoremInt(2) 46. ForallInt(45,"x","x") 47. ForallElim(46,"a") 48. ForallInt(47,"y","y") 49. ForallElim(48,"b") 50. ForallInt(49,"u","u") 51. ForallElim(50,"x") 52. ForallInt(51,"v","v") 53. ForallElim(52,"z") 54. ImpElim(42,53) 55. ForallInt(47,"y","y") 56. ForallElim(55,"c") 57. ForallInt(56,"v","v") 58. ForallElim(57,"w") 59. ImpElim(44,58) 60. AndElimL(54) 61. AndElimR(54) 62. AndElimL(59) 63. AndElimR(59) 64. AndElimL(17) 65. AndElimL(21) 66. AndElimR(64) 67. AndElimR(65) 68. AndElimL(64) 69. AndElimL(65) 70. EqualitySub(62,60,[0]) 71. EqualitySub(68,70,[0]) 72. DefExp(2,"Function",[0]) 73. AndElimR(72) 74. ForallElim(73,"u") 75. ForallElim(74,"y") 76. ForallElim(75,"v") 77. AndInt(71,69) 78. ImpElim(77,76) 79. EqualitySub(66,78,[0]) 80. DefExp(1,"Function",[0]) 81. AndElimR(80) 82. ForallElim(81,"v") 83. ForallElim(82,"z") 84. ForallElim(83,"w") 85. AndInt(79,67) 86. ImpElim(85,84) 87. EqualitySub(61,86,[0]) 88. Symmetry(63) 89. EqualitySub(87,88,[0]) 90. ExistsElim(20,21,89,"w") 91. ExistsElim(19,20,90,"v") 92. ExistsElim(18,19,91,"u") 93. ExistsElim(16,17,92,"z") 94. ExistsElim(15,16,93,"y") 95. ExistsElim(14,15,94,"x") 96. ImpInt(95,4) 97. ImpInt(96,3) 98. Hyp("(A -> (B -> C))") 99. Hyp("(A & B)") 100. AndElimL(99) 101. ImpElim(100,98) 102. AndElimR(99) 103. ImpElim(102,101) 104. ImpInt(103,99) 105. ImpInt(104,98) 106. PolySub(105,"A","Elem(orderedpair(a,b),comp(f,g))") 107. PolySub(106,"B","Elem(orderedpair(a,c), comp(f,g))") 108. PolySub(107,"C","(b = c)") 109. ImpElim(97,108) 110. AndElimL(72) 111. AndElimL(80) 112. Hyp("Elem(z,comp(f,g))") 113. EqualitySub(112,9,[0]) 114. ClassElim(113) 115. AndElimR(114) 116. ExistsInst(115,"x") 117. ExistsInst(116,"y") 118. ExistsInst(117,"l") 119. AndElimR(118) 120. ExistsInt(119,"l","l",[0]) 121. ExistsInt(120,"x","x",[0]) 122. ExistsElim(117,118,121,"l") 123. ExistsElim(116,117,122,"y") 124. ExistsElim(115,116,123,"x") 125. ImpInt(124,112) 126. ForallInt(125,"z","z") 127. DefSub(126,"Relation",["comp(f,g)"],[0]) 128. ForallInt(109,"c","c") 129. ForallInt(128,"b","b") 130. ForallInt(129,"a","a") 131. AndInt(127,130) 132. DefSub(131,"Function",["comp(f,g)"],[0]) 133. ImpInt(132,0)

Th67. (domain(U) = U) & (range(U) = U)

0. Hyp("Elem(z,domain(U))") 1. ExistsInt(0,"domain(U)","w",[0]) 2. DefSub(1,"Set",["z"],[0]) 3. TheoremInt(1) 4. EquivExp(3) 5. AndElimR(4) 6. ForallInt(5,"x","x") 7. ForallElim(6,"z") 8. ImpElim(2,7) 9. ImpInt(8,0) 10. Hyp("Elem(z,U)") 11. EquivExp(4) 12. AndElimL(11) 13. ForallInt(12,"x","x") 14. ForallElim(13,"z") 15. ImpElim(10,14) 16. TheoremInt(2) 17. AndElimL(16) 18. ForallInt(17,"x","x") 19. ForallElim(18,"z") 20. TheoremInt(3) 21. ForallInt(20,"x","x") 22. ForallElim(21,"z") 23. ForallInt(22,"y","y") 24. ForallElim(23,"0") 25. AndInt(15,19) 26. ImpElim(25,24) 27. TheoremInt(4) 28. AndElimL(27) 29. EquivExp(28) 30. AndElimL(29) 31. ForallInt(30,"x","x") 32. ForallElim(31,"z") 33. ForallInt(32,"y","y") 34. ForallElim(33,"0") 35. DefEqInt(16) 36. AndInt(15,26) 37. ImpElim(36,34) 38. AndElimR(11) 39. ForallInt(38,"x","x") 40. ForallElim(39,"orderedpair(z,0)") 41. ImpElim(37,40) 42. ExistsInt(41,"0","w",[0]) 43. AndInt(15,42) 44. ClassInt(43,"w") 45. Symmetry(35) 46. ForallInt(45,"f","f") 47. ForallElim(46,"U") 48. EqualitySub(44,47,[0]) 49. DefEqInt(17) 50. ForallInt(30,"x","x") 51. ForallElim(50,"0") 52. ForallInt(51,"y","y") 53. ForallElim(52,"z") 54. AndInt(26,15) 55. ImpElim(54,53) 56. ForallInt(38,"x","x") 57. ForallElim(56,"orderedpair(0,z)") 58. ImpElim(55,57) 59. ExistsInt(58,"0","w",[0]) 60. DefEqInt(17) 61. Symmetry(60) 62. ForallInt(61,"f","f") 63. ForallElim(62,"U") 64. AndInt(15,59) 65. ClassInt(64,"w") 66. EqualitySub(65,63,[0]) 67. ImpInt(48,10) 68. ImpInt(66,10) 69. Hyp("Elem(z,range(U))") 70. ExistsInt(69,"range(U)","w",[0]) 71. DefSub(70,"Set",["z"],[0]) 72. ImpElim(71,7) 73. ImpInt(72,69) 74. AndInt(9,67) 75. EquivConst(74) 76. ForallInt(75,"z","z") 77. AndInt(73,68) 78. EquivConst(77) 79. ForallInt(78,"z","z") 80. AxInt(0) 81. ForallElim(80,"domain(U)") 82. ForallElim(81,"U") 83. EquivExp(82) 84. AndElimR(83) 85. ImpElim(76,84) 86. ForallElim(80,"range(U)") 87. ForallElim(86,"U") 88. EquivExp(87) 89. AndElimR(88) 90. ImpElim(79,89) 91. AndInt(85,90)

Th69. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U))

0. Hyp("neg Elem(z,domain(f))") 1. Hyp("Elem(a, extension y. Elem(orderedpair(z,y),f))") 2. ClassElim(1) 3. AndElimR(2) 4. ExistsInt(3,"a","w",[0]) 5. ExistsInt(3,"f","v",[0]) 6. DefSub(5,"Set", ["orderedpair(z,a)"],[0]) 7. TheoremInt(1) 8. AndElimL(7) 9. EquivExp(8) 10. AndElimR(9) 11. ForallInt(10,"x","x") 12. ForallElim(11,"z") 13. ForallInt(12,"y","y") 14. ForallElim(13,"a") 15. ImpElim(6,14) 16. AndElimL(15) 17. AndInt(16,4) 18. ClassInt(17,"w") 19. DefEqInt(16) 20. Symmetry(19) 21. EqualitySub(18,20,[0]) 22. ImpElim(21,0) 23. ImpInt(22,1) 24. ForallInt(23,"a","a") 25. Hyp("Elem(b,0)") 26. DefEqInt(4) 27. EqualitySub(25,26,[0]) 28. ClassElim(27) 29. AndElimR(28) 30. Identity("b") 31. ImpElim(30,29) 32. AbsI(31,"Elem(b, extension y. Elem(orderedpair(z,y),f))") 33. ImpInt(32,25) 34. Hyp("Elem(b, extension y. Elem(orderedpair(z,y),f))") 35. ForallElim(24,"b") 36. ImpElim(34,35) 37. AbsI(36,"Elem(b,0)") 38. ImpInt(37,34) 39. AndInt(38,33) 40. EquivConst(39) 41. ForallInt(40,"b","b") 42. AxInt(0) 43. ForallElim(42,"extension y. Elem(orderedpair(z,y), f)") 44. ForallElim(43,"0") 45. EquivExp(44) 46. AndElimR(45) 47. ImpElim(41,46) 48. TheoremInt(2) 49. AndElimL(48) 50. Symmetry(47) 51. EqualitySub(49,50,[0]) 52. DefEqInt(18) 53. ForallInt(52,"x","x") 54. ForallElim(53,"z") 55. Symmetry(54) 56. EqualitySub(51,55,[0]) 57. ImpInt(56,0) 58. Hyp("Elem(z,domain(f))") 59. EqualitySub(58,19,[0]) 60. ClassElim(59) 61. AndElimL(60) 62. AndElimR(60) 63. Hyp("(extension a. Elem(orderedpair(z,a), f) = 0)") 64. ExistsInst(62,"y") 65. ExistsInt(64,"f","v",[0]) 66. DefSub(65,"Set",["orderedpair(z,y)"],[0]) 67. TheoremInt(3) 68. AndElimL(67) 69. EquivExp(68) 70. AndElimR(69) 71. ForallInt(70,"x","x") 72. ForallElim(71,"z") 73. ImpElim(66,72) 74. AndElimR(73) 75. AndInt(74,64) 76. ClassInt(75,"w") 77. EqualitySub(76,63,[0]) 78. DefEqInt(4) 79. EqualitySub(77,78,[0]) 80. ClassElim(79) 81. AndElimR(80) 82. Identity("y") 83. ImpElim(82,81) 84. ImpInt(83,63) 85. TheoremInt(4) 86. ForallInt(85,"x","x") 87. ForallElim(86,"extension a. Elem(orderedpair(z,a), f)") 88. ImpElim(84,87) 89. DefEqInt(18) 90. ForallInt(89,"x","x") 91. ForallElim(90,"z") 92. Symmetry(91) 93. EqualitySub(88,92,[0]) 94. TheoremInt(5) 95. EquivExp(94) 96. AndElimR(95) 97. ForallInt(96,"x","x") 98. ForallElim(97,"app(f,z)") 99. ImpElim(93,98) 100. ExistsElim(62,64,99,"y") 101. ImpInt(100,58) 102. AndInt(57,101)

