```
$ python3.9 -i proofenvironment.py

Welcome to PyLog 1.0

Natural Deduction Proof Assistant and Proof Checker

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>>> Load("Kelley-Morse")
True
>>> ShowAxioms()
0. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))
1. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y)))
2. (Set(x) & Set(y)) -> Set((x ∪ y))
3. (Function(f) & Set(domain(f))) -> Set(range(f))
4. Set(x) -> Set(∪x)
5. ¬(x = 0) -> ∃y.((y ε x) & ((y ∩ x) = 0))
6. ∃y.((Set(y) & (0 ε y)) & ∀x.((x ε y) -> (suc x ε y)))
7. ∃f.(Choice(f) & (domain(f) = (U ~ {0})))
>>> ShowDefinitions()
Set(x) <-> ∃y.(x ε y)
(x ⊂ y) <-> ∀z.((z ε x) -> (z ε y))
Relation(r) <-> ∀z.((z ε r) -> ∃x.∃y.(z = (x,y)))
Function(f) <-> (Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)))
Trans(r) <-> ∀x.∀y.∀z.((((x,y) ε r) & ((y,z) ε r)) -> ((x,z) ε r))
Connects(r,x) <-> ∀y.∀z.(((y ε x) & (z ε x)) -> ((y = z) v (((y,z) ε r) v ((z,y) ε r))))
Asymmetric(r,x) <-> ∀y.∀z.(((y ε x) & (z ε x)) -> (((y,z) ε r) -> ¬((z,y) ε r)))
First(r,x,z) <-> ((z ε x) & ∀y.((y ε x) -> ¬((y,z) ε r)))
WellOrders(r,x) <-> (Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z)))
Section(r,x,y) <-> (((y ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε y)) & ((u,v) ε
r)) -> (u ε y)))
OrderPreserving(f,r,s) <-> ((Function(f) & (WellOrders(r,domain(f)) &
WellOrders(s,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) ->
(((f'u),(f'v)) ε s)))
1-to-1(f) <-> (Function(f) & Function((f)⁻¹))
Full(x) <-> ∀y.((y ε x) -> (y ⊂ x))
Ordinal(x) <-> (Full(x) & Connects(E,x))
Integer(x) <-> (Ordinal(x) & WellOrders((E)⁻¹,x))
Choice(f) <-> (Function(f) & ∀y.((y ε domain(f)) -> ((f'y) ε y)))
Equi(x,y) <-> ∃f.(1-to-1(f) & ((domain(f) = x) & (range(f) = y)))
Card(x) <-> (Ordinal(x) & ∀y.(((y ε x) & (y ε ord)) -> ¬Equi(y,x)))
TransIn(r,x) <-> ∀u.∀v.∀w.(((u ε x) & ((v ε x) & (w ε x))) -> ((((u,v) ε r) & ((v,w) ε
r)) -> ((u,w) ε r)))
>>> ShowDefEquations()
0. (x ∪ y) = {z: ((z ε x) v (z ε y))}
1. (x ∩ y) = {z: ((z ε x) & (z ε y))}
2. ~x = {y: ¬(y ε x)}
3. (x ~ y) = (x ∩ ~y)
4. 0 = {x: ¬(x = x)}
5. U = {x: (x = x)}
6. ∪x = {z: ∃y.((y ε x) & (z ε y))}
7. ∩x = {z: ∀y.((y ε x) -> (z ε y))}
8. Px = {y: (y ⊂ x)}
9. {x} = {z: ((x ε U) -> (z = x))}
10. {x,y} = ({x} ∪ {y})
11. (x,y) = {{x},{x,y}}
12. proj1(x) = ∩∩x
13. proj2(x) = (∩∪x ∪ (∪∪x ~ ∪∩x))
14. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}
15. (r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}
16. domain(f) = {x: ∃y.((x,y) ε f)}
17. range(f) = {y: ∃x.((x,y) ε f)}
18. (f'x) = ∩{y: ((x,y) ε f)}
19. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}
20. func(x,y) = {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))}
21. E = {z: ∃x.∃y.((z = (x,y)) & (x ε y))}
22. ord = {x: Ordinal(x)}
23. suc x = (x ∪ {x})
24. (f|x) = (f ∩ (x X U))
25. ω = {x: Integer(x)}
>>> Test()
```

Th4. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

```
0. z ε (x ∪ y)  Hyp
1. (x ∪ y) = {z: ((z ε x) v (z ε y))}  DefEqInt
2. z ε {z: ((z ε x) v (z ε y))}  EqualitySub 0 1
3. Set(z) & ((z ε x) v (z ε y))  ClassElim 2
4. (z ε x) v (z ε y)  AndElimR 3
5. (z ε (x ∪ y)) -> ((z ε x) v (z ε y))  ImpInt 4
6. (z ε x) v (z ε y)  Hyp
7. z ε x  Hyp
8. ∃x.(z ε x)  ExistsInt 7
9. Set(z)  DefSub 8
10. z ε y  Hyp
11. ∃y.(z ε y)  ExistsInt 10
12. Set(z)  DefSub 11
13. Set(z)  OrElim 6 7 9 10 12
14. Set(z) & ((z ε x) v (z ε y))  AndInt 13 6
15. z ε {z: ((z ε x) v (z ε y))}  ClassInt 14
16. {z: ((z ε x) v (z ε y))} = (x ∪ y)  Symmetry 1
17. z ε (x ∪ y)  EqualitySub 15 16
18. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  ImpInt 17
19. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
AndInt 5 18
20. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  EquivConst 19
21. z ε (x ∩ y)  Hyp
22. (x ∩ y) = {z: ((z ε x) & (z ε y))}  DefEqInt
23. z ε {z: ((z ε x) & (z ε y))}  EqualitySub 21 22
24. Set(z) & ((z ε x) & (z ε y))  ClassElim 23
25. (z ε x) & (z ε y)  AndElimR 24
26. (z ε (x ∩ y)) -> ((z ε x) & (z ε y))  ImpInt 25
27. (z ε x) & (z ε y)  Hyp
28. z ε x  AndElimL 27
29. ∃x.(z ε x)  ExistsInt 28
30. Set(z)  DefSub 29
31. Set(z) & ((z ε x) & (z ε y))  AndInt 30 27
32. z ε {z: ((z ε x) & (z ε y))}  ClassInt 31
33. {z: ((z ε x) & (z ε y))} = (x ∩ y)  Symmetry 22
34. z ε (x ∩ y)  EqualitySub 32 33
35. ((z ε x) & (z ε y)) -> (z ε (x ∩ y))  ImpInt 34
36. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
AndInt 26 35
37. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  EquivConst 36
38. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
AndInt 20 37 Qed
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Used Theorems

Th5. ((x ∪ x) = x) & ((x ∩ x) = x)

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0. z ε (x ∪ x)  Hyp
1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
2. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  AndElimL 1
3. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 2
4. (z ε (x ∪ y)) -> ((z ε x) v (z ε y))  AndElimL 3
5. ∀y.((z ε (x ∪ y)) -> ((z ε x) v (z ε y)))  ForallInt 4
6. (z ε (x ∪ x)) -> ((z ε x) v (z ε x))  ForallElim 5
7. (z ε x) v (z ε x)  ImpElim 0 6
8. z ε x  Hyp
9. z ε x  Hyp
10. z ε x  OrElim 7 8 8 9 9
11. (z ε (x ∪ x)) -> (z ε x)  ImpInt 10
12. z ε x  Hyp
13. (z ε x) v (z ε x)  OrIntL 12
14. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  AndElimR 3
15. ∀y.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 14
16. ((z ε x) v (z ε x)) -> (z ε (x ∪ x))  ForallElim 15
17. z ε (x ∪ x)  ImpElim 13 16
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18. (z ε x) -> (z ε (x ∪ x)) ImpInt 17
19. ((z ε (x ∪ x)) -> (z ε x)) & ((z ε x) -> (z ε (x ∪ x))) AndInt 11 18
20. (z ε (x ∪ x)) <-> (z ε x) EquivConst 19
21. ∀z.((z ε (x ∪ x)) <-> (z ε x)) ForallInt 20
22. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
23. ∀y.(((x ∪ x) = y) <-> ∀z.((z ε (x ∪ x)) <-> (z ε y))) ForallElim 22
24. ((x ∪ x) = x) <-> ∀z.((z ε (x ∪ x)) <-> (z ε x)) ForallElim 23
25. (((x ∪ x) = x) -> ∀z.((z ε (x ∪ x)) <-> (z ε x))) & (∀z.((z ε (x ∪ x)) <-> (z ε x)) -> ((x ∪ x) = x)) EquivExp 24
26. ∀z.((z ε (x ∪ x)) <-> (z ε x)) -> ((x ∪ x) = x) AndElimR 25
27. (x ∪ x) = x ImpElim 21 26
28. z ε (x ∩ x) Hyp
29. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 1
30. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 29
31. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 30
32. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) ForallInt 31
33. (z ε (x ∩ x)) -> ((z ε x) & (z ε x)) ForallElim 32
34. (z ε x) & (z ε x) ImpElim 28 33
35. z ε x AndElimR 34
36. (z ε (x ∩ x)) -> (z ε x) ImpInt 35
37. z ε x Hyp
38. (z ε x) & (z ε x) AndInt 37 37
39. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 30
40. ∀y.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 39
41. ((z ε x) & (z ε x)) -> (z ε (x ∩ x)) ForallElim 40
42. z ε (x ∩ x) ImpElim 38 41
43. (z ε x) -> (z ε (x ∩ x)) ImpInt 42
44. ((z ε (x ∩ x)) -> (z ε x)) & ((z ε x) -> (z ε (x ∩ x))) AndInt 36 43
45. (z ε (x ∩ x)) <-> (z ε x) EquivConst 44
46. ∀y.(((x ∩ x) = y) <-> ∀z.((z ε (x ∩ x)) <-> (z ε y))) ForallElim 22
47. ((x ∩ x) = x) <-> ∀z.((z ε (x ∩ x)) <-> (z ε x)) ForallElim 46
48. (((x ∩ x) = x) -> ∀z.((z ε (x ∩ x)) <-> (z ε x))) & (∀z.((z ε (x ∩ x)) <-> (z ε x)) -> ((x ∩ x) = x)) EquivExp 47
49. ∀z.((z ε (x ∩ x)) <-> (z ε x)) -> ((x ∩ x) = x) AndElimR 48
50. ∀z.((z ε (x ∩ x)) <-> (z ε x)) ForallInt 45
51. (x ∩ x) = x ImpElim 50 49
52. ((x ∪ x) = x) & ((x ∩ x) = x) AndInt 27 51 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

Th6. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))

0. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
1. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 0
2. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 1
3. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 2
4. z ε (x ∪ y) Hyp
5. (z ε x) v (z ε y) ImpElim 4 3
6. (A v B) -> (B v A) TheoremInt
7. ((z ε x) v B) -> (B v (z ε x)) PolySub 6
8. ((z ε x) v (z ε y)) -> ((z ε y) v (z ε x)) PolySub 7
9. (z ε y) v (z ε x) ImpElim 5 8
10. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 2
11. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 10
12. ((z ε w) v (z ε y)) -> (z ε (w ∪ y)) ForallElim 11
13. ∀y.(((z ε w) v (z ε y)) -> (z ε (w ∪ y))) ForallInt 12
14. ((z ε w) v (z ε x)) -> (z ε (w ∪ x)) ForallElim 13
15. ∀w.(((z ε w) v (z ε x)) -> (z ε (w ∪ x))) ForallInt 14
16. ((z ε y) v (z ε x)) -> (z ε (y ∪ x)) ForallElim 15
17. z ε (y ∪ x) ImpElim 9 16
18. (z ε (x ∪ y)) -> (z ε (y ∪ x)) ImpInt 17
19. ∀x.((z ε (x ∪ y)) -> (z ε (y ∪ x))) ForallInt 18
20. (z ε (w ∪ y)) -> (z ε (y ∪ w)) ForallElim 19
21. ∀y.((z ε (w ∪ y)) -> (z ε (y ∪ w))) ForallInt 20
22. (z ε (w ∪ v)) -> (z ε (v ∪ w)) ForallElim 21
23. ∀w.((z ε (w ∪ v)) -> (z ε (v ∪ w))) ForallInt 22

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24. (z ε (y ∪ v)) -> (z ε (v ∪ y))  ForallElim 23
25. ∀v.((z ε (y ∪ v)) -> (z ε (v ∪ y)))  ForallInt 24
26. (z ε (y ∪ x)) -> (z ε (x ∪ y))  ForallElim 25
27. ((z ε (x ∪ y)) -> (z ε (y ∪ x))) & ((z ε (y ∪ x)) -> (z ε (x ∪ y)))  AndInt 18 26
28. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
29. ∀e.(((x ∪ y) = e) <-> ∀z.((z ε (x ∪ y)) <-> (z ε e)))  ForallElim 28
30. ((x ∪ y) = (y ∪ x)) <-> ∀z.((z ε (x ∪ y)) <-> (z ε (y ∪ x)))  ForallElim 29
31. (((x ∪ y) = (y ∪ x)) -> ∀z.((z ε (x ∪ y)) <-> (z ε (y ∪ x)))) & (∀z.((z ε (x ∪ y)) <-
> (z ε (y ∪ x))) -> ((x ∪ y) = (y ∪ x)))  EquivExp 30
32. ∀z.((z ε (x ∪ y)) <-> (z ε (y ∪ x))) -> ((x ∪ y) = (y ∪ x))  AndElimR 31
33. (z ε (x ∪ y)) <-> (z ε (y ∪ x))  EquivConst 27
34. ∀z.((z ε (x ∪ y)) <-> (z ε (y ∪ x)))  ForallInt 33
35. (x ∪ y) = (y ∪ x)  ImpElim 34 32
36. z ε (x ∩ y)  Hyp
37. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 0
38. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 37
39. (z ε (x ∩ y)) -> ((z ε x) & (z ε y))  AndElimL 38
40. (z ε x) & (z ε y)  ImpElim 36 39
41. (A & B) -> (B & A)  TheoremInt
42. ((z ε x) & B) -> (B & (z ε x))  PolySub 41
43. ((z ε x) & (z ε y)) -> ((z ε y) & (z ε x))  PolySub 42
44. (z ε y) & (z ε x)  ImpElim 40 43
45. ((z ε x) & (z ε y)) -> (z ε (x ∩ y))  AndElimR 38
46. ∀w.(((z ε w) & (z ε y)) -> (z ε (w ∩ y)))  ForallInt 45
47. ∀v.∀w.(((z ε w) & (z ε v)) -> (z ε (w ∩ v)))  ForallInt 46
48. ∀w.(((z ε w) & (z ε x)) -> (z ε (w ∩ x)))  ForallElim 47
49. ((z ε y) & (z ε x)) -> (z ε (y ∩ x))  ForallElim 48
50. z ε (y ∩ x)  ImpElim 44 49
51. (z ε (x ∩ y)) -> (z ε (y ∩ x))  ImpInt 50
52. ∀v.((z ε (v ∩ y)) -> (z ε (y ∩ v)))  ForallInt 51
53. ∀w.∀v.((z ε (v ∩ w)) -> (z ε (w ∩ v)))  ForallInt 52
54. ∀v.((z ε (v ∩ x)) -> (z ε (x ∩ v)))  ForallElim 53
55. (z ε (y ∩ x)) -> (z ε (x ∩ y))  ForallElim 54
56. ((z ε (x ∩ y)) -> (z ε (y ∩ x))) & ((z ε (y ∩ x)) -> (z ε (x ∩ y)))  AndInt 51 55
57. ∀g.(((x ∩ y) = g) <-> ∀z.((z ε (x ∩ y)) <-> (z ε g)))  ForallElim 28
58. ((x ∩ y) = (y ∩ x)) <-> ∀z.((z ε (x ∩ y)) <-> (z ε (y ∩ x)))  ForallElim 57
59. (((x ∩ y) = (y ∩ x)) -> ∀z.((z ε (x ∩ y)) <-> (z ε (y ∩ x)))) & (∀z.((z ε (x ∩ y)) <-
> (z ε (y ∩ x))) -> ((x ∩ y) = (y ∩ x)))  EquivExp 58
60. ∀z.((z ε (x ∩ y)) <-> (z ε (y ∩ x))) -> ((x ∩ y) = (y ∩ x))  AndElimR 59
61. (z ε (x ∩ y)) <-> (z ε (y ∩ x))  EquivConst 56
62. ∀z.((z ε (x ∩ y)) <-> (z ε (y ∩ x)))  ForallInt 61
63. (x ∩ y) = (y ∩ x)  ImpElim 62 60
64. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))  AndInt 35 63 Qed
```

Used Theorems

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2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
1. (A v B) -> (B v A)
3. (A & B) -> (B & A)
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Th7. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))

```
0. w ε ((x ∪ y) ∪ z)  Hyp
1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
2. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  AndElimL 1
3. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 2
4. (z ε (x ∪ y)) -> ((z ε x) v (z ε y))  AndElimL 3
5. ∀z.((z ε (x ∪ y)) -> ((z ε x) v (z ε y)))  ForallInt 4
6. (w ε (x ∪ y)) -> ((w ε x) v (w ε y))  ForallElim 5
7. ∀x.((w ε (x ∪ y)) -> ((w ε x) v (w ε y)))  ForallInt 6
8. (w ε (a ∪ y)) -> ((w ε a) v (w ε y))  ForallElim 7
9. ∀y.((w ε (a ∪ y)) -> ((w ε a) v (w ε y)))  ForallInt 8
10. (w ε (a ∪ z)) -> ((w ε a) v (w ε z))  ForallElim 9
11. ∀a.((w ε (a ∪ z)) -> ((w ε a) v (w ε z)))  ForallInt 10
12. (w ε ((x ∪ y) ∪ z)) -> ((w ε (x ∪ y)) v (w ε z))  ForallElim 11
13. (w ε (x ∪ y)) v (w ε z)  ImpElim 0 12
14. w ε (x ∪ y)  Hyp
15. (w ε x) v (w ε y)  ImpElim 14 6
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16. ((w ε x) v (w ε y)) v (w ε z) OrIntR 15
17. w ε z Hyp
18. ((w ε x) v (w ε y)) v (w ε z) OrIntL 17
19. ((w ε x) v (w ε y)) v (w ε z) OrElim 13 14 16 17 18
20. ((A v B) v C) <-> (A v (B v C)) TheoremInt
21. (((w ε x) v B) v C) <-> ((w ε x) v (B v C)) PolySub 20
22. (((w ε x) v (w ε y)) v C) <-> ((w ε x) v ((w ε y) v C)) PolySub 21
23. (((w ε x) v (w ε y)) v (w ε z)) <-> ((w ε x) v ((w ε y) v (w ε z))) PolySub 22
24. ((((w ε x) v (w ε y)) v (w ε z)) -> ((w ε x) v ((w ε y) v (w ε z)))) & (((w ε x) v ((w ε y) v (w ε z))) -> (((w ε x) v (w ε y)) v (w ε z))) EquivExp 23
25. (((w ε x) v (w ε y)) v (w ε z)) -> ((w ε x) v ((w ε y) v (w ε z))) AndElimL 24
26. (w ε x) v ((w ε y) v (w ε z)) ImpElim 19 25
27. ((z ε x) v (z ε y)) -> (z ε (x U y)) AndElimR 3
28. ∀z.(((z ε x) v (z ε y)) -> (z ε (x U y))) ForallInt 27
29. ((w ε x) v (w ε y)) -> (w ε (x U y)) ForallElim 28
30. ∀x.(((w ε x) v (w ε y)) -> (w ε (x U y))) ForallInt 29
31. ((w ε a) v (w ε y)) -> (w ε (a U y)) ForallElim 30
32. ∀y.(((w ε a) v (w ε y)) -> (w ε (a U y))) ForallInt 31
33. ((w ε a) v (w ε z)) -> (w ε (a U z)) ForallElim 32
34. ∀a.(((w ε a) v (w ε z)) -> (w ε (a U z))) ForallInt 33
35. ((w ε y) v (w ε z)) -> (w ε (y U z)) ForallElim 34
36. (w ε y) v (w ε z) Hyp
37. w ε (y U z) ImpElim 36 35
38. (w ε x) v (w ε (y U z)) OrIntL 37
39. ∀y.(((w ε a) v (w ε y)) -> (w ε (a U y))) ForallInt 31
40. ((w ε a) v (w ε (y U z))) -> (w ε (a U (y U z))) ForallElim 32
41. ∀a.(((w ε a) v (w ε (y U z))) -> (w ε (a U (y U z)))) ForallInt 40
42. ((w ε x) v (w ε (y U z))) -> (w ε (x U (y U z))) ForallElim 41
43. w ε (x U (y U z)) ImpElim 38 42
44. w ε x Hyp
45. (w ε x) v (w ε (y U z)) OrIntR 44
46. ∀y.(((w ε a) v (w ε y)) -> (w ε (a U y))) ForallInt 31
47. ((w ε a) v (w ε (y U z))) -> (w ε (a U (y U z))) ForallElim 32
48. ∀a.(((w ε a) v (w ε (y U z))) -> (w ε (a U (y U z)))) ForallInt 47
49. ((w ε x) v (w ε (y U z))) -> (w ε (x U (y U z))) ForallElim 48
50. w ε (x U (y U z)) ImpElim 45 49
51. w ε (x U (y U z)) OrElim 26 44 50 36 43
52. (w ε ((x U y) U z)) -> (w ε (x U (y U z))) ImpInt 51
53. w ε (x U (y U z)) Hyp
54. ∀y.((w ε (a U y)) -> ((w ε a) v (w ε y))) ForallInt 8
55. (w ε (a U (y U z))) -> ((w ε a) v (w ε (y U z))) ForallElim 9
56. ∀a.((w ε (a U (y U z))) -> ((w ε a) v (w ε (y U z)))) ForallInt 55
57. (w ε (x U (y U z))) -> ((w ε x) v (w ε (y U z))) ForallElim 56
58. (w ε x) v (w ε (y U z)) ImpElim 53 57
59. w ε x Hyp
60. (w ε x) v ((w ε y) v (w ε z)) OrIntR 59
61. w ε (y U z) Hyp
62. ∀a.((w ε (a U z)) -> ((w ε a) v (w ε z))) ForallInt 10
63. (w ε (y U z)) -> ((w ε y) v (w ε z)) ForallElim 11
64. (w ε y) v (w ε z) ImpElim 61 63
65. (w ε x) v ((w ε y) v (w ε z)) OrIntL 64
66. (w ε x) v ((w ε y) v (w ε z)) OrElim 58 59 60 61 65
67. ((w ε x) v ((w ε y) v (w ε z))) -> (((w ε x) v (w ε y)) v (w ε z)) AndElimR 24
68. ((w ε x) v (w ε y)) v (w ε z) ImpElim 66 67
69. (w ε x) v (w ε y) Hyp
70. ∀z.(((z ε x) v (z ε y)) -> (z ε (x U y))) ForallInt 27
71. ((w ε x) v (w ε y)) -> (w ε (x U y)) ForallElim 28
72. w ε (x U y) ImpElim 69 71
73. (w ε (x U y)) v (w ε z) OrIntR 72
74. w ε z Hyp
75. (w ε (x U y)) v (w ε z) OrIntL 74
76. (w ε (x U y)) v (w ε z) OrElim 68 69 73 74 75
77. ∀a.(((w ε a) v (w ε z)) -> (w ε (a U z))) ForallInt 33
78. ((w ε (x U y)) v (w ε z)) -> (w ε ((x U y) U z)) ForallElim 34
79. w ε ((x U y) U z) ImpElim 76 78
80. (w ε (x U (y U z))) -> (w ε ((x U y) U z)) ImpInt 79
81. ((w ε ((x U y) U z)) -> (w ε (x U (y U z)))) & ((w ε (x U (y U z))) -> (w ε ((x U y) U z))) AndInt 52 80
82. (w ε ((x U y) U z)) <-> (w ε (x U (y U z))) EquivConst 81
83. w ε ((x ∩ y) ∩ z) Hyp
84. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 1

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85. ∀z.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))  ForallInt 84
86. (w ε (x ∩ y)) <-> ((w ε x) & (w ε y))  ForallElim 85
87. ∀x.((w ε (x ∩ y)) <-> ((w ε x) & (w ε y)))  ForallInt 86
88. (w ε (a ∩ y)) <-> ((w ε a) & (w ε y))  ForallElim 87
89. ∀y.((w ε (a ∩ y)) <-> ((w ε a) & (w ε y)))  ForallInt 88
90. (w ε (a ∩ b)) <-> ((w ε a) & (w ε b))  ForallElim 89
91. ∀a.((w ε (a ∩ b)) <-> ((w ε a) & (w ε b)))  ForallInt 90
92. (w ε ((x ∩ y) ∩ b)) <-> ((w ε (x ∩ y)) & (w ε b))  ForallElim 91
93. ∀b.((w ε ((x ∩ y) ∩ b)) <-> ((w ε (x ∩ y)) & (w ε b)))  ForallInt 92
94. (w ε ((x ∩ y) ∩ z)) <-> ((w ε (x ∩ y)) & (w ε z))  ForallElim 93
95. ((w ε ((x ∩ y) ∩ z)) -> ((w ε (x ∩ y)) & (w ε z))) & (((w ε (x ∩ y)) & (w ε z)) -> (w
ε ((x ∩ y) ∩ z)))  EquivExp 94
96. (w ε ((x ∩ y) ∩ z)) -> ((w ε (x ∩ y)) & (w ε z))  AndElimL 95
97. (w ε (x ∩ y)) & (w ε z)  ImpElim 83 96
98. w ε (x ∩ y)  AndElimL 97
99. ((w ε (x ∩ y)) -> ((w ε x) & (w ε y))) & (((w ε x) & (w ε y)) -> (w ε (x ∩ y)))
EquivExp 86
100. (w ε (x ∩ y)) -> ((w ε x) & (w ε y))  AndElimL 99
101. (w ε x) & (w ε y)  ImpElim 98 100
102. w ε z  AndElimR 97
103. w ε x  AndElimL 101
104. w ε y  AndElimR 101
105. (w ε y) & (w ε z)  AndInt 104 102
106. ((w ε (a ∩ b)) -> ((w ε a) & (w ε b))) & (((w ε a) & (w ε b)) -> (w ε (a ∩ b)))
EquivExp 90
107. ((w ε a) & (w ε b)) -> (w ε (a ∩ b))  AndElimR 106
108. ∀a.(((w ε a) & (w ε b)) -> (w ε (a ∩ b)))  ForallInt 107
109. ((w ε y) & (w ε b)) -> (w ε (y ∩ b))  ForallElim 108
110. ∀b.(((w ε y) & (w ε b)) -> (w ε (y ∩ b)))  ForallInt 109
111. ((w ε y) & (w ε z)) -> (w ε (y ∩ z))  ForallElim 110
112. w ε (y ∩ z)  ImpElim 105 111
113. (w ε x) & (w ε (y ∩ z))  AndInt 103 112
114. ∀a.(((w ε a) & (w ε b)) -> (w ε (a ∩ b)))  ForallInt 107
115. ((w ε x) & (w ε b)) -> (w ε (x ∩ b))  ForallElim 108
116. ∀b.(((w ε x) & (w ε b)) -> (w ε (x ∩ b)))  ForallInt 115
117. ((w ε x) & (w ε (y ∩ z))) -> (w ε (x ∩ (y ∩ z)))  ForallElim 116
118. w ε (x ∩ (y ∩ z))  ImpElim 113 117
119. (w ε ((x ∩ y) ∩ z)) -> (w ε (x ∩ (y ∩ z)))  ImpInt 118
120. w ε (x ∩ (y ∩ z))  Hyp
121. (w ε (a ∩ b)) -> ((w ε a) & (w ε b))  AndElimL 106
122. ∀a.((w ε (a ∩ b)) -> ((w ε a) & (w ε b)))  ForallInt 121
123. (w ε (x ∩ b)) -> ((w ε x) & (w ε b))  ForallElim 122
124. ∀b.((w ε (x ∩ b)) -> ((w ε x) & (w ε b)))  ForallInt 123
125. ∀b.((w ε (x ∩ b)) -> ((w ε x) & (w ε b)))  ForallInt 123
126. (w ε (x ∩ (y ∩ z))) -> ((w ε x) & (w ε (y ∩ z)))  ForallElim 124
127. (w ε x) & (w ε (y ∩ z))  ImpElim 120 126
128. w ε (y ∩ z)  AndElimR 127
129. w ε x  AndElimL 127
130. ∀a.((w ε (a ∩ b)) -> ((w ε a) & (w ε b)))  ForallInt 121
131. (w ε (y ∩ b)) -> ((w ε y) & (w ε b))  ForallElim 122
132. ∀b.((w ε (y ∩ b)) -> ((w ε y) & (w ε b)))  ForallInt 131
133. (w ε (y ∩ z)) -> ((w ε y) & (w ε z))  ForallElim 132
134. (w ε y) & (w ε z)  ImpElim 128 133
135. w ε y  AndElimL 134
136. w ε z  AndElimR 134
137. (w ε x) & (w ε y)  AndInt 129 135
138. ((w ε x) & (w ε y)) -> (w ε (x ∩ y))  AndElimR 99
139. w ε (x ∩ y)  ImpElim 137 138
140. (w ε (x ∩ y)) & (w ε z)  AndInt 139 136
141. ∀a.((w ε (a ∩ b)) -> ((w ε a) & (w ε b)))  ForallInt 121
142. ∀a.(((w ε a) & (w ε b)) -> (w ε (a ∩ b)))  ForallInt 107
143. ((w ε (x ∩ y)) & (w ε b)) -> (w ε ((x ∩ y) ∩ b))  ForallElim 108
144. ∀b.(((w ε (x ∩ y)) & (w ε b)) -> (w ε ((x ∩ y) ∩ b)))  ForallInt 143
145. ((w ε (x ∩ y)) & (w ε z)) -> (w ε ((x ∩ y) ∩ z))  ForallElim 144
146. w ε ((x ∩ y) ∩ z)  ImpElim 140 145
147. (w ε (x ∩ (y ∩ z))) -> (w ε ((x ∩ y) ∩ z))  ImpInt 146
148. ((w ε ((x ∩ y) ∩ z)) -> (w ε (x ∩ (y ∩ z)))) & ((w ε (x ∩ (y ∩ z))) -> (w ε ((x ∩ y)
∩ z)))  AndInt 119 147
149. (w ε ((x ∩ y) ∩ z)) <-> (w ε (x ∩ (y ∩ z)))  EquivConst 148
150. ((w ε ((x ∪ y) ∪ z)) <-> (w ε (x ∪ (y ∪ z)))) & ((w ε ((x ∩ y) ∩ z)) <-> (w ε (x ∩
(y ∩ z))))  AndInt 82 149
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151. (w ε ((x ∩ y) ∩ z)) <-> (w ε (x ∩ (y ∩ z))) AndElimR 150
152. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
153. ∀h.((((x ∩ y) ∩ z) = h) <-> ∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε h))) ForallElim 152
154. (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) <-> ∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε (x ∩ (y ∩ z)))) ForallElim 153
155. ∀w.((w ε ((x ∩ y) ∩ z)) <-> (w ε (x ∩ (y ∩ z)))) ForallInt 151
156. ((((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) -> ∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε (x ∩ (y ∩ z))))) & (∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε (x ∩ (y ∩ z)))) -> (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))) EquivExp 154
157. ∀i.((i ε ((x ∩ y) ∩ z)) <-> (i ε (x ∩ (y ∩ z)))) -> (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) AndElimR 156
158. ((x ∩ y) ∩ z) = (x ∩ (y ∩ z)) ImpElim 155 157
159. ∀j.((((x ∪ y) ∪ z) = j) <-> ∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε j))) ForallElim 152
160. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) <-> ∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε (x ∪ (y ∪ z)))) ForallElim 159
161. ((((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) -> ∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε (x ∪ (y ∪ z))))) & (∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε (x ∪ (y ∪ z)))) -> (((x ∪ y) ∪ z) = (x ∪ (y ∪ z)))) EquivExp 160
162. ∀k.((k ε ((x ∪ y) ∪ z)) <-> (k ε (x ∪ (y ∪ z)))) -> (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) AndElimR 161
163. (w ε ((x ∪ y) ∪ z)) <-> (w ε (x ∪ (y ∪ z))) AndElimL 150
164. ∀w.((w ε ((x ∪ y) ∪ z)) <-> (w ε (x ∪ (y ∪ z)))) ForallInt 163
165. ((x ∪ y) ∪ z) = (x ∪ (y ∪ z)) ImpElim 164 162
166. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) AndInt 165 158
Qed

Used Theorems

3. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
1. ((A v B) v C) <-> (A v (B v C))

Th8. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))

0. w ε (x ∩ (y ∪ z)) Hyp
1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
2. ∀z.(((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))) ForallInt 1
3. ((w ε (x ∪ y)) <-> ((w ε x) v (w ε y))) & ((w ε (x ∩ y)) <-> ((w ε x) & (w ε y))) ForallElim 2
4. ∀y.(((w ε (x ∪ y)) <-> ((w ε x) v (w ε y))) & ((w ε (x ∩ y)) <-> ((w ε x) & (w ε y)))) ForallInt 3
5. ((w ε (x ∪ a)) <-> ((w ε x) v (w ε a))) & ((w ε (x ∩ a)) <-> ((w ε x) & (w ε a))) ForallElim 4
6. (w ε (x ∩ a)) <-> ((w ε x) & (w ε a)) AndElimR 5
7. ((w ε (x ∩ a)) -> ((w ε x) & (w ε a))) & (((w ε x) & (w ε a)) -> (w ε (x ∩ a))) EquivExp 6
8. (w ε (x ∩ a)) -> ((w ε x) & (w ε a)) AndElimL 7
9. ∀a.((w ε (x ∩ a)) -> ((w ε x) & (w ε a))) ForallInt 8
10. (w ε (x ∩ (y ∪ z))) -> ((w ε x) & (w ε (y ∪ z))) ForallElim 9
11. (w ε x) & (w ε (y ∪ z)) ImpElim 0 10
12. w ε (y ∪ z) AndElimR 11
13. w ε x AndElimL 11
14. (w ε (x ∪ a)) <-> ((w ε x) v (w ε a)) AndElimL 5
15. ∀x.((w ε (x ∪ a)) <-> ((w ε x) v (w ε a))) ForallInt 14
16. (w ε (b ∪ a)) <-> ((w ε b) v (w ε a)) ForallElim 15
17. ∀b.((w ε (b ∪ a)) <-> ((w ε b) v (w ε a))) ForallInt 16
18. (w ε (y ∪ a)) <-> ((w ε y) v (w ε a)) ForallElim 17
19. ∀a.((w ε (y ∪ a)) <-> ((w ε y) v (w ε a))) ForallInt 18
20. (w ε (y ∪ z)) <-> ((w ε y) v (w ε z)) ForallElim 19
21. ((w ε (y ∪ z)) -> ((w ε y) v (w ε z))) & (((w ε y) v (w ε z)) -> (w ε (y ∪ z))) EquivExp 20
22. (w ε (y ∪ z)) -> ((w ε y) v (w ε z)) AndElimL 21
23. (w ε y) v (w ε z) ImpElim 12 22
24. (w ε x) & ((w ε y) v (w ε z)) AndInt 13 23
25. (A & (B v C)) <-> ((A & B) v (A & C)) TheoremInt
26. ((w ε x) & (B v C)) <-> (((w ε x) & B) v ((w ε x) & C)) PolySub 25
27. ((w ε x) & ((w ε y) v C)) <-> (((w ε x) & (w ε y)) v ((w ε x) & C)) PolySub 26
28. ((w ε x) & ((w ε y) v (w ε z))) <-> (((w ε x) & (w ε y)) v ((w ε x) & (w ε z))) PolySub 27
29. (((w ε x) & ((w ε y) v (w ε z))) -> (((w ε x) & (w ε y)) v ((w ε x) & (w ε z)))) &

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((((w ε x) & (w ε y)) v ((w ε x) & (w ε z))) -> ((w ε x) & ((w ε y) v (w ε z))))
EquivExp 28
30. ((w ε x) & ((w ε y) v (w ε z))) -> (((w ε x) & (w ε y)) v ((w ε x) & (w ε z)))
AndElimL 29
31. ((w ε x) & (w ε y)) v ((w ε x) & (w ε z))  ImpElim 24 30
32. (w ε x) & (w ε y)  Hyp
33. (w ε (x ∩ y)) <-> ((w ε x) & (w ε y))  AndElimR 3
34. ((w ε (x ∩ y)) -> ((w ε x) & (w ε y))) & (((w ε x) & (w ε y)) -> (w ε (x ∩ y)))
EquivExp 33
35. ((w ε x) & (w ε y)) -> (w ε (x ∩ y))  AndElimR 34
36. w ε (x ∩ y)  ImpElim 32 35
37. (w ε (x ∩ y)) v (w ε (x ∩ z))  OrIntR 36
38. (w ε x) & (w ε z)  Hyp
39. ∀y.(((w ε x) & (w ε y)) -> (w ε (x ∩ y)))  ForallInt 35
40. ((w ε x) & (w ε z)) -> (w ε (x ∩ z))  ForallElim 39
41. w ε (x ∩ z)  ImpElim 38 40
42. (w ε (x ∩ y)) v (w ε (x ∩ z))  OrIntL 41
43. (w ε (x ∩ y)) v (w ε (x ∩ z))  OrElim 31 32 37 38 42
44. ((w ε (b U a)) -> ((w ε b) v (w ε a))) & (((w ε b) v (w ε a)) -> (w ε (b U a)))
EquivExp 16
45. ((w ε b) v (w ε a)) -> (w ε (b U a))  AndElimR 44
46. ∀b.(((w ε b) v (w ε a)) -> (w ε (b U a)))  ForallInt 45
47. ((w ε (x ∩ y)) v (w ε a)) -> (w ε ((x ∩ y) U a))  ForallElim 46
48. ∀a.(((w ε (x ∩ y)) v (w ε a)) -> (w ε ((x ∩ y) U a)))  ForallInt 47
49. ((w ε (x ∩ y)) v (w ε (x ∩ z))) -> (w ε ((x ∩ y) U (x ∩ z)))  ForallElim 48
50. w ε ((x ∩ y) U (x ∩ z))  ImpElim 43 49
51. (w ε (x ∩ (y U z))) -> (w ε ((x ∩ y) U (x ∩ z)))  ImpInt 50
52. w ε ((x ∩ y) U (x ∩ z))  Hyp
53. (w ε (b U a)) -> ((w ε b) v (w ε a))  AndElimL 44
54. ∀b.((w ε (b U a)) -> ((w ε b) v (w ε a)))  ForallInt 53
55. (w ε ((x ∩ y) U a)) -> ((w ε (x ∩ y)) v (w ε a))  ForallElim 54
56. ∀a.((w ε ((x ∩ y) U a)) -> ((w ε (x ∩ y)) v (w ε a)))  ForallInt 55
57. (w ε ((x ∩ y) U (x ∩ z))) -> ((w ε (x ∩ y)) v (w ε (x ∩ z)))  ForallElim 56
58. (w ε (x ∩ y)) v (w ε (x ∩ z))  ImpElim 52 57
59. ∀a.((w ε (x ∩ a)) -> ((w ε x) & (w ε a)))  ForallInt 8
60. (w ε (x ∩ y)) -> ((w ε x) & (w ε y))  ForallElim 9
61. ∀a.((w ε (x ∩ a)) -> ((w ε x) & (w ε a)))  ForallInt 8
62. (w ε (x ∩ z)) -> ((w ε x) & (w ε z))  ForallElim 9
63. w ε (x ∩ y)  Hyp
64. (w ε x) & (w ε y)  ImpElim 63 60
65. w ε y  AndElimR 64
66. (w ε y) v (w ε z)  OrIntR 65
67. ((w ε b) v (w ε a)) -> (w ε (b U a))  AndElimR 44
68. ∀b.(((w ε b) v (w ε a)) -> (w ε (b U a)))  ForallInt 67
69. ((w ε y) v (w ε a)) -> (w ε (y U a))  ForallElim 68
70. ∀a.(((w ε y) v (w ε a)) -> (w ε (y U a)))  ForallInt 69
71. ((w ε y) v (w ε z)) -> (w ε (y U z))  ForallElim 70
72. w ε (y U z)  ImpElim 66 71
73. w ε x  AndElimL 64
74. (w ε x) & (w ε (y U z))  AndInt 73 72
75. ((w ε x) & (w ε a)) -> (w ε (x ∩ a))  AndElimR 7
76. ∀a.(((w ε x) & (w ε a)) -> (w ε (x ∩ a)))  ForallInt 75
77. ((w ε x) & (w ε (y U z))) -> (w ε (x ∩ (y U z)))  ForallElim 76
78. w ε (x ∩ (y U z))  ImpElim 74 77
79. w ε (x ∩ z)  Hyp
80. (w ε x) & (w ε z)  ImpElim 79 62
81. w ε x  AndElimL 80
82. w ε z  AndElimR 80
83. (w ε y) v (w ε z)  OrIntL 82
84. w ε (y U z)  ImpElim 83 71
85. (w ε x) & (w ε (y U z))  AndInt 81 84
86. w ε (x ∩ (y U z))  ImpElim 85 77
87. w ε (x ∩ (y U z))  OrElim 58 63 78 79 86
88. (w ε ((x ∩ y) U (x ∩ z))) -> (w ε (x ∩ (y U z)))  ImpInt 87
89. ((w ε (x ∩ (y U z))) -> (w ε ((x ∩ y) U (x ∩ z)))) & ((w ε ((x ∩ y) U (x ∩ z))) -> (w
ε (x ∩ (y U z))))  AndInt 51 88
90. (w ε (x ∩ (y U z))) <-> (w ε ((x ∩ y) U (x ∩ z)))  EquivConst 89
91. w ε (x U (y ∩ z))  Hyp
92. ((w ε (b U a)) -> ((w ε b) v (w ε a))) & (((w ε b) v (w ε a)) -> (w ε (b U a)))
EquivExp 16
93. ∀b.(((w ε (b U a)) -> ((w ε b) v (w ε a))) & (((w ε b) v (w ε a)) -> (w ε (b U a))))
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ForallInt 92
94. ((w ε (x U a)) -> ((w ε x) v (w ε a))) & (((w ε x) v (w ε a)) -> (w ε (x U a))) ForallElim 93
95. ∀a.(((w ε (x U a)) -> ((w ε x) v (w ε a))) & (((w ε x) v (w ε a)) -> (w ε (x U a)))) ForallInt 94
96. ((w ε (x U (y ∩ z))) -> ((w ε x) v (w ε (y ∩ z)))) & (((w ε x) v (w ε (y ∩ z))) -> (w ε (x U (y ∩ z)))) ForallElim 95
97. (w ε (x U (y ∩ z))) -> ((w ε x) v (w ε (y ∩ z))) AndElimL 96
98. (w ε x) v (w ε (y ∩ z)) ImpElim 91 97
99. w ε x Hyp
100. (w ε x) v (w ε y) OrIntR 99
101. ((w ε b) v (w ε a)) -> (w ε (b U a)) AndElimR 92
102. ∀b.(((w ε b) v (w ε a)) -> (w ε (b U a))) ForallInt 101
103. ((w ε x) v (w ε a)) -> (w ε (x U a)) ForallElim 102
104. ∀a.(((w ε x) v (w ε a)) -> (w ε (x U a))) ForallInt 103
105. ((w ε x) v (w ε y)) -> (w ε (x U y)) ForallElim 104
106. w ε (x U y) ImpElim 100 105
107. (w ε x) v (w ε z) OrIntR 99
108. ∀a.(((w ε x) v (w ε a)) -> (w ε (x U a))) ForallInt 103
109. ((w ε x) v (w ε z)) -> (w ε (x U z)) ForallElim 104
110. w ε (x U z) ImpElim 107 109
111. (w ε (x U y)) & (w ε (x U z)) AndInt 106 110
112. ∀x.((w ε (x ∩ a)) <-> ((w ε x) & (w ε a))) ForallInt 6
113. (w ε (b ∩ a)) <-> ((w ε b) & (w ε a)) ForallElim 112
114. ((w ε (b ∩ a)) -> ((w ε b) & (w ε a))) & (((w ε b) & (w ε a)) -> (w ε (b ∩ a))) EquivExp 113
115. ((w ε b) & (w ε a)) -> (w ε (b ∩ a)) AndElimR 114
116. ∀b.(((w ε b) & (w ε a)) -> (w ε (b ∩ a))) ForallInt 115
117. ((w ε (x U y)) & (w ε a)) -> (w ε ((x U y) ∩ a)) ForallElim 116
118. ∀a.(((w ε (x U y)) & (w ε a)) -> (w ε ((x U y) ∩ a))) ForallInt 117
119. ((w ε (x U y)) & (w ε (x U z))) -> (w ε ((x U y) ∩ (x U z))) ForallElim 118
120. w ε ((x U y) ∩ (x U z)) ImpElim 111 119
121. w ε (y ∩ z) Hyp
122. (w ε (b ∩ a)) -> ((w ε b) & (w ε a)) AndElimL 114
123. ∀b.((w ε (b ∩ a)) -> ((w ε b) & (w ε a))) ForallInt 122
124. (w ε (y ∩ a)) -> ((w ε y) & (w ε a)) ForallElim 123
125. ∀a.((w ε (y ∩ a)) -> ((w ε y) & (w ε a))) ForallInt 124
126. (w ε (y ∩ z)) -> ((w ε y) & (w ε z)) ForallElim 125
127. (w ε y) & (w ε z) ImpElim 121 126
128. w ε y AndElimL 127
129. w ε z AndElimR 127
130. (w ε x) v (w ε y) OrIntL 128
131. (w ε x) v (w ε z) OrIntL 129
132. w ε (x U z) ImpElim 131 109
133. (z ε (x U y)) <-> ((z ε x) v (z ε y)) AndElimL 1
134. ((z ε (x U y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x U y))) EquivExp 133
135. ((z ε x) v (z ε y)) -> (z ε (x U y)) AndElimR 134
136. ∀z.(((z ε x) v (z ε y)) -> (z ε (x U y))) ForallInt 135
137. ((w ε x) v (w ε y)) -> (w ε (x U y)) ForallElim 136
138. w ε (x U y) ImpElim 130 137
139. (w ε (x U y)) & (w ε (x U z)) AndInt 138 132
140. w ε ((x U y) ∩ (x U z)) ImpElim 139 119
141. w ε ((x U y) ∩ (x U z)) OrElim 98 99 120 121 140
142. (w ε (x U (y ∩ z))) -> (w ε ((x U y) ∩ (x U z))) ImpInt 141
143. w ε ((x U y) ∩ (x U z)) Hyp
144. (w ε (b ∩ a)) -> ((w ε b) & (w ε a)) AndElimL 114
145. ∀b.(((w ε (b ∩ a)) -> ((w ε b) & (w ε a))) & (((w ε b) & (w ε a)) -> (w ε (b ∩ a)))) ForallInt 114
146. ((w ε ((x U y) ∩ a)) -> ((w ε (x U y)) & (w ε a))) & (((w ε (x U y)) & (w ε a)) -> (w ε ((x U y) ∩ a))) ForallElim 145
147. ∀a.(((w ε ((x U y) ∩ a)) -> ((w ε (x U y)) & (w ε a))) & (((w ε (x U y)) & (w ε a)) -> (w ε ((x U y) ∩ a)))) ForallInt 146
148. ((w ε ((x U y) ∩ (x U z))) -> ((w ε (x U y)) & (w ε (x U z)))) & (((w ε (x U y)) & (w ε (x U z))) -> (w ε ((x U y) ∩ (x U z)))) ForallElim 147
149. (w ε ((x U y) ∩ (x U z))) -> ((w ε (x U y)) & (w ε (x U z))) AndElimL 148
150. (w ε (x U y)) & (w ε (x U z)) ImpElim 143 149
151. w ε (x U y) AndElimL 150
152. w ε (x U z) AndElimR 150
153. (z ε (x U y)) -> ((z ε x) v (z ε y)) AndElimL 134
154. ∀z.((z ε (x U y)) -> ((z ε x) v (z ε y))) ForallInt 153

155. (w ε (x ∪ y)) -> ((w ε x) v (w ε y)) ForallElim 154
156. ∀y.((w ε (x ∪ y)) -> ((w ε x) v (w ε y))) ForallInt 155
157. (w ε (x ∪ z)) -> ((w ε x) v (w ε z)) ForallElim 156
158. (w ε x) v (w ε y) ImpElim 151 155
159. (w ε x) v (w ε z) ImpElim 152 157
160. w ε x Hyp
161. (w ε x) v (w ε (y ∩ z)) OrIntR 160
162. ((w ε (x ∪ a)) -> ((w ε x) v (w ε a))) & (((w ε x) v (w ε a)) -> (w ε (x ∪ a))) EquivExp 14
163. ((w ε x) v (w ε a)) -> (w ε (x ∪ a)) AndElimR 162
164. ∀a.(((w ε x) v (w ε a)) -> (w ε (x ∪ a))) ForallInt 163
165. ((w ε x) v (w ε (y ∩ z))) -> (w ε (x ∪ (y ∩ z))) ForallElim 164
166. w ε (x ∪ (y ∩ z)) ImpElim 161 165
167. (w ε x) -> (w ε (x ∪ (y ∩ z))) ImpInt 166
168. w ε y Hyp
169. w ε x Hyp
170. w ε (x ∪ (y ∩ z)) ImpElim 169 167
171. w ε z Hyp
172. (w ε y) & (w ε z) AndInt 168 171
173. ∀a.(((w ε b) & (w ε a)) -> (w ε (b ∩ a))) ForallInt 115
174. ((w ε y) & (w ε a)) -> (w ε (y ∩ a)) ForallElim 116
175. ∀a.(((w ε y) & (w ε a)) -> (w ε (y ∩ a))) ForallInt 174
176. ((w ε y) & (w ε z)) -> (w ε (y ∩ z)) ForallElim 175
177. w ε (y ∩ z) ImpElim 172 176
178. (w ε x) v (w ε (y ∩ z)) OrIntL 177
179. w ε (x ∪ (y ∩ z)) ImpElim 178 165
180. w ε (x ∪ (y ∩ z)) OrElim 159 169 170 171 179
181. w ε (x ∪ (y ∩ z)) OrElim 158 160 166 168 180
182. (w ε ((x ∪ y) ∩ (x ∪ z))) -> (w ε (x ∪ (y ∩ z))) ImpInt 181
183. ((w ε (x ∪ (y ∩ z))) -> (w ε ((x ∪ y) ∩ (x ∪ z)))) & ((w ε ((x ∪ y) ∩ (x ∪ z))) -> (w ε (x ∪ (y ∩ z)))) AndInt 142 182
184. (w ε (x ∪ (y ∩ z))) <-> (w ε ((x ∪ y) ∩ (x ∪ z))) EquivConst 183
185. ((w ε (x ∩ (y ∪ z))) <-> (w ε ((x ∩ y) ∪ (x ∩ z)))) & ((w ε (x ∪ (y ∩ z))) <-> (w ε ((x ∪ y) ∩ (x ∪ z)))) AndInt 90 184
186. (w ε (x ∪ (y ∩ z))) <-> (w ε ((x ∪ y) ∩ (x ∪ z))) AndElimR 185
187. (w ε (x ∩ (y ∪ z))) <-> (w ε ((x ∩ y) ∪ (x ∩ z))) AndElimL 185
188. ∀w.((w ε (x ∪ (y ∩ z))) <-> (w ε ((x ∪ y) ∩ (x ∪ z)))) ForallInt 186
189. ∀w.((w ε (x ∩ (y ∪ z))) <-> (w ε ((x ∩ y) ∪ (x ∩ z)))) ForallInt 187
190. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
191. ∀j.(((x ∩ (y ∪ z)) = j) <-> ∀k.((k ε (x ∩ (y ∪ z))) <-> (k ε j))) ForallElim 190
192. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) <-> ∀k.((k ε (x ∩ (y ∪ z))) <-> (k ε ((x ∩ y) ∪ (x ∩ z)))) ForallElim 191
193. (((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) -> ∀k.((k ε (x ∩ (y ∪ z))) <-> (k ε ((x ∩ y) ∪ (x ∩ z))))) & (∀k.((k ε (x ∩ (y ∪ z))) <-> (k ε ((x ∩ y) ∪ (x ∩ z)))) -> ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z)))) EquivExp 192
194. ∀k.((k ε (x ∩ (y ∪ z))) <-> (k ε ((x ∩ y) ∪ (x ∩ z)))) -> ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) AndElimR 193
195. (x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z)) ImpElim 189 194
196. ∀l.(((x ∪ (y ∩ z)) = l) <-> ∀m.((m ε (x ∪ (y ∩ z))) <-> (m ε l))) ForallElim 190
197. ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))) <-> ∀m.((m ε (x ∪ (y ∩ z))) <-> (m ε ((x ∪ y) ∩ (x ∪ z)))) ForallElim 196
198. (((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))) -> ∀m.((m ε (x ∪ (y ∩ z))) <-> (m ε ((x ∪ y) ∩ (x ∪ z))))) & (∀m.((m ε (x ∪ (y ∩ z))) <-> (m ε ((x ∪ y) ∩ (x ∪ z)))) -> ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))) EquivExp 197
199. ∀m.((m ε (x ∪ (y ∩ z))) <-> (m ε ((x ∪ y) ∩ (x ∪ z)))) -> ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))) AndElimR 198
200. (x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)) ImpElim 188 199
201. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))) AndInt 195 200 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
2. (A & (B v C)) <-> ((A & B) v (A & C))

Th11. ~~x = x

0. z ε ~~x Hyp
1. ~x = {y: ¬(y ε x)} DefEqInt
2. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1
3. ~~x = {y: ¬(y ε ~x)} ForallElim 2

4. z ε {y: ¬(y ε ~x)} EqualitySub 0 3
5. Set(z) & ¬(z ε ~x) ClassElim 4
6. ¬(z ε ~x) AndElimR 5
7. ¬(z ε x) Hyp
8. Set(z) AndElimL 5
9. Set(z) & ¬(z ε x) AndInt 8 7
10. z ε {y: ¬(y ε x)} ClassInt 9
11. {y: ¬(y ε x)} = ~x Symmetry 1
12. z ε ~x EqualitySub 10 11
13. _|_ ImpElim 12 6
14. ¬¬(z ε x) ImpInt 13
15. D <-> ¬¬D TheoremInt
16. (z ε x) <-> ¬¬(z ε x) PolySub 15
17. ((z ε x) -> ¬¬(z ε x)) & (¬¬(z ε x) -> (z ε x)) EquivExp 16
18. ¬¬(z ε x) -> (z ε x) AndElimR 17
19. z ε x ImpElim 14 18
20. (z ε ~~x) -> (z ε x) ImpInt 19
21. z ε x Hyp
22. (z ε x) -> ¬¬(z ε x) AndElimL 17
23. ¬¬(z ε x) ImpElim 21 22
24. z ε ~x Hyp
25. z ε {y: ¬(y ε x)} EqualitySub 24 1
26. Set(z) & ¬(z ε x) ClassElim 25
27. ¬(z ε x) AndElimR 26
28. _|_ ImpElim 27 23
29. ¬(z ε ~x) ImpInt 28
30. ∃y.(z ε y) ExistsInt 21
31. Set(z) DefSub 30
32. Set(z) & ¬(z ε ~x) AndInt 31 29
33. z ε {y: ¬(y ε ~x)} ClassInt 32
34. {y: ¬(y ε ~x)} = ~~x Symmetry 3
35. z ε ~~x EqualitySub 33 34
36. (z ε x) -> (z ε ~~x) ImpInt 35
37. ((z ε ~~x) -> (z ε x)) & ((z ε x) -> (z ε ~~x)) AndInt 20 36
38. (z ε ~~x) <-> (z ε x) EquivConst 37
39. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
40. ∀y.((~~x = y) <-> ∀z.((z ε ~~x) <-> (z ε y))) ForallElim 39
41. (~~x = x) <-> ∀z.((z ε ~~x) <-> (z ε x)) ForallElim 40
42. ((~~x = x) -> ∀z.((z ε ~~x) <-> (z ε x))) & (∀z.((z ε ~~x) <-> (z ε x)) -> (~~x = x))
EquivExp 41
43. ∀z.((z ε ~~x) <-> (z ε x)) -> (~~x = x) AndElimR 42
44. ∀z.((z ε ~~x) <-> (z ε x)) ForallInt 38
45. ~~x = x ImpElim 44 43 Qed

Used Theorems

1. D <-> ¬¬D

Th12. (~(x ∪ y) = (~x ∩ ~y)) & (~(x ∩ y) = (~x ∪ ~y))

0. z ε ~(x ∪ y) Hyp
1. ~x = {y: ¬(y ε x)} DefEqInt
2. ∀a.(~a = {y: ¬(y ε a)}) ForallInt 1
3. ~(x ∪ y) = {t: ¬(t ε (x ∪ y))} ForallElim 2
4. z ε {t: ¬(t ε (x ∪ y))} EqualitySub 0 3
5. Set(z) & ¬(z ε (x ∪ y)) ClassElim 4
6. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
7. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 6
8. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 7
9. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 8
10. (A -> B) -> (¬B -> ¬A) TheoremInt
11. (((z ε x) v (z ε y)) -> B) -> (¬B -> ¬((z ε x) v (z ε y))) PolySub 10
12. (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) -> (¬(z ε (x ∪ y)) -> ¬((z ε x) v (z ε y)))
PolySub 11
13. ¬(z ε (x ∪ y)) -> ¬((z ε x) v (z ε y)) ImpElim 9 12
14. ¬(z ε (x ∪ y)) AndElimR 5
15. ¬((z ε x) v (z ε y)) ImpElim 14 13
16. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B)) TheoremInt
17. (¬((z ε x) v B) <-> (¬(z ε x) & ¬B)) & (¬((z ε x) & B) <-> (¬(z ε x) v ¬B)) PolySub

18. (¬((z ε x) v (z ε y)) <-> (¬(z ε x) & ¬(z ε y))) & (¬((z ε x) & (z ε y)) <-> (¬(z ε x) v ¬(z ε y))) PolySub 17
19. ¬((z ε x) v (z ε y)) <-> (¬(z ε x) & ¬(z ε y)) AndElimL 18
20. (¬((z ε x) v (z ε y)) -> (¬(z ε x) & ¬(z ε y))) & ((¬(z ε x) & ¬(z ε y)) -> ¬((z ε x) v (z ε y))) EquivExp 19
21. ¬((z ε x) v (z ε y)) -> (¬(z ε x) & ¬(z ε y)) AndElimL 20
22. ¬(z ε x) & ¬(z ε y) ImpElim 15 21
23. Set(z) AndElimL 5
24. ¬(z ε x) AndElimL 22
25. ¬(z ε y) AndElimR 22
26. Set(z) & ¬(z ε y) AndInt 23 25
27. z ε {z: ¬(z ε y)} ClassInt 26
28. Set(z) & ¬(z ε x) AndInt 23 24
29. z ε {z: ¬(z ε x)} ClassInt 28
30. ~x = {y: ¬(y ε x)} DefEqInt
31. {y: ¬(y ε x)} = ~x Symmetry 30
32. z ε ~x EqualitySub 29 31
33. ∀w.(~w = {y: ¬(y ε w)}) ForallInt 30
34. ~y = {x_0: ¬(x_0 ε y)} ForallElim 33
35. {x_0: ¬(x_0 ε y)} = ~y Symmetry 34
36. z ε ~y EqualitySub 27 35
37. (z ε ~x) & (z ε ~y) AndInt 32 36
38. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 6
39. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 38
40. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 39
41. ∀x.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 40
42. ((z ε ~x) & (z ε y)) -> (z ε (~x ∩ y)) ForallElim 41
43. ∀y.(((z ε ~x) & (z ε y)) -> (z ε (~x ∩ y))) ForallInt 42
44. ((z ε ~x) & (z ε ~y)) -> (z ε (~x ∩ ~y)) ForallElim 43
45. z ε (~x ∩ ~y) ImpElim 37 44
46. (z ε ~(x ∪ y)) -> (z ε (~x ∩ ~y)) ImpInt 45
47. z ε (~x ∩ ~y) Hyp
48. ∀x.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) ForallInt 38
49. (z ε (~x ∩ y)) <-> ((z ε ~x) & (z ε y)) ForallElim 48
50. ∀y.((z ε (~x ∩ y)) <-> ((z ε ~x) & (z ε y))) ForallInt 49
51. (z ε (~x ∩ ~y)) <-> ((z ε ~x) & (z ε ~y)) ForallElim 50
52. ((z ε (~x ∩ ~y)) -> ((z ε ~x) & (z ε ~y))) & (((z ε ~x) & (z ε ~y)) -> (z ε (~x ∩ ~y))) EquivExp 51
53. (z ε (~x ∩ ~y)) -> ((z ε ~x) & (z ε ~y)) AndElimL 52
54. (z ε ~x) & (z ε ~y) ImpElim 47 53
55. z ε ~y AndElimR 54
56. z ε ~x AndElimL 54
57. z ε {y: ¬(y ε x)} EqualitySub 56 30
58. z ε {x_0: ¬(x_0 ε y)} EqualitySub 55 34
59. Set(z) & ¬(z ε x) ClassElim 57
60. Set(z) & ¬(z ε y) ClassElim 58
61. ¬(z ε x) AndElimR 59
62. ¬(z ε y) AndElimR 60
63. ¬(z ε x) & ¬(z ε y) AndInt 61 62
64. (¬(z ε x) & ¬(z ε y)) -> ¬((z ε x) v (z ε y)) AndElimR 20
65. ¬((z ε x) v (z ε y)) ImpElim 63 64
66. z ε (x ∪ y) Hyp
67. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 8
68. (z ε x) v (z ε y) ImpElim 66 67
69. _|_ ImpElim 68 65
70. ¬(z ε (x ∪ y)) ImpInt 69
71. Set(z) AndElimL 59
72. Set(z) & ¬(z ε (x ∪ y)) AndInt 71 70
73. z ε {w: ¬(w ε (x ∪ y))} ClassInt 72
74. ∀y.({x_0: ¬(x_0 ε y)} = ~y) ForallInt 35
75. {x_0: ¬(x_0 ε (x ∪ y))} = ~(x ∪ y) ForallElim 74
76. z ε ~(x ∪ y) EqualitySub 73 75
77. (z ε (~x ∩ ~y)) -> (z ε ~(x ∪ y)) ImpInt 76
78. ((z ε ~(x ∪ y)) -> (z ε (~x ∩ ~y))) & ((z ε (~x ∩ ~y)) -> (z ε ~(x ∪ y))) AndInt 46 77
79. (z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y)) EquivConst 78
80. z ε ~(x ∩ y) Hyp
81. ∀y.(~y = {x_0: ¬(x_0 ε y)}) ForallInt 34
82. ~(x ∩ y) = {x_0: ¬(x_0 ε (x ∩ y))} ForallElim 81

83. z ε {x_0: ¬(x_0 ε (x ∩ y))} EqualitySub 80 82
84. Set(z) & ¬(z ε (x ∩ y)) ClassElim 83
85. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 39
86. (((z ε x) & (z ε y)) -> B) -> (¬B -> ¬((z ε x) & (z ε y))) PolySub 10
87. (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) -> (¬(z ε (x ∩ y)) -> ¬((z ε x) & (z ε y))) PolySub 86
88. ¬(z ε (x ∩ y)) -> ¬((z ε x) & (z ε y)) ImpElim 85 87
89. ¬(z ε (x ∩ y)) AndElimR 84
90. ¬((z ε x) & (z ε y)) ImpElim 89 88
91. ¬(A & B) <-> (¬A v ¬B) AndElimR 16
92. ¬((z ε x) & B) <-> (¬(z ε x) v ¬B) PolySub 91
93. ¬((z ε x) & (z ε y)) <-> (¬(z ε x) v ¬(z ε y)) PolySub 92
94. (¬((z ε x) & (z ε y)) -> (¬(z ε x) v ¬(z ε y))) & ((¬(z ε x) v ¬(z ε y)) -> ¬((z ε x) & (z ε y))) EquivExp 93
95. ¬((z ε x) & (z ε y)) -> (¬(z ε x) v ¬(z ε y)) AndElimL 94
96. ¬(z ε x) v ¬(z ε y) ImpElim 90 95
97. ¬(z ε x) Hyp
98. Set(z) AndElimL 84
99. Set(z) & ¬(z ε x) AndInt 98 97
100. z ε {w: ¬(w ε x)} ClassInt 99
101. (z ε {w: ¬(w ε x)}) v (z ε {w: ¬(w ε y)}) OrIntR 100
102. {y: ¬(y ε x)} = ~x Symmetry 30
103. ∀x.({y: ¬(y ε x)} = ~x) ForallInt 102
104. {x_1: ¬(x_1 ε y)} = ~y ForallElim 103
105. (z ε ~x) v (z ε {w: ¬(w ε y)}) EqualitySub 101 102
106. (z ε ~x) v (z ε ~y) EqualitySub 105 104
107. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 9
108. ((z ε ~x) v (z ε y)) -> (z ε (~x ∪ y)) ForallElim 107
109. ∀y.(((z ε ~x) v (z ε y)) -> (z ε (~x ∪ y))) ForallInt 108
110. ((z ε ~x) v (z ε ~y)) -> (z ε (~x ∪ ~y)) ForallElim 109
111. z ε (~x ∪ ~y) ImpElim 106 110
112. ¬(z ε y) Hyp
113. Set(z) & ¬(z ε y) AndInt 98 112
114. z ε {z: ¬(z ε y)} ClassInt 113
115. (z ε {z: ¬(z ε x)}) v (z ε {z: ¬(z ε y)}) OrIntL 114
116. (z ε ~x) v (z ε {z: ¬(z ε y)}) EqualitySub 115 102
117. (z ε ~x) v (z ε ~y) EqualitySub 116 104
118. z ε (~x ∪ ~y) ImpElim 117 110
119. z ε (~x ∪ ~y) OrElim 96 97 111 112 118
120. (z ε ~(x ∩ y)) -> (z ε (~x ∪ ~y)) ImpInt 119
121. z ε (~x ∪ ~y) Hyp
122. ∃w.(z ε w) ExistsInt 121
123. Set(z) DefSub 122
124. x = x Identity
125. x = x Identity
126. x = x Identity
127. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 8
128. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 127
129. (z ε (~x ∪ y)) -> ((z ε ~x) v (z ε y)) ForallElim 128
130. ∀y.((z ε (~x ∪ y)) -> ((z ε ~x) v (z ε y))) ForallInt 129
131. (z ε (~x ∪ ~y)) -> ((z ε ~x) v (z ε ~y)) ForallElim 130
132. (z ε ~x) v (z ε ~y) ImpElim 121 131
133. z ε ~x Hyp
134. z ε {y: ¬(y ε x)} EqualitySub 133 30
135. Set(z) & ¬(z ε x) ClassElim 134
136. ¬(z ε x) AndElimR 135
137. z ε ~y Hyp
138. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 30
139. ~y = {x_3: ¬(x_3 ε y)} ForallElim 138
140. z ε {x_3: ¬(x_3 ε y)} EqualitySub 137 139
141. Set(z) & ¬(z ε y) ClassElim 140
142. ¬(z ε y) AndElimR 141
143. ¬(z ε x) v ¬(z ε y) OrIntR 136
144. ¬(z ε x) v ¬(z ε y) OrIntL 142
145. ¬(z ε x) v ¬(z ε y) OrElim 132 133 143 137 144
146. ¬(A & B) <-> (¬A v ¬B) AndElimR 16
147. (¬(A & B) -> (¬A v ¬B)) & ((¬A v ¬B) -> ¬(A & B)) EquivExp 146
148. (¬A v ¬B) -> ¬(A & B) AndElimR 147
149. (¬(z ε x) v ¬B) -> ¬((z ε x) & B) PolySub 148
150. (¬(z ε x) v ¬(z ε y)) -> ¬((z ε x) & (z ε y)) PolySub 149
151. ¬((z ε x) & (z ε y)) ImpElim 145 150

152. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 6
153. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 152
154. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 153
155. ((z ε (x ∩ y)) -> B) -> (¬B -> ¬(z ε (x ∩ y))) PolySub 10
156. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) -> (¬((z ε x) & (z ε y)) -> ¬(z ε (x ∩ y)))
PolySub 155
157. ¬((z ε x) & (z ε y)) -> ¬(z ε (x ∩ y)) ImpElim 154 156
158. ¬(z ε (x ∩ y)) ImpElim 151 157
159. Set(z) DefSub 122
160. Set(z) & ¬(z ε (x ∩ y)) AndInt 159 158
161. z ε {w: ¬(w ε (x ∩ y))} ClassInt 160
162. ∀x.({y: ¬(y ε x)} = ~x) ForallInt 31
163. {x_5: ¬(x_5 ε (x ∩ y))} = ~(x ∩ y) ForallElim 162
164. z ε ~(x ∩ y) EqualitySub 161 163
165. (z ε (~x ∪ ~y)) -> (z ε ~(x ∩ y)) ImpInt 164
166. ((z ε ~(x ∩ y)) -> (z ε (~x ∪ ~y))) & ((z ε (~x ∪ ~y)) -> (z ε ~(x ∩ y))) AndInt
120 165
167. (z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y)) EquivConst 166
168. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
169. ∀x_6.((~(x ∪ y) = x_6) <-> ∀z.((z ε ~(x ∪ y)) <-> (z ε x_6))) ForallElim 168
170. (~(x ∪ y) = (~x ∩ ~y)) <-> ∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y))) ForallElim 169
171. ∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y))) ForallInt 79
172. ((~(x ∪ y) = (~x ∩ ~y)) -> ∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y)))) & (∀z.((z ε ~(x
∪ y)) <-> (z ε (~x ∩ ~y))) -> (~(x ∪ y) = (~x ∩ ~y))) EquivExp 170
173. ∀z.((z ε ~(x ∪ y)) <-> (z ε (~x ∩ ~y))) -> (~(x ∪ y) = (~x ∩ ~y)) AndElimR 172
174. ~(x ∪ y) = (~x ∩ ~y) ImpElim 171 173
175. ∀x_7.((~(x ∩ y) = x_7) <-> ∀z.((z ε ~(x ∩ y)) <-> (z ε x_7))) ForallElim 168
176. (~(x ∩ y) = (~x ∪ ~y)) <-> ∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y))) ForallElim 175
177. ((~(x ∩ y) = (~x ∪ ~y)) -> ∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y)))) & (∀z.((z ε ~(x
∩ y)) <-> (z ε (~x ∪ ~y))) -> (~(x ∩ y) = (~x ∪ ~y))) EquivExp 176
178. ∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y))) -> (~(x ∩ y) = (~x ∪ ~y)) AndElimR 177
179. ∀z.((z ε ~(x ∩ y)) <-> (z ε (~x ∪ ~y))) ForallInt 167
180. ~(x ∩ y) = (~x ∪ ~y) ImpElim 179 178
181. (~(x ∪ y) = (~x ∩ ~y)) & (~(x ∩ y) = (~x ∪ ~y)) AndInt 174 180 Qed

Used Theorems

2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
3. (A -> B) -> (¬B -> ¬A)
1. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))

Th14. (x ∩ (y ~ z)) = ((x ∩ y) ∩ ~z)

0. (x ~ y) = (x ∩ ~y) DefEqInt
1. ∀a.((a ~ y) = (a ∩ ~y)) ForallInt 0
2. ∀b.∀a.((a ~ b) = (a ∩ ~b)) ForallInt 1
3. ∀a.((a ~ z) = (a ∩ ~z)) ForallElim 2
4. (y ~ z) = (y ∩ ~z) ForallElim 3
5. (x ∩ (y ~ z)) = (x ∩ (y ~ z)) Identity
6. (x ∩ (y ~ z)) = (x ∩ (y ∩ ~z)) EqualitySub 5 4
7. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z))) TheoremInt
8. ((x ∩ y) ∩ z) = (x ∩ (y ∩ z)) AndElimR 7
9. (x ∩ (y ∩ z)) = ((x ∩ y) ∩ z) Symmetry 8
10. ∀z.((x ∩ (y ∩ z)) = ((x ∩ y) ∩ z)) ForallInt 9
11. (x ∩ (y ∩ ~z)) = ((x ∩ y) ∩ ~z) ForallElim 10
12. (x ∩ (y ~ z)) = ((x ∩ y) ∩ ~z) EqualitySub 6 11 Qed

Used Theorems

4. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))

Th16. ¬(x ε 0)

0. x ε 0 Hyp
1. 0 = {x: ¬(x = x)} DefEqInt
2. x ε {x: ¬(x = x)} EqualitySub 0 1
3. Set(x) & ¬(x = x) ClassElim 2
4. ¬(x = x) AndElimR 3
5. x = x Identity
6. _|_ ImpElim 5 4

7. ¬(x ε 0) ImpInt 6 Qed

Used Theorems

Th17. ((0 U x) = x) & ((0 ∩ x) = 0)

0. z ε (0 U x) Hyp
1. (x U y) = {z: ((z ε x) v (z ε y))} DefEqInt
2. ∀x.((x U y) = {z: ((z ε x) v (z ε y))}) ForallInt 1
3. (0 U y) = {z: ((z ε 0) v (z ε y))} ForallElim 2
4. ∀y.((0 U y) = {z: ((z ε 0) v (z ε y))}) ForallInt 3
5. (0 U x) = {z: ((z ε 0) v (z ε x))} ForallElim 4
6. z ε {z: ((z ε 0) v (z ε x))} EqualitySub 0 5
7. Set(z) & ((z ε 0) v (z ε x)) ClassElim 6
8. (z ε 0) v (z ε x) AndElimR 7
9. z ε 0 Hyp
10. ¬(x ε 0) TheoremInt
11. ∀x.¬(x ε 0) ForallInt 10
12. ¬(z ε 0) ForallElim 11
13. _|_ ImpElim 9 12
14. z ε x AbsI 13
15. z ε x Hyp
16. z ε x OrElim 8 9 14 15 15
17. (z ε (0 U x)) -> (z ε x) ImpInt 16
18. z ε x Hyp
19. (z ε 0) v (z ε x) OrIntL 18
20. ∃x.(z ε x) ExistsInt 18
21. Set(z) DefSub 20
22. Set(z) & ((z ε 0) v (z ε x)) AndInt 21 19
23. z ε {z: ((z ε 0) v (z ε x))} ClassInt 22
24. {z: ((z ε 0) v (z ε x))} = (0 U x) Symmetry 5
25. z ε (0 U x) EqualitySub 23 24
26. (z ε x) -> (z ε (0 U x)) ImpInt 25
27. ((z ε (0 U x)) -> (z ε x)) & ((z ε x) -> (z ε (0 U x))) AndInt 17 26
28. (z ε (0 U x)) <-> (z ε x) EquivConst 27
29. ∀z.((z ε (0 U x)) <-> (z ε x)) ForallInt 28
30. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
31. ∀y.(((0 U x) = y) <-> ∀z.((z ε (0 U x)) <-> (z ε y))) ForallElim 30
32. ((0 U x) = x) <-> ∀z.((z ε (0 U x)) <-> (z ε x)) ForallElim 31
33. (((0 U x) = x) -> ∀z.((z ε (0 U x)) <-> (z ε x))) & (∀z.((z ε (0 U x)) <-> (z ε x)) -
> ((0 U x) = x)) EquivExp 32
34. ∀z.((z ε (0 U x)) <-> (z ε x)) -> ((0 U x) = x) AndElimR 33
35. (0 U x) = x ImpElim 29 34
36. z ε (0 ∩ x) Hyp
37. (x ∩ y) = {z: ((z ε x) & (z ε y))} DefEqInt
38. ∀x.((x ∩ y) = {z: ((z ε x) & (z ε y))}) ForallInt 37
39. (0 ∩ y) = {z: ((z ε 0) & (z ε y))} ForallElim 38
40. ∀y.((0 ∩ y) = {z: ((z ε 0) & (z ε y))}) ForallInt 39
41. (0 ∩ x) = {z: ((z ε 0) & (z ε x))} ForallElim 40
42. z ε {z: ((z ε 0) & (z ε x))} EqualitySub 36 41
43. Set(z) & ((z ε 0) & (z ε x)) ClassElim 42
44. (z ε 0) & (z ε x) AndElimR 43
45. z ε 0 AndElimL 44
46. (z ε (0 ∩ x)) -> (z ε 0) ImpInt 45
47. z ε 0 Hyp
48. _|_ ImpElim 47 12
49. z ε (0 ∩ x) AbsI 48
50. (z ε 0) -> (z ε (0 ∩ x)) ImpInt 49
51. ((z ε (0 ∩ x)) -> (z ε 0)) & ((z ε 0) -> (z ε (0 ∩ x))) AndInt 46 50
52. (z ε (0 ∩ x)) <-> (z ε 0) EquivConst 51
53. ∀z.((z ε (0 ∩ x)) <-> (z ε 0)) ForallInt 52
54. ∀y.(((0 ∩ x) = y) <-> ∀z.((z ε (0 ∩ x)) <-> (z ε y))) ForallElim 30
55. ((0 ∩ x) = 0) <-> ∀z.((z ε (0 ∩ x)) <-> (z ε 0)) ForallElim 54
56. (((0 ∩ x) = 0) -> ∀z.((z ε (0 ∩ x)) <-> (z ε 0))) & (∀z.((z ε (0 ∩ x)) <-> (z ε 0)) -
> ((0 ∩ x) = 0)) EquivExp 55
57. ∀z.((z ε (0 ∩ x)) <-> (z ε 0)) -> ((0 ∩ x) = 0) AndElimR 56
58. (0 ∩ x) = 0 ImpElim 53 57
59. ((0 U x) = x) & ((0 ∩ x) = 0) AndInt 35 58 Qed

Used Theorems

