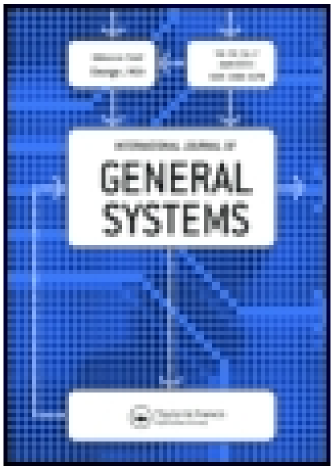

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OBJECTS

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OBJECTS

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This paper formalizes and studies the notion of an *object*, in the sense of a coherent collection of observations over time of some attributes. "Coherence" of observations turns out to be closely related to the mathematical theory of sheaves, which makes possible the importation of some non-trivial results. The spaces over which observations are parameterized can be more general than those usually associated with time, and it is possible to take account of linear, topological, or other structure of the sets of observations in a systematic way. This permits a comprehensive approach and provides a firm foundation for a general systems theory.

INDEX TERMS Object, system, sheaf, structure, category

This paper formalizes and studies the notion of an *object* in the sense of a coherent collection of observations over time of some attributes. This notion is close to many authors' idea of a *system*, but we reserve that word for "higher level" situations involving interactions of a number of components. Our approach is mathematical, so that none of our results actually depend on any assumptions about the empirical content of the "observations," which could even be given *a priori* (as in our examples); that is, the theory is logically independent of any foundation or philosophy of science. "Coherence" of observations turns out to be closely related to the mathematical theory of "sheaves," and this makes possible the importation of some rather deep results, from sheaf theory to object theory. The observations may actually range over much more general sets than those usually associated with time, and this makes possible some applications of a somewhat unexpected character. Finally, note that the choice of attributes is assumed to have been already made, and is given for each object along with the observations comprising it (the problem of choosing good attributes to observe is possibly one of the hardest available in system theory, and we are happy to sweep it under the rug).

We will be using modest amounts of topology and modern algebra, for which reference may be made, for example, (respectively) to Kelley[5] or Singer-Thorpe,[13] and to Mac Lane-Birkhoff[8] or Goguen-Thatcher-Wagner-Wright.[3] Also, we will be drawing illustrative examples from a rather wide range of subjects, precisely to demonstrate the generality of the concepts we are discussing.

Our notation is generally rather standard, and is exactly that of Goguen-Thatcher-Wagner-Wright.[3] We shall use the symbol "□" as a terminator for examples, as well as proofs. Material preceded with a "*" is more complex and skippable, especially at a first reading. The font $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, etc. is used for categories.

The author would like to thank Mario Bunge, Bill Lawvere, and Susanna Ginali for their valuable comments, and L. A. Zadeh for getting him interested in this field.

The set over which observations range will be denoted T. Often T is the set $\mathbb{R}^+$ of positive reals, $\mathbb{R}$ of all reals, or ω of nonnegative integers, and we assume it always comes equipped with an associated set $\mathscr{I}$ of subsets, usually intervals, over which observations may range. Hereafter T also designates the pair $\langle T, \mathscr{I}\rangle$. Some typical simple examples are: all open intervals (a, b) in $\mathbb{R}^+$; and all initial segments $[0, t]$ in ω.

*Example 0. Binary trees. Somewhat more complexly, we can consider a "time set" in which binary choices are mandated at each of a (countably) infinite sequence of instants, $i = 1, 2, 3, \ldots$ Let $T = \{0, 1\}^*$ be the set of all finite sequences $a_1 \ldots a_n$ of *binary* numbers, i.e., with $a_i \in \{0, 1\}$. Think of each of these as a sequence of binary choices, say of left or right (for 0 or 1 respectively), and therefore as a description of a path in an infinitely bifurcating tree (see Figure 1). Each $a_1 \ldots a_n$ in

$\{0, 1\}^*$ also names a unique *node*, the end point of the path of choices: e.g. $a = 011$. Thus the empty sequence of binary choices, denoted Λ, represents the *root* of the tree.

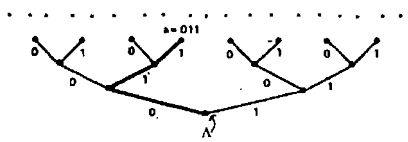

FIGURE 1 A binary tree.