Th70. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

0. Hyp("Function(f)") 1. Hyp("Elem(z,f)") 2. DefExp(0,"Function",[0]) 3. AndElimL(2) 4. DefExp(3,"Relation",[0]) 5. ForallElim(4,"z") 6. ImpElim(1,5) 7. ExistsInst(6,"x") 8. ExistsInst(7,"y") 9. AndElimR(2) 10. DefEqInt(18) 11. Hyp("Elem(a, extension y. Elem(orderedpair(x,y),f))") 12. ClassElim(11) 13. AndElimR(12) 14. ForallElim(9,"x") 15. ForallElim(14,"y") 16. ForallElim(15,"a") 17. EqualitySub(1,8,[0]) 18. AndInt(17,13) 19. ImpElim(18,16) 20. DefEqInt(9) 21. ForallInt(20,"x","x") 22. ForallElim(21,"y") 23. ImpInt(19,11) 24. ExistsInt(1,"f","w",[0]) 25. DefSub(24,"Set",["z"],[0]) 26. EqualitySub(25,8,[0]) 27. TheoremInt(2) 28. AndElimL(27) 29. EquivExp(28) 30. AndElimR(29) 31. ImpElim(26,30) 32. AndElimR(31) 33. TheoremInt(3) 34. ForallInt(33,"y","y") 35. ForallElim(34,"a") 36. ForallInt(35,"x","x") 37. ForallElim(36,"y") 38. ImpElim(32,37) 39. EquivExp(38) 40. AndElimR(39) 41. Symmetry(19) 42. ImpElim(41,40) 43. ImpInt(42,11) 44. Hyp("Elem(a, singleton(y))") 45. EquivExp(38) 46. AndElimL(45) 47. ImpElim(44,46) 48. Symmetry(47) 49. EqualitySub(1,8,[0]) 50. EqualitySub(49,48,[0]) 51. EqualitySub(32,48,[0]) 52. AndInt(51,50) 53. ClassInt(52,"y") 54. ImpInt(53,44) 55. AndInt(43,54) 56. EquivConst(55) 57. ForallInt(56,"a","a") 58. AxInt(0) 59. ForallElim(58,"extension y. Elem(orderedpair(x,y),f)") 60. ForallElim(59,"singleton(y)") 61. EquivExp(60) 62. AndElimR(61) 63. ImpElim(57,62) 64. EqualitySub(10,63,[0]) 65. TheoremInt(4) 66. AndElimL(65) 67. ForallInt(66,"x","x") 68. ForallElim(67,"y") 69. ImpElim(32,68) 70. AndElimL(69) 71. EqualitySub(64,70,[0]) 72. AndInt(8,71) 73. ExistsInt(72,"y","y",[0,1]) 74. ExistsInt(73,"x","x",[0,1]) 75. AndInt(25,74) 76. ClassInt(75,"w") 77. ExistsElim(7,8,76,"y") 78. ExistsElim(6,7,77,"x") 79. ImpInt(78,1) 80. Hyp("Elem(z, extension w. exists x. exists y. ((w = orderedpair(x,y)) & (app(f,x) = y)))") 81. ClassElim(80) 82. AndElimL(81) 83. AndElimR(81) 84. ExistsInst(83,"x") 85. ExistsInst(84,"y") 86. AndElimL(85) 87. AndElimR(85) 88. EqualitySub(87,10,[0]) 89. EqualitySub(82,86,[0]) 90. ImpElim(89,30) 91. AndElimR(90) 92. Symmetry(87) 93. EqualitySub(91,92,[0]) 94. Hyp("(app(f,x) = U)") 95. TheoremInt(5) 96. EqualitySub(93,94,[0]) 97. ImpElim(96,95) 98. ImpInt(97,94) 99. TheoremInt(6) 100. AndElimL(99) 101. TheoremInt(7) 102.

PolySub(101,"A","neg Elem(z,domain(f))") 103. PolySub(102,"B","(app(f,z) = U)") 104. ImpElim(100,103) 105. TheoremInt(8) 106. EquivExp(105) 107. AndElimR(106) 108. PolySub(107,"D","Elem(z,domain(f))") 109. Hyp("neg (app(f,z) = U)") 110. ImpElim(109,104) 111. ImpElim(110,108) 112. ImpInt(111,109) 113. ForallInt(112,"z","z") 114. ForallElim(113,"x") 115. ImpElim(98,114) 116. DefEqInt(16) 117. EqualitySub(115,116,[0]) 118. ClassElim(117) 119. AndElimR(118) 120. ExistsInst(119,"b") 121. Hyp("Elem(e, singleton(b))") 122. ExistsInt(120,"f","w",[0]) 123. DefSub(122,"Set",["orderedpair(x,b)"],[0]) 124. ForallInt(30,"y","y") 125. ForallElim(124,"b") 126. ImpElim(123,125) 127. AndElimR(126) 128. TheoremInt(3) 129. ForallInt(128,"x","x") 130. ForallElim(129,"b") 131. ImpElim(127,130) 132. ForallInt(131,"y","y") 133. ForallElim(132,"e") 134. EquivExp(133) 135. AndElimL(134) 136. ImpElim(121,135) 137. Symmetry(136) 138. EqualitySub(120,137,[0]) 139. EqualitySub(127,137,[0]) 140. AndInt(139,138) 141. ClassInt(140,"y") 142. Hyp("Elem(e, extension y. Elem(orderedpair(x,y), f))") 143. ClassElim(142) 144. AndElimR(143) 145. DefExp(0,"Function",[0]) 146. AndElimR(145) 147. ImpInt(141,121) 148. AndInt(120,144) 149. ForallElim(146,"x") 150. ForallElim(149,"b") 151. ForallElim(150,"e") 152. ImpElim(148,151) 153. EquivExp(131) 154. EquivExp(133) 155. AndElimR(154) 156. Symmetry(152) 157. ImpElim(156,155) 158. ImpInt(157,142) 159. AndInt(147,158) 160. EquivConst(159) 161. ForallInt(160,"e","e") 162. AxInt(0) 163. ForallElim(162,"singleton(b)") 164. ForallElim(163,"extension y. Elem(orderedpair(x,y), f)") 165. EquivExp(164) 166. AndElimR(165) 167. ImpElim(161,166) 168. Symmetry(167) 169. EqualitySub(88,168,[0]) 170. TheoremInt(4) 171. AndElimL(170) 172. ForallInt(171,"x","x") 173. ForallElim(172,"b") 174. ImpElim(127,173) 175. AndElimL(174) 176. EqualitySub(169,175,[0]) 177. EqualitySub(120,176,[0]) 178. EqualitySub(120,176,[0]) 179. Symmetry(86) 180. EqualitySub(178,179,[0]) 181. Identity("x") 182. ExistsElim(119,120,180,"b") 183. ExistsElim(84,85,182,"y") 184. ExistsElim(83,84,183,"x") 185. ImpInt(184,80) 186. AndInt(79,185) 187. EquivConst(186) 188. ForallInt(187,"z","z") 189. AxInt(0) 190. ForallElim(189,"f") 191. ForallElim(190,"extension w. exists x. exists y. ((w = orderedpair(x,y)) & (app(f,x) = y))") 192. EquivExp(191) 193. AndElimR(192) 194. ImpElim(188,193) 195. ImpInt(194,0)

Th71. (Function(f) & Function(g)) -> ((f = g) <-> ∀z.((f'z) = (g'z)))