```
2. ¬(x ε 0)
```

Th19. (x ε U) <-> Set(x)

```
0. x ε U  Hyp
1. U = {x: (x = x)}  DefEqInt
2. x ε {x: (x = x)}  EqualitySub 0 1
3. Set(x) & (x = x)  ClassElim 2
4. Set(x)  AndElimL 3
5. (x ε U) -> Set(x)  ImpInt 4
6. Set(x)  Hyp
7. x = x  Identity
8. Set(x) & (x = x)  AndInt 6 7
9. x ε {x: (x = x)}  ClassInt 8
10. {x: (x = x)} = U  Symmetry 1
11. x ε U  EqualitySub 9 10
12. Set(x) -> (x ε U)  ImpInt 11
13. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  AndInt 5 12
14. (x ε U) <-> Set(x)  EquivConst 13 Qed
```

Used Theorems

Th20. ((x ∪ U) = U) & ((x ∩ U) = x)

```
0. z ε (x ∪ U)  Hyp
1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
2. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  AndElimL 1
3. ∀y.((z ε (x ∪ y)) <-> ((z ε x) v (z ε y)))  ForallInt 2
4. (z ε (x ∪ U)) <-> ((z ε x) v (z ε U))  ForallElim 3
5. ((z ε (x ∪ U)) -> ((z ε x) v (z ε U))) & (((z ε x) v (z ε U)) -> (z ε (x ∪ U)))
EquivExp 4
6. (z ε (x ∪ U)) -> ((z ε x) v (z ε U))  AndElimL 5
7. (z ε x) v (z ε U)  ImpElim 0 6
8. z ε x  Hyp
9. ∃y.(z ε y)  ExistsInt 8
10. Set(z)  DefSub 9
11. (x ε U) <-> Set(x)  TheoremInt
12. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 11
13. Set(x) -> (x ε U)  AndElimR 12
14. ∀x.(Set(x) -> (x ε U))  ForallInt 13
15. Set(z) -> (z ε U)  ForallElim 14
16. z ε U  ImpElim 10 15
17. z ε U  Hyp
18. z ε U  OrElim 7 8 16 17 17
19. (z ε (x ∪ U)) -> (z ε U)  ImpInt 18
20. z ε U  Hyp
21. (z ε x) v (z ε U)  OrIntL 20
22. ((z ε x) v (z ε U)) -> (z ε (x ∪ U))  AndElimR 5
23. z ε (x ∪ U)  ImpElim 21 22
24. (z ε U) -> (z ε (x ∪ U))  ImpInt 23
25. ((z ε (x ∪ U)) -> (z ε U)) & ((z ε U) -> (z ε (x ∪ U)))  AndInt 19 24
26. (z ε (x ∪ U)) <-> (z ε U)  EquivConst 25
27. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
28. ∀y.(((x ∪ U) = y) <-> ∀z.((z ε (x ∪ U)) <-> (z ε y)))  ForallElim 27
29. ((x ∪ U) = U) <-> ∀z.((z ε (x ∪ U)) <-> (z ε U))  ForallElim 28
30. ∀z.((z ε (x ∪ U)) <-> (z ε U))  ForallInt 26
31. (((x ∪ U) = U) -> ∀z.((z ε (x ∪ U)) <-> (z ε U))) & (∀z.((z ε (x ∪ U)) <-> (z ε U)) -
> ((x ∪ U) = U))  EquivExp 29
32. ∀z.((z ε (x ∪ U)) <-> (z ε U)) -> ((x ∪ U) = U)  AndElimR 31
33. (x ∪ U) = U  ImpElim 30 32
34. z ε (x ∩ U)  Hyp
35. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 1
36. ∀y.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))  ForallInt 35
37. (z ε (x ∩ U)) <-> ((z ε x) & (z ε U))  ForallElim 36
38. ((z ε (x ∩ U)) -> ((z ε x) & (z ε U))) & (((z ε x) & (z ε U)) -> (z ε (x ∩ U)))
EquivExp 37
39. (z ε (x ∩ U)) -> ((z ε x) & (z ε U))  AndElimL 38
40. (z ε x) & (z ε U)  ImpElim 34 39
```

41. z ε x AndElimL 40
42. (z ε (x ∩ U)) -> (z ε x) ImpInt 41
43. z ε x Hyp
44. ∃y.(z ε y) ExistsInt 43
45. Set(z) DefSub 44
46. z ε U ImpElim 45 15
47. (z ε x) & (z ε U) AndInt 43 46
48. ((z ε x) & (z ε U)) -> (z ε (x ∩ U)) AndElimR 38
49. z ε (x ∩ U) ImpElim 47 48
50. (z ε x) -> (z ε (x ∩ U)) ImpInt 49
51. ((z ε (x ∩ U)) -> (z ε x)) & ((z ε x) -> (z ε (x ∩ U))) AndInt 42 50
52. (z ε (x ∩ U)) <-> (z ε x) EquivConst 51
53. ∀z.((z ε (x ∩ U)) <-> (z ε x)) ForallInt 52
54. ∀y.(((x ∩ U) = y) <-> ∀z.((z ε (x ∩ U)) <-> (z ε y))) ForallElim 27
55. ((x ∩ U) = x) <-> ∀z.((z ε (x ∩ U)) <-> (z ε x)) ForallElim 54
56. (((x ∩ U) = x) -> ∀z.((z ε (x ∩ U)) <-> (z ε x))) & (∀z.((z ε (x ∩ U)) <-> (z ε x)) -> ((x ∩ U) = x)) EquivExp 55
57. ∀z.((z ε (x ∩ U)) <-> (z ε x)) -> ((x ∩ U) = x) AndElimR 56
58. (x ∩ U) = x ImpElim 53 57
59. ((x ∪ U) = U) & ((x ∩ U) = x) AndInt 33 58 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
2. (x ε U) <-> Set(x)

Th21. (~0 = U) & (~U = 0)

0. z ε ~0 Hyp
1. ~x = {y: ¬(y ε x)} DefEqInt
2. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1
3. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1
4. ~0 = {y: ¬(y ε 0)} ForallElim 3
5. z ε {y: ¬(y ε 0)} EqualitySub 0 4
6. Set(z) & ¬(z ε 0) ClassElim 5
7. Set(z) AndElimL 6
8. (x ε U) <-> Set(x) TheoremInt
9. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 8
10. Set(x) -> (x ε U) AndElimR 9
11. ∀x.(Set(x) -> (x ε U)) ForallInt 10
12. Set(z) -> (z ε U) ForallElim 11
13. z ε U ImpElim 7 12
14. (z ε ~0) -> (z ε U) ImpInt 13
15. z ε U Hyp
16. (x ε U) -> Set(x) AndElimL 9
17. ∀x.((x ε U) -> Set(x)) ForallInt 16
18. (z ε U) -> Set(z) ForallElim 17
19. Set(z) ImpElim 15 18
20. ¬(x ε 0) TheoremInt
21. ∀x.¬(x ε 0) ForallInt 20
22. ¬(z ε 0) ForallElim 21
23. Set(z) & ¬(z ε 0) AndInt 19 22
24. z ε {y: ¬(y ε 0)} ClassInt 23
25. {y: ¬(y ε 0)} = ~0 Symmetry 4
26. z ε ~0 EqualitySub 24 25
27. (z ε U) -> (z ε ~0) ImpInt 26
28. ((z ε ~0) -> (z ε U)) & ((z ε U) -> (z ε ~0)) AndInt 14 27
29. (z ε ~0) <-> (z ε U) EquivConst 28
30. ∀z.((z ε ~0) <-> (z ε U)) ForallInt 29
31. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
32. ∀y.((~0 = y) <-> ∀z.((z ε ~0) <-> (z ε y))) ForallElim 31
33. (~0 = U) <-> ∀z.((z ε ~0) <-> (z ε U)) ForallElim 32
34. ((~0 = U) -> ∀z.((z ε ~0) <-> (z ε U))) & (∀z.((z ε ~0) <-> (z ε U)) -> (~0 = U)) EquivExp 33
35. ∀z.((z ε ~0) <-> (z ε U)) -> (~0 = U) AndElimR 34
36. ~0 = U ImpElim 30 35
37. z ε ~U Hyp
38. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1
39. ~U = {y: ¬(y ε U)} ForallElim 38
40. z ε {y: ¬(y ε U)} EqualitySub 37 39
41. Set(z) & ¬(z ε U) ClassElim 40

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42. ¬(z ε U)  AndElimR 41
43. Set(z)  AndElimL 41
44. z ε U  ImpElim 43 12
45. _|_  ImpElim 44 42
46. z ε 0  AbsI 45
47. (z ε ~U) -> (z ε 0)  ImpInt 46
48. z ε 0  Hyp
49. 0 = {x: ¬(x = x)}  DefEqInt
50. z ε {x: ¬(x = x)}  EqualitySub 48 49
51. Set(z) & ¬(z = z)  ClassElim 50
52. Set(z)  AndElimL 51
53. ¬(z = z)  AndElimR 51
54. z = z  Identity
55. _|_  ImpElim 54 53
56. z ε ~U  AbsI 55
57. (z ε 0) -> (z ε ~U)  ImpInt 56
58. ((z ε ~U) -> (z ε 0)) & ((z ε 0) -> (z ε ~U))  AndInt 47 57
59. (z ε ~U) <-> (z ε 0)  EquivConst 58
60. ∀z.((z ε ~U) <-> (z ε 0))  ForallInt 59
61. ∀y.((~U = y) <-> ∀z.((z ε ~U) <-> (z ε y)))  ForallElim 31
62. (~U = 0) <-> ∀z.((z ε ~U) <-> (z ε 0))  ForallElim 61
63. ((~U = 0) -> ∀z.((z ε ~U) <-> (z ε 0))) & (∀z.((z ε ~U) <-> (z ε 0)) -> (~U = 0))
EquivExp 62
64. ∀z.((z ε ~U) <-> (z ε 0)) -> (~U = 0)  AndElimR 63
65. ~U = 0  ImpElim 60 64
66. (~0 = U) & (~U = 0)  AndInt 36 65 Qed
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Used Theorems

```
1. (x ε U) <-> Set(x)
2. ¬(x ε 0)
```

Th24. (∩0 = U) & (∪0 = 0)

```
0. x ε ∩0  Hyp
1. ∩x = {z: ∀y.((y ε x) -> (z ε y))}  DefEqInt
2. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))})  ForallInt 1
3. ∩0 = {z: ∀y.((y ε 0) -> (z ε y))}  ForallElim 2
4. x ε {z: ∀y.((y ε 0) -> (z ε y))}  EqualitySub 0 3
5. Set(x) & ∀y.((y ε 0) -> (x ε y))  ClassElim 4
6. Set(x)  AndElimL 5
7. (x ε U) <-> Set(x)  TheoremInt
8. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 7
9. Set(x) -> (x ε U)  AndElimR 8
10. x ε U  ImpElim 6 9
11. (x ε ∩0) -> (x ε U)  ImpInt 10
12. x ε U  Hyp
13. y ε 0  Hyp
14. ¬(x ε 0)  TheoremInt
15. ∀x.¬(x ε 0)  ForallInt 14
16. ¬(y ε 0)  ForallElim 15
17. _|_  ImpElim 13 16
18. x ε y  AbsI 17
19. (y ε 0) -> (x ε y)  ImpInt 18
20. ∀y.((y ε 0) -> (x ε y))  ForallInt 19
21. (x ε U) -> Set(x)  AndElimL 8
22. Set(x)  ImpElim 12 21
23. Set(x) & ∀y.((y ε 0) -> (x ε y))  AndInt 22 20
24. x ε {z: ∀y.((y ε 0) -> (z ε y))}  ClassInt 23
25. {z: ∀y.((y ε 0) -> (z ε y))} = ∩0  Symmetry 3
26. x ε ∩0  EqualitySub 24 25
27. (x ε U) -> (x ε ∩0)  ImpInt 26
28. ((x ε ∩0) -> (x ε U)) & ((x ε U) -> (x ε ∩0))  AndInt 11 27
29. (x ε ∩0) <-> (x ε U)  EquivConst 28
30. ∀z.((z ε ∩0) <-> (z ε U))  ForallInt 29
31. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
32. ∀y.((∩0 = y) <-> ∀z.((z ε ∩0) <-> (z ε y)))  ForallElim 31
33. (∩0 = U) <-> ∀z.((z ε ∩0) <-> (z ε U))  ForallElim 32
34. ((∩0 = U) -> ∀z.((z ε ∩0) <-> (z ε U))) & (∀z.((z ε ∩0) <-> (z ε U)) -> (∩0 = U))
EquivExp 33
35. ∀z.((z ε ∩0) <-> (z ε U)) -> (∩0 = U)  AndElimR 34
```

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36. ∩0 = U  ImpElim 30 35
37. z ε ∪0  Hyp
38. ∪x = {z: ∃y.((y ε x) & (z ε y))}  DefEqInt
39. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))})  ForallInt 38
40. ∪0 = {z: ∃y.((y ε 0) & (z ε y))}  ForallElim 39
41. z ε {z: ∃y.((y ε 0) & (z ε y))}  EqualitySub 37 40
42. Set(z) & ∃y.((y ε 0) & (z ε y))  ClassElim 41
43. ∃y.((y ε 0) & (z ε y))  AndElimR 42
44. (a ε 0) & (z ε a)  Hyp
45. ∀x.¬(x ε 0)  ForallInt 14
46. ¬(a ε 0)  ForallElim 45
47. a ε 0  AndElimL 44
48. _|_  ImpElim 47 46
49. z ε 0  AbsI 48
50. z ε 0  ExistsElim 43 44 49
51. (z ε ∪0) -> (z ε 0)  ImpInt 50
52. z ε 0  Hyp
53. ∀x.¬(x ε 0)  ForallInt 14
54. ¬(z ε 0)  ForallElim 53
55. _|_  ImpElim 52 54
56. z ε ∪0  AbsI 55
57. (z ε 0) -> (z ε ∪0)  ImpInt 56
58. ((z ε ∪0) -> (z ε 0)) & ((z ε 0) -> (z ε ∪0))  AndInt 51 57
59. (z ε ∪0) <-> (z ε 0)  EquivConst 58
60. ∀z.((z ε ∪0) <-> (z ε 0))  ForallInt 59
61. ∀y.((∪0 = y) <-> ∀z.((z ε ∪0) <-> (z ε y)))  ForallElim 31
62. (∪0 = 0) <-> ∀z.((z ε ∪0) <-> (z ε 0))  ForallElim 61
63. ((∪0 = 0) -> ∀z.((z ε ∪0) <-> (z ε 0))) & (∀z.((z ε ∪0) <-> (z ε 0)) -> (∪0 = 0))
EquivExp 62
64. ∀z.((z ε ∪0) <-> (z ε 0)) -> (∪0 = 0)  AndElimR 63
65. ∪0 = 0  ImpElim 60 64
66. (∩0 = U) & (∪0 = 0)  AndInt 36 65 Qed
```

Used Theorems

```
1. (x ε U) <-> Set(x)
2. ¬(x ε 0)
```

Th26. (0 ⊂ x) & (x ⊂ U)

```
0. z ε 0  Hyp
1. ¬(x ε 0)  TheoremInt
2. ∀x.¬(x ε 0)  ForallInt 1
3. ¬(z ε 0)  ForallElim 2
4. _|_  ImpElim 0 3
5. z ε x  AbsI 4
6. (z ε 0) -> (z ε x)  ImpInt 5
7. ∀z.((z ε 0) -> (z ε x))  ForallInt 6
8. 0 ⊂ x  DefSub 7
9. z ε x  Hyp
10. ∃y.(z ε y)  ExistsInt 9
11. Set(z)  DefSub 10
12. (x ε U) <-> Set(x)  TheoremInt
13. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 12
14. Set(x) -> (x ε U)  AndElimR 13
15. ∀x.(Set(x) -> (x ε U))  ForallInt 14
16. Set(z) -> (z ε U)  ForallElim 15
17. z ε U  ImpElim 11 16
18. (z ε x) -> (z ε U)  ImpInt 17
19. ∀z.((z ε x) -> (z ε U))  ForallInt 18
20. x ⊂ U  DefSub 19
21. (0 ⊂ x) & (x ⊂ U)  AndInt 8 20 Qed
```

Used Theorems

```
1. ¬(x ε 0)
2. (x ε U) <-> Set(x)
```

Th27. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

```
0. a = b  Hyp
```

1. z ε a Hyp
2. z ε b EqualitySub 1 0
3. (z ε a) -> (z ε b) ImpInt 2
4. ∀z.((z ε a) -> (z ε b)) ForallInt 3
5. a ⊂ b DefSub 4
6. z ε b Hyp
7. b = a Symmetry 0
8. z ε a EqualitySub 6 7
9. (z ε b) -> (z ε a) ImpInt 8
10. ∀z.((z ε b) -> (z ε a)) ForallInt 9
11. b ⊂ a DefSub 10
12. (a ⊂ b) & (b ⊂ a) AndInt 5 11
13. (a = b) -> ((a ⊂ b) & (b ⊂ a)) ImpInt 12
14. (a ⊂ b) & (b ⊂ a) Hyp
15. a ⊂ b AndElimL 14
16. b ⊂ a AndElimR 14
17. z ε a Hyp
18. ∀z.((z ε a) -> (z ε b)) DefExp 15
19. (z ε a) -> (z ε b) ForallElim 18
20. z ε b ImpElim 17 19
21. (z ε a) -> (z ε b) ImpInt 20
22. z ε b Hyp
23. ∀z.((z ε b) -> (z ε a)) DefExp 16
24. (z ε b) -> (z ε a) ForallElim 23
25. z ε a ImpElim 22 24
26. (z ε b) -> (z ε a) ImpInt 25
27. ((z ε a) -> (z ε b)) & ((z ε b) -> (z ε a)) AndInt 21 26
28. (z ε a) <-> (z ε b) EquivConst 27
29. ∀z.((z ε a) <-> (z ε b)) ForallInt 28
30. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
31. ∀y.((a = y) <-> ∀z.((z ε a) <-> (z ε y))) ForallElim 30
32. (a = b) <-> ∀z.((z ε a) <-> (z ε b)) ForallElim 31
33. ((a = b) -> ∀z.((z ε a) <-> (z ε b))) & (∀z.((z ε a) <-> (z ε b)) -> (a = b))
EquivExp 32
34. ∀z.((z ε a) <-> (z ε b)) -> (a = b) AndElimR 33
35. a = b ImpElim 29 34
36. ((a ⊂ b) & (b ⊂ a)) -> (a = b) ImpInt 35
37. ((a = b) -> ((a ⊂ b) & (b ⊂ a))) & (((a ⊂ b) & (b ⊂ a)) -> (a = b)) AndInt 13 36
38. (a = b) <-> ((a ⊂ b) & (b ⊂ a)) EquivConst 37
39. ∀a.((a = b) <-> ((a ⊂ b) & (b ⊂ a))) ForallInt 38
40. (x = b) <-> ((x ⊂ b) & (b ⊂ x)) ForallElim 39
41. ∀b.((x = b) <-> ((x ⊂ b) & (b ⊂ x))) ForallInt 40
42. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) ForallElim 41 Qed

Used Theorems

Th28. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)

0. (a ⊂ b) & (b ⊂ c) Hyp
1. b ⊂ c AndElimR 0
2. a ⊂ b AndElimL 0
3. ∀z.((z ε b) -> (z ε c)) DefExp 1
4. ∀z.((z ε a) -> (z ε b)) DefExp 2
5. (z ε b) -> (z ε c) ForallElim 3
6. (z ε a) -> (z ε b) ForallElim 4
7. z ε a Hyp
8. z ε b ImpElim 7 6
9. z ε c ImpElim 8 5
10. (z ε a) -> (z ε c) ImpInt 9
11. ∀z.((z ε a) -> (z ε c)) ForallInt 10
12. a ⊂ c DefSub 11
13. ((a ⊂ b) & (b ⊂ c)) -> (a ⊂ c) ImpInt 12
14. ∀a.(((a ⊂ b) & (b ⊂ c)) -> (a ⊂ c)) ForallInt 13
15. ((x ⊂ b) & (b ⊂ c)) -> (x ⊂ c) ForallElim 14
16. ∀b.(((x ⊂ b) & (b ⊂ c)) -> (x ⊂ c)) ForallInt 15
17. ((x ⊂ y) & (y ⊂ c)) -> (x ⊂ c) ForallElim 16
18. ∀c.(((x ⊂ y) & (y ⊂ c)) -> (x ⊂ c)) ForallInt 17
19. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z) ForallElim 18 Qed

Used Theorems

Th29. (x ⊂ y) <-> ((x ∪ y) = y)

0. a ⊂ b Hyp
1. z ε (a ∪ b) Hyp
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
3. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 2
4. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 3
5. ∀x.(((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))) ForallInt 4
6. ((z ε (a ∪ y)) -> ((z ε a) v (z ε y))) & (((z ε a) v (z ε y)) -> (z ε (a ∪ y))) ForallElim 5
7. ∀y.(((z ε (a ∪ y)) -> ((z ε a) v (z ε y))) & (((z ε a) v (z ε y)) -> (z ε (a ∪ y)))) ForallInt 6
8. ((z ε (a ∪ b)) -> ((z ε a) v (z ε b))) & (((z ε a) v (z ε b)) -> (z ε (a ∪ b))) ForallElim 7
9. (z ε (a ∪ b)) -> ((z ε a) v (z ε b)) AndElimL 8
10. (z ε a) v (z ε b) ImpElim 1 9
11. z ε a Hyp
12. ∀z.((z ε a) -> (z ε b)) DefExp 0
13. (z ε a) -> (z ε b) ForallElim 12
14. z ε b ImpElim 11 13
15. z ε b Hyp
16. z ε b OrElim 10 11 14 15 15
17. (z ε (a ∪ b)) -> (z ε b) ImpInt 16
18. z ε b Hyp
19. (z ε a) v (z ε b) OrIntL 18
20. ((z ε a) v (z ε b)) -> (z ε (a ∪ b)) AndElimR 8
21. z ε (a ∪ b) ImpElim 19 20
22. (z ε b) -> (z ε (a ∪ b)) ImpInt 21
23. ((z ε (a ∪ b)) -> (z ε b)) & ((z ε b) -> (z ε (a ∪ b))) AndInt 17 22
24. (z ε (a ∪ b)) <-> (z ε b) EquivConst 23
25. ∀z.((z ε (a ∪ b)) <-> (z ε b)) ForallInt 24
26. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
27. ∀y.(((a ∪ b) = y) <-> ∀z.((z ε (a ∪ b)) <-> (z ε y))) ForallElim 26
28. ((a ∪ b) = b) <-> ∀z.((z ε (a ∪ b)) <-> (z ε b)) ForallElim 27
29. (((a ∪ b) = b) -> ∀z.((z ε (a ∪ b)) <-> (z ε b))) & (∀z.((z ε (a ∪ b)) <-> (z ε b)) -> ((a ∪ b) = b)) EquivExp 28
30. ∀z.((z ε (a ∪ b)) <-> (z ε b)) -> ((a ∪ b) = b) AndElimR 29
31. (a ∪ b) = b ImpElim 25 30
32. (a ⊂ b) -> ((a ∪ b) = b) ImpInt 31
33. (a ∪ b) = b Hyp
34. z ε a Hyp
35. (z ε a) v (z ε b) OrIntR 34
36. ((z ε a) v (z ε b)) -> (z ε (a ∪ b)) AndElimR 8
37. z ε (a ∪ b) ImpElim 35 36
38. z ε b EqualitySub 37 33
39. (z ε a) -> (z ε b) ImpInt 38
40. ∀z.((z ε a) -> (z ε b)) ForallInt 39
41. a ⊂ b DefSub 40
42. ((a ∪ b) = b) -> (a ⊂ b) ImpInt 41
43. ((a ⊂ b) -> ((a ∪ b) = b)) & (((a ∪ b) = b) -> (a ⊂ b)) AndInt 32 42
44. (a ⊂ b) <-> ((a ∪ b) = b) EquivConst 43
45. ∀a.((a ⊂ b) <-> ((a ∪ b) = b)) ForallInt 44
46. (x ⊂ b) <-> ((x ∪ b) = b) ForallElim 45
47. ∀b.((x ⊂ b) <-> ((x ∪ b) = b)) ForallInt 46
48. (x ⊂ y) <-> ((x ∪ y) = y) ForallElim 47 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

Th30. (x ⊂ y) <-> ((x ∩ y) = x)

0. a ⊂ b Hyp
1. z ε (a ∩ b) Hyp
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt

3. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 2
4. ∀x.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) ForallInt 3
5. (z ε (a ∩ y)) <-> ((z ε a) & (z ε y)) ForallElim 4
6. ∀y.((z ε (a ∩ y)) <-> ((z ε a) & (z ε y))) ForallInt 5
7. (z ε (a ∩ b)) <-> ((z ε a) & (z ε b)) ForallElim 6
8. ((z ε (a ∩ b)) -> ((z ε a) & (z ε b))) & (((z ε a) & (z ε b)) -> (z ε (a ∩ b))) EquivExp 7
9. (z ε (a ∩ b)) -> ((z ε a) & (z ε b)) AndElimL 8
10. (z ε a) & (z ε b) ImpElim 1 9
11. z ε a AndElimL 10
12. (z ε (a ∩ b)) -> (z ε a) ImpInt 11
13. z ε a Hyp
14. ∀z.((z ε a) -> (z ε b)) DefExp 0
15. (z ε a) -> (z ε b) ForallElim 14
16. z ε b ImpElim 13 15
17. (z ε a) & (z ε b) AndInt 13 16
18. ((z ε a) & (z ε b)) -> (z ε (a ∩ b)) AndElimR 8
19. z ε (a ∩ b) ImpElim 17 18
20. (z ε a) -> (z ε (a ∩ b)) ImpInt 19
21. ((z ε (a ∩ b)) -> (z ε a)) & ((z ε a) -> (z ε (a ∩ b))) AndInt 12 20
22. (z ε (a ∩ b)) <-> (z ε a) EquivConst 21
23. ∀z.((z ε (a ∩ b)) <-> (z ε a)) ForallInt 22
24. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
25. ∀y.(((a ∩ b) = y) <-> ∀z.((z ε (a ∩ b)) <-> (z ε y))) ForallElim 24
26. ((a ∩ b) = a) <-> ∀z.((z ε (a ∩ b)) <-> (z ε a)) ForallElim 25
27. (((a ∩ b) = a) -> ∀z.((z ε (a ∩ b)) <-> (z ε a))) & (∀z.((z ε (a ∩ b)) <-> (z ε a)) -> ((a ∩ b) = a)) EquivExp 26
28. ∀z.((z ε (a ∩ b)) <-> (z ε a)) -> ((a ∩ b) = a) AndElimR 27
29. (a ∩ b) = a ImpElim 23 28
30. (a ⊂ b) -> ((a ∩ b) = a) ImpInt 29
31. (a ∩ b) = a Hyp
32. z ε a Hyp
33. a = (a ∩ b) Symmetry 31
34. z ε (a ∩ b) EqualitySub 32 33
35. (z ε a) & (z ε b) ImpElim 34 9
36. z ε b AndElimR 35
37. (z ε a) -> (z ε b) ImpInt 36
38. ∀z.((z ε a) -> (z ε b)) ForallInt 37
39. a ⊂ b DefSub 38
40. ((a ∩ b) = a) -> (a ⊂ b) ImpInt 39
41. ((a ⊂ b) -> ((a ∩ b) = a)) & (((a ∩ b) = a) -> (a ⊂ b)) AndInt 30 40
42. (a ⊂ b) <-> ((a ∩ b) = a) EquivConst 41
43. ∀a.((a ⊂ b) <-> ((a ∩ b) = a)) ForallInt 42
44. (x ⊂ b) <-> ((x ∩ b) = x) ForallElim 43
45. ∀b.((x ⊂ b) <-> ((x ∩ b) = x)) ForallInt 44
46. (x ⊂ y) <-> ((x ∩ y) = x) ForallElim 45 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

Th31. (x ⊂ y) -> ((∪x ⊂ ∪y) & (∩y ⊂ ∩x))

0. a ⊂ b Hyp
1. z ε ∪a Hyp
2. ∪x = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
3. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 2
4. ∪a = {z: ∃y.((y ε a) & (z ε y))} ForallElim 3
5. z ε {z: ∃y.((y ε a) & (z ε y))} EqualitySub 1 4
6. Set(z) & ∃y.((y ε a) & (z ε y)) ClassElim 5
7. ∃y.((y ε a) & (z ε y)) AndElimR 6
8. (y ε a) & (z ε y) Hyp
9. ∀z.((z ε a) -> (z ε b)) DefExp 0
10. (y ε a) -> (y ε b) ForallElim 9
11. y ε a AndElimL 8
12. y ε b ImpElim 11 10
13. z ε y AndElimR 8
14. (y ε b) & (z ε y) AndInt 12 13
15. ∃y.((y ε b) & (z ε y)) ExistsInt 14
16. Set(z) AndElimL 6
17. Set(z) & ∃y.((y ε b) & (z ε y)) AndInt 16 15

18. z ε {z: ∃y.((y ε b) & (z ε y))} ClassInt 17
19. ∀x.(Ux = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 2
20. Ub = {z: ∃y.((y ε b) & (z ε y))} ForallElim 19
21. {z: ∃y.((y ε b) & (z ε y))} = Ub Symmetry 20
22. z ε Ub EqualitySub 18 21
23. z ε Ub ExistsElim 7 8 22
24. (z ε Ua) -> (z ε Ub) ImpInt 23
25. ∀z.((z ε Ua) -> (z ε Ub)) ForallInt 24
26. Ua ⊂ Ub DefSub 25
27. z ε ∩b Hyp
28. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
29. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 28
30. ∩b = {z: ∀y.((y ε b) -> (z ε y))} ForallElim 29
31. z ε {z: ∀y.((y ε b) -> (z ε y))} EqualitySub 27 30
32. Set(z) & ∀y.((y ε b) -> (z ε y)) ClassElim 31
33. Set(z) AndElimL 32
34. ∀y.((y ε b) -> (z ε y)) AndElimR 32
35. (y ε b) -> (z ε y) ForallElim 34
36. y ε a Hyp
37. y ε b ImpElim 36 10
38. z ε y ImpElim 37 35
39. (y ε a) -> (z ε y) ImpInt 38
40. ∀y.((y ε a) -> (z ε y)) ForallInt 39
41. Set(z) & ∀y.((y ε a) -> (z ε y)) AndInt 33 40
42. z ε {z: ∀y.((y ε a) -> (z ε y))} ClassInt 41
43. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 28
44. ∩a = {z: ∀y.((y ε a) -> (z ε y))} ForallElim 43
45. {z: ∀y.((y ε a) -> (z ε y))} = ∩a Symmetry 44
46. z ε ∩a EqualitySub 42 45
47. (z ε ∩b) -> (z ε ∩a) ImpInt 46
48. ∀z.((z ε ∩b) -> (z ε ∩a)) ForallInt 47
49. ∩b ⊂ ∩a DefSub 48
50. (Ua ⊂ Ub) & (∩b ⊂ ∩a) AndInt 26 49
51. (a ⊂ b) -> ((Ua ⊂ Ub) & (∩b ⊂ ∩a)) ImpInt 50
52. ∀a.((a ⊂ b) -> ((Ua ⊂ Ub) & (∩b ⊂ ∩a))) ForallInt 51
53. (x ⊂ b) -> ((Ux ⊂ Ub) & (∩b ⊂ ∩x)) ForallElim 52
54. ∀b.((x ⊂ b) -> ((Ux ⊂ Ub) & (∩b ⊂ ∩x))) ForallInt 53
55. (x ⊂ y) -> ((Ux ⊂ Uy) & (∩y ⊂ ∩x)) ForallElim 54 Qed

Used Theorems

Th32. (x ε y) -> ((x ⊂ Uy) & (∩y ⊂ x))

0. a ε b Hyp
1. x ε a Hyp
2. (a ε b) & (x ε a) AndInt 0 1
3. ∃y.((y ε b) & (x ε y)) ExistsInt 2
4. ∃y.(x ε y) ExistsInt 1
5. Set(x) DefSub 4
6. Set(x) & ∃y.((y ε b) & (x ε y)) AndInt 5 3
7. x ε {z: ∃y.((y ε b) & (z ε y))} ClassInt 6
8. Ux = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
9. {z: ∃y.((y ε x) & (z ε y))} = Ux Symmetry 8
10. ∀x.({z: ∃y.((y ε x) & (z ε y))} = Ux) ForallInt 9
11. {z: ∃y.((y ε b) & (z ε y))} = Ub ForallElim 10
12. x ε Ub EqualitySub 7 11
13. (x ε a) -> (x ε Ub) ImpInt 12
14. ∀z.((z ε a) -> (z ε Ub)) ForallInt 13
15. a ⊂ Ub DefSub 14
16. x ε ∩b Hyp
17. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
18. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 17
19. ∩b = {z: ∀y.((y ε b) -> (z ε y))} ForallElim 18
20. x ε {z: ∀y.((y ε b) -> (z ε y))} EqualitySub 16 19
21. Set(x) & ∀y.((y ε b) -> (x ε y)) ClassElim 20
22. ∀y.((y ε b) -> (x ε y)) AndElimR 21
23. (a ε b) -> (x ε a) ForallElim 22
24. x ε a ImpElim 0 23
25. (x ε ∩b) -> (x ε a) ImpInt 24
26. ∀z.((z ε ∩b) -> (z ε a)) ForallInt 25

27. ∩b ⊂ a DefSub 26
28. (a ⊂ ∪b) & (∩b ⊂ a) AndInt 15 27
29. (a ε b) -> ((a ⊂ ∪b) & (∩b ⊂ a)) ImpInt 28
30. ∀a.((a ε b) -> ((a ⊂ ∪b) & (∩b ⊂ a))) ForallInt 29
31. (x ε b) -> ((x ⊂ ∪b) & (∩b ⊂ x)) ForallElim 30
32. ∀b.((x ε b) -> ((x ⊂ ∪b) & (∩b ⊂ x))) ForallInt 31
33. (x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x)) ForallElim 32 Qed

Used Theorems

Th33. (Set(x) & (y ⊂ x)) -> Set(y)

0. Set(a) & (b ⊂ a) Hyp
1. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y))) AxInt
2. ∀x.(Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y)))) ForallInt 1
3. Set(a) -> ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ε y))) ForallElim 2
4. Set(a) AndElimL 0
5. ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ε y))) ImpElim 4 3
6. Set(w) & ∀z.((z ⊂ a) -> (z ε w)) Hyp
7. ∀z.((z ⊂ a) -> (z ε w)) AndElimR 6
8. (b ⊂ a) -> (b ε w) ForallElim 7
9. b ⊂ a AndElimR 0
10. b ε w ImpElim 9 8
11. ∃z.(b ε z) ExistsInt 10
12. Set(b) DefSub 11
13. Set(b) ExistsElim 5 6 12
14. (Set(a) & (b ⊂ a)) -> Set(b) ImpInt 13
15. ∀a.((Set(a) & (b ⊂ a)) -> Set(b)) ForallInt 14
16. (Set(x) & (b ⊂ x)) -> Set(b) ForallElim 15
17. ∀b.((Set(x) & (b ⊂ x)) -> Set(b)) ForallInt 16
18. (Set(x) & (y ⊂ x)) -> Set(y) ForallElim 17 Qed

Used Theorems

Th34. (0 = ∩U) & (U = ∪U)

0. z ε 0 Hyp
1. 0 = {x: ¬(x = x)} DefEqInt
2. z ε {x: ¬(x = x)} EqualitySub 0 1
3. Set(z) & ¬(z = z) ClassElim 2
4. ¬(z = z) AndElimR 3
5. z = z Identity
6. _|_ ImpElim 5 4
7. z ε ∩U AbsI 6
8. (z ε 0) -> (z ε ∩U) ImpInt 7
9. z ε ∩U Hyp
10. U = {x: (x = x)} DefEqInt
11. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
12. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 11
13. ∩U = {z: ∀y.((y ε U) -> (z ε y))} ForallElim 12
14. z ε {z: ∀y.((y ε U) -> (z ε y))} EqualitySub 9 13
15. Set(z) & ∀y.((y ε U) -> (z ε y)) ClassElim 14
16. ∀y.((y ε U) -> (z ε y)) AndElimR 15
17. (0 ε U) -> (z ε 0) ForallElim 16
18. (0 ⊂ x) & (x ⊂ U) TheoremInt
19. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
20. 0 ⊂ x AndElimL 18
21. ∀x.(0 ⊂ x) ForallInt 20
22. 0 ⊂ z ForallElim 21
23. ∀x.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 19
24. (Set(z) & (y ⊂ z)) -> Set(y) ForallElim 23
25. ∀y.((Set(z) & (y ⊂ z)) -> Set(y)) ForallInt 24
26. (Set(z) & (0 ⊂ z)) -> Set(0) ForallElim 25
27. Set(z) AndElimL 15
28. Set(z) & (0 ⊂ z) AndInt 27 22
29. Set(0) ImpElim 28 26
30. (x ε U) <-> Set(x) TheoremInt
31. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 30
32. Set(x) -> (x ε U) AndElimR 31

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33. ∀x.(Set(x) -> (x ε U))  ForallInt 32
34. Set(0) -> (0 ε U)  ForallElim 33
35. 0 ε U  ImpElim 29 34
36. z ε 0  ImpElim 35 17
37. (z ε ∩U) -> (z ε 0)  ImpInt 36
38. ((z ε 0) -> (z ε ∩U)) & ((z ε ∩U) -> (z ε 0))  AndInt 8 37
39. (z ε 0) <-> (z ε ∩U)  EquivConst 38
40. ∀z.((z ε 0) <-> (z ε ∩U))  ForallInt 39
41. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
42. ∀y.((0 = y) <-> ∀z.((z ε 0) <-> (z ε y)))  ForallElim 41
43. (0 = ∩U) <-> ∀z.((z ε 0) <-> (z ε ∩U))  ForallElim 42
44. ((0 = ∩U) -> ∀z.((z ε 0) <-> (z ε ∩U))) & (∀z.((z ε 0) <-> (z ε ∩U)) -> (0 = ∩U))
EquivExp 43
45. ∀z.((z ε 0) <-> (z ε ∩U)) -> (0 = ∩U)  AndElimR 44
46. 0 = ∩U  ImpElim 40 45
47. z ε U  Hyp
48. ∪x = {z: ∃y.((y ε x) & (z ε y))}  DefEqInt
49. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))})  ForallInt 48
50. ∪U = {z: ∃y.((y ε U) & (z ε y))}  ForallElim 49
51. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y)))  AxInt
52. (x ε U) -> Set(x)  AndElimL 31
53. ∀x.((x ε U) -> Set(x))  ForallInt 52
54. (z ε U) -> Set(z)  ForallElim 53
55. Set(z)  ImpElim 47 54
56. ∀x.(Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y))))  ForallInt 51
57. Set(z) -> ∃y.(Set(y) & ∀i.((i ⊂ z) -> (i ε y)))  ForallElim 56
58. ∃y.(Set(y) & ∀i.((i ⊂ z) -> (i ε y)))  ImpElim 55 57
59. Set(a) & ∀i.((i ⊂ z) -> (i ε a))  Hyp
60. z = z  Identity
61. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
62. ∀x.((x = y) <-> ((x ⊂ y) & (y ⊂ x)))  ForallInt 61
63. (z = y) <-> ((z ⊂ y) & (y ⊂ z))  ForallElim 62
64. ∀y.((z = y) <-> ((z ⊂ y) & (y ⊂ z)))  ForallInt 63
65. (z = z) <-> ((z ⊂ z) & (z ⊂ z))  ForallElim 64
66. ((z = z) -> ((z ⊂ z) & (z ⊂ z))) & (((z ⊂ z) & (z ⊂ z)) -> (z = z))  EquivExp 65
67. (z = z) -> ((z ⊂ z) & (z ⊂ z))  AndElimL 66
68. (z ⊂ z) & (z ⊂ z)  ImpElim 60 67
69. z ⊂ z  AndElimL 68
70. ∀i.((i ⊂ z) -> (i ε a))  AndElimR 59
71. (z ⊂ z) -> (z ε a)  ForallElim 70
72. z ε a  ImpElim 69 71
73. Set(a)  AndElimL 59
74. ∀x.(Set(x) -> (x ε U))  ForallInt 32
75. Set(a) -> (a ε U)  ForallElim 74
76. a ε U  ImpElim 73 75
77. (a ε U) & (z ε a)  AndInt 76 72
78. ∃y.((y ε U) & (z ε y))  ExistsInt 77
79. ∃y.((y ε U) & (z ε y))  ExistsElim 58 59 78
80. Set(z) & ∃y.((y ε U) & (z ε y))  AndInt 55 79
81. z ε {y: ∃j.((j ε U) & (y ε j))}  ClassInt 80
82. {z: ∃y.((y ε U) & (z ε y))} = ∪U  Symmetry 50
83. z ε ∪U  EqualitySub 81 82
84. (z ε U) -> (z ε ∪U)  ImpInt 83
85. z ε ∪U  Hyp
86. ∃y.(z ε y)  ExistsInt 85
87. Set(z)  DefSub 86
88. ∀x.(Set(x) -> (x ε U))  ForallInt 32
89. Set(z) -> (z ε U)  ForallElim 88
90. z ε U  ImpElim 87 89
91. (z ε ∪U) -> (z ε U)  ImpInt 90
92. ((z ε U) -> (z ε ∪U)) & ((z ε ∪U) -> (z ε U))  AndInt 84 91
93. (z ε U) <-> (z ε ∪U)  EquivConst 92
94. ∀z.((z ε U) <-> (z ε ∪U))  ForallInt 93
95. ∀y.((U = y) <-> ∀z.((z ε U) <-> (z ε y)))  ForallElim 41
96. (U = ∪U) <-> ∀z.((z ε U) <-> (z ε ∪U))  ForallElim 95
97. ((U = ∪U) -> ∀z.((z ε U) <-> (z ε ∪U))) & (∀z.((z ε U) <-> (z ε ∪U)) -> (U = ∪U))
EquivExp 96
98. ∀z.((z ε U) <-> (z ε ∪U)) -> (U = ∪U)  AndElimR 97
99. U = ∪U  ImpElim 94 98
100. (0 = ∩U) & (U = ∪U)  AndInt 46 99 Qed
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Used Theorems

1. (0 ⊂ x) & (x ⊂ U)
2. (Set(x) & (y ⊂ x)) -> Set(y)
3. (x ε U) <-> Set(x)
4. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

Th35. ¬(x = 0) -> Set(∩x)

0. ∀z.¬(z ε a) Hyp
1. z ε a Hyp
2. ¬(z ε a) ForallElim 0
3. _|_ ImpElim 1 2
4. z ε 0 AbsI 3
5. (z ε a) -> (z ε 0) ImpInt 4
6. z ε 0 Hyp
7. 0 = {x: ¬(x = x)} DefEqInt
8. z ε {x: ¬(x = x)} EqualitySub 6 7
9. Set(z) & ¬(z = z) ClassElim 8
10. ¬(z = z) AndElimR 9
11. z = z Identity
12. _|_ ImpElim 11 10
13. z ε a AbsI 12
14. (z ε 0) -> (z ε a) ImpInt 13
15. ((z ε a) -> (z ε 0)) & ((z ε 0) -> (z ε a)) AndInt 5 14
16. (z ε a) <-> (z ε 0) EquivConst 15
17. ∀z.((z ε a) <-> (z ε 0)) ForallInt 16
18. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
19. ∀y.((a = y) <-> ∀z.((z ε a) <-> (z ε y))) ForallElim 18
20. (a = 0) <-> ∀z.((z ε a) <-> (z ε 0)) ForallElim 19
21. ((a = 0) -> ∀z.((z ε a) <-> (z ε 0))) & (∀z.((z ε a) <-> (z ε 0)) -> (a = 0)) EquivExp 20
22. ∀z.((z ε a) <-> (z ε 0)) -> (a = 0) AndElimR 21
23. a = 0 ImpElim 17 22
24. ∀z.¬(z ε a) -> (a = 0) ImpInt 23
25. (A -> B) -> (¬B -> ¬A) TheoremInt
26. (∀z.¬(z ε a) -> B) -> (¬B -> ¬∀z.¬(z ε a)) PolySub 25
27. (∀z.¬(z ε a) -> (a = 0)) -> (¬(a = 0) -> ¬∀z.¬(z ε a)) PolySub 26
28. ¬(a = 0) -> ¬∀z.¬(z ε a) ImpElim 24 27
29. ¬∀z.¬(z ε a) Hyp
30. ¬∃z.(z ε a) Hyp
31. z ε a Hyp
32. ∃z.(z ε a) ExistsInt 31
33. _|_ ImpElim 32 30
34. ¬(z ε a) ImpInt 33
35. ∀z.¬(z ε a) ForallInt 34
36. ¬∃z.(z ε a) -> ∀z.¬(z ε a) ImpInt 35
37. (A -> B) -> (¬B -> ¬A) TheoremInt
38. (¬∃z.(z ε a) -> B) -> (¬B -> ¬¬∃z.(z ε a)) PolySub 37
39. (¬∃x_0.(x_0 ε a) -> ∀z.¬(z ε a)) -> (¬∀z.¬(z ε a) -> ¬¬∃x_0.(x_0 ε a)) PolySub 38
40. ¬∀z.¬(z ε a) -> ¬¬∃x_0.(x_0 ε a) ImpElim 36 39
41. D <-> ¬¬D TheoremInt
42. ∃l.(l ε a) <-> ¬¬∃l.(l ε a) PolySub 41
43. (∃l.(l ε a) -> ¬¬∃l.(l ε a)) & (¬¬∃l.(l ε a) -> ∃l.(l ε a)) EquivExp 42
44. ¬¬∃l.(l ε a) -> ∃l.(l ε a) AndElimR 43
45. ¬(a = 0) Hyp
46. ¬∀z.¬(z ε a) ImpElim 45 28
47. ¬¬∃x_0.(x_0 ε a) ImpElim 46 40
48. ∃l.(l ε a) ImpElim 47 44
49. ¬(a = 0) -> ∃l.(l ε a) ImpInt 48
50. ∃l.(l ε a) Hyp
51. b ε a Hyp
52. (x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x)) TheoremInt
53. ∀x.((x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x))) ForallInt 52
54. (b ε y) -> ((b ⊂ ∪y) & (∩y ⊂ b)) ForallElim 53
55. ∀y.((b ε y) -> ((b ⊂ ∪y) & (∩y ⊂ b))) ForallInt 54
56. (b ε a) -> ((b ⊂ ∪a) & (∩a ⊂ b)) ForallElim 55
57. (b ⊂ ∪a) & (∩a ⊂ b) ImpElim 51 56
58. ∩a ⊂ b AndElimR 57
59. ∃y.(b ε y) ExistsInt 51
60. Set(b) DefSub 59

61. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
62. ∀x.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 61
63. (Set(b) & (y ⊂ b)) -> Set(y) ForallElim 62
64. ∀y.((Set(b) & (y ⊂ b)) -> Set(y)) ForallInt 63
65. (Set(b) & (∩a ⊂ b)) -> Set(∩a) ForallElim 64
66. Set(b) & (∩a ⊂ b) AndInt 60 58
67. Set(∩a) ImpElim 66 65
68. Set(∩a) ExistsElim 50 51 67
69. ∃l.(l ε a) -> Set(∩a) ImpInt 68
70. ¬(a = 0) Hyp
71. ∃l.(l ε a) ImpElim 70 49
72. Set(∩a) ImpElim 71 69
73. ¬(a = 0) -> Set(∩a) ImpInt 72
74. ∀a.(¬(a = 0) -> Set(∩a)) ForallInt 73
75. ¬(x = 0) -> Set(∩x) ForallElim 74 Qed

Used Theorems

1. (A -> B) -> (¬B -> ¬A)
2. D <-> ¬¬D
4. (x ε y) -> ((x ⊂ ∪y) & (∩y ⊂ x))
5. (Set(x) & (y ⊂ x)) -> Set(y)

Th37. U = PU

0. x ε U Hyp
1. (0 ⊂ x) & (x ⊂ U) TheoremInt
2. x ⊂ U AndElimR 1
3. Px = {y: (y ⊂ x)} DefEqInt
4. ∀x.(Px = {y: (y ⊂ x)}) ForallInt 3
5. PU = {y: (y ⊂ U)} ForallElim 4
6. ∃y.(x ε y) ExistsInt 0
7. Set(x) DefSub 6
8. Set(x) & (x ⊂ U) AndInt 7 2
9. x ε {y: (y ⊂ U)} ClassInt 8
10. {y: (y ⊂ U)} = PU Symmetry 5
11. x ε PU EqualitySub 9 10
12. (x ε U) -> (x ε PU) ImpInt 11
13. x ε PU Hyp
14. ∃y.(x ε y) ExistsInt 13
15. Set(x) DefSub 14
16. (x ε U) <-> Set(x) TheoremInt
17. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 16
18. Set(x) -> (x ε U) AndElimR 17
19. x ε U ImpElim 15 18
20. (x ε PU) -> (x ε U) ImpInt 19
21. ((x ε U) -> (x ε PU)) & ((x ε PU) -> (x ε U)) AndInt 12 20
22. (x ε U) <-> (x ε PU) EquivConst 21
23. ∀z.((z ε U) <-> (z ε PU)) ForallInt 22
24. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
25. ∀y.((U = y) <-> ∀z.((z ε U) <-> (z ε y))) ForallElim 24
26. (U = PU) <-> ∀z.((z ε U) <-> (z ε PU)) ForallElim 25
27. ((U = PU) -> ∀z.((z ε U) <-> (z ε PU))) & (∀z.((z ε U) <-> (z ε PU)) -> (U = PU))
EquivExp 26
28. ∀z.((z ε U) <-> (z ε PU)) -> (U = PU) AndElimR 27
29. U = PU ImpElim 23 28 Qed

Used Theorems

1. (0 ⊂ x) & (x ⊂ U)
2. (x ε U) <-> Set(x)

Th38. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))

0. Set(a) Hyp
1. Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y))) AxInt
2. ∀x.(Set(x) -> ∃y.(Set(y) & ∀z.((z ⊂ x) -> (z ε y)))) ForallInt 1
3. Set(a) -> ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ε y))) ForallElim 2
4. ∃y.(Set(y) & ∀z.((z ⊂ a) -> (z ε y))) ImpElim 0 3
5. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
6. ∀y.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 5

7. (Set(x) & (Pa ⊂ x)) -> Set(Pa) ForallElim 6
8. Set(b) & ∀z.((z ⊂ a) -> (z ε b)) Hyp
9. ∀x.((Set(x) & (Pa ⊂ x)) -> Set(Pa)) ForallInt 7
10. (Set(b) & (Pa ⊂ b)) -> Set(Pa) ForallElim 9
11. z ε Pa Hyp
12. Px = {y: (y ⊂ x)} DefEqInt
13. ∀x.(Px = {y: (y ⊂ x)}) ForallInt 12
14. Pa = {y: (y ⊂ a)} ForallElim 13
15. z ε {y: (y ⊂ a)} EqualitySub 11 14
16. Set(z) & (z ⊂ a) ClassElim 15
17. ∀z.((z ⊂ a) -> (z ε b)) AndElimR 8
18. z ⊂ a AndElimR 16
19. (z ⊂ a) -> (z ε b) ForallElim 17
20. z ε b ImpElim 18 19
21. (z ε Pa) -> (z ε b) ImpInt 20
22. ∀z.((z ε Pa) -> (z ε b)) ForallInt 21
23. Pa ⊂ b DefSub 22
24. Set(b) AndElimL 8
25. Set(b) & (Pa ⊂ b) AndInt 24 23
26. Set(Pa) ImpElim 25 10
27. Set(Pa) ExistsElim 4 8 26
28. z ⊂ a Hyp
29. Set(a) & (z ⊂ a) AndInt 0 28
30. ∀x.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 5
31. (Set(a) & (y ⊂ a)) -> Set(y) ForallElim 30
32. ∀y.((Set(a) & (y ⊂ a)) -> Set(y)) ForallInt 31
33. (Set(a) & (z ⊂ a)) -> Set(z) ForallElim 32
34. Set(z) ImpElim 29 33
35. Set(z) & (z ⊂ a) AndInt 34 28
36. z ε {y: (y ⊂ a)} ClassInt 35
37. {y: (y ⊂ a)} = Pa Symmetry 14
38. z ε Pa EqualitySub 36 37
39. (z ⊂ a) -> (z ε Pa) ImpInt 38
40. z ε Pa Hyp
41. z ε {y: (y ⊂ a)} EqualitySub 40 14
42. Set(z) & (z ⊂ a) ClassElim 41
43. z ⊂ a AndElimR 42
44. (z ε Pa) -> (z ⊂ a) ImpInt 43
45. ((z ⊂ a) -> (z ε Pa)) & ((z ε Pa) -> (z ⊂ a)) AndInt 39 44
46. (z ⊂ a) <-> (z ε Pa) EquivConst 45
47. Set(Pa) & ((z ⊂ a) <-> (z ε Pa)) AndInt 27 46
48. Set(a) -> (Set(Pa) & ((z ⊂ a) <-> (z ε Pa))) ImpInt 47
49. ∀a.(Set(a) -> (Set(Pa) & ((z ⊂ a) <-> (z ε Pa)))) ForallInt 48
50. Set(x) -> (Set(Px) & ((z ⊂ x) <-> (z ε Px))) ForallElim 49
51. ∀z.(Set(x) -> (Set(Px) & ((z ⊂ x) <-> (z ε Px)))) ForallInt 50
52. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px))) ForallElim 51 Qed

Used Theorems

1. (Set(x) & (y ⊂ x)) -> Set(y)

Th39. ¬Set(U)

0. rus = {z: ¬(z ε z)} DefEqInt
1. rus ε rus Hyp
2. rus ε {z: ¬(z ε z)} EqualitySub 1 0
3. Set(rus) & ¬(rus ε rus) ClassElim 2
4. ¬(rus ε rus) AndElimR 3
5. _|_ ImpElim 1 4
6. ¬Set(rus) AbsI 5
7. ¬(rus ε rus) Hyp
8. Set(rus) Hyp
9. Set(rus) & ¬(rus ε rus) AndInt 8 7
10. rus ε {z: ¬(z ε z)} ClassInt 9
11. {z: ¬(z ε z)} = rus Symmetry 0
12. rus ε rus EqualitySub 10 11
13. _|_ ImpElim 12 7
14. ¬Set(rus) ImpInt 13
15. A v ¬A TheoremInt
16. (rus ε rus) v ¬(rus ε rus) PolySub 15
17. ¬Set(rus) OrElim 16 1 6 7 14

18. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
19. (0 ⊂ x) & (x ⊂ U) TheoremInt
20. x ⊂ U AndElimR 19
21. Set(U) Hyp
22. ∀x.(x ⊂ U) ForallInt 20
23. rus ⊂ U ForallElim 22
24. Set(U) & (rus ⊂ U) AndInt 21 23
25. ∀x.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 18
26. (Set(U) & (y ⊂ U)) -> Set(y) ForallElim 25
27. ∀y.((Set(U) & (y ⊂ U)) -> Set(y)) ForallInt 26
28. (Set(U) & (rus ⊂ U)) -> Set(rus) ForallElim 27
29. Set(rus) ImpElim 24 28
30. _|_ ImpElim 29 17
31. ¬Set(U) ImpInt 30 Qed

Used Theorems

1. A v ¬A
2. (Set(x) & (y ⊂ x)) -> Set(y)
3. (0 ⊂ x) & (x ⊂ U)

Th41. Set(x) -> ((y ε {x}) <-> (y = x))

0. Set(x) Hyp
1. y ε {x} Hyp
2. {x} = {z: ((x ε U) -> (z = x))} DefEqInt
3. y ε {z: ((x ε U) -> (z = x))} EqualitySub 1 2
4. Set(y) & ((x ε U) -> (y = x)) ClassElim 3
5. (x ε U) <-> Set(x) TheoremInt
6. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 5
7. Set(x) -> (x ε U) AndElimR 6
8. x ε U ImpElim 0 7
9. (x ε U) -> (y = x) AndElimR 4
10. y = x ImpElim 8 9
11. (y ε {x}) -> (y = x) ImpInt 10
12. y = x Hyp
13. x = y Symmetry 12
14. Set(y) EqualitySub 0 13
15. y = x Hyp
16. x ε U Hyp
17. (x ε U) -> (y = x) ImpInt 15
18. (y = x) -> ((x ε U) -> (y = x)) ImpInt 17
19. (x ε U) -> (y = x) ImpElim 12 18
20. Set(y) & ((x ε U) -> (y = x)) AndInt 14 19
21. y ε {z: ((x ε U) -> (z = x))} ClassInt 20
22. {z: ((x ε U) -> (z = x))} = {x} Symmetry 2
23. y ε {x} EqualitySub 21 22
24. (y = x) -> (y ε {x}) ImpInt 23
25. ((y ε {x}) -> (y = x)) & ((y = x) -> (y ε {x})) AndInt 11 24
26. (y ε {x}) <-> (y = x) EquivConst 25
27. Set(x) -> ((y ε {x}) <-> (y = x)) ImpInt 26 Qed

Used Theorems

1. (x ε U) <-> Set(x)

Th42. Set(x) -> Set({x})

0. Set(x) Hyp
1. z ε {x} Hyp
2. {x} = {z: ((x ε U) -> (z = x))} DefEqInt
3. z ε {z: ((x ε U) -> (z = x))} EqualitySub 1 2
4. Set(z) & ((x ε U) -> (z = x)) ClassElim 3
5. (x ε U) -> (z = x) AndElimR 4
6. (x ε U) <-> Set(x) TheoremInt
7. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 6
8. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 6
9. Set(x) -> (x ε U) AndElimR 8
10. x ε U ImpElim 0 9
11. z = x ImpElim 10 5
12. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt

13. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y)) EquivExp 12
14. (x = y) -> ((x ⊂ y) & (y ⊂ x)) AndElimL 13
15. ∀x.((x = y) -> ((x ⊂ y) & (y ⊂ x))) ForallInt 14
16. (z = y) -> ((z ⊂ y) & (y ⊂ z)) ForallElim 15
17. ∀y.((z = y) -> ((z ⊂ y) & (y ⊂ z))) ForallInt 16
18. (z = x) -> ((z ⊂ x) & (x ⊂ z)) ForallElim 17
19. (z ⊂ x) & (x ⊂ z) ImpElim 11 18
20. z ⊂ x AndElimL 19
21. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px))) TheoremInt
22. Set(Px) & ((y ⊂ x) <-> (y ε Px)) ImpElim 0 21
23. (y ⊂ x) <-> (y ε Px) AndElimR 22
24. ((y ⊂ x) -> (y ε Px)) & ((y ε Px) -> (y ⊂ x)) EquivExp 23
25. (y ⊂ x) -> (y ε Px) AndElimL 24
26. ∀y.((y ⊂ x) -> (y ε Px)) ForallInt 25
27. (z ⊂ x) -> (z ε Px) ForallElim 26
28. z ε Px ImpElim 20 27
29. (z ε {x}) -> (z ε Px) ImpInt 28
30. ∀z.((z ε {x}) -> (z ε Px)) ForallInt 29
31. {x} ⊂ Px DefSub 30
32. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
33. ∀x.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 32
34. (Set(Px) & (y ⊂ Px)) -> Set(y) ForallElim 33
35. ∀y.((Set(Px) & (y ⊂ Px)) -> Set(y)) ForallInt 34
36. (Set(Px) & ({x} ⊂ Px)) -> Set({x}) ForallElim 35
37. Set(Px) AndElimL 22
38. Set(Px) & ({x} ⊂ Px) AndInt 37 31
39. Set({x}) ImpElim 38 36
40. Set(x) -> Set({x}) ImpInt 39 Qed

Used Theorems

3. (x ε U) <-> Set(x)
2. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
1. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))
4. (Set(x) & (y ⊂ x)) -> Set(y)

Th43. ({x} = U) <-> ¬Set(x)

0. Set(x) Hyp
1. Set(x) -> Set({x}) TheoremInt
2. Set({x}) ImpElim 0 1
3. ¬Set(U) TheoremInt
4. {x} = U Hyp
5. Set(U) EqualitySub 2 4
6. _|_ ImpElim 5 3
7. ¬({x} = U) ImpInt 6
8. ¬Set(x) Hyp
9. x ε U Hyp
10. ∃y.(x ε y) ExistsInt 9
11. Set(x) DefSub 10
12. _|_ ImpElim 11 8
13. ¬(x ε U) ImpInt 12
14. x ε U Hyp
15. _|_ ImpElim 14 13
16. y = x AbsI 15
17. (x ε U) -> (y = x) ImpInt 16
18. y ε U Hyp
19. (x ε U) <-> Set(x) TheoremInt
20. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 19
21. (x ε U) -> Set(x) AndElimL 20
22. ∀x.((x ε U) -> Set(x)) ForallInt 21
23. (y ε U) -> Set(y) ForallElim 22
24. Set(y) ImpElim 18 23
25. Set(y) & ((x ε U) -> (y = x)) AndInt 24 17
26. y ε {z: ((x ε U) -> (z = x))} ClassInt 25
27. {x} = {z: ((x ε U) -> (z = x))} DefEqInt
28. {z: ((x ε U) -> (z = x))} = {x} Symmetry 27
29. y ε {x} EqualitySub 26 28
30. (y ε U) -> (y ε {x}) ImpInt 29
31. ∀z.((z ε U) -> (z ε {x})) ForallInt 30
32. U ⊂ {x} DefSub 31

33. (0 ⊂ x) & (x ⊂ U) TheoremInt
34. ∀x.((0 ⊂ x) & (x ⊂ U)) ForallInt 33
35. (0 ⊂ {x}) & ({x} ⊂ U) ForallElim 34
36. {x} ⊂ U AndElimR 35
37. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
38. ∀x.((x = y) <-> ((x ⊂ y) & (y ⊂ x))) ForallInt 37
39. ({x} = y) <-> (({x} ⊂ y) & (y ⊂ {x})) ForallElim 38
40. ∀y.(({x} = y) <-> (({x} ⊂ y) & (y ⊂ {x}))) ForallInt 39
41. ({x} = U) <-> (({x} ⊂ U) & (U ⊂ {x})) ForallElim 40
42. (({x} = U) -> (({x} ⊂ U) & (U ⊂ {x}))) & ((({x} ⊂ U) & (U ⊂ {x})) -> ({x} = U)) EquivExp 41
43. (({x} = U) -> (({x} ⊂ U) & (U ⊂ {x}))) & ((({x} ⊂ U) & (U ⊂ {x})) -> ({x} = U)) EquivExp 41
44. (({x} ⊂ U) & (U ⊂ {x})) -> ({x} = U) AndElimR 43
45. ({x} ⊂ U) & (U ⊂ {x}) AndInt 36 32
46. {x} = U ImpElim 45 44
47. ¬Set(x) -> ({x} = U) ImpInt 46
48. Set(x) -> ¬({x} = U) ImpInt 7
49. (A -> B) -> (¬B -> ¬A) TheoremInt
50. (Set(x) -> B) -> (¬B -> ¬Set(x)) PolySub 49
51. (Set(x) -> ¬({x} = U)) -> (¬¬({x} = U) -> ¬Set(x)) PolySub 50
52. ¬¬({x} = U) -> ¬Set(x) ImpElim 48 51
53. D <-> ¬¬D TheoremInt
54. (D -> ¬¬D) & (¬¬D -> D) EquivExp 53
55. D -> ¬¬D AndElimL 54
56. ({x} = U) -> ¬¬({x} = U) PolySub 55
57. {x} = U Hyp
58. ¬¬({x} = U) ImpElim 57 56
59. ¬Set(x) ImpElim 58 52
60. ({x} = U) -> ¬Set(x) ImpInt 59
61. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U)) AndInt 60 47
62. ({x} = U) <-> ¬Set(x) EquivConst 61 Qed

Used Theorems

1. Set(x) -> Set({x})
2. ¬Set(U)
3. (x ε U) <-> Set(x)
4. (0 ⊂ x) & (x ⊂ U)
6. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
10. (A -> B) -> (¬B -> ¬A)
9. D <-> ¬¬D

Th44. (Set(x) -> ((∩{x} = x) & (⋃{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (⋃{x} = U)))

0. z ε ∩{x} Hyp
1. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
2. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 1
3. ∩{x} = {z: ∀y.((y ε {x}) -> (z ε y))} ForallElim 2
4. z ε {z: ∀y.((y ε {x}) -> (z ε y))} EqualitySub 0 3
5. Set(z) & ∀y.((y ε {x}) -> (z ε y)) ClassElim 4
6. ∀y.((y ε {x}) -> (z ε y)) AndElimR 5
7. Set(x) Hyp
8. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
9. (y ε {x}) <-> (y = x) ImpElim 7 8
10. ((y ε {x}) -> (y = x)) & ((y = x) -> (y ε {x})) EquivExp 9
11. (y = x) -> (y ε {x}) AndElimR 10
12. ∀y.((y = x) -> (y ε {x})) ForallInt 11
13. (x = x) -> (x ε {x}) ForallElim 12
14. x = x Identity
15. x ε {x} ImpElim 14 13
16. (x ε {x}) -> (z ε x) ForallElim 6
17. z ε x ImpElim 15 16
18. (z ε ∩{x}) -> (z ε x) ImpInt 17
19. z ε x Hyp
20. y ε {x} Hyp
21. (y ε {x}) -> (y = x) AndElimL 10
22. y = x ImpElim 20 21
23. x = y Symmetry 22
24. z ε y EqualitySub 19 23
25. (y ε {x}) -> (z ε y) ImpInt 24

26. ∀y.((y ε {x}) -> (z ε y)) ForallInt 25
27. ∃x.(z ε x) ExistsInt 19
28. Set(z) DefSub 27
29. Set(z) & ∀y.((y ε {x}) -> (z ε y)) AndInt 28 26
30. z ε {z: ∀y.((y ε {x}) -> (z ε y))} ClassInt 29
31. {z: ∀y.((y ε {x}) -> (z ε y))} = ∩{x} Symmetry 3
32. z ε ∩{x} EqualitySub 30 31
33. (z ε x) -> (z ε ∩{x}) ImpInt 32
34. ((z ε ∩{x}) -> (z ε x)) & ((z ε x) -> (z ε ∩{x})) AndInt 18 33
35. (z ε ∩{x}) <-> (z ε x) EquivConst 34
36. ∀z.((z ε ∩{x}) <-> (z ε x)) ForallInt 35
37. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
38. ∀y.((∩{x} = y) <-> ∀z.((z ε ∩{x}) <-> (z ε y))) ForallElim 37
39. (∩{x} = x) <-> ∀z.((z ε ∩{x}) <-> (z ε x)) ForallElim 38
40. ((∩{x} = x) -> ∀z.((z ε ∩{x}) <-> (z ε x))) & (∀z.((z ε ∩{x}) <-> (z ε x)) -> (∩{x} = x)) EquivExp 39
41. ∀z.((z ε ∩{x}) <-> (z ε x)) -> (∩{x} = x) AndElimR 40
42. ∩{x} = x ImpElim 36 41
43. z ε ∪{x} Hyp
44. ∪x = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
45. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 44
46. ∪{x} = {z: ∃y.((y ε {x}) & (z ε y))} ForallElim 45
47. z ε {z: ∃y.((y ε {x}) & (z ε y))} EqualitySub 43 46
48. Set(z) & ∃y.((y ε {x}) & (z ε y)) ClassElim 47
49. ∃y.((y ε {x}) & (z ε y)) AndElimR 48
50. (a ε {x}) & (z ε a) Hyp
51. ∀y.((y ε {x}) -> (y = x)) ForallInt 21
52. (a ε {x}) -> (a = x) ForallElim 51
53. a ε {x} AndElimL 50
54. a = x ImpElim 53 52
55. z ε a AndElimR 50
56. z ε x EqualitySub 55 54
57. z ε x ExistsElim 49 50 56
58. (z ε ∪{x}) -> (z ε x) ImpInt 57
59. z ε x Hyp
60. (y = x) -> (y ε {x}) AndElimR 10
61. ∀y.((y = x) -> (y ε {x})) ForallInt 60
62. (x = x) -> (x ε {x}) ForallElim 61
63. x ε {x} ImpElim 14 62
64. (x ε {x}) & (z ε x) AndInt 63 59
65. ∃y.((y ε {x}) & (z ε y)) ExistsInt 64
66. ∃y.(z ε y) ExistsInt 59
67. Set(z) DefSub 66
68. Set(z) & ∃y.((y ε {x}) & (z ε y)) AndInt 67 65
69. z ε {z: ∃y.((y ε {x}) & (z ε y))} ClassInt 68
70. {z: ∃y.((y ε {x}) & (z ε y))} = ∪{x} Symmetry 46
71. z ε ∪{x} EqualitySub 69 70
72. (z ε x) -> (z ε ∪{x}) ImpInt 71
73. ((z ε ∪{x}) -> (z ε x)) & ((z ε x) -> (z ε ∪{x})) AndInt 58 72
74. (z ε ∪{x}) <-> (z ε x) EquivConst 73
75. ∀z.((z ε ∪{x}) <-> (z ε x)) ForallInt 74
76. ∀y.((∪{x} = y) <-> ∀z.((z ε ∪{x}) <-> (z ε y))) ForallElim 37
77. (∪{x} = x) <-> ∀z.((z ε ∪{x}) <-> (z ε x)) ForallElim 76
78. ((∪{x} = x) -> ∀z.((z ε ∪{x}) <-> (z ε x))) & (∀z.((z ε ∪{x}) <-> (z ε x)) -> (∪{x} = x)) EquivExp 77
79. ∀z.((z ε ∪{x}) <-> (z ε x)) -> (∪{x} = x) AndElimR 78
80. ∪{x} = x ImpElim 75 79
81. (∩{x} = x) & (∪{x} = x) AndInt 42 80
82. Set(x) -> ((∩{x} = x) & (∪{x} = x)) ImpInt 81
83. ¬Set(x) Hyp
84. ({x} = U) <-> ¬Set(x) TheoremInt
85. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U)) EquivExp 84
86. ¬Set(x) -> ({x} = U) AndElimR 85
87. {x} = U ImpElim 83 86
88. (0 = ∩U) & (U = ∪U) TheoremInt
89. U = {x} Symmetry 87
90. (0 = ∩{x}) & (U = ∪{x}) EqualitySub 88 89
91. 0 = ∩{x} AndElimL 90
92. U = ∪{x} AndElimR 90
93. ∩{x} = 0 Symmetry 91
94. ∪{x} = U Symmetry 92