A *binary forest* is a subset $F \subseteq \{0, 1\}^*$ such that $a_1 \ldots a_n$ and $a_1 \ldots a_n\, a_{n+1} \ldots a_{n+m}$ in F imply that $a_1 \ldots a_n\, a_{n+1} \ldots a_{n+i} \in F$ for $i = 1, \ldots, m$. An *initial* (binary) *subtree* of $\{0, 1\}^*$ is a forest F such that $\Lambda \in F$. For example, Figure 2 shows the binary subtree $\{\Lambda, 0, 1, 00, 01, 10, 11, 000, 001, 100, 101, 111\}$. Now let $\mathscr{I}$ be the set of all initial subtrees of $T = \{0, 1\}^*$. Then $\langle\{0, 1\}^*, \mathscr{I}\rangle$ is useful for tree automata and arithmetic terms, among other things. □

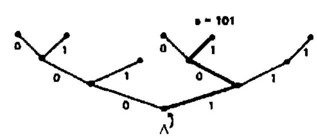

FIGURE 2 A finite binary subtree.

For X, Y sets, let $Pfn(X, Y)$ denote the set of all *partial* functions from X to Y, and let $def(f)$ denote the *set of definition* of $f \in Pfn(X, Y)$; in the usual set-theoretic model with $f \subseteq X \times Y$, $def(f) = \{x \mid \langle x, y\rangle \in f\}$.

DEFINITION 1. A *quasi-object* $\mathbf{A}$ is a subset of $Pfn(T, A)$, for some set A called the *attribute set* of $\mathbf{A}$, such that $f \in \mathbf{A}$ imples $def(f) \in \mathscr{I}$. Thus, a function $def\colon \mathbf{A} \to \mathscr{I}$ is well-defined on any quasi-object.

Now some illustrations:

EXAMPLE 1. *Free monoid quasi-objects.* Let $T = \omega$ and let $\mathscr{I}$ consist of all initial segments $[0, t]$ of ω, and also the empty segment $\varnothing$. Let A be a set. Then the set of all conceivable finite observations starting at $t = 0$ of a source producing words from A as alphabet is

$$\mathbf{A}^* = \{x\colon [0, t] \to A \mid t \in \omega\} \bigcup \{\Lambda\}$$

(where $\Lambda\colon \varnothing \to A$ is the null observation). $\mathbf{A}^*$ is the largest quasi-object having this particular $\langle T, \mathscr{I}\rangle$ and A, and is the underlying set of the free monoid generated by A. □

EXAMPLE 2. *Attribute quasi-objects.* Given some particular $\langle T, \mathscr{I}\rangle$ and A, call

$$\mathbf{A}^* = \{f \in Pfn(T, A) \mid def(f) \in \mathscr{I}\}$$

the $\langle T, \mathscr{I}\rangle$-attribute quasi-object for A, usually omitting the prefix $\langle T, \mathscr{I}\rangle$. Especially when $T = \mathbb{R}$ or $\mathbb{R}^+$ and $A = \mathbb{R}^n$, we may admit only continuous, continuously differentiable, or infinitely many times continuously differentiable, (indicated C^0, C^1, C^∞ respectively) functions, and speak of C^0-, or continuous attribute objects, etc. □

Example 1 arises from the construction o Example 2 with $T = \omega$ and $\mathscr{I} = \{[0, t] \mid t \in \omega\} \bigcup \{\varnothing\}$.

Very often A is a product $A_1 \times \ldots \times A_n$ of other sets, so that each attribute $a \in A$ is a compound $\langle a_1, \ldots, a_n\rangle$ of attributes $a_i \in A_i$. We shall call $\mathbf{A}^*$ an attribute quasi-object with attributes $A_1, \ldots, A_n$ in this case.

PROPOSITION 1. If $A = A_1 \times \ldots \times A_n$, then $\mathbf{A}^*$ is (naturally isomorphic to) a relation among the attribute objects $\mathbf{A}_1^*, \mathbf{A}_2^*, \ldots, \mathbf{A}_n$.