0. Hyp("(Function(f) & Function(g))") 1. Hyp("forall z. (app(f,z) = app(g,z))") 2. Hyp("Elem(e,f)") 3. TheoremInt(1) 4. AndElimL(0) 5. AndElimR(0) 6. ImpElim(4,3) 7. EqualitySub(2,6,[0]) 8. ClassElim(7) 9. AndElimL(8) 10. AndElimR(8) 11. ExistsInst(10,"x") 12. ExistsInst(11,"y") 13. ForallElim(1,"x") 14. EqualitySub(12,13,[0]) 15. ExistsInt(14,"y","y",[0,1]) 16. ExistsInt(15,"x","x",[0,1]) 17. AndInt(9,16) 18. ClassInt(17,"w") 19. ForallInt(3,"f","f") 20. ForallElim(19,"g") 21. ImpElim(5,20) 22. Symmetry(21) 23. EqualitySub(18,22,[0]) 24. ExistsElim(11,12,23,"y") 25. ExistsElim(10,11,24,"x") 26. ImpInt(25,2) 27. Hyp("Elem(e,g)") 28. EqualitySub(27,21,[0]) 29. ClassElim(28) 30. AndElimL(29) 31. AndElimR(29) 32. ExistsInst(31,"x") 33. ExistsInst(32,"y") 34. Symmetry(13) 35. EqualitySub(33,34,[0]) 36. ExistsInt(35,"y","y",[0,1]) 37. ExistsInt(36,"x","x",[0,1]) 38. AndInt(30,37) 39. ClassInt(38,"w") 40. Symmetry(6) 41. EqualitySub(39,40,[0]) 42. ExistsElim(32,33,41,"y") 43. ExistsElim(31,32,42,"x") 44. ImpInt(43,27) 45. AndInt(26,44) 46. EquivConst(45) 47. ForallInt(46,"e","e") 48. AxInt(0) 49. ForallElim(48,"f") 50. ForallElim(49,"g") 51. EquivExp(50) 52. AndElimR(51) 53. ImpElim(47,52) 54. ImpInt(53,1) 55. Hyp("(f = g)") 56. Identity("app(f,z)") 57. EqualitySub(56,55,[1]) 58. ForallInt(57,"z","z") 59. ImpInt(58,55) 60. AndInt(59,54) 61. EquivConst(60) 62. ImpInt(61,0)

Th73. (Set(u) & Set(y)) -> Set(({u} X y))

0. Hyp("(Set(u) & Set(y))") 1. Hyp("(f = extension a. exists w. exists z. ((a = orderedpair(w,z)) & Elem(w,y) & (z = orderedpair(u,w))))") 2. Hyp("Elem(x,domain(f))") 3. DefEqInt(16) 4. EqualitySub(2,3,[0]) 5. ClassElim(4) 6. EqualitySub(5,1,[0]) 7. AndElimL(6) 8. AndElimR(6) 9. ExistsInst(8,"c") 10. ClassElim(9) 11. AndElimL(10) 12. AndElimR(10) 13. ExistsInst(12,"w") 14. ExistsInst(13,"z") 15. AndElimL(14) 16. TheoremInt(1) 17. AndElimL(16) 18. EquivExp(17) 19. AndElimR(18) 20. ForallInt(19,"y","y") 21. ForallElim(20,"c") 22. ImpElim(11,21) 23. TheoremInt(2) 24. ForallInt(23,"y","y") 25. ForallElim(24,"c") 26. ForallInt(25,"u","u") 27. ForallElim(26,"w") 28. ForallInt(27,"v","v") 29. ForallElim(28,"z") 30. AndInt(22,15) 31. ImpElim(30,29) 32. AndElimL(31) 33. AndElimR(14) 34. AndElimL(33) 35.

Symmetry(32) 36. EqualitySub(34,35,[0]) 37. ExistsElim(13,14,36,"z") 38. ExistsElim(12,13,37,"w") 39. ExistsElim(8,9,38,"c") 40. ImpInt(39,2) 41. Hyp("Elem(x,y)") 42. Hyp("(z = orderedpair(u,x))") 43. Hyp("(a = orderedpair(x,z))") 44. AndInt(43,42) 45. ExistsInt(44,"z","z",[0,1]) 46. ExistsInt(45,"x","x",[0,1]) 47. ExistsInt(41,"y","y",[0]) 48. DefSub(47,"Set", ["x"],[0]) 49. AndElimL(0) 50. AndInt(49,48) 51. EquivExp(17) 52. AndElimL(51) 53. ForallInt(52,"x","x") 54. ForallElim(53,"u") 55. ForallInt(54,"y","y") 56. ForallElim(55,"x") 57. ImpElim(50,56) 58. Symmetry(42) 59. EqualitySub(57,58,[0]) 60. AndInt(48,59) 61. ForallInt(51,"y","y") 62. ForallInt(52,"y","y") 63. ForallElim(62,"z") 64. ImpElim(60,63) 65. Symmetry(43) 66. EqualitySub(64,65,[0]) 67. AndInt(66,46) 68. Symmetry(1) 69. ClassInt(67,"a") 70. AndInt(41,42) 71. AndInt(43,70) 72. ExistsInt(71,"z","z", [0,1]) 73. ExistsInt(72,"x","x",[0,1]) 74. AndInt(66,73) 75. ClassInt(74,"a") 76. EqualitySub(75,68,[0]) 77. EqualitySub(76,43,[0]) 78. ExistsInt(77,"z","z",[0]) 79. AndInt(48,78) 80. ClassInt(79,"w") 81. Symmetry(3) 82. EqualitySub(80,81, [0]) 83. ImpInt(82,43) 84. ForallInt(83,"a","a") 85. ForallElim(84,"orderedpair(x,z)") 86. Identity("orderedpair(x,z)") 87. ImpElim(86,85) 88. ImpInt(87,42) 89. ForallInt(88,"z","z") 90. ForallElim(89,"orderedpair(u,x)") 91. Identity("orderedpair(u,x)") 92. ImpElim(91,90) 93. ImpInt(92,41) 94. AndInt(40,93) 95. EquivConst(94) 96. ForallInt(95,"x","x") 97. AxInt(0) 98. ForallElim(97,"domain(f)") 99. ForallElim(98,"y") 100. EquivExp(99) 101. AndElimR(100) 102. ImpElim(96,101) 103. Hyp("Elem(x,range(f))") 104. DefEqInt(17) 105. EqualitySub(103,104,[0]) 106. ClassElim(105) 107. AndElimR(106) 108. EqualitySub(107,1,[0]) 109. ExistsInst(108,"c") 110. ClassElim(109) 111. AndElimR(110) 112. ExistsInst(111,"w") 113. ExistsInst(112,"z") 114. AndElimL(110) 115. ForallInt(19,"x","x") 116. ForallElim(115,"c") 117. ForallInt(116,"y","y") 118. ForallElim(117,"x") 119. ImpElim(114,118) 120. ForallInt(23,"x","x") 121. ForallElim(120,"c") 122. ForallInt(121,"y","y") 123. ForallElim(122,"x") 124. ForallInt(123,"u","u") 125. ForallElim(124,"w") 126. ForallInt(125,"v","v") 127. ForallElim(126,"z") 128. AndElimL(113) 129. AndInt(119,128) 130. ImpElim(129,127) 131. AndElimR(113) 132. AndElimL(131) 133. AndElimR(131) 134. AndElimR(130) 135. Symmetry(134) 136. EqualitySub(133,135,[0]) 137. AndElimL(119) 138. AndElimL(130) 139. EqualitySub(137,138,[0]) 140. TheoremInt(3) 141. AndElimL(0) 142. ForallInt(140,"x","x") 143. ForallElim(142,"u") 144. ForallInt(143,"y","y") 145. ForallElim(144,"u") 146. ImpElim(141,145) 147. EquivExp(146) 148. AndElimR(147) 149. Identity("u") 150. ImpElim(149,148) 151. AndInt(150,132) 152. AndInt(136,151) 153. AndElimR(119) 154. ExistsInt(152,"w","w",[0,1]) 155. ExistsInt(154,"u","b",[0,1]) 156. AndInt(153,155) 157. ClassInt(156,"e") 158. DefEqInt(19) 159. ForallInt(158,"x","x") 160. ForallElim(159,"singleton(u)") 161. Symmetry(160) 162. EqualitySub(157,161,[0]) 163. ExistsElim(112,113,162,"z") 164. Hyp("Elem(x, prod(singleton(u),y))") 165. EqualitySub(164,160,[0]) 166. ClassElim(165) 167. AndElimR(166) 168. ExistsElim(111,112,163,"w") 169. ExistsElim(108,109,168,"c") 170. ImpInt(169,103) 171. ExistsInst(167,"a") 172. ExistsInst(171,"b") 173. AndElimL(172) 174. AndElimR(172) 175. AndElimL(174) 176. AndElimR(174) 177. ForallInt(143,"y","y") 178. ForallElim(177,"a") 179. AndElimL(0) 180. ImpElim(179,178) 181. EquivExp(180) 182. AndElimL(181) 183. ImpElim(175,182) 184. EqualitySub(173,183,[0]) 185. Hyp("(c = orderedpair(b,x))") 186. AndInt(176,184) 187. AndInt(185,186) 188. ExistsInt(187,"x","x",[0,1]) 189. ExistsInt(188,"b","b",[0,1]) 190. AndElimL(166) 191. ExistsInt(176,"y","y",[0]) 192. DefSub(191,"Set",["b"],[0]) 193. ForallInt(52,"x","x") 194. ForallElim(193,"b") 195. ForallInt(194,"y","y") 196. ForallElim(195,"x") 197. AndInt(192,190) 198. ImpElim(197,196) 199. Symmetry(185) 200. EqualitySub(198,199,[0]) 201. AndInt(200,189) 202. ClassInt(201,"w") 203. Symmetry(1) 204. EqualitySub(202,203,[0]) 205. EqualitySub(204,185,[0]) 206. ExistsInt(205,"b","b",[0]) 207. AndInt(190,206) 208. ClassInt(207,"w") 209. Symmetry(104) 210. EqualitySub(208,209,[0]) 211. ImpInt(210,185) 212. ForallInt(211,"c","c") 213. ForallElim(212,"orderedpair(b,x)") 214. Identity("orderedpair(b,x)") 215. ImpElim(214,213) 216. ExistsElim(171,172,215,"b") 217. ExistsElim(167,171,216,"a") 218. ImpInt(217,164) 219. AndInt(170,218) 220. EquivConst(219) 221. ForallInt(220,"x","x") 222. AxInt(0) 223. ForallElim(222,"range(f)") 224. ForallElim(223,"prod(singleton(u), y)") 225. EquivExp(224) 226. AndElimR(225) 227. ImpElim(221,226) 228. AxInt(3) 229. AndElimR(0) 230. Symmetry(102) 231. EqualitySub(229,230,[0]) 232. Hyp("Elem(x,f)") 233. EqualitySub(232,1,[0]) 234. ClassElim(233) 235. AndElimR(234) 236. ExistsInst(235,"w") 237. ExistsInst(236,"z") 238. AndElimL(237) 239. ExistsInt(238,"z","z",[0]) 240. ExistsInt(239,"w","w",[0]) 241. ExistsElim(236,237,240,"z") 242. ExistsElim(235,236,241,"w") 243. ImpInt(242,232) 244. ForallInt(243,"x","x") 245. DefSub(244,"Relation",["f"],[0]) 246. Hyp("Elem(orderedpair(a,b),f)") 247. Hyp("Elem(orderedpair(a,c),f)") 248.