95. (∩{x} = 0) & (∪{x} = U) AndInt 93 94
96. ¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)) ImpInt 95
97. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))
AndInt 82 96 Qed

Used Theorems

1. Set(x) -> ((y ε {x}) <-> (y = x))
2. ({x} = U) <-> ¬Set(x)
3. (0 = ∩U) & (U = ∪U)

Th46. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) &
(({x,y} = U) <-> (¬Set(x) v ¬Set(y)))

0. Set(x) & Set(y) Hyp
1. Set(x) -> Set({x}) TheoremInt
2. Set(x) AndElimL 0
3. Set(y) AndElimR 0
4. Set({x}) ImpElim 2 1
5. ∀x.(Set(x) -> Set({x})) ForallInt 1
6. Set(y) -> Set({y}) ForallElim 5
7. Set({y}) ImpElim 3 6
8. (Set(x) & Set(y)) -> Set((x ∪ y)) AxInt
9. ∀x.((Set(x) & Set(y)) -> Set((x ∪ y))) ForallInt 8
10. (Set({x}) & Set(y)) -> Set(({x} ∪ y)) ForallElim 9
11. ∀y.((Set({x}) & Set(y)) -> Set(({x} ∪ y))) ForallInt 10
12. (Set({x}) & Set({y})) -> Set(({x} ∪ {y})) ForallElim 11
13. Set({x}) & Set({y}) AndInt 4 7
14. Set(({x} ∪ {y})) ImpElim 13 12
15. {x,y} = ({x} ∪ {y}) DefEqInt
16. ({x} ∪ {y}) = {x,y} Symmetry 15
17. Set({x,y}) EqualitySub 14 16
18. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
19. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 18
20. z ε {x,y} Hyp
21. z ε ({x} ∪ {y}) EqualitySub 20 15
22. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 19
23. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 22
24. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 23
25. (z ε ({x} ∪ y)) -> ((z ε {x}) v (z ε y)) ForallElim 24
26. ∀y.((z ε ({x} ∪ y)) -> ((z ε {x}) v (z ε y))) ForallInt 25
27. (z ε ({x} ∪ {y})) -> ((z ε {x}) v (z ε {y})) ForallElim 26
28. (z ε {x}) v (z ε {y}) ImpElim 21 27
29. z ε {x} Hyp
30. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
31. ∀y.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 30
32. Set(x) -> ((z ε {x}) <-> (z = x)) ForallElim 31
33. ∀x.(Set(x) -> ((z ε {x}) <-> (z = x))) ForallInt 32
34. Set(y) -> ((z ε {y}) <-> (z = y)) ForallElim 33
35. (z ε {x}) <-> (z = x) ImpElim 2 32
36. ((z ε {x}) -> (z = x)) & ((z = x) -> (z ε {x})) EquivExp 35
37. (z ε {x}) -> (z = x) AndElimL 36
38. z = x ImpElim 29 37
39. (z = x) v (z = y) OrIntR 38
40. z ε {y} Hyp
41. (z ε {y}) <-> (z = y) ImpElim 3 34
42. ((z ε {y}) -> (z = y)) & ((z = y) -> (z ε {y})) EquivExp 41
43. (z ε {y}) -> (z = y) AndElimL 42
44. z = y ImpElim 40 43
45. (z = x) v (z = y) OrIntL 44
46. (z = x) v (z = y) OrElim 28 29 39 40 45
47. (z ε {x,y}) -> ((z = x) v (z = y)) ImpInt 46
48. (z = x) v (z = y) Hyp
49. z = x Hyp
50. (z = x) -> (z ε {x}) AndElimR 36
51. z ε {x} ImpElim 49 50
52. (z ε {x}) v (z ε {y}) OrIntR 51
53. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 22
54. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 53

55. ((z ε {x}) v (z ε y)) -> (z ε ({x} ∪ y)) ForallElim 54
56. ∀y.(((z ε {x}) v (z ε y)) -> (z ε ({x} ∪ y))) ForallInt 55
57. ((z ε {x}) v (z ε {y})) -> (z ε ({x} ∪ {y})) ForallElim 56
58. z ε ({x} ∪ {y}) ImpElim 52 57
59. z = y Hyp
60. (z = y) -> (z ε {y}) AndElimR 42
61. z ε {y} ImpElim 59 60
62. (z ε {x}) v (z ε {y}) OrIntL 61
63. z ε ({x} ∪ {y}) ImpElim 62 57
64. z ε ({x} ∪ {y}) OrElim 48 49 58 59 63
65. ((z = x) v (z = y)) -> (z ε ({x} ∪ {y})) ImpInt 64
66. ((z = x) v (z = y)) -> (z ε {x,y}) EqualitySub 65 16
67. ((z ε {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ε {x,y})) AndInt 47 66
68. (z ε {x,y}) <-> ((z = x) v (z = y)) EquivConst 67
69. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))) AndInt 17 68
70. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))) ImpInt 69
71. {x,y} = U Hyp
72. ({x} ∪ {y}) = U EqualitySub 71 15
73. ¬Set(U) TheoremInt
74. U = ({x} ∪ {y}) Symmetry 72
75. ¬Set(({x} ∪ {y})) EqualitySub 73 74
76. (Set(x) & Set(y)) -> Set((x ∪ y)) AxInt
77. (A -> B) -> (¬B -> ¬A) TheoremInt
78. ((Set(x) & Set(y)) -> B) -> (¬B -> ¬(Set(x) & Set(y))) PolySub 77
79. ((Set(x) & Set(y)) -> Set((x ∪ y))) -> (¬Set((x ∪ y)) -> ¬(Set(x) & Set(y))) PolySub 78
80. ¬Set((x ∪ y)) -> ¬(Set(x) & Set(y)) ImpElim 76 79
81. ∀x.(¬Set((x ∪ y)) -> ¬(Set(x) & Set(y))) ForallInt 80
82. ¬Set(({x} ∪ y)) -> ¬(Set({x}) & Set(y)) ForallElim 81
83. ∀y.(¬Set(({x} ∪ y)) -> ¬(Set({x}) & Set(y))) ForallInt 82
84. ¬Set(({x} ∪ {y})) -> ¬(Set({x}) & Set({y})) ForallElim 83
85. ¬(Set({x}) & Set({y})) ImpElim 75 84
86. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B)) TheoremInt
87. ¬(A & B) <-> (¬A v ¬B) AndElimR 86
88. ¬(Set({x}) & B) <-> (¬Set({x}) v ¬B) PolySub 87
89. ¬(Set({x}) & Set({y})) <-> (¬Set({x}) v ¬Set({y})) PolySub 88
90. (¬(Set({x}) & Set({y})) -> (¬Set({x}) v ¬Set({y}))) & ((¬Set({x}) v ¬Set({y})) -> ¬(Set({x}) & Set({y}))) EquivExp 89
91. ¬(Set({x}) & Set({y})) -> (¬Set({x}) v ¬Set({y})) AndElimL 90
92. ¬Set({x}) v ¬Set({y}) ImpElim 85 91
93. ¬Set({x}) Hyp
94. Set(x) -> Set({x}) TheoremInt
95. (Set(x) -> B) -> (¬B -> ¬Set(x)) PolySub 77
96. (Set(x) -> Set({x})) -> (¬Set({x}) -> ¬Set(x)) PolySub 95
97. ¬Set({x}) -> ¬Set(x) ImpElim 94 96
98. ¬Set(x) ImpElim 93 97
99. ¬Set({x}) -> ¬Set(x) ImpInt 98
100. ∀a.(¬Set({a}) -> ¬Set(a)) ForallInt 99
101. ¬Set({y}) Hyp
102. ¬Set({y}) -> ¬Set(y) ForallElim 100
103. ¬Set(y) ImpElim 101 102
104. ¬Set(x) v ¬Set(y) OrIntR 98
105. ¬Set(x) v ¬Set(y) OrIntL 103
106. ¬Set(x) v ¬Set(y) OrElim 92 93 104 101 105
107. ({x,y} = U) -> (¬Set(x) v ¬Set(y)) ImpInt 106
108. ¬Set(x) v ¬Set(y) Hyp
109. ¬Set(x) Hyp
110. ({x} = U) <-> ¬Set(x) TheoremInt
111. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U)) EquivExp 110
112. ¬Set(x) -> ({x} = U) AndElimR 111
113. {x} = U ImpElim 109 112
114. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
115. (x ∪ U) = U AndElimL 114
116. ∀x.((x ∪ U) = U) ForallInt 115
117. ({y} ∪ U) = U ForallElim 116
118. U = {x} Symmetry 113
119. ({y} ∪ {x}) = U EqualitySub 117 118
120. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
121. (x ∪ y) = (y ∪ x) AndElimL 120
122. ∀x.((x ∪ y) = (y ∪ x)) ForallInt 121

123. ({x} ∪ y) = (y ∪ {x}) ForallElim 122
124. ∀y.(({x} ∪ y) = (y ∪ {x})) ForallInt 123
125. ({x} ∪ {y}) = ({y} ∪ {x}) ForallElim 124
126. ({y} ∪ {x}) = ({x} ∪ {y}) Symmetry 125
127. ({x} ∪ {y}) = U EqualitySub 119 126
128. {x,y} = U EqualitySub 127 16
129. ¬Set(x) -> ({x,y} = U) ImpInt 128
130. ∀a.(¬Set(a) -> ({a,y} = U)) ForallInt 129
131. ∀b.∀a.(¬Set(a) -> ({a,b} = U)) ForallInt 130
132. ¬Set(y) Hyp
133. ∀a.(¬Set(a) -> ({a,z} = U)) ForallElim 131
134. ¬Set(y) -> ({y,z} = U) ForallElim 133
135. ∀z.(¬Set(y) -> ({y,z} = U)) ForallInt 134
136. ¬Set(y) -> ({y,x} = U) ForallElim 135
137. ∀x.({x,y} = ({x} ∪ {y})) ForallInt 15
138. {a,y} = ({a} ∪ {y}) ForallElim 137
139. ∀y.({a,y} = ({a} ∪ {y})) ForallInt 138
140. {a,b} = ({a} ∪ {b}) ForallElim 139
141. ∀a.({a,b} = ({a} ∪ {b})) ForallInt 140
142. {y,b} = ({y} ∪ {b}) ForallElim 141
143. ∀b.({y,b} = ({y} ∪ {b})) ForallInt 142
144. {y,x} = ({y} ∪ {x}) ForallElim 143
145. {y,x} = ({x} ∪ {y}) EqualitySub 144 126
146. {y,x} = {x,y} EqualitySub 145 16
147. ¬Set(y) -> ({x,y} = U) EqualitySub 136 146
148. {x,y} = U ImpElim 132 147
149. {x,y} = U OrElim 108 109 128 132 148
150. (¬Set(x) v ¬Set(y)) -> ({x,y} = U) ImpInt 149
151. (({x,y} = U) -> (¬Set(x) v ¬Set(y))) & ((¬Set(x) v ¬Set(y)) -> ({x,y} = U)) AndInt 107 150
152. ({x,y} = U) <-> (¬Set(x) v ¬Set(y)) EquivConst 151
153. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) AndInt 70 152 Qed

Used Theorems

1. Set(x) -> Set({x})
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
3. Set(x) -> ((y ε {x}) <-> (y = x))
4. ¬Set(U)
5. (A -> B) -> (¬B -> ¬A)
6. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
7. ({x} = U) <-> ¬Set(x)
8. ((x ∪ U) = U) & ((x ∩ U) = x)
10. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))

Th47. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})))

0. Set(x) & Set(y) Hyp
1. z ε ∩{x,y} Hyp
2. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
3. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 2
4. ∩{x,y} = {z: ∀x_0.((x_0 ε {x,y}) -> (z ε x_0))} ForallElim 3
5. z ε {z: ∀x_0.((x_0 ε {x,y}) -> (z ε x_0))} EqualitySub 1 4
6. Set(z) & ∀x_0.((x_0 ε {x,y}) -> (z ε x_0)) ClassElim 5
7. ∀x_0.((x_0 ε {x,y}) -> (z ε x_0)) AndElimR 6
8. (x ε {x,y}) -> (z ε x) ForallElim 7
9. (y ε {x,y}) -> (z ε y) ForallElim 7
10. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
11. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))) AndElimL 10
12. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))) ImpElim 0 11
13. (z ε {x,y}) <-> ((z = x) v (z = y)) AndElimR 12
14. ((z ε {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ε {x,y})) EquivExp 13
15. ((z = x) v (z = y)) -> (z ε {x,y}) AndElimR 14
16. ∀z.(((z = x) v (z = y)) -> (z ε {x,y})) ForallInt 15
17. ((x = x) v (x = y)) -> (x ε {x,y}) ForallElim 16
18. ∀z.(((z = x) v (z = y)) -> (z ε {x,y})) ForallInt 15

19. ((y = x) v (y = y)) -> (y ε {x,y}) ForallElim 18
20. x = x Identity
21. y = y Identity
22. (x = x) v (x = y) OrIntR 20
23. x ε {x,y} ImpElim 22 17
24. z ε x ImpElim 23 8
25. (y = x) v (y = y) OrIntL 21
26. y ε {x,y} ImpElim 25 19
27. z ε y ImpElim 26 9
28. (z ε x) & (z ε y) AndInt 24 27
29. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
30. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 29
31. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 30
32. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 31
33. z ε (x ∩ y) ImpElim 28 32
34. (z ε ∩{x,y}) -> (z ε (x ∩ y)) ImpInt 33
35. z ε (x ∩ y) Hyp
36. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 31
37. (z ε x) & (z ε y) ImpElim 35 36
38. c ε {x,y} Hyp
39. (z ε {x,y}) -> ((z = x) v (z = y)) AndElimL 14
40. ∀z.((z ε {x,y}) -> ((z = x) v (z = y))) ForallInt 39
41. (c ε {x,y}) -> ((c = x) v (c = y)) ForallElim 40
42. (c = x) v (c = y) ImpElim 38 41
43. c = x Hyp
44. z ε x AndElimL 37
45. x = c Symmetry 43
46. z ε c EqualitySub 44 45
47. c = y Hyp
48. z ε y AndElimR 37
49. y = c Symmetry 47
50. z ε c EqualitySub 48 49
51. z ε c OrElim 42 43 46 47 50
52. (c ε {x,y}) -> (z ε c) ImpInt 51
53. ∀c.((c ε {x,y}) -> (z ε c)) ForallInt 52
54. ∃c.(z ε c) ExistsInt 35
55. Set(z) DefSub 54
56. Set(z) & ∀c.((c ε {x,y}) -> (z ε c)) AndInt 55 53
57. z ε {c: ∀x_4.((x_4 ε {x,y}) -> (c ε x_4))} ClassInt 56
58. {z: ∀x_0.((x_0 ε {x,y}) -> (z ε x_0))} = ∩{x,y} Symmetry 4
59. z ε ∩{x,y} EqualitySub 57 58
60. (z ε (x ∩ y)) -> (z ε ∩{x,y}) ImpInt 59
61. ((z ε ∩{x,y}) -> (z ε (x ∩ y))) & ((z ε (x ∩ y)) -> (z ε ∩{x,y})) AndInt 34 60
62. (z ε ∩{x,y}) <-> (z ε (x ∩ y)) EquivConst 61
63. ∀z.((z ε ∩{x,y}) <-> (z ε (x ∩ y))) ForallInt 62
64. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
65. ∀x_6.((∩{x,y} = x_6) <-> ∀z.((z ε ∩{x,y}) <-> (z ε x_6))) ForallElim 64
66. (∩{x,y} = (x ∩ y)) <-> ∀z.((z ε ∩{x,y}) <-> (z ε (x ∩ y))) ForallElim 65
67. ((∩{x,y} = (x ∩ y)) -> ∀z.((z ε ∩{x,y}) <-> (z ε (x ∩ y)))) & (∀z.((z ε ∩{x,y}) <->
(z ε (x ∩ y))) -> (∩{x,y} = (x ∩ y))) EquivExp 66
68. ∀z.((z ε ∩{x,y}) <-> (z ε (x ∩ y))) -> (∩{x,y} = (x ∩ y)) AndElimR 67
69. ∩{x,y} = (x ∩ y) ImpElim 63 68
70. z ε ∪{x,y} Hyp
71. ∪x = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
72. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 71
73. ∪{x,y} = {z: ∃x_8.((x_8 ε {x,y}) & (z ε x_8))} ForallElim 72
74. z ε {z: ∃x_8.((x_8 ε {x,y}) & (z ε x_8))} EqualitySub 70 73
75. Set(z) & ∃x_8.((x_8 ε {x,y}) & (z ε x_8)) ClassElim 74
76. ∃x_8.((x_8 ε {x,y}) & (z ε x_8)) AndElimR 75
77. (u ε {x,y}) & (z ε u) Hyp
78. u ε {x,y} AndElimL 77
79. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
80. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))) AndElimL
79
81. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))) ImpElim 0 80
82. (z ε {x,y}) <-> ((z = x) v (z = y)) AndElimR 81
83. ((z ε {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ε {x,y})) EquivExp
82

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84. (z ε {x,y}) -> ((z = x) v (z = y))  AndElimL 83
85. ∀z.((z ε {x,y}) -> ((z = x) v (z = y)))  ForallInt 84
86. (u ε {x,y}) -> ((u = x) v (u = y))  ForallElim 85
87. (u = x) v (u = y)  ImpElim 78 86
88. u = x  Hyp
89. z ε u  AndElimR 77
90. z ε x  EqualitySub 89 88
91. (z ε x) v (z ε y)  OrIntR 90
92. u = y  Hyp
93. z ε y  EqualitySub 89 92
94. (z ε x) v (z ε y)  OrIntL 93
95. (z ε x) v (z ε y)  OrElim 87 88 91 92 94
96. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
97. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  AndElimL 96
98. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 97
99. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  AndElimR 98
100. z ε (x ∪ y)  ImpElim 95 99
101. z ε (x ∪ y)  ExistsElim 76 77 100
102. (z ε ∪{x,y}) -> (z ε (x ∪ y))  ImpInt 101
103. z ε (x ∪ y)  Hyp
104. (z ε (x ∪ y)) -> ((z ε x) v (z ε y))  AndElimL 98
105. (z ε x) v (z ε y)  ImpElim 103 104
106. z ε x  Hyp
107. ((z ε {x,y}) -> ((z = x) v (z = y))) & (((z = x) v (z = y)) -> (z ε {x,y}))
EquivExp 82
108. ((z = x) v (z = y)) -> (z ε {x,y})  AndElimR 107
109. ∀z.(((z = x) v (z = y)) -> (z ε {x,y}))  ForallInt 108
110. ((x = x) v (x = y)) -> (x ε {x,y})  ForallElim 109
111. x = x  Identity
112. (x = x) v (x = y)  OrIntR 111
113. x ε {x,y}  ImpElim 112 110
114. (x ε {x,y}) & (z ε x)  AndInt 113 106
115. ∃a.((a ε {x,y}) & (z ε a))  ExistsInt 114
116. ∃y.(z ε y)  ExistsInt 106
117. Set(z)  DefSub 116
118. Set(z) & ∃a.((a ε {x,y}) & (z ε a))  AndInt 117 115
119. z ε {b: ∃a.((a ε {x,y}) & (b ε a))}  ClassInt 118
120. {z: ∃x_8.((x_8 ε {x,y}) & (z ε x_8))} = ∪{x,y}  Symmetry 73
121. z ε ∪{x,y}  EqualitySub 119 120
122. z ε y  Hyp
123. y = y  Identity
124. ∀z.(((z = x) v (z = y)) -> (z ε {x,y}))  ForallInt 108
125. ((y = x) v (y = y)) -> (y ε {x,y})  ForallElim 124
126. (y = x) v (y = y)  OrIntL 123
127. y ε {x,y}  ImpElim 126 125
128. (y ε {x,y}) & (z ε y)  AndInt 127 122
129. ∃a.((a ε {x,y}) & (z ε a))  ExistsInt 128
130. ∃y.(z ε y)  ExistsInt 122
131. Set(z)  DefSub 130
132. Set(z) & ∃a.((a ε {x,y}) & (z ε a))  AndInt 131 129
133. z ε {b: ∃a.((a ε {x,y}) & (b ε a))}  ClassInt 132
134. z ε ∪{x,y}  EqualitySub 133 120
135. z ε ∪{x,y}  OrElim 105 106 121 122 134
136. (z ε (x ∪ y)) -> (z ε ∪{x,y})  ImpInt 135
137. ((z ε ∪{x,y}) -> (z ε (x ∪ y))) & ((z ε (x ∪ y)) -> (z ε ∪{x,y}))  AndInt 102 136
138. (z ε ∪{x,y}) <-> (z ε (x ∪ y))  EquivConst 137
139. ∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y)))  ForallInt 138
140. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
141. ∀x_14.((∪{x,y} = x_14) <-> ∀z.((z ε ∪{x,y}) <-> (z ε x_14)))  ForallElim 140
142. (∪{x,y} = (x ∪ y)) <-> ∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y)))  ForallElim 141
143. ((∪{x,y} = (x ∪ y)) -> ∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y)))) & (∀z.((z ε ∪{x,y}) <->
(z ε (x ∪ y))) -> (∪{x,y} = (x ∪ y)))  EquivExp 142
144. ∀z.((z ε ∪{x,y}) <-> (z ε (x ∪ y))) -> (∪{x,y} = (x ∪ y))  AndElimR 143
145. ∪{x,y} = (x ∪ y)  ImpElim 139 144
146. (∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y))  AndInt 69 145
147. (Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))  ImpInt 146
148. ¬Set(x) v ¬Set(y)  Hyp
149. ({x} = U) <-> ¬Set(x)  TheoremInt
150. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U))  EquivExp 149
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151. ¬Set(x) -> ({x} = U) AndElimR 150
152. ¬Set(x) Hyp
153. {x} = U ImpElim 152 151
154. {x,y} = ({x} ∪ {y}) DefEqInt
155. {x,y} = (U ∪ {y}) EqualitySub 154 153
156. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
157. (x ∪ U) = U AndElimL 156
158. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
159. (x ∪ y) = (y ∪ x) AndElimL 158
160. ∀y.((x ∪ y) = (y ∪ x)) ForallInt 159
161. (x ∪ U) = (U ∪ x) ForallElim 160
162. (U ∪ x) = U EqualitySub 157 161
163. ∀x.((U ∪ x) = U) ForallInt 162
164. (U ∪ {y}) = U ForallElim 163
165. {x,y} = U EqualitySub 155 164
166. (0 = ∩U) & (U = ∪U) TheoremInt
167. U = {x,y} Symmetry 165
168. (0 = ∩{x,y}) & (U = ∪{x,y}) EqualitySub 166 167
169. ¬Set(y) Hyp
170. ∀x.(¬Set(x) -> ({x} = U)) ForallInt 151
171. ¬Set(y) -> ({y} = U) ForallElim 170
172. {y} = U ImpElim 169 171
173. {x,y} = ({x} ∪ U) EqualitySub 154 172
174. ∀x.((x ∪ U) = U) ForallInt 157
175. ({x} ∪ U) = U ForallElim 174
176. {x,y} = U EqualitySub 173 175
177. U = {x,y} Symmetry 176
178. (0 = ∩{x,y}) & (U = ∪{x,y}) EqualitySub 166 177
179. (0 = ∩{x,y}) & (U = ∪{x,y}) OrElim 148 152 168 169 178
180. (¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})) ImpInt 179
181. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y}))) AndInt 147 180 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
3. ({x} = U) <-> ¬Set(x)
4. ((x ∪ U) = U) & ((x ∩ U) = x)
5. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
6. (0 = ∩U) & (U = ∪U)

Th49. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

0. Set(x) & Set(y) Hyp
1. Set(x) AndElimL 0
2. Set(x) -> Set({x}) TheoremInt
3. Set({x}) ImpElim 1 2
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
5. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))) AndElimL 4
6. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))) ImpElim 0 5
7. Set({x,y}) AndElimL 6
8. ∀x.((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) ForallInt 5
9. (Set({x}) & Set(y)) -> (Set({{x},y}) & ((z ε {{x},y}) <-> ((z = {x}) v (z = y)))) ForallElim 8
10. ∀y.((Set({x}) & Set(y)) -> (Set({{x},y}) & ((z ε {{x},y}) <-> ((z = {x}) v (z = y))))) ForallInt 9
11. (Set({x}) & Set({x,y})) -> (Set({{x},{x,y}}) & ((z ε {{x},{x,y}}) <-> ((z = {x}) v (z = {x,y})))) ForallElim 10
12. Set({x}) & Set({x,y}) AndInt 3 7
13. Set({{x},{x,y}}) & ((z ε {{x},{x,y}}) <-> ((z = {x}) v (z = {x,y}))) ImpElim 12 11
14. Set({{x},{x,y}}) AndElimL 13
15. (x,y) = {{x},{x,y}} DefEqInt
16. {{x},{x,y}} = (x,y) Symmetry 15
17. Set((x,y)) EqualitySub 14 16
18. (Set(x) & Set(y)) -> Set((x,y)) ImpInt 17
19. ¬Set(x) v ¬Set(y) Hyp
20. ¬Set(x) Hyp

21. ({x} = U) <-> ¬Set(x) TheoremInt
22. (({x} = U) -> ¬Set(x)) & (¬Set(x) -> ({x} = U)) EquivExp 21
23. ¬Set(x) -> ({x} = U) AndElimR 22
24. {x} = U ImpElim 20 23
25. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
26. ({x,y} = U) <-> (¬Set(x) v ¬Set(y)) AndElimR 25
27. (({x,y} = U) -> (¬Set(x) v ¬Set(y))) & ((¬Set(x) v ¬Set(y)) -> ({x,y} = U)) EquivExp 26
28. (¬Set(x) v ¬Set(y)) -> ({x,y} = U) AndElimR 27
29. ¬Set(x) v ¬Set(y) OrIntR 20
30. {x,y} = U ImpElim 29 28
31. ¬Set(U) TheoremInt
32. U = {x} Symmetry 24
33. ¬Set({x}) EqualitySub 31 32
34. ∀x.(¬Set(x) -> ({x} = U)) ForallInt 23
35. ¬Set({x}) -> ({{x}} = U) ForallElim 34
36. {{x}} = U ImpElim 33 35
37. {x,y} = ({x} ∪ {y}) DefEqInt
38. ∀x.({x,y} = ({x} ∪ {y})) ForallInt 37
39. {{x},y} = ({{x}} ∪ {y}) ForallElim 38
40. ∀y.({{x},y} = ({{x}} ∪ {y})) ForallInt 39
41. {{x},{x,y}} = ({{x}} ∪ {{x,y}}) ForallElim 40
42. U = {x,y} Symmetry 30
43. ¬Set({x,y}) EqualitySub 31 42
44. ∀x.(¬Set(x) -> ({x} = U)) ForallInt 23
45. ¬Set({x,y}) -> ({{x,y}} = U) ForallElim 44
46. {{x,y}} = U ImpElim 43 45
47. {{x},{x,y}} = ({{x}} ∪ U) EqualitySub 41 46
48. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
49. (x ∪ U) = U AndElimL 48
50. ∀x.((x ∪ U) = U) ForallInt 49
51. ({{x}} ∪ U) = U ForallElim 50
52. {{x},{x,y}} = U EqualitySub 47 51
53. (x,y) = U EqualitySub 15 52
54. U = (x,y) Symmetry 53
55. ¬Set((x,y)) EqualitySub 31 54
56. ¬Set(y) Hyp
57. ¬Set(x) v ¬Set(y) OrIntL 56
58. {x,y} = U ImpElim 57 28
59. U = {x,y} Symmetry 58
60. ¬Set({x,y}) EqualitySub 31 59
61. {{x,y}} = U ImpElim 60 45
62. {{x},{x,y}} = ({{x}} ∪ U) EqualitySub 41 61
63. {{x},{x,y}} = U EqualitySub 62 51
64. (x,y) = U EqualitySub 15 63
65. U = (x,y) Symmetry 64
66. ¬Set((x,y)) EqualitySub 31 65
67. ¬Set((x,y)) OrElim 19 20 55 56 66
68. (¬Set(x) v ¬Set(y)) -> ¬Set((x,y)) ImpInt 67
69. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B)) TheoremInt
70. ¬(A & B) <-> (¬A v ¬B) AndElimR 69
71. (¬(A & B) -> (¬A v ¬B)) & ((¬A v ¬B) -> ¬(A & B)) EquivExp 70
72. ¬(A & B) -> (¬A v ¬B) AndElimL 71
73. ¬(Set(x) & B) -> (¬Set(x) v ¬B) PolySub 72
74. ¬(Set(x) & Set(y)) -> (¬Set(x) v ¬Set(y)) PolySub 73
75. ¬(Set(x) & Set(y)) Hyp
76. ¬Set(x) v ¬Set(y) ImpElim 75 74
77. ¬Set((x,y)) ImpElim 76 68
78. ¬(Set(x) & Set(y)) -> ¬Set((x,y)) ImpInt 77
79. (A -> B) -> (¬B -> ¬A) TheoremInt
80. (¬(Set(x) & Set(y)) -> B) -> (¬B -> ¬¬(Set(x) & Set(y))) PolySub 79
81. (¬(Set(x) & Set(y)) -> ¬Set((x,y))) -> (¬¬Set((x,y)) -> ¬¬(Set(x) & Set(y))) PolySub 80
82. ¬¬Set((x,y)) -> ¬¬(Set(x) & Set(y)) ImpElim 78 81
83. D <-> ¬¬D TheoremInt
84. (D -> ¬¬D) & (¬¬D -> D) EquivExp 83
85. D -> ¬¬D AndElimL 84
86. (D -> ¬¬D) & (¬¬D -> D) EquivExp 83
87. ¬¬D -> D AndElimR 86
88. Set((x,y)) -> ¬¬Set((x,y)) PolySub 85

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89. ¬¬(Set(x) & Set(y)) -> (Set(x) & Set(y))  PolySub 87
90. Set((x,y))  Hyp
91. ¬¬Set((x,y))  ImpElim 90 88
92. ¬¬(Set(x) & Set(y))  ImpElim 91 82
93. Set(x) & Set(y)  ImpElim 92 89
94. Set((x,y)) -> (Set(x) & Set(y))  ImpInt 93
95. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  AndInt 18 94
96. (Set(x) & Set(y)) <-> Set((x,y))  EquivConst 95
97. ¬Set((x,y))  Hyp
98. ((Set(x) & Set(y)) -> B) -> (¬B -> ¬(Set(x) & Set(y)))  PolySub 79
99. ((Set(x) & Set(y)) -> Set((x,y))) -> (¬Set((x,y)) -> ¬(Set(x) & Set(y)))  PolySub 98
100. ¬Set((x,y)) -> ¬(Set(x) & Set(y))  ImpElim 18 99
101. ¬(Set(x) & Set(y))  ImpElim 97 100
102. ¬Set(x) v ¬Set(y)  ImpElim 101 74
103. ¬Set(x)  Hyp
104. {x} = U  ImpElim 103 23
105. U = {x}  Symmetry 104
106. ¬Set({x})  EqualitySub 31 105
107. {{x}} = U  ImpElim 106 35
108. {{x},{x,y}} = (U ∪ {{x,y}})  EqualitySub 41 107
109. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))  TheoremInt
110. (x ∪ y) = (y ∪ x)  AndElimL 109
111. ∀x.((x ∪ y) = (y ∪ x))  ForallInt 110
112. (U ∪ y) = (y ∪ U)  ForallElim 111
113. ∀y.((U ∪ y) = (y ∪ U))  ForallInt 112
114. (U ∪ {{x,y}}) = ({{x,y}} ∪ U)  ForallElim 113
115. {{x},{x,y}} = ({{x,y}} ∪ U)  EqualitySub 108 114
116. ((x ∪ U) = U) & ((x ∩ U) = x)  TheoremInt
117. (x ∪ U) = U  AndElimL 116
118. ∀x.((x ∪ U) = U)  ForallInt 117
119. ({{x,y}} ∪ U) = U  ForallElim 118
120. (U ∪ {{x,y}}) = U  EqualitySub 114 119
121. {{x},{x,y}} = U  EqualitySub 108 120
122. (x,y) = U  EqualitySub 15 121
123. ¬Set(y)  Hyp
124. ({x,y} = U) <-> (¬Set(x) v ¬Set(y))  AndElimR 25
125. (({x,y} = U) -> (¬Set(x) v ¬Set(y))) & ((¬Set(x) v ¬Set(y)) -> ({x,y} = U))
EquivExp 124
126. (¬Set(x) v ¬Set(y)) -> ({x,y} = U)  AndElimR 125
127. ¬Set(x) v ¬Set(y)  OrIntL 123
128. {x,y} = U  ImpElim 127 126
129. U = {x,y}  Symmetry 128
130. ¬Set({x,y})  EqualitySub 31 129
131. {{x,y}} = U  ImpElim 130 45
132. {{x},{x,y}} = ({{x}} ∪ U)  EqualitySub 41 131
133. ∀x.((x ∪ U) = U)  ForallInt 117
134. ({{x}} ∪ U) = U  ForallElim 133
135. {{x},{x,y}} = U  EqualitySub 132 134
136. (x,y) = U  EqualitySub 15 135
137. (x,y) = U  OrElim 102 103 122 123 136
138. ¬Set((x,y)) -> ((x,y) = U)  ImpInt 137
139. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  AndInt 96 138 Qed
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Used Theorems

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1. Set(x) -> Set({x})
2. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))
3. ({x} = U) <-> ¬Set(x)
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))
5. ¬Set(U)
6. ((x ∪ U) = U) & ((x ∩ U) = x)
9. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
7. (A -> B) -> (¬B -> ¬A)
8. D <-> ¬¬D
10. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
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Th50. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) &
(∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) ->
(((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))
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0. Set(x) & Set(y)  Hyp
1. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})))  TheoremInt
2. (Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))  AndElimL 1
3. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))  TheoremInt
4. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))  AndElimL 3
5. Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))  ImpElim 0 4
6. Set({x,y})  AndElimL 5
7. Set(x) -> Set({x})  TheoremInt
8. Set(x)  AndElimL 0
9. Set({x})  ImpElim 8 7
10. ∀x.(((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y}))))  ForallInt 1
11. ((Set({x}) & Set(y)) -> ((∩{{x},y} = ({x} ∩ y)) & (∪{{x},y} = ({x} ∪ y)))) & ((¬Set({x}) v ¬Set(y)) -> ((0 = ∩{{x},y}) & (U = ∪{{x},y})))  ForallElim 10
12. ∀y.(((Set({x}) & Set(y)) -> ((∩{{x},y} = ({x} ∩ y)) & (∪{{x},y} = ({x} ∪ y)))) & ((¬Set({x}) v ¬Set(y)) -> ((0 = ∩{{x},y}) & (U = ∪{{x},y}))))  ForallInt 11
13. ((Set({x}) & Set({x,y})) -> ((∩{{x},{x,y}} = ({x} ∩ {x,y})) & (∪{{x},{x,y}} = ({x} ∪ {x,y})))) & ((¬Set({x}) v ¬Set({x,y})) -> ((0 = ∩{{x},{x,y}}) & (U = ∪{{x},{x,y}})))  ForallElim 12
14. Set({x}) & Set({x,y})  AndInt 9 6
15. (Set({x}) & Set({x,y})) -> ((∩{{x},{x,y}} = ({x} ∩ {x,y})) & (∪{{x},{x,y}} = ({x} ∪ {x,y})))  AndElimL 13
16. (∩{{x},{x,y}} = ({x} ∩ {x,y})) & (∪{{x},{x,y}} = ({x} ∪ {x,y}))  ImpElim 14 15
17. {x,y} = ({x} ∪ {y})  DefEqInt
18. (∩{{x},{x,y}} = ({x} ∩ ({x} ∪ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))  EqualitySub 16 17
19. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))  TheoremInt
20. ∀x.(((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))))  ForallInt 19
21. (({x} ∩ (y ∪ z)) = (({x} ∩ y) ∪ ({x} ∩ z))) & (({x} ∪ (y ∩ z)) = (({x} ∪ y) ∩ ({x} ∪ z)))  ForallElim 20
22. ∀y.((({x} ∩ (y ∪ z)) = (({x} ∩ y) ∪ ({x} ∩ z))) & (({x} ∪ (y ∩ z)) = (({x} ∪ y) ∩ ({x} ∪ z))))  ForallInt 21
23. (({x} ∩ ({x} ∪ z)) = (({x} ∩ {x}) ∪ ({x} ∩ z))) & (({x} ∪ ({x} ∩ z)) = (({x} ∪ {x}) ∩ ({x} ∪ z)))  ForallElim 22
24. ∀z.((({x} ∩ ({x} ∪ z)) = (({x} ∩ {x}) ∪ ({x} ∩ z))) & (({x} ∪ ({x} ∩ z)) = (({x} ∪ {x}) ∩ ({x} ∪ z))))  ForallInt 23
25. (({x} ∩ ({x} ∪ {y})) = (({x} ∩ {x}) ∪ ({x} ∩ {y}))) & (({x} ∪ ({x} ∩ {y})) = (({x} ∪ {x}) ∩ ({x} ∪ {y})))  ForallElim 24
26. ((x ∪ x) = x) & ((x ∩ x) = x)  TheoremInt
27. ∀x.(((x ∪ x) = x) & ((x ∩ x) = x))  ForallInt 26
28. (({x} ∪ {x}) = {x}) & (({x} ∩ {x}) = {x})  ForallElim 27
29. ({x} ∪ {x}) = {x}  AndElimL 28
30. ({x} ∩ {x}) = {x}  AndElimR 28
31. ({x} ∩ ({x} ∪ {y})) = (({x} ∩ {x}) ∪ ({x} ∩ {y}))  AndElimL 25
32. ({x} ∪ ({x} ∩ {y})) = (({x} ∪ {x}) ∩ ({x} ∪ {y}))  AndElimR 25
33. (∩{{x},{x,y}} = (({x} ∩ {x}) ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))  EqualitySub 18 31
34. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ ({x} ∪ {y})))  EqualitySub 33 30
35. (((x ∪ y) ∪ z) = (x ∪ (y ∪ z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))  TheoremInt
36. ((x ∪ y) ∪ z) = (x ∪ (y ∪ z))  AndElimL 35
37. ∀x.(((x ∪ y) ∪ z) = (x ∪ (y ∪ z)))  ForallInt 36
38. (({x} ∪ y) ∪ z) = ({x} ∪ (y ∪ z))  ForallElim 37
39. ∀y.((({x} ∪ y) ∪ z) = ({x} ∪ (y ∪ z)))  ForallInt 38
40. (({x} ∪ {x}) ∪ z) = ({x} ∪ ({x} ∪ z))  ForallElim 39
41. ∀z.((({x} ∪ {x}) ∪ z) = ({x} ∪ ({x} ∪ z)))  ForallInt 40
42. (({x} ∪ {x}) ∪ {y}) = ({x} ∪ ({x} ∪ {y}))  ForallElim 41
43. ({x} ∪ ({x} ∪ {y})) = (({x} ∪ {x}) ∪ {y})  Symmetry 42
44. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = (({x} ∪ {x}) ∪ {y}))  EqualitySub 34 43
45. (∩{{x},{x,y}} = ({x} ∪ ({x} ∩ {y}))) & (∪{{x},{x,y}} = ({x} ∪ {y}))  EqualitySub 44 29
46. z ε ({x} ∩ {y})  Hyp
47. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))  TheoremInt
48. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 47
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49. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 48
50. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 49
51. ∀x.((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) ForallInt 50
52. (z ε ({x} ∩ y)) -> ((z ε {x}) & (z ε y)) ForallElim 51
53. ∀y.((z ε ({x} ∩ y)) -> ((z ε {x}) & (z ε y))) ForallInt 52
54. (z ε ({x} ∩ {y})) -> ((z ε {x}) & (z ε {y})) ForallElim 53
55. (z ε {x}) & (z ε {y}) ImpElim 46 54
56. z ε {x} AndElimL 55
57. (z ε ({x} ∩ {y})) -> (z ε {x}) ImpInt 56
58. ∀z.((z ε ({x} ∩ {y})) -> (z ε {x})) ForallInt 57
59. ∀x.∀z.((z ε ({x} ∩ {y})) -> (z ε {x})) ForallInt 58
60. ∀z.((z ε ({a} ∩ {y})) -> (z ε {a})) ForallElim 59
61. ∀y.∀z.((z ε ({a} ∩ {y})) -> (z ε {a})) ForallInt 60
62. ∀z.((z ε ({a} ∩ {b})) -> (z ε {a})) ForallElim 61
63. ({a} ∩ {b}) ⊂ {a} DefSub 62
64. (x ⊂ y) <-> ((x ∪ y) = y) TheoremInt
65. ∀x.((x ⊂ y) <-> ((x ∪ y) = y)) ForallInt 64
66. (({a} ∩ {b}) ⊂ y) <-> ((({a} ∩ {b}) ∪ y) = y) ForallElim 65
67. ∀y.((({a} ∩ {b}) ⊂ y) <-> ((({a} ∩ {b}) ∪ y) = y)) ForallInt 66
68. (({a} ∩ {b}) ⊂ {a}) <-> ((({a} ∩ {b}) ∪ {a}) = {a}) ForallElim 67
69. ((({a} ∩ {b}) ⊂ {a}) -> ((({a} ∩ {b}) ∪ {a}) = {a})) & (((({a} ∩ {b}) ∪ {a}) = {a}) -
> (({a} ∩ {b}) ⊂ {a})) EquivExp 68
70. (({a} ∩ {b}) ⊂ {a}) -> ((({a} ∩ {b}) ∪ {a}) = {a}) AndElimL 69
71. (({a} ∩ {b}) ∪ {a}) = {a} ImpElim 63 70
72. ∀a.((({a} ∩ {b}) ∪ {a}) = {a}) ForallInt 71
73. (({x} ∩ {b}) ∪ {x}) = {x} ForallElim 72
74. ∀b.((({x} ∩ {b}) ∪ {x}) = {x}) ForallInt 73
75. (({x} ∩ {y}) ∪ {x}) = {x} ForallElim 74
76. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
77. (x ∪ y) = (y ∪ x) AndElimL 76
78. ∀x.((x ∪ y) = (y ∪ x)) ForallInt 77
79. (({x} ∩ {a}) ∪ y) = (y ∪ ({x} ∩ {a})) ForallElim 78
80. ∀y.((({x} ∩ {a}) ∪ y) = (y ∪ ({x} ∩ {a}))) ForallInt 79
81. (({x} ∩ {a}) ∪ {x}) = ({x} ∪ ({x} ∩ {a})) ForallElim 80
82. ∀a.((({x} ∩ {a}) ∪ {x}) = ({x} ∪ ({x} ∩ {a}))) ForallInt 81
83. (({x} ∩ {y}) ∪ {x}) = ({x} ∪ ({x} ∩ {y})) ForallElim 82
84. ({x} ∪ ({x} ∩ {y})) = {x} EqualitySub 75 83
85. (∩{{x},{x,y}} = {x}) & (∪{{x},{x,y}} = ({x} ∪ {y})) EqualitySub 45 84
86. ({x} ∪ {y}) = {x,y} Symmetry 17
87. (∩{{x},{x,y}} = {x}) & (∪{{x},{x,y}} = {x,y}) EqualitySub 85 86
88. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))
TheoremInt
89. Set(x) -> ((∩{x} = x) & (∪{x} = x)) AndElimL 88
90. (∩{x} = x) & (∪{x} = x) ImpElim 8 89
91. (x,y) = {{x},{x,y}} DefEqInt
92. {{x},{x,y}} = (x,y) Symmetry 91
93. (∩(x,y) = {x}) & (∪(x,y) = {x,y}) EqualitySub 87 92
94. ∩(x,y) = {x} AndElimL 93
95. ∪(x,y) = {x,y} AndElimR 93
96. {x} = ∩(x,y) Symmetry 94
97. {x,y} = ∪(x,y) Symmetry 95
98. ∩{x} = x AndElimL 90
99. ∩∩(x,y) = x EqualitySub 98 96
100. ∪{x} = x AndElimR 90
101. ∪∩(x,y) = x EqualitySub 100 96
102. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)))) & ((¬Set(x) v
¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y}))) TheoremInt
103. (Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y))) AndElimL 102
104. (∩{x,y} = (x ∩ y)) & (∪{x,y} = (x ∪ y)) ImpElim 0 103
105. ∩{x,y} = (x ∩ y) AndElimL 104
106. ∪{x,y} = (x ∪ y) AndElimR 104
107. ∩∪(x,y) = (x ∩ y) EqualitySub 105 97
108. ∪∪(x,y) = (x ∪ y) EqualitySub 106 97
109. (¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = ∪{x,y})) AndElimR 102
110. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
111. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 110
112. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 111
113. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 112
114. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B)) TheoremInt
115. ¬(A & B) <-> (¬A v ¬B) AndElimR 114

116. (¬(A & B) -> (¬A v ¬B)) & ((¬A v ¬B) -> ¬(A & B)) EquivExp 115
117. (¬A v ¬B) -> ¬(A & B) AndElimR 116
118. (¬Set(x) v ¬B) -> ¬(Set(x) & B) PolySub 117
119. (¬Set(x) v ¬Set(y)) -> ¬(Set(x) & Set(y)) PolySub 118
120. (A -> B) -> (¬B -> ¬A) TheoremInt
121. (Set((x,y)) -> B) -> (¬B -> ¬Set((x,y))) PolySub 120
122. (Set((x,y)) -> (Set(x) & Set(y))) -> (¬(Set(x) & Set(y)) -> ¬Set((x,y))) PolySub 121
123. ¬(Set(x) & Set(y)) -> ¬Set((x,y)) ImpElim 113 122
124. ¬Set((x,y)) -> ((x,y) = U) AndElimR 110
125. ¬Set(x) v ¬Set(y) Hyp
126. ¬(Set(x) & Set(y)) ImpElim 125 119
127. ¬Set((x,y)) ImpElim 126 123
128. (x,y) = U ImpElim 127 124
129. U = (x,y) Symmetry 128
130. (0 = ∩U) & (U = **U**U) TheoremInt
131. (0 = ∩(x,y)) & (U = **U**(x,y)) EqualitySub 130 129
132. U = **U**(x,y) AndElimR 131
133. 0 = ∩(x,y) AndElimL 131
134. (∩0 = U) & (**U**0 = 0) TheoremInt
135. (0 = ∩**U**(x,y)) & (U = **UU**(x,y)) EqualitySub 130 132
136. (∩∩(x,y) = U) & (**U**∩(x,y) = 0) EqualitySub 134 133
137. 0 = ∩**U**(x,y) AndElimL 135
138. U = **UU**(x,y) AndElimR 135
139. ∩**U**(x,y) = 0 Symmetry 137
140. **UU**(x,y) = U Symmetry 138
141. (**UU**(x,y) = U) & (∩**U**(x,y) = 0) AndInt 140 139
142. ∩∩(x,y) = U AndElimL 136
143. **U**∩(x,y) = 0 AndElimR 136
144. (**U**∩(x,y) = 0) & (∩∩(x,y) = U) AndInt 143 142
145. ((**U**∩(x,y) = 0) & (∩∩(x,y) = U)) & ((**UU**(x,y) = U) & (∩**U**(x,y) = 0)) AndInt 144 141
146. (¬Set(x) v ¬Set(y)) -> (((**U**∩(x,y) = 0) & (∩∩(x,y) = U)) & ((**UU**(x,y) = U) & (∩**U**(x,y) = 0))) ImpInt 145
147. (**U**(x,y) = {x,y}) & (∩(x,y) = {x}) AndInt 95 94
148. (**U**∩(x,y) = x) & (∩∩(x,y) = x) AndInt 101 99
149. (**UU**(x,y) = (x **U** y)) & (∩**U**(x,y) = (x ∩ y)) AndInt 108 107
150. ((**U**(x,y) = {x,y}) & (∩(x,y) = {x})) & ((**U**∩(x,y) = x) & (∩∩(x,y) = x)) AndInt 147 148
151. (((**U**(x,y) = {x,y}) & (∩(x,y) = {x})) & ((**U**∩(x,y) = x) & (∩∩(x,y) = x))) & ((**UU**(x,y) = (x **U** y)) & (∩**U**(x,y) = (x ∩ y))) AndInt 150 149
152. (Set(x) & Set(y)) -> ((((**U**(x,y) = {x,y}) & (∩(x,y) = {x})) & ((**U**∩(x,y) = x) & (∩∩(x,y) = x))) & ((**UU**(x,y) = (x **U** y)) & (∩**U**(x,y) = (x ∩ y)))) ImpInt 151
153. ((Set(x) & Set(y)) -> ((((**U**(x,y) = {x,y}) & (∩(x,y) = {x})) & ((**U**∩(x,y) = x) & (∩∩(x,y) = x))) & ((**UU**(x,y) = (x **U** y)) & (∩**U**(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((**U**∩(x,y) = 0) & (∩∩(x,y) = U)) & ((**UU**(x,y) = U) & (∩**U**(x,y) = 0)))) AndInt 152 146 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> ((∩{x,y} = (x ∩ y)) & (**U**{x,y} = (x **U** y)))) & ((¬Set(x) v ¬Set(y)) -> ((0 = ∩{x,y}) & (U = **U**{x,y})))
2. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y)))
3. Set(x) -> Set({x})
4. ((x ∩ (y **U** z)) = ((x ∩ y) **U** (x ∩ z))) & ((x **U** (y ∩ z)) = ((x **U** y) ∩ (x **U** z)))
5. ((x **U** x) = x) & ((x ∩ x) = x)
6. (((x **U** y) **U** z) = (x **U** (y **U** z))) & (((x ∩ y) ∩ z) = (x ∩ (y ∩ z)))
7. ((z ε (x **U** y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
9. (x ⊂ y) <-> ((x **U** y) = y)
10. ((x **U** y) = (y **U** x)) & ((x ∩ y) = (y ∩ x))
11. (Set(x) -> ((∩{x} = x) & (**U**{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (**U**{x} = U)))
12. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
13. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
14. (A -> B) -> (¬B -> ¬A)
15. (0 = ∩U) & (U = **U**U)
16. (∩0 = U) & (**U**0 = 0)

Th53. proj2(U) = U

0. proj2(x) = (∩**U**x **U** (**UU**x ~ **U**∩x)) DefEqInt
1. ∀x.(proj2(x) = (∩**U**x **U** (**UU**x ~ **U**∩x))) ForallInt 0
2. proj2(U) = (∩**U**U **U** (**UU**U ~ **U**∩U)) ForallElim 1

3. (0 = ∩U) & (U = ∪U) TheoremInt
4. (∩0 = U) & (∪0 = 0) TheoremInt
5. 0 = ∩U AndElimL 3
6. U = ∪U AndElimR 3
7. ∩0 = U AndElimL 4
8. ∪0 = 0 AndElimR 4
9. ∩U = 0 Symmetry 5
10. ∪U = U Symmetry 6
11. proj2(U) = (∩U ∪ (∪U ~ ∪∩U)) EqualitySub 2 10
12. proj2(U) = (0 ∪ (∪U ~ ∪0)) EqualitySub 11 9
13. proj2(U) = (0 ∪ (U ~ ∪0)) EqualitySub 12 10
14. proj2(U) = (0 ∪ (U ~ 0)) EqualitySub 13 8
15. ((0 ∪ x) = x) & ((0 ∩ x) = 0) TheoremInt
16. (0 ∪ x) = x AndElimL 15
17. ∀x.((0 ∪ x) = x) ForallInt 16
18. (0 ∪ (U ~ 0)) = (U ~ 0) ForallElim 17
19. proj2(U) = (U ~ 0) EqualitySub 14 18
20. (x ~ y) = (x ∩ ~y) DefEqInt
21. ∀x.((x ~ y) = (x ∩ ~y)) ForallInt 20
22. (U ~ y) = (U ∩ ~y) ForallElim 21
23. ∀y.((U ~ y) = (U ∩ ~y)) ForallInt 22
24. (U ~ 0) = (U ∩ ~0) ForallElim 23
25. (~0 = U) & (~U = 0) TheoremInt
26. ~0 = U AndElimL 25
27. (U ~ 0) = (U ∩ U) EqualitySub 24 26
28. ((x ∪ x) = x) & ((x ∩ x) = x) TheoremInt
29. (x ∩ x) = x AndElimR 28
30. ∀x.((x ∩ x) = x) ForallInt 29
31. (U ∩ U) = U ForallElim 30
32. (U ~ 0) = U EqualitySub 27 31
33. proj2(U) = U EqualitySub 19 32 Qed

Used Theorems

1. (0 = ∩U) & (U = ∪U)
2. (∩0 = U) & (∪0 = 0)
3. ((0 ∪ x) = x) & ((0 ∩ x) = 0)
5. (~0 = U) & (~U = 0)
6. ((x ∪ x) = x) & ((x ∩ x) = x)

Th54. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)))

0. Set(x) & Set(y) Hyp
1. proj1(x) = ∩∩x DefEqInt
2. proj2(x) = (∩∪x ∪ (∪∪x ~ ∪∩x)) DefEqInt
3. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0)))) TheoremInt
4. (Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y)))) AndElimL 3
5. (((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))) ImpElim 0 4
6. ((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x)) AndElimL 5
7. (∪∩(x,y) = x) & (∩∩(x,y) = x) AndElimR 6
8. ∩∩(x,y) = x AndElimR 7
9. ∀x.(proj1(x) = ∩∩x) ForallInt 1
10. ∀x.(proj1(x) = ∩∩x) ForallInt 1
11. proj1((x,y)) = ∩∩(x,y) ForallElim 10
12. proj1((x,y)) = x EqualitySub 11 8
13. ∀x.(proj2(x) = (∩∪x ∪ (∪∪x ~ ∪∩x))) ForallInt 2
14. proj2((x,y)) = (∩∪(x,y) ∪ (∪∪(x,y) ~ ∪∩(x,y))) ForallElim 13
15. ∪∩(x,y) = x AndElimL 7
16. (∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y)) AndElimR 5
17. ∪∪(x,y) = (x ∪ y) AndElimL 16
18. ∩∪(x,y) = (x ∩ y) AndElimR 16
19. proj2((x,y)) = (∩∪(x,y) ∪ ((x ∪ y) ~ ∪∩(x,y))) EqualitySub 14 17
20. proj2((x,y)) = ((x ∩ y) ∪ ((x ∪ y) ~ ∪∩(x,y))) EqualitySub 19 18
21. proj2((x,y)) = ((x ∩ y) ∪ ((x ∪ y) ~ x)) EqualitySub 20 15
22. z ε ((x ∪ y) ~ x) Hyp
23. (x ~ y) = (x ∩ ~y) DefEqInt

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24. ∀x.((x ~ y) = (x ∩ ~y))  ForallInt 23
25. (a ~ y) = (a ∩ ~y)  ForallElim 24
26. ∀y.((a ~ y) = (a ∩ ~y))  ForallInt 25
27. (a ~ b) = (a ∩ ~b)  ForallElim 26
28. ∀a.((a ~ b) = (a ∩ ~b))  ForallInt 27
29. ((x U y) ~ b) = ((x U y) ∩ ~b)  ForallElim 28
30. ∀b.(((x U y) ~ b) = ((x U y) ∩ ~b))  ForallInt 29
31. ((x U y) ~ x) = ((x U y) ∩ ~x)  ForallElim 30
32. z ε ((x U y) ∩ ~x)  EqualitySub 22 31
33. ((z ε (x U y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
34. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 33
35. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 34
36. (z ε (x ∩ y)) -> ((z ε x) & (z ε y))  AndElimL 35
37. ∀x.((z ε (x ∩ y)) -> ((z ε x) & (z ε y)))  ForallInt 36
38. (z ε (a ∩ y)) -> ((z ε a) & (z ε y))  ForallElim 37
39. ∀y.((z ε (a ∩ y)) -> ((z ε a) & (z ε y)))  ForallInt 38
40. (z ε (a ∩ b)) -> ((z ε a) & (z ε b))  ForallElim 39
41. ∀a.((z ε (a ∩ b)) -> ((z ε a) & (z ε b)))  ForallInt 40
42. (z ε ((x U y) ∩ b)) -> ((z ε (x U y)) & (z ε b))  ForallElim 41
43. ∀b.((z ε ((x U y) ∩ b)) -> ((z ε (x U y)) & (z ε b)))  ForallInt 42
44. (z ε ((x U y) ∩ ~x)) -> ((z ε (x U y)) & (z ε ~x))  ForallElim 43
45. (z ε (x U y)) & (z ε ~x)  ImpElim 32 44
46. z ε (x U y)  AndElimL 45
47. (z ε (x U y)) <-> ((z ε x) v (z ε y))  AndElimL 33
48. ((z ε (x U y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x U y)))
EquivExp 47
49. (z ε (x U y)) -> ((z ε x) v (z ε y))  AndElimL 48
50. (z ε x) v (z ε y)  ImpElim 46 49
51. z ε ~x  AndElimR 45
52. ~x = {y: ¬(y ε x)}  DefEqInt
53. z ε {y: ¬(y ε x)}  EqualitySub 51 52
54. Set(z) & ¬(z ε x)  ClassElim 53
55. ¬(z ε x)  AndElimR 54
56. z ε x  Hyp
57. _|_  ImpElim 56 55
58. z ε (y ∩ ~x)  AbsI 57
59. z ε y  Hyp
60. (z ε y) & (z ε ~x)  AndInt 59 51
61. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 34
62. ((z ε x) & (z ε y)) -> (z ε (x ∩ y))  AndElimR 61
63. ∀y.(((z ε x) & (z ε y)) -> (z ε (x ∩ y)))  ForallInt 62
64. ((z ε x) & (z ε a)) -> (z ε (x ∩ a))  ForallElim 63
65. ∀x.(((z ε x) & (z ε a)) -> (z ε (x ∩ a)))  ForallInt 64
66. ((z ε y) & (z ε a)) -> (z ε (y ∩ a))  ForallElim 65
67. ∀a.(((z ε y) & (z ε a)) -> (z ε (y ∩ a)))  ForallInt 66
68. ∀a.(((z ε y) & (z ε a)) -> (z ε (y ∩ a)))  ForallInt 66
69. ((z ε y) & (z ε ~x)) -> (z ε (y ∩ ~x))  ForallElim 68
70. z ε (y ∩ ~x)  ImpElim 60 69
71. z ε (y ∩ ~x)  OrElim 50 56 58 59 70
72. (z ε ((x U y) ~ x)) -> (z ε (y ∩ ~x))  ImpInt 71
73. z ε (y ∩ ~x)  Hyp
74. (z ε (x ∩ y)) -> ((z ε x) & (z ε y))  AndElimL 61
75. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y)))  ForallInt 74
76. (z ε (x ∩ a)) -> ((z ε x) & (z ε a))  ForallElim 75
77. ∀x.((z ε (x ∩ a)) -> ((z ε x) & (z ε a)))  ForallInt 76
78. (z ε (y ∩ a)) -> ((z ε y) & (z ε a))  ForallElim 77
79. ∀a.((z ε (y ∩ a)) -> ((z ε y) & (z ε a)))  ForallInt 78
80. (z ε (y ∩ ~x)) -> ((z ε y) & (z ε ~x))  ForallElim 79
81. (z ε y) & (z ε ~x)  ImpElim 73 80
82. z ε y  AndElimL 81
83. (z ε x) v (z ε y)  OrIntL 82
84. ((z ε x) v (z ε y)) -> (z ε (x U y))  AndElimR 48
85. z ε (x U y)  ImpElim 83 84
86. z ε ~x  AndElimR 81
87. (z ε (x U y)) & (z ε ~x)  AndInt 85 86
88. ((z ε x) & (z ε y)) -> (z ε (x ∩ y))  AndElimR 35
89. ∀y.(((z ε x) & (z ε y)) -> (z ε (x ∩ y)))  ForallInt 88
90. ((z ε x) & (z ε a)) -> (z ε (x ∩ a))  ForallElim 89
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91. ∀x.(((z ε x) & (z ε a)) -> (z ε (x ∩ a))) ForallInt 90
92. ((z ε (x ∪ y)) & (z ε a)) -> (z ε ((x ∪ y) ∩ a)) ForallElim 91
93. ∀a.(((z ε (x ∪ y)) & (z ε a)) -> (z ε ((x ∪ y) ∩ a))) ForallInt 92
94. ((z ε (x ∪ y)) & (z ε ~x)) -> (z ε ((x ∪ y) ∩ ~x)) ForallElim 93
95. z ε ((x ∪ y) ∩ ~x) ImpElim 87 94
96. ((x ∪ y) ∩ ~x) = ((x ∪ y) ~ x) Symmetry 31
97. z ε ((x ∪ y) ~ x) EqualitySub 95 96
98. (z ε (y ∩ ~x)) -> (z ε ((x ∪ y) ~ x)) ImpInt 97
99. ((z ε ((x ∪ y) ~ x)) -> (z ε (y ∩ ~x))) & ((z ε (y ∩ ~x)) -> (z ε ((x ∪ y) ~ x))) AndInt 72 98
100. (z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x)) EquivConst 99
101. ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x))) ForallInt 100
102. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
103. ∀o.((((x ∪ y) ~ x) = o) <-> ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε o))) ForallElim 102
104. (((x ∪ y) ~ x) = (y ∩ ~x)) <-> ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x))) ForallElim 103
105. ((((x ∪ y) ~ x) = (y ∩ ~x)) -> ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x)))) & (∀z. ((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x))) -> (((x ∪ y) ~ x) = (y ∩ ~x))) EquivExp 104
106. ∀z.((z ε ((x ∪ y) ~ x)) <-> (z ε (y ∩ ~x))) -> (((x ∪ y) ~ x) = (y ∩ ~x)) AndElimR 105
107. ((x ∪ y) ~ x) = (y ∩ ~x) ImpElim 101 106
108. proj2((x,y)) = ((x ∩ y) ∪ (y ∩ ~x)) EqualitySub 21 107
109. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x)) TheoremInt
110. (x ∩ y) = (y ∩ x) AndElimR 109
111. proj2((x,y)) = ((y ∩ x) ∪ (y ∩ ~x)) EqualitySub 108 110
112. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z))) TheoremInt
113. (x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z)) AndElimL 112
114. ((x ∩ y) ∪ (x ∩ z)) = (x ∩ (y ∪ z)) Symmetry 113
115. ∀x.(((x ∩ y) ∪ (x ∩ z)) = (x ∩ (y ∪ z))) ForallInt 114
116. ((a ∩ y) ∪ (a ∩ z)) = (a ∩ (y ∪ z)) ForallElim 115
117. ∀y.(((a ∩ y) ∪ (a ∩ z)) = (a ∩ (y ∪ z))) ForallInt 116
118. ((a ∩ b) ∪ (a ∩ z)) = (a ∩ (b ∪ z)) ForallElim 117
119. ∀a.(((a ∩ b) ∪ (a ∩ z)) = (a ∩ (b ∪ z))) ForallInt 118
120. ((y ∩ b) ∪ (y ∩ z)) = (y ∩ (b ∪ z)) ForallElim 119
121. ∀b.(((y ∩ b) ∪ (y ∩ z)) = (y ∩ (b ∪ z))) ForallInt 120
122. ((y ∩ x) ∪ (y ∩ z)) = (y ∩ (x ∪ z)) ForallElim 121
123. ∀z.(((y ∩ x) ∪ (y ∩ z)) = (y ∩ (x ∪ z))) ForallInt 122
124. ((y ∩ x) ∪ (y ∩ ~x)) = (y ∩ (x ∪ ~x)) ForallElim 123
125. proj2((x,y)) = (y ∩ (x ∪ ~x)) EqualitySub 111 124
126. z ε U Hyp
127. A v ¬A TheoremInt
128. (z ε x) v ¬(z ε x) PolySub 127
129. z ε x Hyp
130. (z ε x) v (z ε ~x) OrIntR 129
131. ∀y.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 84
132. ((z ε x) v (z ε ~x)) -> (z ε (x ∪ ~x)) ForallElim 131
133. z ε (x ∪ ~x) ImpElim 130 132
134. ¬(z ε x) Hyp
135. ∃y.(z ε y) ExistsInt 126
136. Set(z) DefSub 135
137. ¬(z ε x) & Set(z) AndInt 134 136
138. z ε {z: ¬(z ε x)} ClassInt 137
139. {y: ¬(y ε x)} = ~x Symmetry 52
140. z ε ~x EqualitySub 138 139
141. (z ε x) v (z ε ~x) OrIntL 140
142. z ε (x ∪ ~x) ImpElim 141 132
143. z ε (x ∪ ~x) OrElim 128 129 133 134 142
144. (z ε U) -> (z ε (x ∪ ~x)) ImpInt 143
145. ∀z.((z ε U) -> (z ε (x ∪ ~x))) ForallInt 144
146. U ⊂ (x ∪ ~x) DefSub 145
147. (0 ⊂ x) & (x ⊂ U) TheoremInt
148. x ⊂ U AndElimR 147
149. ∀x.(x ⊂ U) ForallInt 148
150. (x ∪ ~x) ⊂ U ForallElim 149
151. (U ⊂ (x ∪ ~x)) & ((x ∪ ~x) ⊂ U) AndInt 146 150
152. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
153. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y)) EquivExp 152
154. ((x ⊂ y) & (y ⊂ x)) -> (x = y) AndElimR 153
155. ∀x.(((x ⊂ y) & (y ⊂ x)) -> (x = y)) ForallInt 154
156. ((U ⊂ y) & (y ⊂ U)) -> (U = y) ForallElim 155

157. ∀y.(((U ⊂ y) & (y ⊂ U)) -> (U = y)) ForallInt 156
158. ((U ⊂ (x ∪ ~x)) & ((x ∪ ~x) ⊂ U)) -> (U = (x ∪ ~x)) ForallElim 157
159. U = (x ∪ ~x) ImpElim 151 158
160. (x ∪ ~x) = U Symmetry 159
161. proj2((x,y)) = (y ∩ U) EqualitySub 125 160
162. ((x ∪ U) = U) & ((x ∩ U) = x) TheoremInt
163. (x ∩ U) = x AndElimR 162
164. ∀x.((x ∩ U) = x) ForallInt 163
165. (y ∩ U) = y ForallElim 164
166. proj2((x,y)) = y EqualitySub 161 165
167. (proj1((x,y)) = x) & (proj2((x,y)) = y) AndInt 12 166
168. (Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y)) ImpInt 167
169. ¬Set(x) v ¬Set(y) Hyp
170. (¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))) AndElimR 3
171. ((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0)) ImpElim 169 170
172. (∪∩(x,y) = 0) & (∩∩(x,y) = U) AndElimL 171
173. ∩∩(x,y) = U AndElimR 172
174. proj1((x,y)) = U EqualitySub 11 173
175. (∪∪(x,y) = U) & (∩∪(x,y) = 0) AndElimR 171
176. ∩∪(x,y) = 0 AndElimR 175
177. ∪∪(x,y) = U AndElimL 175
178. ∪∩(x,y) = 0 AndElimL 172
179. proj2((x,y)) = (∩∪(x,y) ∪ (U ~ ∪∩(x,y))) EqualitySub 14 177
180. proj2((x,y)) = (∩∪(x,y) ∪ (U ~ 0)) EqualitySub 179 178
181. proj2((x,y)) = (0 ∪ (U ~ 0)) EqualitySub 180 176
182. ((0 ∪ x) = x) & ((0 ∩ x) = 0) TheoremInt
183. (0 ∪ x) = x AndElimL 182
184. ∀x.((0 ∪ x) = x) ForallInt 183
185. (0 ∪ (U ~ 0)) = (U ~ 0) ForallElim 184
186. proj2((x,y)) = (U ~ 0) EqualitySub 181 185
187. ∀x.((x ~ y) = (x ∩ ~y)) ForallInt 23
188. (U ~ y) = (U ∩ ~y) ForallElim 187
189. ∀y.((U ~ y) = (U ∩ ~y)) ForallInt 188
190. (U ~ 0) = (U ∩ ~0) ForallElim 189
191. proj2((x,y)) = (U ∩ ~0) EqualitySub 186 190
192. (~0 = U) & (~U = 0) TheoremInt
193. ~0 = U AndElimL 192
194. proj2((x,y)) = (U ∩ U) EqualitySub 191 193
195. ((x ∪ x) = x) & ((x ∩ x) = x) TheoremInt
196. (x ∩ x) = x AndElimR 195
197. ∀x.((x ∩ x) = x) ForallInt 196
198. (U ∩ U) = U ForallElim 197
199. proj2((x,y)) = U EqualitySub 194 198
200. (proj1((x,y)) = U) & (proj2((x,y)) = U) AndInt 174 199
201. (¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)) ImpInt 200
202. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U))) AndInt 168 201 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> ((((∪(x,y) = {x,y}) & (∩(x,y) = {x})) & ((∪∩(x,y) = x) & (∩∩(x,y) = x))) & ((∪∪(x,y) = (x ∪ y)) & (∩∪(x,y) = (x ∩ y))))) & ((¬Set(x) v ¬Set(y)) -> (((∪∩(x,y) = 0) & (∩∩(x,y) = U)) & ((∪∪(x,y) = U) & (∩∪(x,y) = 0))))
2. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
3. ((x ∪ y) = (y ∪ x)) & ((x ∩ y) = (y ∩ x))
4. ((x ∩ (y ∪ z)) = ((x ∩ y) ∪ (x ∩ z))) & ((x ∪ (y ∩ z)) = ((x ∪ y) ∩ (x ∪ z)))
0. A v ¬A
5. (0 ⊂ x) & (x ⊂ U)
6. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
8. ((x ∪ U) = U) & ((x ∩ U) = x)
7. ((0 ∪ x) = x) & ((0 ∩ x) = 0)
9. (~0 = U) & (~U = 0)
10. ((x ∪ x) = x) & ((x ∩ x) = x)

Th55. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

0. (Set(x) & Set(y)) & ((x,y) = (u,v)) Hyp
1. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U))) TheoremInt
2. (Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y)) AndElimL 1

3. Set(x) & Set(y) AndElimL 0
4. (proj1((x,y)) = x) & (proj2((x,y)) = y) ImpElim 3 2
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
6. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 5
7. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 6
8. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 7
9. Set((x,y)) ImpElim 3 8
10. (x,y) = (u,v) AndElimR 0
11. Set((u,v)) EqualitySub 9 10
12. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 6
13. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 12
14. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 13
15. Set((u,y)) -> (Set(u) & Set(y)) ForallElim 14
16. ∀y.(Set((u,y)) -> (Set(u) & Set(y))) ForallInt 15
17. Set((u,v)) -> (Set(u) & Set(v)) ForallElim 16
18. Set(u) & Set(v) ImpElim 11 17
19. ∀x.((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) ForallInt 2
20. (Set(u) & Set(y)) -> ((proj1((u,y)) = u) & (proj2((u,y)) = y)) ForallElim 19
21. ∀y.((Set(u) & Set(y)) -> ((proj1((u,y)) = u) & (proj2((u,y)) = y))) ForallInt 20
22. (Set(u) & Set(v)) -> ((proj1((u,v)) = u) & (proj2((u,v)) = v)) ForallElim 21
23. (proj1((u,v)) = u) & (proj2((u,v)) = v) ImpElim 18 22
24. proj1((x,y)) = x AndElimL 4
25. proj2((x,y)) = y AndElimR 4
26. proj1((u,v)) = u AndElimL 23
27. proj2((u,v)) = v AndElimR 23
28. proj1((u,v)) = x EqualitySub 24 10
29. u = x EqualitySub 28 26
30. proj2((u,v)) = y EqualitySub 25 10
31. v = y EqualitySub 30 27
32. x = u Symmetry 29
33. y = v Symmetry 31
34. (x = u) & (y = v) AndInt 32 33
35. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) ImpInt 34 Qed