PROOF. $\mathbf{A}^* = \{f\colon U \to A_1 \times \ldots \times A_n \mid U \in \mathscr{I}\}$ For $f \in \mathbf{A}^*$, write $f = \langle f_1, \ldots, f_n\rangle$, where $f_i\colon U \to A_i$. Then $\mathbf{A}^* = \{\langle f_1, \ldots, f_n\rangle \mid f_i\colon U \to A_i$, for some $U \in \mathscr{I}\}$. On the other hand, $\mathbf{A}_1^* \times \mathbf{A}_2^* \times \ldots \times \mathbf{A}_n^* = \{\langle f_1, \ldots, f_n\rangle \mid f_i\colon U_i \to A_i$, for some $U_i \in \mathscr{I}\}$ (here $\langle f_1, \ldots, f_n\rangle$ denotes an actual n-tuple). Then evidently, $\mathbf{A}^*$ is (isomorphic to) that proper subset of $\mathbf{A}_1^* \times \ldots \times \mathbf{A}_n^*$ consisting of those $\langle f_1, \ldots, f_n\rangle$ such that $def(f_1) = \ldots = def(f_n)$. (We have isomorphism rather than equality because of the two different senses of $\langle f_1, \ldots, f_n\rangle$). □

A system is often defined to be a relation among (i.e., subset of the set product of) attribute objects; Mesarovic[9,10] has been advocating such an approach for some time. However, as the above proof shows, when we allow a multitude of intervals in attribute objects, care must be taken that inconsistent intervals do not occur in multi-attribute objects. This can be arranged by using a different notion of product.

A serious problem with the framework of this paper is that there is no direct method for treating interconnections of objects, which we have elsewhere claimed characterize the notion of system (see Goguen[1,2]). However, the present paper supplies the rich stock of objects necessary to support an interconnection based system theory.

EXAMPLE 3. *A capacitor*. The set of all possible observations of the voltages v_1, v_2 at the terminals of, and current k through, a capacitor of C farads is

$$C = \{\langle v_1, v_2, k\rangle \mid v_1, v_2, k\colon U \to \mathbb{R} \text{ are } C^\infty \text{ on some } U \in \mathscr{I} \text{ and } k = C\cdot\frac{d}{dt}(v_1 - v_2)\},$$

where $T = \mathbb{R}^+$ and $\mathscr{I}$ contains all open intervals of T. Here $A = \mathbb{R}^3$ and $f\colon U \to A$ appears as a triple $\langle v_1, v_2, k\rangle$ of functions $U \to \mathbb{R}$ with the same domain. □

EXAMPLE 4. *A finite state machine*. The set of all possible input-output observations of $M = \langle X, S, Y, \delta, \beta\rangle$, where X, S, Y are the finite sets of inputs, states, and outputs, and $\delta\colon S \times X \to S$, $\beta\colon S \to Y$ are the transition and output functions, is

$$\mathbf{E} = \left\{ \langle x, y\rangle \;\middle|\; \begin{array}{l} x\colon U \to X,\ y\colon U \to Y, \text{ and there is} \\ \text{some } s\colon U \to S, \text{ for some } U \in \mathscr{I}, \\ \text{such that } y(t) = \beta(s(t)) \text{ for all } t \in U \\ \text{and } s(t) = \delta(s(t-1), x(t)) \text{ whenever} \\ t,\ t-1 \in U \end{array} \right\}$$

where $\langle T, \mathscr{I}\rangle$ is as in Example 1. □

EXAMPLE 4*a*. The above can be modified by allowing $\mathscr{I}$ to contain all intervals $[m, n]$ with $n \geqq m$, $m, n \in \omega$, possibly plus $\varnothing$. □

EXAMPLE 4*b*. The example can also be modified by adding to M a fixed starting state $s_0 \in S$, and adding to E the corresponding equation $s(0) = s_0$. □

EXAMPLE 5. *A delayor*. The set of all possible observations of a unit delayor over an alphabet A, with initial condition $a_0 \in A$, is

$$\mathbf{D} = \left\{ \langle x, x'\rangle \;\middle|\; \begin{array}{l} x, x'\colon U \to A,\ x'(0) = a_0, \text{and} \\ x'(t) = x(t-1) \text{ for } t,\ t-1 \in U, \\ \text{and } V \in \mathscr{I} \end{array} \right\}$$

with $\langle T, \mathscr{I}\rangle$ as in Example 1. The delayor is also a special case of Example 4b, with $X = S = Y = A$, $\beta(a) = \delta(a, b) = a$, and $s_0 = a_0$. □