EqualitySub(246,1,[0]) 249. EqualitySub(247,1,[0]) 250. ClassElim(248) 251. ClassElim(249) 252. AndElimR(250) 253. AndElimR(251) 254. ExistsInst(252,"x1") 255. ExistsInst(254,"y1") 256. ExistsInst(253,"x2") 257. ExistsInst(256,"y2") 258. AndElimL(255) 259. AndElimL(257) 260. TheoremInt(1) 261. AndElimL(260) 262. EquivExp(261) 263. AndElimR(262) 264. AndElimL(250) 265. AndElimL(251) 266. ForallInt(263,"x","x") 267. ForallElim(266,"a") 268. ForallInt(267,"y","y") 269. ForallElim(268,"b") 270. ForallInt(267,"y","y") 271. ForallElim(270,"c") 272. ImpElim(264,269) 273. ImpElim(265,271) 274. TheoremInt(2) 275. ForallInt(274,"x","x") 276. ForallElim(275,"a") 277. ForallInt(276,"y","y") 278. ForallElim(277,"b") 279. ForallInt(278,"u","u") 280. ForallElim(279,"x1") 281. ForallInt(280,"v","v") 282. ForallElim(281,"y1") 283. AndInt(272,258) 284. ImpElim(283,282) 285. AndInt(273,259) 286. ForallInt(276,"y","y") 287. ForallElim(286,"c") 288. ForallInt(287,"u","u") 289. ForallElim(288,"x2") 290. ForallInt(289,"v","v") 291. ForallElim(290,"y2") 292. ImpElim(285,291) 293. AndElimR(255) 294. AndElimR(257) 295. AndElimL(284) 296. AndElimL(292) 297. EqualitySub(296,295,[0]) 298. AndElimR(293) 299. AndElimR(294) 300. Symmetry(297) 301. EqualitySub(299,300,[0]) 302. Symmetry(301) 303. EqualitySub(298,302,[0]) 304. EqualitySub(258,297,[0]) 305. EqualitySub(304,303,[0]) 306. Symmetry(259) 307. EqualitySub(305,306,[0]) 308. AndInt(272,307) 309. ForallInt(278,"u","u") 310. ForallElim(309,"a") 311. ForallInt(310,"v","v") 312. ForallElim(311,"c") 313. ImpElim(308,312) 314. AndElimR(313) 315. ExistsElim(256,257,314,"y2") 316. ExistsElim(253,256,315,"x2") 317. ExistsElim(254,255,316,"y1") 318. ExistsElim(252,254,317,"x1") 319. ImpInt(318,247) 320. ImpInt(319,246) 321. Hyp("(A -> (B -> C))") 322. Hyp("(A & B)") 323. AndElimL(322) 324. ImpElim(323,321) 325. AndElimR(322) 326. ImpElim(325,324) 327. ImpInt(326,322) 328. ImpInt(327,321) 329. PolySub(328,"A","Elem(orderedpair(a,b),f)") 330. PolySub(329,"B","Elem(orderedpair(a,c),f)") 331. PolySub(330,"C","(b = c)") 332. ImpElim(320,331) 333. ForallInt(332,"c","c") 334. ForallInt(333,"b","b") 335. ForallInt(334,"a","a") 336. AndInt(245,335) 337. DefSub(336,"Function",["f"],[0]) 338. AndInt(337,231) 339. AxInt(3) 340. ImpElim(338,339) 341. EqualitySub(340,227,[0]) 342. ImpInt(341,1) 343. ForallInt(342,"f","f") 344. ForallElim(343,"extension a.exists w.exists z.((a = orderedpair(w,z)) & (Elem(w,y) & (z = orderedpair(u,w)))) ") 345. Identity(" extension a.exists w.exists z.((a = orderedpair(w,z)) & (Elem(w,y) & (z = orderedpair(u,w)))) ") 346. ImpElim(345,344) 347. ImpInt(346,0)

Th74. (Set(x) & Set(y)) -> Set((x X y))

0. Hyp("(f = extension a. exists u. exists z. ((a = orderedpair(u,z)) & Elem(u,x) & (z = prod(singleton(u),y)))) ") 1. Hyp("Elem(c,f)") 2. EqualitySub(1,0,[0]) 3. ClassElim(2) 4. AndElimR(3) 5. ExistsInst(4,"u") 6. ExistsInst(5,"z") 7. AndElimL(6) 8. ExistsInt(7,"z","z",[0]) 9. ExistsInt(8,"u","u",[0]) 10. ExistsElim(5,6,9,"z") 11. ExistsElim(4,5,10,"u") 12. ImpInt(11,1) 13. ForallInt(12,"c","c") 14. DefSub(13,"Relation",["f"],[0]) 15. Hyp("(Elem(orderedpair(a,b),f) & Elem(orderedpair(a,c),f))") 16. AndElimL(15) 17. AndElimR(15) 18. EqualitySub(16,0,[0]) 19. EqualitySub(17,0,[0]) 20. ClassElim(18) 21. ClassElim(19) 22. AndElimR(20) 23. AndElimR(21) 24. ExistsInst(22,"x1") 25. ExistsInst(24,"y1") 26. ExistsInst(23,"x2") 27. ExistsInst(26,"y2") 28. AndElimL(20) 29. AndElimL(21) 30. TheoremInt(1) 31. AndElimL(30) 32. EquivExp(31) 33. AndElimR(32) 34. ForallInt(33,"x","x") 35. ForallElim(34,"a") 36. ForallInt(35,"y","y") 37. ForallElim(36,"b") 38. ForallInt(35,"y","y") 39. ForallElim(38,"c") 40. ImpElim(28,37) 41. ImpElim(29,39) 42. TheoremInt(2) 43. ForallInt(42,"x","x") 44. ForallElim(43,"a") 45. ForallInt(44,"x","x") 46. ForallInt(44,"y","y") 47. ForallElim(46,"b") 48. AndElimL(25) 49. AndElimL(27) 50. ForallInt(47,"u","u") 51. ForallElim(50,"x1") 52. ForallInt(51,"v","v") 53. ForallElim(52,"y1") 54. AndInt(40,48) 55. ImpElim(54,53) 56. ForallInt(44,"y","y") 57. ForallElim(56,"c") 58. ForallInt(57,"u","u") 59. ForallElim(58,"x2") 60. ForallInt(59,"v","v") 61. ForallElim(60,"y2") 62. AndInt(41,49) 63. ImpElim(62,61) 64. AndElimL(55) 65. AndElimL(63) 66. EqualitySub(64,65,[0]) 67. AndElimR(25) 68. AndElimR(27) 69. AndElimR(67) 70. AndElimR(68) 71. EqualitySub(70,66,[0]) 72. Symmetry(71) 73. EqualitySub(69,72,[0]) 74. AndElimR(55) 75. AndElimR(63) 76. EqualitySub(74,73,[0]) 77. Symmetry(76) 78. EqualitySub(75,77,[0]) 79. ExistsElim(26,27,78,"y2") 80. ExistsElim(23,26,79,"x2") 81. ExistsElim(24,25,80,"y1") 82. ExistsElim(22,24,81,"x1") 83. Symmetry(82) 84. ImpInt(83,15) 85. ForallInt(84,"c","c") 86. ForallInt(85,"b","b") 87. ForallInt(86,"a","a") 88. AndInt(14,87) 89. DefSub(88,"Function",["f"],[0]) 90. Hyp("Elem(a,x)") 91. Hyp("(b = prod(singleton(a),y))") 92. AndInt(90,91) 93. Hyp("(c = orderedpair(a,b))") 94. AndInt(93,92) 95. ExistsInt(94,"b","b",[0,1])