Used Theorems

1. ((Set(x) & Set(y)) -> ((proj1((x,y)) = x) & (proj2((x,y)) = y))) & ((¬Set(x) v ¬Set(y)) -> ((proj1((x,y)) = U) & (proj2((x,y)) = U)))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th58. ((r∘s)∘t) = (r∘(s∘t))

0. z ε ((r∘s)∘t) Hyp
1. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))} DefEqInt
2. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
3. ((r∘s)∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (r∘s))) & (w = (x,z)))} ForallElim 2
4. ∀b.(((r∘s)∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (r∘s))) & (w = (x,z)))}) ForallInt 3
5. ((r∘s)∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε (r∘s))) & (w = (x,z)))} ForallElim 4
6. z ε {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε (r∘s))) & (w = (x,z)))} EqualitySub 0 5
7. Set(z) & ∃x.∃y.∃x_1.((((x,y) ε t) & ((y,x_1) ε (r∘s))) & (z = (x,x_1))) ClassElim 6
8. ∃x.∃y.∃x_1.((((x,y) ε t) & ((y,x_1) ε (r∘s))) & (z = (x,x_1))) AndElimR 7
9. ∃y.∃x_1.((((x,y) ε t) & ((y,x_1) ε (r∘s))) & (z = (x,x_1))) Hyp
10. ∃x_1.((((x,y) ε t) & ((y,x_1) ε (r∘s))) & (z = (x,x_1))) Hyp
11. (((x,y) ε t) & ((y,c) ε (r∘s))) & (z = (x,c)) Hyp
12. ((x,y) ε t) & ((y,c) ε (r∘s)) AndElimL 11
13. (y,c) ε (r∘s) AndElimR 12
14. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
15. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 14
16. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}) ForallInt 15
17. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 16
18. (y,c) ε {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} EqualitySub 13 17
19. Set((y,c)) & ∃x.∃x_2.∃z.((((x,x_2) ε s) & ((x_2,z) ε r)) & ((y,c) = (x,z))) ClassElim 18
20. ∃x.∃x_2.∃z.((((x,x_2) ε s) & ((x_2,z) ε r)) & ((y,c) = (x,z))) AndElimR 19
21. ∃x_2.∃z.((((a,x_2) ε s) & ((x_2,z) ε r)) & ((y,c) = (a,z))) Hyp
22. ∃z.((((a,b) ε s) & ((b,z) ε r)) & ((y,c) = (a,z))) Hyp
23. (((a,b) ε s) & ((b,d) ε r)) & ((y,c) = (a,d)) Hyp
24. ((a,b) ε s) & ((b,d) ε r) AndElimL 23

25. (x,y) ε t AndElimL 12
26. (a,b) ε s AndElimL 24
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
28. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 27
29. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 28
30. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 29
31. ∀y.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 30
32. Set((x,c)) -> (Set(x) & Set(c)) ForallElim 31
33. ∀x.(Set((x,c)) -> (Set(x) & Set(c))) ForallInt 32
34. Set((y,c)) -> (Set(y) & Set(c)) ForallElim 33
35. Set((y,c)) AndElimL 19
36. Set(y) & Set(c) ImpElim 35 34
37. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
38. ∀y.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 37
39. ((Set(x) & Set(c)) & ((x,c) = (u,v))) -> ((x = u) & (c = v)) ForallElim 38
40. ∀x.(((Set(x) & Set(c)) & ((x,c) = (u,v))) -> ((x = u) & (c = v))) ForallInt 39
41. ((Set(y) & Set(c)) & ((y,c) = (u,v))) -> ((y = u) & (c = v)) ForallElim 40
42. ∀u.(((Set(y) & Set(c)) & ((y,c) = (u,v))) -> ((y = u) & (c = v))) ForallInt 41
43. ((Set(y) & Set(c)) & ((y,c) = (a,v))) -> ((y = a) & (c = v)) ForallElim 42
44. ∀v.(((Set(y) & Set(c)) & ((y,c) = (a,v))) -> ((y = a) & (c = v))) ForallInt 43
45. ((Set(y) & Set(c)) & ((y,c) = (a,d))) -> ((y = a) & (c = d)) ForallElim 44
46. (y,c) = (a,d) AndElimR 23
47. (Set(y) & Set(c)) & ((y,c) = (a,d)) AndInt 36 46
48. (y = a) & (c = d) ImpElim 47 45
49. y = a AndElimL 48
50. c = d AndElimR 48
51. (x,a) ε t EqualitySub 25 49
52. ((x,a) ε t) & ((a,b) ε s) AndInt 51 26
53. (b,d) ε r AndElimR 24
54. g = (x,b) Hyp
55. (((x,a) ε t) & ((a,b) ε s)) & (g = (x,b)) AndInt 52 54
56. ∃b.((((x,a) ε t) & ((a,b) ε s)) & (g = (x,b))) ExistsInt 55
57. ∃a.∃b.((((x,a) ε t) & ((a,b) ε s)) & (g = (x,b))) ExistsInt 56
58. ∃x.∃a.∃b.((((x,a) ε t) & ((a,b) ε s)) & (g = (x,b))) ExistsInt 57
59. ∃r.((b,d) ε r) ExistsInt 53
60. Set((b,d)) DefSub 59
61. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 30
62. Set((b,y)) -> (Set(b) & Set(y)) ForallElim 61
63. ∀y.(Set((b,y)) -> (Set(b) & Set(y))) ForallInt 62
64. Set((b,d)) -> (Set(b) & Set(d)) ForallElim 63
65. Set(b) & Set(d) ImpElim 60 64
66. Set(b) AndElimL 65
67. ∃t.((x,a) ε t) ExistsInt 51
68. Set((x,a)) DefSub 67
69. ∀y.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 30
70. Set((x,a)) -> (Set(x) & Set(a)) ForallElim 69
71. Set(x) & Set(a) ImpElim 68 70
72. Set(x) AndElimL 71
73. Set(x) & Set(b) AndInt 72 66
74. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 28
75. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 74
76. ∀y.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 75
77. (Set(x) & Set(b)) -> Set((x,b)) ForallElim 76
78. Set((x,b)) ImpElim 73 77
79. (x,b) = g Symmetry 54
80. Set(g) EqualitySub 78 79
81. Set(g) & ∃x.∃a.∃b.((((x,a) ε t) & ((a,b) ε s)) & (g = (x,b))) AndInt 80 58
82. g ε {w: ∃x.∃a.∃b.((((x,a) ε t) & ((a,b) ε s)) & (w = (x,b)))} ClassInt 81
83. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
84. (s∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε s)) & (w = (x,z)))} ForallElim 83
85. ∀b.((s∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε s)) & (w = (x,z)))}) ForallInt 84
86. (s∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε s)) & (w = (x,z)))} ForallElim 85
87. {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε s)) & (w = (x,z)))} = (s∘t) Symmetry 86
88. g ε (s∘t) EqualitySub 82 87
89. (x,b) ε (s∘t) EqualitySub 88 54
90. (g = (x,b)) -> ((x,b) ε (s∘t)) ImpInt 89
91. ∀g.((g = (x,b)) -> ((x,b) ε (s∘t))) ForallInt 90
92. ((x,b) = (x,b)) -> ((x,b) ε (s∘t)) ForallElim 91
93. (x,b) = (x,b) Identity
94. (x,b) ε (s∘t) ImpElim 93 92
95. ((b,d) ε r) & ((x,b) ε (s∘t)) AndInt 53 94

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96. d = c  Symmetry 50
97. z = (x,c)  AndElimR 11
98. ((x,b) ε (s∘t)) & ((b,d) ε r)  AndInt 94 53
99. (((x,b) ε (s∘t)) & ((b,d) ε r)) & (z = (x,c))  AndInt 98 97
100. (((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c))  EqualitySub 99 96
101. ∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c)))  ExistsInt 100
102. ∃b.∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c)))  ExistsInt 101
103. ∃x.∃b.∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c)))  ExistsInt 102
104. Set(z)  AndElimL 7
105. Set(z) & ∃x.∃b.∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (z = (x,c)))  AndInt 104 103
106. z ε {w: ∃x.∃b.∃c.((((x,b) ε (s∘t)) & ((b,c) ε r)) & (w = (x,c)))}  ClassInt 105
107. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
108. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 107
109. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 108
110. (r∘(s∘t)) = {w: ∃x.∃y.∃z.((((x,y) ε (s∘t)) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 109
111. {w: ∃x.∃y.∃z.((((x,y) ε (s∘t)) & ((y,z) ε r)) & (w = (x,z)))} = (r∘(s∘t))  Symmetry 110
112. z ε (r∘(s∘t))  EqualitySub 106 111
113. z ε (r∘(s∘t))  ExistsElim 22 23 112
114. z ε (r∘(s∘t))  ExistsElim 21 22 113
115. z ε (r∘(s∘t))  ExistsElim 20 21 114
116. z ε (r∘(s∘t))  ExistsElim 10 11 115
117. z ε (r∘(s∘t))  ExistsElim 9 10 116
118. z ε (r∘(s∘t))  ExistsElim 8 9 117
119. (z ε ((r∘s)∘t)) -> (z ε (r∘(s∘t)))  ImpInt 118
120. z ε (r∘(s∘t))  Hyp
121. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
122. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 121
123. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 122
124. (r∘(s∘t)) = {w: ∃x.∃y.∃z.((((x,y) ε (s∘t)) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 123
125. z ε {w: ∃x.∃y.∃z.((((x,y) ε (s∘t)) & ((y,z) ε r)) & (w = (x,z)))}  EqualitySub 120 124
126. Set(z) & ∃x.∃y.∃x_7.((((x,y) ε (s∘t)) & ((y,x_7) ε r)) & (z = (x,x_7)))  ClassElim 125
127. ∃x.∃y.∃x_7.((((x,y) ε (s∘t)) & ((y,x_7) ε r)) & (z = (x,x_7)))  AndElimR 126
128. ∃y.∃x_7.((((x,y) ε (s∘t)) & ((y,x_7) ε r)) & (z = (x,x_7)))  Hyp
129. ∃x_7.((((x,y) ε (s∘t)) & ((y,x_7) ε r)) & (z = (x,x_7)))  Hyp
130. (((x,y) ε (s∘t)) & ((y,c) ε r)) & (z = (x,c))  Hyp
131. z = (x,c)  AndElimR 130
132. ((x,y) ε (s∘t)) & ((y,c) ε r)  AndElimL 130
133. (x,y) ε (s∘t)  AndElimL 132
134. (y,c) ε r  AndElimR 132
135. (x,y) ε {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε s)) & (w = (x,z)))}  EqualitySub 133 86
136. Set((x,y)) & ∃x_8.∃x_9.∃z.((((x_8,x_9) ε t) & ((x_9,z) ε s)) & ((x,y) = (x_8,z))) ClassElim 135
137. Set((x,y))  AndElimL 136
138. ∃x_8.∃x_9.∃z.((((x_8,x_9) ε t) & ((x_9,z) ε s)) & ((x,y) = (x_8,z)))  AndElimR 136
139. ∃x_9.∃z.((((a,x_9) ε t) & ((x_9,z) ε s)) & ((x,y) = (a,z)))  Hyp
140. ∃z.((((a,b) ε t) & ((b,z) ε s)) & ((x,y) = (a,z)))  Hyp
141. (((a,b) ε t) & ((b,d) ε s)) & ((x,y) = (a,d))  Hyp
142. (x,y) = (a,d)  AndElimR 141
143. Set((a,d))  EqualitySub 137 142
144. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 74
145. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 144
146. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 145
147. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 146
148. Set((a,d)) -> (Set(a) & Set(d))  ForallElim 147
149. Set(a) & Set(d)  ImpElim 143 148
150. Set(a)  AndElimL 149
151. Set(d)  AndElimR 149
152. ((a,b) ε t) & ((b,d) ε s)  AndElimL 141
153. (b,d) ε s  AndElimR 152
154. ((b,d) ε s) & ((y,c) ε r)  AndInt 153 134
155. Set(x) & Set(y)  ImpElim 137 144
156. (Set(x) & Set(y)) & ((x,y) = (a,d))  AndInt 155 142
157. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
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158. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 157
159. ((Set(x) & Set(y)) & ((x,y) = (a,v))) -> ((x = a) & (y = v))  ForallElim 158
160. ∀v.(((Set(x) & Set(y)) & ((x,y) = (a,v))) -> ((x = a) & (y = v)))  ForallInt 159
161. ((Set(x) & Set(y)) & ((x,y) = (a,d))) -> ((x = a) & (y = d))  ForallElim 160
162. (x = a) & (y = d)  ImpElim 156 161
163. y = d  AndElimR 162
164. d = y  Symmetry 163
165. ((b,y) ε s) & ((y,c) ε r)  EqualitySub 154 164
166. h = (b,c)  Hyp
167. ∃w.((b,d) ε w)  ExistsInt 153
168. ∃w.((y,c) ε w)  ExistsInt 134
169. Set((b,d))  DefSub 167
170. Set((y,c))  DefSub 168
171. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 144
172. Set((b,y)) -> (Set(b) & Set(y))  ForallElim 171
173. ∀y.(Set((b,y)) -> (Set(b) & Set(y)))  ForallInt 172
174. Set((b,d)) -> (Set(b) & Set(d))  ForallElim 173
175. ∀y.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 144
176. Set((x,c)) -> (Set(x) & Set(c))  ForallElim 175
177. ∀x.(Set((x,c)) -> (Set(x) & Set(c)))  ForallInt 176
178. Set((y,c)) -> (Set(y) & Set(c))  ForallElim 177
179. Set(b) & Set(d)  ImpElim 169 174
180. Set(y) & Set(c)  ImpElim 170 178
181. Set(b)  AndElimL 179
182. Set(c)  AndElimR 180
183. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 74
184. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 183
185. (Set(b) & Set(y)) -> Set((b,y))  ForallElim 184
186. ∀y.((Set(b) & Set(y)) -> Set((b,y)))  ForallInt 185
187. (Set(b) & Set(c)) -> Set((b,c))  ForallElim 186
188. Set(b) & Set(c)  AndInt 181 182
189. Set((b,c))  ImpElim 188 187
190. (b,c) = h  Symmetry 166
191. Set(h)  EqualitySub 189 190
192. (((b,y) ε s) & ((y,c) ε r)) & (h = (b,c))  AndInt 165 166
193. ∃c.((((b,y) ε s) & ((y,c) ε r)) & (h = (b,c)))  ExistsInt 192
194. ∃y.∃c.((((b,y) ε s) & ((y,c) ε r)) & (h = (b,c)))  ExistsInt 193
195. ∃b.∃y.∃c.((((b,y) ε s) & ((y,c) ε r)) & (h = (b,c)))  ExistsInt 194
196. Set(h) & ∃b.∃y.∃c.((((b,y) ε s) & ((y,c) ε r)) & (h = (b,c)))  AndInt 191 195
197. h ε {w: ∃b.∃y.∃c.((((b,y) ε s) & ((y,c) ε r)) & (w = (b,c)))}  ClassInt 196
198. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
199. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 198
200. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt
199
201. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 200
202. {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} = (r∘s)  Symmetry 201
203. h ε (r∘s)  EqualitySub 197 202
204. (b,c) ε (r∘s)  EqualitySub 203 166
205. (h = (b,c)) -> ((b,c) ε (r∘s))  ImpInt 204
206. ∀h.((h = (b,c)) -> ((b,c) ε (r∘s)))  ForallInt 205
207. ((b,c) = (b,c)) -> ((b,c) ε (r∘s))  ForallElim 206
208. (b,c) = (b,c)  Identity
209. (b,c) ε (r∘s)  ImpElim 208 207
210. (a,b) ε t  AndElimL 152
211. x = a  AndElimL 162
212. a = x  Symmetry 211
213. (x,b) ε t  EqualitySub 210 212
214. ((x,b) ε t) & ((b,c) ε (r∘s))  AndInt 213 209
215. (((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c))  AndInt 214 131
216. ∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c)))  ExistsInt 215
217. ∃b.∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c)))  ExistsInt 216
218. ∃x.∃b.∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c)))  ExistsInt 217
219. Set(z)  AndElimL 126
220. Set(z) & ∃x.∃b.∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (z = (x,c)))  AndInt 219 218
221. z ε {w: ∃x.∃b.∃c.((((x,b) ε t) & ((b,c) ε (r∘s))) & (w = (x,c)))}  ClassInt 220
222. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
223. ((r∘s)∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (r∘s))) & (w = (x,z)))}
ForallElim 222
224. ∀b.(((r∘s)∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (r∘s))) & (w = (x,z)))})
ForallInt 223
225. ((r∘s)∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε (r∘s))) & (w = (x,z)))}
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ForallElim 224
226. {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε (r∘s))) & (w = (x,z)))} = ((r∘s)∘t) Symmetry 225
227. z ε ((r∘s)∘t) EqualitySub 221 226
228. z ε ((r∘s)∘t) ExistsElim 140 141 227
229. z ε ((r∘s)∘t) ExistsElim 139 140 228
230. z ε ((r∘s)∘t) ExistsElim 138 139 229
231. z ε ((r∘s)∘t) ExistsElim 129 130 230
232. z ε ((r∘s)∘t) ExistsElim 128 129 231
233. z ε ((r∘s)∘t) ExistsElim 127 128 232
234. (z ε (r∘(s∘t))) -> (z ε ((r∘s)∘t)) ImpInt 233
235. ((z ε ((r∘s)∘t)) -> (z ε (r∘(s∘t)))) & ((z ε (r∘(s∘t))) -> (z ε ((r∘s)∘t))) AndInt 119 234
236. (z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t))) EquivConst 235
237. ∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t)))) ForallInt 236
238. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
239. ∀y.((((r∘s)∘t) = y) <-> ∀z.((z ε ((r∘s)∘t)) <-> (z ε y))) ForallElim 238
240. (((r∘s)∘t) = (r∘(s∘t))) <-> ∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t)))) ForallElim 239
241. ((((r∘s)∘t) = (r∘(s∘t))) -> ∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t))))) & (∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t)))) -> (((r∘s)∘t) = (r∘(s∘t)))) EquivExp 240
242. ∀z.((z ε ((r∘s)∘t)) <-> (z ε (r∘(s∘t)))) -> (((r∘s)∘t) = (r∘(s∘t))) AndElimR 241
243. ((r∘s)∘t) = (r∘(s∘t)) ImpElim 237 242 Qed

Used Theorems

2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
1. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th59. ((r∘(s ∪ t)) = ((r∘s) ∪ (r∘t))) & ((r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t)))

0. z ε (r∘(s ∪ t)) Hyp
1. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))} DefEqInt
2. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
3. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 2
4. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}) ForallInt 3
5. (r∘(s ∪ t)) = {w: ∃x.∃y.∃z.((((x,y) ε (s ∪ t)) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 4
6. z ε {w: ∃x.∃y.∃z.((((x,y) ε (s ∪ t)) & ((y,z) ε r)) & (w = (x,z)))} EqualitySub 0 5
7. Set(z) & ∃x.∃y.∃x_1.((((x,y) ε (s ∪ t)) & ((y,x_1) ε r)) & (z = (x,x_1))) ClassElim 6
8. ∃x.∃y.∃x_1.((((x,y) ε (s ∪ t)) & ((y,x_1) ε r)) & (z = (x,x_1))) AndElimR 7
9. ∃y.∃x_1.((((x,y) ε (s ∪ t)) & ((y,x_1) ε r)) & (z = (x,x_1))) Hyp
10. ∃x_1.((((x,y) ε (s ∪ t)) & ((y,x_1) ε r)) & (z = (x,x_1))) Hyp
11. (((x,y) ε (s ∪ t)) & ((y,c) ε r)) & (z = (x,c)) Hyp
12. ((x,y) ε (s ∪ t)) & ((y,c) ε r) AndElimL 11
13. (x,y) ε (s ∪ t) AndElimL 12
14. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
15. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 14
16. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 15
17. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 16
18. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 17
19. (z ε (s ∪ y)) -> ((z ε s) v (z ε y)) ForallElim 18
20. ∀y.((z ε (s ∪ y)) -> ((z ε s) v (z ε y))) ForallInt 19
21. (z ε (s ∪ t)) -> ((z ε s) v (z ε t)) ForallElim 20
22. ∀z.((z ε (s ∪ t)) -> ((z ε s) v (z ε t))) ForallInt 21
23. ((x,y) ε (s ∪ t)) -> (((x,y) ε s) v ((x,y) ε t)) ForallElim 22
24. ((x,y) ε s) v ((x,y) ε t) ImpElim 13 23
25. (x,y) ε s Hyp
26. (y,c) ε r AndElimR 12
27. ((x,y) ε s) & ((y,c) ε r) AndInt 25 26
28. z = (x,c) AndElimR 11
29. (((x,y) ε s) & ((y,c) ε r)) & (z = (x,c)) AndInt 27 28
30. ∃c.((((x,y) ε s) & ((y,c) ε r)) & (z = (x,c))) ExistsInt 29
31. ∃y.∃c.((((x,y) ε s) & ((y,c) ε r)) & (z = (x,c))) ExistsInt 30
32. ∃x.∃y.∃c.((((x,y) ε s) & ((y,c) ε r)) & (z = (x,c))) ExistsInt 31
33. Set(z) AndElimL 7
34. Set(z) & ∃x.∃y.∃c.((((x,y) ε s) & ((y,c) ε r)) & (z = (x,c))) AndInt 33 32
35. z ε {w: ∃x.∃y.∃c.((((x,y) ε s) & ((y,c) ε r)) & (w = (x,c)))} ClassInt 34
36. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
37. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 36

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38. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 37
39. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 38
40. {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} = (r∘s)  Symmetry 39
41. z ε (r∘s)  EqualitySub 35 40
42. (z ε (r∘s)) v (z ε (r∘t))  OrIntR 41
43. ((z ε x) v (z ε y)) -> (z ε (x U y))  AndElimR 16
44. ∀x.(((z ε x) v (z ε y)) -> (z ε (x U y)))  ForallInt 43
45. ((z ε (r∘s)) v (z ε y)) -> (z ε ((r∘s) U y))  ForallElim 44
46. ∀y.(((z ε (r∘s)) v (z ε y)) -> (z ε ((r∘s) U y)))  ForallInt 45
47. ((z ε (r∘s)) v (z ε (r∘t))) -> (z ε ((r∘s) U (r∘t)))  ForallElim 46
48. z ε ((r∘s) U (r∘t))  ImpElim 42 47
49. (x,y) ε t  Hyp
50. ((x,y) ε t) & ((y,c) ε r)  AndInt 49 26
51. (((x,y) ε t) & ((y,c) ε r)) & (z = (x,c))  AndInt 50 28
52. ∃c.((((x,y) ε t) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 51
53. ∃y.∃c.((((x,y) ε t) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 52
54. ∃x.∃y.∃c.((((x,y) ε t) & ((y,c) ε r)) & (z = (x,c)))  ExistsInt 53
55. Set(z) & ∃x.∃y.∃c.((((x,y) ε t) & ((y,c) ε r)) & (z = (x,c)))  AndInt 33 54
56. z ε {w: ∃x.∃y.∃c.((((x,y) ε t) & ((y,c) ε r)) & (w = (x,c)))}  ClassInt 55
57. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
58. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 57
59. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 58
60. (r∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 59
61. {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε r)) & (w = (x,z)))} = (r∘t)  Symmetry 60
62. z ε (r∘t)  EqualitySub 56 61
63. (z ε (r∘s)) v (z ε (r∘t))  OrIntL 62
64. z ε ((r∘s) U (r∘t))  ImpElim 63 47
65. z ε ((r∘s) U (r∘t))  OrElim 24 25 48 49 64
66. z ε ((r∘s) U (r∘t))  ExistsElim 10 11 65
67. z ε ((r∘s) U (r∘t))  ExistsElim 9 10 66
68. z ε ((r∘s) U (r∘t))  ExistsElim 8 9 67
69. (z ε (r∘(s U t))) -> (z ε ((r∘s) U (r∘t)))  ImpInt 68
70. z ε ((r∘s) U (r∘t))  Hyp
71. ∀x.((z ε (x U y)) -> ((z ε x) v (z ε y)))  ForallInt 17
72. (z ε ((r∘s) U y)) -> ((z ε (r∘s)) v (z ε y))  ForallElim 71
73. ∀y.((z ε ((r∘s) U y)) -> ((z ε (r∘s)) v (z ε y)))  ForallInt 72
74. (z ε ((r∘s) U (r∘t))) -> ((z ε (r∘s)) v (z ε (r∘t)))  ForallElim 73
75. (z ε (r∘s)) v (z ε (r∘t))  ImpElim 70 74
76. z ε (r∘s)  Hyp
77. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 1
78. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 77
79. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 78
80. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 79
81. z ε {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  EqualitySub 76 80
82. Set(z) & ∃x.∃y.∃x_2.((((x,y) ε s) & ((y,x_2) ε r)) & (z = (x,x_2)))  ClassElim 81
83. ∃x.∃y.∃x_2.((((x,y) ε s) & ((y,x_2) ε r)) & (z = (x,x_2)))  AndElimR 82
84. ∃y.∃x_2.((((x,y) ε s) & ((y,x_2) ε r)) & (z = (x,x_2)))  Hyp
85. ∃x_2.((((x,y) ε s) & ((y,x_2) ε r)) & (z = (x,x_2)))  Hyp
86. (((x,y) ε s) & ((y,m) ε r)) & (z = (x,m))  Hyp
87. ((x,y) ε s) & ((y,m) ε r)  AndElimL 86
88. (x,y) ε s  AndElimL 87
89. ((x,y) ε s) v ((x,y) ε t)  OrIntR 88
90. (y,m) ε r  AndElimR 87
91. ((z ε (x U y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x U y)))
EquivExp 15
92. ((z ε x) v (z ε y)) -> (z ε (x U y))  AndElimR 91
93. ∀x.(((z ε x) v (z ε y)) -> (z ε (x U y)))  ForallInt 92
94. ((z ε s) v (z ε y)) -> (z ε (s U y))  ForallElim 93
95. ∀y.(((z ε s) v (z ε y)) -> (z ε (s U y)))  ForallInt 94
96. ((z ε s) v (z ε t)) -> (z ε (s U t))  ForallElim 95
97. ∀z.(((z ε s) v (z ε t)) -> (z ε (s U t)))  ForallInt 96
98. (((x,y) ε s) v ((x,y) ε t)) -> ((x,y) ε (s U t))  ForallElim 97
99. (x,y) ε (s U t)  ImpElim 89 98
100. ((x,y) ε (s U t)) & ((y,m) ε r)  AndInt 99 90
101. z = (x,m)  AndElimR 86
102. (((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m))  AndInt 100 101
103. ∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m)))  ExistsInt 102
104. ∃y.∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m)))  ExistsInt 103
105. ∃x.∃y.∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m)))  ExistsInt 104
106. Set(z)  AndElimL 82
107. Set(z) & ∃x.∃y.∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (z = (x,m)))  AndInt 106 105
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108. z ε {w: ∃x.∃y.∃m.((((x,y) ε (s U t)) & ((y,m) ε r)) & (w = (x,m)))} ClassInt 107
109. {w: ∃x.∃y.∃z.((((x,y) ε (s U t)) & ((y,z) ε r)) & (w = (x,z)))} = (r∘(s U t)) Symmetry 5
110. z ε (r∘(s U t)) EqualitySub 108 109
111. z ε (r∘(s U t)) ExistsElim 85 86 110
112. z ε (r∘(s U t)) ExistsElim 84 85 111
113. z ε (r∘(s U t)) ExistsElim 83 84 112
114. z ε (r∘t) Hyp
115. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}) ForallInt 78
116. (r∘t) = {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 115
117. z ε {w: ∃x.∃y.∃z.((((x,y) ε t) & ((y,z) ε r)) & (w = (x,z)))} EqualitySub 114 116
118. Set(z) & ∃x.∃y.∃x_4.((((x,y) ε t) & ((y,x_4) ε r)) & (z = (x,x_4))) ClassElim 117
119. ∃x.∃y.∃x_4.((((x,y) ε t) & ((y,x_4) ε r)) & (z = (x,x_4))) AndElimR 118
120. ∃y.∃x_4.((((x,y) ε t) & ((y,x_4) ε r)) & (z = (x,x_4))) Hyp
121. ∃x_4.((((x,y) ε t) & ((y,x_4) ε r)) & (z = (x,x_4))) Hyp
122. (((x,y) ε t) & ((y,e) ε r)) & (z = (x,e)) Hyp
123. ((x,y) ε t) & ((y,e) ε r) AndElimL 122
124. (x,y) ε t AndElimL 123
125. ((x,y) ε s) v ((x,y) ε t) OrIntL 124
126. (x,y) ε (s U t) ImpElim 125 98
127. (y,e) ε r AndElimR 123
128. ((x,y) ε (s U t)) & ((y,e) ε r) AndInt 126 127
129. z = (x,e) AndElimR 122
130. (((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e)) AndInt 128 129
131. ∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 130
132. ∃y.∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 131
133. ∃x.∃y.∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 132
134. Set(z) AndElimL 118
135. Set(z) & ∃x.∃y.∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (z = (x,e))) AndInt 134 133
136. z ε {w: ∃x.∃y.∃e.((((x,y) ε (s U t)) & ((y,e) ε r)) & (w = (x,e)))} ClassInt 135
137. z ε (r∘(s U t)) EqualitySub 136 109
138. z ε (r∘(s U t)) ExistsElim 121 122 137
139. z ε (r∘(s U t)) ExistsElim 120 121 138
140. z ε (r∘(s U t)) ExistsElim 119 120 139
141. z ε (r∘(s U t)) OrElim 75 76 113 114 140
142. (z ε ((r∘s) U (r∘t))) -> (z ε (r∘(s U t))) ImpInt 141
143. ((z ε (r∘(s U t))) -> (z ε ((r∘s) U (r∘t)))) & ((z ε ((r∘s) U (r∘t))) -> (z ε (r∘(s U t)))) AndInt 69 142
144. (z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t))) EquivConst 143
145. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
146. ∀y.(((r∘(s U t)) = y) <-> ∀z.((z ε (r∘(s U t))) <-> (z ε y))) ForallElim 145
147. ((r∘(s U t)) = ((r∘s) U (r∘t))) <-> ∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t)))) ForallElim 146
148. (((r∘(s U t)) = ((r∘s) U (r∘t))) -> ∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t))))) & (∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t)))) -> ((r∘(s U t)) = ((r∘s) U (r∘t)))) EquivExp 147
149. ∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t)))) -> ((r∘(s U t)) = ((r∘s) U (r∘t))) AndElimR 148
150. ∀z.((z ε (r∘(s U t))) <-> (z ε ((r∘s) U (r∘t)))) ForallInt 144
151. (r∘(s U t)) = ((r∘s) U (r∘t)) ImpElim 150 149
152. z ε (r∘(s ∩ t)) Hyp
153. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 1
154. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 153
155. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}) ForallInt 154
156. (r∘(s ∩ t)) = {w: ∃x.∃y.∃z.((((x,y) ε (s ∩ t)) & ((y,z) ε r)) & (w = (x,z)))} ForallElim 155
157. z ε {w: ∃x.∃y.∃z.((((x,y) ε (s ∩ t)) & ((y,z) ε r)) & (w = (x,z)))} EqualitySub 152 156
158. Set(z) & ∃x.∃y.∃x_5.((((x,y) ε (s ∩ t)) & ((y,x_5) ε r)) & (z = (x,x_5))) ClassElim 157
159. ∃x.∃y.∃x_5.((((x,y) ε (s ∩ t)) & ((y,x_5) ε r)) & (z = (x,x_5))) AndElimR 158
160. ∃y.∃x_5.((((x,y) ε (s ∩ t)) & ((y,x_5) ε r)) & (z = (x,x_5))) Hyp
161. ∃x_5.((((x,y) ε (s ∩ t)) & ((y,x_5) ε r)) & (z = (x,x_5))) Hyp
162. (((x,y) ε (s ∩ t)) & ((y,e) ε r)) & (z = (x,e)) Hyp
163. ((x,y) ε (s ∩ t)) & ((y,e) ε r) AndElimL 162
164. (x,y) ε (s ∩ t) AndElimL 163
165. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 14
166. ∀x.((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) ForallInt 165
167. (z ε (s ∩ y)) <-> ((z ε s) & (z ε y)) ForallElim 166
168. ∀y.((z ε (s ∩ y)) <-> ((z ε s) & (z ε y))) ForallInt 167

169. (z ε (s ∩ t)) <-> ((z ε s) & (z ε t)) ForallElim 168
170. ∀z.((z ε (s ∩ t)) <-> ((z ε s) & (z ε t))) ForallInt 169
171. ((x,y) ε (s ∩ t)) <-> (((x,y) ε s) & ((x,y) ε t)) ForallElim 170
172. (((x,y) ε (s ∩ t)) -> (((x,y) ε s) & ((x,y) ε t))) & ((((x,y) ε s) & ((x,y) ε t)) -> ((x,y) ε (s ∩ t))) EquivExp 171
173. ((x,y) ε (s ∩ t)) -> (((x,y) ε s) & ((x,y) ε t)) AndElimL 172
174. ((x,y) ε s) & ((x,y) ε t) ImpElim 164 173
175. (x,y) ε s AndElimL 174
176. (y,e) ε r AndElimR 163
177. ((x,y) ε s) & ((y,e) ε r) AndInt 175 176
178. z = (x,e) AndElimR 162
179. (((x,y) ε s) & ((y,e) ε r)) & (z = (x,e)) AndInt 177 178
180. ∃e.((((x,y) ε s) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 179
181. ∃y.∃e.((((x,y) ε s) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 180
182. ∃x.∃y.∃e.((((x,y) ε s) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 181
183. Set(z) AndElimL 158
184. Set(z) & ∃x.∃y.∃e.((((x,y) ε s) & ((y,e) ε r)) & (z = (x,e))) AndInt 183 182
185. z ε {w: ∃x.∃y.∃e.((((x,y) ε s) & ((y,e) ε r)) & (w = (x,e)))} ClassInt 184
186. z ε (r∘s) EqualitySub 185 40
187. (x,y) ε t AndElimR 174
188. ((x,y) ε t) & ((y,e) ε r) AndInt 187 176
189. (((x,y) ε t) & ((y,e) ε r)) & (z = (x,e)) AndInt 188 178
190. ∃e.((((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 189
191. ∃y.∃e.((((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 190
192. ∃x.∃y.∃e.((((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))) ExistsInt 191
193. Set(z) & ∃x.∃y.∃e.((((x,y) ε t) & ((y,e) ε r)) & (z = (x,e))) AndInt 183 192
194. z ε {w: ∃x.∃y.∃e.((((x,y) ε t) & ((y,e) ε r)) & (w = (x,e)))} ClassInt 193
195. z ε (r∘t) EqualitySub 194 61
196. (z ε (r∘s)) & (z ε (r∘t)) AndInt 186 195
197. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 165
198. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 197
199. ∀x.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 198
200. ((z ε (r∘s)) & (z ε y)) -> (z ε ((r∘s) ∩ y)) ForallElim 199
201. ∀y.(((z ε (r∘s)) & (z ε y)) -> (z ε ((r∘s) ∩ y))) ForallInt 200
202. ((z ε (r∘s)) & (z ε (r∘t))) -> (z ε ((r∘s) ∩ (r∘t))) ForallElim 201
203. z ε ((r∘s) ∩ (r∘t)) ImpElim 196 202
204. z ε ((r∘s) ∩ (r∘t)) ExistsElim 161 162 203
205. z ε ((r∘s) ∩ (r∘t)) ExistsElim 160 161 204
206. z ε ((r∘s) ∩ (r∘t)) ExistsElim 159 160 205
207. (z ε (r∘(s ∩ t))) -> (z ε ((r∘s) ∩ (r∘t))) ImpInt 206
208. ∀z.((z ε (r∘(s ∩ t))) -> (z ε ((r∘s) ∩ (r∘t)))) ForallInt 207
209. (r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t)) DefSub 208
210. ((r∘(s ∪ t)) = ((r∘s) ∪ (r∘t))) & ((r∘(s ∩ t)) ⊂ ((r∘s) ∩ (r∘t))) AndInt 151 209
Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

Th61. Relation(r) -> (((r)$^{-1}$)$^{-1}$ = r)

0. z ε ((r)$^{-1}$)$^{-1}$ Hyp
1. (r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} DefEqInt
2. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 1
3. ((r)$^{-1}$)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x)))} ForallElim 2
4. z ε {z: ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x)))} EqualitySub 0 3
5. Set(z) & ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x))) ClassElim 4
6. ∃x.∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x))) AndElimR 5
7. ∃y.(((x,y) ε (r)$^{-1}$) & (z = (y,x))) Hyp
8. ((x,y) ε (r)$^{-1}$) & (z = (y,x)) Hyp
9. (x,y) ε (r)$^{-1}$ AndElimL 8
10. (x,y) ε {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} EqualitySub 9 1
11. Set((x,y)) & ∃x_0.∃x_2.(((x_0,x_2) ε r) & ((x,y) = (x_2,x_0))) ClassElim 10
12. ∃x_0.∃x_2.(((x_0,x_2) ε r) & ((x,y) = (x_2,x_0))) AndElimR 11
13. ∃x_2.(((c,x_2) ε r) & ((x,y) = (x_2,c))) Hyp
14. ((c,d) ε r) & ((x,y) = (d,c)) Hyp
15. z = (y,x) AndElimR 8
16. Set(z) AndElimL 5
17. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
18. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt

19. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 18
20. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 19
21. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 20
22. Set((y,x)) EqualitySub 16 15
23. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 21
24. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 23
25. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 24
26. Set((a,x)) -> (Set(a) & Set(x)) ForallElim 25
27. ∀a.(Set((a,x)) -> (Set(a) & Set(x))) ForallInt 26
28. Set((y,x)) -> (Set(y) & Set(x)) ForallElim 27
29. Set(y) & Set(x) ImpElim 22 28
30. Set(y) AndElimL 29
31. Set(x) AndElimR 29
32. Set(x) & Set(y) AndInt 31 30
33. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 17
34. ((Set(x) & Set(y)) & ((x,y) = (d,v))) -> ((x = d) & (y = v)) ForallElim 33
35. ∀v.(((Set(x) & Set(y)) & ((x,y) = (d,v))) -> ((x = d) & (y = v))) ForallInt 34
36. ((Set(x) & Set(y)) & ((x,y) = (d,c))) -> ((x = d) & (y = c)) ForallElim 35
37. (x,y) = (d,c) AndElimR 14
38. (Set(x) & Set(y)) & ((x,y) = (d,c)) AndInt 32 37
39. (x = d) & (y = c) ImpElim 38 36
40. x = d AndElimL 39
41. y = c AndElimR 39
42. (c,d) ε r AndElimL 14
43. d = x Symmetry 40
44. c = y Symmetry 41
45. (c,x) ε r EqualitySub 42 43
46. (y,x) ε r EqualitySub 45 44
47. (y,x) ε r ExistsElim 13 14 46
48. (y,x) ε r ExistsElim 12 13 47
49. (y,x) = z Symmetry 15
50. z ε r EqualitySub 48 49
51. z ε r ExistsElim 7 8 50
52. z ε r ExistsElim 6 7 51
53. (z ε $((r)^{-1})^{-1}$) -> (z ε r) ImpInt 52
54. Relation(r) Hyp
55. z ε r Hyp
56. ∀z.((z ε r) -> ∃x.∃y.(z = (x,y))) DefExp 54
57. (z ε r) -> ∃x.∃y.(z = (x,y)) ForallElim 56
58. ∃x.∃y.(z = (x,y)) ImpElim 55 57
59. ∃y.(z = (x,y)) Hyp
60. z = (x,y) Hyp
61. f = (y,x) Hyp
62. (x,y) ε r EqualitySub 55 60
63. ((x,y) ε r) & (f = (y,x)) AndInt 62 61
64. Set((y,x)) EqualitySub 16 15
65. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
66. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 65
67. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 66
68. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 67
69. ∃w.(z ε w) ExistsInt 55
70. Set(z) DefSub 69
71. Set((x,y)) EqualitySub 70 60
72. Set(x) & Set(y) ImpElim 71 68
73. Set(x) AndElimL 72
74. Set(y) AndElimR 72
75. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 66
76. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 75
77. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 76
78. (Set(a) & Set(y)) -> Set((a,y)) ForallElim 77
79. ∀y.((Set(a) & Set(y)) -> Set((a,y))) ForallInt 78
80. (Set(a) & Set(x)) -> Set((a,x)) ForallElim 79
81. ∀a.((Set(a) & Set(x)) -> Set((a,x))) ForallInt 80
82. (Set(y) & Set(x)) -> Set((y,x)) ForallElim 81
83. Set(y) & Set(x) AndInt 74 73
84. Set((y,x)) ImpElim 83 82
85. (y,x) = f Symmetry 61
86. Set(f) EqualitySub 84 85
87. ∃y.(((x,y) ε r) & (f = (y,x))) ExistsInt 63
88. ∃x.∃y.(((x,y) ε r) & (f = (y,x))) ExistsInt 87
89. Set(f) & ∃x.∃y.(((x,y) ε r) & (f = (y,x))) AndInt 86 88

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90. f ε {w: ∃x.∃y.(((x,y) ε r) & (w = (y,x)))}  ClassInt 89
91. {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} = (r)⁻¹  Symmetry 1
92. f ε (r)⁻¹  EqualitySub 90 91
93. (y,x) ε (r)⁻¹  EqualitySub 92 61
94. (f = (y,x)) -> ((y,x) ε (r)⁻¹)  ImpInt 93
95. ∀f.((f = (y,x)) -> ((y,x) ε (r)⁻¹))  ForallInt 94
96. ((y,x) = (y,x)) -> ((y,x) ε (r)⁻¹)  ForallElim 95
97. (y,x) = (y,x)  Identity
98. (y,x) ε (r)⁻¹  ImpElim 97 96
99. ((y,x) ε (r)⁻¹) & (z = (x,y))  AndInt 98 60
100. ∃x.(((y,x) ε (r)⁻¹) & (z = (x,y)))  ExistsInt 99
101. ∃y.∃x.(((y,x) ε (r)⁻¹) & (z = (x,y)))  ExistsInt 100
102. Set(z) & ∃y.∃x.(((y,x) ε (r)⁻¹) & (z = (x,y)))  AndInt 70 101
103. z ε {w: ∃y.∃x.(((y,x) ε (r)⁻¹) & (w = (x,y)))}  ClassInt 102
104. ∀r.((r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))})  ForallInt 1
105. ((r)⁻¹)⁻¹ = {z: ∃x.∃y.(((x,y) ε (r)⁻¹) & (z = (y,x)))}  ForallElim 104
106. {z: ∃x.∃y.(((x,y) ε (r)⁻¹) & (z = (y,x)))} = ((r)⁻¹)⁻¹  Symmetry 105
107. z ε ((r)⁻¹)⁻¹  EqualitySub 103 106
108. z ε ((r)⁻¹)⁻¹  ExistsElim 59 60 107
109. z ε ((r)⁻¹)⁻¹  ExistsElim 58 59 108
110. (z ε r) -> (z ε ((r)⁻¹)⁻¹)  ImpInt 109
111. ((z ε ((r)⁻¹)⁻¹) -> (z ε r)) & ((z ε r) -> (z ε ((r)⁻¹)⁻¹))  AndInt 53 110
112. (z ε ((r)⁻¹)⁻¹) <-> (z ε r)  EquivConst 111
113. ∀z.((z ε ((r)⁻¹)⁻¹) <-> (z ε r))  ForallInt 112
114. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
115. ∀y.((((r)⁻¹)⁻¹ = y) <-> ∀z.((z ε ((r)⁻¹)⁻¹) <-> (z ε y)))  ForallElim 114
116. (((r)⁻¹)⁻¹ = r) <-> ∀z.((z ε ((r)⁻¹)⁻¹) <-> (z ε r))  ForallElim 115
117. ((((r)⁻¹)⁻¹ = r) -> ∀z.((z ε ((r)⁻¹)⁻¹) <-> (z ε r))) & (∀z.((z ε ((r)⁻¹)⁻¹) <-> (z ε r)) -> (((r)⁻¹)⁻¹ = r))  EquivExp 116
118. ∀z.((z ε ((r)⁻¹)⁻¹) <-> (z ε r)) -> (((r)⁻¹)⁻¹ = r)  AndElimR 117
119. ((r)⁻¹)⁻¹ = r  ImpElim 113 118
120. Relation(r) -> (((r)⁻¹)⁻¹ = r)  ImpInt 119 Qed

Used Theorems

1. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th62. ((r∘s))⁻¹ = ((s)⁻¹∘(r)⁻¹)

0. z ε ((r∘s))⁻¹  Hyp
1. (r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}  DefEqInt
2. ∀r.((r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))})  ForallInt 1
3. ((r∘s))⁻¹ = {z: ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))}  ForallElim 2
4. z ε {z: ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))}  EqualitySub 0 3
5. Set(z) & ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))  ClassElim 4
6. ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))  AndElimR 5
7. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}  DefEqInt
8. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 7
9. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 8
10. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt 9
11. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 10
12. ∃y.(((x,y) ε (r∘s)) & (z = (y,x)))  Hyp
13. ((x,y) ε (r∘s)) & (z = (y,x))  Hyp
14. (x,y) ε (r∘s)  AndElimL 13
15. (x,y) ε {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  EqualitySub 14 11
16. Set((x,y)) & ∃x_0.∃x_1.∃z.((((x_0,x_1) ε s) & ((x_1,z) ε r)) & ((x,y) = (x_0,z))) ClassElim 15
17. ∃x_0.∃x_1.∃z.((((x_0,x_1) ε s) & ((x_1,z) ε r)) & ((x,y) = (x_0,z)))  AndElimR 16
18. ∃x_1.∃z.((((c,x_1) ε s) & ((x_1,z) ε r)) & ((x,y) = (c,z)))  Hyp
19. ∃z.((((c,d) ε s) & ((d,z) ε r)) & ((x,y) = (c,z)))  Hyp
20. (((c,d) ε s) & ((d,b) ε r)) & ((x,y) = (c,b))  Hyp
21. ∃w.((x,y) ε w)  ExistsInt 14
22. Set((x,y))  DefSub 21
23. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
24. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 23
25. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 24
26. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 25
27. Set(x) & Set(y)  ImpElim 22 26
28. (x,y) = (c,b)  AndElimR 20
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29. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
30. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 29
31. ((Set(x) & Set(y)) & ((x,y) = (c,v))) -> ((x = c) & (y = v)) ForallElim 30
32. ∀v.(((Set(x) & Set(y)) & ((x,y) = (c,v))) -> ((x = c) & (y = v))) ForallInt 31
33. ((Set(x) & Set(y)) & ((x,y) = (c,b))) -> ((x = c) & (y = b)) ForallElim 32
34. (Set(x) & Set(y)) & ((x,y) = (c,b)) AndInt 27 28
35. (x = c) & (y = b) ImpElim 34 33
36. x = c AndElimL 35
37. y = b AndElimR 35
38. c = x Symmetry 36
39. b = y Symmetry 37
40. (((x,d) ε s) & ((d,b) ε r)) & ((x,y) = (x,b)) EqualitySub 20 38
41. (((x,d) ε s) & ((d,y) ε r)) & ((x,y) = (x,y)) EqualitySub 40 39
42. ((x,d) ε s) & ((d,y) ε r) AndElimL 41
43. h = (d,x) Hyp
44. (x,d) ε s AndElimL 42
45. ((x,d) ε s) & (h = (d,x)) AndInt 44 43
46. ∃d.(((x,d) ε s) & (h = (d,x))) ExistsInt 45
47. ∃x.∃d.(((x,d) ε s) & (h = (d,x))) ExistsInt 46
48. (x,d) ε s AndElimL 45
49. ∃w.((x,d) ε w) ExistsInt 48
50. Set((x,d)) DefSub 49
51. ∀y.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 26
52. Set((x,d)) -> (Set(x) & Set(d)) ForallElim 51
53. Set(x) & Set(d) ImpElim 50 52
54. Set(d) AndElimR 53
55. Set(x) AndElimL 53
56. Set(x) & Set(d) AndInt 55 54
57. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 25
58. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 57
59. (Set(d) & Set(y)) -> Set((d,y)) ForallElim 58
60. ∀y.((Set(d) & Set(y)) -> Set((d,y))) ForallInt 59
61. (Set(d) & Set(x)) -> Set((d,x)) ForallElim 60
62. Set(d) & Set(x) AndInt 54 55
63. Set((d,x)) ImpElim 62 61
64. (d,x) = h Symmetry 43
65. Set(h) EqualitySub 63 64
66. Set(h) & ∃x.∃d.(((x,d) ε s) & (h = (d,x))) AndInt 65 47
67. h ε {w: ∃x.∃d.(((x,d) ε s) & (w = (d,x)))} ClassInt 66
68. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 1
69. (s)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε s) & (z = (y,x)))} ForallElim 68
70. {z: ∃x.∃y.(((x,y) ε s) & (z = (y,x)))} = (s)$^{-1}$ Symmetry 69
71. h ε (s)$^{-1}$ EqualitySub 67 70
72. (d,x) ε (s)$^{-1}$ EqualitySub 71 43
73. (h = (d,x)) -> ((d,x) ε (s)$^{-1}$) ImpInt 72
74. ∀h.((h = (d,x)) -> ((d,x) ε (s)$^{-1}$)) ForallInt 73
75. ((d,x) = (d,x)) -> ((d,x) ε (s)$^{-1}$) ForallElim 74
76. (d,x) = (d,x) Identity
77. (d,x) ε (s)$^{-1}$ ImpElim 76 75
78. f = (y,d) Hyp
79. (d,y) ε r AndElimR 42
80. ((d,y) ε r) & (f = (y,d)) AndInt 79 78
81. ∃y.(((d,y) ε r) & (f = (y,d))) ExistsInt 80
82. ∃d.∃y.(((d,y) ε r) & (f = (y,d))) ExistsInt 81
83. Set(y) AndElimR 27
84. Set(y) & Set(d) AndInt 83 54
85. ∀y.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 57
86. (Set(x) & Set(d)) -> Set((x,d)) ForallElim 85
87. ∀x.((Set(x) & Set(d)) -> Set((x,d))) ForallInt 86
88. (Set(y) & Set(d)) -> Set((y,d)) ForallElim 87
89. Set((y,d)) ImpElim 84 88
90. (y,d) = f Symmetry 78
91. Set(f) EqualitySub 89 90
92. Set(f) & ∃d.∃y.(((d,y) ε r) & (f = (y,d))) AndInt 91 82
93. f ε {w: ∃d.∃y.(((d,y) ε r) & (w = (y,d)))} ClassInt 92
94. {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} = (r)$^{-1}$ Symmetry 1
95. f ε (r)$^{-1}$ EqualitySub 93 94
96. (y,d) ε (r)$^{-1}$ EqualitySub 95 78
97. (f = (y,d)) -> ((y,d) ε (r)$^{-1}$) ImpInt 96
98. ∀f.((f = (y,d)) -> ((y,d) ε (r)$^{-1}$)) ForallInt 97
99. ((y,d) = (y,d)) -> ((y,d) ε (r)$^{-1}$) ForallElim 98

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100. (y,d) = (y,d)  Identity
101. (y,d) ε (r)⁻¹  ImpElim 100 99
102. ((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)  AndInt 101 77
103. z = (y,x)  AndElimR 13
104. (((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x))  AndInt 102 103
105. ∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x)))  ExistsInt 104
106. ∃d.∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x)))  ExistsInt 105
107. ∃y.∃d.∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x)))  ExistsInt 106
108. Set(z)  AndElimL 5
109. Set(z) & ∃y.∃d.∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (z = (y,x)))  AndInt 108 107
110. z ε {w: ∃y.∃d.∃x.((((y,d) ε (r)⁻¹) & ((d,x) ε (s)⁻¹)) & (w = (y,x)))}  ClassInt 109
111. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 7
112. ((s)⁻¹∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))} ForallElim 111
113. ∀b.(((s)⁻¹∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))}) ForallInt 112
114. ((s)⁻¹∘(r)⁻¹) = {w: ∃x.∃y.∃z.((((x,y) ε (r)⁻¹) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))} ForallElim 113
115. {w: ∃x.∃y.∃z.((((x,y) ε (r)⁻¹) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))} = ((s)⁻¹∘(r)⁻¹) Symmetry 114
116. z ε ((s)⁻¹∘(r)⁻¹)  EqualitySub 110 115
117. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 19 20 116
118. (h = (d,x)) -> (z ε ((s)⁻¹∘(r)⁻¹))  ImpInt 117
119. ∀h.((h = (d,x)) -> (z ε ((s)⁻¹∘(r)⁻¹)))  ForallInt 118
120. ((d,x) = (d,x)) -> (z ε ((s)⁻¹∘(r)⁻¹))  ForallElim 119
121. (d,x) = (d,x)  Identity
122. z ε ((s)⁻¹∘(r)⁻¹)  ImpElim 121 120
123. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 18 19 122
124. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 17 18 123
125. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 12 13 124
126. z ε ((s)⁻¹∘(r)⁻¹)  ExistsElim 6 12 125
127. (z ε ((r∘s))⁻¹) -> (z ε ((s)⁻¹∘(r)⁻¹))  ImpInt 126
128. z ε ((s)⁻¹∘(r)⁻¹)  Hyp
129. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 7
130. ((s)⁻¹∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))} ForallElim 129
131. ∀b.(((s)⁻¹∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))}) ForallInt 130
132. ((s)⁻¹∘(r)⁻¹) = {w: ∃x.∃y.∃z.((((x,y) ε (r)⁻¹) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))} ForallElim 131
133. z ε {w: ∃x.∃y.∃z.((((x,y) ε (r)⁻¹) & ((y,z) ε (s)⁻¹)) & (w = (x,z)))}  EqualitySub 128 132
134. Set(z) & ∃x.∃y.∃x_9.((((x,y) ε (r)⁻¹) & ((y,x_9) ε (s)⁻¹)) & (z = (x,x_9))) ClassElim 133
135. Set(z)  AndElimL 134
136. ∃x.∃y.∃x_9.((((x,y) ε (r)⁻¹) & ((y,x_9) ε (s)⁻¹)) & (z = (x,x_9)))  AndElimR 134
137. ∃y.∃x_9.((((x,y) ε (r)⁻¹) & ((y,x_9) ε (s)⁻¹)) & (z = (x,x_9)))  Hyp
138. ∃x_9.((((x,y) ε (r)⁻¹) & ((y,x_9) ε (s)⁻¹)) & (z = (x,x_9)))  Hyp
139. (((x,y) ε (r)⁻¹) & ((y,a) ε (s)⁻¹)) & (z = (x,a))  Hyp
140. z = (x,a)  AndElimR 139
141. ((x,y) ε (r)⁻¹) & ((y,a) ε (s)⁻¹)  AndElimL 139
142. (x,y) ε (r)⁻¹  AndElimL 141
143. (y,a) ε (s)⁻¹  AndElimR 141
144. ∀r.((r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))})  ForallInt 1
145. (s)⁻¹ = {z: ∃x.∃y.(((x,y) ε s) & (z = (y,x)))}  ForallElim 144
146. (x,y) ε {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}  EqualitySub 142 1
147. (y,a) ε {z: ∃x.∃y.(((x,y) ε s) & (z = (y,x)))}  EqualitySub 143 145
148. Set((x,y)) & ∃x_10.∃x_11.(((x_10,x_11) ε r) & ((x,y) = (x_11,x_10)))  ClassElim 146
149. Set((y,a)) & ∃x.∃x_12.(((x,x_12) ε s) & ((y,a) = (x_12,x)))  ClassElim 147
150. Set((x,y))  AndElimL 148
151. ∃x_10.∃x_11.(((x_10,x_11) ε r) & ((x,y) = (x_11,x_10)))  AndElimR 148
152. Set((y,a))  AndElimL 149
153. ∃x.∃x_12.(((x,x_12) ε s) & ((y,a) = (x_12,x)))  AndElimR 149
154. ∃x_11.(((b,x_11) ε r) & ((x,y) = (x_11,b)))  Hyp
155. ((b,c) ε r) & ((x,y) = (c,b))  Hyp
156. ∃x_12.(((d,x_12) ε s) & ((y,a) = (x_12,d)))  Hyp
157. ((d,e) ε s) & ((y,a) = (e,d))  Hyp
158. (b,c) ε r  AndElimL 155
159. (d,e) ε s  AndElimL 157
160. (x,y) = (c,b)  AndElimR 155
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161. (y,a) = (e,d)  AndElimR 157
162. Set(x) & Set(y)  ImpElim 150 26
163. (Set(x) & Set(y)) & ((x,y) = (c,b))  AndInt 162 160
164. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 29
165. ((Set(x) & Set(y)) & ((x,y) = (c,v))) -> ((x = c) & (y = v))  ForallElim 164
166. ∀v.(((Set(x) & Set(y)) & ((x,y) = (c,v))) -> ((x = c) & (y = v)))  ForallInt 165
167. ((Set(x) & Set(y)) & ((x,y) = (c,b))) -> ((x = c) & (y = b))  ForallElim 166
168. (x = c) & (y = b)  ImpElim 163 167
169. x = c  AndElimL 168
170. y = b  AndElimR 168
171. c = x  Symmetry 169
172. b = y  Symmetry 170
173. ∀y.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 26
174. Set((x,a)) -> (Set(x) & Set(a))  ForallElim 173
175. ∀x.(Set((x,a)) -> (Set(x) & Set(a)))  ForallInt 174
176. Set((y,a)) -> (Set(y) & Set(a))  ForallElim 175
177. Set(y) & Set(a)  ImpElim 152 176
178. ((d,e) ε s) & ((b,c) ε r)  AndInt 159 158
179. ((d,e) ε s) & ((b,x) ε r)  EqualitySub 178 171
180. (Set(y) & Set(a)) & ((y,a) = (e,d))  AndInt 177 161
181. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 29
182. ((Set(x) & Set(y)) & ((x,y) = (e,v))) -> ((x = e) & (y = v))  ForallElim 181
183. ∀y.(((Set(x) & Set(y)) & ((x,y) = (e,v))) -> ((x = e) & (y = v)))  ForallInt 182
184. ((Set(x) & Set(a)) & ((x,a) = (e,v))) -> ((x = e) & (a = v))  ForallElim 183
185. ∀x.(((Set(x) & Set(a)) & ((x,a) = (e,v))) -> ((x = e) & (a = v)))  ForallInt 184
186. ((Set(y) & Set(a)) & ((y,a) = (e,v))) -> ((y = e) & (a = v))  ForallElim 185
187. ∀v.(((Set(y) & Set(a)) & ((y,a) = (e,v))) -> ((y = e) & (a = v)))  ForallInt 186
188. ((Set(y) & Set(a)) & ((y,a) = (e,d))) -> ((y = e) & (a = d))  ForallElim 187
189. (y = e) & (a = d)  ImpElim 180 188
190. y = e  AndElimL 189
191. a = d  AndElimR 189
192. e = y  Symmetry 190
193. ((d,y) ε s) & ((b,x) ε r)  EqualitySub 179 192
194. ((d,y) ε s) & ((y,x) ε r)  EqualitySub 193 172
195. d = a  Symmetry 191
196. ((a,y) ε s) & ((y,x) ε r)  EqualitySub 194 195
197. h = (a,x)  Hyp
198. Set(a)  AndElimR 177
199. Set(x)  AndElimL 162
200. Set(a) & Set(x)  AndInt 198 199
201. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 57
202. (Set(a) & Set(y)) -> Set((a,y))  ForallElim 201
203. ∀y.((Set(a) & Set(y)) -> Set((a,y)))  ForallInt 202
204. (Set(a) & Set(x)) -> Set((a,x))  ForallElim 203
205. Set((a,x))  ImpElim 200 204
206. (a,x) = h  Symmetry 197
207. Set(h)  EqualitySub 205 206
208. (((a,y) ε s) & ((y,x) ε r)) & (h = (a,x))  AndInt 196 197
209. ∃x.((((a,y) ε s) & ((y,x) ε r)) & (h = (a,x)))  ExistsInt 208
210. ∃y.∃x.((((a,y) ε s) & ((y,x) ε r)) & (h = (a,x)))  ExistsInt 209
211. ∃a.∃y.∃x.((((a,y) ε s) & ((y,x) ε r)) & (h = (a,x)))  ExistsInt 210
212. Set(h) & ∃a.∃y.∃x.((((a,y) ε s) & ((y,x) ε r)) & (h = (a,x)))  AndInt 207 211
213. h ε {w: ∃a.∃y.∃x.((((a,y) ε s) & ((y,x) ε r)) & (w = (a,x)))}  ClassInt 212
214. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))})  ForallInt 7
215. (r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 214
216. ∀b.((r∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε r)) & (w = (x,z)))})  ForallInt
215
217. (r∘s) = {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))}  ForallElim 216
218. {w: ∃x.∃y.∃z.((((x,y) ε s) & ((y,z) ε r)) & (w = (x,z)))} = (r∘s)  Symmetry 217
219. h ε (r∘s)  EqualitySub 213 218
220. (a,x) ε (r∘s)  EqualitySub 219 197
221. (h = (a,x)) -> ((a,x) ε (r∘s))  ImpInt 220
222. ∀h.((h = (a,x)) -> ((a,x) ε (r∘s)))  ForallInt 221
223. ((a,x) = (a,x)) -> ((a,x) ε (r∘s))  ForallElim 222
224. (a,x) = (a,x)  Identity
225. (a,x) ε (r∘s)  ImpElim 224 223
226. f = (x,a)  Hyp
227. (x,a) = f  Symmetry 226
228. Set((x,a))  EqualitySub 135 140
229. Set(f)  EqualitySub 228 227
230. ((a,x) ε (r∘s)) & (f = (x,a))  AndInt 220 226
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231. ∃x.(((a,x) ε (r∘s)) & (f = (x,a))) ExistsInt 230
232. ∃a.∃x.(((a,x) ε (r∘s)) & (f = (x,a))) ExistsInt 231
233. Set(f) & ∃a.∃x.(((a,x) ε (r∘s)) & (f = (x,a))) AndInt 229 232
234. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 1
235. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 1
236. ((r∘s))$^{-1}$ = {z: ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))} ForallElim 235
237. {z: ∃x.∃y.(((x,y) ε (r∘s)) & (z = (y,x)))} = ((r∘s))$^{-1}$ Symmetry 236
238. f ε {w: ∃a.∃x.(((a,x) ε (r∘s)) & (w = (x,a)))} ClassInt 233
239. f ε ((r∘s))$^{-1}$ EqualitySub 238 237
240. (x,a) ε ((r∘s))$^{-1}$ EqualitySub 239 226
241. (f = (x,a)) -> ((x,a) ε ((r∘s))$^{-1}$) ImpInt 240
242. ∀f.((f = (x,a)) -> ((x,a) ε ((r∘s))$^{-1}$)) ForallInt 241
243. ((x,a) = (x,a)) -> ((x,a) ε ((r∘s))$^{-1}$) ForallElim 242
244. (x,a) = (x,a) Identity
245. (x,a) ε ((r∘s))$^{-1}$ ImpElim 244 243
246. f ε ((r∘s))$^{-1}$ EqualitySub 245 227
247. f ε ((r∘s))$^{-1}$ ExistsElim 156 157 246
248. f ε ((r∘s))$^{-1}$ ExistsElim 153 156 247
249. f ε ((r∘s))$^{-1}$ ExistsElim 154 155 248
250. f ε ((r∘s))$^{-1}$ ExistsElim 151 154 249
251. f ε ((r∘s))$^{-1}$ ExistsElim 154 155 250
252. (h = (a,x)) -> (f ε ((r∘s))$^{-1}$) ImpInt 251
253. ∀h.((h = (a,x)) -> (f ε ((r∘s))$^{-1}$)) ForallInt 252
254. ∀h.((h = (a,x)) -> (f ε ((r∘s))$^{-1}$)) ForallInt 252
255. ((a,x) = (a,x)) -> (f ε ((r∘s))$^{-1}$) ForallElim 254
256. (a,x) = (a,x) Identity
257. f ε ((r∘s))$^{-1}$ ImpElim 256 255
258. (x,a) ε ((r∘s))$^{-1}$ EqualitySub 257 226
259. (f = (x,a)) -> ((x,a) ε ((r∘s))$^{-1}$) ImpInt 258
260. ∀f.((f = (x,a)) -> ((x,a) ε ((r∘s))$^{-1}$)) ForallInt 259
261. ((x,a) = (x,a)) -> ((x,a) ε ((r∘s))$^{-1}$) ForallElim 260
262. (x,a) = (x,a) Identity
263. (x,a) ε ((r∘s))$^{-1}$ ImpElim 262 261
264. (x,a) = z Symmetry 140
265. z ε ((r∘s))$^{-1}$ EqualitySub 263 264
266. z ε ((r∘s))$^{-1}$ ExistsElim 151 154 265
267. z ε ((r∘s))$^{-1}$ ExistsElim 138 139 266
268. z ε ((r∘s))$^{-1}$ ExistsElim 137 138 267
269. z ε ((r∘s))$^{-1}$ ExistsElim 136 137 268
270. (z ε ((s)$^{-1}$∘(r)$^{-1}$)) -> (z ε ((r∘s))$^{-1}$) ImpInt 269
271. ((z ε ((r∘s))$^{-1}$) -> (z ε ((s)$^{-1}$∘(r)$^{-1}$))) & ((z ε ((s)$^{-1}$∘(r)$^{-1}$)) -> (z ε ((r∘s))$^{-1}$)) AndInt 127 270
272. (z ε ((r∘s))$^{-1}$) <-> (z ε ((s)$^{-1}$∘(r)$^{-1}$)) EquivConst 271
273. ∀z.((z ε ((r∘s))$^{-1}$) <-> (z ε ((s)$^{-1}$∘(r)$^{-1}$))) ForallInt 272
274. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
275. ∀y.((((r∘s))$^{-1}$ = y) <-> ∀z.((z ε ((r∘s))$^{-1}$) <-> (z ε y))) ForallElim 274
276. (((r∘s))$^{-1}$ = ((s)$^{-1}$∘(r)$^{-1}$)) <-> ∀z.((z ε ((r∘s))$^{-1}$) <-> (z ε ((s)$^{-1}$∘(r)$^{-1}$))) ForallElim 275
277. ((((r∘s))$^{-1}$ = ((s)$^{-1}$∘(r)$^{-1}$)) -> ∀z.((z ε ((r∘s))$^{-1}$) <-> (z ε ((s)$^{-1}$∘(r)$^{-1}$)))) & (∀z.((z ε ((r∘s))$^{-1}$) <-> (z ε ((s)$^{-1}$∘(r)$^{-1}$))) -> (((r∘s))$^{-1}$ = ((s)$^{-1}$∘(r)$^{-1}$))) EquivExp 276
278. ∀z.((z ε ((r∘s))$^{-1}$) <-> (z ε ((s)$^{-1}$∘(r)$^{-1}$))) -> (((r∘s))$^{-1}$ = ((s)$^{-1}$∘(r)$^{-1}$)) AndElimR 277
279. ((r∘s))$^{-1}$ = ((s)$^{-1}$∘(r)$^{-1}$) ImpElim 273 278 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th64. (Function(f) & Function(g)) -> Function((f∘g))

0. Function(f) & Function(g) Hyp
1. Function(f) AndElimL 0
2. Function(g) AndElimR 0
3. (a,b) ε (f∘g) Hyp
4. (a,c) ε (f∘g) Hyp
5. (a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))} DefEqInt
6. ∀a.((a∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε a)) & (w = (x,z)))}) ForallInt 5
7. (f∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε f)) & (w = (x,z)))} ForallElim 6
8. ∀b.((f∘b) = {w: ∃x.∃y.∃z.((((x,y) ε b) & ((y,z) ε f)) & (w = (x,z)))}) ForallInt 7
9. (f∘g) = {w: ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & (w = (x,z)))} ForallElim 8

10. (a,b) ε {w: ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & (w = (x,z)))} EqualitySub 3 9
11. (a,c) ε {w: ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & (w = (x,z)))} EqualitySub 4 9
12. Set((a,b)) & ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z))) ClassElim 10
13. Set((a,c)) & ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,c) = (x,z))) ClassElim 11
14. ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z))) AndElimR 12
15. ∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z))) Hyp
16. ∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z))) Hyp
17. (((x,y) ε g) & ((y,z) ε f)) & ((a,b) = (x,z)) Hyp
18. ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & ((a,c) = (x,z))) AndElimR 13
19. ∃y.∃z.((((u,y) ε g) & ((y,z) ε f)) & ((a,c) = (u,z))) Hyp
20. ∃z.((((u,v) ε g) & ((v,z) ε f)) & ((a,c) = (u,z))) Hyp
21. (((u,v) ε g) & ((v,w) ε f)) & ((a,c) = (u,w)) Hyp
22. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
23. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 22
24. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 23
25. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 24
26. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 25
27. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 26
28. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 27
29. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 28
30. Set((a,b)) AndElimL 12
31. Set(a) & Set(b) ImpElim 30 29
32. Set(a) AndElimL 31
33. Set(b) AndElimR 31
34. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 25
35. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 34
36. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 35
37. Set((a,c)) -> (Set(a) & Set(c)) ForallElim 36
38. Set((a,c)) AndElimL 13
39. Set(a) & Set(c) ImpElim 38 37
40. Set(c) AndElimR 39
41. (a,b) = (x,z) AndElimR 17
42. (Set(a) & Set(b)) & ((a,b) = (x,z)) AndInt 31 41
43. (a,c) = (u,w) AndElimR 21
44. (Set(a) & Set(c)) & ((a,c) = (u,w)) AndInt 39 43
45. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
46. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 45
47. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)) ForallElim 46
48. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 47
49. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)) ForallElim 48
50. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))) ForallInt 49
51. ((Set(a) & Set(b)) & ((a,b) = (x,v))) -> ((a = x) & (b = v)) ForallElim 50
52. ∀v.(((Set(a) & Set(b)) & ((a,b) = (x,v))) -> ((a = x) & (b = v))) ForallInt 51
53. ((Set(a) & Set(b)) & ((a,b) = (x,z))) -> ((a = x) & (b = z)) ForallElim 52
54. (a = x) & (b = z) ImpElim 42 53
55. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 47
56. ((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v)) ForallElim 55
57. ∀v.(((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v))) ForallInt 56
58. ((Set(a) & Set(c)) & ((a,c) = (u,w))) -> ((a = u) & (c = w)) ForallElim 57
59. (a = u) & (c = w) ImpElim 44 58
60. a = x AndElimL 54
61. b = z AndElimR 54
62. a = u AndElimL 59
63. c = w AndElimR 59
64. ((x,y) ε g) & ((y,z) ε f) AndElimL 17
65. ((u,v) ε g) & ((v,w) ε f) AndElimL 21
66. (y,z) ε f AndElimR 64
67. (v,w) ε f AndElimR 65
68. (x,y) ε g AndElimL 64
69. (u,v) ε g AndElimL 65
70. x = u EqualitySub 62 60
71. (u,y) ε g EqualitySub 68 70
72. Relation(g) & ∀x.∀y.∀z.((((x,y) ε g) & ((x,z) ε g)) -> (y = z)) DefExp 2
73. ∀x.∀y.∀z.((((x,y) ε g) & ((x,z) ε g)) -> (y = z)) AndElimR 72
74. ∀y.∀z.((((u,y) ε g) & ((u,z) ε g)) -> (y = z)) ForallElim 73
75. ∀z.((((u,y) ε g) & ((u,z) ε g)) -> (y = z)) ForallElim 74
76. (((u,y) ε g) & ((u,v) ε g)) -> (y = v) ForallElim 75
77. ((u,y) ε g) & ((u,v) ε g) AndInt 71 69
78. y = v ImpElim 77 76
79. (v,z) ε f EqualitySub 66 78
80. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 1

81. ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) AndElimR 80
82. ∀y.∀z.((((v,y) ε f) & ((v,z) ε f)) -> (y = z)) ForallElim 81
83. ∀x_3.((((v,z) ε f) & ((v,x_3) ε f)) -> (z = x_3)) ForallElim 82
84. (((v,z) ε f) & ((v,w) ε f)) -> (z = w) ForallElim 83
85. ((v,z) ε f) & ((v,w) ε f) AndInt 79 67
86. z = w ImpElim 85 84
87. b = w EqualitySub 61 86
88. w = c Symmetry 63
89. b = c EqualitySub 87 88
90. b = c ExistsElim 20 21 89
91. b = c ExistsElim 19 20 90
92. b = c ExistsElim 18 19 91
93. b = c ExistsElim 16 17 92
94. b = c ExistsElim 15 16 93
95. b = c ExistsElim 14 15 94
96. ((a,c) ε (f∘g)) -> (b = c) ImpInt 95
97. ((a,b) ε (f∘g)) -> (((a,c) ε (f∘g)) -> (b = c)) ImpInt 96
98. A -> (B -> C) Hyp
99. A & B Hyp
100. A AndElimL 99
101. B -> C ImpElim 100 98
102. B AndElimR 99
103. C ImpElim 102 101
104. (A & B) -> C ImpInt 103
105. (A -> (B -> C)) -> ((A & B) -> C) ImpInt 104
106. (((a,b) ε (f∘g)) -> (B -> C)) -> ((((a,b) ε (f∘g)) & B) -> C) PolySub 105
107. (((a,b) ε (f∘g)) -> (((a,c) ε (f∘g)) -> C)) -> ((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> C) PolySub 106
108. (((a,b) ε (f∘g)) -> (((a,c) ε (f∘g)) -> (b = c))) -> ((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) PolySub 107
109. (((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c) ImpElim 97 108
110. Relation(g) AndElimL 72
111. Relation(f) AndElimL 80
112. z ε (f∘g) Hyp
113. z ε {w: ∃x.∃y.∃z.((((x,y) ε g) & ((y,z) ε f)) & (w = (x,z)))} EqualitySub 112 9
114. Set(z) & ∃x.∃y.∃x_4.((((x,y) ε g) & ((y,x_4) ε f)) & (z = (x,x_4))) ClassElim 113
115. ∃x.∃y.∃x_4.((((x,y) ε g) & ((y,x_4) ε f)) & (z = (x,x_4))) AndElimR 114
116. ∃y.∃x_4.((((x,y) ε g) & ((y,x_4) ε f)) & (z = (x,x_4))) Hyp
117. ∃x_4.((((x,y) ε g) & ((y,x_4) ε f)) & (z = (x,x_4))) Hyp
118. (((x,y) ε g) & ((y,l) ε f)) & (z = (x,l)) Hyp
119. z = (x,l) AndElimR 118
120. ∃l.(z = (x,l)) ExistsInt 119
121. ∃x.∃l.(z = (x,l)) ExistsInt 120
122. ∃x.∃l.(z = (x,l)) ExistsElim 117 118 121
123. ∃x.∃l.(z = (x,l)) ExistsElim 116 117 122
124. ∃x.∃l.(z = (x,l)) ExistsElim 115 116 123
125. (z ε (f∘g)) -> ∃x.∃l.(z = (x,l)) ImpInt 124
126. ∀z.((z ε (f∘g)) -> ∃x.∃l.(z = (x,l))) ForallInt 125
127. Relation((f∘g)) DefSub 126
128. ∀c.((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) ForallInt 109
129. ∀b.∀c.((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) ForallInt 128
130. ∀a.∀b.∀c.((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) ForallInt 129
131. Relation((f∘g)) & ∀a.∀b.∀c.((((a,b) ε (f∘g)) & ((a,c) ε (f∘g))) -> (b = c)) AndInt 127 130
132. Function((f∘g)) DefSub 131
133. (Function(f) & Function(g)) -> Function((f∘g)) ImpInt 132 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th67. (domain(U) = U) & (range(U) = U)

0. z ε domain(U) Hyp
1. ∃w.(z ε w) ExistsInt 0
2. Set(z) DefSub 1
3. (x ε U) <-> Set(x) TheoremInt
4. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 3
5. Set(x) -> (x ε U) AndElimR 4
6. ∀x.(Set(x) -> (x ε U)) ForallInt 5

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7. Set(z) -> (z ε U)  ForallElim 6
8. z ε U  ImpElim 2 7
9. (z ε domain(U)) -> (z ε U)  ImpInt 8
10. z ε U  Hyp
11. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U))  EquivExp 4
12. (x ε U) -> Set(x)  AndElimL 11
13. ∀x.((x ε U) -> Set(x))  ForallInt 12
14. (z ε U) -> Set(z)  ForallElim 13
15. Set(z)  ImpElim 10 14
16. (0 ⊂ x) & (x ⊂ U)  TheoremInt
17. 0 ⊂ x  AndElimL 16
18. ∀x.(0 ⊂ x)  ForallInt 17
19. 0 ⊂ z  ForallElim 18
20. (Set(x) & (y ⊂ x)) -> Set(y)  TheoremInt
21. ∀x.((Set(x) & (y ⊂ x)) -> Set(y))  ForallInt 20
22. (Set(z) & (y ⊂ z)) -> Set(y)  ForallElim 21
23. ∀y.((Set(z) & (y ⊂ z)) -> Set(y))  ForallInt 22
24. (Set(z) & (0 ⊂ z)) -> Set(0)  ForallElim 23
25. Set(z) & (0 ⊂ z)  AndInt 15 19
26. Set(0)  ImpElim 25 24
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
28. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 27
29. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 28
30. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 29
31. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 30
32. (Set(z) & Set(y)) -> Set((z,y))  ForallElim 31
33. ∀y.((Set(z) & Set(y)) -> Set((z,y)))  ForallInt 32
34. (Set(z) & Set(0)) -> Set((z,0))  ForallElim 33
35. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
36. Set(z) & Set(0)  AndInt 15 26
37. Set((z,0))  ImpElim 36 34
38. Set(x) -> (x ε U)  AndElimR 11
39. ∀x.(Set(x) -> (x ε U))  ForallInt 38
40. Set((z,0)) -> ((z,0) ε U)  ForallElim 39
41. (z,0) ε U  ImpElim 37 40
42. ∃w.((z,w) ε U)  ExistsInt 41
43. Set(z) & ∃w.((z,w) ε U)  AndInt 15 42
44. z ε {w: ∃i.((w,i) ε U)}  ClassInt 43
45. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 35
46. ∀f.({x: ∃y.((x,y) ε f)} = domain(f))  ForallInt 45
47. {x: ∃y.((x,y) ε U)} = domain(U)  ForallElim 46
48. z ε domain(U)  EqualitySub 44 47
49. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
50. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 30
51. (Set(0) & Set(y)) -> Set((0,y))  ForallElim 50
52. ∀y.((Set(0) & Set(y)) -> Set((0,y)))  ForallInt 51
53. (Set(0) & Set(z)) -> Set((0,z))  ForallElim 52
54. Set(0) & Set(z)  AndInt 26 15
55. Set((0,z))  ImpElim 54 53
56. ∀x.(Set(x) -> (x ε U))  ForallInt 38
57. Set((0,z)) -> ((0,z) ε U)  ForallElim 56
58. (0,z) ε U  ImpElim 55 57
59. ∃w.((w,z) ε U)  ExistsInt 58
60. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
61. {y: ∃x.((x,y) ε f)} = range(f)  Symmetry 60
62. ∀f.({y: ∃x.((x,y) ε f)} = range(f))  ForallInt 61
63. {y: ∃x.((x,y) ε U)} = range(U)  ForallElim 62
64. Set(z) & ∃w.((w,z) ε U)  AndInt 15 59
65. z ε {w: ∃j.((j,w) ε U)}  ClassInt 64
66. z ε range(U)  EqualitySub 65 63
67. (z ε U) -> (z ε domain(U))  ImpInt 48
68. (z ε U) -> (z ε range(U))  ImpInt 66
69. z ε range(U)  Hyp
70. ∃w.(z ε w)  ExistsInt 69
71. Set(z)  DefSub 70
72. z ε U  ImpElim 71 7
73. (z ε range(U)) -> (z ε U)  ImpInt 72
74. ((z ε domain(U)) -> (z ε U)) & ((z ε U) -> (z ε domain(U)))  AndInt 9 67
75. (z ε domain(U)) <-> (z ε U)  EquivConst 74
76. ∀z.((z ε domain(U)) <-> (z ε U))  ForallInt 75
77. ((z ε range(U)) -> (z ε U)) & ((z ε U) -> (z ε range(U)))  AndInt 73 68
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78. (z ε range(U)) <-> (z ε U) EquivConst 77
79. ∀z.((z ε range(U)) <-> (z ε U)) ForallInt 78
80. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
81. ∀y.((domain(U) = y) <-> ∀z.((z ε domain(U)) <-> (z ε y))) ForallElim 80
82. (domain(U) = U) <-> ∀z.((z ε domain(U)) <-> (z ε U)) ForallElim 81
83. ((domain(U) = U) -> ∀z.((z ε domain(U)) <-> (z ε U))) & (∀z.((z ε domain(U)) <-> (z ε U)) -> (domain(U) = U)) EquivExp 82
84. ∀z.((z ε domain(U)) <-> (z ε U)) -> (domain(U) = U) AndElimR 83
85. domain(U) = U ImpElim 76 84
86. ∀y.((range(U) = y) <-> ∀z.((z ε range(U)) <-> (z ε y))) ForallElim 80
87. (range(U) = U) <-> ∀z.((z ε range(U)) <-> (z ε U)) ForallElim 86
88. ((range(U) = U) -> ∀z.((z ε range(U)) <-> (z ε U))) & (∀z.((z ε range(U)) <-> (z ε U)) -> (range(U) = U)) EquivExp 87
89. ∀z.((z ε range(U)) <-> (z ε U)) -> (range(U) = U) AndElimR 88
90. range(U) = U ImpElim 79 89
91. (domain(U) = U) & (range(U) = U) AndInt 85 90 Qed

Used Theorems

1. (x ε U) <-> Set(x)
2. (0 ⊂ x) & (x ⊂ U)
3. (Set(x) & (y ⊂ x)) -> Set(y)
4. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th69. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U))

0. ¬(z ε domain(f)) Hyp
1. a ε {y: ((z,y) ε f)} Hyp
2. Set(a) & ((z,a) ε f) ClassElim 1
3. (z,a) ε f AndElimR 2
4. ∃w.((z,w) ε f) ExistsInt 3
5. ∃v.((z,a) ε v) ExistsInt 3
6. Set((z,a)) DefSub 5
7. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
8. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 7
9. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 8
10. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 9
11. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 10
12. Set((z,y)) -> (Set(z) & Set(y)) ForallElim 11
13. ∀y.(Set((z,y)) -> (Set(z) & Set(y))) ForallInt 12
14. Set((z,a)) -> (Set(z) & Set(a)) ForallElim 13
15. Set(z) & Set(a) ImpElim 6 14
16. Set(z) AndElimL 15
17. Set(z) & ∃w.((z,w) ε f) AndInt 16 4
18. z ε {w: ∃x_1.((w,x_1) ε f)} ClassInt 17
19. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
20. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 19
21. z ε domain(f) EqualitySub 18 20
22. _|_ ImpElim 21 0
23. ¬(a ε {y: ((z,y) ε f)}) ImpInt 22
24. ∀a.¬(a ε {y: ((z,y) ε f)}) ForallInt 23
25. b ε 0 Hyp
26. 0 = {x: ¬(x = x)} DefEqInt
27. b ε {x: ¬(x = x)} EqualitySub 25 26
28. Set(b) & ¬(b = b) ClassElim 27
29. ¬(b = b) AndElimR 28
30. b = b Identity
31. _|_ ImpElim 30 29
32. b ε {y: ((z,y) ε f)} AbsI 31
33. (b ε 0) -> (b ε {y: ((z,y) ε f)}) ImpInt 32
34. b ε {y: ((z,y) ε f)} Hyp
35. ¬(b ε {y: ((z,y) ε f)}) ForallElim 24
36. _|_ ImpElim 34 35
37. b ε 0 AbsI 36
38. (b ε {y: ((z,y) ε f)}) -> (b ε 0) ImpInt 37
39. ((b ε {y: ((z,y) ε f)}) -> (b ε 0)) & ((b ε 0) -> (b ε {y: ((z,y) ε f)})) AndInt 38 33
40. (b ε {y: ((z,y) ε f)}) <-> (b ε 0) EquivConst 39
41. ∀b.((b ε {y: ((z,y) ε f)}) <-> (b ε 0)) ForallInt 40
42. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
43. ∀x_2.(({y: ((z,y) ε f)} = x_2) <-> ∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε x_2)))

ForallElim 42
44. ({y: ((z,y) ε f)} = 0) <-> ∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε 0)) ForallElim 43
45. (({y: ((z,y) ε f)} = 0) -> ∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε 0))) & (∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε 0)) -> ({y: ((z,y) ε f)} = 0)) EquivExp 44
46. ∀x_3.((x_3 ε {y: ((z,y) ε f)}) <-> (x_3 ε 0)) -> ({y: ((z,y) ε f)} = 0) AndElimR 45
47. {y: ((z,y) ε f)} = 0 ImpElim 41 46
48. (∩0 = U) & (∪0 = 0) TheoremInt
49. ∩0 = U AndElimL 48
50. 0 = {y: ((z,y) ε f)} Symmetry 47
51. ∩{y: ((z,y) ε f)} = U EqualitySub 49 50
52. (f'x) = ∩{y: ((x,y) ε f)} DefEqInt
53. ∀x.((f'x) = ∩{y: ((x,y) ε f)}) ForallInt 52
54. (f'z) = ∩{y: ((z,y) ε f)} ForallElim 53
55. ∩{y: ((z,y) ε f)} = (f'z) Symmetry 54
56. (f'z) = U EqualitySub 51 55
57. ¬(z ε domain(f)) -> ((f'z) = U) ImpInt 56
58. z ε domain(f) Hyp
59. z ε {x: ∃y.((x,y) ε f)} EqualitySub 58 19
60. Set(z) & ∃y.((z,y) ε f) ClassElim 59
61. Set(z) AndElimL 60
62. ∃y.((z,y) ε f) AndElimR 60
63. {a: ((z,a) ε f)} = 0 Hyp
64. (z,y) ε f Hyp
65. ∃v.((z,y) ε v) ExistsInt 64
66. Set((z,y)) DefSub 65
67. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
68. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 67
69. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 68
70. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 69
71. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 70
72. Set((z,y)) -> (Set(z) & Set(y)) ForallElim 71
73. Set(z) & Set(y) ImpElim 66 72
74. Set(y) AndElimR 73
75. Set(y) & ((z,y) ε f) AndInt 74 64
76. y ε {w: ((z,w) ε f)} ClassInt 75
77. y ε 0 EqualitySub 76 63
78. 0 = {x: ¬(x = x)} DefEqInt
79. y ε {x: ¬(x = x)} EqualitySub 77 78
80. Set(y) & ¬(y = y) ClassElim 79
81. ¬(y = y) AndElimR 80
82. y = y Identity
83. _|_ ImpElim 82 81
84. ¬({a: ((z,a) ε f)} = 0) ImpInt 83
85. ¬(x = 0) -> Set(∩x) TheoremInt
86. ∀x.(¬(x = 0) -> Set(∩x)) ForallInt 85
87. ¬({a: ((z,a) ε f)} = 0) -> Set(∩{a: ((z,a) ε f)}) ForallElim 86
88. Set(∩{a: ((z,a) ε f)}) ImpElim 84 87
89. (f'x) = ∩{y: ((x,y) ε f)} DefEqInt
90. ∀x.((f'x) = ∩{y: ((x,y) ε f)}) ForallInt 89
91. (f'z) = ∩{y: ((z,y) ε f)} ForallElim 90
92. ∩{y: ((z,y) ε f)} = (f'z) Symmetry 91
93. Set((f'z)) EqualitySub 88 92
94. (x ε U) <-> Set(x) TheoremInt
95. ((x ε U) -> Set(x)) & (Set(x) -> (x ε U)) EquivExp 94
96. Set(x) -> (x ε U) AndElimR 95
97. ∀x.(Set(x) -> (x ε U)) ForallInt 96
98. Set((f'z)) -> ((f'z) ε U) ForallElim 97
99. (f'z) ε U ImpElim 93 98
100. (f'z) ε U ExistsElim 62 64 99
101. (z ε domain(f)) -> ((f'z) ε U) ImpInt 100
102. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U)) AndInt 57 101
Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. (∩0 = U) & (∪0 = 0)
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
4. ¬(x = 0) -> Set(∩x)
5. (x ε U) <-> Set(x)

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Th70. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

0. Function(f)  Hyp
1. z ε f  Hyp
2. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  DefExp 0
3. Relation(f)  AndElimL 2
4. ∀z.((z ε f) -> ∃x.∃y.(z = (x,y)))  DefExp 3
5. (z ε f) -> ∃x.∃y.(z = (x,y))  ForallElim 4
6. ∃x.∃y.(z = (x,y))  ImpElim 1 5
7. ∃y.(z = (x,y))  Hyp
8. z = (x,y)  Hyp
9. ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  AndElimR 2
10. (f'x) = ∩{y: ((x,y) ε f)}  DefEqInt
11. a ε {y: ((x,y) ε f)}  Hyp
12. Set(a) & ((x,a) ε f)  ClassElim 11
13. (x,a) ε f  AndElimR 12
14. ∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  ForallElim 9
15. ∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  ForallElim 14
16. (((x,y) ε f) & ((x,a) ε f)) -> (y = a)  ForallElim 15
17. (x,y) ε f  EqualitySub 1 8
18. ((x,y) ε f) & ((x,a) ε f)  AndInt 17 13
19. y = a  ImpElim 18 16
20. {x} = {z: ((x ε U) -> (z = x))}  DefEqInt
21. ∀x.({x} = {z: ((x ε U) -> (z = x))})  ForallInt 20
22. {y} = {z: ((y ε U) -> (z = y))}  ForallElim 21
23. (a ε {y: ((x,y) ε f)}) -> (y = a)  ImpInt 19
24. ∃w.(z ε w)  ExistsInt 1
25. Set(z)  DefSub 24
26. Set((x,y))  EqualitySub 25 8
27. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
28. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 27
29. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 28
30. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 29
31. Set(x) & Set(y)  ImpElim 26 30
32. Set(y)  AndElimR 31
33. Set(x) -> ((y ε {x}) <-> (y = x))  TheoremInt
34. ∀y.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 33
35. Set(x) -> ((a ε {x}) <-> (a = x))  ForallElim 34
36. ∀x.(Set(x) -> ((a ε {x}) <-> (a = x)))  ForallInt 35
37. Set(y) -> ((a ε {y}) <-> (a = y))  ForallElim 36
38. (a ε {y}) <-> (a = y)  ImpElim 32 37
39. ((a ε {y}) -> (a = y)) & ((a = y) -> (a ε {y}))  EquivExp 38
40. (a = y) -> (a ε {y})  AndElimR 39
41. a = y  Symmetry 19
42. a ε {y}  ImpElim 41 40
43. (a ε {y: ((x,y) ε f)}) -> (a ε {y})  ImpInt 42
44. a ε {y}  Hyp
45. ((a ε {y}) -> (a = y)) & ((a = y) -> (a ε {y}))  EquivExp 38
46. (a ε {y}) -> (a = y)  AndElimL 45
47. a = y  ImpElim 44 46
48. y = a  Symmetry 47
49. (x,y) ε f  EqualitySub 1 8
50. (x,a) ε f  EqualitySub 49 48
51. Set(a)  EqualitySub 32 48
52. Set(a) & ((x,a) ε f)  AndInt 51 50
53. a ε {y: ((x,y) ε f)}  ClassInt 52
54. (a ε {y}) -> (a ε {y: ((x,y) ε f)})  ImpInt 53
55. ((a ε {y: ((x,y) ε f)}) -> (a ε {y})) & ((a ε {y}) -> (a ε {y: ((x,y) ε f)}))  AndInt
43 54
56. (a ε {y: ((x,y) ε f)}) <-> (a ε {y})  EquivConst 55
57. ∀a.((a ε {y: ((x,y) ε f)}) <-> (a ε {y}))  ForallInt 56
58. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
59. ∀x_5.(({y: ((x,y) ε f)} = x_5) <-> ∀z.((z ε {y: ((x,y) ε f)}) <-> (z ε x_5)))
ForallElim 58
60. ({x_6: ((x,x_6) ε f)} = {y}) <-> ∀z.((z ε {x_6: ((x,x_6) ε f)}) <-> (z ε {y}))
ForallElim 59
61. (({x_6: ((x,x_6) ε f)} = {y}) -> ∀z.((z ε {x_6: ((x,x_6) ε f)}) <-> (z ε {y}))) &
(∀z.((z ε {x_6: ((x,x_6) ε f)}) <-> (z ε {y})) -> ({x_6: ((x,x_6) ε f)} = {y}))  EquivExp
60
62. ∀z.((z ε {x_6: ((x,x_6) ε f)}) <-> (z ε {y})) -> ({x_6: ((x,x_6) ε f)} = {y})
```

AndElimR 61
63. {x_6: ((x,x_6) ε f)} = {y} ImpElim 57 62
64. (f'x) = ∩{y} EqualitySub 10 63
65. (Set(x) -> ((∩{x} = x) & (U{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (U{x} = U)))
TheoremInt
66. Set(x) -> ((∩{x} = x) & (U{x} = x)) AndElimL 65
67. ∀x.(Set(x) -> ((∩{x} = x) & (U{x} = x))) ForallInt 66
68. Set(y) -> ((∩{y} = y) & (U{y} = y)) ForallElim 67
69. (∩{y} = y) & (U{y} = y) ImpElim 32 68
70. ∩{y} = y AndElimL 69
71. (f'x) = y EqualitySub 64 70
72. (z = (x,y)) & ((f'x) = y) AndInt 8 71
73. ∃y.((z = (x,y)) & ((f'x) = y)) ExistsInt 72
74. ∃x.∃y.((z = (x,y)) & ((f'x) = y)) ExistsInt 73
75. Set(z) & ∃x.∃y.((z = (x,y)) & ((f'x) = y)) AndInt 25 74
76. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ClassInt 75
77. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ExistsElim 7 8 76
78. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ExistsElim 6 7 77
79. (z ε f) -> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) ImpInt 78
80. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} Hyp
81. Set(z) & ∃x.∃y.((z = (x,y)) & ((f'x) = y)) ClassElim 80
82. Set(z) AndElimL 81
83. ∃x.∃y.((z = (x,y)) & ((f'x) = y)) AndElimR 81
84. ∃y.((z = (x,y)) & ((f'x) = y)) Hyp
85. (z = (x,y)) & ((f'x) = y) Hyp
86. z = (x,y) AndElimL 85
87. (f'x) = y AndElimR 85
88. ∩{y: ((x,y) ε f)} = y EqualitySub 87 10
89. Set((x,y)) EqualitySub 82 86
90. Set(x) & Set(y) ImpElim 89 30
91. Set(y) AndElimR 90
92. y = (f'x) Symmetry 87
93. Set((f'x)) EqualitySub 91 92
94. (f'x) = U Hyp
95. ¬Set(U) TheoremInt
96. Set(U) EqualitySub 93 94
97. _|_ ImpElim 96 95
98. ¬((f'x) = U) ImpInt 97
99. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U)) TheoremInt
100. ¬(z ε domain(f)) -> ((f'z) = U) AndElimL 99
101. (A -> B) -> (¬B -> ¬A) TheoremInt
102. (¬(z ε domain(f)) -> B) -> (¬B -> ¬¬(z ε domain(f))) PolySub 101
103. (¬(z ε domain(f)) -> ((f'z) = U)) -> (¬((f'z) = U) -> ¬¬(z ε domain(f))) PolySub
102
104. ¬((f'z) = U) -> ¬¬(z ε domain(f)) ImpElim 100 103
105. D <-> ¬¬D TheoremInt
106. (D -> ¬¬D) & (¬¬D -> D) EquivExp 105
107. ¬¬D -> D AndElimR 106
108. ¬¬(z ε domain(f)) -> (z ε domain(f)) PolySub 107
109. ¬((f'z) = U) Hyp
110. ¬¬(z ε domain(f)) ImpElim 109 104
111. z ε domain(f) ImpElim 110 108
112. ¬((f'z) = U) -> (z ε domain(f)) ImpInt 111
113. ∀z.(¬((f'z) = U) -> (z ε domain(f))) ForallInt 112
114. ¬((f'x) = U) -> (x ε domain(f)) ForallElim 113
115. x ε domain(f) ImpElim 98 114
116. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
117. x ε {x: ∃y.((x,y) ε f)} EqualitySub 115 116
118. Set(x) & ∃y.((x,y) ε f) ClassElim 117
119. ∃y.((x,y) ε f) AndElimR 118
120. (x,b) ε f Hyp
121. e ε {b} Hyp
122. ∃w.((x,b) ε w) ExistsInt 120
123. Set((x,b)) DefSub 122
124. ∀y.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 30
125. Set((x,b)) -> (Set(x) & Set(b)) ForallElim 124
126. Set(x) & Set(b) ImpElim 123 125
127. Set(b) AndElimR 126
128. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
129. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 128
130. Set(b) -> ((y ε {b}) <-> (y = b)) ForallElim 129

131. (y ε {b}) <-> (y = b) ImpElim 127 130
132. ∀y.((y ε {b}) <-> (y = b)) ForallInt 131
133. (e ε {b}) <-> (e = b) ForallElim 132
134. ((e ε {b}) -> (e = b)) & ((e = b) -> (e ε {b})) EquivExp 133
135. (e ε {b}) -> (e = b) AndElimL 134
136. e = b ImpElim 121 135
137. b = e Symmetry 136
138. (x,e) ε f EqualitySub 120 137
139. Set(e) EqualitySub 127 137
140. Set(e) & ((x,e) ε f) AndInt 139 138
141. e ε {y: ((x,y) ε f)} ClassInt 140
142. e ε {y: ((x,y) ε f)} Hyp
143. Set(e) & ((x,e) ε f) ClassElim 142
144. (x,e) ε f AndElimR 143
145. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 0
146. ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) AndElimR 145
147. (e ε {b}) -> (e ε {y: ((x,y) ε f)}) ImpInt 141
148. ((x,b) ε f) & ((x,e) ε f) AndInt 120 144
149. ∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) ForallElim 146
150. ∀z.((((x,b) ε f) & ((x,z) ε f)) -> (b = z)) ForallElim 149
151. (((x,b) ε f) & ((x,e) ε f)) -> (b = e) ForallElim 150
152. b = e ImpElim 148 151
153. ((y ε {b}) -> (y = b)) & ((y = b) -> (y ε {b})) EquivExp 131
154. ((e ε {b}) -> (e = b)) & ((e = b) -> (e ε {b})) EquivExp 133
155. (e = b) -> (e ε {b}) AndElimR 154
156. e = b Symmetry 152
157. e ε {b} ImpElim 156 155
158. (e ε {y: ((x,y) ε f)}) -> (e ε {b}) ImpInt 157
159. ((e ε {b}) -> (e ε {y: ((x,y) ε f)})) & ((e ε {y: ((x,y) ε f)}) -> (e ε {b}))
AndInt 147 158
160. (e ε {b}) <-> (e ε {y: ((x,y) ε f)}) EquivConst 159
161. ∀e.((e ε {b}) <-> (e ε {y: ((x,y) ε f)})) ForallInt 160
162. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
163. ∀y.(({b} = y) <-> ∀z.((z ε {b}) <-> (z ε y))) ForallElim 162
164. ({b} = {y: ((x,y) ε f)}) <-> ∀z.((z ε {b}) <-> (z ε {y: ((x,y) ε f)})) ForallElim
163
165. (({b} = {y: ((x,y) ε f)}) -> ∀z.((z ε {b}) <-> (z ε {y: ((x,y) ε f)}))) & (∀z.((z ε
{b}) <-> (z ε {y: ((x,y) ε f)})) -> ({b} = {y: ((x,y) ε f)})) EquivExp 164
166. ∀z.((z ε {b}) <-> (z ε {y: ((x,y) ε f)})) -> ({b} = {y: ((x,y) ε f)}) AndElimR 165
167. {b} = {y: ((x,y) ε f)} ImpElim 161 166
168. {y: ((x,y) ε f)} = {b} Symmetry 167
169. ∩{b} = y EqualitySub 88 168
170. (Set(x) -> ((∩{x} = x) & (∪{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (∪{x} = U)))
TheoremInt
171. Set(x) -> ((∩{x} = x) & (∪{x} = x)) AndElimL 170
172. ∀x.(Set(x) -> ((∩{x} = x) & (∪{x} = x))) ForallInt 171
173. Set(b) -> ((∩{b} = b) & (∪{b} = b)) ForallElim 172
174. (∩{b} = b) & (∪{b} = b) ImpElim 127 173
175. ∩{b} = b AndElimL 174
176. b = y EqualitySub 169 175
177. (x,y) ε f EqualitySub 120 176
178. (x,y) ε f EqualitySub 120 176
179. (x,y) = z Symmetry 86
180. z ε f EqualitySub 178 179
181. x = x Identity
182. z ε f ExistsElim 119 120 180
183. z ε f ExistsElim 84 85 182
184. z ε f ExistsElim 83 84 183
185. (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) -> (z ε f) ImpInt 184
186. ((z ε f) -> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) & ((z ε {w: ∃x.∃y.((w =
(x,y)) & ((f'x) = y))}) -> (z ε f)) AndInt 79 185
187. (z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) EquivConst 186
188. ∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) ForallInt 187
189. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
190. ∀y.((f = y) <-> ∀z.((z ε f) <-> (z ε y))) ForallElim 189
191. (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) <-> ∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w
= (x,y)) & ((f'x) = y))})) ForallElim 190
192. ((f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) -> ∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w
= (x,y)) & ((f'x) = y))}))) & (∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) =
y))})) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) EquivExp 191
193. ∀z.((z ε f) <-> (z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})) -> (f = {w: ∃x.∃y.((w

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= (x,y)) & ((f'x) = y))})  AndElimR 192
194. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ImpElim 188 193
195. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  ImpInt 194 Qed
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Used Theorems

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2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. Set(x) -> ((y ε {x}) <-> (y = x))
4. (Set(x) -> ((∩{x} = x) & (⋃{x} = x))) & (¬Set(x) -> ((∩{x} = 0) & (⋃{x} = U)))
5. ¬Set(U)
6. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U))
7. (A -> B) -> (¬B -> ¬A)
8. D <-> ¬¬D
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Th71. (Function(f) & Function(g)) -> ((f = g) <-> ∀z.((f'z) = (g'z)))

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0. Function(f) & Function(g)  Hyp
1. ∀z.((f'z) = (g'z))  Hyp
2. e ε f  Hyp
3. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  TheoremInt
4. Function(f)  AndElimL 0
5. Function(g)  AndElimR 0
6. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ImpElim 4 3
7. e ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  EqualitySub 2 6
8. Set(e) & ∃x.∃y.((e = (x,y)) & ((f'x) = y))  ClassElim 7
9. Set(e)  AndElimL 8
10. ∃x.∃y.((e = (x,y)) & ((f'x) = y))  AndElimR 8
11. ∃y.((e = (x,y)) & ((f'x) = y))  Hyp
12. (e = (x,y)) & ((f'x) = y)  Hyp
13. (f'x) = (g'x)  ForallElim 1
14. (e = (x,y)) & ((g'x) = y)  EqualitySub 12 13
15. ∃y.((e = (x,y)) & ((g'x) = y))  ExistsInt 14
16. ∃x.∃y.((e = (x,y)) & ((g'x) = y))  ExistsInt 15
17. Set(e) & ∃x.∃y.((e = (x,y)) & ((g'x) = y))  AndInt 9 16
18. e ε {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  ClassInt 17
19. ∀f.(Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}))  ForallInt 3
20. Function(g) -> (g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))})  ForallElim 19
21. g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  ImpElim 5 20
22. {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))} = g  Symmetry 21
23. e ε g  EqualitySub 18 22
24. e ε g  ExistsElim 11 12 23
25. e ε g  ExistsElim 10 11 24
26. (e ε f) -> (e ε g)  ImpInt 25
27. e ε g  Hyp
28. e ε {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  EqualitySub 27 21
29. Set(e) & ∃x.∃y.((e = (x,y)) & ((g'x) = y))  ClassElim 28
30. Set(e)  AndElimL 29
31. ∃x.∃y.((e = (x,y)) & ((g'x) = y))  AndElimR 29
32. ∃y.((e = (x,y)) & ((g'x) = y))  Hyp
33. (e = (x,y)) & ((g'x) = y)  Hyp
34. (g'x) = (f'x)  Symmetry 13
35. (e = (x,y)) & ((f'x) = y)  EqualitySub 33 34
36. ∃y.((e = (x,y)) & ((f'x) = y))  ExistsInt 35
37. ∃x.∃y.((e = (x,y)) & ((f'x) = y))  ExistsInt 36
38. Set(e) & ∃x.∃y.((e = (x,y)) & ((f'x) = y))  AndInt 30 37
39. e ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ClassInt 38
40. {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} = f  Symmetry 6
41. e ε f  EqualitySub 39 40
42. e ε f  ExistsElim 32 33 41
43. e ε f  ExistsElim 31 32 42
44. (e ε g) -> (e ε f)  ImpInt 43
45. ((e ε f) -> (e ε g)) & ((e ε g) -> (e ε f))  AndInt 26 44
46. (e ε f) <-> (e ε g)  EquivConst 45
47. ∀e.((e ε f) <-> (e ε g))  ForallInt 46
48. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
49. ∀y.((f = y) <-> ∀z.((z ε f) <-> (z ε y)))  ForallElim 48
50. (f = g) <-> ∀z.((z ε f) <-> (z ε g))  ForallElim 49
51. ((f = g) -> ∀z.((z ε f) <-> (z ε g))) & (∀z.((z ε f) <-> (z ε g)) -> (f = g))
EquivExp 50
52. ∀z.((z ε f) <-> (z ε g)) -> (f = g)  AndElimR 51
53. f = g  ImpElim 47 52
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54. ∀z.((f'z) = (g'z)) -> (f = g) ImpInt 53
55. f = g Hyp
56. (f'z) = (f'z) Identity
57. (f'z) = (g'z) EqualitySub 56 55
58. ∀z.((f'z) = (g'z)) ForallInt 57
59. (f = g) -> ∀z.((f'z) = (g'z)) ImpInt 58
60. ((f = g) -> ∀z.((f'z) = (g'z))) & (∀z.((f'z) = (g'z)) -> (f = g)) AndInt 59 54
61. (f = g) <-> ∀z.((f'z) = (g'z)) EquivConst 60
62. (Function(f) & Function(g)) -> ((f = g) <-> ∀z.((f'z) = (g'z))) ImpInt 61 Qed

Used Theorems

1. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

Th73. (Set(u) & Set(y)) -> Set(({u} X y))

0. Set(u) & Set(y) Hyp
1. f = {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} Hyp
2. x ε domain(f) Hyp
3. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
4. x ε {x: ∃y.((x,y) ε f)} EqualitySub 2 3
5. Set(x) & ∃y.((x,y) ε f) ClassElim 4
6. Set(x) & ∃x_0.((x,x_0) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))})
EqualitySub 5 1
7. Set(x) AndElimL 6
8. ∃x_0.((x,x_0) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}) AndElimR 6
9. (x,c) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} Hyp
10. Set((x,c)) & ∃w.∃z.(((x,c) = (w,z)) & ((w ε y) & (z = (u,w)))) ClassElim 9
11. Set((x,c)) AndElimL 10
12. ∃w.∃z.(((x,c) = (w,z)) & ((w ε y) & (z = (u,w)))) AndElimR 10
13. ∃z.(((x,c) = (w,z)) & ((w ε y) & (z = (u,w)))) Hyp
14. ((x,c) = (w,z)) & ((w ε y) & (z = (u,w))) Hyp
15. (x,c) = (w,z) AndElimL 14
16. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
17. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 16
18. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 17
19. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 18
20. ∀y.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 19
21. Set((x,c)) -> (Set(x) & Set(c)) ForallElim 20
22. Set(x) & Set(c) ImpElim 11 21
23. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
24. ∀y.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 23
25. ((Set(x) & Set(c)) & ((x,c) = (u,v))) -> ((x = u) & (c = v)) ForallElim 24
26. ∀u.(((Set(x) & Set(c)) & ((x,c) = (u,v))) -> ((x = u) & (c = v))) ForallInt 25
27. ((Set(x) & Set(c)) & ((x,c) = (w,v))) -> ((x = w) & (c = v)) ForallElim 26
28. ∀v.(((Set(x) & Set(c)) & ((x,c) = (w,v))) -> ((x = w) & (c = v))) ForallInt 27
29. ((Set(x) & Set(c)) & ((x,c) = (w,z))) -> ((x = w) & (c = z)) ForallElim 28
30. (Set(x) & Set(c)) & ((x,c) = (w,z)) AndInt 22 15
31. (x = w) & (c = z) ImpElim 30 29
32. x = w AndElimL 31
33. (w ε y) & (z = (u,w)) AndElimR 14
34. w ε y AndElimL 33
35. w = x Symmetry 32
36. x ε y EqualitySub 34 35
37. x ε y ExistsElim 13 14 36
38. x ε y ExistsElim 12 13 37
39. x ε y ExistsElim 8 9 38
40. (x ε domain(f)) -> (x ε y) ImpInt 39
41. x ε y Hyp
42. z = (u,x) Hyp
43. a = (x,z) Hyp
44. (a = (x,z)) & (z = (u,x)) AndInt 43 42
45. ∃z.((a = (x,z)) & (z = (u,x))) ExistsInt 44
46. ∃x.∃z.((a = (x,z)) & (z = (u,x))) ExistsInt 45
47. ∃y.(x ε y) ExistsInt 41
48. Set(x) DefSub 47
49. Set(u) AndElimL 0
50. Set(u) & Set(x) AndInt 49 48
51. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 17
52. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 51
53. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 52

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54. (Set(u) & Set(y)) -> Set((u,y))  ForallElim 53
55. ∀y.((Set(u) & Set(y)) -> Set((u,y)))  ForallInt 54
56. (Set(u) & Set(x)) -> Set((u,x))  ForallElim 55
57. Set((u,x))  ImpElim 50 56
58. (u,x) = z  Symmetry 42
59. Set(z)  EqualitySub 57 58
60. Set(x) & Set(z)  AndInt 48 59
61. ∀y.(((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))))  ForallInt
51
62. ∀y.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 52
63. (Set(x) & Set(z)) -> Set((x,z))  ForallElim 62
64. Set((x,z))  ImpElim 60 63
65. (x,z) = a  Symmetry 43
66. Set(a)  EqualitySub 64 65
67. Set(a) & ∃x.∃z.((a = (x,z)) & (z = (u,x)))  AndInt 66 46
68. {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} = f  Symmetry 1
69. a ε {a: ∃x.∃z.((a = (x,z)) & (z = (u,x)))}  ClassInt 67
70. (x ε y) & (z = (u,x))  AndInt 41 42
71. (a = (x,z)) & ((x ε y) & (z = (u,x)))  AndInt 43 70
72. ∃z.((a = (x,z)) & ((x ε y) & (z = (u,x))))  ExistsInt 71
73. ∃x.∃z.((a = (x,z)) & ((x ε y) & (z = (u,x))))  ExistsInt 72
74. Set(a) & ∃x.∃z.((a = (x,z)) & ((x ε y) & (z = (u,x))))  AndInt 66 73
75. a ε {a: ∃x.∃z.((a = (x,z)) & ((x ε y) & (z = (u,x))))}  ClassInt 74
76. a ε f  EqualitySub 75 68
77. (x,z) ε f  EqualitySub 76 43
78. ∃z.((x,z) ε f)  ExistsInt 77
79. Set(x) & ∃z.((x,z) ε f)  AndInt 48 78
80. x ε {w: ∃z.((w,z) ε f)}  ClassInt 79
81. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 3
82. x ε domain(f)  EqualitySub 80 81
83. (a = (x,z)) -> (x ε domain(f))  ImpInt 82
84. ∀a.((a = (x,z)) -> (x ε domain(f)))  ForallInt 83
85. ((x,z) = (x,z)) -> (x ε domain(f))  ForallElim 84
86. (x,z) = (x,z)  Identity
87. x ε domain(f)  ImpElim 86 85
88. (z = (u,x)) -> (x ε domain(f))  ImpInt 87
89. ∀z.((z = (u,x)) -> (x ε domain(f)))  ForallInt 88
90. ((u,x) = (u,x)) -> (x ε domain(f))  ForallElim 89
91. (u,x) = (u,x)  Identity
92. x ε domain(f)  ImpElim 91 90
93. (x ε y) -> (x ε domain(f))  ImpInt 92
94. ((x ε domain(f)) -> (x ε y)) & ((x ε y) -> (x ε domain(f)))  AndInt 40 93
95. (x ε domain(f)) <-> (x ε y)  EquivConst 94
96. ∀x.((x ε domain(f)) <-> (x ε y))  ForallInt 95
97. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
98. ∀y.((domain(f) = y) <-> ∀z.((z ε domain(f)) <-> (z ε y)))  ForallElim 97
99. (domain(f) = y) <-> ∀z.((z ε domain(f)) <-> (z ε y))  ForallElim 98
100. ((domain(f) = y) -> ∀z.((z ε domain(f)) <-> (z ε y))) & (∀z.((z ε domain(f)) <-> (z
ε y)) -> (domain(f) = y))  EquivExp 99
101. ∀z.((z ε domain(f)) <-> (z ε y)) -> (domain(f) = y)  AndElimR 100
102. domain(f) = y  ImpElim 96 101
103. x ε range(f)  Hyp
104. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
105. x ε {y: ∃x.((x,y) ε f)}  EqualitySub 103 104
106. Set(x) & ∃x_4.((x_4,x) ε f)  ClassElim 105
107. ∃x_4.((x_4,x) ε f)  AndElimR 106
108. ∃x_4.((x_4,x) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))})  EqualitySub 107
1
109. (c,x) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}  Hyp
110. Set((c,x)) & ∃w.∃z.(((c,x) = (w,z)) & ((w ε y) & (z = (u,w))))  ClassElim 109
111. ∃w.∃z.(((c,x) = (w,z)) & ((w ε y) & (z = (u,w))))  AndElimR 110
112. ∃z.(((c,x) = (w,z)) & ((w ε y) & (z = (u,w))))  Hyp
113. ((c,x) = (w,z)) & ((w ε y) & (z = (u,w)))  Hyp
114. Set((c,x))  AndElimL 110
115. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 19
116. Set((c,y)) -> (Set(c) & Set(y))  ForallElim 115
117. ∀y.(Set((c,y)) -> (Set(c) & Set(y)))  ForallInt 116
118. Set((c,x)) -> (Set(c) & Set(x))  ForallElim 117
119. Set(c) & Set(x)  ImpElim 114 118
120. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 23
121. ((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v))  ForallElim 120
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122. ∀y.(((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v)))  ForallInt 121
123. ((Set(c) & Set(x)) & ((c,x) = (u,v))) -> ((c = u) & (x = v))  ForallElim 122
124. ∀u.(((Set(c) & Set(x)) & ((c,x) = (u,v))) -> ((c = u) & (x = v)))  ForallInt 123
125. ((Set(c) & Set(x)) & ((c,x) = (w,v))) -> ((c = w) & (x = v))  ForallElim 124
126. ∀v.(((Set(c) & Set(x)) & ((c,x) = (w,v))) -> ((c = w) & (x = v)))  ForallInt 125
127. ((Set(c) & Set(x)) & ((c,x) = (w,z))) -> ((c = w) & (x = z))  ForallElim 126
128. (c,x) = (w,z)  AndElimL 113
129. (Set(c) & Set(x)) & ((c,x) = (w,z))  AndInt 119 128
130. (c = w) & (x = z)  ImpElim 129 127
131. (w ε y) & (z = (u,w))  AndElimR 113
132. w ε y  AndElimL 131
133. z = (u,w)  AndElimR 131
134. x = z  AndElimR 130
135. z = x  Symmetry 134
136. x = (u,w)  EqualitySub 133 135
137. Set(c)  AndElimL 119
138. c = w  AndElimL 130
139. Set(w)  EqualitySub 137 138
140. Set(x) -> ((y ε {x}) <-> (y = x))  TheoremInt
141. Set(u)  AndElimL 0
142. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 140
143. Set(u) -> ((y ε {u}) <-> (y = u))  ForallElim 142
144. ∀y.(Set(u) -> ((y ε {u}) <-> (y = u)))  ForallInt 143
145. Set(u) -> ((u ε {u}) <-> (u = u))  ForallElim 144
146. (u ε {u}) <-> (u = u)  ImpElim 141 145
147. ((u ε {u}) -> (u = u)) & ((u = u) -> (u ε {u}))  EquivExp 146
148. (u = u) -> (u ε {u})  AndElimR 147
149. u = u  Identity
150. u ε {u}  ImpElim 149 148
151. (u ε {u}) & (w ε y)  AndInt 150 132
152. (x = (u,w)) & ((u ε {u}) & (w ε y))  AndInt 136 151
153. Set(x)  AndElimR 119
154. ∃w.((x = (u,w)) & ((u ε {u}) & (w ε y)))  ExistsInt 152
155. ∃b.∃w.((x = (b,w)) & ((b ε {u}) & (w ε y)))  ExistsInt 154
156. Set(x) & ∃b.∃w.((x = (b,w)) & ((b ε {u}) & (w ε y)))  AndInt 153 155
157. x ε {e: ∃b.∃w.((e = (b,w)) & ((b ε {u}) & (w ε y)))}  ClassInt 156
158. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}  DefEqInt
159. ∀x.((x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))})  ForallInt 158
160. ({u} X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))}  ForallElim 159
161. {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))} = ({u} X y)  Symmetry 160
162. x ε ({u} X y)  EqualitySub 157 161
163. x ε ({u} X y)  ExistsElim 112 113 162
164. x ε ({u} X y)  Hyp
165. x ε {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))}  EqualitySub 164 160
166. Set(x) & ∃a.∃b.((x = (a,b)) & ((a ε {u}) & (b ε y)))  ClassElim 165
167. ∃a.∃b.((x = (a,b)) & ((a ε {u}) & (b ε y)))  AndElimR 166
168. x ε ({u} X y)  ExistsElim 111 112 163
169. x ε ({u} X y)  ExistsElim 108 109 168
170. (x ε range(f)) -> (x ε ({u} X y))  ImpInt 169
171. ∃b.((x = (a,b)) & ((a ε {u}) & (b ε y)))  Hyp
172. (x = (a,b)) & ((a ε {u}) & (b ε y))  Hyp
173. x = (a,b)  AndElimL 172
174. (a ε {u}) & (b ε y)  AndElimR 172
175. a ε {u}  AndElimL 174
176. b ε y  AndElimR 174
177. ∀y.(Set(u) -> ((y ε {u}) <-> (y = u)))  ForallInt 143
178. Set(u) -> ((a ε {u}) <-> (a = u))  ForallElim 177
179. Set(u)  AndElimL 0
180. (a ε {u}) <-> (a = u)  ImpElim 179 178
181. ((a ε {u}) -> (a = u)) & ((a = u) -> (a ε {u}))  EquivExp 180
182. (a ε {u}) -> (a = u)  AndElimL 181
183. a = u  ImpElim 175 182
184. x = (u,b)  EqualitySub 173 183
185. c = (b,x)  Hyp
186. (b ε y) & (x = (u,b))  AndInt 176 184
187. (c = (b,x)) & ((b ε y) & (x = (u,b)))  AndInt 185 186
188. ∃x.((c = (b,x)) & ((b ε y) & (x = (u,b))))  ExistsInt 187
189. ∃b.∃x.((c = (b,x)) & ((b ε y) & (x = (u,b))))  ExistsInt 188
190. Set(x)  AndElimL 166
191. ∃y.(b ε y)  ExistsInt 176
192. Set(b)  DefSub 191
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193. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 52
194. (Set(b) & Set(y)) -> Set((b,y)) ForallElim 193
195. ∀y.((Set(b) & Set(y)) -> Set((b,y))) ForallInt 194
196. (Set(b) & Set(x)) -> Set((b,x)) ForallElim 195
197. Set(b) & Set(x) AndInt 192 190
198. Set((b,x)) ImpElim 197 196
199. (b,x) = c Symmetry 185
200. Set(c) EqualitySub 198 199
201. Set(c) & ∃b.∃x.((c = (b,x)) & ((b ε y) & (x = (u,b)))) AndInt 200 189
202. c ε {w: ∃b.∃x.((w = (b,x)) & ((b ε y) & (x = (u,b))))} ClassInt 201
203. {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} = f Symmetry 1
204. c ε f EqualitySub 202 203
205. (b,x) ε f EqualitySub 204 185
206. ∃b.((b,x) ε f) ExistsInt 205
207. Set(x) & ∃b.((b,x) ε f) AndInt 190 206
208. x ε {w: ∃b.((b,w) ε f)} ClassInt 207
209. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 104
210. x ε range(f) EqualitySub 208 209
211. (c = (b,x)) -> (x ε range(f)) ImpInt 210
212. ∀c.((c = (b,x)) -> (x ε range(f))) ForallInt 211
213. ((b,x) = (b,x)) -> (x ε range(f)) ForallElim 212
214. (b,x) = (b,x) Identity
215. x ε range(f) ImpElim 214 213
216. x ε range(f) ExistsElim 171 172 215
217. x ε range(f) ExistsElim 167 171 216
218. (x ε ({u} X y)) -> (x ε range(f)) ImpInt 217
219. ((x ε range(f)) -> (x ε ({u} X y))) & ((x ε ({u} X y)) -> (x ε range(f))) AndInt 170 218
220. (x ε range(f)) <-> (x ε ({u} X y)) EquivConst 219
221. ∀x.((x ε range(f)) <-> (x ε ({u} X y))) ForallInt 220
222. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
223. ∀y.((range(f) = y) <-> ∀z.((z ε range(f)) <-> (z ε y))) ForallElim 222
224. (range(f) = ({u} X y)) <-> ∀z.((z ε range(f)) <-> (z ε ({u} X y))) ForallElim 223
225. ((range(f) = ({u} X y)) -> ∀z.((z ε range(f)) <-> (z ε ({u} X y)))) & (∀z.((z ε range(f)) <-> (z ε ({u} X y))) -> (range(f) = ({u} X y))) EquivExp 224
226. ∀z.((z ε range(f)) <-> (z ε ({u} X y))) -> (range(f) = ({u} X y)) AndElimR 225
227. range(f) = ({u} X y) ImpElim 221 226
228. (Function(f) & Set(domain(f))) -> Set(range(f)) AxInt
229. Set(y) AndElimR 0
230. y = domain(f) Symmetry 102
231. Set(domain(f)) EqualitySub 229 230
232. x ε f Hyp
233. x ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} EqualitySub 232 1
234. Set(x) & ∃w.∃z.((x = (w,z)) & ((w ε y) & (z = (u,w)))) ClassElim 233
235. ∃w.∃z.((x = (w,z)) & ((w ε y) & (z = (u,w)))) AndElimR 234
236. ∃z.((x = (w,z)) & ((w ε y) & (z = (u,w)))) Hyp
237. (x = (w,z)) & ((w ε y) & (z = (u,w))) Hyp
238. x = (w,z) AndElimL 237
239. ∃z.(x = (w,z)) ExistsInt 238
240. ∃w.∃z.(x = (w,z)) ExistsInt 239
241. ∃w.∃z.(x = (w,z)) ExistsElim 236 237 240
242. ∃w.∃z.(x = (w,z)) ExistsElim 235 236 241
243. (x ε f) -> ∃w.∃z.(x = (w,z)) ImpInt 242
244. ∀x.((x ε f) -> ∃w.∃z.(x = (w,z))) ForallInt 243
245. Relation(f) DefSub 244
246. (a,b) ε f Hyp
247. (a,c) ε f Hyp
248. (a,b) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} EqualitySub 246 1
249. (a,c) ε {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} EqualitySub 247 1
250. Set((a,b)) & ∃w.∃z.(((a,b) = (w,z)) & ((w ε y) & (z = (u,w)))) ClassElim 248
251. Set((a,c)) & ∃w.∃z.(((a,c) = (w,z)) & ((w ε y) & (z = (u,w)))) ClassElim 249
252. ∃w.∃z.(((a,b) = (w,z)) & ((w ε y) & (z = (u,w)))) AndElimR 250
253. ∃w.∃z.(((a,c) = (w,z)) & ((w ε y) & (z = (u,w)))) AndElimR 251
254. ∃z.(((a,b) = (x1,z)) & ((x1 ε y) & (z = (u,x1)))) Hyp
255. ((a,b) = (x1,y1)) & ((x1 ε y) & (y1 = (u,x1))) Hyp
256. ∃z.(((a,c) = (x2,z)) & ((x2 ε y) & (z = (u,x2)))) Hyp
257. ((a,c) = (x2,y2)) & ((x2 ε y) & (y2 = (u,x2))) Hyp
258. (a,b) = (x1,y1) AndElimL 255
259. (a,c) = (x2,y2) AndElimL 257
260. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
261. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 260

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262. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 261
263. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 262
264. Set((a,b))  AndElimL 250
265. Set((a,c))  AndElimL 251
266. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 263
267. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 266
268. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 267
269. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 268
270. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 267
271. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 270
272. Set(a) & Set(b)  ImpElim 264 269
273. Set(a) & Set(c)  ImpElim 265 271
274. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
275. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 274
276. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))  ForallElim 275
277. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 276
278. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))  ForallElim 277
279. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)))  ForallInt 278
280. ((Set(a) & Set(b)) & ((a,b) = (x1,v))) -> ((a = x1) & (b = v))  ForallElim 279
281. ∀v.(((Set(a) & Set(b)) & ((a,b) = (x1,v))) -> ((a = x1) & (b = v)))  ForallInt 280
282. ((Set(a) & Set(b)) & ((a,b) = (x1,y1))) -> ((a = x1) & (b = y1))  ForallElim 281
283. (Set(a) & Set(b)) & ((a,b) = (x1,y1))  AndInt 272 258
284. (a = x1) & (b = y1)  ImpElim 283 282
285. (Set(a) & Set(c)) & ((a,c) = (x2,y2))  AndInt 273 259
286. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 276
287. ((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v))  ForallElim 286
288. ∀u.(((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v)))  ForallInt 287
289. ((Set(a) & Set(c)) & ((a,c) = (x2,v))) -> ((a = x2) & (c = v))  ForallElim 288
290. ∀v.(((Set(a) & Set(c)) & ((a,c) = (x2,v))) -> ((a = x2) & (c = v)))  ForallInt 289
291. ((Set(a) & Set(c)) & ((a,c) = (x2,y2))) -> ((a = x2) & (c = y2))  ForallElim 290
292. (a = x2) & (c = y2)  ImpElim 285 291
293. (x1 ε y) & (y1 = (u,x1))  AndElimR 255
294. (x2 ε y) & (y2 = (u,x2))  AndElimR 257
295. a = x1  AndElimL 284
296. a = x2  AndElimL 292
297. x1 = x2  EqualitySub 296 295
298. y1 = (u,x1)  AndElimR 293
299. y2 = (u,x2)  AndElimR 294
300. x2 = x1  Symmetry 297
301. y2 = (u,x1)  EqualitySub 299 300
302. (u,x1) = y2  Symmetry 301
303. y1 = y2  EqualitySub 298 302
304. (a,b) = (x2,y1)  EqualitySub 258 297
305. (a,b) = (x2,y2)  EqualitySub 304 303
306. (x2,y2) = (a,c)  Symmetry 259
307. (a,b) = (a,c)  EqualitySub 305 306
308. (Set(a) & Set(b)) & ((a,b) = (a,c))  AndInt 272 307
309. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)))  ForallInt 278
310. ((Set(a) & Set(b)) & ((a,b) = (a,v))) -> ((a = a) & (b = v))  ForallElim 309
311. ∀v.(((Set(a) & Set(b)) & ((a,b) = (a,v))) -> ((a = a) & (b = v)))  ForallInt 310
312. ((Set(a) & Set(b)) & ((a,b) = (a,c))) -> ((a = a) & (b = c))  ForallElim 311
313. (a = a) & (b = c)  ImpElim 308 312
314. b = c  AndElimR 313
315. b = c  ExistsElim 256 257 314
316. b = c  ExistsElim 253 256 315
317. b = c  ExistsElim 254 255 316
318. b = c  ExistsElim 252 254 317
319. ((a,c) ε f) -> (b = c)  ImpInt 318
320. ((a,b) ε f) -> (((a,c) ε f) -> (b = c))  ImpInt 319
321. A -> (B -> C)  Hyp
322. A & B  Hyp
323. A  AndElimL 322
324. B -> C  ImpElim 323 321
325. B  AndElimR 322
326. C  ImpElim 325 324
327. (A & B) -> C  ImpInt 326
328. (A -> (B -> C)) -> ((A & B) -> C)  ImpInt 327
329. (((a,b) ε f) -> (B -> C)) -> ((((a,b) ε f) & B) -> C)  PolySub 328
330. (((a,b) ε f) -> (((a,c) ε f) -> C)) -> ((((a,b) ε f) & ((a,c) ε f)) -> C)  PolySub
329
331. (((a,b) ε f) -> (((a,c) ε f) -> (b = c))) -> ((((a,b) ε f) & ((a,c) ε f)) -> (b =
```

c)) PolySub 330
332. (((a,b) ε f) & ((a,c) ε f)) -> (b = c) ImpElim 320 331
333. ∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 332
334. ∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 333
335. ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 334
336. Relation(f) & ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) AndInt 245 335
337. Function(f) DefSub 336
338. Function(f) & Set(domain(f)) AndInt 337 231
339. (Function(f) & Set(domain(f))) -> Set(range(f)) AxInt
340. Set(range(f)) ImpElim 338 339
341. Set(({u} X y)) EqualitySub 340 227
342. (f = {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}) -> Set(({u} X y)) ImpInt 341
343. ∀f.((f = {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))}) -> Set(({u} X y))) ForallInt 342
344. ({a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} = {x_8: ∃x_9.∃x_10.((x_8 = (x_9,x_10)) & ((x_9 ε y) & (x_10 = (u,x_9))))}) -> Set(({u} X y)) ForallElim 343
345. {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} = {a: ∃w.∃z.((a = (w,z)) & ((w ε y) & (z = (u,w))))} Identity
346. Set(({u} X y)) ImpElim 345 344
347. (Set(u) & Set(y)) -> Set(({u} X y)) ImpInt 346 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
3. Set(x) -> ((y ε {x}) <-> (y = x))

Th74. (Set(x) & Set(y)) -> Set((x X y))

0. f = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} Hyp
1. c ε f Hyp
2. c ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} EqualitySub 1 0
3. Set(c) & ∃u.∃z.((c = (u,z)) & ((u ε x) & (z = ({u} X y)))) ClassElim 2
4. ∃u.∃z.((c = (u,z)) & ((u ε x) & (z = ({u} X y)))) AndElimR 3
5. ∃z.((c = (u,z)) & ((u ε x) & (z = ({u} X y)))) Hyp
6. (c = (u,z)) & ((u ε x) & (z = ({u} X y))) Hyp
7. c = (u,z) AndElimL 6
8. ∃z.(c = (u,z)) ExistsInt 7
9. ∃u.∃z.(c = (u,z)) ExistsInt 8
10. ∃u.∃z.(c = (u,z)) ExistsElim 5 6 9
11. ∃u.∃z.(c = (u,z)) ExistsElim 4 5 10
12. (c ε f) -> ∃u.∃z.(c = (u,z)) ImpInt 11
13. ∀c.((c ε f) -> ∃u.∃z.(c = (u,z))) ForallInt 12
14. Relation(f) DefSub 13
15. ((a,b) ε f) & ((a,c) ε f) Hyp
16. (a,b) ε f AndElimL 15
17. (a,c) ε f AndElimR 15
18. (a,b) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} EqualitySub 16 0
19. (a,c) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} EqualitySub 17 0
20. Set((a,b)) & ∃u.∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y)))) ClassElim 18
21. Set((a,c)) & ∃u.∃z.(((a,c) = (u,z)) & ((u ε x) & (z = ({u} X y)))) ClassElim 19
22. ∃u.∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y)))) AndElimR 20
23. ∃u.∃z.(((a,c) = (u,z)) & ((u ε x) & (z = ({u} X y)))) AndElimR 21
24. ∃z.(((a,b) = (x1,z)) & ((x1 ε x) & (z = ({x1} X y)))) Hyp
25. ((a,b) = (x1,y1)) & ((x1 ε x) & (y1 = ({x1} X y))) Hyp
26. ∃z.(((a,c) = (x2,z)) & ((x2 ε x) & (z = ({x2} X y)))) Hyp
27. ((a,c) = (x2,y2)) & ((x2 ε x) & (y2 = ({x2} X y))) Hyp
28. Set((a,b)) AndElimL 20
29. Set((a,c)) AndElimL 21
30. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
31. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 30
32. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 31
33. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 32
34. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 33
35. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 34
36. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 35
37. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 36
38. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 35
39. Set((a,c)) -> (Set(a) & Set(c)) ForallElim 38
40. Set(a) & Set(b) ImpElim 28 37

41. Set(a) & Set(c) ImpElim 29 39
42. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
43. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 42
44. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)) ForallElim 43
45. ∀x.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 44
46. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 44
47. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)) ForallElim 46
48. (a,b) = (x1,y1) AndElimL 25
49. (a,c) = (x2,y2) AndElimL 27
50. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))) ForallInt 47
51. ((Set(a) & Set(b)) & ((a,b) = (x1,v))) -> ((a = x1) & (b = v)) ForallElim 50
52. ∀v.(((Set(a) & Set(b)) & ((a,b) = (x1,v))) -> ((a = x1) & (b = v))) ForallInt 51
53. ((Set(a) & Set(b)) & ((a,b) = (x1,y1))) -> ((a = x1) & (b = y1)) ForallElim 52
54. (Set(a) & Set(b)) & ((a,b) = (x1,y1)) AndInt 40 48
55. (a = x1) & (b = y1) ImpElim 54 53
56. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 44
57. ((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v)) ForallElim 56
58. ∀u.(((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v))) ForallInt 57
59. ((Set(a) & Set(c)) & ((a,c) = (x2,v))) -> ((a = x2) & (c = v)) ForallElim 58
60. ∀v.(((Set(a) & Set(c)) & ((a,c) = (x2,v))) -> ((a = x2) & (c = v))) ForallInt 59
61. ((Set(a) & Set(c)) & ((a,c) = (x2,y2))) -> ((a = x2) & (c = y2)) ForallElim 60
62. (Set(a) & Set(c)) & ((a,c) = (x2,y2)) AndInt 41 49
63. (a = x2) & (c = y2) ImpElim 62 61
64. a = x1 AndElimL 55
65. a = x2 AndElimL 63
66. x2 = x1 EqualitySub 64 65
67. (x1 ε x) & (y1 = ({x1} X y)) AndElimR 25
68. (x2 ε x) & (y2 = ({x2} X y)) AndElimR 27
69. y1 = ({x1} X y) AndElimR 67
70. y2 = ({x2} X y) AndElimR 68
71. y2 = ({x1} X y) EqualitySub 70 66
72. ({x1} X y) = y2 Symmetry 71
73. y1 = y2 EqualitySub 69 72
74. b = y1 AndElimR 55
75. c = y2 AndElimR 63
76. b = y2 EqualitySub 74 73
77. y2 = b Symmetry 76
78. c = b EqualitySub 75 77
79. c = b ExistsElim 26 27 78
80. c = b ExistsElim 23 26 79
81. c = b ExistsElim 24 25 80
82. c = b ExistsElim 22 24 81
83. b = c Symmetry 82
84. (((a,b) ε f) & ((a,c) ε f)) -> (b = c) ImpInt 83
85. ∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 84
86. ∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 85
87. ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 86
88. Relation(f) & ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) AndInt 14 87
89. Function(f) DefSub 88
90. a ε x Hyp
91. b = ({a} X y) Hyp
92. (a ε x) & (b = ({a} X y)) AndInt 90 91
93. c = (a,b) Hyp
94. (c = (a,b)) & ((a ε x) & (b = ({a} X y))) AndInt 93 92
95. ∃b.((c = (a,b)) & ((a ε x) & (b = ({a} X y)))) ExistsInt 94
96. ∃a.∃b.((c = (a,b)) & ((a ε x) & (b = ({a} X y)))) ExistsInt 95
97. Set(x) & Set(y) Hyp
98. ∃w.(a ε w) ExistsInt 90
99. Set(a) DefSub 98
100. Set(x) -> Set({x}) TheoremInt
101. ∀x.(Set(x) -> Set({x})) ForallInt 100
102. Set(a) -> Set({a}) ForallElim 101
103. Set({a}) ImpElim 99 102
104. Set(y) AndElimR 97
105. (Set(u) & Set(y)) -> Set(({u} X y)) TheoremInt
106. ∀u.((Set(u) & Set(y)) -> Set(({u} X y))) ForallInt 105
107. (Set(a) & Set(y)) -> Set(({a} X y)) ForallElim 106
108. Set(a) & Set(y) AndInt 99 104
109. Set(({a} X y)) ImpElim 108 107
110. ({a} X y) = b Symmetry 91
111. Set(b) EqualitySub 109 110

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112. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
113. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 112
114. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 113
115. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 114
116. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 115
117. (Set(a) & Set(y)) -> Set((a,y))  ForallElim 116
118. ∀y.((Set(a) & Set(y)) -> Set((a,y)))  ForallInt 117
119. (Set(a) & Set(b)) -> Set((a,b))  ForallElim 118
120. Set(a) & Set(b)  AndInt 99 111
121. Set((a,b))  ImpElim 120 119
122. (a,b) = c  Symmetry 93
123. Set(c)  EqualitySub 121 122
124. Set(c) & ∃a.∃b.((c = (a,b)) & ((a ε x) & (b = ({a} X y))))  AndInt 123 96
125. c ε {w: ∃a.∃b.((w = (a,b)) & ((a ε x) & (b = ({a} X y))))}  ClassInt 124
126. (a,b) ε {w: ∃x_6.∃x_8.((w = (x_6,x_8)) & ((x_6 ε x) & (x_8 = ({x_6} X y))))}
EqualitySub 125 93
127. {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = f  Symmetry 0
128. (a,b) ε f  EqualitySub 126 127
129. ∃b.((a,b) ε f)  ExistsInt 128
130. Set(a) & ∃b.((a,b) ε f)  AndInt 99 129
131. a ε {w: ∃b.((w,b) ε f)}  ClassInt 130
132. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
133. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 132
134. a ε domain(f)  EqualitySub 131 133
135. (c = (a,b)) -> (a ε domain(f))  ImpInt 134
136. ∀c.((c = (a,b)) -> (a ε domain(f)))  ForallInt 135
137. ((a,b) = (a,b)) -> (a ε domain(f))  ForallElim 136
138. (a,b) = (a,b)  Identity
139. a ε domain(f)  ImpElim 138 137
140. (b = ({a} X y)) -> (a ε domain(f))  ImpInt 139
141. ∀b.((b = ({a} X y)) -> (a ε domain(f)))  ForallInt 140
142. (({a} X y) = ({a} X y)) -> (a ε domain(f))  ForallElim 141
143. ({a} X y) = ({a} X y)  Identity
144. a ε domain(f)  ImpElim 143 142
145. (a ε x) -> (a ε domain(f))  ImpInt 144
146. a ε domain(f)  Hyp
147. a ε {x: ∃y.((x,y) ε f)}  EqualitySub 146 132
148. Set(a) & ∃y.((a,y) ε f)  ClassElim 147
149. ∃y.((a,y) ε f)  AndElimR 148
150. (a,b) ε f  Hyp
151. (a,b) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}  EqualitySub 150 0
152. Set((a,b)) & ∃u.∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y))))  ClassElim 151
153. Set((a,b))  AndElimL 152
154. ∃u.∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y))))  AndElimR 152
155. ∃z.(((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y))))  Hyp
156. ((a,b) = (u,z)) & ((u ε x) & (z = ({u} X y)))  Hyp
157. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
158. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 157
159. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 158
160. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 159
161. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 160
162. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 161
163. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 162
164. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 163
165. Set(a) & Set(b)  ImpElim 153 164
166. (a,b) = (u,z)  AndElimL 156
167. (Set(a) & Set(b)) & ((a,b) = (u,z))  AndInt 165 166
168. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
169. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 168
170. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))  ForallElim 169
171. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 170
172. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))  ForallElim 171
173. ∀v.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)))  ForallInt 172
174. ((Set(a) & Set(b)) & ((a,b) = (u,z))) -> ((a = u) & (b = z))  ForallElim 173
175. (a = u) & (b = z)  ImpElim 167 174
176. a = u  AndElimL 175
177. (u ε x) & (z = ({u} X y))  AndElimR 156
178. u ε x  AndElimL 177
179. u = a  Symmetry 176
180. a ε x  EqualitySub 178 179
181. a ε x  ExistsElim 155 156 180
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182. a ε x ExistsElim 154 155 181
183. a ε x ExistsElim 149 150 182
184. (a ε domain(f)) -> (a ε x) ImpInt 183
185. ((a ε x) -> (a ε domain(f))) & ((a ε domain(f)) -> (a ε x)) AndInt 145 184
186. (a ε x) <-> (a ε domain(f)) EquivConst 185
187. ∀a.((a ε x) <-> (a ε domain(f))) ForallInt 186
188. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
189. ∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) ForallElim 188
190. (x = domain(f)) <-> ∀z.((z ε x) <-> (z ε domain(f))) ForallElim 189
191. ((x = domain(f)) -> ∀z.((z ε x) <-> (z ε domain(f)))) & (∀z.((z ε x) <-> (z ε domain(f))) -> (x = domain(f))) EquivExp 190
192. ∀z.((z ε x) <-> (z ε domain(f))) -> (x = domain(f)) AndElimR 191
193. x = domain(f) ImpElim 187 192
194. Function(f) & (x = domain(f)) AndInt 89 193
195. (f = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) -> (Function(f) & (x = domain(f))) ImpInt 194
196. ({a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) -> (Function(f) & (x = domain(f))) EqualitySub 195 0
197. {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} Identity
198. Function(f) & (x = domain(f)) ImpElim 197 196
199. x = domain(f) AndElimR 198
200. Set(x) AndElimL 97
201. Set(domain(f)) EqualitySub 200 199
202. Function(f) AndElimL 198
203. Function(f) & Set(domain(f)) AndInt 202 201
204. (Function(f) & Set(domain(f))) -> Set(range(f)) AxInt
205. Set(range(f)) ImpElim 203 204
206. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
207. range(f) = {x_10: ∃x_11.((x_11,x_10) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))})} EqualitySub 206 0
208. e ε range(f) Hyp
209. e ε {x_10: ∃x_11.((x_11,x_10) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))})} EqualitySub 208 207
210. Set(e) & ∃x_11.((x_11,e) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) ClassElim 209
211. ∃x_11.((x_11,e) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) AndElimR 210
212. (c,e) ε {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} Hyp
213. Set((c,e)) & ∃u.∃z.(((c,e) = (u,z)) & ((u ε x) & (z = ({u} X y)))) ClassElim 212
214. ∃u.∃z.(((c,e) = (u,z)) & ((u ε x) & (z = ({u} X y)))) AndElimR 213
215. ∃z.(((c,e) = (u,z)) & ((u ε x) & (z = ({u} X y)))) Hyp
216. ((c,e) = (u,z)) & ((u ε x) & (z = ({u} X y))) Hyp
217. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
218. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 217
219. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 218
220. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 219
221. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 220
222. Set((c,y)) -> (Set(c) & Set(y)) ForallElim 221
223. ∀y.(Set((c,y)) -> (Set(c) & Set(y))) ForallInt 222
224. Set((c,e)) -> (Set(c) & Set(e)) ForallElim 223
225. Set((c,e)) AndElimL 213
226. Set(c) & Set(e) ImpElim 225 224
227. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
228. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 227
229. ((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v)) ForallElim 228
230. ∀y.(((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v))) ForallInt 229
231. ((Set(c) & Set(e)) & ((c,e) = (u,v))) -> ((c = u) & (e = v)) ForallElim 230
232. (c,e) = (u,z) AndElimL 216
233. (Set(c) & Set(e)) & ((c,e) = (u,z)) AndInt 226 232
234. ∀v.(((Set(c) & Set(e)) & ((c,e) = (u,v))) -> ((c = u) & (e = v))) ForallInt 231
235. ((Set(c) & Set(e)) & ((c,e) = (u,z))) -> ((c = u) & (e = z)) ForallElim 234
236. (c = u) & (e = z) ImpElim 233 235
237. (u ε x) & (z = ({u} X y)) AndElimR 216
238. z = ({u} X y) AndElimR 237
239. e = z AndElimR 236
240. z = e Symmetry 239
241. e = ({u} X y) EqualitySub 238 240
242. u ε x AndElimL 237
243. (u ε x) & (e = ({u} X y)) AndInt 242 241
244. ∃u.((u ε x) & (e = ({u} X y))) ExistsInt 243