EXAMPLE 6. *An anticipator*. To show that objects need not necessarily be conventional causal systems, we consider a device whose output anticipates its input; thus,

$$\left\{ \langle x, y\rangle \;\middle|\; \begin{array}{l} x, y\colon U \to A,\ V \in \mathscr{I} \text{ and} \\ y(t) = x(t+1) \text{ for } t,\ t+1 \in U \end{array} \right\}. \;\square$$

EXAMPLE 7. *Temperature of a thin stone slab*. Consider observations of temperature of a fixed slab, as time functions over various fixed portions of its surface S. Then $T = S \times \mathbb{R}^+$ and $\mathscr{I}$ is some collection of sets of the form $U \times I$, with $U \subseteq S$ and $I \subseteq \mathbb{R}^+$. Also $A = \mathbb{R}$ and observations are solutions of a certain well-known partial differential equation over the given surface-time domains. We omit explicit details, because our purpose is only to show that there are interesting T which are not just semigroups or simply ordered sets. T might even be the surface of a sphere. □

EXAMPLE 7*a*. The above can be modified to assume a fixed surface and a fixed temperature distribution at $t = 0$. Account can then be taken of possible contractions and expansions, so that T now becomes a surface-time 3-submanifold of space-time $\mathbb{R}^4$ (the usual way to describe an embedded time-varying surface). $\mathscr{I}$ might be an atlas of charts for S, perhaps all simply connected open sets, or perhaps even all open sets. For this example, there will be just one possible observation for each $U \in \mathscr{I}$, uniquely determined by the initial conditions. □

*EXAMPLE 8. *Labelled trees*. Let T be $\{0, 1\}^*$ and let $\mathscr{I}$ be all finite initial binary subtrees (see Example 0). The quasi-object

$$\{f\colon U \to A \mid U \in \mathscr{I}\}$$

is the set of all A-labelled (finite) binary trees. □

*EXAMPLE 8*a*. *Binary terms*. Let $\langle T, \mathscr{I}\rangle$ be as in Example 8, and let A_0, A_1, A_2 be sets, whose members will be thought of as operator symbols of rank 0, 1, 2 respectively (rank 0 symbols are constants). Let $A = A_0 \bigcup A_1 \bigcup A_2$. Then

$$\left\{ f\colon U \to A \;\middle|\; \begin{array}{l} f(a) \in A_2 \text{ if } a0, a1 \in U; f(a) \in A_1 \\ \text{if } a0 \in I \text{ or } a1 \in U, \text{ but not both;} \\ f(a) \in A_0 \text{ if neither } a0, a1 \in U; \\ \text{and } U \in \mathscr{I} \end{array} \right\}$$

is the set of all terms which can be constructed from A_0, A_1, A_2. Here the three conditions on i with respect to U mean that i has two, one, or no successors in U. □

*EXAMPLE 8*b*. *Arithmetic functions*. In the previous example, let $A_0 = \{X_0, X_1, X_2\}$, $A_1 = \{-\}$, $A_2 = \{+, \times, \uparrow\}$. Then the trees (or terms) we get are more conventionally represented as expressions such as $(-X_0 + X_1) \times (X_1 \uparrow (X_2 + X_1))$ and $((x_0 \uparrow X_0) + (X_0 \uparrow X_0)) + (-X_0)$. (The arrow "↑"

is computer science notation for exponentiation.) See Figure 3. □

It can be verified that each of these examples satisfies the following property, a generalization of one called "CUS" by Zadeh[15]. For $f\colon U \to A$ and $V \subseteq U$, we let $f \mid V$ denote the restriction of f to V.

DEFINITION 2. A quasi-object A is *closed under restriction* iff whenever $f \in \mathrm{A}$, $U \subseteq V = def(f)$, and $U \in \mathscr{I}$, then $f \mid U \in A$.

Examples 8a and 8b show that interesting quasi-objects are not always closed under restriction. For, unless $A_0 \subseteq A_1 \subseteq A_2$, the restricted trees will not be properly labelled. This is, in particular, illustrated in Example 8b.

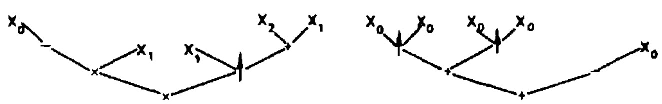

FIGURE 3 Two terms.