96. ExistsInt(95,"a","a",[0,1]) 97. Hyp("(Set(x) & Set(y))") 98. ExistsInt(90,"x","w",[0]) 99. DefSub(98,"Set",["a"],[0]) 100. TheoremInt(3) 101. ForallInt(100,"x","x") 102. ForallElim(101,"a") 103. ImpElim(99,102) 104. AndElimR(97) 105. TheoremInt(4) 106. ForallInt(105,"u","u") 107. ForallElim(106,"a") 108. AndInt(99,104) 109. ImpElim(108,107) 110. Symmetry(91) 111. EqualitySub(109,110,[0]) 112. TheoremInt(5) 113. AndElimL(112) 114. EquivExp(113) 115. AndElimL(114) 116. ForallInt(115,"x","x") 117. ForallElim(116,"a") 118. ForallInt(117,"y","y") 119. ForallElim(118,"b") 120. AndInt(99,111) 121. ImpElim(120,119) 122. Symmetry(93) 123. EqualitySub(121,122,[0]) 124. AndInt(123,96) 125. ClassInt(124,"w") 126. EqualitySub(125,93,[0]) 127. Symmetry(0) 128. EqualitySub(126,127,[0]) 129. ExistsInt(128,"b","b",[0]) 130. AndInt(99,129) 131. ClassInt(130,"w") 132. DefEqInt(16) 133. Symmetry(132) 134. EqualitySub(131,133,[0]) 135. ImpInt(134,93) 136. ForallInt(135,"c","c") 137. ForallElim(136,"orderedpair(a,b)") 138. Identity("orderedpair(a,b)") 139. ImpElim(138,137) 140. ImpInt(139,91) 141. ForallInt(140,"b","b") 142. ForallElim(141,"prod(singleton(a),y)") 143. Identity("prod(singleton(a),y)") 144. ImpElim(143,142) 145. ImpInt(144,90) 146. Hyp("Elem(a, domain(f))") 147. EqualitySub(146,132,[0]) 148. ClassElim(147) 149. AndElimR(148) 150. ExistsInst(149,"b") 151. EqualitySub(150,0,[0]) 152. ClassElim(151) 153. AndElimL(152) 154. AndElimR(152) 155. ExistsInst(154,"u") 156. ExistsInst(155,"z") 157. TheoremInt(5) 158. AndElimL(157) 159. EquivExp(158) 160. AndElimR(159) 161. ForallInt(160,"x","x") 162. ForallElim(161,"a") 163. ForallInt(162,"y","y") 164. ForallElim(163,"b") 165. ImpElim(153,164) 166. AndElimL(156) 167. AndInt(165,166) 168. TheoremInt(2) 169. ForallInt(168,"x","x") 170. ForallElim(169,"a") 171. ForallInt(170,"y","y") 172. ForallElim(171,"b") 173. ForallInt(172,"v","v") 174. ForallElim(173,"z") 175. ImpElim(167,174) 176. AndElimL(175) 177. AndElimR(156) 178. AndElimL(177) 179. Symmetry(176) 180. EqualitySub(178,179,[0]) 181. ExistsElim(155,156,180,"z") 182. ExistsElim(154,155,181,"u") 183. ExistsElim(149,150,182,"b") 184. ImpInt(183,146) 185. AndInt(145,184) 186. EquivConst(185) 187. ForallInt(186,"a","a") 188. AxInt(0) 189. ForallElim(188,"x") 190. ForallElim(189,"domain(f)") 191. EquivExp(190) 192. AndElimR(191) 193. ImpElim(187,192) 194. AndInt(89,193) 195. ImpInt(194,0) 196. EqualitySub(195,0,[0]) 197. Identity(" extension a.exists u.exists z.((a = orderedpair(u,z)) & (Elem(u,x) & (z = prod(singleton(u),y)))) ") 198. ImpElim(197,196) 199. AndElimR(198) 200. AndElimL(97) 201. EqualitySub(200,199,[0]) 202. AndElimL(198) 203. AndInt(202,201) 204. AxInt(3) 205. ImpElim(203,204) 206. DefEqInt(17) 207. EqualitySub(206,0,[1]) 208. Hyp("Elem(e, range(f))") 209. EqualitySub(208,207,[0]) 210. ClassElim(209) 211. AndElimR(210) 212. ExistsInst(211,"c") 213. ClassElim(212) 214. AndElimR(213) 215. ExistsInst(214,"u") 216. ExistsInst(215,"z") 217. TheoremInt(1) 218. AndElimL(217) 219. EquivExp(218) 220. AndElimR(219) 221. ForallInt(220,"x","x") 222. ForallElim(221,"c") 223. ForallInt(222,"y","y") 224. ForallElim(223,"e") 225. AndElimL(213) 226. ImpElim(225,224) 227. TheoremInt(2) 228. ForallInt(227,"x","x") 229. ForallElim(228,"c") 230. ForallInt(229,"y","y") 231. ForallElim(230,"e") 232. AndElimL(216) 233. AndInt(226,232) 234. ForallInt(231,"v","v") 235. ForallElim(234,"z") 236. ImpElim(233,235) 237. AndElimR(216) 238. AndElimR(237) 239. AndElimR(236) 240. Symmetry(239) 241. EqualitySub(238,240,[0]) 242. AndElimL(237) 243. AndInt(242,241) 244. ExistsInt(243,"u","u",[0,1]) 245. AndElimR(226) 246. AndInt(245,244) 247. ClassInt(246,"w") 248. ExistsElim(215,216,247,"z") 249. ExistsElim(214,215,248,"u") 250. ExistsElim(211,212,249,"c") 251. ImpInt(250,208) 252. Hyp("Elem(e,extension w.exists u.(Elem(u,x) & (w = prod(singleton(u),y)))) ") 253. ClassElim(252) 254. AndElimL(253) 255. AndElimR(253) 256. ExistsInst(255,"u") 257. Identity("orderedpair(u,e)") 258. AndInt(257,256) 259. ExistsInt(258,"e","b",[1,2]) 260. ExistsInt(259,"u","v",[1,2,3]) 261. AndElimL(256) 262. ExistsInt(261,"x","w",[0]) 263. DefSub(262,"Set",["u"],[0]) 264. AndInt(263,254) 265. AndElimL(219) 266. ForallInt(265,"x","x") 267. ForallElim(266,"u") 268. ForallInt(267,"y","y") 269. ForallElim(268,"e") 270. ImpElim(264,269) 271. AndInt(270,260) 272. Hyp("(c = orderedpair(u,e))") 273. Symmetry(272) 274. EqualitySub(271,273,[0,1]) 275. ClassInt(274,"w") 276. EqualitySub(275,272,[0]) 277. ImpInt(276,272) 278. ForallInt(277,"c","c") 279. ForallElim(278,"orderedpair(u,e)") 280. Identity("orderedpair(u,e)") 281. ImpElim(280,279) 282. Symmetry(0) 283. EqualitySub(281,282,[0]) 284. ExistsInt(283,"u","u",[0]) 285. ExistsElim(255,256,284,"u") 286. AndInt(254,285) 287. ClassInt(286,"w") 288. DefEqInt(17) 289. Symmetry(288) 290. EqualitySub(287,289,[0]) 291. ImpInt(290,252) 292. AndInt(251,291) 293. EquivConst(292) 294. ForallInt(293,"e","e") 295. AxInt(0) 296. ForallElim(295,"range(f)") 297. ForallElim(296,"extension w.exists u.(Elem(u,x) & (w = prod(singleton(u),y))) ") 298. EquivExp(297) 299. AndElimR(298) 300. ImpElim(294,299) 301. Hyp("Elem(e, bigunion(range(f)))") 302. EqualitySub(301,300,[0]) 303. DefEqInt(6) 304.