245. Set(e) AndElimR 226
246. Set(e) & ∃u.((u ε x) & (e = ({u} X y))) AndInt 245 244
247. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} ClassInt 246
248. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} ExistsElim 215 216 247
249. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} ExistsElim 214 215 248
250. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} ExistsElim 211 212 249
251. (e ε range(f)) -> (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) ImpInt 250
252. e ε {w: ∃u.((u ε x) & (w = ({u} X y)))} Hyp
253. Set(e) & ∃u.((u ε x) & (e = ({u} X y))) ClassElim 252
254. Set(e) AndElimL 253
255. ∃u.((u ε x) & (e = ({u} X y))) AndElimR 253
256. (u ε x) & (e = ({u} X y)) Hyp
257. (u,e) = (u,e) Identity
258. ((u,e) = (u,e)) & ((u ε x) & (e = ({u} X y))) AndInt 257 256
259. ∃b.(((u,e) = (u,b)) & ((u ε x) & (b = ({u} X y)))) ExistsInt 258
260. ∃v.∃b.(((u,e) = (v,b)) & ((v ε x) & (b = ({v} X y)))) ExistsInt 259
261. u ε x AndElimL 256
262. ∃w.(u ε w) ExistsInt 261
263. Set(u) DefSub 262
264. Set(u) & Set(e) AndInt 263 254
265. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 219
266. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 265
267. (Set(u) & Set(y)) -> Set((u,y)) ForallElim 266
268. ∀y.((Set(u) & Set(y)) -> Set((u,y))) ForallInt 267
269. (Set(u) & Set(e)) -> Set((u,e)) ForallElim 268
270. Set((u,e)) ImpElim 264 269
271. Set((u,e)) & ∃v.∃b.(((u,e) = (v,b)) & ((v ε x) & (b = ({v} X y)))) AndInt 270 260
272. c = (u,e) Hyp
273. (u,e) = c Symmetry 272
274. Set(c) & ∃v.∃b.((c = (v,b)) & ((v ε x) & (b = ({v} X y)))) EqualitySub 271 273
275. c ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))} ClassInt 274
276. (u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))} EqualitySub 275 272
277. (c = (u,e)) -> ((u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))})
ImpInt 276
278. ∀c.((c = (u,e)) -> ((u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))}))
ForallInt 277
279. ((u,e) = (u,e)) -> ((u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))})
ForallElim 278
280. (u,e) = (u,e) Identity
281. (u,e) ε {w: ∃v.∃b.((w = (v,b)) & ((v ε x) & (b = ({v} X y))))} ImpElim 280 279
282. {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = f Symmetry 0
283. (u,e) ε f EqualitySub 281 282
284. ∃u.((u,e) ε f) ExistsInt 283
285. ∃u.((u,e) ε f) ExistsElim 255 256 284
286. Set(e) & ∃u.((u,e) ε f) AndInt 254 285
287. e ε {w: ∃u.((u,w) ε f)} ClassInt 286
288. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
289. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 288
290. e ε range(f) EqualitySub 287 289
291. (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) -> (e ε range(f)) ImpInt 290
292. ((e ε range(f)) -> (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))})) & ((e ε {w: ∃u.((u ε
x) & (w = ({u} X y)))}) -> (e ε range(f))) AndInt 251 291
293. (e ε range(f)) <-> (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) EquivConst 292
294. ∀e.((e ε range(f)) <-> (e ε {w: ∃u.((u ε x) & (w = ({u} X y)))})) ForallInt 293
295. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
296. ∀y.((range(f) = y) <-> ∀z.((z ε range(f)) <-> (z ε y))) ForallElim 295
297. (range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))}) <-> ∀z.((z ε range(f)) <-> (z ε {w:
∃u.((u ε x) & (w = ({u} X y)))})) ForallElim 296
298. ((range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))}) -> ∀z.((z ε range(f)) <-> (z ε {w:
∃u.((u ε x) & (w = ({u} X y)))}))) & (∀z.((z ε range(f)) <-> (z ε {w: ∃u.((u ε x) & (w =
({u} X y)))})) -> (range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))})) EquivExp 297
299. ∀z.((z ε range(f)) <-> (z ε {w: ∃u.((u ε x) & (w = ({u} X y)))})) -> (range(f) = {w:
∃u.((u ε x) & (w = ({u} X y)))}) AndElimR 298
300. range(f) = {w: ∃u.((u ε x) & (w = ({u} X y)))} ImpElim 294 299
301. e ε ∪range(f) Hyp
302. e ε ∪{w: ∃u.((u ε x) & (w = ({u} X y)))} EqualitySub 301 300
303. ∪x = {z: ∃y.((y ε x) & (z ε y))} DefEqInt
304. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))}) ForallInt 303
305. ∪range(f) = {z: ∃y.((y ε range(f)) & (z ε y))} ForallElim 304
306. ∪range(f) = {z: ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (z ε x_13))}
EqualitySub 305 300

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307. e ε {z: ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (z ε x_13))}
EqualitySub 301 306
308. Set(e) & ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (e ε x_13))
ClassElim 307
309. ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (e ε x_13))  AndElimR 308
310. (x_5 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (e ε x_5)  Hyp
311. e ε x_5  AndElimR 310
312. x_5 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}  AndElimL 310
313. Set(x_5) & ∃u.((u ε x) & (x_5 = ({u} X y)))  ClassElim 312
314. Set(x_5)  AndElimL 313
315. ∃u.((u ε x) & (x_5 = ({u} X y)))  AndElimR 313
316. (u ε x) & (x_5 = ({u} X y))  Hyp
317. x_5 = ({u} X y)  AndElimR 316
318. e ε ({u} X y)  EqualitySub 311 317
319. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}  DefEqInt
320. ∀x.((x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))})  ForallInt 319
321. ({u} X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))}  ForallElim 320
322. e ε {z: ∃a.∃b.((z = (a,b)) & ((a ε {u}) & (b ε y)))}  EqualitySub 318 321
323. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ε {u}) & (b ε y)))  ClassElim 322
324. ∃a.∃b.((e = (a,b)) & ((a ε {u}) & (b ε y)))  AndElimR 323
325. ∃b.((e = (a,b)) & ((a ε {u}) & (b ε y)))  Hyp
326. (e = (a,b)) & ((a ε {u}) & (b ε y))  Hyp
327. (a ε {u}) & (b ε y)  AndElimR 326
328. a ε {u}  AndElimL 327
329. Set(x) -> ((y ε {x}) <-> (y = x))  TheoremInt
330. u ε x  AndElimL 316
331. ∃w.(u ε w)  ExistsInt 330
332. Set(u)  DefSub 331
333. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 329
334. Set(u) -> ((y ε {u}) <-> (y = u))  ForallElim 333
335. ∀y.(Set(u) -> ((y ε {u}) <-> (y = u)))  ForallInt 334
336. Set(u) -> ((a ε {u}) <-> (a = u))  ForallElim 335
337. (a ε {u}) <-> (a = u)  ImpElim 332 336
338. ((a ε {u}) -> (a = u)) & ((a = u) -> (a ε {u}))  EquivExp 337
339. (a ε {u}) -> (a = u)  AndElimL 338
340. a = u  ImpElim 328 339
341. u = a  Symmetry 340
342. a ε x  EqualitySub 330 341
343. b ε y  AndElimR 327
344. (a ε x) & (b ε y)  AndInt 342 343
345. e = (a,b)  AndElimL 326
346. (e = (a,b)) & ((a ε x) & (b ε y))  AndInt 345 344
347. ∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  ExistsInt 346
348. ∃a.∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  ExistsInt 347
349. Set(e)  AndElimL 323
350. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  AndInt 349 348
351. e ε {w: ∃a.∃b.((w = (a,b)) & ((a ε x) & (b ε y)))}  ClassInt 350
352. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}  DefEqInt
353. {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} = (x X y)  Symmetry 352
354. e ε (x X y)  EqualitySub 351 353
355. e ε (x X y)  ExistsElim 325 326 354
356. e ε (x X y)  ExistsElim 324 325 355
357. e ε (x X y)  ExistsElim 315 316 356
358. e ε (x X y)  ExistsElim 309 310 357
359. (e ε Urange(f)) -> (e ε (x X y))  ImpInt 358
360. e ε (x X y)  Hyp
361. e ε {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}  EqualitySub 360 352
362. Set(e) & ∃a.∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  ClassElim 361
363. Set(e)  AndElimL 362
364. ∃a.∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  AndElimR 362
365. ∃b.((e = (a,b)) & ((a ε x) & (b ε y)))  Hyp
366. (e = (a,b)) & ((a ε x) & (b ε y))  Hyp
367. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 218
368. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 367
369. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 368
370. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 369
371. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 370
372. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 371
373. e = (a,b)  AndElimL 366
374. Set((a,b))  EqualitySub 363 373
375. Set(a) & Set(b)  ImpElim 374 372
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376. Set(a)  AndElimL 375
377. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 329
378. Set(a) -> ((y ε {a}) <-> (y = a))  ForallElim 377
379. ∀y.(Set(a) -> ((y ε {a}) <-> (y = a)))  ForallInt 378
380. Set(a) -> ((a ε {a}) <-> (a = a))  ForallElim 379
381. (a ε {a}) <-> (a = a)  ImpElim 376 380
382. ((a ε {a}) -> (a = a)) & ((a = a) -> (a ε {a}))  EquivExp 381
383. (a = a) -> (a ε {a})  AndElimR 382
384. a = a  Identity
385. a ε {a}  ImpElim 384 383
386. e = (a,b)  AndElimL 366
387. (a ε x) & (b ε y)  AndElimR 366
388. a ε x  AndElimL 387
389. b ε y  AndElimR 387
390. (a ε {a}) & (b ε y)  AndInt 385 389
391. (e = (a,b)) & ((a ε {a}) & (b ε y))  AndInt 386 390
392. ∃u.((e = (a,u)) & ((a ε {a}) & (u ε y)))  ExistsInt 391
393. ∃v.∃u.((e = (v,u)) & ((v ε {a}) & (u ε y)))  ExistsInt 392
394. Set(e) & ∃v.∃u.((e = (v,u)) & ((v ε {a}) & (u ε y)))  AndInt 363 393
395. e ε {w: ∃v.∃u.((w = (v,u)) & ((v ε {a}) & (u ε y)))}  ClassInt 394
396. ∀x.((x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))})  ForallInt 319
397. ({a} X y) = {z: ∃x_15.∃b.((z = (x_15,b)) & ((x_15 ε {a}) & (b ε y)))}  ForallElim 396
398. {z: ∃x_15.∃b.((z = (x_15,b)) & ((x_15 ε {a}) & (b ε y)))} = ({a} X y)  Symmetry 397
399. e ε ({a} X y)  EqualitySub 395 398
400. g = ({a} X y)  Hyp
401. ({a} X y) = g  Symmetry 400
402. (a ε x) & (g = ({a} X y))  AndInt 388 400
403. ∃a.((a ε x) & (g = ({a} X y)))  ExistsInt 402
404. (Set(u) & Set(y)) -> Set(({u} X y))  TheoremInt
405. ∀u.((Set(u) & Set(y)) -> Set(({u} X y)))  ForallInt 404
406. (Set(a) & Set(y)) -> Set(({a} X y))  ForallElim 405
407. Set(y)  AndElimR 97
408. Set(a) & Set(y)  AndInt 376 407
409. Set(({a} X y))  ImpElim 408 406
410. Set(g)  EqualitySub 409 401
411. Set(g) & ∃a.((a ε x) & (g = ({a} X y)))  AndInt 410 403
412. g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}  ClassInt 411
413. e ε g  EqualitySub 399 401
414. (g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}) & (e ε g)  AndInt 412 413
415. ∃g.((g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}) & (e ε g))  ExistsInt 414
416. Set(e) & ∃g.((g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}) & (e ε g))  AndInt 363 415
417. e ε {d: ∃g.((g ε {w: ∃a.((a ε x) & (w = ({a} X y)))}) & (d ε g))}  ClassInt 416
418. {z: ∃x_13.((x_13 ε {w: ∃u.((u ε x) & (w = ({u} X y)))}) & (z ε x_13))} = Urange(f)  Symmetry 306
419. e ε Urange(f)  EqualitySub 417 418
420. (g = ({a} X y)) -> (e ε Urange(f))  ImpInt 419
421. ∀g.((g = ({a} X y)) -> (e ε Urange(f)))  ForallInt 420
422. (({a} X y) = ({a} X y)) -> (e ε Urange(f))  ForallElim 421
423. ({a} X y) = ({a} X y)  Identity
424. e ε Urange(f)  ImpElim 423 422
425. e ε Urange(f)  ExistsElim 365 366 424
426. e ε Urange(f)  ExistsElim 364 365 425
427. (e ε (x X y)) -> (e ε Urange(f))  ImpInt 426
428. ((e ε Urange(f)) -> (e ε (x X y))) & ((e ε (x X y)) -> (e ε Urange(f)))  AndInt 359 427
429. (e ε Urange(f)) <-> (e ε (x X y))  EquivConst 428
430. ∀e.((e ε Urange(f)) <-> (e ε (x X y)))  ForallInt 429
431. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
432. ∀y.((Urange(f) = y) <-> ∀z.((z ε Urange(f)) <-> (z ε y)))  ForallElim 431
433. (Urange(f) = (x X y)) <-> ∀z.((z ε Urange(f)) <-> (z ε (x X y)))  ForallElim 432
434. ((Urange(f) = (x X y)) -> ∀z.((z ε Urange(f)) <-> (z ε (x X y)))) & (∀z.((z ε Urange(f)) <-> (z ε (x X y))) -> (Urange(f) = (x X y)))  EquivExp 433
435. ∀z.((z ε Urange(f)) <-> (z ε (x X y))) -> (Urange(f) = (x X y))  AndElimR 434
436. Urange(f) = (x X y)  ImpElim 430 435
437. Set(x) -> Set(Ux)  AxInt
438. ∀x.(Set(x) -> Set(Ux))  ForallInt 437
439. Set(range(f)) -> Set(Urange(f))  ForallElim 438
440. Set(Urange(f))  ImpElim 205 439
441. Set((x X y))  EqualitySub 440 436
442. (Set(x) & Set(y)) -> Set((x X y))  ImpInt 441
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443. (f = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) -> ((Set(x) & Set(y)) -> Set((x X y))) ImpInt 442
444. ∀f.((f = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))}) -> ((Set(x) & Set(y)) -> Set((x X y)))) ForallInt 443
445. ({a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = {x_16: ∃x_17.∃x_18.((x_16 = (x_17,x_18)) & ((x_17 ε x) & (x_18 = ({x_17} X y))))}) -> ((Set(x) & Set(y)) -> Set((x X y))) ForallElim 444
446. {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} = {a: ∃u.∃z.((a = (u,z)) & ((u ε x) & (z = ({u} X y))))} Identity
447. (Set(x) & Set(y)) -> Set((x X y)) ImpElim 446 445 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
3. Set(x) -> Set({x})
4. (Set(u) & Set(y)) -> Set(({u} X y))
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
6. Set(x) -> ((y ε {x}) <-> (y = x))
7. (Set(u) & Set(y)) -> Set(({u} X y))

Th75. (Function(f) & Set(domain(f))) -> (f ⊂ (domain(f) X range(f)))

0. Function(f) & Set(domain(f)) Hyp
1. z ε f Hyp
2. Function(f) AndElimL 0
3. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 2
4. Relation(f) AndElimL 3
5. ∀z.((z ε f) -> ∃x.∃y.(z = (x,y))) DefExp 4
6. (z ε f) -> ∃x.∃y.(z = (x,y)) ForallElim 5
7. ∃x.∃y.(z = (x,y)) ImpElim 1 6
8. ∃y.(z = (x,y)) Hyp
9. z = (x,y) Hyp
10. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
11. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
12. ∃y.(z = (x,y)) ExistsInt 9
13. ∃f.(z ε f) ExistsInt 1
14. Set(z) DefSub 13
15. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
16. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 15
17. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 16
18. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 17
19. Set((x,y)) EqualitySub 14 9
20. Set(x) & Set(y) ImpElim 19 18
21. Set(x) AndElimL 20
22. (x,y) ε f EqualitySub 1 9
23. ∃y.((x,y) ε f) ExistsInt 22
24. Set(x) & ∃y.((x,y) ε f) AndInt 21 23
25. x ε {w: ∃y.((w,y) ε f)} ClassInt 24
26. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 10
27. x ε domain(f) EqualitySub 25 26
28. ∃x.((x,y) ε f) ExistsInt 22
29. Set(y) AndElimR 20
30. Set(y) & ∃x.((x,y) ε f) AndInt 29 28
31. y ε {w: ∃x.((x,w) ε f)} ClassInt 30
32. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 11
33. y ε range(f) EqualitySub 31 32
34. (x ε domain(f)) & (y ε range(f)) AndInt 27 33
35. (z = (x,y)) & ((x ε domain(f)) & (y ε range(f))) AndInt 9 34
36. ∃y.((z = (x,y)) & ((x ε domain(f)) & (y ε range(f)))) ExistsInt 35
37. ∃x.∃y.((z = (x,y)) & ((x ε domain(f)) & (y ε range(f)))) ExistsInt 36
38. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} DefEqInt
39. ∀x.((x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))}) ForallInt 38
40. (domain(f) X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε domain(f)) & (b ε y)))} ForallElim 39
41. ∀y.((domain(f) X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε domain(f)) & (b ε y)))}) ForallInt 40
42. (domain(f) X range(f)) = {z: ∃a.∃b.((z = (a,b)) & ((a ε domain(f)) & (b ε range(f))))} ForallElim 41
43. Set(z) & ∃x.∃y.((z = (x,y)) & ((x ε domain(f)) & (y ε range(f)))) AndInt 14 37
44. z ε {w: ∃x.∃y.((w = (x,y)) & ((x ε domain(f)) & (y ε range(f))))} ClassInt 43

45. {z: ∃a.∃b.((z = (a,b)) & ((a ε domain(f)) & (b ε range(f))))} = (domain(f) X range(f)) Symmetry 42
46. z ε (domain(f) X range(f)) EqualitySub 44 45
47. z ε (domain(f) X range(f)) ExistsElim 8 9 46
48. z ε (domain(f) X range(f)) ExistsElim 7 8 47
49. (z ε f) -> (z ε (domain(f) X range(f))) ImpInt 48
50. ∀z.((z ε f) -> (z ε (domain(f) X range(f)))) ForallInt 49
51. f ⊂ (domain(f) X range(f)) DefSub 50
52. (Function(f) & Set(domain(f))) -> (f ⊂ (domain(f) X range(f))) ImpInt 51 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th77. (Set(x) & Set(y)) -> Set(func(x,y))

0. Set(x) & Set(y) Hyp
1. f ε func(x,y) Hyp
2. func(x,y) = {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))} DefEqInt
3. f ε {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))} EqualitySub 1 2
4. Set(f) & (Function(f) & ((domain(f) = x) & (range(f) = y))) ClassElim 3
5. Set(f) AndElimL 4
6. Function(f) & ((domain(f) = x) & (range(f) = y)) AndElimR 4
7. Function(f) AndElimL 6
8. (domain(f) = x) & (range(f) = y) AndElimR 6
9. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 7
10. Relation(f) AndElimL 9
11. ∀z.((z ε f) -> ∃x.∃y.(z = (x,y))) DefExp 10
12. z ε f Hyp
13. (z ε f) -> ∃x.∃y.(z = (x,y)) ForallElim 11
14. ∃x.∃y.(z = (x,y)) ImpElim 12 13
15. ∃y.(z = (a,y)) Hyp
16. z = (a,b) Hyp
17. (x X y) = {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} DefEqInt
18. (a,b) ε f EqualitySub 12 16
19. ∃w.((a,w) ε f) ExistsInt 18
20. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
21. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
22. ∃w.((a,b) ε w) ExistsInt 18
23. Set((a,b)) DefSub 22
24. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
25. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 24
26. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 25
27. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 26
28. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 27
29. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 28
30. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 29
31. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 30
32. Set(a) & Set(b) ImpElim 23 31
33. Set(a) AndElimL 32
34. Set(a) & ∃w.((a,w) ε f) AndInt 33 19
35. a ε {w: ∃x_5.((w,x_5) ε f)} ClassInt 34
36. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 20
37. a ε domain(f) EqualitySub 35 36
38. domain(f) = x AndElimL 8
39. a ε x EqualitySub 37 38
40. ∃w.((w,b) ε f) ExistsInt 18
41. Set(b) AndElimR 32
42. Set(b) & ∃w.((w,b) ε f) AndInt 41 40
43. b ε {w: ∃x_8.((x_8,w) ε f)} ClassInt 42
44. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 21
45. b ε range(f) EqualitySub 43 44
46. range(f) = y AndElimR 8
47. b ε y EqualitySub 45 46
48. (a ε x) & (b ε y) AndInt 39 47
49. (z = (a,b)) & ((a ε x) & (b ε y)) AndInt 16 48
50. (a,b) = z Symmetry 16
51. Set(z) EqualitySub 23 50
52. ∃b.((z = (a,b)) & ((a ε x) & (b ε y))) ExistsInt 49
53. ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y))) ExistsInt 52
54. Set(z) & ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y))) AndInt 51 53

55. z ε {w: ∃a.∃b.((w = (a,b)) & ((a ε x) & (b ε y)))} ClassInt 54
56. {z: ∃a.∃b.((z = (a,b)) & ((a ε x) & (b ε y)))} = (x X y) Symmetry 17
57. z ε (x X y) EqualitySub 55 56
58. z ε (x X y) ExistsElim 15 16 57
59. z ε (x X y) ExistsElim 14 15 58
60. (z ε f) -> (z ε (x X y)) ImpInt 59
61. ∀z.((z ε f) -> (z ε (x X y))) ForallInt 60
62. f ⊂ (x X y) DefSub 61
63. (Set(x) & Set(y)) -> Set((x X y)) TheoremInt
64. Set((x X y)) ImpElim 0 63
65. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px))) TheoremInt
66. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
67. ∀y.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 66
68. (Set(x) & (c ⊂ x)) -> Set(c) ForallElim 67
69. ∀x.((Set(x) & (c ⊂ x)) -> Set(c)) ForallInt 68
70. (Set((x X y)) & (c ⊂ (x X y))) -> Set(c) ForallElim 69
71. ∀c.((Set((x X y)) & (c ⊂ (x X y))) -> Set(c)) ForallInt 70
72. (Set((x X y)) & (f ⊂ (x X y))) -> Set(f) ForallElim 71
73. Set((x X y)) & (f ⊂ (x X y)) AndInt 64 62
74. Set(f) ImpElim 73 72
75. ∀y.(Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))) ForallInt 65
76. Set(x) -> (Set(Px) & ((f ⊂ x) <-> (f ε Px))) ForallElim 75
77. ∀x.(Set(x) -> (Set(Px) & ((f ⊂ x) <-> (f ε Px)))) ForallInt 76
78. Set((x X y)) -> (Set(P(x X y)) & ((f ⊂ (x X y)) <-> (f ε P(x X y)))) ForallElim 77
79. Set(P(x X y)) & ((f ⊂ (x X y)) <-> (f ε P(x X y))) ImpElim 64 78
80. Set(P(x X y)) AndElimL 79
81. (f ⊂ (x X y)) <-> (f ε P(x X y)) AndElimR 79
82. ((f ⊂ (x X y)) -> (f ε P(x X y))) & ((f ε P(x X y)) -> (f ⊂ (x X y))) EquivExp 81
83. (f ⊂ (x X y)) -> (f ε P(x X y)) AndElimL 82
84. f ε P(x X y) ImpElim 62 83
85. (f ε func(x,y)) -> (f ε P(x X y)) ImpInt 84
86. ∀f.((f ε func(x,y)) -> (f ε P(x X y))) ForallInt 85
87. func(x,y) ⊂ P(x X y) DefSub 86
88. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
89. ∀y.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 88
90. (Set(x) & (c ⊂ x)) -> Set(c) ForallElim 89
91. ∀x.((Set(x) & (c ⊂ x)) -> Set(c)) ForallInt 90
92. (Set(P(x X y)) & (c ⊂ P(x X y))) -> Set(c) ForallElim 91
93. ∀c.((Set(P(x X y)) & (c ⊂ P(x X y))) -> Set(c)) ForallInt 92
94. (Set(P(x X y)) & (func(x,y) ⊂ P(x X y))) -> Set(func(x,y)) ForallElim 93
95. Set(P(x X y)) & (func(x,y) ⊂ P(x X y)) AndInt 80 87
96. Set(func(x,y)) ImpElim 95 94
97. (Set(x) & Set(y)) -> Set(func(x,y)) ImpInt 96 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. (Set(x) & Set(y)) -> Set((x X y))
3. Set(x) -> (Set(Px) & ((y ⊂ x) <-> (y ε Px)))
4. (Set(x) & (y ⊂ x)) -> Set(y)

Th88. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))

0. WellOrders(r,x) Hyp
1. (u ε x) & ((v ε x) & (w ε x)) Hyp
2. ((u,v) ε r) & ((v,w) ε r) Hyp
3. z ε {u,v} Hyp
4. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
5. (Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y)))) AndElimL 4
6. ∀x.((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) ForallInt 5
7. (Set(c) & Set(y)) -> (Set({c,y}) & ((z ε {c,y}) <-> ((z = c) v (z = y)))) ForallElim 6
8. ∀y.((Set(c) & Set(y)) -> (Set({c,y}) & ((z ε {c,y}) <-> ((z = c) v (z = y))))) ForallInt 7
9. (Set(c) & Set(d)) -> (Set({c,d}) & ((z ε {c,d}) <-> ((z = c) v (z = d)))) ForallElim 8
10. ∀z.((Set(c) & Set(d)) -> (Set({c,d}) & ((z ε {c,d}) <-> ((z = c) v (z = d))))) ForallInt 9
11. (Set(c) & Set(d)) -> (Set({c,d}) & ((e ε {c,d}) <-> ((e = c) v (e = d)))) ForallElim

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10
12. u ε x  AndElimL 1
13. (v ε x) & (w ε x)  AndElimR 1
14. v ε x  AndElimL 13
15. ∃x.(u ε x)  ExistsInt 12
16. Set(u)  DefSub 15
17. ∃x.(v ε x)  ExistsInt 14
18. Set(v)  DefSub 17
19. ∀c.((Set(c) & Set(d)) -> (Set({c,d}) & ((e ε {c,d}) <-> ((e = c) v (e = d)))))
ForallInt 11
20. (Set(u) & Set(d)) -> (Set({u,d}) & ((e ε {u,d}) <-> ((e = u) v (e = d))))  ForallElim
19
21. ∀d.((Set(u) & Set(d)) -> (Set({u,d}) & ((e ε {u,d}) <-> ((e = u) v (e = d)))))
ForallInt 20
22. (Set(u) & Set(v)) -> (Set({u,v}) & ((e ε {u,v}) <-> ((e = u) v (e = v))))  ForallElim
21
23. Set(u) & Set(v)  AndInt 16 18
24. Set({u,v}) & ((e ε {u,v}) <-> ((e = u) v (e = v)))  ImpElim 23 22
25. (e ε {u,v}) <-> ((e = u) v (e = v))  AndElimR 24
26. ∀e.((e ε {u,v}) <-> ((e = u) v (e = v)))  ForallInt 25
27. (z ε {u,v}) <-> ((z = u) v (z = v))  ForallElim 26
28. ((z ε {u,v}) -> ((z = u) v (z = v))) & (((z = u) v (z = v)) -> (z ε {u,v}))  EquivExp
27
29. (z ε {u,v}) -> ((z = u) v (z = v))  AndElimL 28
30. (z = u) v (z = v)  ImpElim 3 29
31. z = u  Hyp
32. u ε x  AndElimL 1
33. u = z  Symmetry 31
34. z ε x  EqualitySub 32 33
35. z = v  Hyp
36. (v ε x) & (w ε x)  AndElimR 1
37. v ε x  AndElimL 36
38. v = z  Symmetry 35
39. z ε x  EqualitySub 37 38
40. z ε x  OrElim 30 31 34 35 39
41. (z ε {u,v}) -> (z ε x)  ImpInt 40
42. ∀z.((z ε {u,v}) -> (z ε x))  ForallInt 41
43. {u,v} ⊂ x  DefSub 42
44. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  DefExp 0
45. ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  AndElimR 44
46. (({u,v} ⊂ x) & ¬({u,v} = 0)) -> ∃z.First(r,{u,v},z)  ForallElim 45
47. u = u  Identity
48. (u = u) v (v = v)  OrIntR 47
49. ((e ε {u,v}) -> ((e = u) v (e = v))) & (((e = u) v (e = v)) -> (e ε {u,v}))  EquivExp
25
50. ((e = u) v (e = v)) -> (e ε {u,v})  AndElimR 49
51. ∀e.(((e = u) v (e = v)) -> (e ε {u,v}))  ForallInt 50
52. ((u = u) v (u = v)) -> (u ε {u,v})  ForallElim 51
53. (u = u) v (u = v)  OrIntR 47
54. u ε {u,v}  ImpElim 53 52
55. {u,v} = 0  Hyp
56. u ε 0  EqualitySub 54 55
57. ¬(x ε 0)  TheoremInt
58. ∀x.¬(x ε 0)  ForallInt 57
59. ¬(u ε 0)  ForallElim 58
60. _|_  ImpElim 56 59
61. ¬({u,v} = 0)  ImpInt 60
62. ({u,v} ⊂ x) & ¬({u,v} = 0)  AndInt 43 61
63. ∃z.First(r,{u,v},z)  ImpElim 62 46
64. First(r,{u,v},f)  Hyp
65. (f ε {u,v}) & ∀y.((y ε {u,v}) -> ¬((y,f) ε r))  DefExp 64
66. f ε {u,v}  AndElimL 65
67. ((e ε {u,v}) -> ((e = u) v (e = v))) & (((e = u) v (e = v)) -> (e ε {u,v}))  EquivExp
25
68. (e ε {u,v}) -> ((e = u) v (e = v))  AndElimL 67
69. ∀e.((e ε {u,v}) -> ((e = u) v (e = v)))  ForallInt 68
70. (f ε {u,v}) -> ((f = u) v (f = v))  ForallElim 69
71. (f = u) v (f = v)  ImpElim 66 70
72. ∀y.((y ε {u,v}) -> ¬((y,f) ε r))  AndElimR 65
73. (u ε {u,v}) -> ¬((u,f) ε r)  ForallElim 72
74. (v ε {u,v}) -> ¬((v,f) ε r)  ForallElim 72
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75. f = u  Hyp
76. ∀e.(((e = u) v (e = v)) -> (e ε {u,v}))  ForallInt 50
77. ((v = u) v (v = v)) -> (v ε {u,v})  ForallElim 76
78. v = v  Identity
79. (v = u) v (v = v)  OrIntL 78
80. v ε {u,v}  ImpElim 79 77
81. ¬((v,f) ε r)  ImpElim 80 74
82. ¬((v,u) ε r)  EqualitySub 81 75
83. ¬((v,u) ε r) v ¬((u,v) ε r)  OrIntR 82
84. f = v  Hyp
85. ∀e.(((e = u) v (e = v)) -> (e ε {u,v}))  ForallInt 50
86. ((u = u) v (u = v)) -> (u ε {u,v})  ForallElim 85
87. u = u  Identity
88. (u = u) v (u = v)  OrIntR 87
89. u ε {u,v}  ImpElim 88 86
90. (u ε {u,v}) -> ¬((u,f) ε r)  ForallElim 72
91. ¬((u,f) ε r)  ImpElim 89 90
92. ¬((u,v) ε r)  EqualitySub 91 84
93. ¬((v,u) ε r) v ¬((u,v) ε r)  OrIntL 92
94. ¬((v,u) ε r) v ¬((u,v) ε r)  OrElim 71 75 83 84 93
95. ¬((v,u) ε r) v ¬((u,v) ε r)  ExistsElim 63 64 94
96. (B v ¬A) -> (A -> B)  TheoremInt
97. (¬((v,u) ε r) v ¬A) -> (A -> ¬((v,u) ε r))  PolySub 96
98. (¬((v,u) ε r) v ¬((u,v) ε r)) -> (((u,v) ε r) -> ¬((v,u) ε r))  PolySub 97
99. ((u,v) ε r) -> ¬((v,u) ε r)  ImpElim 95 98
100. ((u ε x) & ((v ε x) & (w ε x))) -> (((u,v) ε r) -> ¬((v,u) ε r))  ImpInt 99
101. ∀w.(((u ε x) & ((v ε x) & (w ε x))) -> (((u,v) ε r) -> ¬((v,u) ε r)))  ForallInt 100
102. ((u ε x) & ((v ε x) & (v ε x))) -> (((u,v) ε r) -> ¬((v,u) ε r))  ForallElim 101
103. (u ε x) & (v ε x)  Hyp
104. (u,v) ε r  Hyp
105. u ε x  AndElimL 103
106. v ε x  AndElimR 103
107. (v ε x) & (v ε x)  AndInt 106 106
108. (u ε x) & ((v ε x) & (v ε x))  AndInt 105 107
109. ((u,v) ε r) -> ¬((v,u) ε r)  ImpElim 108 102
110. ¬((v,u) ε r)  ImpElim 104 109
111. ((u,v) ε r) -> ¬((v,u) ε r)  ImpInt 110
112. ((u ε x) & (v ε x)) -> (((u,v) ε r) -> ¬((v,u) ε r))  ImpInt 111
113. ∀z.(((u ε x) & (z ε x)) -> (((u,z) ε r) -> ¬((z,u) ε r)))  ForallInt 112
114. ∀y.∀z.(((y ε x) & (z ε x)) -> (((y,z) ε r) -> ¬((z,y) ε r)))  ForallInt 113
115. Asymmetric(r,x)  DefSub 114
116. ¬TransIn(r,x)  Hyp
117. ¬∀u.∀v.∀w.(((u ε x) & ((v ε x) & (w ε x))) -> ((((u,v) ε r) & ((v,w) ε r)) -> ((u,w) ε r)))  DefExp 116
118. ¬∀i.P(i) -> ∃c.¬P(c)  TheoremInt
119. ¬∀i.∀v.∀w.(((i ε x) & ((v ε x) & (w ε x))) -> ((((i,v) ε r) & ((v,w) ε r)) -> ((i,w) ε r))) -> ∃c.¬∀v.∀w.(((c ε x) & ((v ε x) & (w ε x))) -> ((((c,v) ε r) & ((v,w) ε r)) -> ((c,w) ε r)))  PredSub 118
120. ∃c.¬∀v.∀w.(((c ε x) & ((v ε x) & (w ε x))) -> ((((c,v) ε r) & ((v,w) ε r)) -> ((c,w) ε r)))  ImpElim 117 119
121. ¬∀v.∀w.(((k ε x) & ((v ε x) & (w ε x))) -> ((((k,v) ε r) & ((v,w) ε r)) -> ((k,w) ε r)))  Hyp
122. ¬∀i.∀w.(((k ε x) & ((i ε x) & (w ε x))) -> ((((k,i) ε r) & ((i,w) ε r)) -> ((k,w) ε r))) -> ∃c.¬∀w.(((k ε x) & ((c ε x) & (w ε x))) -> ((((k,c) ε r) & ((c,w) ε r)) -> ((k,w) ε r)))  PredSub 118
123. ∃c.¬∀w.(((k ε x) & ((c ε x) & (w ε x))) -> ((((k,c) ε r) & ((c,w) ε r)) -> ((k,w) ε r)))  ImpElim 121 122
124. ¬∀w.(((k ε x) & ((p ε x) & (w ε x))) -> ((((k,p) ε r) & ((p,w) ε r)) -> ((k,w) ε r)))  Hyp
125. ¬∀i.(((k ε x) & ((p ε x) & (i ε x))) -> ((((k,p) ε r) & ((p,i) ε r)) -> ((k,i) ε r))) -> ∃c.¬(((k ε x) & ((p ε x) & (c ε x))) -> ((((k,p) ε r) & ((p,c) ε r)) -> ((k,c) ε r)))  PredSub 118
126. ∃c.¬(((k ε x) & ((p ε x) & (c ε x))) -> ((((k,p) ε r) & ((p,c) ε r)) -> ((k,c) ε r)))  ImpElim 124 125
127. ¬(((k ε x) & ((p ε x) & (q ε x))) -> ((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)))  Hyp
128. (A -> B) -> (¬B -> ¬A)  TheoremInt
129. (A -> C) -> (¬C -> ¬A)  PolySub 128
130. ((B v ¬A) -> C) -> (¬C -> ¬(B v ¬A))  PolySub 129
131. ((B v ¬A) -> (A -> B)) -> (¬(A -> B) -> ¬(B v ¬A))  PolySub 130
132. (B v ¬A) -> (A -> B)  TheoremInt
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133. ¬(A -> B) -> ¬(B v ¬A)  ImpElim 132 131
134. ¬(((k ε x) & ((p ε x) & (q ε x))) -> B) -> ¬(B v ¬((k ε x) & ((p ε x) & (q ε x))))
PolySub 133
135. ¬(((k ε x) & ((p ε x) & (q ε x))) -> ((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r))) -
> ¬(((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) v ¬((k ε x) & ((p ε x) & (q ε x))))
PolySub 134
136. ¬(((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) v ¬((k ε x) & ((p ε x) & (q ε x))))
ImpElim 127 135
137. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
138. ¬(A v B) <-> (¬A & ¬B)  AndElimL 137
139. ¬(A v C) <-> (¬A & ¬C)  PolySub 138
140. ¬(B v C) <-> (¬B & ¬C)  PolySub 139
141. ¬(B v ¬A) <-> (¬B & ¬¬A)  PolySub 140
142. (¬(B v ¬A) -> (¬B & ¬¬A)) & ((¬B & ¬¬A) -> ¬(B v ¬A))  EquivExp 141
143. ¬(B v ¬A) -> (¬B & ¬¬A)  AndElimL 142
144. D <-> ¬¬D  TheoremInt
145. (D -> ¬¬D) & (¬¬D -> D)  EquivExp 144
146. ¬¬D -> D  AndElimR 145
147. ¬¬A -> A  PolySub 146
148. ¬(B v ¬A)  Hyp
149. ¬B & ¬¬A  ImpElim 148 143
150. ¬B  AndElimL 149
151. ¬¬A  AndElimR 149
152. A  ImpElim 151 147
153. ¬B & A  AndInt 150 152
154. ¬(B v ¬A) -> (¬B & A)  ImpInt 153
155. ¬(A -> B)  Hyp
156. ¬(B v ¬A)  ImpElim 155 133
157. ¬B & A  ImpElim 156 154
158. ¬(A -> B) -> (¬B & A)  ImpInt 157
159. ¬(((k ε x) & ((p ε x) & (q ε x))) -> B) -> (¬B & ((k ε x) & ((p ε x) & (q ε x))))
PolySub 158
160. ¬(((k ε x) & ((p ε x) & (q ε x))) -> ((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r))) -
> (¬((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) & ((k ε x) & ((p ε x) & (q ε x))))
PolySub 159
161. ¬((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) & ((k ε x) & ((p ε x) & (q ε x)))
ImpElim 127 160
162. ¬((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r))  AndElimL 161
163. (k ε x) & ((p ε x) & (q ε x))  AndElimR 161
164. ¬((((k,p) ε r) & ((p,q) ε r)) -> B) -> (¬B & (((k,p) ε r) & ((p,q) ε r)))  PolySub
158
165. ¬((((k,p) ε r) & ((p,q) ε r)) -> ((k,q) ε r)) -> (¬((k,q) ε r) & (((k,p) ε r) &
((p,q) ε r)))  PolySub 164
166. ¬((k,q) ε r) & (((k,p) ε r) & ((p,q) ε r))  ImpElim 162 165
167. ¬((k,q) ε r)  AndElimL 166
168. k ε x  AndElimL 163
169. (p ε x) & (q ε x)  AndElimR 163
170. q ε x  AndElimR 169
171. Connects(r,x)  AndElimL 44
172. ∀y.∀z.(((y ε x) & (z ε x)) -> ((y = z) v (((y,z) ε r) v ((z,y) ε r))))  DefExp 171
173. ∀z.(((k ε x) & (z ε x)) -> ((k = z) v (((k,z) ε r) v ((z,k) ε r))))  ForallElim 172
174. ((k ε x) & (q ε x)) -> ((k = q) v (((k,q) ε r) v ((q,k) ε r)))  ForallElim 173
175. (k ε x) & (q ε x)  AndInt 168 170
176. (k = q) v (((k,q) ε r) v ((q,k) ε r))  ImpElim 175 174
177. k = q  Hyp
178. ((k,p) ε r) & ((p,q) ε r)  AndElimR 166
179. ((q,p) ε r) & ((p,q) ε r)  EqualitySub 178 177
180. ∀z.(((q ε x) & (z ε x)) -> (((q,z) ε r) -> ¬((z,q) ε r)))  ForallElim 114
181. ((q ε x) & (p ε x)) -> (((q,p) ε r) -> ¬((p,q) ε r))  ForallElim 180
182. p ε x  AndElimL 169
183. (q ε x) & (p ε x)  AndInt 170 182
184. ((q,p) ε r) -> ¬((p,q) ε r)  ImpElim 183 181
185. (q,p) ε r  AndElimL 179
186. ¬((p,q) ε r)  ImpElim 185 184
187. (p,q) ε r  AndElimR 178
188. _|_  ImpElim 187 186
189. (q,k) ε r  AbsI 188
190. ((k,q) ε r) v ((q,k) ε r)  Hyp
191. (k,q) ε r  Hyp
192. _|_  ImpElim 191 167
193. (q,k) ε r  AbsI 192
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194. (q,k) ε r Hyp
195. (q,k) ε r OrElim 190 191 193 194 194
196. (q,k) ε r OrElim 176 177 189 190 195
197. ((q,k) ε r) & (((k,p) ε r) & ((p,q) ε r)) AndInt 196 178
198. cyc = {p,{q,k}} Hyp
199. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y} = U) <-> (¬Set(x) v ¬Set(y))) TheoremInt
200. k ε x AndElimL 163
201. ∃w.(k ε w) ExistsInt 200
202. Set(k) DefSub 201
203. (p ε x) & (q ε x) AndElimR 163
204. q ε x AndElimR 203
205. ∃w.(q ε w) ExistsInt 204
206. Set(q) DefSub 205
207. p ε x AndElimL 203
208. ∃w.(p ε w) ExistsInt 207
209. Set(p) DefSub 208
210. triad = ({p} ∪ ({q} ∪ {k})) Hyp
211. z ε triad Hyp
212. Set(x) -> Set({x}) TheoremInt
213. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
214. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 213
215. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 214
216. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 215
217. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 216
218. (z ε ({p} ∪ y)) -> ((z ε {p}) v (z ε y)) ForallElim 217
219. ∀y.((z ε ({p} ∪ y)) -> ((z ε {p}) v (z ε y))) ForallInt 218
220. (z ε ({p} ∪ ({q} ∪ {k}))) -> ((z ε {p}) v (z ε ({q} ∪ {k}))) ForallElim 219
221. z ε ({p} ∪ ({q} ∪ {k})) EqualitySub 211 210
222. (z ε {p}) v (z ε ({q} ∪ {k})) ImpElim 221 220
223. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
224. z ε {p} Hyp
225. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 223
226. Set(p) -> ((y ε {p}) <-> (y = p)) ForallElim 225
227. (y ε {p}) <-> (y = p) ImpElim 209 226
228. ((y ε {p}) -> (y = p)) & ((y = p) -> (y ε {p})) EquivExp 227
229. (y ε {p}) -> (y = p) AndElimL 228
230. ∀y.((y ε {p}) -> (y = p)) ForallInt 229
231. (z ε {p}) -> (z = p) ForallElim 230
232. z = p ImpElim 224 231
233. p = z Symmetry 232
234. z ε x EqualitySub 207 233
235. z ε ({q} ∪ {k}) Hyp
236. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 216
237. (z ε ({q} ∪ y)) -> ((z ε {q}) v (z ε y)) ForallElim 236
238. ∀y.((z ε ({q} ∪ y)) -> ((z ε {q}) v (z ε y))) ForallInt 237
239. (z ε ({q} ∪ {k})) -> ((z ε {q}) v (z ε {k})) ForallElim 238
240. (z ε {q}) v (z ε {k}) ImpElim 235 239
241. z ε {q} Hyp
242. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 223
243. Set(q) -> ((y ε {q}) <-> (y = q)) ForallElim 242
244. (y ε {q}) <-> (y = q) ImpElim 206 243
245. ((y ε {q}) -> (y = q)) & ((y = q) -> (y ε {q})) EquivExp 244
246. (y ε {q}) -> (y = q) AndElimL 245
247. ∀y.((y ε {q}) -> (y = q)) ForallInt 246
248. (z ε {q}) -> (z = q) ForallElim 247
249. z = q ImpElim 241 248
250. q = z Symmetry 249
251. z ε x EqualitySub 204 250
252. z ε {k} Hyp
253. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 223
254. Set(k) -> ((y ε {k}) <-> (y = k)) ForallElim 253
255. (y ε {k}) <-> (y = k) ImpElim 202 254
256. ((y ε {k}) -> (y = k)) & ((y = k) -> (y ε {k})) EquivExp 255
257. (y ε {k}) -> (y = k) AndElimL 256
258. ∀y.((y ε {k}) -> (y = k)) ForallInt 257
259. (z ε {k}) -> (z = k) ForallElim 258
260. z = k ImpElim 252 259
261. k = z Symmetry 260

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262. z ε x  EqualitySub 200 261
263. z ε x  OrElim 240 241 251 252 262
264. z ε x  OrElim 222 224 234 235 263
265. (z ε triad) -> (z ε x)  ImpInt 264
266. ∀z.((z ε triad) -> (z ε x))  ForallInt 265
267. triad ⊂ x  DefSub 266
268. ((triad ⊂ x) & ¬(triad = 0)) -> ∃z.First(r,triad,z)  ForallElim 45
269. ∀y.((y ε {p}) <-> (y = p))  ForallInt 227
270. (p ε {p}) <-> (p = p)  ForallElim 269
271. ((p ε {p}) -> (p = p)) & ((p = p) -> (p ε {p}))  EquivExp 270
272. (p = p) -> (p ε {p})  AndElimR 271
273. p = p  Identity
274. p ε {p}  ImpElim 273 272
275. (p ε {p}) v (p ε ({q} ∪ {k}))  OrIntR 274
276. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 214
277. ((z ε x) v (z ε y)) -> (z ε (x ∪ y))  AndElimR 276
278. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 277
279. ((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y))  ForallElim 278
280. ∀y.(((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y)))  ForallInt 279
281. ((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k})))  ForallElim 280
282. ∀z.(((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k}))))  ForallInt 281
283. ((p ε {p}) v (p ε ({q} ∪ {k}))) -> (p ε ({p} ∪ ({q} ∪ {k})))  ForallElim 282
284. p ε ({p} ∪ ({q} ∪ {k}))  ImpElim 275 283
285. ({p} ∪ ({q} ∪ {k})) = triad  Symmetry 210
286. p ε triad  EqualitySub 284 285
287. ¬(x ε 0)  TheoremInt
288. triad = 0  Hyp
289. 0 = triad  Symmetry 288
290. p ε 0  EqualitySub 286 288
291. ∀x.¬(x ε 0)  ForallInt 287
292. ¬(p ε 0)  ForallElim 291
293. _|_  ImpElim 290 292
294. ¬(triad = 0)  ImpInt 293
295. (triad ⊂ x) & ¬(triad = 0)  AndInt 267 294
296. ∃z.First(r,triad,z)  ImpElim 295 268
297. First(r,triad,l)  Hyp
298. (l ε triad) & ∀y.((y ε triad) -> ¬((y,l) ε r))  DefExp 297
299. l ε triad  AndElimL 298
300. l ε ({p} ∪ ({q} ∪ {k}))  EqualitySub 299 210
301. ∀z.((z ε ({p} ∪ ({q} ∪ {k}))) -> ((z ε {p}) v (z ε ({q} ∪ {k}))))  ForallInt 220
302. (l ε ({p} ∪ ({q} ∪ {k}))) -> ((l ε {p}) v (l ε ({q} ∪ {k})))  ForallElim 301
303. (l ε {p}) v (l ε ({q} ∪ {k}))  ImpElim 300 302
304. l ε {p}  Hyp
305. ∀y.((y ε {p}) -> (y = p))  ForallInt 229
306. (l ε {p}) -> (l = p)  ForallElim 305
307. l = p  ImpElim 304 306
308. Set(x) -> ((y ε {x}) <-> (y = x))  TheoremInt
309. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 308
310. Set(k) -> ((y ε {k}) <-> (y = k))  ForallElim 309
311. k = k  Identity
312. (y ε {k}) <-> (y = k)  ImpElim 202 310
313. ∀y.((y ε {k}) <-> (y = k))  ForallInt 312
314. (k ε {k}) <-> (k = k)  ForallElim 313
315. ((k ε {k}) -> (k = k)) & ((k = k) -> (k ε {k}))  EquivExp 314
316. (k = k) -> (k ε {k})  AndElimR 315
317. k ε {k}  ImpElim 311 316
318. (k ε {q}) v (k ε {k})  OrIntL 317
319. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 277
320. ((z ε {q}) v (z ε y)) -> (z ε ({q} ∪ y))  ForallElim 319
321. ∀y.(((z ε {q}) v (z ε y)) -> (z ε ({q} ∪ y)))  ForallInt 320
322. ((z ε {q}) v (z ε {k})) -> (z ε ({q} ∪ {k}))  ForallElim 321
323. ∀z.(((z ε {q}) v (z ε {k})) -> (z ε ({q} ∪ {k})))  ForallInt 322
324. ((k ε {q}) v (k ε {k})) -> (k ε ({q} ∪ {k}))  ForallElim 323
325. k ε ({q} ∪ {k})  ImpElim 318 324
326. (k ε {p}) v (k ε ({q} ∪ {k}))  OrIntL 325
327. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y)))  ForallInt 277
328. ((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y))  ForallElim 327
329. ∀y.(((z ε {p}) v (z ε y)) -> (z ε ({p} ∪ y)))  ForallInt 328
330. ((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k})))  ForallElim 329
331. ∀z.(((z ε {p}) v (z ε ({q} ∪ {k}))) -> (z ε ({p} ∪ ({q} ∪ {k}))))  ForallInt 330
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332. ((k ε {p}) v (k ε ({q} U {k}))) -> (k ε ({p} U ({q} U {k})))  ForallElim 331
333. k ε ({p} U ({q} U {k}))  ImpElim 326 332
334. ({p} U ({q} U {k})) = triad  Symmetry 210
335. k ε triad  EqualitySub 333 334
336. ∀y.((y ε triad) -> ¬((y,l) ε r))  AndElimR 298
337. ∀y.((y ε triad) -> ¬((y,p) ε r))  EqualitySub 336 307
338. (k ε triad) -> ¬((k,p) ε r)  ForallElim 337
339. ¬((k,p) ε r)  ImpElim 335 338
340. ((k,p) ε r) & ((p,q) ε r)  AndElimR 197
341. (k,p) ε r  AndElimL 340
342. _|_  ImpElim 341 339
343. l ε ({q} U {k})  Hyp
344. (z ε (x U y)) -> ((z ε x) v (z ε y))  AndElimL 276
345. ∀x.((z ε (x U y)) -> ((z ε x) v (z ε y)))  ForallInt 344
346. (z ε ({q} U y)) -> ((z ε {q}) v (z ε y))  ForallElim 345
347. ∀y.((z ε ({q} U y)) -> ((z ε {q}) v (z ε y)))  ForallInt 346
348. (z ε ({q} U {k})) -> ((z ε {q}) v (z ε {k}))  ForallElim 347
349. ∀z.((z ε ({q} U {k})) -> ((z ε {q}) v (z ε {k})))  ForallInt 348
350. (l ε ({q} U {k})) -> ((l ε {q}) v (l ε {k}))  ForallElim 349
351. (l ε {q}) v (l ε {k})  ImpElim 343 350
352. l ε {q}  Hyp
353. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 308
354. Set(q) -> ((y ε {q}) <-> (y = q))  ForallElim 353
355. ∀y.(Set(q) -> ((y ε {q}) <-> (y = q)))  ForallInt 354
356. Set(q) -> ((l ε {q}) <-> (l = q))  ForallElim 355
357. (l ε {q}) <-> (l = q)  ImpElim 206 356
358. ((l ε {q}) -> (l = q)) & ((l = q) -> (l ε {q}))  EquivExp 357
359. (l ε {q}) -> (l = q)  AndElimL 358
360. l = q  ImpElim 352 359
361. ∀y.((y ε triad) -> ¬((y,l) ε r))  AndElimR 298
362. ∀y.((y ε triad) -> ¬((y,q) ε r))  EqualitySub 361 360
363. (p ε triad) -> ¬((p,q) ε r)  ForallElim 362
364. ¬((p,q) ε r)  ImpElim 286 363
365. (p,q) ε r  AndElimR 340
366. _|_  ImpElim 365 364
367. l ε {k}  Hyp
368. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 308
369. Set(k) -> ((y ε {k}) <-> (y = k))  ForallElim 368
370. (y ε {k}) <-> (y = k)  ImpElim 202 369
371. ∀y.((y ε {k}) <-> (y = k))  ForallInt 370
372. (l ε {k}) <-> (l = k)  ForallElim 371
373. ((l ε {k}) -> (l = k)) & ((l = k) -> (l ε {k}))  EquivExp 372
374. (l ε {k}) -> (l = k)  AndElimL 373
375. l = k  ImpElim 367 374
376. ∀y.((y ε triad) -> ¬((y,k) ε r))  EqualitySub 361 375
377. (q ε triad) -> ¬((q,k) ε r)  ForallElim 376
378. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 308
379. Set(q) -> ((y ε {q}) <-> (y = q))  ForallElim 378
380. (y ε {q}) <-> (y = q)  ImpElim 206 379
381. ∀y.((y ε {q}) <-> (y = q))  ForallInt 380
382. (q ε {q}) <-> (q = q)  ForallElim 381
383. q = q  Identity
384. ((q ε {q}) -> (q = q)) & ((q = q) -> (q ε {q}))  EquivExp 382
385. (q = q) -> (q ε {q})  AndElimR 384
386. q ε {q}  ImpElim 383 385
387. (q ε {q}) v (q ε {k})  OrIntR 386
388. ∀x.(((z ε x) v (z ε y)) -> (z ε (x U y)))  ForallInt 277
389. ((z ε {q}) v (z ε y)) -> (z ε ({q} U y))  ForallElim 388
390. ∀y.(((z ε {q}) v (z ε y)) -> (z ε ({q} U y)))  ForallInt 389
391. ((z ε {q}) v (z ε {k})) -> (z ε ({q} U {k}))  ForallElim 390
392. ∀z.(((z ε {q}) v (z ε {k})) -> (z ε ({q} U {k})))  ForallInt 391
393. ((q ε {q}) v (q ε {k})) -> (q ε ({q} U {k}))  ForallElim 392
394. q ε ({q} U {k})  ImpElim 387 393
395. (q ε {p}) v (q ε ({q} U {k}))  OrIntL 394
396. ∀x.(((z ε x) v (z ε y)) -> (z ε (x U y)))  ForallInt 277
397. ((z ε {p}) v (z ε y)) -> (z ε ({p} U y))  ForallElim 396
398. ∀y.(((z ε {p}) v (z ε y)) -> (z ε ({p} U y)))  ForallInt 397
399. ((z ε {p}) v (z ε ({q} U {k}))) -> (z ε ({p} U ({q} U {k})))  ForallElim 398
400. ∀z.(((z ε {p}) v (z ε ({q} U {k}))) -> (z ε ({p} U ({q} U {k}))))  ForallInt 399
401. ((q ε {p}) v (q ε ({q} U {k}))) -> (q ε ({p} U ({q} U {k})))  ForallElim 400
402. q ε ({p} U ({q} U {k}))  ImpElim 395 401
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403. ({p} ∪ ({q} ∪ {k})) = triad  Symmetry 210
404. q ε triad  EqualitySub 402 403
405. ∀y.((y ε triad) -> ¬((y,k) ε r))  EqualitySub 361 375
406. (q ε triad) -> ¬((q,k) ε r)  ForallElim 405
407. ¬((q,k) ε r)  ImpElim 404 406
408. (q,k) ε r  AndElimL 197
409. _|_  ImpElim 408 407
410. _|_  OrElim 351 352 366 367 409
411. _|_  OrElim 303 304 342 343 410
412. _|_  ExistsElim 296 297 411
413. ¬(triad = ({p} ∪ ({q} ∪ {k})))  ImpInt 412
414. ∀triad.¬(triad = ({p} ∪ ({q} ∪ {k})))  ForallInt 413
415. ¬(({p} ∪ ({q} ∪ {k})) = ({p} ∪ ({q} ∪ {k})))  ForallElim 414
416. ({p} ∪ ({q} ∪ {k})) = ({p} ∪ ({q} ∪ {k}))  Identity
417. _|_  ImpElim 416 415
418. _|_  ExistsElim 126 127 417
419. _|_  ExistsElim 123 124 418
420. _|_  ExistsElim 120 121 419
421. ¬¬TransIn(r,x)  ImpInt 420
422. D <-> ¬¬D  TheoremInt
423. (D -> ¬¬D) & (¬¬D -> D)  EquivExp 422
424. ¬¬D -> D  AndElimR 423
425. ¬¬TransIn(r,x) -> TransIn(r,x)  PolySub 424
426. TransIn(r,x)  ImpElim 421 425
427. Asymmetric(r,x) & TransIn(r,x)  AndInt 115 426
428. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  ImpInt 427 Qed
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Used Theorems

```
1. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))
2. ¬(x ε 0)
3. (B v ¬A) -> (A -> B)
5. ¬∀i.P(i) -> ∃c.¬P(c)
7. (A -> B) -> (¬B -> ¬A)
6. (B v ¬A) -> (A -> B)
8. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
9. D <-> ¬¬D
10. ((Set(x) & Set(y)) -> (Set({x,y}) & ((z ε {x,y}) <-> ((z = x) v (z = y))))) & (({x,y}
= U) <-> (¬Set(x) v ¬Set(y)))
11. Set(x) -> Set({x})
12. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
13. Set(x) -> ((y ε {x}) <-> (y = x))
14. ¬(x ε 0)
```