$V \subseteq U$ in $\mathscr{I}$) is real linear. These spaces can also be given linear topological structure. In Example 1, if A is a vector space (over some field K, such as the reals) then each $\mathrm{A}^*(U)$ and each restriction are also K-linear. In Examples 4 and 4a (without $\varnothing$) each $\mathbf{E}(U)$ and restriction are K-linear if each X, S, Y and δ, β are K-linear. In Example 4b this holds only if $S_0 = 0$. In Example 5, the $\mathbf{D}(U)$ and restrictions are K-linear if A is and if $a_0 = 0$. Example 6 is similar. In Example 7 each $\mathrm{A}(U)$ has real linear topological structure. As originally described, each example had "discrete", i.e. set, structure. The systematic way to allow for whatever structure the $\mathrm{A}(U)$ and restrictions have, is to generalize the category $\mathscr{SET}$ which was target of the preobject as functor, to an arbitrary "structure category" $\mathscr{S}$. In

For a given a quasi-object A, we are often interested in the collection of all observations defined on some $U \in \mathscr{I}$. For this purpose, given $U \in \mathscr{I}$ let $\mathrm{A}(U)$ denote the set $def^{-1}(U)$ of all $f \in \mathrm{A}$ such that $def(f) = U$. Thus, a quasi-object A determines on $\mathscr{I}$ a set-valued function, also denoted A, sending $U \in \mathscr{I}$ to the set $\mathrm{A}(U)$. Whenever U, $V \in \mathscr{I}$ and $V \subseteq U$, if A is closed under restriction, a function $\mathrm{A}(U) \to \mathrm{A}(V)$ is defined by sending $f \in \mathrm{A}(U)$ to $f \mid V \in \mathrm{A}(V)$. If i denotes the inclusion function $i\colon V \hookrightarrow U$, then $\mathrm{A}(i)\colon \mathrm{A}(U) \to \mathrm{A}(V)$ will denote the restriction function. For $W \subseteq V \subseteq U$ with $U, V, W \in \mathscr{I}$, letting $j\colon W \hookrightarrow V$ and $i\colon V \hookrightarrow U$ be the inclusions, we have $ji\colon W \hookrightarrow U$ also an inclusion, and also $\mathrm{A}(ji) = \mathrm{A}(i)\,\mathrm{A}(j)$. A quick way of summarizing these facts is to say that a quasi-object A closed under restriction has an *associated* functor $\mathrm{A}\colon \mathscr{I}^{OP} \to \mathscr{SET}$, where $\mathscr{I}^{OP}$ is the "opposite" category, to $\mathscr{I}$ viewed as a category with objects elements of $\mathscr{I}$ as a set, with morphisms inclusion functions $V \hookrightarrow U$, and with the obvious identities and composition. (In $\mathscr{I}^{OP}$ the morphisms can be thought of converses of inclusions.) We call this functor the *preobject associated to* the quasi-object A.

Before giving a general definition of preobject we must discuss the question of the structure of objects. Notice that in Example 3 each $\mathbf{C}(U)$ is a real linear space, and restriction $\mathbf{C}(U) \to \mathbf{C}(V)$ (for

the above discussions $\mathscr{S}$ was: the category $\mathscr{LIN}$ of real linear spaces and maps; or $\mathscr{LIN}_K$ of K-linear spaces and maps; or $\mathscr{LINTOP}$ of linear topological spaces and maps; or just $\mathscr{SET}$.

DEFINITION 3. A *preobject* A with structure category $\mathscr{S}$ is a functor from an *underlying* category $\mathscr{U}$ to $\mathscr{S}$.

In the above examples, $\mathscr{U}$ was of course $\mathscr{I}^{OP}$. This gives a rather rich collection of examples of preobjects, with a variety of structure categories. Incidentally, one can associate a preobject to a quasi-object not closed under restriction by letting $\mathscr{U} = |\mathscr{I}|$, the category having only identity morphisms for the objects in $\mathscr{I}$. Preobjects of observations over a single set T will have $\mathscr{U} = \{T\}$, a one object one morphism category.

There certainly are preobjects not associated to any quasi-object, since in the definition $\mathscr{U}$ need not arise from a partially ordered set $\mathscr{I}$, and the morphisms $\mathrm{A}(i)$ need not be restrictions. Indeed, the objects $\mathrm{A}(I)$ need not even contain functions to restrict; $\mathscr{S}$ might even be an abstract category whose objects do not have elements at all. We shall not discuss general conditions on $\mathscr{S}$ and A such that restriction makes sense. Hereafter we shall more often be concerned with preobjects than with the quasi-objects which may have produced them.