ForallInt(303,"x","x")   305. ForallElim(304,"range(f)")   306. EqualitySub(305,300,[1])   307. EqualitySub(301,306,[0])   308. ClassElim(307)   309. AndElimR(308)   310. ExistsInst(309,"x_5")   311. AndElimR(310)   312. AndElimL(310)   313. ClassElim(312)   314. AndElimL(313)   315. AndElimR(313)   316. ExistsInst(315,"u")   317. AndElimR(316)   318. EqualitySub(311,317,[0])   319. DefEqInt(19)   320. ForallInt(319,"x","x")   321. ForallElim(320,"singleton(u)")   322. EqualitySub(318,321,[0])   323. ClassElim(322)   324. AndElimR(323)   325. ExistsInst(324,"a")   326. ExistsInst(325,"b")   327. AndElimR(326)   328. AndElimL(327)   329. TheoremInt(6)   330. AndElimL(316)   331. ExistsInt(330,"x","w",[0])   332. DefSub(331,"Set",["u"],[0])   333. ForallInt(329,"x","x")   334. ForallElim(333,"u")   335. ForallInt(334,"y","y")   336. ForallElim(335,"a")   337. ImpElim(332,336)   338. EquivExp(337)   339. AndElimL(338)   340. ImpElim(328,339)   341. Symmetry(340)   342. EqualitySub(330,341,[0])   343. AndElimR(327)   344. AndInt(342,343)   345. AndElimL(326)   346. AndInt(345,344)   347. ExistsInt(346,"b","b",[0,1])   348. ExistsInt(347,"a","a",[0,1])   349. AndElimL(323)   350. AndInt(349,348)   351. ClassInt(350,"w")   352. DefEqInt(19)   353. Symmetry(352)   354. EqualitySub(351,353,[0])   355. ExistsElim(325,326,354,"b")   356. ExistsElim(324,325,355,"a")   357. ExistsElim(315,316,356,"u")   358. ExistsElim(309,310,357,"x_5")   359. ImpInt(358,301)   360. Hyp("Elem(e,prod(x,y))")   361. EqualitySub(360,352,[0])   362. ClassElim(361)   363. AndElimL(362)   364. AndElimR(362)   365. ExistsInst(364,"a")   366. ExistsInst(365,"b")   367. EquivExp(218)   368. AndElimR(367)   369. ForallInt(368,"x","x")   370. ForallElim(369,"a")   371. ForallInt(370,"y","y")   372. ForallElim(371,"b")   373. AndElimL(366)   374. EqualitySub(363,373,[0])   375. ImpElim(374,372)   376. AndElimL(375)   377. ForallInt(329,"x","x")   378. ForallElim(377,"a")   379. ForallInt(378,"y","y")   380. ForallElim(379,"a")   381. ImpElim(376,380)   382. EquivExp(381)   383. AndElimR(382)   384. Identity("a")   385. ImpElim(384,383)   386. AndElimL(366)   387. AndElimR(366)   388. AndElimL(387)   389. AndElimR(387)   390. AndInt(385,389)   391. AndInt(386,390)   392. ExistsInt(391,"b","u",[0,1])   393. ExistsInt(392,"a","v",[0,1])   394. AndInt(363,393)   395. ClassInt(394,"w")   396. ForallInt(319,"x","x")   397. ForallElim(396,"singleton(a)")   398. Symmetry(397)   399. EqualitySub(395,398,[0])   400. Hyp("(g = prod(singleton(a),y))")   401. Symmetry(400)   402. AndInt(388,400)   403. ExistsInt(402,"a","a",[0,1])   404. TheoremInt(7)   405. ForallInt(404,"u","u")   406. ForallElim(405,"a")   407. AndElimR(97)   408. AndInt(376,407)   409. ImpElim(408,406)   410. EqualitySub(409,401,[0])   411. AndInt(410,403)   412. ClassInt(411,"w")   413. EqualitySub(399,401,[0])   414. AndInt(412,413)   415. ExistsInt(414,"g","g",[0,1])   416. AndInt(363,415)   417. ClassInt(416,"d")   418. Symmetry(306)   419. EqualitySub(417,418,[0])   420. ImpInt(419,400)   421. ForallInt(420,"g","g")   422. ForallElim(421,"prod(singleton(a),y)")   423. Identity("prod(singleton(a),y)")   424. ImpElim(423,422)   425. ExistsElim(365,366,424,"b")   426. ExistsElim(364,365,425,"a")   427. ImpInt(426,360)   428. AndInt(359,427)   429. EquivConst(428)   430. ForallInt(429,"e","e")   431. AxInt(0)   432. ForallElim(431,"bigunion(range(f))")   433. ForallElim(432,"prod(x,y)")   434. EquivExp(433)   435. AndElimR(434)   436. ImpElim(430,435)   437. AxInt(4)   438. ForallInt(437,"x","x")   439. ForallElim(438,"range(f)")   440. ImpElim(205,439)   441. EqualitySub(440,436,[0])   442. ImpInt(441,97)   443. ImpInt(442,0)   444. ForallInt(443,"f","f")   445. ForallElim(444," extension a.exists u.exists z.((a = orderedpair(u,z)) & (Elem(u,x) & (z = prod(singleton(u),y)))) ")   446. Identity(" extension a.exists u.exists z.((a = orderedpair(u,z)) & (Elem(u,x) & (z = prod(singleton(u),y)))) ")   447. ImpElim(446,445)

Th75. (Function(f) & Set(domain(f))) -> (f ⊂ (domain(f) X range(f)))

0. Hyp("(Function(f) & Set(domain(f)))")   1. Hyp("Elem(z,f)")   2. AndElimL(0)   3. DefExp(2,"Function",[0])   4. AndElimL(3)   5. DefExp(4,"Relation",[0])   6. ForallElim(5,"z")   7. ImpElim(1,6)   8. ExistsInst(7,"x")   9. ExistsInst(8,"y")   10. DefEqInt(16)   11. DefEqInt(17)   12. ExistsInt(9,"y","y",[0])   13. ExistsInt(1,"f","f",[0])   14. DefSub(13,"Set",["z"],[0])   15. TheoremInt(1)   16. AndElimL(15)   17. EquivExp(16)   18. AndElimR(17)   19. EqualitySub(14,9,[0])   20. ImpElim(19,18)   21. AndElimL(20)   22. EqualitySub(1,9,[0])   23. ExistsInt(22,"y","y",[0])   24. AndInt(21,23)   25. ClassInt(24,"w")   26. Symmetry(10)   27. EqualitySub(25,26,[0])   28. ExistsInt(22,"x","x",[0])   29. AndElimR(20)   30. AndInt(29,28)   31. ClassInt(30,"w")   32. Symmetry(11)   33. EqualitySub(31,32,[0])   34. AndInt(27,33)   35. AndInt(9,34)   36. ExistsInt(35,"y","y",[0,1])   37. ExistsInt(36,"x","x",[0,1])   38. DefEqInt(19)   39. ForallInt(38,"x","x")   40. ForallElim(39,"domain(f)")   41. ForallInt(40,"y","y")   42. ForallElim(41,"range(f)")   43. AndInt(14,37)   44. ClassInt(43,"w")   45. Symmetry(42)   46. EqualitySub(44,45,[0])   47. ExistsElim(8,9,46,"y")   48.

ExistsElim(7,8,47,"x") 49. ImpInt(48,1) 50. ForallInt(49,"z","z") 51. DefSub(50,"Contains",["f","prod(domain(f),range(f))"],[0]) 52. ImpInt(51,0)

Th77. (Set(x) & Set(y)) -> Set(func(x,y))

0. Hyp("(Set(x) & Set(y))") 1. Hyp("Elem(f, func(x,y))") 2. DefEqInt(20) 3. EqualitySub(1,2,[0]) 4. ClassElim(3) 5. AndElimL(4) 6. AndElimR(4) 7. AndElimL(6) 8. AndElimR(6) 9. DefExp(7,"Function",[0]) 10. AndElimL(9) 11. DefExp(10,"Relation",[0]) 12. Hyp("Elem(z,f)") 13. ForallElim(11,"z") 14. ImpElim(12,13) 15. ExistsInst(14,"a") 16. ExistsInst(15,"b") 17. DefEqInt(19) 18. EqualitySub(12,16,[0]) 19. ExistsInt(18,"b","w",[0]) 20. DefEqInt(16) 21. DefEqInt(17) 22. ExistsInt(18,"f","w",[0]) 23. DefSub(22,"Set",["orderedpair(a,b)"],[0]) 24. TheoremInt(1) 25. AndElimL(24) 26. EquivExp(25) 27. AndElimR(26) 28. ForallInt(27,"x","x") 29. ForallElim(28,"a") 30. ForallInt(29,"y","y") 31. ForallElim(30,"b") 32. ImpElim(23,31) 33. AndElimL(32) 34. AndInt(33,19) 35. ClassInt(34,"w") 36. Symmetry(20) 37. EqualitySub(35,36,[0]) 38. AndElimL(8) 39. EqualitySub(37,38,[0]) 40. ExistsInt(18,"a","w",[0]) 41. AndElimR(32) 42. AndInt(41,40) 43. ClassInt(42,"w") 44. Symmetry(21) 45. EqualitySub(43,44,[0]) 46. AndElimR(8) 47. EqualitySub(45,46,[0]) 48. AndInt(39,47) 49. AndInt(16,48) 50. Symmetry(16) 51. EqualitySub(23,50,[0]) 52. ExistsInt(49,"b","b",[0,1]) 53. ExistsInt(52,"a","a",[0,1]) 54. AndInt(51,53) 55. ClassInt(54,"w") 56. Symmetry(17) 57. EqualitySub(55,56,[0]) 58. ExistsElim(15,16,57,"b") 59. ExistsElim(14,15,58,"a") 60. ImpInt(59,12) 61. ForallInt(60,"z","z") 62. DefSub(61,"Contains",["f","prod(x,y)"],[0]) 63. TheoremInt(2) 64. ImpElim(0,63) 65. TheoremInt(3) 66. TheoremInt(4) 67. ForallInt(66,"y","y") 68. ForallElim(67,"c") 69. ForallInt(68,"x","x") 70. ForallElim(69,"prod(x,y)") 71. ForallInt(70,"c","c") 72. ForallElim(71,"f") 73. AndInt(64,62) 74. ImpElim(73,72) 75. ForallInt(65,"y","y") 76. ForallElim(75,"f") 77. ForallInt(76,"x","x") 78. ForallElim(77,"prod(x,y)") 79. ImpElim(64,78) 80. AndElimL(79) 81. AndElimR(79) 82. EquivExp(81) 83. AndElimL(82) 84. ImpElim(62,83) 85. ImpInt(84,1) 86. ForallInt(85,"f","f") 87. DefSub(86,"Contains",["func(x,y)","parts(prod(x,y))"],[0]) 88. TheoremInt(4) 89. ForallInt(88,"y","y") 90. ForallElim(89,"c") 91. ForallInt(90,"x","x") 92. ForallElim(91,"parts(prod(x,y))") 93. ForallInt(92,"c","c") 94. ForallElim(93,"func(x,y)") 95. AndInt(80,87) 96. ImpElim(95,94) 97. ImpInt(96,0)