Th90. (¬(n = 0) & ∀y.((y ε n) -> Section(r,x,y))) -> (Section(r,x,∪n) & Section(r,x,∩n))

```
0. ¬(n = 0) & ∀y.((y ε n) -> Section(r,x,y))  Hyp
1. z ε ∪n  Hyp
2. ∪x = {z: ∃y.((y ε x) & (z ε y))}  DefEqInt
3. ∀x.(∪x = {z: ∃y.((y ε x) & (z ε y))})  ForallInt 2
4. ∪n = {z: ∃y.((y ε n) & (z ε y))}  ForallElim 3
5. z ε {z: ∃y.((y ε n) & (z ε y))}  EqualitySub 1 4
6. Set(z) & ∃y.((y ε n) & (z ε y))  ClassElim 5
7. ∀y.((y ε n) -> Section(r,x,y))  AndElimR 0
8. ∃y.((y ε n) & (z ε y))  AndElimR 6
9. (m ε n) & (z ε m)  Hyp
10. (m ε n) -> Section(r,x,m)  ForallElim 7
11. m ε n  AndElimL 9
12. Section(r,x,m)  ImpElim 11 10
13. ((m ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m))
DefExp 12
14. (m ⊂ x) & WellOrders(r,x)  AndElimL 13
15. m ⊂ x  AndElimL 14
16. ∀z.((z ε m) -> (z ε x))  DefExp 15
17. (z ε m) -> (z ε x)  ForallElim 16
18. z ε m  AndElimR 9
19. z ε x  ImpElim 18 17
20. z ε x  ExistsElim 8 9 19
21. (z ε ∪n) -> (z ε x)  ImpInt 20
22. ∀z.((z ε ∪n) -> (z ε x))  ForallInt 21
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23. Un ⊂ x DefSub 22
24. WellOrders(r,x) AndElimR 14
25. (u ε x) & ((v ε Un) & ((u,v) ε r)) Hyp
26. (v ε Un) & ((u,v) ε r) AndElimR 25
27. v ε Un AndElimL 26
28. v ε {z: ∃y.((y ε n) & (z ε y))} EqualitySub 27 4
29. Set(v) & ∃y.((y ε n) & (v ε y)) ClassElim 28
30. ∃y.((y ε n) & (v ε y)) AndElimR 29
31. (m ε n) & (v ε m) Hyp
32. ∀y.((y ε n) -> Section(r,x,y)) AndElimR 0
33. (m ε n) -> Section(r,x,m) ForallElim 32
34. m ε n AndElimL 31
35. Section(r,x,m) ImpElim 34 33
36. ((m ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m)) DefExp 35
37. ∀u.∀v.((((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m)) AndElimR 36
38. ∀v.((((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m)) ForallElim 37
39. (((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m) ForallElim 38
40. (v ε Un) & ((u,v) ε r) AndElimR 25
41. (u,v) ε r AndElimR 40
42. u ε x AndElimL 25
43. v ε m AndElimR 31
44. (u ε x) & (v ε m) AndInt 42 43
45. ((u ε x) & (v ε m)) & ((u,v) ε r) AndInt 44 41
46. u ε m ImpElim 45 39
47. (m ε n) & (u ε m) AndInt 34 46
48. ∃m.((m ε n) & (u ε m)) ExistsInt 47
49. ∃w.(u ε w) ExistsInt 46
50. Set(u) DefSub 49
51. Set(u) & ∃m.((m ε n) & (u ε m)) AndInt 50 48
52. u ε {u: ∃m.((m ε n) & (u ε m))} ClassInt 51
53. {z: ∃y.((y ε n) & (z ε y))} = Un Symmetry 4
54. u ε Un EqualitySub 52 53
55. u ε Un ExistsElim 30 31 54
56. ((u ε x) & ((v ε Un) & ((u,v) ε r))) -> (u ε Un) ImpInt 55
57. ((u ε x) & (v ε Un)) & ((u,v) ε r) Hyp
58. (u ε x) & (v ε Un) AndElimL 57
59. (u,v) ε r AndElimR 57
60. u ε x AndElimL 58
61. v ε Un AndElimR 58
62. (v ε Un) & ((u,v) ε r) AndInt 61 59
63. (u ε x) & ((v ε Un) & ((u,v) ε r)) AndInt 60 62
64. u ε Un ImpElim 63 56
65. (((u ε x) & (v ε Un)) & ((u,v) ε r)) -> (u ε Un) ImpInt 64
66. ∀v.((((u ε x) & (v ε Un)) & ((u,v) ε r)) -> (u ε Un)) ForallInt 65
67. ∀u.∀v.((((u ε x) & (v ε Un)) & ((u,v) ε r)) -> (u ε Un)) ForallInt 66
68. ∃w.(w ε n) Hyp
69. a ε n Hyp
70. ∀y.((y ε n) -> Section(r,x,y)) AndElimR 0
71. (a ε n) -> Section(r,x,a) ForallElim 70
72. Section(r,x,a) ImpElim 69 71
73. ((a ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε a)) & ((u,v) ε r)) -> (u ε a)) DefExp 72
74. (a ⊂ x) & WellOrders(r,x) AndElimL 73
75. WellOrders(r,x) AndElimR 74
76. WellOrders(r,x) ExistsElim 68 69 75
77. ∃w.(w ε n) -> WellOrders(r,x) ImpInt 76
78. ¬(n = 0) AndElimL 0
79. ¬∃i.P(i) -> ∀j.¬P(j) TheoremInt
80. ¬∃w.(w ε n) Hyp
81. ¬∃i.(i ε n) -> ∀j.¬(j ε n) PredSub 79
82. ∀j.¬(j ε n) ImpElim 80 81
83. b ε n Hyp
84. ¬(b ε n) ForallElim 82
85. _|_ ImpElim 83 84
86. b ε 0 AbsI 85
87. (b ε n) -> (b ε 0) ImpInt 86
88. b ε 0 Hyp
89. 0 = {x: ¬(x = x)} DefEqInt
90. b ε {x: ¬(x = x)} EqualitySub 88 89
91. Set(b) & ¬(b = b) ClassElim 90

92. ¬(b = b) AndElimR 91
93. b = b Identity
94. _|_ ImpElim 93 92
95. b ε n AbsI 94
96. (b ε 0) -> (b ε n) ImpInt 95
97. ((b ε n) -> (b ε 0)) & ((b ε 0) -> (b ε n)) AndInt 87 96
98. (b ε n) <-> (b ε 0) EquivConst 97
99. ∀b.((b ε n) <-> (b ε 0)) ForallInt 98
100. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y))) AxInt
101. ∀y.((n = y) <-> ∀z.((z ε n) <-> (z ε y))) ForallElim 100
102. (n = 0) <-> ∀z.((z ε n) <-> (z ε 0)) ForallElim 101
103. ((n = 0) -> ∀z.((z ε n) <-> (z ε 0))) & (∀z.((z ε n) <-> (z ε 0)) -> (n = 0)) EquivExp 102
104. ∀z.((z ε n) <-> (z ε 0)) -> (n = 0) AndElimR 103
105. n = 0 ImpElim 99 104
106. _|_ ImpElim 105 78
107. ¬¬∃w.(w ε n) ImpInt 106
108. D <-> ¬¬D TheoremInt
109. (D -> ¬¬D) & (¬¬D -> D) EquivExp 108
110. ¬¬D -> D AndElimR 109
111. ¬¬∃w.(w ε n) -> ∃w.(w ε n) PolySub 110
112. ∃w.(w ε n) ImpElim 107 111
113. WellOrders(r,x) ImpElim 112 77
114. (∪n ⊂ x) & WellOrders(r,x) AndInt 23 113
115. ((∪n ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε ∪n)) & ((u,v) ε r)) -> (u ε ∪n)) AndInt 114 67
116. Section(r,x,∪n) DefSub 115
117. z ε ∩n Hyp
118. ∩x = {z: ∀y.((y ε x) -> (z ε y))} DefEqInt
119. ∀x.(∩x = {z: ∀y.((y ε x) -> (z ε y))}) ForallInt 118
120. ∩n = {z: ∀y.((y ε n) -> (z ε y))} ForallElim 119
121. z ε {z: ∀y.((y ε n) -> (z ε y))} EqualitySub 117 120
122. Set(z) & ∀y.((y ε n) -> (z ε y)) ClassElim 121
123. ∀y.((y ε n) -> (z ε y)) AndElimR 122
124. m ε n Hyp
125. (m ε n) -> (z ε m) ForallElim 123
126. z ε m ImpElim 124 125
127. (m ε n) -> Section(r,x,m) ForallElim 7
128. Section(r,x,m) ImpElim 124 127
129. ((m ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m)) DefExp 128
130. (m ⊂ x) & WellOrders(r,x) AndElimL 129
131. m ⊂ x AndElimL 130
132. ∀z.((z ε m) -> (z ε x)) DefExp 131
133. (z ε m) -> (z ε x) ForallElim 132
134. z ε x ImpElim 126 133
135. (z ε ∩n) -> (z ε x) ImpInt 134
136. (z ε ∩n) -> (z ε x) ExistsElim 112 124 135
137. ∀z.((z ε ∩n) -> (z ε x)) ForallInt 136
138. ∩n ⊂ x DefSub 137
139. (∩n ⊂ x) & WellOrders(r,x) AndInt 138 113
140. ((u ε x) & (v ε ∩n)) & ((u,v) ε r) Hyp
141. (u ε x) & (v ε ∩n) AndElimL 140
142. v ε ∩n AndElimR 141
143. v ε {z: ∀y.((y ε n) -> (z ε y))} EqualitySub 142 120
144. Set(v) & ∀y.((y ε n) -> (v ε y)) ClassElim 143
145. ∀y.((y ε n) -> (v ε y)) AndElimR 144
146. (m ε n) -> (v ε m) ForallElim 145
147. v ε m ImpElim 124 146
148. ∀u.∀v.((((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m)) AndElimR 129
149. ∀v.((((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m)) ForallElim 148
150. (((u ε x) & (v ε m)) & ((u,v) ε r)) -> (u ε m) ForallElim 149
151. (u,v) ε r AndElimR 140
152. (u ε x) & (v ε ∩n) AndElimL 140
153. u ε x AndElimL 152
154. (u ε x) & (v ε m) AndInt 153 147
155. ((u ε x) & (v ε m)) & ((u,v) ε r) AndInt 154 151
156. u ε m ImpElim 155 150
157. (m ε n) -> (u ε m) ImpInt 156
158. ∀m.((m ε n) -> (u ε m)) ForallInt 157
159. ∃w.(u ε w) ExistsInt 153

160. Set(u) DefSub 159
161. Set(u) & ∀m.((m ε n) -> (u ε m)) AndInt 160 158
162. u ε {w: ∀m.((m ε n) -> (w ε m))} ClassInt 161
163. {z: ∀y.((y ε n) -> (z ε y))} = ∩n Symmetry 120
164. u ε ∩n EqualitySub 162 163
165. (((u ε x) & (v ε ∩n)) & ((u,v) ε r)) -> (u ε ∩n) ImpInt 164
166. ∀v.((((u ε x) & (v ε ∩n)) & ((u,v) ε r)) -> (u ε ∩n)) ForallInt 165
167. ∀u.∀v.((((u ε x) & (v ε ∩n)) & ((u,v) ε r)) -> (u ε ∩n)) ForallInt 166
168. ((∩n ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε ∩n)) & ((u,v) ε r)) -> (u ε ∩n)) AndInt 139 167
169. Section(r,x,∩n) DefSub 168
170. Section(r,x,∪n) & Section(r,x,∩n) AndInt 116 169
171. (¬(n = 0) & ∀y.((y ε n) -> Section(r,x,y))) -> (Section(r,x,∪n) & Section(r,x,∩n)) ImpInt 170 Qed

Used Theorems

2. ¬∃i.P(i) -> ∀j.¬P(j)
3. D <-> ¬¬D

Th91. (Section(r,x,y) & ¬(y = x)) -> ∃v.((v ε x) & (y = {u: ((u ε x) & ((u,v) ε r))}))

0. Section(r,x,y) & ¬(y = x) Hyp
1. Section(r,x,y) AndElimL 0
2. ¬(y = x) AndElimR 0
3. ((y ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε y)) & ((u,v) ε r)) -> (u ε y)) DefExp 1
4. (y ⊂ x) & WellOrders(r,x) AndElimL 3
5. y ⊂ x AndElimL 4
6. (x ~ y) = (x ∩ ~y) DefEqInt
7. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
8. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y)) EquivExp 7
9. ((x ⊂ y) & (y ⊂ x)) -> (x = y) AndElimR 8
10. (A -> B) -> (¬B -> ¬A) TheoremInt
11. (((x ⊂ y) & (y ⊂ x)) -> B) -> (¬B -> ¬((x ⊂ y) & (y ⊂ x))) PolySub 10
12. (((x ⊂ y) & (y ⊂ x)) -> (x = y)) -> (¬(x = y) -> ¬((x ⊂ y) & (y ⊂ x))) PolySub 11
13. ¬(x = y) -> ¬((x ⊂ y) & (y ⊂ x)) ImpElim 9 12
14. ∀y.(¬(x = y) -> ¬((x ⊂ y) & (y ⊂ x))) ForallInt 13
15. ¬(x = a) -> ¬((x ⊂ a) & (a ⊂ x)) ForallElim 14
16. ∀x.(¬(x = a) -> ¬((x ⊂ a) & (a ⊂ x))) ForallInt 15
17. ¬(y = a) -> ¬((y ⊂ a) & (a ⊂ y)) ForallElim 16
18. ∀a.(¬(y = a) -> ¬((y ⊂ a) & (a ⊂ y))) ForallInt 17
19. ¬(y = x) -> ¬((y ⊂ x) & (x ⊂ y)) ForallElim 18
20. ¬((y ⊂ x) & (x ⊂ y)) ImpElim 2 19
21. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B)) TheoremInt
22. ¬(A & B) <-> (¬A v ¬B) AndElimR 21
23. ¬((y ⊂ x) & B) <-> (¬(y ⊂ x) v ¬B) PolySub 22
24. ¬((y ⊂ x) & (x ⊂ y)) <-> (¬(y ⊂ x) v ¬(x ⊂ y)) PolySub 23
25. (¬((y ⊂ x) & (x ⊂ y)) -> (¬(y ⊂ x) v ¬(x ⊂ y))) & ((¬(y ⊂ x) v ¬(x ⊂ y)) -> ¬((y ⊂ x) & (x ⊂ y))) EquivExp 24
26. ¬((y ⊂ x) & (x ⊂ y)) -> (¬(y ⊂ x) v ¬(x ⊂ y)) AndElimL 25
27. ¬(y ⊂ x) v ¬(x ⊂ y) ImpElim 20 26
28. ¬(y ⊂ x) Hyp
29. _|_ ImpElim 5 28
30. ¬(x ⊂ y) AbsI 29
31. ¬(x ⊂ y) Hyp
32. ¬(x ⊂ y) OrElim 27 28 30 31 31
33. ¬∀z.((z ε x) -> (z ε y)) DefExp 32
34. ¬∀i.P(i) -> ∃c.¬P(c) TheoremInt
35. ¬∀i.((i ε x) -> (i ε y)) -> ∃c.¬((c ε x) -> (c ε y)) PredSub 34
36. ∃c.¬((c ε x) -> (c ε y)) ImpElim 33 35
37. (A -> B) -> (¬B -> ¬A) TheoremInt
38. (C -> B) -> (¬B -> ¬C) PolySub 37
39. (C -> D) -> (¬D -> ¬C) PolySub 38
40. ((B v ¬A) -> D) -> (¬D -> ¬(B v ¬A)) PolySub 39
41. ((B v ¬A) -> (A -> B)) -> (¬(A -> B) -> ¬(B v ¬A)) PolySub 40
42. (B v ¬A) -> (A -> B) TheoremInt
43. ¬(A -> B) -> ¬(B v ¬A) ImpElim 42 41
44. ¬((c ε x) -> (c ε y)) Hyp
45. ¬((c ε x) -> B) -> ¬(B v ¬(c ε x)) PolySub 43
46. ¬((c ε x) -> (c ε y)) -> ¬((c ε y) v ¬(c ε x)) PolySub 45

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47. ¬((c ε y) v ¬(c ε x))  ImpElim 44 46
48. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
49. ¬(A v B) <-> (¬A & ¬B)  AndElimL 48
50. (¬(A v B) -> (¬A & ¬B)) & ((¬A & ¬B) -> ¬(A v B))  EquivExp 49
51. ¬(A v B) -> (¬A & ¬B)  AndElimL 50
52. ¬((c ε y) v B) -> (¬(c ε y) & ¬B)  PolySub 51
53. ¬((c ε y) v ¬(c ε x)) -> (¬(c ε y) & ¬¬(c ε x))  PolySub 52
54. ¬(c ε y) & ¬¬(c ε x)  ImpElim 47 53
55. ¬(c ε y)  AndElimL 54
56. ¬¬(c ε x)  AndElimR 54
57. D <-> ¬¬D  TheoremInt
58. (D -> ¬¬D) & (¬¬D -> D)  EquivExp 57
59. ¬¬D -> D  AndElimR 58
60. ¬¬(c ε x) -> (c ε x)  PolySub 59
61. c ε x  ImpElim 56 60
62. ~x = {y: ¬(y ε x)}  DefEqInt
63. ∃w.(c ε w)  ExistsInt 61
64. Set(c)  DefSub 63
65. Set(c) & ¬(c ε y)  AndInt 64 55
66. c ε {w: ¬(w ε y)}  ClassInt 65
67. {y: ¬(y ε x)} = ~x  Symmetry 62
68. c ε {w: ¬(w ε y)}  EqualitySub 66 67
69. ∀x.({y: ¬(y ε x)} = ~x)  ForallInt 67
70. {x_14: ¬(x_14 ε y)} = ~y  ForallElim 69
71. c ε ~y  EqualitySub 66 70
72. (c ε x) & (c ε ~y)  AndInt 61 71
73. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
74. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 73
75. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 74
76. ((z ε x) & (z ε y)) -> (z ε (x ∩ y))  AndElimR 75
77. ∀z.(((z ε x) & (z ε y)) -> (z ε (x ∩ y)))  ForallInt 76
78. ((c ε x) & (c ε y)) -> (c ε (x ∩ y))  ForallElim 77
79. ∀y.(((c ε x) & (c ε y)) -> (c ε (x ∩ y)))  ForallInt 78
80. ((c ε x) & (c ε ~y)) -> (c ε (x ∩ ~y))  ForallElim 79
81. c ε (x ∩ ~y)  ImpElim 72 80
82. (x ∩ ~y) = (x ~ y)  Symmetry 6
83. c ε (x ~ y)  EqualitySub 81 82
84. (x ~ y) = 0  Hyp
85. c ε 0  EqualitySub 83 84
86. ¬(x ε 0)  TheoremInt
87. ∀x.¬(x ε 0)  ForallInt 86
88. ¬(c ε 0)  ForallElim 87
89. _|_  ImpElim 85 88
90. ¬((x ~ y) = 0)  ImpInt 89
91. ¬((x ~ y) = 0)  ExistsElim 36 44 90
92. ((y ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε y)) & ((u,v) ε r)) -> (u ε y))
DefExp 1
93. (y ⊂ x) & WellOrders(r,x)  AndElimL 92
94. WellOrders(r,x)  AndElimR 93
95. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  DefExp 94
96. ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  AndElimR 95
97. (((x ~ y) ⊂ x) & ¬((x ~ y) = 0)) -> ∃z.First(r,(x ~ y),z)  ForallElim 96
98. (x ~ y) = (x ∩ ~y)  Symmetry 82
99. z ε (x ~ y)  Hyp
100. z ε (x ∩ ~y)  EqualitySub 99 98
101. (z ε (x ∩ y)) -> ((z ε x) & (z ε y))  AndElimL 75
102. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y)))  ForallInt 101
103. (z ε (x ∩ ~y)) -> ((z ε x) & (z ε ~y))  ForallElim 102
104. (z ε x) & (z ε ~y)  ImpElim 100 103
105. z ε x  AndElimL 104
106. (z ε (x ~ y)) -> (z ε x)  ImpInt 105
107. ∀z.((z ε (x ~ y)) -> (z ε x))  ForallInt 106
108. (x ~ y) ⊂ x  DefSub 107
109. ((x ~ y) ⊂ x) & ¬((x ~ y) = 0)  AndInt 108 91
110. ∃z.First(r,(x ~ y),z)  ImpElim 109 97
111. First(r,(x ~ y),v)  Hyp
112. (v ε (x ~ y)) & ∀x_25.((x_25 ε (x ~ y)) -> ¬((x_25,v) ε r))  DefExp 111
113. z ε {u: ((u ε x) & ((u,v) ε r))}  Hyp
114. Set(z) & ((z ε x) & ((z,v) ε r))  ClassElim 113
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115. ∀x_25.((x_25 ε (x ~ y)) -> ¬((x_25,v) ε r))  AndElimR 112
116. (z ε (x ~ y)) -> ¬((z,v) ε r)  ForallElim 115
117. (z ε x) & ((z,v) ε r)  AndElimR 114
118. (z,v) ε r  AndElimR 117
119. v ε (x ~ y)  AndElimL 112
120. (A -> B) -> (¬B -> ¬A)  TheoremInt
121. ((z ε (x ~ y)) -> B) -> (¬B -> ¬(z ε (x ~ y)))  PolySub 120
122. ((z ε (x ~ y)) -> ¬((z,v) ε r)) -> (¬¬((z,v) ε r) -> ¬(z ε (x ~ y)))  PolySub 121
123. ¬¬((z,v) ε r) -> ¬(z ε (x ~ y))  ImpElim 116 122
124. D <-> ¬¬D  TheoremInt
125. (D -> ¬¬D) & (¬¬D -> D)  EquivExp 124
126. D -> ¬¬D  AndElimL 125
127. ((z,v) ε r) -> ¬¬((z,v) ε r)  PolySub 126
128. ¬¬((z,v) ε r)  ImpElim 118 127
129. ¬(z ε (x ~ y))  ImpElim 128 123
130. ¬(z ε (x ∩ ~y))  EqualitySub 129 98
131. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
132. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y))  AndElimR 131
133. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 132
134. ((z ε x) & (z ε y)) -> (z ε (x ∩ y))  AndElimR 133
135. ∀y.(((z ε x) & (z ε y)) -> (z ε (x ∩ y)))  ForallInt 134
136. ((z ε x) & (z ε ~y)) -> (z ε (x ∩ ~y))  ForallElim 135
137. (((z ε x) & (z ε ~y)) -> B) -> (¬B -> ¬((z ε x) & (z ε ~y)))  PolySub 120
138. (((z ε x) & (z ε ~y)) -> (z ε (x ∩ ~y))) -> (¬(z ε (x ∩ ~y)) -> ¬((z ε x) & (z ε ~y)))  PolySub 137
139. ¬(z ε (x ∩ ~y)) -> ¬((z ε x) & (z ε ~y))  ImpElim 136 138
140. ¬((z ε x) & (z ε ~y))  ImpElim 130 139
141. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))  TheoremInt
142. ¬(A & B) <-> (¬A v ¬B)  AndElimR 141
143. ¬((z ε x) & B) <-> (¬(z ε x) v ¬B)  PolySub 142
144. ¬((z ε x) & (z ε ~y)) <-> (¬(z ε x) v ¬(z ε ~y))  PolySub 143
145. (¬((z ε x) & (z ε ~y)) -> (¬(z ε x) v ¬(z ε ~y))) & ((¬(z ε x) v ¬(z ε ~y)) -> ¬((z ε x) & (z ε ~y)))  EquivExp 144
146. ¬((z ε x) & (z ε ~y)) -> (¬(z ε x) v ¬(z ε ~y))  AndElimL 145
147. ¬(z ε x) v ¬(z ε ~y)  ImpElim 140 146
148. ¬(z ε x)  Hyp
149. z ε x  AndElimL 117
150. _|_  ImpElim 149 148
151. ¬(z ε ~y)  AbsI 150
152. ¬(z ε ~y)  Hyp
153. ¬(z ε ~y)  OrElim 147 148 151 152 152
154. Set(z)  AndElimL 114
155. Set(z) & ¬(z ε ~y)  AndInt 154 153
156. z ε {w: ¬(w ε ~y)}  ClassInt 155
157. ~x = {y: ¬(y ε x)}  DefEqInt
158. ∀x.(~x = {y: ¬(y ε x)})  ForallInt 157
159. ~~y = {x_26: ¬(x_26 ε ~y)}  ForallElim 158
160. {x_26: ¬(x_26 ε ~y)} = ~~y  Symmetry 159
161. z ε ~~y  EqualitySub 156 160
162. ~~x = x  TheoremInt
163. ∀x.(~~x = x)  ForallInt 162
164. ~~y = y  ForallElim 163
165. z ε y  EqualitySub 161 164
166. (z ε {u: ((u ε x) & ((u,v) ε r))}) -> (z ε y)  ImpInt 165
167. z ε y  Hyp
168. ∀z.((z ε y) -> (z ε x))  DefExp 5
169. (z ε y) -> (z ε x)  ForallElim 168
170. z ε x  ImpElim 167 169
171. x = x  Identity
172. x = x  Identity
173. x = x  Identity
174. x = x  Identity
175. x = x  Identity
176. x = x  Identity
177. ∀z.((z ε (x ∩ ~y)) -> ((z ε x) & (z ε ~y)))  ForallInt 103
178. (v ε (x ∩ ~y)) -> ((v ε x) & (v ε ~y))  ForallElim 177
179. v ε (x ~ y)  AndElimL 112
180. v ε (x ∩ ~y)  EqualitySub 179 6
181. (v ε x) & (v ε ~y)  ImpElim 180 178
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182. v ε ~y  AndElimR 181
183. (v,z) ε r  Hyp
184. ∀u.∀v.((((u ε x) & (v ε y)) & ((u,v) ε r)) -> (u ε y))  AndElimR 3
185. ∀x_29.((((v ε x) & (x_29 ε y)) & ((v,x_29) ε r)) -> (v ε y))  ForallElim 184
186. (((v ε x) & (z ε y)) & ((v,z) ε r)) -> (v ε y)  ForallElim 185
187. v ε x  AndElimL 181
188. (v ε x) & (z ε y)  AndInt 187 167
189. ((v ε x) & (z ε y)) & ((v,z) ε r)  AndInt 188 183
190. v ε y  ImpElim 189 186
191. ~x = {y: ¬(y ε x)}  DefEqInt
192. ∀x.(~x = {y: ¬(y ε x)})  ForallInt 191
193. ~y = {x_30: ¬(x_30 ε y)}  ForallElim 192
194. v ε {x_30: ¬(x_30 ε y)}  EqualitySub 182 193
195. Set(v) & ¬(v ε y)  ClassElim 194
196. ¬(v ε y)  AndElimR 195
197. _|_  ImpElim 190 196
198. ¬((v,z) ε r)  ImpInt 197
199. WellOrders(r,x)  AndElimR 4
200. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  TheoremInt
201. Asymmetric(r,x) & TransIn(r,x)  ImpElim 199 200
202. Asymmetric(r,x)  AndElimL 201
203. ∀y.∀z.(((y ε x) & (z ε x)) -> (((y,z) ε r) -> ¬((z,y) ε r)))  DefExp 202
204. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z))  DefExp 199
205. Connects(r,x)  AndElimL 204
206. ∀y.∀z.(((y ε x) & (z ε x)) -> ((y = z) v (((y,z) ε r) v ((z,y) ε r))))  DefExp 205
207. ∀z.(((v ε x) & (z ε x)) -> ((v = z) v (((v,z) ε r) v ((z,v) ε r))))  ForallElim 206
208. ((v ε x) & (z ε x)) -> ((v = z) v (((v,z) ε r) v ((z,v) ε r)))  ForallElim 207
209. (v ε x) & (z ε x)  AndInt 187 170
210. (v = z) v (((v,z) ε r) v ((z,v) ε r))  ImpElim 209 208
211. ∀z.(((v ε x) & (z ε x)) -> (((v,z) ε r) -> ¬((z,v) ε r)))  ForallElim 203
212. ((v ε x) & (z ε x)) -> (((v,z) ε r) -> ¬((z,v) ε r))  ForallElim 211
213. ((v,z) ε r) -> ¬((z,v) ε r)  ImpElim 209 212
214. v = z  Hyp
215. ¬(z ε y)  EqualitySub 196 214
216. _|_  ImpElim 167 215
217. (z,v) ε r  AbsI 216
218. ((v,z) ε r) v ((z,v) ε r)  Hyp
219. (v,z) ε r  Hyp
220. _|_  ImpElim 219 198
221. (z,v) ε r  AbsI 220
222. (z,v) ε r  Hyp
223. (z,v) ε r  OrElim 218 219 221 222 222
224. (z,v) ε r  OrElim 210 214 217 218 223
225. (z ε x) & ((z,v) ε r)  AndInt 170 224
226. ∃w.(z ε w)  ExistsInt 167
227. Set(z)  DefSub 226
228. Set(z) & ((z ε x) & ((z,v) ε r))  AndInt 227 225
229. z ε {w: ((w ε x) & ((w,v) ε r))}  ClassInt 228
230. (z ε y) -> (z ε {w: ((w ε x) & ((w,v) ε r))})  ImpInt 229
231. ((z ε y) -> (z ε {w: ((w ε x) & ((w,v) ε r))})) & ((z ε {u: ((u ε x) & ((u,v) ε
r))}) -> (z ε y))  AndInt 230 166
232. (z ε y) <-> (z ε {w: ((w ε x) & ((w,v) ε r))})  EquivConst 231
233. ∀x.∀y.((x = y) <-> ∀z.((z ε x) <-> (z ε y)))  AxInt
234. ∀x_38.((y = x_38) <-> ∀z.((z ε y) <-> (z ε x_38)))  ForallElim 233
235. (y = {u: ((u ε x) & ((u,v) ε r))}) <-> ∀z.((z ε y) <-> (z ε {u: ((u ε x) & ((u,v) ε
r))}))  ForallElim 234
236. ((y = {u: ((u ε x) & ((u,v) ε r))}) -> ∀z.((z ε y) <-> (z ε {u: ((u ε x) & ((u,v) ε
r))}))) & (∀z.((z ε y) <-> (z ε {u: ((u ε x) & ((u,v) ε r))})) -> (y = {u: ((u ε x) &
((u,v) ε r))}))  EquivExp 235
237. ∀z.((z ε y) <-> (z ε {u: ((u ε x) & ((u,v) ε r))})) -> (y = {u: ((u ε x) & ((u,v) ε
r))})  AndElimR 236
238. ∀z.((z ε y) <-> (z ε {w: ((w ε x) & ((w,v) ε r))}))  ForallInt 232
239. y = {u: ((u ε x) & ((u,v) ε r))}  ImpElim 238 237
240. (v ε x) & (y = {u: ((u ε x) & ((u,v) ε r))})  AndInt 187 239
241. ∃v.((v ε x) & (y = {u: ((u ε x) & ((u,v) ε r))}))  ExistsInt 240
242. ∃v.((v ε x) & (y = {u: ((u ε x) & ((u,v) ε r))}))  ExistsElim 110 111 241
243. (Section(r,x,y) & ¬(y = x)) -> ∃v.((v ε x) & (y = {u: ((u ε x) & ((u,v) ε r))}))
ImpInt 242 Qed
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Used Theorems

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1. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
2. (A -> B) -> (¬B -> ¬A)
4. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
3. ¬∀i.P(i) -> ∃c.¬P(c)
5. (B v ¬A) -> (A -> B)
6. D <-> ¬¬D
7. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
8. ¬(x ε 0)
9. ~~x = x
10. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))

Th92. (Section(r,z,a) & Section(r,z,b)) -> ((a ⊂ b) v (b ⊂ a))

0. Section(r,z,a) & Section(r,z,b)  Hyp
1. (Section(r,x,y) & ¬(y = x)) -> ∃v.((v ε x) & (y = {u: ((u ε x) & ((u,v) ε r))}))
TheoremInt
2. ∀x.((Section(r,x,y) & ¬(y = x)) -> ∃v.((v ε x) & (y = {u: ((u ε x) & ((u,v) ε r))})))
ForallInt 1
3. (Section(r,z,y) & ¬(y = z)) -> ∃v.((v ε z) & (y = {u: ((u ε z) & ((u,v) ε r))}))
ForallElim 2
4. ∀y.((Section(r,z,y) & ¬(y = z)) -> ∃v.((v ε z) & (y = {u: ((u ε z) & ((u,v) ε r))})))
ForallInt 3
5. (Section(r,z,a) & ¬(a = z)) -> ∃v.((v ε z) & (a = {u: ((u ε z) & ((u,v) ε r))}))
ForallElim 4
6. ∀y.((Section(r,z,y) & ¬(y = z)) -> ∃v.((v ε z) & (y = {u: ((u ε z) & ((u,v) ε r))})))
ForallInt 3
7. (Section(r,z,b) & ¬(b = z)) -> ∃v.((v ε z) & (b = {u: ((u ε z) & ((u,v) ε r))}))
ForallElim 6
8. ¬(a = z)  Hyp
9. ¬(b = z)  Hyp
10. Section(r,z,a)  AndElimL 0
11. Section(r,z,b)  AndElimR 0
12. Section(r,z,a) & ¬(a = z)  AndInt 10 8
13. Section(r,z,b) & ¬(b = z)  AndInt 11 9
14. ∃v.((v ε z) & (a = {u: ((u ε z) & ((u,v) ε r))}))  ImpElim 12 5
15. ∃v.((v ε z) & (b = {u: ((u ε z) & ((u,v) ε r))}))  ImpElim 13 7
16. (u ε z) & (a = {x_1: ((x_1 ε z) & ((x_1,u) ε r))})  Hyp
17. (v ε z) & (b = {u: ((u ε z) & ((u,v) ε r))})  Hyp
18. ((a ⊂ z) & WellOrders(r,z)) & ∀u.∀v.((((u ε z) & (v ε a)) & ((u,v) ε r)) -> (u ε a))
DefExp 10
19. (a ⊂ z) & WellOrders(r,z)  AndElimL 18
20. WellOrders(r,z)  AndElimR 19
21. Connects(r,z) & ∀y.(((y ⊂ z) & ¬(y = 0)) -> ∃x_11.First(r,y,x_11))  DefExp 20
22. Connects(r,z)  AndElimL 21
23. ∀y.∀x_14.(((y ε z) & (x_14 ε z)) -> ((y = x_14) v (((y,x_14) ε r) v ((x_14,y) ε r))))
DefExp 22
24. ∀x_14.(((u ε z) & (x_14 ε z)) -> ((u = x_14) v (((u,x_14) ε r) v ((x_14,u) ε r))))
ForallElim 23
25. ((u ε z) & (v ε z)) -> ((u = v) v (((u,v) ε r) v ((v,u) ε r)))  ForallElim 24
26. u ε z  AndElimL 16
27. v ε z  AndElimL 17
28. (u ε z) & (v ε z)  AndInt 26 27
29. (u = v) v (((u,v) ε r) v ((v,u) ε r))  ImpElim 28 25
30. u = v  Hyp
31. a = {x_1: ((x_1 ε z) & ((x_1,u) ε r))}  AndElimR 16
32. b = {u: ((u ε z) & ((u,v) ε r))}  AndElimR 17
33. a = {x_1: ((x_1 ε z) & ((x_1,v) ε r))}  EqualitySub 31 30
34. {x_1: ((x_1 ε z) & ((x_1,v) ε r))} = a  Symmetry 33
35. b = a  EqualitySub 32 34
36. a = b  Symmetry 35
37. (x = y) <-> ((x ⊂ y) & (y ⊂ x))  TheoremInt
38. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y))  EquivExp 37
39. (x = y) -> ((x ⊂ y) & (y ⊂ x))  AndElimL 38
40. ∀x.((x = y) -> ((x ⊂ y) & (y ⊂ x)))  ForallInt 39
41. (a = y) -> ((a ⊂ y) & (y ⊂ a))  ForallElim 40
42. ∀y.((a = y) -> ((a ⊂ y) & (y ⊂ a)))  ForallInt 41
43. (a = b) -> ((a ⊂ b) & (b ⊂ a))  ForallElim 42
44. (a ⊂ b) & (b ⊂ a)  ImpElim 36 43
45. a ⊂ b  AndElimL 44
46. (a ⊂ b) v (b ⊂ a)  OrIntR 45
47. ((u,v) ε r) v ((v,u) ε r)  Hyp
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48. (u,v) ε r  Hyp
49. x ε a  Hyp
50. x ε {x_1: ((x_1 ε z) & ((x_1,u) ε r))}  EqualitySub 49 31
51. Set(x) & ((x ε z) & ((x,u) ε r))  ClassElim 50
52. (x ε z) & ((x,u) ε r)  AndElimR 51
53. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))  TheoremInt
54. ∀x.(WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x)))  ForallInt 53
55. WellOrders(r,z) -> (Asymmetric(r,z) & TransIn(r,z))  ForallElim 54
56. Asymmetric(r,z) & TransIn(r,z)  ImpElim 20 55
57. TransIn(r,z)  AndElimR 56
58. ∀u.∀v.∀w.(((u ε z) & ((v ε z) & (w ε z))) -> ((((u,v) ε r) & ((v,w) ε r)) -> ((u,w) ε r)))  DefExp 57
59. x ε z  AndElimL 52
60. ∀v.∀w.(((x ε z) & ((v ε z) & (w ε z))) -> ((((x,v) ε r) & ((v,w) ε r)) -> ((x,w) ε r)))  ForallElim 58
61. ∀w.(((x ε z) & ((u ε z) & (w ε z))) -> ((((x,u) ε r) & ((u,w) ε r)) -> ((x,w) ε r)))  ForallElim 60
62. ((x ε z) & ((u ε z) & (v ε z))) -> ((((x,u) ε r) & ((u,v) ε r)) -> ((x,v) ε r))  ForallElim 61
63. (u ε z) & (v ε z)  AndInt 26 27
64. (x ε z) & ((u ε z) & (v ε z))  AndInt 59 63
65. (((x,u) ε r) & ((u,v) ε r)) -> ((x,v) ε r)  ImpElim 64 62
66. (x,u) ε r  AndElimR 52
67. ((x,u) ε r) & ((u,v) ε r)  AndInt 66 48
68. (x,v) ε r  ImpElim 67 65
69. (x ε z) & ((x,v) ε r)  AndInt 59 68
70. ∃w.(x ε w)  ExistsInt 49
71. Set(x)  DefSub 70
72. Set(x) & ((x ε z) & ((x,v) ε r))  AndInt 71 69
73. x ε {w: ((w ε z) & ((w,v) ε r))}  ClassInt 72
74. {u: ((u ε z) & ((u,v) ε r))} = b  Symmetry 32
75. x ε b  EqualitySub 73 74
76. (x ε a) -> (x ε b)  ImpInt 75
77. ∀x.((x ε a) -> (x ε b))  ForallInt 76
78. a ⊂ b  DefSub 77
79. (a ⊂ b) v (b ⊂ a)  OrIntR 78
80. (v,u) ε r  Hyp
81. x ε b  Hyp
82. x ε {u: ((u ε z) & ((u,v) ε r))}  EqualitySub 81 32
83. Set(x) & ((x ε z) & ((x,v) ε r))  ClassElim 82
84. (x ε z) & ((x,v) ε r)  AndElimR 83
85. (x,v) ε r  AndElimR 84
86. ∀w.(((x ε z) & ((v ε z) & (w ε z))) -> ((((x,v) ε r) & ((v,w) ε r)) -> ((x,w) ε r)))  ForallElim 60
87. ((x ε z) & ((v ε z) & (u ε z))) -> ((((x,v) ε r) & ((v,u) ε r)) -> ((x,u) ε r))  ForallElim 86
88. (v ε z) & (u ε z)  AndInt 27 26
89. x ε z  AndElimL 84
90. (x ε z) & ((v ε z) & (u ε z))  AndInt 89 88
91. (((x,v) ε r) & ((v,u) ε r)) -> ((x,u) ε r)  ImpElim 90 87
92. ((x,v) ε r) & ((v,u) ε r)  AndInt 85 80
93. (x,u) ε r  ImpElim 92 91
94. (x ε z) & ((x,u) ε r)  AndInt 89 93
95. ∃w.(x ε w)  ExistsInt 81
96. Set(x)  DefSub 95
97. Set(x) & ((x ε z) & ((x,u) ε r))  AndInt 96 94
98. x ε {w: ((w ε z) & ((w,u) ε r))}  ClassInt 97
99. {x_1: ((x_1 ε z) & ((x_1,u) ε r))} = a  Symmetry 31
100. x ε a  EqualitySub 98 99
101. (x ε b) -> (x ε a)  ImpInt 100
102. ∀x.((x ε b) -> (x ε a))  ForallInt 101
103. b ⊂ a  DefSub 102
104. (a ⊂ b) v (b ⊂ a)  OrIntL 103
105. (a ⊂ b) v (b ⊂ a)  OrElim 47 48 79 80 104
106. (a ⊂ b) v (b ⊂ a)  OrElim 29 30 46 47 105
107. (a ⊂ b) v (b ⊂ a)  ExistsElim 15 17 106
108. (a ⊂ b) v (b ⊂ a)  ExistsElim 14 16 107
109. b = z  Hyp
110. z = b  Symmetry 109
111. ((a ⊂ b) & WellOrders(r,b)) & ∀u.∀v.((((u ε b) & (v ε a)) & ((u,v) ε r)) -> (u ε a))  EqualitySub 18 110
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112. (a ⊂ b) & WellOrders(r,b) AndElimL 111
113. a ⊂ b AndElimL 112
114. (a ⊂ b) v (b ⊂ a) OrIntR 113
115. A v ¬A TheoremInt
116. (b = z) v ¬(b = z) PolySub 115
117. (a ⊂ b) v (b ⊂ a) OrElim 116 109 114 9 108
118. a = z Hyp
119. z = a Symmetry 118
120. ((b ⊂ z) & WellOrders(r,z)) & ∀u.∀v.((((u ε z) & (v ε b)) & ((u,v) ε r)) -> (u ε b)) DefExp 11
121. (b ⊂ z) & WellOrders(r,z) AndElimL 120
122. b ⊂ z AndElimL 121
123. b ⊂ a EqualitySub 122 119
124. (a ⊂ b) v (b ⊂ a) OrIntL 123
125. (a = z) v ¬(a = z) PolySub 115
126. (a ⊂ b) v (b ⊂ a) OrElim 125 118 124 8 117
127. (Section(r,z,a) & Section(r,z,b)) -> ((a ⊂ b) v (b ⊂ a)) ImpInt 126 Qed

Used Theorems

1. (Section(r,x,y) & ¬(y = x)) -> ∃v.((v ε x) & (y = {u: ((u ε x) & ((u,v) ε r))}))
2. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
3. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
0. A v ¬A

FunctionApp. ((f ε func(x,y)) & (a ε x)) -> ((f'a) ε y)

0. (f ε func(x,y)) & (a ε x) Hyp
1. f ε func(x,y) AndElimL 0
2. func(x,y) = {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))} DefEqInt
3. f ε {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))} EqualitySub 1 2
4. Set(f) & (Function(f) & ((domain(f) = x) & (range(f) = y))) ClassElim 3
5. Function(f) & ((domain(f) = x) & (range(f) = y)) AndElimR 4
6. u = (a,(f'a)) Hyp
7. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
8. Function(f) AndElimL 5
9. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 8 7
10. (f'a) = (f'a) Identity
11. (u = (a,(f'a))) & ((f'a) = (f'a)) AndInt 6 10
12. ∃w.((u = (a,w)) & ((f'a) = w)) ExistsInt 11
13. ∃b.∃w.((u = (b,w)) & ((f'b) = w)) ExistsInt 12
14. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U)) TheoremInt
15. (z ε domain(f)) -> ((f'z) ε U) AndElimR 14
16. ∀z.((z ε domain(f)) -> ((f'z) ε U)) ForallInt 15
17. (a ε domain(f)) -> ((f'a) ε U) ForallElim 16
18. a ε x AndElimR 0
19. (domain(f) = x) & (range(f) = y) AndElimR 5
20. domain(f) = x AndElimL 19
21. x = domain(f) Symmetry 20
22. a ε domain(f) EqualitySub 18 21
23. (f'a) ε U ImpElim 22 17
24. ∃w.((f'a) ε w) ExistsInt 23
25. Set((f'a)) DefSub 24
26. ∃w.(a ε w) ExistsInt 18
27. Set(a) DefSub 26
28. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
29. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 28
30. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 29
31. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 30
32. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 31
33. (Set(a) & Set(y)) -> Set((a,y)) ForallElim 32
34. ∀y.((Set(a) & Set(y)) -> Set((a,y))) ForallInt 33
35. (Set(a) & Set((f'a))) -> Set((a,(f'a))) ForallElim 34
36. Set(a) & Set((f'a)) AndInt 27 25
37. Set((a,(f'a))) ImpElim 36 35
38. (a,(f'a)) = u Symmetry 6
39. Set(u) EqualitySub 37 38
40. Set(u) & ∃b.∃w.((u = (b,w)) & ((f'b) = w)) AndInt 39 13
41. u ε {w: ∃b.∃j.((w = (b,j)) & ((f'b) = j))} ClassInt 40
42. {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} = f Symmetry 9
43. u ε f EqualitySub 41 42

44. (a,(f'a)) ε f EqualitySub 43 6
45. (u = (a,(f'a))) -> ((a,(f'a)) ε f) ImpInt 44
46. ∀u.((u = (a,(f'a))) -> ((a,(f'a)) ε f)) ForallInt 45
47. ((a,(f'a)) = (a,(f'a))) -> ((a,(f'a)) ε f) ForallElim 46
48. (a,(f'a)) = (a,(f'a)) Identity
49. (a,(f'a)) ε f ImpElim 48 47
50. ∃u.((u,(f'a)) ε f) ExistsInt 49
51. Set((f'a)) & ∃u.((u,(f'a)) ε f) AndInt 25 50
52. u = (f'a) Hyp
53. (f'a) = u Symmetry 52
54. Set(u) & ∃k.((k,u) ε f) EqualitySub 51 53
55. u ε {w: ∃k.((k,w) ε f)} ClassInt 54
56. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
57. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 56
58. u ε range(f) EqualitySub 55 57
59. (f'a) ε range(f) EqualitySub 58 52
60. (u = (f'a)) -> ((f'a) ε range(f)) ImpInt 59
61. ∀u.((u = (f'a)) -> ((f'a) ε range(f))) ForallInt 60
62. ((f'a) = (f'a)) -> ((f'a) ε range(f)) ForallElim 61
63. (f'a) = (f'a) Identity
64. (f'a) ε range(f) ImpElim 63 62
65. (domain(f) = x) & (range(f) = y) AndElimR 5
66. range(f) = y AndElimR 65
67. (f'a) ε y EqualitySub 64 66
68. ((f ε func(x,y)) & (a ε x)) -> ((f'a) ε y) ImpInt 67 Qed

Used Theorems

1. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
2. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

Th94. (Section(r,z,a) & ((f ε func(a,z)) & OrderPreserving(f,r,r))) -> ((x ε a) -> ¬(((f'x),x) ε r))

0. Section(r,z,a) & ((f ε func(a,z)) & OrderPreserving(f,r,r)) Hyp
1. u ε a Hyp
2. c = {u: ((u ε a) & (((f'u),u) ε r))} Hyp
3. Section(r,z,a) AndElimL 0
4. ((a ⊂ z) & WellOrders(r,z)) & ∀u.∀v.((((u ε z) & (v ε a)) & ((u,v) ε r)) -> (u ε a)) DefExp 3
5. (a ⊂ z) & WellOrders(r,z) AndElimL 4
6. WellOrders(r,z) AndElimR 5
7. Connects(r,z) & ∀y.(((y ⊂ z) & ¬(y = 0)) -> ∃x_8.First(r,y,x_8)) DefExp 6
8. ∀y.(((y ⊂ z) & ¬(y = 0)) -> ∃x_8.First(r,y,x_8)) AndElimR 7
9. ((c ⊂ z) & ¬(c = 0)) -> ∃x_8.First(r,c,x_8) ForallElim 8
10. ¬(c = 0) Hyp
11. x ε c Hyp
12. x ε {u: ((u ε a) & (((f'u),u) ε r))} EqualitySub 11 2
13. Set(x) & ((x ε a) & (((f'x),x) ε r)) ClassElim 12
14. (x ε a) & (((f'x),x) ε r) AndElimR 13
15. x ε a AndElimL 14
16. (x ε c) -> (x ε a) ImpInt 15
17. ∀x.((x ε c) -> (x ε a)) ForallInt 16
18. c ⊂ a DefSub 17
19. a ⊂ z AndElimL 5
20. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z) TheoremInt
21. ∀x.(((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)) ForallInt 20
22. ((c ⊂ y) & (y ⊂ z)) -> (c ⊂ z) ForallElim 21
23. ∀y.(((c ⊂ y) & (y ⊂ z)) -> (c ⊂ z)) ForallInt 22
24. ((c ⊂ a) & (a ⊂ z)) -> (c ⊂ z) ForallElim 23
25. (c ⊂ a) & (a ⊂ z) AndInt 18 19
26. c ⊂ z ImpElim 25 24
27. (c ⊂ z) & ¬(c = 0) AndInt 26 10
28. ∃x_8.First(r,c,x_8) ImpElim 27 9
29. First(r,c,k) Hyp
30. (k ε c) & ∀y.((y ε c) -> ¬((y,k) ε r)) DefExp 29
31. k ε c AndElimL 30
32. k ε {u: ((u ε a) & (((f'u),u) ε r))} EqualitySub 31 2
33. Set(k) & ((k ε a) & (((f'k),k) ε r)) ClassElim 32
34. (k ε a) & (((f'k),k) ε r) AndElimR 33

35. ((f'k),k) ε r AndElimR 34
36. (f ε func(a,z)) & OrderPreserving(f,r,r) AndElimR 0
37. OrderPreserving(f,r,r) AndElimR 36
38. (Function(f) & (WellOrders(r,domain(f)) & WellOrders(r,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε r)) DefExp 37
39. ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε r)) AndElimR 38
40. f ε func(a,z) AndElimL 36
41. func(x,y) = {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))} DefEqInt
42. ∀x.(func(x,y) = {f: (Function(f) & ((domain(f) = x) & (range(f) = y)))}) ForallInt 41
43. func(a,y) = {f: (Function(f) & ((domain(f) = a) & (range(f) = y)))} ForallElim 42
44. ∀y.(func(a,y) = {f: (Function(f) & ((domain(f) = a) & (range(f) = y)))}) ForallInt 43
45. func(a,z) = {f: (Function(f) & ((domain(f) = a) & (range(f) = z)))} ForallElim 44
46. f ε {f: (Function(f) & ((domain(f) = a) & (range(f) = z)))} EqualitySub 40 45
47. Set(f) & (Function(f) & ((domain(f) = a) & (range(f) = z))) ClassElim 46
48. Function(f) & ((domain(f) = a) & (range(f) = z)) AndElimR 47
49. (domain(f) = a) & (range(f) = z) AndElimR 48
50. domain(f) = a AndElimL 49
51. ∀z.((z ε c) -> (z ε a)) DefExp 18
52. (k ε c) -> (k ε a) ForallElim 51
53. k ε a ImpElim 31 52
54. ((f ε func(x,y)) & (a ε x)) -> ((f'a) ε y) TheoremInt
55. ∀a.(((f ε func(x,y)) & (a ε x)) -> ((f'a) ε y)) ForallInt 54
56. ((f ε func(x,y)) & (k ε x)) -> ((f'k) ε y) ForallElim 55
57. ∀x.(((f ε func(x,y)) & (k ε x)) -> ((f'k) ε y)) ForallInt 56
58. ((f ε func(a,y)) & (k ε a)) -> ((f'k) ε y) ForallElim 57
59. ∀y.(((f ε func(a,y)) & (k ε a)) -> ((f'k) ε y)) ForallInt 58
60. ((f ε func(a,z)) & (k ε a)) -> ((f'k) ε z) ForallElim 59
61. (f ε func(a,z)) & (k ε a) AndInt 40 53
62. (f'k) ε z ImpElim 61 60
63. ∀u.∀v.((((u ε z) & (v ε a)) & ((u,v) ε r)) -> (u ε a)) AndElimR 4
64. ∀v.(((((f'k) ε z) & (v ε a)) & (((f'k),v) ε r)) -> ((f'k) ε a)) ForallElim 63
65. ((((f'k) ε z) & (k ε a)) & (((f'k),k) ε r)) -> ((f'k) ε a) ForallElim 64
66. ((f'k) ε z) & (k ε a) AndInt 62 53
67. (((f'k) ε z) & (k ε a)) & (((f'k),k) ε r) AndInt 66 35
68. (f'k) ε a ImpElim 67 65
69. a = domain(f) Symmetry 50
70. k ε domain(f) EqualitySub 53 69
71. (f'k) ε domain(f) EqualitySub 68 69
72. ∀v.(((((f'k) ε domain(f)) & (v ε domain(f))) & (((f'k),v) ε r)) -> (((f'(f'k)),(f'v)) ε r)) ForallElim 39
73. ((((f'k) ε domain(f)) & (k ε domain(f))) & (((f'k),k) ε r)) -> (((f'(f'k)),(f'k)) ε r) ForallElim 72
74. ((f'k) ε domain(f)) & (k ε domain(f)) AndInt 71 70
75. (((f'k) ε domain(f)) & (k ε domain(f))) & (((f'k),k) ε r) AndInt 74 35
76. ((f'(f'k)),(f'k)) ε r ImpElim 75 73
77. u = (f'k) Hyp
78. (f'k) = u Symmetry 77
79. ((f'u),u) ε r EqualitySub 76 78
80. u ε a EqualitySub 68 78
81. (u ε a) & (((f'u),u) ε r) AndInt 80 79
82. ∃w.((f'k) ε w) ExistsInt 68
83. Set((f'k)) DefSub 82
84. Set(u) EqualitySub 83 78
85. Set(u) & ((u ε a) & (((f'u),u) ε r)) AndInt 84 81
86. u ε {w: ((w ε a) & (((f'w),w) ε r))} ClassInt 85
87. (f'k) ε {w: ((w ε a) & (((f'w),w) ε r))} EqualitySub 86 77
88. {u: ((u ε a) & (((f'u),u) ε r))} = c Symmetry 2
89. (f'k) ε c EqualitySub 87 88
90. (u = (f'k)) -> ((f'k) ε c) ImpInt 89
91. ∀u.((u = (f'k)) -> ((f'k) ε c)) ForallInt 90
92. ((f'k) = (f'k)) -> ((f'k) ε c) ForallElim 91
93. (f'k) = (f'k) Identity
94. (f'k) ε c ImpElim 93 92
95. ∀y.((y ε c) -> ¬((y,k) ε r)) AndElimR 30
96. ((f'k) ε c) -> ¬(((f'k),k) ε r) ForallElim 95
97. ¬(((f'k),k) ε r) ImpElim 94 96
98. _|_ ImpElim 35 97
99. _|_ ExistsElim 28 29 98

100. ¬¬(c = 0) ImpInt 99
101. D <-> ¬¬D TheoremInt
102. (D -> ¬¬D) & (¬¬D -> D) EquivExp 101
103. ¬¬D -> D AndElimR 102
104. ¬¬(c = 0) -> (c = 0) PolySub 103
105. c = 0 ImpElim 100 104
106. {u: ((u ε a) & (((f'u),u) ε r))} = 0 EqualitySub 105 2
107. (c = {u: ((u ε a) & (((f'u),u) ε r))}) -> ({u: ((u ε a) & (((f'u),u) ε r))} = 0) ImpInt 106
108. ∀c.((c = {u: ((u ε a) & (((f'u),u) ε r))}) -> ({u: ((u ε a) & (((f'u),u) ε r))} = 0)) ForallInt 107
109. ({u: ((u ε a) & (((f'u),u) ε r))} = {x_20: ((x_20 ε a) & (((f'x_20),x_20) ε r))}) -> ({x_20: ((x_20 ε a) & (((f'x_20),x_20) ε r))} = 0) ForallElim 108
110. {u: ((u ε a) & (((f'u),u) ε r))} = {u: ((u ε a) & (((f'u),u) ε r))} Identity
111. {x_20: ((x_20 ε a) & (((f'x_20),x_20) ε r))} = 0 ImpElim 110 109
112. x ε a Hyp
113. ((f'x),x) ε r Hyp
114. (x ε a) & (((f'x),x) ε r) AndInt 112 113
115. ∃w.(x ε w) ExistsInt 112
116. Set(x) DefSub 115
117. Set(x) & ((x ε a) & (((f'x),x) ε r)) AndInt 116 114
118. x ε {w: ((w ε a) & (((f'w),w) ε r))} ClassInt 117
119. x ε 0 EqualitySub 118 111
120. ¬(x ε 0) TheoremInt
121. _|_ ImpElim 119 120
122. ¬(((f'x),x) ε r) ImpInt 121
123. (x ε a) -> ¬(((f'x),x) ε r) ImpInt 122
124. (Section(r,z,a) & ((f ε func(a,z)) & OrderPreserving(f,r,r))) -> ((x ε a) -> ¬(((f'x),x) ε r)) ImpInt 123 Qed

Used Theorems

1. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)
2. ((f ε func(x,y)) & (a ε x)) -> ((f'a) ε y)
3. D <-> ¬¬D
4. ¬(x ε 0)

1-to-1. 1-to-1(f) <-> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))

0. 1-to-1(f) Hyp
1. Function(f) & Function((f)$^{-1}$) DefExp 0
2. (x ε domain(f)) & ((y ε domain(f)) & ¬(x = y)) Hyp
3. Function(f) AndElimL 1
4. Function((f)$^{-1}$) AndElimR 1
5. Relation((f)$^{-1}$) & ∀x.∀y.∀z.((((x,y) ε (f)$^{-1}$) & ((x,z) ε (f)$^{-1}$)) -> (y = z)) DefExp 4
6. ∀x.∀y.∀z.((((x,y) ε (f)$^{-1}$) & ((x,z) ε (f)$^{-1}$)) -> (y = z)) AndElimR 5
7. (f'x) = (f'y) Hyp
8. ∀y.∀z.(((((f'x),y) ε (f)$^{-1}$) & (((f'x),z) ε (f)$^{-1}$)) -> (y = z)) ForallElim 6
9. ∀z.(((((f'x),x) ε (f)$^{-1}$) & (((f'x),z) ε (f)$^{-1}$)) -> (x = z)) ForallElim 8
10. ((((f'x),x) ε (f)$^{-1}$) & (((f'x),y) ε (f)$^{-1}$)) -> (x = y) ForallElim 9
11. (y ε domain(f)) & ¬(x = y) AndElimR 2
12. ¬(x = y) AndElimR 11
13. (r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} DefEqInt
14. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 13
15. (f)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} ForallElim 14
16. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
17. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 3 16
18. (x,(f'x)) = (x,(f'x)) Identity
19. (f'x) = (f'x) Identity
20. ((x,(f'x)) = (x,(f'x))) & ((f'x) = (f'x)) AndInt 18 19
21. ∃w.((w = (x,(f'x))) & ((f'x) = (f'x))) ExistsInt 20
22. (w = (x,(f'x))) & ((f'x) = (f'x)) Hyp
23. ∃a.((w = (x,a)) & ((f'x) = a)) ExistsInt 22
24. ∃b.∃a.((w = (b,a)) & ((f'b) = a)) ExistsInt 23
25. w = (x,(f'x)) AndElimL 22
26. x ε domain(f) AndElimL 2
27. ∃w.(x ε w) ExistsInt 26
28. Set(x) DefSub 27
29. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U)) TheoremInt
30. (z ε domain(f)) -> ((f'z) ε U) AndElimR 29

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31. ∀z.((z ε domain(f)) -> ((f'z) ε U))  ForallInt 30
32. (x ε domain(f)) -> ((f'x) ε U)  ForallElim 31
33. (f'x) ε U  ImpElim 26 32
34. ∃w.((f'x) ε w)  ExistsInt 33
35. ∃w.((f'x) ε w)  DefSub 34
36. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
37. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 36
38. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 37
39. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 38
40. ∀y.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 39
41. (Set(x) & Set((f'x))) -> Set((x,(f'x)))  ForallElim 40
42. Set((f'x))  DefSub 34
43. Set(x) & Set((f'x))  AndInt 28 42
44. Set((x,(f'x)))  ImpElim 43 41
45. w = (x,(f'x))  AndElimL 22
46. (x,(f'x)) = w  Symmetry 45
47. Set(w)  EqualitySub 44 46
48. Set(w) & ∃b.∃a.((w = (b,a)) & ((f'b) = a))  AndInt 47 24
49. w ε {w: ∃b.∃a.((w = (b,a)) & ((f'b) = a))}  ClassInt 48
50. {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} = f  Symmetry 17
51. w ε f  EqualitySub 49 50
52. (x,(f'x)) ε f  EqualitySub 51 25
53. (x,(f'x)) ε f  ExistsElim 21 22 52
54. (x,(f'y)) ε f  EqualitySub 53 7
55. ((f'x),x) = ((f'x),x)  Identity
56. ((x,(f'x)) ε f) & (((f'x),x) = ((f'x),x))  AndInt 52 55
57. ∃w.(((x,(f'x)) ε f) & (w = ((f'x),x)))  ExistsInt 56
58. ((x,(f'x)) ε f) & (w = ((f'x),x))  Hyp
59. Set((f'x)) & Set(x)  AndInt 42 28
60. ∀x.(((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)))  ForallInt 36
61. ((Set((f'x)) & Set(y)) <-> Set(((f'x),y))) & (¬Set(((f'x),y)) -> (((f'x),y) = U))  ForallElim 60
62. ∀y.(((Set((f'x)) & Set(y)) <-> Set(((f'x),y))) & (¬Set(((f'x),y)) -> (((f'x),y) = U)))  ForallInt 61
63. ((Set((f'x)) & Set(x)) <-> Set(((f'x),x))) & (¬Set(((f'x),x)) -> (((f'x),x) = U))  ForallElim 62
64. (Set((f'x)) & Set(x)) <-> Set(((f'x),x))  AndElimL 63
65. ((Set((f'x)) & Set(x)) -> Set(((f'x),x))) & (Set(((f'x),x)) -> (Set((f'x)) & Set(x)))  EquivExp 64
66. (Set((f'x)) & Set(x)) -> Set(((f'x),x))  AndElimL 65
67. Set(((f'x),x))  ImpElim 59 66
68. w = ((f'x),x)  AndElimR 58
69. ((f'x),x) = w  Symmetry 68
70. Set(w)  EqualitySub 67 69
71. ∃y.(((x,y) ε f) & (w = (y,x)))  ExistsInt 58
72. ∃x.∃y.(((x,y) ε f) & (w = (y,x)))  ExistsInt 71
73. Set(w) & ∃x.∃y.(((x,y) ε f) & (w = (y,x)))  AndInt 70 72
74. w ε {w: ∃x.∃y.(((x,y) ε f) & (w = (y,x)))}  ClassInt 73
75. {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} = (f)⁻¹  Symmetry 15
76. w ε (f)⁻¹  EqualitySub 74 75
77. ((f'x),x) ε (f)⁻¹  EqualitySub 76 68
78. ((f'x),x) ε (f)⁻¹  ExistsElim 57 58 77
79. ((f'x),x) ε (f)⁻¹  ExistsElim 21 22 78
80. (y,(f'y)) = (y,(f'y))  Identity
81. (f'y) = (f'y)  Identity
82. ((y,(f'y)) = (y,(f'y))) & ((f'y) = (f'y))  AndInt 80 81
83. ∃w.((w = (y,(f'y))) & ((f'y) = (f'y)))  ExistsInt 82
84. (w = (y,(f'y))) & ((f'y) = (f'y))  Hyp
85. ∃a.((w = (y,a)) & ((f'y) = a))  ExistsInt 84
86. ∃b.∃a.((w = (b,a)) & ((f'b) = a))  ExistsInt 85
87. (y ε domain(f)) & ¬(x = y)  AndElimR 2
88. y ε domain(f)  AndElimL 87
89. ∃w.(y ε w)  ExistsInt 88
90. Set(y)  DefSub 89
91. ∀z.((z ε domain(f)) -> ((f'z) ε U))  ForallInt 30
92. (y ε domain(f)) -> ((f'y) ε U)  ForallElim 91
93. (f'y) ε U  ImpElim 88 92
94. ∃w.((f'y) ε w)  ExistsInt 93
95. Set((f'y))  DefSub 94
96. Set(y) & Set((f'y))  AndInt 90 95
97. ∀x.((Set((f'x)) & Set(x)) -> Set(((f'x),x)))  ForallInt 66
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98. ∀y.(((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)))  ForallInt 36
99. ((Set(x) & Set((f'y))) <-> Set((x,(f'y)))) & (¬Set((x,(f'y))) -> ((x,(f'y)) = U)) ForallElim 98
100. ∀x.(((Set(x) & Set((f'y))) <-> Set((x,(f'y)))) & (¬Set((x,(f'y))) -> ((x,(f'y)) = U)))  ForallInt 99
101. ((Set(y) & Set((f'y))) <-> Set((y,(f'y)))) & (¬Set((y,(f'y))) -> ((y,(f'y)) = U)) ForallElim 100
102. ((Set(y) & Set((f'y))) <-> Set((y,(f'y)))) & (¬Set((y,(f'y))) -> ((y,(f'y)) = U)) EquivExp 101
103. (Set(y) & Set((f'y))) <-> Set((y,(f'y)))  AndElimL 102
104. ((Set(y) & Set((f'y))) -> Set((y,(f'y)))) & (Set((y,(f'y))) -> (Set(y) & Set((f'y))))  EquivExp 103
105. (Set(y) & Set((f'y))) -> Set((y,(f'y)))  AndElimL 104
106. Set((y,(f'y)))  ImpElim 96 105
107. w = (y,(f'y))  AndElimL 84
108. (y,(f'y)) = w  Symmetry 107
109. Set(w)  EqualitySub 106 108
110. Set(w) & ∃b.∃a.((w = (b,a)) & ((f'b) = a))  AndInt 109 86
111. w ε {w: ∃b.∃a.((w = (b,a)) & ((f'b) = a))}  ClassInt 110
112. {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} = f  Symmetry 17
113. w ε f  EqualitySub 111 112
114. (y,(f'y)) ε f  EqualitySub 113 107
115. (y,(f'y)) ε f  ExistsElim 83 84 114
116. ((f'y),y) = ((f'y),y)  Identity
117. ((y,(f'y)) ε f) & (((f'y),y) = ((f'y),y))  AndInt 115 116
118. ∃w.(((y,(f'y)) ε f) & (w = ((f'y),y)))  ExistsInt 117
119. ((y,(f'y)) ε f) & (w = ((f'y),y))  Hyp
120. ∃b.(((y,b) ε f) & (w = (b,y)))  ExistsInt 119
121. ∃a.∃b.(((a,b) ε f) & (w = (b,a)))  ExistsInt 120
122. Set(y) & Set((f'y))  AndInt 90 95
123. w = ((f'y),y)  AndElimR 119
124. Set((f'y)) & Set(y)  AndInt 95 90
125. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 36
126. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 125
127. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 126
128. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 127
129. (Set((f'y)) & Set(y)) -> Set(((f'y),y))  ForallElim 128
130. Set(((f'y),y))  ImpElim 124 129
131. ((f'y),y) = w  Symmetry 123
132. Set(w)  EqualitySub 130 131
133. Set(w) & ∃a.∃b.(((a,b) ε f) & (w = (b,a)))  AndInt 132 121
134. w ε {w: ∃a.∃b.(((a,b) ε f) & (w = (b,a)))}  ClassInt 133
135. {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} = (f)⁻¹  Symmetry 15
136. w ε (f)⁻¹  EqualitySub 134 135
137. ((f'y),y) ε (f)⁻¹  EqualitySub 136 123
138. (f'y) = (f'x)  Symmetry 7
139. ((f'y),y) ε (f)⁻¹  ExistsElim 118 119 137
140. ((f'x),y) ε (f)⁻¹  EqualitySub 139 138
141. (((f'x),x) ε (f)⁻¹) & (((f'x),y) ε (f)⁻¹)  AndInt 79 140
142. x = y  ImpElim 141 10
143. _|_  ImpElim 142 12
144. ¬((f'x) = (f'y))  ImpInt 143
145. ((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))  ImpInt 144
146. Function(f)  AndElimL 1
147. ∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))  ForallInt 145
148. ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) ForallInt 147
149. Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))  AndInt 146 148
150. x = x  Identity
151. Function(f) & (((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) AndInt 146 145
152. 1-to-1(f) -> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))  ImpInt 149
153. Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))  Hyp
154. ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) AndElimR 153
155. ((x,y) ε (f)⁻¹) & ((x,z) ε (f)⁻¹)  Hyp
156. (x,y) ε (f)⁻¹  AndElimL 155
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157. (x,z) ε (f)^-1  AndElimR 155
158. (x,y) ε {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))}  EqualitySub 156 15
159. (x,z) ε {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))}  EqualitySub 157 15
160. Set((x,y)) & ∃x_17.∃x_18.(((x_17,x_18) ε f) & ((x,y) = (x_18,x_17)))  ClassElim 158
161. Set((x,z)) & ∃x_20.∃y.(((x_20,y) ε f) & ((x,z) = (y,x_20)))  ClassElim 159
162. ∃x_17.∃x_18.(((x_17,x_18) ε f) & ((x,y) = (x_18,x_17)))  AndElimR 160
163. ∃x_20.∃y.(((x_20,y) ε f) & ((x,z) = (y,x_20)))  AndElimR 161
164. ∃x_18.(((a,x_18) ε f) & ((x,y) = (x_18,a)))  Hyp
165. ((a,b) ε f) & ((x,y) = (b,a))  Hyp
166. ∃y.(((c,y) ε f) & ((x,z) = (y,c)))  Hyp
167. ((c,d) ε f) & ((x,z) = (d,c))  Hyp
168. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
169. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
170. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 169
171. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 170
172. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 171
173. ∀y.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 172
174. Set((x,z)) -> (Set(x) & Set(z))  ForallElim 173
175. Set((x,y))  AndElimL 160
176. Set((x,z))  AndElimL 161
177. Set(x) & Set(y)  ImpElim 175 172
178. Set(x) & Set(z)  ImpElim 176 174
179. (x,y) = (b,a)  AndElimR 165
180. (Set(x) & Set(y)) & ((x,y) = (b,a))  AndInt 177 179
181. ∀u.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 168
182. ((Set(x) & Set(y)) & ((x,y) = (b,v))) -> ((x = b) & (y = v))  ForallElim 181
183. ∀v.(((Set(x) & Set(y)) & ((x,y) = (b,v))) -> ((x = b) & (y = v)))  ForallInt 182
184. ((Set(x) & Set(y)) & ((x,y) = (b,a))) -> ((x = b) & (y = a))  ForallElim 183
185. (x = b) & (y = a)  ImpElim 180 184
186. (x,z) = (d,c)  AndElimR 167
187. ∀y.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 168
188. ((Set(x) & Set(z)) & ((x,z) = (u,v))) -> ((x = u) & (z = v))  ForallElim 187
189. ∀u.(((Set(x) & Set(z)) & ((x,z) = (u,v))) -> ((x = u) & (z = v)))  ForallInt 188
190. ((Set(x) & Set(z)) & ((x,z) = (d,v))) -> ((x = d) & (z = v))  ForallElim 189
191. ∀v.(((Set(x) & Set(z)) & ((x,z) = (d,v))) -> ((x = d) & (z = v)))  ForallInt 190
192. ((Set(x) & Set(z)) & ((x,z) = (d,c))) -> ((x = d) & (z = c))  ForallElim 191
193. (Set(x) & Set(z)) & ((x,z) = (d,c))  AndInt 178 186
194. (x = d) & (z = c)  ImpElim 193 192
195. (a,b) ε f  AndElimL 165
196. (c,d) ε f  AndElimL 167
197. x = b  AndElimL 185
198. x = d  AndElimL 194
199. b = x  Symmetry 197
200. b = d  EqualitySub 199 198
201. (a,d) ε f  EqualitySub 195 200
202. ∃d.((a,d) ε f)  ExistsInt 201
203. Set(y)  AndElimR 177
204. y = a  AndElimR 185
205. Set(a)  EqualitySub 203 204
206. Set(a) & ∃d.((a,d) ε f)  AndInt 205 202
207. a ε {w: ∃d.((w,d) ε f)}  ClassInt 206
208. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
209. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 208
210. a ε domain(f)  EqualitySub 207 209
211. ∃d.((c,d) ε f)  ExistsInt 196
212. Set(z)  AndElimR 178
213. z = c  AndElimR 194
214. Set(c)  EqualitySub 212 213
215. Set(c) & ∃d.((c,d) ε f)  AndInt 214 211
216. c ε {w: ∃d.((w,d) ε f)}  ClassInt 215
217. c ε domain(f)  EqualitySub 216 209
218. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  TheoremInt
219. Function(f)  AndElimL 153
220. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  ImpElim 219 218
221. (c,d) ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  EqualitySub 196 220
222. Set((c,d)) & ∃x.∃y.(((c,d) = (x,y)) & ((f'x) = y))  ClassElim 221
223. (a,d) ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}  EqualitySub 201 220
224. Set((a,d)) & ∃x.∃y.(((a,d) = (x,y)) & ((f'x) = y))  ClassElim 223
225. ∃x.∃y.(((c,d) = (x,y)) & ((f'x) = y))  AndElimR 222
226. ∃x.∃y.(((a,d) = (x,y)) & ((f'x) = y))  AndElimR 224
227. ∃y.(((c,d) = (c1,y)) & ((f'c1) = y))  Hyp
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228. ((c,d) = (c1,d1)) & ((f'c1) = d1) Hyp
229. ∃y.(((a,d) = (a1,y)) & ((f'a1) = y)) Hyp
230. ((a,d) = (a1,d2)) & ((f'a1) = d2) Hyp
231. Set((c,d)) AndElimL 222
232. Set((a,d)) AndElimL 224
233. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 172
234. Set((c,y)) -> (Set(c) & Set(y)) ForallElim 233
235. ∀y.(Set((c,y)) -> (Set(c) & Set(y))) ForallInt 234
236. Set((c,d)) -> (Set(c) & Set(d)) ForallElim 235
237. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 172
238. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 237
239. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 238
240. Set((a,d)) -> (Set(a) & Set(d)) ForallElim 239
241. Set(c) & Set(d) ImpElim 231 236
242. Set(a) & Set(d) ImpElim 232 240
243. (c,d) = (c1,d1) AndElimL 228
244. (a,d) = (a1,d2) AndElimL 230
245. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 168
246. ((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v)) ForallElim 245
247. ∀y.(((Set(c) & Set(y)) & ((c,y) = (u,v))) -> ((c = u) & (y = v))) ForallInt 246
248. ((Set(c) & Set(d)) & ((c,d) = (u,v))) -> ((c = u) & (d = v)) ForallElim 247
249. ∀u.(((Set(c) & Set(d)) & ((c,d) = (u,v))) -> ((c = u) & (d = v))) ForallInt 248
250. ((Set(c) & Set(d)) & ((c,d) = (c1,v))) -> ((c = c1) & (d = v)) ForallElim 249
251. ∀v.(((Set(c) & Set(d)) & ((c,d) = (c1,v))) -> ((c = c1) & (d = v))) ForallInt 250
252. ((Set(c) & Set(d)) & ((c,d) = (c1,d1))) -> ((c = c1) & (d = d1)) ForallElim 251
253. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 168
254. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)) ForallElim 253
255. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 254
256. ((Set(a) & Set(d)) & ((a,d) = (u,v))) -> ((a = u) & (d = v)) ForallElim 255
257. ∀u.(((Set(a) & Set(d)) & ((a,d) = (u,v))) -> ((a = u) & (d = v))) ForallInt 256
258. ((Set(a) & Set(d)) & ((a,d) = (a1,v))) -> ((a = a1) & (d = v)) ForallElim 257
259. ∀v.(((Set(a) & Set(d)) & ((a,d) = (a1,v))) -> ((a = a1) & (d = v))) ForallInt 258
260. ((Set(a) & Set(d)) & ((a,d) = (a1,d2))) -> ((a = a1) & (d = d2)) ForallElim 259
261. (Set(c) & Set(d)) & ((c,d) = (c1,d1)) AndInt 241 243
262. (Set(a) & Set(d)) & ((a,d) = (a1,d2)) AndInt 242 244
263. (c = c1) & (d = d1) ImpElim 261 252
264. (a = a1) & (d = d2) ImpElim 262 260
265. c = c1 AndElimL 263
266. d = d1 AndElimR 263
267. a = a1 AndElimL 264
268. d = d2 AndElimR 264
269. (f'c1) = d1 AndElimR 228
270. (f'a1) = d2 AndElimR 230
271. c1 = c Symmetry 265
272. a1 = a Symmetry 267
273. (f'c) = d1 EqualitySub 269 271
274. (f'a) = d2 EqualitySub 270 272
275. d2 = d1 EqualitySub 266 268
276. (f'a) = d1 EqualitySub 274 275
277. d1 = (f'c) Symmetry 273
278. (f'a) = (f'c) EqualitySub 276 277
279. a = y Symmetry 204
280. c = z Symmetry 213
281. (f'y) = (f'c) EqualitySub 278 279
282. (f'y) = (f'z) EqualitySub 281 280
283. y ε domain(f) EqualitySub 210 279
284. z ε domain(f) EqualitySub 217 280
285. ¬(y = z) Hyp
286. ∀x_24.(((y ε domain(f)) & ((x_24 ε domain(f)) & ¬(y = x_24))) -> ¬((f'y) = (f'x_24))) ForallElim 154
287. ((y ε domain(f)) & ((z ε domain(f)) & ¬(y = z))) -> ¬((f'y) = (f'z)) ForallElim 286
288. (z ε domain(f)) & ¬(y = z) AndInt 284 285
289. (y ε domain(f)) & ((z ε domain(f)) & ¬(y = z)) AndInt 283 288
290. ¬((f'y) = (f'z)) ImpElim 289 287
291. _|_ ImpElim 282 290
292. ¬¬(y = z) ImpInt 291
293. D <-> ¬¬D TheoremInt
294. (D -> ¬¬D) & (¬¬D -> D) EquivExp 293
295. ¬¬D -> D AndElimR 294
296. ¬¬(y = z) -> (y = z) PolySub 295
297. y = z ImpElim 292 296

298. y = z ExistsElim 229 230 297
299. y = z ExistsElim 226 229 298
300. y = z ExistsElim 227 228 299
301. y = z ExistsElim 225 227 300
302. y = z ExistsElim 166 167 301
303. y = z ExistsElim 163 166 302
304. y = z ExistsElim 164 165 303
305. y = z ExistsElim 162 164 304
306. (((x,y) ε (f)$^{-1}$) & ((x,z) ε (f)$^{-1}$)) -> (y = z) ImpInt 305
307. ∀z.((((x,y) ε (f)$^{-1}$) & ((x,z) ε (f)$^{-1}$)) -> (y = z)) ForallInt 306
308. ∀y.∀z.((((x,y) ε (f)$^{-1}$) & ((x,z) ε (f)$^{-1}$)) -> (y = z)) ForallInt 307
309. ∀x.∀y.∀z.((((x,y) ε (f)$^{-1}$) & ((x,z) ε (f)$^{-1}$)) -> (y = z)) ForallInt 308
310. Function(f) AndElimL 153
311. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z)) DefExp 310
312. Relation(f) AndElimL 311
313. z ε (f)$^{-1}$ Hyp
314. (r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} DefEqInt
315. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 314
316. (f)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} ForallElim 315
317. ∀z.((z ε f) -> ∃x.∃y.(z = (x,y))) DefExp 312
318. z ε {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} EqualitySub 313 316
319. Set(z) & ∃x.∃y.(((x,y) ε f) & (z = (y,x))) ClassElim 318
320. ∃x.∃y.(((x,y) ε f) & (z = (y,x))) AndElimR 319
321. ∃y.(((x,y) ε f) & (z = (y,x))) Hyp
322. ((x,y) ε f) & (z = (y,x)) Hyp
323. z = (y,x) AndElimR 322
324. ∃x.(z = (y,x)) ExistsInt 323
325. ∃y.∃x.(z = (y,x)) ExistsInt 324
326. ∃y.∃x.(z = (y,x)) ExistsElim 321 322 325
327. ∃y.∃x.(z = (y,x)) ExistsElim 320 321 326
328. (z ε (f)$^{-1}$) -> ∃y.∃x.(z = (y,x)) ImpInt 327
329. ∀z.((z ε (f)$^{-1}$) -> ∃y.∃x.(z = (y,x))) ForallInt 328
330. Relation((f)$^{-1}$) DefSub 329
331. Relation((f)$^{-1}$) & ∀x.∀y.∀z.((((x,y) ε (f)$^{-1}$) & ((x,z) ε (f)$^{-1}$)) -> (y = z)) AndInt 330 309
332. Function((f)$^{-1}$) DefSub 331
333. Function(f) & Function((f)$^{-1}$) AndInt 310 332
334. 1-to-1(f) DefSub 333
335. (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))) -> 1-to-1(f) ImpInt 334
336. (1-to-1(f) -> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))) & ((Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))) -> 1-to-1(f)) AndInt 152 335
337. 1-to-1(f) <-> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))) EquivConst 336 Qed

Used Theorems

1. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
2. (¬(z ε domain(f)) -> ((f'z) = U)) & ((z ε domain(f)) -> ((f'z) ε U))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
4. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
6. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
8. D <-> ¬¬D

FunctionRange. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))

0. Function(f) & (a ε domain(f)) Hyp
1. Function(f) AndElimL 0
2. a ε domain(f) AndElimR 0
3. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
4. a ε {x: ∃y.((x,y) ε f)} EqualitySub 2 3
5. Set(a) & ∃y.((a,y) ε f) ClassElim 4
6. Set(a) AndElimL 5
7. ∃y.((a,y) ε f) AndElimR 5
8. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
9. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 1 8
10. (a,y) ε f Hyp
11. (a,y) ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 10 9
12. Set((a,y)) & ∃x.∃x_0.(((a,y) = (x,x_0)) & ((f'x) = x_0)) ClassElim 11

13. Set((a,y)) AndElimL 12
14. ∃x.∃x_0.(((a,y) = (x,x_0)) & ((f'x) = x_0)) AndElimR 12
15. ∃x_0.(((a,y) = (b,x_0)) & ((f'b) = x_0)) Hyp
16. ((a,y) = (b,c)) & ((f'b) = c) Hyp
17. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
18. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 17
19. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 18
20. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 19
21. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 20
22. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 21
23. Set(a) & Set(y) ImpElim 13 22
24. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
25. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 24
26. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)) ForallElim 25
27. ∀u.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 26
28. ((Set(a) & Set(y)) & ((a,y) = (b,v))) -> ((a = b) & (y = v)) ForallElim 27
29. ∀v.(((Set(a) & Set(y)) & ((a,y) = (b,v))) -> ((a = b) & (y = v))) ForallInt 28
30. ((Set(a) & Set(y)) & ((a,y) = (b,c))) -> ((a = b) & (y = c)) ForallElim 29
31. (a,y) = (b,c) AndElimL 16
32. (Set(a) & Set(y)) & ((a,y) = (b,c)) AndInt 23 31
33. (a = b) & (y = c) ImpElim 32 30
34. a = b AndElimL 33
35. y = c AndElimR 33
36. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
37. (f'b) = c AndElimR 16
38. c = y Symmetry 35
39. (f'b) = y EqualitySub 37 38
40. y = (f'b) Symmetry 39
41. (a,(f'b)) ε f EqualitySub 10 40
42. ∃a.((a,(f'b)) ε f) ExistsInt 41
43. Set(y) AndElimR 23
44. ∃a.((a,y) ε f) ExistsInt 10
45. Set(y) & ∃a.((a,y) ε f) AndInt 43 44
46. y ε {w: ∃a.((a,w) ε f)} ClassInt 45
47. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 36
48. y ε range(f) EqualitySub 46 47
49. (f'b) ε range(f) EqualitySub 48 40
50. (f'b) ε range(f) ExistsElim 15 16 49
51. b = a Symmetry 34
52. (f'a) ε range(f) EqualitySub 50 51
53. (f'a) ε range(f) ExistsElim 15 16 52
54. (f'a) ε range(f) ExistsElim 14 15 53
55. (f'a) ε range(f) ExistsElim 7 10 54
56. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f)) ImpInt 55 Qed

Used Theorems

4. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
6. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

Th96. OrderPreserving(f,r,s) -> (1-to-1(f) & OrderPreserving((f)$^{-1}$,s,r))

0. OrderPreserving(f,r,s) Hyp
1. (x ε domain(f)) & ((y ε domain(f)) & ¬(x = y)) Hyp
2. (Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) DefExp 0
3. (f'x) = (f'y) Hyp
4. Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f))) AndElimL 2
5. WellOrders(r,domain(f)) & WellOrders(s,range(f)) AndElimR 4
6. WellOrders(r,domain(f)) AndElimL 5
7. Connects(r,domain(f)) & ∀y.(((y ⊂ domain(f)) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 6
8. Connects(r,domain(f)) AndElimL 7
9. ∀y.∀z.(((y ε domain(f)) & (z ε domain(f))) -> ((y = z) v (((y,z) ε r) v ((z,y) ε r)))) DefExp 8
10. ∀z.(((x ε domain(f)) & (z ε domain(f))) -> ((x = z) v (((x,z) ε r) v ((z,x) ε r)))) ForallElim 9
11. ((x ε domain(f)) & (y ε domain(f))) -> ((x = y) v (((x,y) ε r) v ((y,x) ε r))) ForallElim 10
12. x ε domain(f) AndElimL 1
13. (y ε domain(f)) & ¬(x = y) AndElimR 1

14. y ε domain(f) AndElimL 13
15. (x ε domain(f)) & (y ε domain(f)) AndInt 12 14
16. (x = y) v (((x,y) ε r) v ((y,x) ε r)) ImpElim 15 11