Since we now have this structure "preobject," we ought to look for the morphisms which preserve that structure.

DEFINITION 4. Let **A** and **B** be preobjects with structure $\mathscr{S}$ and underlying category $\mathscr{U}$. Then a *morphism* $\mathbf{A} \to \mathbf{B}$ is a natural transformation φ: $\mathbf{A} \Rightarrow \mathbf{B}$ of functors $\mathscr{U} \to \mathscr{S}$. This means that we are given a family φ_U: $\mathbf{A}(U) \to \mathbf{B}(U)$ of morphisms in $\mathscr{S}$, one for each $U \in |\mathscr{U}|$, such that if i: $U \to V$ in $\mathscr{U}$, then

$$\begin{tikzcd} \mathbf{A}(U) \arrow[r, "\varphi_U"] \arrow[d, "A(i)"'] & \mathbf{B}(U) \arrow[d, "B(i)"] \\ \mathbf{A}(V) \arrow[r, "\varphi_V"] & \mathbf{B}(V) \end{tikzcd}$$

commutes in $\mathscr{S}$.

In a preobject associated to a quasi-object, this means that the morphisms φ_U respect (commute with) restriction, and in effect, together define a global function on the quasi-object. Of course, the φ_U are in $\mathscr{S}$ and therefore preserve the structure of the $\mathscr{S}$-object $\mathbf{A}(U)$.

Thus, the category of *all* preobjects with structure $\mathscr{U}$ and underlying $\mathscr{S}$ is the functor category $\mathscr{NAT}$ $(\mathscr{U}, \mathscr{S})$. Now some examples of preobject morphisms.

EXAMPLE 9. Let f: $A \to B$ be a (set) function and let $\mathbf{A}^*$ and $\mathbf{B}^*$ denote the preobjects associated to the quasi-objects of Example 2. Now define f^*: $\mathbf{A}^* \to \mathbf{B}^*$ by letting $f^*_U : \mathbf{A}^*(U) \to \mathbf{B}^*(U)$ (for $U \in |\mathscr{I}|$) send a: $U \to A$ to the composite af: $U \to B$. We check that f^* is natural. Let i: $V \to U$ in $\mathscr{I}$. Then $\mathbf{A}^*(i)$: $\mathbf{A}^*(U) \to \mathbf{A}^*(V)$ and $\mathbf{B}^*(i)$: $\mathbf{B}^*(U) \to \mathbf{B}^*(V)$, by restriction. Let $a \in \mathbf{A}^*(U)$. Then for $v \in V$, $v(a\mathbf{A}^*(i)\, f^*_V) = vaf$ in $\mathbf{B}^*(V)$; and $v(af^*_U\mathbf{B}^*(i))$ equals the same thing. The assignment of $\mathbf{A}^*$ to $A \in |\mathscr{SET}|$ and of f^* to $f \in \mathscr{SET}$ defines a functor $*$: $\mathscr{SET} \to \mathscr{NAT}$ $(\mathscr{I}^{OP}, \mathscr{SET})$ whose values are preobjects. This functor is a sort of generalization of the free monoid functor. Moreover, if $\mathscr{S}$ is $\mathscr{LIN}_K$ rather than $\mathscr{SET}$, i.e., if A, B, and f are K-linear, so are $\mathbf{A}^*$, $\mathbf{B}^*$ and f^*, so that we get a functor $*$: $\mathscr{LIN}_K \to \mathscr{NAT}(\mathscr{I}^{OP}, \mathscr{LIN}_K)$. Similarly for $\mathscr{S} = \mathscr{LINTOP}$, etc. These considerations apply to any $\langle T, \mathscr{I}\rangle$ whatsoever. □