Th88. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))

0. Hyp("WellOrders(r,x)") 1. Hyp("(Elem(u,x) & Elem(v,x) & Elem(w,x))") 2. Hyp("(Elem(orderedpair(u,v),r) & Elem(orderedpair(v,w), r))") 3. Hyp("Elem(z,pair(u,v))") 4. TheoremInt(1) 5. AndElimL(4) 6. ForallInt(5,"x","x") 7. ForallElim(6,"c") 8. ForallInt(7,"y","y") 9. ForallElim(8,"d") 10. ForallInt(9,"z","z") 11. ForallElim(10,"e") 12. AndElimL(1) 13. AndElimR(1) 14. AndElimL(13) 15. ExistsInt(12,"x","x",[0]) 16. DefSub(15,"Set",["u"],[0]) 17. ExistsInt(14,"x","x",[0]) 18. DefSub(17,"Set",["v"],[0]) 19. ForallInt(11,"c","c") 20. ForallElim(19,"u") 21. ForallInt(20,"d","d") 22. ForallElim(21,"v") 23. AndInt(16,18) 24. ImpElim(23,22) 25. AndElimR(24) 26. ForallInt(25,"e","e") 27. ForallElim(26,"z") 28. EquivExp(27) 29. AndElimL(28) 30. ImpElim(3,29) 31. Hyp("(z = u)") 32. AndElimL(1) 33. Symmetry(31) 34. EqualitySub(32,33,[0]) 35. Hyp("(z = v)") 36. AndElimR(1) 37. AndElimL(36) 38. Symmetry(35) 39. EqualitySub(37,38,[0]) 40. OrElim(30,31,34,35,39) 41. ImpInt(40,3) 42. ForallInt(41,"z","z") 43. DefSub(42,"Contains",["pair(u,v)","x"],[0]) 44. DefExp(0,"WellOrders",[0]) 45. AndElimR(44) 46. ForallElim(45,"pair(u,v)") 47. Identity("u") 48. OrIntR(47,"(v = v)") 49. EquivExp(25) 50. AndElimR(49) 51. ForallInt(50,"e","e") 52. ForallElim(51,"u") 53. OrIntR(47,"(u = v)") 54. ImpElim(53,52) 55. Hyp("(pair(u,v) = 0)") 56. EqualitySub(54,55,[0]) 57. TheoremInt(2) 58. ForallInt(57,"x","x") 59. ForallElim(58,"u") 60. ImpElim(56,59) 61. ImpInt(60,55) 62. AndInt(43,61) 63. ImpElim(62,46) 64. ExistsInst(63,"f") 65. DefExp(64,"First",[0]) 66. AndElimL(65) 67. EquivExp(25) 68. AndElimL(67) 69. ForallInt(68,"e","e") 70. ForallElim(69,"f") 71. ImpElim(66,70) 72. AndElimR(65) 73. ForallElim(72,"u") 74. ForallElim(72,"v") 75. Hyp("(f = u)") 76. ForallInt(50,"e","e") 77. ForallElim(76,"v") 78. Identity("v") 79. OrIntL(78,"(v = u)") 80. ImpElim(79,77) 81. ImpElim(80,74) 82. EqualitySub(81,75,[0]) 83. OrIntR(82,"neg Elem(orderedpair(u,v),r)") 84. Hyp("(f = v)") 85. ForallInt(50,"e","e") 86. ForallElim(85,"u") 87. Identity("u") 88. OrIntR(87,"(u = v)") 89. ImpElim(88,86) 90. ForallElim(72,"u") 91. ImpElim(89,90) 92. EqualitySub(91,84,[0]) 93. OrIntL(92,"neg Elem(orderedpair(v,u),r)") 94. OrElim(71,75,83,84,93) 95. ExistsElim(63,64,94,"f") 96. TheoremInt(3) 97. PolySub(96,"B","neg

Elem(orderedpair(v,u),r)")   98. PolySub(97,"A","Elem(orderedpair(u,v),r)")   99. ImpElim(95,98)   100. ImpInt(99,1)   101. ForallInt(100,"w","w")   102. ForallElim(101,"v")   103. Hyp("(Elem(u,x) & Elem(v,x))")   104. Hyp("Elem(orderedpair(u,v), r)")   105. AndElimL(103)   106. AndElimR(103)   107. AndInt(106,106)   108. AndInt(105,107)   109. ImpElim(108,102)   110. ImpElim(104,109)   111. ImpInt(110,104)   112. ImpInt(111,103)   113. ForallInt(112,"v","z")   114. ForallInt(113,"u","y")   115. DefSub(114,"Asymmetric",["r","x"],[0])   116. Hyp("neg TransIn(r,x)")   117. DefExp(116,"TransIn",[0])   118. TheoremInt(5)   119. PredSub(118,"P",["u"],"forall v.forall w.((Elem(u,x) & (Elem(v,x) & Elem(w,x))) -> ((Elem(orderedpair(u,v),r) & Elem(orderedpair(v,w),r)) -> Elem(orderedpair(u,w),r))) ", [0,1])   120. ImpElim(117,119)   121. ExistsInst(120,"k")   122. PredSub(118,"P", ["v"],"forall w.((Elem(k,x) & (Elem(v,x) & Elem(w,x))) -> ((Elem(orderedpair(k,v),r) & Elem(orderedpair(v,w),r)) -> Elem(orderedpair(k,w),r))) ",[0,1])   123. ImpElim(121,122)   124. ExistsInst(123,"p")   125. PredSub(118,"P",["w"],"((Elem(k,x) & (Elem(p,x) & Elem(w,x))) -> ((Elem(orderedpair(k,p),r) & Elem(orderedpair(p,w),r)) -> Elem(orderedpair(k,w),r))) ",[0,1])   126. ImpElim(124,125)   127. ExistsInst(126,"q")   128. TheoremInt(7)   129. PolySub(128,"B","C")   130. PolySub(129,"A","(B v neg A)")   131. PolySub(130,"C","(A -> B)")   132. TheoremInt(6)   133. ImpElim(132,131)   134. PolySub(133,"A"," (Elem(k,x) & (Elem(p,x) & Elem(q,x))) ")   135. PolySub(134,"B"," ((Elem(orderedpair(k,p),r) & Elem(orderedpair(p,q),r)) -> Elem(orderedpair(k,q),r)) ")   136. ImpElim(127,135)   137. TheoremInt(8)   138. AndElimL(137)   139. PolySub(138,"B","C")   140. PolySub(139,"A","B")   141. PolySub(140,"C","neg A")   142. EquivExp(141)   143. AndElimL(142)   144. TheoremInt(9)   145. EquivExp(144)   146. AndElimR(145)   147. PolySub(146,"D","A")   148. Hyp("neg (B v neg A)")   149. ImpElim(148,143)   150. AndElimL(149)   151. AndElimR(149)   152. ImpElim(151,147)   153. AndInt(150,152)   154. ImpInt(153,148)   155. Hyp("neg (A -> B)")   156. ImpElim(155,133)   157. ImpElim(156,154)   158. ImpInt(157,155)   159. PolySub(158,"A"," (Elem(k,x) & (Elem(p,x) & Elem(q,x))) ")   160. PolySub(159,"B"," ((Elem(orderedpair(k,p),r) & Elem(orderedpair(p,q),r)) -> Elem(orderedpair(k,q),r)) ")   161. ImpElim(127,160)   162. AndElimL(161)   163. AndElimR(161)   164. PolySub(158,"A"," (Elem(orderedpair(k,p),r) & Elem(orderedpair(p,q),r))")   165. PolySub(164,"B"," Elem(orderedpair(k,q),r) ")   166. ImpElim(162,165)   167. AndElimL(166)   168. AndElimL(163)   169. AndElimR(163)   170. AndElimR(169)   171. AndElimL(44)   172. DefExp(171,"Connects",[0])   173. ForallElim(172,"k")   174. ForallElim(173,"q")   175. AndInt(168,170)   176. ImpElim(175,174)   177. Hyp("(k = q)")   178. AndElimR(166)   179. EqualitySub(178,177,[0])   180. ForallElim(114,"q")   181. ForallElim(180,"p")   182. AndElimL(169)   183. AndInt(170,182)   184. ImpElim(183,181)   185. AndElimL(179)   186. ImpElim(185,184)   187. AndElimR(178)   188. ImpElim(187,186)   189. AbsI(188,"Elem(orderedpair(q,k),r)")   190. Hyp("(Elem(orderedpair(k,q),r) v Elem(orderedpair(q,k),r))")   191. Hyp("Elem(orderedpair(k,q),r)")   192. ImpElim(191,167)   193. AbsI(192,"Elem(orderedpair(q,k),r)")   194. Hyp("Elem(orderedpair(q,k),r)")   195. OrElim(190,191,193,194,194)   196. OrElim(176,177,189,190,195)   197. AndInt(196,178)   198. Hyp("(cyc = pair(p,pair(q,k)))")   199. TheoremInt(10)   200. AndElimL(163)   201. ExistsInt(200,"x","w",[0])   202. DefSub(201,"Set",["k"],[0])   203. AndElimR(163)   204. AndElimR(203)   205. ExistsInt(204,"x","w",[0])   206. DefSub(205,"Set",["q"],[0])   207. AndElimL(203)   208. ExistsInt(207,"x","w",[0])   209. DefSub(208,"Set",["p"],[0])   210. Hyp("(triad = union(singleton(p), union(singleton(q), singleton(k)))) ")   211. Hyp("Elem(z, triad)")   212. TheoremInt(11)   213. TheoremInt(12)   214. AndElimL(213)   215. EquivExp(214)   216. AndElimL(215)   217. ForallInt(216,"x","x")   218. ForallElim(217,"singleton(p)")   219. ForallInt(218,"y","y")   220. ForallElim(219,"union(singleton(q),singleton(k))")   221. EqualitySub(211,210,[0])   222. ImpElim(221,220)   223. TheoremInt(13)   224. Hyp("Elem(z,singleton(p))")   225. ForallInt(223,"x","x")   226. ForallElim(225,"p")   227. ImpElim(209,226)   228. EquivExp(227)   229. AndElimL(228)   230. ForallInt(229,"y","y")   231. ForallElim(230,"z")   232. ImpElim(224,231)   233. Symmetry(232)   234. EqualitySub(207,233,[0])   235. Hyp("Elem(z,union(singleton(q),singleton(k)))")   236. ForallInt(216,"x","x")   237. ForallElim(236,"singleton(q)")   238. ForallInt(237,"y","y")   239. ForallElim(238,"singleton(k)")   240. ImpElim(235,239)   241. Hyp("Elem(z,singleton(q))")   242. ForallInt(223,"x","x")   243. ForallElim(242,"q")   244. ImpElim(206,243)   245. EquivExp(244)   246. AndElimL(245)   247. ForallInt(246,"y","y")   248. ForallElim(247,"z")   249. ImpElim(241,248)   250. Symmetry(249)   251. EqualitySub(204,250,[0])   252. Hyp("Elem(z,singleton(k))")   253. ForallInt(223,"x","x")   254. ForallElim(253,"k")   255. ImpElim(202,254)   256. EquivExp(255)   257. AndElimL(256)   258. ForallInt(257,"y","y")   259. ForallElim(258,"z")   260. ImpElim(252,259)   261. Symmetry(260)   262. EqualitySub(200,261,[0])   263. OrElim(240,241,251,252,262)   264. OrElim(222,224,234,235,263)   265. ImpInt(264,211)   266. ForallInt(265,"z","z")   267. DefSub(266,"Contains",["triad","x"],[0])   268. ForallElim(45,"triad")   269. ForallInt(227,"y","y")   270. ForallElim(269,"p")   271. EquivExp(270)   272.

AndElimR(271) 273. Identity("p") 274. ImpElim(273,272) 275. OrIntR(274,"Elem(p,union(singleton(q),singleton(k)))") 276. EquivExp(214) 277. AndElimR(276) 278. ForallInt(277,"x","x") 279. ForallElim(278,"singleton(p)") 280. ForallInt(279,"y","y") 281. ForallElim(280,"union(singleton(q),singleton(k))") 282. ForallInt(281,"z","z") 283. ForallElim(282,"p") 284. ImpElim(275,283) 285. Symmetry(210) 286. EqualitySub(284,285,[0]) 287. TheoremInt(14) 288. Hyp("(triad = 0)") 289. Symmetry(288) 290. EqualitySub(286,288,[0]) 291. ForallInt(287,"x","x") 292. ForallElim(291,"p") 293. ImpElim(290,292) 294. ImpInt(293,288) 295. AndInt(267,294) 296. ImpElim(295,268) 297. ExistsInst(296,"l") 298. DefExp(297,"First",[0]) 299. AndElimL(298) 300. EqualitySub(299,210,[0]) 301. ForallInt(220,"z","z") 302. ForallElim(301,"l") 303. ImpElim(300,302) 304. Hyp("Elem(l,singleton(p))") 305. ForallInt(229,"y","y") 306. ForallElim(305,"l") 307. ImpElim(304,306) 308. TheoremInt(13) 309. ForallInt(308,"x","x") 310. ForallElim(309,"k") 311. Identity("k") 312. ImpElim(202,310) 313. ForallInt(312,"y","y") 314. ForallElim(313,"k") 315. EquivExp(314) 316. AndElimR(315) 317. ImpElim(311,316) 318. OrIntL(317,"Elem(k,singleton(q))") 319. ForallInt(277,"x","x") 320. ForallElim(319,"singleton(q)") 321. ForallInt(320,"y","y") 322. ForallElim(321,"singleton(k)") 323. ForallInt(322,"z","z") 324. ForallElim(323,"k") 325. ImpElim(318,324) 326. OrIntL(325,"Elem(k,singleton(p))") 327. ForallInt(277,"x","x") 328. ForallElim(327,"singleton(p)") 329. ForallInt(328,"y","y") 330. ForallElim(329,"union(singleton(q),singleton(k))") 331. ForallInt(330,"z","z") 332. ForallElim(331,"k") 333. ImpElim(326,332) 334. Symmetry(210) 335. EqualitySub(333,334,[0]) 336. AndElimR(298) 337. EqualitySub(336,307,[0]) 338. ForallElim(337,"k") 339. ImpElim(335,338) 340. AndElimR(197) 341. AndElimL(340) 342. ImpElim(341,339) 343. Hyp("Elem(l, union(singleton(q), singleton(k)))") 344. AndElimL(276) 345. ForallInt(344,"x","x") 346. ForallElim(345,"singleton(q)") 347. ForallInt(346,"y","y") 348. ForallElim(347,"singleton(k)") 349. ForallInt(348,"z","z") 350. ForallElim(349,"l") 351. ImpElim(343,350) 352. Hyp("Elem(l,singleton(q))") 353. ForallInt(308,"x","x") 354. ForallElim(353,"q") 355. ForallInt(354,"y","y") 356. ForallElim(355,"l") 357. ImpElim(206,356) 358. EquivExp(357) 359. AndElimL(358) 360. ImpElim(352,359) 361. AndElimR(298) 362. EqualitySub(361,360,[0]) 363. ForallElim(362,"p") 364. ImpElim(286,363) 365. AndElimR(340) 366. ImpElim(365,364) 367. Hyp("Elem(l, singleton(k))") 368. ForallInt(308,"x","x") 369. ForallElim(368,"k") 370. ImpElim(202,369) 371. ForallInt(370,"y","y") 372. ForallElim(371,"l") 373. EquivExp(372) 374. AndElimL(373) 375. ImpElim(367,374) 376. EqualitySub(361,375,[0]) 377. ForallElim(376,"q") 378. ForallInt(308,"x","x") 379. ForallElim(378,"q") 380. ImpElim(206,379) 381. ForallInt(380,"y","y") 382. ForallElim(381,"q") 383. Identity("q") 384. EquivExp(382) 385. AndElimR(384) 386. ImpElim(383,385) 387. OrIntR(386,"Elem(q,singleton(k))") 388. ForallInt(277,"x","x") 389. ForallElim(388,"singleton(q)") 390. ForallInt(389,"y","y") 391. ForallElim(390,"singleton(k)") 392. ForallInt(391,"z","z") 393. ForallElim(392,"q") 394. ImpElim(387,393) 395. OrIntL(394,"Elem(q,singleton(p))") 396. ForallInt(277,"x","x") 397. ForallElim(396,"singleton(p)") 398. ForallInt(397,"y","y") 399. ForallElim(398,"union(singleton(q),singleton(k))") 400. ForallInt(399,"z","z") 401. ForallElim(400,"q") 402. ImpElim(395,401) 403. Symmetry(210) 404. EqualitySub(402,403,[0]) 405. EqualitySub(361,375,[0]) 406. ForallElim(405,"q") 407. ImpElim(404,406) 408. AndElimL(197) 409. ImpElim(408,407) 410. OrElim(351,352,366,367,409) 411. OrElim(303,304,342,343,410) 412. ExistsElim(296,297,411,"l") 413. ImpInt(412,210) 414. ForallInt(413,"triad","triad") 415. ForallElim(414,"union(singleton(p),union(singleton(q),singleton(k)))") 416. Identity("union(singleton(p),union(singleton(q),singleton(k)))") 417. ImpElim(416,415) 418. ExistsElim(126,127,417,"q") 419. ExistsElim(123,124,418,"p") 420. ExistsElim(120,121,419,"k") 421. ImpInt(420,116) 422. TheoremInt(9) 423. EquivExp(422) 424. AndElimR(423) 425. PolySub(424,"D","TransIn(r,x)") 426. ImpElim(421,425) 427. AndInt(115,426) 428. ImpInt(427,0)