EXAMPLE 10. Consider the preobject **C** associated to Example 3 with $\mathscr{S} = \mathscr{LINTOP}$, and let $\mathbf{V}_1^*$, $\mathbf{V}_2^*$, $\mathbf{K}^*$ denote now preobjects associated with C^∞—attribute objects for the reals $\mathbb{R}$ as attribute space; i.e., $\mathbf{V}_1^*(u)$ is the space of all C^∞ real functions on u. Then the projections sending $\langle v_1, v_2, k\rangle$ to v_1, v_2, and k respectively, define preobject morphisms from **C** to $\mathbf{V}_1^*$, $\mathbf{V}_2^*$, $\mathbf{K}^*$ respectively. Similarly, a linear "change of scale," sending $\langle v_1, v_2, k\rangle$ to av_1, $a \in \mathbb{R}$ (for example) also defines a preobject morphism; and so does the "voltage drop" calculation, sending $\langle v_1, v_2, k\rangle$ to $v_1 - v_2$. In much the same style, the usual objects, measurements, and relationships of linear system theory can all be constructed. However, in order to consider the systems, in the sense of the objects being actually interrelated, it is necessary to introduce the general apparatus of interconnection theory.[1,2] □

EXAMPLE 11. The preobjects associated with Example 4 have the category $\mathscr{FIN}$ of finite sets as structure category; even though the quasi-object **E** is infinite, each $\mathbf{E}(U)$ will be finite. The projections sending $\langle x, y\rangle$ to x and y respectively, define preobject morphisms $\mathbf{E} \to \mathbf{X}^*$ and $\mathbf{E} \to \mathbf{Y}^*$, respectively. The same applies to 4a and 4b, and to 5, 6 and 8 if A is finite. □

EXAMPLE 12. If $\mathscr{I} = \{T\}$, then $\mathscr{NAT}$ $(\mathscr{I}^{OP}, \mathscr{S})$ is isomorphic to $\mathscr{S}$, since $\mathbf{A}$: $\mathscr{I}^{OP} \to \mathscr{S}$ is just a single object $\mathbf{A}(T)$ in $\mathscr{S}$, and a natural transformation φ: $\mathbf{A} \to \mathbf{B}$ is just a single morphism φ_T: $\mathbf{A}(T) \to \mathbf{B}(T)$ in $\mathscr{S}$. □

Because the preobjects constitute a functor category, we know quite a bit about them just from general categorical principles. Many of these are included in the general principle that the category of functors takes on properties of the target category. One good example is the following, proofs of which can be found in most texts on category theory.[9,11]

PROPOSITION 2. If $\mathscr{S}$ is complete, i.e., has inverse limits over all small categories, so is the category $\mathscr{NAT}$ $(\mathscr{U}, \mathscr{S})$ of all preobjects with structure $\mathscr{S}$, for any underlying category $\mathscr{U}$. Moreover, limits are computed "object-wise," in the sense that if D: $\mathscr{X} \to \mathscr{NAT}$ $(\mathscr{U}, \mathscr{S})$ is a diagram of preobjects, and if $U \in |\mathscr{U}|$, then the value of the preobject $\overleftarrow{\lim} D$ at U is $\overleftarrow{\lim}_{x \in X} (D(x))$, a limit computed in $\mathscr{S}$. The values of $\overleftarrow{\lim} D$ on morphisms in $\mathscr{X}$ are those induced by the universal properties of the limits assigned to objects in $\mathscr{X}$, in the usual way. □

Preobjects of observations of some coherent physical process will in general have properties going well beyond mere closure under restriction. In particular, functions which agree with each other on sets covering a given set should agree on that set; and it should also be possible to piece together functions agreeing on their overlaps. The general intuition here is that an object *is* a reasonably coherent set of observations. In a more philosophical language, it is perhaps as if objects do not exist *per se* outside ourselves, but only when we have sufficiently organized some set of observations within ourselves. (This may involve some "interpolation," pattern recognition, statistical methods, or whatever. The study of the construction of objects would seem to have interesting connections with artificial intelligence, cognitive psychology, and analytic philosophy, among other things. We do not pursue that study here.) The proper approach seems to avoid taking specific account of the "function-like" character of the elements of each $\mathbf{A}(U)$; but it is necessary to have elements available. So in the following definition, we assume that $\mathscr{S}$ is a *concrete* category, in the rather weak sense that it has given a faithful functor $V\colon \mathscr{S} \to \mathscr{SET}$. Then $f \in A$, for $A \in |\mathscr{S}|$, will mean that $f \in V(A)$. For simplicity we assume that $\mathscr{U} = \mathscr{I}^{OP}$, although this could be generalized.

DEFINITION 4. Assume that $\mathscr{I}$ is closed under finite intersections. Then a functor $\mathbf{A}\colon \mathscr{I}^{OP} \to \mathscr{S}$ is an *object* iff whenever $I \in |\mathscr{I}|$ and $\bigcup_\alpha I_\alpha = I \in |\mathscr{I}|$ (where α ranges over some index set), then:

1) if $f, g \in \mathbf{A}(I)$ agree on each U_α—that is, if $\mathbf{A}(i_\alpha)(f) = \mathbf{A}(i_\alpha)(g)$ for all α, where $i_\alpha\colon I_\alpha \to I$ is the inclusion—then $f = g$; and

2) if $f_\alpha \in \mathbf{A}(I_\alpha)$ agree on their overlaps—that is, if $\mathbf{A}(i_{\alpha\beta})(f_\alpha) = \mathbf{A}(i_{\beta\alpha})(f_\beta)$ whenever $I_\alpha \cap I_\beta \neq \varnothing$, where $i_{\alpha\beta}$ and $i_{\beta\alpha}$ are the inclusions of $I_\alpha \cap I_\beta$ into I_α and I_β respectively—then there is an $f \in \mathbf{A}(I)$ such that $\mathbf{A}(i_\alpha)(f) = f_\alpha$ for all α.

The preobjects associated to Examples 1, 2, 3, 4, 4b, 5, 6, 7, 7a, and 8 all give objects. But 4a does not, because there can exist pairs $\langle x, y\rangle \in \mathbf{E}([l, m])$ and $\langle x', y'\rangle \in \mathbf{E}([m, n])$ such that $x(m) = x'(m)$ and $y(m) = y'(m)$, but which cannot be spliced together as required in condition (2) above, because their corresponding states at time m are different, $s(m) \neq s'(m)$. A simple example arises from the machine in Example 5 which produces the delayor, but allows all intervals $[m, n]$ with $n \geqq m$. For an obvious subjective example, a shadow might be a monster if we are in a state of fear, but just a shadow (say from a street sign) if we are in a state of clarity.

One great advantage of the approach we have taken here is that it leads to objects familiar from topology. In fact, Definition 4 is exactly the definition of a *sheaf*, except that we did not require $\langle T, \mathscr{I}\rangle$ to necessarily be a topological space (see Seebach-Seebach-Steen[12] for an elementary survey of sheaf theory). Moverover, we have the following rather deep theorem of John Gray[4] for $\langle T, \mathscr{I}\rangle$ a topology (question: for what more general structures does it hold?). First, notice that since objects are in particular preobjects, we already have a notion of morphisms available; indeed, let the category $\mathscr{OBJ}(\mathscr{I}, \mathscr{S})$ of objects be the full subcategory of $\mathscr{NAT}(\mathscr{I}^{OP}, \mathscr{S})$ with objects as objects. (We leave some of the terms of this result undefined.)

THEOREM 3. If $\mathscr{S}$ is complete and has enough small objects, then any inverse limit (in the preobject category) of objects is an object; and moreover, the objects constitute a (full) reflective subcategory of the preobjects (the reflection functor gives a general notion of an object associated to a preobject, or of a sheaf associated to a presheaf). □

The proof is quite technical, but details can be found in Gray[4] and in the last chapter of Mitchell.[11] The reader not particularly familiar with inverse limits and reflective subcategories may not be particularly impressed by these results, but in fact, inverse limits extract the overall behavior of a system of objects,[1] and reflective subcategories extract the "best possible" special version of a structure.[2,3]

In this paper, we hope to have provided a rich stock of objects from which to build systems, and a powerful mathematical theory to back up our use of them. We hope that our examples have shown that this notion of object is widely applicable, captures the essence of its applications, and yet has an ample theory of its own.

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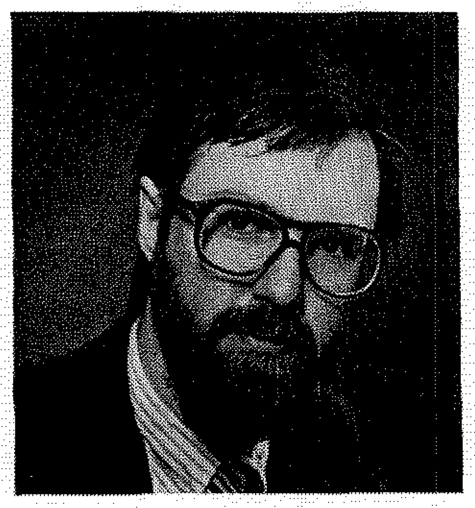

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