17. ¬(x = y) AndElimR 13
18. x = y Hyp
19. _|_ ImpElim 18 17
20. ((x,y) ε r) v ((y,x) ε r) AbsI 19
21. ((x,y) ε r) v ((y,x) ε r) Hyp
22. ((x,y) ε r) v ((y,x) ε r) OrElim 16 18 20 21 21
23. ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s))
AndElimR 2
24. ∀v.((((x ε domain(f)) & (v ε domain(f))) & ((x,v) ε r)) -> (((f'x),(f'v)) ε s))
ForallElim 23
25. (((x ε domain(f)) & (y ε domain(f))) & ((x,y) ε r)) -> (((f'x),(f'y)) ε s)
ForallElim 24
26. x = x Identity
27. x = x Identity
28. ((x,y) ε r) v ((y,x) ε r) AbsI 19
29. ((x,y) ε r) v ((y,x) ε r) Hyp
30. ((x,y) ε r) v ((y,x) ε r) OrElim 16 18 28 29 29
31. (x,y) ε r Hyp
32. WellOrders(s,range(f)) AndElimR 5
33. ((x ε domain(f)) & (y ε domain(f))) & ((x,y) ε r) AndInt 15 31
34. ((f'x),(f'y)) ε s ImpElim 33 25
35. WellOrders(s,range(f)) AndElimR 5
36. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x)) TheoremInt
37. ∀r.(WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))) ForallInt 36
38. WellOrders(s,x) -> (Asymmetric(s,x) & TransIn(s,x)) ForallElim 37
39. ∀x.(WellOrders(s,x) -> (Asymmetric(s,x) & TransIn(s,x))) ForallInt 38
40. WellOrders(s,range(f)) -> (Asymmetric(s,range(f)) & TransIn(s,range(f))) ForallElim
39
41. Asymmetric(s,range(f)) & TransIn(s,range(f)) ImpElim 35 40
42. Asymmetric(s,range(f)) AndElimL 41
43. ∀y.∀z.(((y ε range(f)) & (z ε range(f))) -> (((y,z) ε s) -> ¬((z,y) ε s))) DefExp 42
44. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f)) TheoremInt
45. Function(f) AndElimL 4
46. Function(f) & (x ε domain(f)) AndInt 45 12
47. ∀a.((Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))) ForallInt 44
48. (Function(f) & (x ε domain(f))) -> ((f'x) ε range(f)) ForallElim 47
49. (f'x) ε range(f) ImpElim 46 48
50. ∀z.((((f'x) ε range(f)) & (z ε range(f))) -> ((((f'x),z) ε s) -> ¬((z,(f'x)) ε s)))
ForallElim 43
51. (((f'x) ε range(f)) & ((f'x) ε range(f))) -> ((((f'x),(f'x)) ε s) -> ¬(((f'x),(f'x))
ε s)) ForallElim 50
52. ((f'x) ε range(f)) & ((f'x) ε range(f)) AndInt 49 49
53. (((f'x),(f'x)) ε s) -> ¬(((f'x),(f'x)) ε s) ImpElim 52 51
54. (f'y) = (f'x) Symmetry 3
55. ((f'x),(f'x)) ε s EqualitySub 34 54
56. ¬(((f'x),(f'x)) ε s) ImpElim 55 53
57. _|_ ImpElim 55 56
58. (y,x) ε r Hyp
59. ∀v.((((y ε domain(f)) & (v ε domain(f))) & ((y,v) ε r)) -> (((f'y),(f'v)) ε s))
ForallElim 23
60. (((y ε domain(f)) & (x ε domain(f))) & ((y,x) ε r)) -> (((f'y),(f'x)) ε s)
ForallElim 59
61. (y ε domain(f)) & (x ε domain(f)) AndInt 14 12
62. ((y ε domain(f)) & (x ε domain(f))) & ((y,x) ε r) AndInt 61 58
63. ((f'y),(f'x)) ε s ImpElim 62 60
64. ((f'x),(f'x)) ε s EqualitySub 63 54
65. ¬(((f'x),(f'x)) ε s) ImpElim 64 53
66. _|_ ImpElim 64 65
67. _|_ OrElim 30 31 57 58 66
68. ¬((f'x) = (f'y)) ImpInt 67
69. ((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)) ImpInt 68
70. ∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) ForallInt
69
71. ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))
ForallInt 70
72. 1-to-1(f) <-> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y)))
-> ¬((f'x) = (f'y)))) TheoremInt
73. (1-to-1(f) -> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y)))

-> ¬((f'x) = (f'y))))) & ((Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))) -> 1-to-1(f)) EquivExp 72
74. (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))) -> 1-to-1(f) AndElimR 73
75. Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))) AndInt 45 71
76. 1-to-1(f) ImpElim 75 74
77. OrderPreserving(f,r,s) -> 1-to-1(f) ImpInt 76
78. (x ε domain(f)) & (y ε domain(f)) Hyp
79. ((f'x),(f'y)) ε s Hyp
80. x = y Hyp
81. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x)) TheoremInt
82. ∀r.(WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))) ForallInt 81
83. WellOrders(s,x) -> (Asymmetric(s,x) & TransIn(s,x)) ForallElim 82
84. (Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) DefExp 0
85. Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f))) AndElimL 84
86. WellOrders(r,domain(f)) & WellOrders(s,range(f)) AndElimR 85
87. WellOrders(s,range(f)) AndElimR 86
88. ∀x.(WellOrders(s,x) -> (Asymmetric(s,x) & TransIn(s,x))) ForallInt 83
89. WellOrders(s,range(f)) -> (Asymmetric(s,range(f)) & TransIn(s,range(f))) ForallElim 88
90. Asymmetric(s,range(f)) & TransIn(s,range(f)) ImpElim 87 89
91. Asymmetric(s,range(f)) AndElimL 90
92. ∀y.∀z.(((y ε range(f)) & (z ε range(f))) -> (((y,z) ε s) -> ¬((z,y) ε s))) DefExp 91
93. ∀z.((((f'x) ε range(f)) & (z ε range(f))) -> ((((f'x),z) ε s) -> ¬((z,(f'x)) ε s))) ForallElim 92
94. (((f'x) ε range(f)) & ((f'y) ε range(f))) -> ((((f'x),(f'y)) ε s) -> ¬(((f'y),(f'x)) ε s)) ForallElim 93
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97. y ε domain(f) AndElimR 78
98. Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f))) AndElimL 2
99. Function(f) AndElimL 98
100. Function(f) & (x ε domain(f)) AndInt 99 96
101. ∀a.((Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))) ForallInt 95
102. (Function(f) & (x ε domain(f))) -> ((f'x) ε range(f)) ForallElim 101
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104. y = x Symmetry 80
105. (((f'x) ε range(f)) & ((f'x) ε range(f))) -> ((((f'x),(f'x)) ε s) -> ¬(((f'x),(f'x)) ε s)) EqualitySub 94 104
106. ((f'x) ε range(f)) & ((f'x) ε range(f)) AndInt 103 103
107. (((f'x),(f'x)) ε s) -> ¬(((f'x),(f'x)) ε s) ImpElim 106 105
108. ((f'x),(f'x)) ε s EqualitySub 79 104
109. ¬(((f'x),(f'x)) ε s) ImpElim 108 107
110. _|_ ImpElim 108 109
111. ¬(x = y) ImpInt 110
112. WellOrders(r,domain(f)) AndElimL 86
113. Connects(r,domain(f)) & ∀y.(((y ⊂ domain(f)) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 112
114. Connects(r,domain(f)) AndElimL 113
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116. ∀z.(((x ε domain(f)) & (z ε domain(f))) -> ((x = z) v (((x,z) ε r) v ((z,x) ε r)))) ForallElim 115
117. ((x ε domain(f)) & (y ε domain(f))) -> ((x = y) v (((x,y) ε r) v ((y,x) ε r))) ForallElim 116
118. (x ε domain(f)) & (y ε domain(f)) AndInt 96 97
119. (x = y) v (((x,y) ε r) v ((y,x) ε r)) ImpElim 118 117
120. x = y Hyp
121. _|_ ImpElim 120 111
122. ((x,y) ε r) v ((y,x) ε r) AbsI 121
123. ((x,y) ε r) v ((y,x) ε r) Hyp
124. ((x,y) ε r) v ((y,x) ε r) OrElim 119 120 122 123 123
125. (x,y) ε r Hyp
126. (y,x) ε r Hyp
127. ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) AndElimR 84
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129. (((y ε domain(f)) & (x ε domain(f))) & ((y,x) ε r)) -> (((f'y),(f'x)) ε s)

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136. (f'y) ε range(f) ImpElim 135 134
137. ((f'y) ε range(f)) & ((f'x) ε range(f)) AndInt 136 103
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ForallElim 92
139. (((f'y) ε range(f)) & ((f'x) ε range(f))) -> ((((f'y),(f'x)) ε s) -> ¬(((f'x),(f'y))
ε s)) ForallElim 138
140. (((f'y),(f'x)) ε s) -> ¬(((f'x),(f'y)) ε s) ImpElim 137 139
141. ¬(((f'x),(f'y)) ε s) ImpElim 132 140
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144. (x,y) ε r OrElim 124 125 125 126 143
145. (((f'x),(f'y)) ε s) -> ((x,y) ε r) ImpInt 144
146. ((x ε domain(f)) & (y ε domain(f))) -> ((((f'x),(f'y)) ε s) -> ((x,y) ε r)) ImpInt
145
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148. Function((f)$^{-1}$) AndElimR 147
149. a ε domain((f)$^{-1}$) Hyp
150. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
151. ∀f.(domain(f) = {x: ∃y.((x,y) ε f)}) ForallInt 150
152. domain((f)$^{-1}$) = {x: ∃y.((x,y) ε (f)$^{-1}$)} ForallElim 151
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155. ∃y.((a,y) ε (f)$^{-1}$) AndElimR 154
156. (a,b) ε (f)$^{-1}$ Hyp
157. (r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} DefEqInt
158. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 157
159. (f)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} ForallElim 158
160. (a,b) ε {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} EqualitySub 156 159
161. Set((a,b)) & ∃x.∃y.(((x,y) ε f) & ((a,b) = (y,x))) ClassElim 160
162. ∃x.∃y.(((x,y) ε f) & ((a,b) = (y,x))) AndElimR 161
163. ∃y.(((x1,y) ε f) & ((a,b) = (y,x1))) Hyp
164. ((x1,y1) ε f) & ((a,b) = (y1,x1)) Hyp
165. Set((a,b)) AndElimL 161
166. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
167. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 166
168. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 167
169. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 168
170. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 169
171. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 170
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174. Set(a) & Set(b) ImpElim 165 173
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176. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 175
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179. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)) ForallElim 178
180. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))) ForallInt 179
181. ((Set(a) & Set(b)) & ((a,b) = (y1,v))) -> ((a = y1) & (b = v)) ForallElim 180
182. ∀v.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))) ForallInt 179
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186. ∀v.(((Set(a) & Set(b)) & ((a,b) = (y1,v))) -> ((a = y1) & (b = v))) ForallInt 181
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240. (a,b) ε {w: ∃x_26.∃x_27.(((x_26,x_27) ε f) & (w = (x_27,x_26)))} EqualitySub 239 218
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255. ∀y.((domain((f)$^{-1}$) = y) <-> ∀z.((z ε domain((f)$^{-1}$)) <-> (z ε y))) ForallElim 254
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257. ((domain((f)$^{-1}$) = range(f)) -> ∀z.((z ε domain((f)$^{-1}$)) <-> (z ε range(f)))) & (∀z.((z ε domain((f)$^{-1}$)) <-> (z ε range(f))) -> (domain((f)$^{-1}$) = range(f))) EquivExp 256
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279. ((x ε domain((f)$^{-1}$)) & (y ε domain((f)$^{-1}$))) & ((x,y) ε s) Hyp
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281. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
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284. y ε range(f) AndElimR 282
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286. x ε {y: ∃x.((x,y) ε f)} EqualitySub 283 285
287. y ε {y: ∃x.((x,y) ε f)} EqualitySub 284 285
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293. (b,y) ε f Hyp
294. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 260 281
295. (a,x) ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 292 294
296. (b,y) ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 293 294
297. Set((a,x)) & ∃x_30.∃y.(((a,x) = (x_30,y)) & ((f'x_30) = y)) ClassElim 295
298. Set((b,y)) & ∃x.∃x_31.(((b,y) = (x,x_31)) & ((f'x) = x_31)) ClassElim 296
299. ∃x_30.∃y.(((a,x) = (x_30,y)) & ((f'x_30) = y)) AndElimR 297
300. ∃x.∃x_31.(((b,y) = (x,x_31)) & ((f'x) = x_31)) AndElimR 298
301. ∃y.(((a,x) = (x1,y)) & ((f'x1) = y)) Hyp
302. ((a,x) = (x1,y1)) & ((f'x1) = y1) Hyp
303. ∃x_31.(((b,y) = (x2,x_31)) & ((f'x2) = x_31)) Hyp
304. ((b,y) = (x2,y2)) & ((f'x2) = y2) Hyp
305. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
306. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 305
307. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 306
308. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 307
309. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 308
310. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 309
311. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 310
312. Set((a,x)) -> (Set(a) & Set(x)) ForallElim 311
313. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 308
314. Set((b,y)) -> (Set(b) & Set(y)) ForallElim 313
315. Set((a,x)) AndElimL 297
316. Set((b,y)) AndElimL 298
317. Set(a) & Set(x) ImpElim 315 312
318. Set(b) & Set(y) ImpElim 316 314
319. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
320. (a,x) = (x1,y1) AndElimL 302
321. (b,y) = (x2,y2) AndElimL 304
322. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 319
323. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)) ForallElim 322
324. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 323
325. ((Set(a) & Set(x)) & ((a,x) = (u,v))) -> ((a = u) & (x = v)) ForallElim 324
326. ∀u.(((Set(a) & Set(x)) & ((a,x) = (u,v))) -> ((a = u) & (x = v))) ForallInt 325
327. ((Set(a) & Set(x)) & ((a,x) = (x1,v))) -> ((a = x1) & (x = v)) ForallElim 326
328. ∀v.(((Set(a) & Set(x)) & ((a,x) = (u,v))) -> ((a = u) & (x = v))) ForallInt 325
329. ((Set(a) & Set(x)) & ((a,x) = (u,y1))) -> ((a = u) & (x = y1)) ForallElim 328
330. ∀u.(((Set(a) & Set(x)) & ((a,x) = (u,y1))) -> ((a = u) & (x = y1))) ForallInt 329
331. ((Set(a) & Set(x)) & ((a,x) = (x1,y1))) -> ((a = x1) & (x = y1)) ForallElim 330
332. (Set(a) & Set(x)) & ((a,x) = (x1,y1)) AndInt 317 320

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333. (a = x1) & (x = y1)  ImpElim 332 331
334. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 319
335. ((Set(b) & Set(y)) & ((b,y) = (u,v))) -> ((b = u) & (y = v))  ForallElim 334
336. ∀u.(((Set(b) & Set(y)) & ((b,y) = (u,v))) -> ((b = u) & (y = v)))  ForallInt 335
337. ((Set(b) & Set(y)) & ((b,y) = (x2,v))) -> ((b = x2) & (y = v))  ForallElim 336
338. ∀v.(((Set(b) & Set(y)) & ((b,y) = (x2,v))) -> ((b = x2) & (y = v)))  ForallInt 337
339. ((Set(b) & Set(y)) & ((b,y) = (x2,y2))) -> ((b = x2) & (y = y2))  ForallElim 338
340. (Set(b) & Set(y)) & ((b,y) = (x2,y2))  AndInt 318 321
341. (b = x2) & (y = y2)  ImpElim 340 339
342. a = x1  AndElimL 333
343. x = y1  AndElimR 333
344. b = x2  AndElimL 341
345. y = y2  AndElimR 341
346. (f'x1) = y1  AndElimR 302
347. (f'x2) = y2  AndElimR 304
348. x1 = a  Symmetry 342
349. x2 = b  Symmetry 344
350. y1 = x  Symmetry 343
351. y2 = y  Symmetry 345
352. (f'a) = y1  EqualitySub 346 348
353. (f'a) = x  EqualitySub 352 350
354. (f'b) = y2  EqualitySub 347 349
355. (f'b) = y  EqualitySub 354 351
356. (x,y) ε s  AndElimR 279
357. x = (f'a)  Symmetry 353
358. y = (f'b)  Symmetry 355
359. ∃x.((a,x) ε f)  ExistsInt 292
360. ∃y.((b,y) ε f)  ExistsInt 293
361. Set(a)  AndElimL 317
362. Set(b)  AndElimL 318
363. Set(a) & ∃x.((a,x) ε f)  AndInt 361 359
364. Set(b) & ∃y.((b,y) ε f)  AndInt 362 360
365. a ε {w: ∃x.((w,x) ε f)}  ClassInt 363
366. b ε {w: ∃y.((w,y) ε f)}  ClassInt 364
367. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
368. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 367
369. a ε domain(f)  EqualitySub 365 368
370. b ε domain(f)  EqualitySub 366 368
371. (x,y) ε s  AndElimR 279
372. ((f'a),y) ε s  EqualitySub 371 357
373. ((f'a),(f'b)) ε s  EqualitySub 372 358
374. (a ε domain(f)) & (b ε domain(f))  AndInt 369 370
375. ∀x.(((x ε domain(f)) & (y ε domain(f))) -> ((((f'x),(f'y)) ε s) -> ((x,y) ε r)))
ForallInt 146
376. ((a ε domain(f)) & (y ε domain(f))) -> ((((f'a),(f'y)) ε s) -> ((a,y) ε r))
ForallElim 375
377. ∀y.(((a ε domain(f)) & (y ε domain(f))) -> ((((f'a),(f'y)) ε s) -> ((a,y) ε r)))
ForallInt 376
378. ((a ε domain(f)) & (b ε domain(f))) -> ((((f'a),(f'b)) ε s) -> ((a,b) ε r))
ForallElim 377
379. (((f'a),(f'b)) ε s) -> ((a,b) ε r)  ImpElim 374 378
380. (a,b) ε r  ImpElim 373 379
381. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})  TheoremInt
382. ∀f.(Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}))  ForallInt 381
383. Function((f)⁻¹) -> ((f)⁻¹ = {w: ∃x.∃y.((w = (x,y)) & (((f)⁻¹'x) = y))})  ForallElim
382
384. (f)⁻¹ = {w: ∃x.∃y.((w = (x,y)) & (((f)⁻¹'x) = y))}  ImpElim 148 383
385. (x,a) = (x,a)  Identity
386. ((a,x) ε f) & ((x,a) = (x,a))  AndInt 292 385
387. (y,b) = (y,b)  Identity
388. ((b,y) ε f) & ((y,b) = (y,b))  AndInt 293 387
389. ∃u.(((a,x) ε f) & (u = (x,a)))  ExistsInt 386
390. ∃v.(((b,y) ε f) & (v = (y,b)))  ExistsInt 388
391. ((a,x) ε f) & (u = (x,a))  Hyp
392. ((b,y) ε f) & (v = (y,b))  Hyp
393. ∃x.(((a,x) ε f) & (u = (x,a)))  ExistsInt 391
394. ∃a.∃x.(((a,x) ε f) & (u = (x,a)))  ExistsInt 393
395. ∃y.(((b,y) ε f) & (v = (y,b)))  ExistsInt 392
396. ∃b.∃y.(((b,y) ε f) & (v = (y,b)))  ExistsInt 395
397. u = (x,a)  AndElimR 391
398. v = (y,b)  AndElimR 392
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399. (x,a) = u  Symmetry 397
400. (y,b) = v  Symmetry 398
401. Set(a)  AndElimL 317
402. Set(x)  AndElimR 317
403. Set(b)  AndElimL 318
404. Set(y)  AndElimR 318
405. Set(x) & Set(a)  AndInt 402 401
406. Set(y) & Set(b)  AndInt 404 403
407. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 307
408. ∀y.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 407
409. (Set(x) & Set(a)) -> Set((x,a))  ForallElim 408
410. ∀y.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 407
411. (Set(x) & Set(b)) -> Set((x,b))  ForallElim 410
412. ∀x.((Set(x) & Set(b)) -> Set((x,b)))  ForallInt 411
413. (Set(y) & Set(b)) -> Set((y,b))  ForallElim 412
414. Set((x,a))  ImpElim 405 409
415. Set((y,b))  ImpElim 406 413
416. Set(u)  EqualitySub 414 399
417. Set(v)  EqualitySub 415 400
418. Set(u) & ∃a.∃x.(((a,x) ε f) & (u = (x,a)))  AndInt 416 394
419. Set(v) & ∃b.∃y.(((b,y) ε f) & (v = (y,b)))  AndInt 417 396
420. u ε {w: ∃a.∃x.(((a,x) ε f) & (w = (x,a)))}  ClassInt 418
421. v ε {w: ∃b.∃y.(((b,y) ε f) & (w = (y,b)))}  ClassInt 419
422. (r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}  DefEqInt
423. ∀r.((r)⁻¹ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))})  ForallInt 422
424. (f)⁻¹ = {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))}  ForallElim 423
425. {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} = (f)⁻¹  Symmetry 424
426. u ε (f)⁻¹  EqualitySub 420 425
427. v ε (f)⁻¹  EqualitySub 421 425
428. (x,a) ε (f)⁻¹  EqualitySub 426 397
429. (y,b) ε (f)⁻¹  EqualitySub 427 398
430. ((y,b) ε (f)⁻¹) & ((x,a) ε (f)⁻¹)  AndInt 429 428
431. ((y,b) ε (f)⁻¹) & ((x,a) ε (f)⁻¹)  ExistsElim 390 392 430
432. ((y,b) ε (f)⁻¹) & ((x,a) ε (f)⁻¹)  ExistsElim 389 391 431
433. (y,b) ε (f)⁻¹  AndElimL 432
434. (x,a) ε (f)⁻¹  AndElimR 432
435. (y,b) ε {w: ∃x.∃y.((w = (x,y)) & (((f)⁻¹'x) = y))}  EqualitySub 433 384
436. (x,a) ε {w: ∃x.∃y.((w = (x,y)) & (((f)⁻¹'x) = y))}  EqualitySub 434 384
437. Set((y,b)) & ∃x.∃x_32.(((y,b) = (x,x_32)) & (((f)⁻¹'x) = x_32))  ClassElim 435
438. Set((x,a)) & ∃x_33.∃y.(((x,a) = (x_33,y)) & (((f)⁻¹'x_33) = y))  ClassElim 436
439. ∃x.∃x_32.(((y,b) = (x,x_32)) & (((f)⁻¹'x) = x_32))  AndElimR 437
440. ∃x_33.∃y.(((x,a) = (x_33,y)) & (((f)⁻¹'x_33) = y))  AndElimR 438
441. ∃x_32.(((y,b) = (n1,x_32)) & (((f)⁻¹'n1) = x_32))  Hyp
442. ((y,b) = (n1,n2)) & (((f)⁻¹'n1) = n2)  Hyp
443. ∃y.(((x,a) = (n3,y)) & (((f)⁻¹'n3) = y))  Hyp
444. ((x,a) = (n3,n4)) & (((f)⁻¹'n3) = n4)  Hyp
445. (y,b) = (n1,n2)  AndElimL 442
446. (x,a) = (n3,n4)  AndElimL 444
447. (Set(y) & Set(b)) & ((y,b) = (n1,n2))  AndInt 406 445
448. (Set(x) & Set(a)) & ((x,a) = (n3,n4))  AndInt 405 446
449. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
450. ∀y.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 449
451. ((Set(x) & Set(b)) & ((x,b) = (u,v))) -> ((x = u) & (b = v))  ForallElim 450
452. ∀x.(((Set(x) & Set(b)) & ((x,b) = (u,v))) -> ((x = u) & (b = v)))  ForallInt 451
453. ((Set(y) & Set(b)) & ((y,b) = (u,v))) -> ((y = u) & (b = v))  ForallElim 452
454. ∀u.(((Set(y) & Set(b)) & ((y,b) = (u,v))) -> ((y = u) & (b = v)))  ForallInt 453
455. ((Set(y) & Set(b)) & ((y,b) = (n1,v))) -> ((y = n1) & (b = v))  ForallElim 454
456. ∀v.(((Set(y) & Set(b)) & ((y,b) = (n1,v))) -> ((y = n1) & (b = v)))  ForallInt 455
457. ((Set(y) & Set(b)) & ((y,b) = (n1,n2))) -> ((y = n1) & (b = n2))  ForallElim 456
458. (y = n1) & (b = n2)  ImpElim 447 457
459. ∀y.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 449
460. ((Set(x) & Set(a)) & ((x,a) = (u,v))) -> ((x = u) & (a = v))  ForallElim 459
461. ∀u.(((Set(x) & Set(a)) & ((x,a) = (u,v))) -> ((x = u) & (a = v)))  ForallInt 460
462. ((Set(x) & Set(a)) & ((x,a) = (n3,v))) -> ((x = n3) & (a = v))  ForallElim 461
463. ∀v.(((Set(x) & Set(a)) & ((x,a) = (n3,v))) -> ((x = n3) & (a = v)))  ForallInt 462
464. ((Set(x) & Set(a)) & ((x,a) = (n3,n4))) -> ((x = n3) & (a = n4))  ForallElim 463
465. (x = n3) & (a = n4)  ImpElim 448 464
466. y = n1  AndElimL 458
467. b = n2  AndElimR 458
468. x = n3  AndElimL 465
469. a = n4  AndElimR 465
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470. ((f)$^{-1}$'n1) = n2 AndElimR 442
471. ((f)$^{-1}$'n3) = n4 AndElimR 444
472. n1 = y Symmetry 466
473. n2 = b Symmetry 467
474. n3 = x Symmetry 468
475. n4 = a Symmetry 469
476. ((f)$^{-1}$'y) = n2 EqualitySub 470 472
477. ((f)$^{-1}$'y) = b EqualitySub 476 473
478. ((f)$^{-1}$'x) = n4 EqualitySub 471 474
479. ((f)$^{-1}$'x) = a EqualitySub 478 475
480. (((f)$^{-1}$'y) = b) & (((f)$^{-1}$'x) = a) AndInt 477 479
481. (((f)$^{-1}$'y) = b) & (((f)$^{-1}$'x) = a) ExistsElim 443 444 480
482. (((f)$^{-1}$'y) = b) & (((f)$^{-1}$'x) = a) ExistsElim 440 443 481
483. (((f)$^{-1}$'y) = b) & (((f)$^{-1}$'x) = a) ExistsElim 441 442 482
484. (((f)$^{-1}$'y) = b) & (((f)$^{-1}$'x) = a) ExistsElim 439 441 483
485. ((f)$^{-1}$'y) = b AndElimL 484
486. ((f)$^{-1}$'x) = a AndElimR 484
487. b = ((f)$^{-1}$'y) Symmetry 485
488. a = ((f)$^{-1}$'x) Symmetry 486
489. (a,((f)$^{-1}$'y)) ε r EqualitySub 380 487
490. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r EqualitySub 489 488
491. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r ExistsElim 303 304 490
492. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r ExistsElim 300 303 491
493. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r ExistsElim 301 302 492
494. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r ExistsElim 299 301 493
495. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r ExistsElim 291 293 494
496. (((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r ExistsElim 290 292 495
497. (((x ε domain((f)$^{-1}$)) & (y ε domain((f)$^{-1}$))) & ((x,y) ε s)) -> ((((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r) ImpInt 496
498. ∀y.((((x ε domain((f)$^{-1}$)) & (y ε domain((f)$^{-1}$))) & ((x,y) ε s)) -> ((((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r)) ForallInt 497
499. ∀x.∀y.((((x ε domain((f)$^{-1}$)) & (y ε domain((f)$^{-1}$))) & ((x,y) ε s)) -> ((((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r)) ForallInt 498
500. (Function((f)$^{-1}$) & (WellOrders(s,domain((f)$^{-1}$)) & WellOrders(r,range((f)$^{-1}$)))) & ∀x.∀y.((((x ε domain((f)$^{-1}$)) & (y ε domain((f)$^{-1}$))) & ((x,y) ε s)) -> ((((f)$^{-1}$'x),((f)$^{-1}$'y)) ε r)) AndInt 278 499
501. OrderPreserving((f)$^{-1}$,s,r) DefSub 500
502. 1-to-1(f) & OrderPreserving((f)$^{-1}$,s,r) AndInt 76 501
503. OrderPreserving(f,r,s) -> (1-to-1(f) & OrderPreserving((f)$^{-1}$,s,r)) ImpInt 502 Qed

Used Theorems

2. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
3. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))
4. 1-to-1(f) <-> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
6. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
7. Relation(r) -> (((r)$^{-1}$)$^{-1}$ = r)
8. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

FunctionApp2. (Function(f) & ((a,b) ε f)) -> ((f'a) = b)

0. Function(f) & ((a,b) ε f) Hyp
1. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
2. Function(f) AndElimL 0
3. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 2 1
4. (a,b) ε f AndElimR 0
5. (a,b) ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 4 3
6. Set((a,b)) & ∃x.∃y.(((a,b) = (x,y)) & ((f'x) = y)) ClassElim 5
7. Set((a,b)) AndElimL 6
8. ∃x.∃y.(((a,b) = (x,y)) & ((f'x) = y)) AndElimR 6
9. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
10. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 9
11. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 10
12. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 11
13. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 12
14. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 13
15. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 14
16. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 15
17. Set(a) & Set(b) ImpElim 7 16

18. ∃x.∃y.(((a,b) = (x,y)) & ((f'x) = y)) AndElimR 6
19. ∃y.(((a,b) = (u,y)) & ((f'u) = y)) Hyp
20. ((a,b) = (u,v)) & ((f'u) = v) Hyp
21. (a,b) = (u,v) AndElimL 20
22. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
23. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 22
24. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)) ForallElim 23
25. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 24
26. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)) ForallElim 25
27. (Set(a) & Set(b)) & ((a,b) = (u,v)) AndInt 17 21
28. (a = u) & (b = v) ImpElim 27 26
29. a = u AndElimL 28
30. b = v AndElimR 28
31. u = a Symmetry 29
32. v = b Symmetry 30
33. (f'u) = v AndElimR 20
34. (f'a) = v EqualitySub 33 31
35. (f'a) = b EqualitySub 34 32
36. (f'a) = b ExistsElim 19 20 35
37. (f'a) = b ExistsElim 18 19 36
38. (Function(f) & ((a,b) ε f)) -> ((f'a) = b) ImpInt 37 Qed

Used Theorems

1. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

FunctionInvApp. (Function(f) & (Function($(f)^{-1}$) & (a ε domain(f)))) -> (((f'a) ε domain($(f)^{-1}$)) & (($(f)^{-1}$'(f'a)) = a))

0. Function(f) & (Function($(f)^{-1}$) & (a ε domain(f))) Hyp
1. Function(f) AndElimL 0
2. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
3. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 1 2
4. s = (a,(f'a)) Hyp
5. (f'a) = (f'a) Identity
6. (s = (a,(f'a))) & ((f'a) = (f'a)) AndInt 4 5
7. ∃u.((s = (a,u)) & ((f'a) = u)) ExistsInt 6
8. ∃v.∃u.((s = (v,u)) & ((f'v) = u)) ExistsInt 7
9. Function($(f)^{-1}$) & (a ε domain(f)) AndElimR 0
10. a ε domain(f) AndElimR 9
11. ∃w.(a ε w) ExistsInt 10
12. Set(a) DefSub 11
13. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f)) TheoremInt
14. Function(f) & (a ε domain(f)) AndInt 1 10
15. (f'a) ε range(f) ImpElim 14 13
16. ∃w.((f'a) ε w) ExistsInt 15
17. Set((f'a)) DefSub 16
18. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
19. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 18
20. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 19
21. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 20
22. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 21
23. (Set(a) & Set(y)) -> Set((a,y)) ForallElim 22
24. ∀y.((Set(a) & Set(y)) -> Set((a,y))) ForallInt 23
25. (Set(a) & Set((f'a))) -> Set((a,(f'a))) ForallElim 24
26. Set(a) & Set((f'a)) AndInt 12 17
27. Set((a,(f'a))) ImpElim 26 25
28. (a,(f'a)) = s Symmetry 4
29. Set(s) EqualitySub 27 28
30. Set(s) & ∃v.∃u.((s = (v,u)) & ((f'v) = u)) AndInt 29 8
31. s ε {w: ∃v.∃u.((w = (v,u)) & ((f'v) = u))} ClassInt 30
32. {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} = f Symmetry 3
33. s ε f EqualitySub 31 32
34. (a,(f'a)) ε f EqualitySub 33 4
35. (s = (a,(f'a))) -> ((a,(f'a)) ε f) ImpInt 34
36. ∀s.((s = (a,(f'a))) -> ((a,(f'a)) ε f)) ForallInt 35
37. ((a,(f'a)) = (a,(f'a))) -> ((a,(f'a)) ε f) ForallElim 36
38. (a,(f'a)) = (a,(f'a)) Identity
39. (a,(f'a)) ε f ImpElim 38 37

40. (r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))} DefEqInt
41. ∀r.((r)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε r) & (z = (y,x)))}) ForallInt 40
42. (f)$^{-1}$ = {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} ForallElim 41
43. ((f'a),a) = ((f'a),a) Identity
44. ((a,(f'a)) ε f) & (((f'a),a) = ((f'a),a)) AndInt 39 43
45. ∃t.(((a,(f'a)) ε f) & (t = ((f'a),a))) ExistsInt 44
46. ((a,(f'a)) ε f) & (t = ((f'a),a)) Hyp
47. ∃u.(((a,u) ε f) & (t = (u,a))) ExistsInt 46
48. ∃v.∃u.(((v,u) ε f) & (t = (u,v))) ExistsInt 47
49. t = ((f'a),a) AndElimR 46
50. Set((f'a)) & Set(a) AndInt 17 12
51. ∀x.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 21
52. (Set((f'a)) & Set(y)) -> Set(((f'a),y)) ForallElim 51
53. ∀y.((Set((f'a)) & Set(y)) -> Set(((f'a),y))) ForallInt 52
54. (Set((f'a)) & Set(a)) -> Set(((f'a),a)) ForallElim 53
55. Set(((f'a),a)) ImpElim 50 54
56. ((f'a),a) = t Symmetry 49
57. Set(t) EqualitySub 55 56
58. Set(t) & ∃v.∃u.(((v,u) ε f) & (t = (u,v))) AndInt 57 48
59. t ε {w: ∃v.∃u.(((v,u) ε f) & (w = (u,v)))} ClassInt 58
60. {z: ∃x.∃y.(((x,y) ε f) & (z = (y,x)))} = (f)$^{-1}$ Symmetry 42
61. t ε (f)$^{-1}$ EqualitySub 59 60
62. ((f'a),a) ε (f)$^{-1}$ EqualitySub 61 49
63. ((f'a),a) ε (f)$^{-1}$ ExistsElim 45 46 62
64. (Function(f) & ((a,b) ε f)) -> ((f'a) = b) TheoremInt
65. ∀a.((Function(f) & ((a,b) ε f)) -> ((f'a) = b)) ForallInt 64
66. (Function(f) & ((x,b) ε f)) -> ((f'x) = b) ForallElim 65
67. ∀b.((Function(f) & ((x,b) ε f)) -> ((f'x) = b)) ForallInt 66
68. (Function(f) & ((x,a) ε f)) -> ((f'x) = a) ForallElim 67
69. ∀f.((Function(f) & ((x,a) ε f)) -> ((f'x) = a)) ForallInt 68
70. (Function((f)$^{-1}$) & ((x,a) ε (f)$^{-1}$)) -> (((f)$^{-1}$'x) = a) ForallElim 69
71. ∀x.((Function((f)$^{-1}$) & ((x,a) ε (f)$^{-1}$)) -> (((f)$^{-1}$'x) = a)) ForallInt 70
72. (Function((f)$^{-1}$) & (((f'a),a) ε (f)$^{-1}$)) -> (((f)$^{-1}$'(f'a)) = a) ForallElim 71
73. Function((f)$^{-1}$) AndElimL 9
74. Function((f)$^{-1}$) & (((f'a),a) ε (f)$^{-1}$) AndInt 73 63
75. ((f)$^{-1}$'(f'a)) = a ImpElim 74 72
76. (Function(f) & (Function((f)$^{-1}$) & (a ε domain(f)))) -> (((f)$^{-1}$'(f'a)) = a) ImpInt 75
77. ∃w.(((f'a),w) ε (f)$^{-1}$) ExistsInt 63
78. x = (f'a) Hyp
79. (f'a) = x Symmetry 78
80. Set(x) EqualitySub 17 79
81. ∃w.((x,w) ε (f)$^{-1}$) EqualitySub 77 79
82. Set(x) & ∃w.((x,w) ε (f)$^{-1}$) AndInt 80 81
83. x ε {w: ∃x_2.((w,x_2) ε (f)$^{-1}$)} ClassInt 82
84. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
85. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 84
86. ∀f.({x: ∃y.((x,y) ε f)} = domain(f)) ForallInt 85
87. {x: ∃y.((x,y) ε (f)$^{-1}$)} = domain((f)$^{-1}$) ForallElim 86
88. x ε domain((f)$^{-1}$) EqualitySub 83 87
89. (f'a) ε domain((f)$^{-1}$) EqualitySub 88 78
90. (x = (f'a)) -> ((f'a) ε domain((f)$^{-1}$)) ImpInt 89
91. ∀x.((x = (f'a)) -> ((f'a) ε domain((f)$^{-1}$))) ForallInt 90
92. ((f'a) = (f'a)) -> ((f'a) ε domain((f)$^{-1}$)) ForallElim 91
93. (f'a) = (f'a) Identity
94. (f'a) ε domain((f)$^{-1}$) ImpElim 93 92
95. ((f'a) ε domain((f)$^{-1}$)) & (((f)$^{-1}$'(f'a)) = a) AndInt 94 75
96. (Function(f) & (Function((f)$^{-1}$) & (a ε domain(f)))) -> (((f'a) ε domain((f)$^{-1}$)) & (((f)$^{-1}$'(f'a)) = a)) ImpInt 95 Qed

Used Theorems

1. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
2. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))
3. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
4. (Function(f) & ((a,b) ε f)) -> ((f'a) = b)

FunctionDomRange. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))

0. (a,b) ε f Hyp
1. ∃w.((a,w) ε f) ExistsInt 0
2. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt

3. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
4. ∃w.((w,b) ε f) ExistsInt 0
5. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
6. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 5
7. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 6
8. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 7
9. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 8
10. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 9
11. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 10
12. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 11
13. ∃w.((a,b) ε w) ExistsInt 0
14. Set((a,b)) DefSub 13
15. Set(a) & Set(b) ImpElim 14 12
16. Set(a) AndElimL 15
17. Set(b) AndElimR 15
18. Set(a) & ∃w.((a,w) ε f) AndInt 16 1
19. a ε {w: ∃h.((w,h) ε f)} ClassInt 18
20. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 2
21. a ε domain(f) EqualitySub 19 20
22. Set(b) & ∃w.((w,b) ε f) AndInt 17 4
23. b ε {w: ∃i.((i,w) ε f)} ClassInt 22
24. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 3
25. b ε range(f) EqualitySub 23 24
26. (a ε domain(f)) & (b ε range(f)) AndInt 21 25
27. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f))) ImpInt 26 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))

FunctionPair. (Function(f) & (x ε domain(f))) -> ((x,(f'x)) ε f)

0. Function(f) & (x ε domain(f)) Hyp
1. z = (x,(f'x)) Hyp
2. (f'x) = (f'x) Identity
3. (z = (x,(f'x))) & ((f'x) = (f'x)) AndInt 1 2
4. ∃b.((z = (x,b)) & (b = (f'x))) ExistsInt 3
5. ∃a.∃b.((z = (a,b)) & (b = (f'a))) ExistsInt 4
6. x ε domain(f) AndElimR 0
7. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f)) TheoremInt
8. ∀a.((Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))) ForallInt 7
9. (Function(f) & (x ε domain(f))) -> ((f'x) ε range(f)) ForallElim 8
10. (f'x) ε range(f) ImpElim 0 9
11. ∃w.(x ε w) ExistsInt 6
12. ∃w.((f'x) ε w) ExistsInt 10
13. Set(x) DefSub 11
14. Set((f'x)) DefSub 12
15. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
16. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 15
17. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 16
18. (Set(x) & Set(y)) -> Set((x,y)) AndElimL 17
19. ∀y.((Set(x) & Set(y)) -> Set((x,y))) ForallInt 18
20. (Set(x) & Set((f'x))) -> Set((x,(f'x))) ForallElim 19
21. Set(x) & Set((f'x)) AndInt 13 14
22. Set((x,(f'x))) ImpElim 21 20
23. (x,(f'x)) = z Symmetry 1
24. Set(z) EqualitySub 22 23
25. Set(z) & ∃a.∃b.((z = (a,b)) & (b = (f'a))) AndInt 24 5
26. z ε {w: ∃a.∃b.((w = (a,b)) & (b = (f'a)))} ClassInt 25
27. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
28. Function(f) AndElimL 0
29. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 28 27
30. {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} = f Symmetry 29
31. ∃b.((z = (x,b)) & ((f'x) = b)) ExistsInt 3
32. ∃a.∃b.((z = (a,b)) & ((f'a) = b)) ExistsInt 31
33. Set(z) & ∃a.∃b.((z = (a,b)) & ((f'a) = b)) AndInt 24 32
34. z ε {w: ∃a.∃b.((w = (a,b)) & ((f'a) = b))} ClassInt 33
35. z ε f EqualitySub 34 30
36. (x,(f'x)) ε f EqualitySub 35 1
37. (z = (x,(f'x))) -> ((x,(f'x)) ε f) ImpInt 36
38. ∀z.((z = (x,(f'x))) -> ((x,(f'x)) ε f)) ForallInt 37

39. ((x,(f'x)) = (x,(f'x))) -> ((x,(f'x)) ε f) ForallElim 38
40. (x,(f'x)) = (x,(f'x)) Identity
41. (x,(f'x)) ε f ImpElim 40 39
42. (Function(f) & (x ε domain(f))) -> ((x,(f'x)) ε f) ImpInt 41 Qed

Used Theorems

1. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})

Th97. (OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))))) -> ((f ⊂ g) v (g ⊂ f))

0. OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g)))))) Hyp
1. (Section(r,z,a) & Section(r,z,b)) -> ((a ⊂ b) v (b ⊂ a)) TheoremInt
2. ∀z.((Section(r,z,a) & Section(r,z,b)) -> ((a ⊂ b) v (b ⊂ a))) ForallInt 1
3. (Section(r,x,a) & Section(r,x,b)) -> ((a ⊂ b) v (b ⊂ a)) ForallElim 2
4. ∀a.((Section(r,x,a) & Section(r,x,b)) -> ((a ⊂ b) v (b ⊂ a))) ForallInt 3
5. (Section(r,x,domain(f)) & Section(r,x,b)) -> ((domain(f) ⊂ b) v (b ⊂ domain(f))) ForallElim 4
6. ∀b.((Section(r,x,domain(f)) & Section(r,x,b)) -> ((domain(f) ⊂ b) v (b ⊂ domain(f)))) ForallInt 5
7. (Section(r,x,domain(f)) & Section(r,x,domain(g))) -> ((domain(f) ⊂ domain(g)) v (domain(g) ⊂ domain(f))) ForallElim 6
8. OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))) AndElimR 0
9. Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g)))) AndElimR 8
10. Section(r,x,domain(f)) AndElimL 9
11. Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))) AndElimR 9
12. Section(r,x,domain(g)) AndElimL 11
13. Section(r,x,domain(f)) & Section(r,x,domain(g)) AndInt 10 12
14. (domain(f) ⊂ domain(g)) v (domain(g) ⊂ domain(f)) ImpElim 13 7
15. domain(f) ⊂ domain(g) Hyp
16. class = {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} Hyp
17. OrderPreserving(f,r,s) AndElimL 0
18. OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))) AndElimR 0
19. Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g)))) AndElimR 18
20. Section(r,x,domain(f)) AndElimL 19
21. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f))) DefExp 20
22. (domain(f) ⊂ x) & WellOrders(r,x) AndElimL 21
23. WellOrders(r,x) AndElimR 22
24. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 23
25. domain(f) ⊂ x AndElimL 22
26. ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z)) AndElimR 24
27. ((class ⊂ x) & ¬(class = 0)) -> ∃z.First(r,class,z) ForallElim 26
28. a ε class Hyp
29. a ε {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} EqualitySub 28 16
30. Set(a) & ((a ε domain(f)) & ((a ε domain(g)) & ¬((g'a) = (f'a)))) ClassElim 29
31. (a ε domain(f)) & ((a ε domain(g)) & ¬((g'a) = (f'a))) AndElimR 30
32. a ε domain(f) AndElimL 31
33. ∀z.((z ε domain(f)) -> (z ε x)) DefExp 25
34. (a ε domain(f)) -> (a ε x) ForallElim 33
35. a ε x ImpElim 32 34
36. (a ε class) -> (a ε x) ImpInt 35
37. ∀a.((a ε class) -> (a ε x)) ForallInt 36
38. class ⊂ x DefSub 37
39. ¬(class = 0) Hyp
40. (class ⊂ x) & ¬(class = 0) AndInt 38 39
41. ∃z.First(r,class,z) ImpElim 40 27
42. First(r,class,u) Hyp
43. (u ε class) & ∀y.((y ε class) -> ¬((y,u) ε r)) DefExp 42
44. u ε class AndElimL 43
45. u ε {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} EqualitySub 44 16
46. Set(u) & ((u ε domain(f)) & ((u ε domain(g)) & ¬((g'u) = (f'u)))) ClassElim 45

47. (u ε domain(f)) & ((u ε domain(g)) & ¬((g'u) = (f'u))) AndElimR 46
48. (u ε domain(g)) & ¬((g'u) = (f'u)) AndElimR 47
49. ¬((g'u) = (f'u)) AndElimR 48
50. OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))) AndElimR 0
51. Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g)))) AndElimR 50
52. Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))) AndElimR 51
53. Section(s,y,range(f)) & Section(s,y,range(g)) AndElimR 52
54. Section(s,y,range(f)) AndElimL 53
55. ((range(f) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε s)) -> (u ε range(f))) DefExp 54
56. (range(f) ⊂ y) & WellOrders(s,y) AndElimL 55
57. WellOrders(s,y) AndElimR 56
58. Connects(s,y) & ∀x_34.(((x_34 ⊂ y) & ¬(x_34 = 0)) -> ∃z.First(s,x_34,z)) DefExp 57
59. Connects(s,y) AndElimL 58
60. ∀x_38.∀z.(((x_38 ε y) & (z ε y)) -> ((x_38 = z) v (((x_38,z) ε s) v ((z,x_38) ε s)))) DefExp 59
61. ∀z.((((g'u) ε y) & (z ε y)) -> (((g'u) = z) v ((((g'u),z) ε s) v ((z,(g'u)) ε s)))) ForallElim 60
62. (((g'u) ε y) & ((f'u) ε y)) -> (((g'u) = (f'u)) v ((((g'u),(f'u)) ε s) v (((f'u),(g'u)) ε s))) ForallElim 61
63. range(f) ⊂ y AndElimL 56
64. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f)) TheoremInt
65. (Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) DefExp 17
66. Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f))) AndElimL 65
67. Function(f) AndElimL 66
68. ∀a.((Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))) ForallInt 64
69. (Function(f) & (u ε domain(f))) -> ((f'u) ε range(f)) ForallElim 68
70. u ε domain(g) AndElimL 48
71. u ε domain(f) AndElimL 47
72. Function(f) & (u ε domain(f)) AndInt 67 71
73. (f'u) ε range(f) ImpElim 72 69
74. ∀f.((Function(f) & (u ε domain(f))) -> ((f'u) ε range(f))) ForallInt 69
75. (Function(g) & (u ε domain(g))) -> ((g'u) ε range(g)) ForallElim 74
76. OrderPreserving(g,r,s) AndElimL 18
77. (Function(g) & (WellOrders(r,domain(g)) & WellOrders(s,range(g)))) & ∀u.∀v.((((u ε domain(g)) & (v ε domain(g))) & ((u,v) ε r)) -> (((g'u),(g'v)) ε s)) DefExp 76
78. Function(g) & (WellOrders(r,domain(g)) & WellOrders(s,range(g))) AndElimL 77
79. Function(g) AndElimL 78
80. Function(g) & (u ε domain(g)) AndInt 79 70
81. (g'u) ε range(g) ImpElim 80 75
82. Section(s,y,range(g)) AndElimR 53
83. ((range(g) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(g))) & ((u,v) ε s)) -> (u ε range(g))) DefExp 82
84. (range(g) ⊂ y) & WellOrders(s,y) AndElimL 83
85. range(g) ⊂ y AndElimL 84
86. ∀z.((z ε range(f)) -> (z ε y)) DefExp 63
87. ∀z.((z ε range(g)) -> (z ε y)) DefExp 85
88. ((f'u) ε range(f)) -> ((f'u) ε y) ForallElim 86
89. ((g'u) ε range(g)) -> ((g'u) ε y) ForallElim 87
90. (f'u) ε y ImpElim 73 88
91. (g'u) ε y ImpElim 81 89
92. ((g'u) ε y) & ((f'u) ε y) AndInt 91 90
93. ((g'u) = (f'u)) v ((((g'u),(f'u)) ε s) v (((f'u),(g'u)) ε s)) ImpElim 92 62
94. (g'u) = (f'u) Hyp
95. _|_ ImpElim 94 49
96. (((g'u),(f'u)) ε s) v (((f'u),(g'u)) ε s) AbsI 95
97. (((g'u),(f'u)) ε s) v (((f'u),(g'u)) ε s) Hyp
98. (((g'u),(f'u)) ε s) v (((f'u),(g'u)) ε s) OrElim 93 94 96 97 97
99. ((f'u),(g'u)) ε s Hyp
100. Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))) AndElimR 19
101. Section(s,y,range(f)) & Section(s,y,range(g)) AndElimR 100
102. Section(s,y,range(g)) AndElimR 101
103. ((range(g) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(g))) & ((u,v) ε s)) -> (u ε range(g))) DefExp 102
104. ∀u.∀v.((((u ε y) & (v ε range(g))) & ((u,v) ε s)) -> (u ε range(g))) AndElimR 103
105. ∀v.((((f'u) ε y) & (v ε range(g))) & (((f'u),v) ε s)) -> ((f'u) ε range(g))) ForallElim 104

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106. ((((f'u) ε y) & ((g'u) ε range(g))) & (((f'u),(g'u)) ε s)) -> ((f'u) ε range(g))
ForallElim 105
107. ((f'u) ε y) & ((g'u) ε range(g))  AndInt 90 81
108. (((f'u) ε y) & ((g'u) ε range(g))) & (((f'u),(g'u)) ε s)  AndInt 107 99
109. (f'u) ε range(g)  ImpElim 108 106
110. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
111. ∀f.(range(f) = {y: ∃x.((x,y) ε f)})  ForallInt 110
112. range(g) = {y: ∃x.((x,y) ε g)}  ForallElim 111
113. (f'u) ε {y: ∃x.((x,y) ε g)}  EqualitySub 109 112
114. Set((f'u)) & ∃x.((x,(f'u)) ε g)  ClassElim 113
115. ∃x.((x,(f'u)) ε g)  AndElimR 114
116. (v,(f'u)) ε g  Hyp
117. (Function(f) & ((a,b) ε f)) -> ((f'a) = b)  TheoremInt
118. ∀a.((Function(f) & ((a,b) ε f)) -> ((f'a) = b))  ForallInt 117
119. (Function(f) & ((v,b) ε f)) -> ((f'v) = b)  ForallElim 118
120. ∀f.((Function(f) & ((v,b) ε f)) -> ((f'v) = b))  ForallInt 119
121. (Function(g) & ((v,b) ε g)) -> ((g'v) = b)  ForallElim 120
122. ∀b.((Function(g) & ((v,b) ε g)) -> ((g'v) = b))  ForallInt 121
123. (Function(g) & ((v,(f'u)) ε g)) -> ((g'v) = (f'u))  ForallElim 122
124. Function(g) & ((v,(f'u)) ε g)  AndInt 79 116
125. (g'v) = (f'u)  ImpElim 124 123
126. (f'u) = (g'v)  Symmetry 125
127. ((g'v),(g'u)) ε s  EqualitySub 99 126
128. OrderPreserving(f,r,s) -> (1-to-1(f) & OrderPreserving((f)⁻¹,s,r))  TheoremInt
129. ∀f.(OrderPreserving(f,r,s) -> (1-to-1(f) & OrderPreserving((f)⁻¹,s,r)))  ForallInt
128
130. OrderPreserving(g,r,s) -> (1-to-1(g) & OrderPreserving((g)⁻¹,s,r))  ForallElim 129
131. OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) &
(Section(s,y,range(f)) & Section(s,y,range(g)))))  AndElimR 0
132. OrderPreserving(g,r,s)  AndElimL 131
133. 1-to-1(g) & OrderPreserving((g)⁻¹,s,r)  ImpElim 132 130
134. OrderPreserving((g)⁻¹,s,r)  AndElimR 133
135. (Function((g)⁻¹) & (WellOrders(s,domain((g)⁻¹)) & WellOrders(r,range((g)⁻¹)))) &
∀u.∀v.((((u ε domain((g)⁻¹)) & (v ε domain((g)⁻¹))) & ((u,v) ε s)) -> ((((g)⁻¹'u),
((g)⁻¹'v)) ε r))  DefExp 134
136. (Function(f) & (Function((f)⁻¹) & (a ε domain(f)))) -> (((f'a) ε domain((f)⁻¹)) &
(((f)⁻¹'(f'a)) = a))  TheoremInt
137. ∀f.((Function(f) & (Function((f)⁻¹) & (a ε domain(f)))) -> (((f'a) ε domain((f)⁻¹))
& (((f)⁻¹'(f'a)) = a)))  ForallInt 136
138. (Function(g) & (Function((g)⁻¹) & (a ε domain(g)))) -> (((g'a) ε domain((g)⁻¹)) &
(((g)⁻¹'(g'a)) = a))  ForallElim 137
139. ∀a.((Function(g) & (Function((g)⁻¹) & (a ε domain(g)))) -> (((g'a) ε domain((g)⁻¹))
& (((g)⁻¹'(g'a)) = a)))  ForallInt 138
140. (Function(g) & (Function((g)⁻¹) & (u ε domain(g)))) -> (((g'u) ε domain((g)⁻¹)) &
(((g)⁻¹'(g'u)) = u))  ForallElim 139
141. u ε domain(g)  AndElimL 48
142. Function(g)  AndElimL 124
143. Function((g)⁻¹) & (WellOrders(s,domain((g)⁻¹)) & WellOrders(r,range((g)⁻¹)))
AndElimL 135
144. Function((g)⁻¹)  AndElimL 143
145. Function((g)⁻¹) & (u ε domain(g))  AndInt 144 141
146. Function(g) & (Function((g)⁻¹) & (u ε domain(g)))  AndInt 142 145
147. ((g'u) ε domain((g)⁻¹)) & (((g)⁻¹'(g'u)) = u)  ImpElim 146 140
148. (g'u) ε domain((g)⁻¹)  AndElimL 147
149. ∃w.((v,w) ε g)  ExistsInt 116
150. ∃w.((v,(f'u)) ε w)  ExistsInt 116
151. Set((v,(f'u)))  DefSub 150
152. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
153. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 152
154. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 153
155. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 154
156. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 155
157. Set((v,y)) -> (Set(v) & Set(y))  ForallElim 156
158. ∀y.(Set((v,y)) -> (Set(v) & Set(y)))  ForallInt 157
159. Set((v,(f'u))) -> (Set(v) & Set((f'u)))  ForallElim 158
160. Set(v) & Set((f'u))  ImpElim 151 159
161. Set(v)  AndElimL 160
162. Set(v) & ∃w.((v,w) ε g)  AndInt 161 149
163. v ε {w: ∃x_59.((w,x_59) ε g)}  ClassInt 162
164. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
165. ∀f.(domain(f) = {x: ∃y.((x,y) ε f)})  ForallInt 164
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166. domain(g) = {x: ∃y.((x,y) ε g)}  ForallElim 165
167. {x: ∃y.((x,y) ε g)} = domain(g)  Symmetry 166
168. v ε domain(g)  EqualitySub 163 167
169. ∀a.((Function(g) & (Function((g)⁻¹) & (a ε domain(g)))) -> (((g'a) ε domain((g)⁻¹))
& (((g)⁻¹'(g'a)) = a)))  ForallInt 138
170. (Function(g) & (Function((g)⁻¹) & (v ε domain(g)))) -> (((g'v) ε domain((g)⁻¹)) &
(((g)⁻¹'(g'v)) = v))  ForallElim 169
171. Function((g)⁻¹) & (v ε domain(g))  AndInt 144 168
172. Function(g) & (Function((g)⁻¹) & (v ε domain(g)))  AndInt 142 171
173. ((g'v) ε domain((g)⁻¹)) & (((g)⁻¹'(g'v)) = v)  ImpElim 172 170
174. (g'v) ε domain((g)⁻¹)  AndElimL 173
175. ((g'u) ε domain((g)⁻¹)) & ((g'v) ε domain((g)⁻¹))  AndInt 148 174
176. ∀u.∀v.((((u ε domain((g)⁻¹)) & (v ε domain((g)⁻¹))) & ((u,v) ε s)) -> ((((g)⁻¹'u),
((g)⁻¹'v)) ε r))  AndElimR 135
177. ∀x_60.(((((g'v) ε domain((g)⁻¹)) & (x_60 ε domain((g)⁻¹))) & (((g'v),x_60) ε s)) ->
((((g)⁻¹'(g'v)),((g)⁻¹'x_60)) ε r))  ForallElim 176
178. ((((g'v) ε domain((g)⁻¹)) & ((g'u) ε domain((g)⁻¹))) & (((g'v),(g'u)) ε s)) ->
((((g)⁻¹'(g'v)),((g)⁻¹'(g'u))) ε r)  ForallElim 177
179. ((g'v) ε domain((g)⁻¹)) & ((g'u) ε domain((g)⁻¹))  AndInt 174 148
180. (((g'v) ε domain((g)⁻¹)) & ((g'u) ε domain((g)⁻¹))) & (((g'v),(g'u)) ε s)  AndInt
179 127
181. (((g)⁻¹'(g'v)),((g)⁻¹'(g'u))) ε r  ImpElim 180 178
182. ((g)⁻¹'(g'v)) = v  AndElimR 173
183. ((g)⁻¹'(g'u)) = u  AndElimR 147
184. (v,((g)⁻¹'(g'u))) ε r  EqualitySub 181 182
185. (v,u) ε r  EqualitySub 184 183
186. (u ε class) & ∀y.((y ε class) -> ¬((y,u) ε r))  DefExp 42
187. ∀y.((y ε class) -> ¬((y,u) ε r))  AndElimR 186
188. (v ε class) -> ¬((v,u) ε r)  ForallElim 187
189. (A -> B) -> (¬B -> ¬A)  TheoremInt
190. ((v ε class) -> B) -> (¬B -> ¬(v ε class))  PolySub 189
191. ((v ε class) -> ¬((v,u) ε r)) -> (¬¬((v,u) ε r) -> ¬(v ε class))  PolySub 190
192. ¬¬((v,u) ε r) -> ¬(v ε class)  ImpElim 188 191
193. D <-> ¬¬D  TheoremInt
194. (D -> ¬¬D) & (¬¬D -> D)  EquivExp 193
195. D -> ¬¬D  AndElimL 194
196. ((v,u) ε r) -> ¬¬((v,u) ε r)  PolySub 195
197. (v,u) ε r  Hyp
198. ¬¬((v,u) ε r)  ImpElim 197 196
199. ¬(v ε class)  ImpElim 198 192
200. ((v,u) ε r) -> ¬(v ε class)  ImpInt 199
201. ¬(v ε class)  ImpElim 185 200
202. Section(r,x,domain(f))  AndElimL 51
203. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε
r)) -> (u ε domain(f)))  DefExp 202
204. ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f)))  AndElimR 203
205. ∀x_67.((((v ε x) & (x_67 ε domain(f))) & ((v,x_67) ε r)) -> (v ε domain(f)))
ForallElim 204
206. (((v ε x) & (u ε domain(f))) & ((v,u) ε r)) -> (v ε domain(f))  ForallElim 205
207. u ε domain(f)  AndElimL 47
208. Section(r,x,domain(g))  AndElimL 52
209. ((domain(g) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(g))) & ((u,v) ε
r)) -> (u ε domain(g)))  DefExp 208
210. (domain(g) ⊂ x) & WellOrders(r,x)  AndElimL 209
211. domain(g) ⊂ x  AndElimL 210
212. ∀z.((z ε domain(g)) -> (z ε x))  DefExp 211
213. (v ε domain(g)) -> (v ε x)  ForallElim 212
214. v ε x  ImpElim 168 213
215. (v ε x) & (u ε domain(f))  AndInt 214 207
216. ((v ε x) & (u ε domain(f))) & ((v,u) ε r)  AndInt 215 197
217. v ε domain(f)  ImpElim 216 206
218. ¬((g'v) = (f'v))  Hyp
219. (v ε domain(g)) & ¬((g'v) = (f'v))  AndInt 168 218
220. (v ε domain(f)) & ((v ε domain(g)) & ¬((g'v) = (f'v)))  AndInt 217 219
221. Set(v) & ((v ε domain(f)) & ((v ε domain(g)) & ¬((g'v) = (f'v))))  AndInt 161 220
222. v ε {w: ((w ε domain(f)) & ((w ε domain(g)) & ¬((g'w) = (f'w))))}  ClassInt 221
223. {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} = class  Symmetry 16
224. v ε class  EqualitySub 222 223
225. _|_  ImpElim 224 201
226. ¬¬((g'v) = (f'v))  ImpInt 225
227. D <-> ¬¬D  TheoremInt
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228. (D -> ¬¬D) & (¬¬D -> D) EquivExp 227
229. ¬¬D -> D AndElimR 228
230. ¬¬((g'v) = (f'v)) -> ((g'v) = (f'v)) PolySub 229
231. (g'v) = (f'v) ImpElim 226 230
232. (f'u) = (f'v) EqualitySub 231 125
233. (Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) DefExp 17
234. ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) AndElimR 233
235. ∀x_76.((((v ε domain(f)) & (x_76 ε domain(f))) & ((v,x_76) ε r)) -> (((f'v),(f'x_76)) ε s)) ForallElim 234
236. (((v ε domain(f)) & (u ε domain(f))) & ((v,u) ε r)) -> (((f'v),(f'u)) ε s) ForallElim 235
237. (v ε domain(f)) & (u ε domain(f)) AndInt 217 207
238. ((v ε domain(f)) & (u ε domain(f))) & ((v,u) ε r) AndInt 237 185
239. ((f'v),(f'u)) ε s ImpElim 238 236
240. ((f'v),(f'v)) ε s EqualitySub 239 232
241. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x)) TheoremInt
242. ∀r.(WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))) ForallInt 241
243. WellOrders(s,x) -> (Asymmetric(s,x) & TransIn(s,x)) ForallElim 242
244. ∀x.(WellOrders(s,x) -> (Asymmetric(s,x) & TransIn(s,x))) ForallInt 243
245. WellOrders(s,y) -> (Asymmetric(s,y) & TransIn(s,y)) ForallElim 244
246. Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))) AndElimR 9
247. Section(s,y,range(f)) & Section(s,y,range(g)) AndElimR 246
248. Section(s,y,range(f)) AndElimL 247
249. ((range(f) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε s)) -> (u ε range(f))) DefExp 248
250. (range(f) ⊂ y) & WellOrders(s,y) AndElimL 249
251. WellOrders(s,y) AndElimR 250
252. Asymmetric(s,y) & TransIn(s,y) ImpElim 251 245
253. Asymmetric(s,y) AndElimL 252
254. ∀x_82.∀z.(((x_82 ε y) & (z ε y)) -> (((x_82,z) ε s) -> ¬((z,x_82) ε s))) DefExp 253
255. ∀z.((((f'v) ε y) & (z ε y)) -> ((((f'v),z) ε s) -> ¬((z,(f'v)) ε s))) ForallElim 254
256. (((f'v) ε y) & ((f'v) ε y)) -> ((((f'v),(f'v)) ε s) -> ¬(((f'v),(f'v)) ε s)) ForallElim 255
257. range(f) ⊂ y AndElimL 250
258. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f)) TheoremInt
259. ∀a.((Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))) ForallInt 258
260. (Function(f) & (v ε domain(f))) -> ((f'v) ε range(f)) ForallElim 259
261. Function(f) & (v ε domain(f)) AndInt 67 217
262. (f'v) ε range(f) ImpElim 261 260
263. ∀z.((z ε range(f)) -> (z ε y)) DefExp 257
264. ((f'v) ε range(f)) -> ((f'v) ε y) ForallElim 263
265. (f'v) ε y ImpElim 262 264
266. ((f'v) ε y) & ((f'v) ε y) AndInt 265 265
267. (((f'v),(f'v)) ε s) -> ¬(((f'v),(f'v)) ε s) ImpElim 266 256
268. ¬(((f'v),(f'v)) ε s) ImpElim 240 267
269. _|_ ImpElim 240 268
270. ¬((v,u) ε r) ImpInt 269
271. _|_ ImpElim 185 270
272. _|_ ExistsElim 115 116 271
273. ((g'u),(f'u)) ε s Hyp
274. Section(s,y,range(f)) AndElimL 101
275. ((range(f) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε s)) -> (u ε range(f))) DefExp 274
276. ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε s)) -> (u ε range(f))) AndElimR 275
277. ∀v.(((((g'u) ε y) & (v ε range(f))) & (((g'u),v) ε s)) -> ((g'u) ε range(f))) ForallElim 276
278. ((((g'u) ε y) & ((f'u) ε range(f))) & (((g'u),(f'u)) ε s)) -> ((g'u) ε range(f)) ForallElim 277
279. u ε domain(f) AndElimL 47
280. u ε domain(g) AndElimL 48
281. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f)) TheoremInt
282. ∀a.((Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))) ForallInt 281
283. (Function(f) & (u ε domain(f))) -> ((f'u) ε range(f)) ForallElim 282
284. Function(f) & (u ε domain(f)) AndInt 67 279
285. (f'u) ε range(f) ImpElim 284 283
286. ∀f.((Function(f) & (u ε domain(f))) -> ((f'u) ε range(f))) ForallInt 283
287. (Function(g) & (u ε domain(g))) -> ((g'u) ε range(g)) ForallElim 286
288. Function(g) & (u ε domain(g)) AndInt 79 280

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289. (g'u) ε range(g)  ImpElim 288 287
290. ∀z.((z ε range(g)) -> (z ε y))  DefExp 85
291. ((g'u) ε range(g)) -> ((g'u) ε y)  ForallElim 290
292. (g'u) ε y  ImpElim 289 291
293. ((g'u) ε y) & ((f'u) ε range(f))  AndInt 292 285
294. (((g'u) ε y) & ((f'u) ε range(f))) & (((g'u),(f'u)) ε s)  AndInt 293 273
295. (g'u) ε range(f)  ImpElim 294 278
296. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
297. (g'u) ε {y: ∃x.((x,y) ε f)}  EqualitySub 295 296
298. Set((g'u)) & ∃x.((x,(g'u)) ε f)  ClassElim 297
299. ∃x.((x,(g'u)) ε f)  AndElimR 298
300. (v,(g'u)) ε f  Hyp
301. (Function(f) & ((a,b) ε f)) -> ((f'a) = b)  TheoremInt
302. ∀a.((Function(f) & ((a,b) ε f)) -> ((f'a) = b))  ForallInt 301
303. (Function(f) & ((v,b) ε f)) -> ((f'v) = b)  ForallElim 302
304. ∀b.((Function(f) & ((v,b) ε f)) -> ((f'v) = b))  ForallInt 303
305. (Function(f) & ((v,(g'u)) ε f)) -> ((f'v) = (g'u))  ForallElim 304
306. Function(f) & ((v,(g'u)) ε f)  AndInt 67 300
307. (f'v) = (g'u)  ImpElim 306 305
308. (g'u) = (f'v)  Symmetry 307
309. ((f'v),(f'u)) ε s  EqualitySub 273 308
310. OrderPreserving(f,r,s)  AndElimL 0
311. OrderPreserving(f,r,s) -> (1-to-1(f) & OrderPreserving((f)⁻¹,s,r))  TheoremInt
312. 1-to-1(f) & OrderPreserving((f)⁻¹,s,r)  ImpElim 310 311
313. OrderPreserving((f)⁻¹,s,r)  AndElimR 312
314. (Function((f)⁻¹) & (WellOrders(s,domain((f)⁻¹)) & WellOrders(r,range((f)⁻¹)))) &
∀u.∀v.((((u ε domain((f)⁻¹)) & (v ε domain((f)⁻¹))) & ((u,v) ε s)) -> ((((f)⁻¹'u),
((f)⁻¹'v)) ε r))  DefExp 313
315. ∀u.∀v.((((u ε domain((f)⁻¹)) & (v ε domain((f)⁻¹))) & ((u,v) ε s)) -> ((((f)⁻¹'u),
((f)⁻¹'v)) ε r))  AndElimR 314
316. ∀x_93.(((((f'v) ε domain((f)⁻¹)) & (x_93 ε domain((f)⁻¹))) & (((f'v),x_93) ε s)) ->
((((f)⁻¹'(f'v)),((f)⁻¹'x_93)) ε r))  ForallElim 315
317. ((((f'v) ε domain((f)⁻¹)) & ((f'u) ε domain((f)⁻¹))) & (((f'v),(f'u)) ε s)) ->
((((f)⁻¹'(f'v)),((f)⁻¹'(f'u))) ε r)  ForallElim 316
318. ∃w.((v,w) ε f)  ExistsInt 300
319. ∃w.((v,(g'u)) ε w)  ExistsInt 300
320. Set((v,(g'u)))  DefSub 319
321. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
322. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 321
323. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 322
324. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 323
325. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 324
326. Set((v,y)) -> (Set(v) & Set(y))  ForallElim 325
327. ∀y.(Set((v,y)) -> (Set(v) & Set(y)))  ForallInt 326
328. Set((v,(g'u))) -> (Set(v) & Set((g'u)))  ForallElim 327
329. Set(v) & Set((g'u))  ImpElim 320 328
330. Set(v)  AndElimL 329
331. Set(v) & ∃w.((v,w) ε f)  AndInt 330 318
332. v ε {w: ∃x_95.((w,x_95) ε f)}  ClassInt 331
333. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
334. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 333
335. v ε domain(f)  EqualitySub 332 334
336. ∀a.((Function(f) & (a ε domain(f))) -> ((f'a) ε range(f)))  ForallInt 281
337. (Function(f) & (v ε domain(f))) -> ((f'v) ε range(f))  ForallElim 336
338. Function(f) & (v ε domain(f))  AndInt 67 335
339. (f'v) ε range(f)  ImpElim 338 337
340. ((f'u) ε range(f)) & ((f'v) ε range(f))  AndInt 285 339
341. (Function(f) & (Function((f)⁻¹) & (a ε domain(f)))) -> (((f'a) ε domain((f)⁻¹)) &
(((f)⁻¹'(f'a)) = a))  TheoremInt
342. OrderPreserving((f)⁻¹,s,r)  AndElimR 312
343. (Function((f)⁻¹) & (WellOrders(s,domain((f)⁻¹)) & WellOrders(r,range((f)⁻¹)))) &
∀u.∀v.((((u ε domain((f)⁻¹)) & (v ε domain((f)⁻¹))) & ((u,v) ε s)) -> ((((f)⁻¹'u),
((f)⁻¹'v)) ε r))  DefExp 342
344. Function((f)⁻¹) & (WellOrders(s,domain((f)⁻¹)) & WellOrders(r,range((f)⁻¹)))
AndElimL 343
345. Function((f)⁻¹)  AndElimL 344
346. ∀a.((Function(f) & (Function((f)⁻¹) & (a ε domain(f)))) -> (((f'a) ε domain((f)⁻¹))
& (((f)⁻¹'(f'a)) = a)))  ForallInt 341
347. (Function(f) & (Function((f)⁻¹) & (v ε domain(f)))) -> (((f'v) ε domain((f)⁻¹)) &
(((f)⁻¹'(f'v)) = v))  ForallElim 346
348. Function((f)⁻¹) & (v ε domain(f))  AndInt 345 335
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349. Function(f) & (Function((f)⁻¹) & (v ε domain(f)))  AndInt 67 348
350. ((f'v) ε domain((f)⁻¹)) & (((f)⁻¹'(f'v)) = v)  ImpElim 349 347
351. Function((f)⁻¹) & (u ε domain(f))  AndInt 345 279
352. Function(f) & (Function((f)⁻¹) & (u ε domain(f)))  AndInt 67 351
353. ∀a.((Function(f) & (Function((f)⁻¹) & (a ε domain(f)))) -> (((f'a) ε domain((f)⁻¹)) & (((f)⁻¹'(f'a)) = a)))  ForallInt 341
354. (Function(f) & (Function((f)⁻¹) & (u ε domain(f)))) -> (((f'u) ε domain((f)⁻¹)) & (((f)⁻¹'(f'u)) = u))  ForallElim 353
355. ((f'u) ε domain((f)⁻¹)) & (((f)⁻¹'(f'u)) = u)  ImpElim 352 354
356. (f'v) ε domain((f)⁻¹)  AndElimL 350
357. (f'u) ε domain((f)⁻¹)  AndElimL 355
358. ((f'v) ε domain((f)⁻¹)) & ((f'u) ε domain((f)⁻¹))  AndInt 356 357
359. (((f'v) ε domain((f)⁻¹)) & ((f'u) ε domain((f)⁻¹))) & (((f'v),(f'u)) ε s)  AndInt 358 309
360. (((f)⁻¹'(f'v)),((f)⁻¹'(f'u))) ε r  ImpElim 359 317
361. ((f)⁻¹'(f'v)) = v  AndElimR 350
362. ((f)⁻¹'(f'u)) = u  AndElimR 355
363. (v,((f)⁻¹'(f'u))) ε r  EqualitySub 360 361
364. (v,u) ε r  EqualitySub 363 362
365. ¬(v ε class)  ImpElim 364 200
366. ¬((g'v) = (f'v))  Hyp
367. (u ε domain(g)) & (v ε domain(f))  AndInt 280 335
368. OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g)))))  AndElimR 0
369. Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))  AndElimR 368
370. Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g)))  AndElimR 369
371. Section(r,x,domain(g))  AndElimL 370
372. ((domain(g) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(g))) & ((u,v) ε r)) -> (u ε domain(g)))  DefExp 371
373. ∀u.∀v.((((u ε x) & (v ε domain(g))) & ((u,v) ε r)) -> (u ε domain(g)))  AndElimR 372
374. ∀x_102.((((v ε x) & (x_102 ε domain(g))) & ((v,x_102) ε r)) -> (v ε domain(g)))  ForallElim 373
375. (((v ε x) & (u ε domain(g))) & ((v,u) ε r)) -> (v ε domain(g))  ForallElim 374
376. Section(r,x,domain(f))  AndElimL 369
377. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f)))  DefExp 376
378. (domain(f) ⊂ x) & WellOrders(r,x)  AndElimL 377
379. domain(f) ⊂ x  AndElimL 378
380. ∀z.((z ε domain(f)) -> (z ε x))  DefExp 379
381. (v ε domain(f)) -> (v ε x)  ForallElim 380
382. v ε domain(f)  AndElimR 367
383. v ε x  ImpElim 382 381
384. u ε domain(g)  AndElimL 367
385. (v ε x) & (u ε domain(g))  AndInt 383 384
386. ((v ε x) & (u ε domain(g))) & ((v,u) ε r)  AndInt 385 364
387. v ε domain(g)  ImpElim 386 375
388. (v ε domain(g)) & ¬((g'v) = (f'v))  AndInt 387 366
389. (v ε domain(f)) & ((v ε domain(g)) & ¬((g'v) = (f'v)))  AndInt 382 388
390. ∃w.(v ε w)  ExistsInt 383
391. Set(v)  DefSub 390
392. Set(v) & ((v ε domain(f)) & ((v ε domain(g)) & ¬((g'v) = (f'v))))  AndInt 391 389
393. v ε {w: ((w ε domain(f)) & ((w ε domain(g)) & ¬((g'w) = (f'w))))}  ClassInt 392
394. {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} = class  Symmetry 16
395. v ε class  EqualitySub 393 394
396. _|_  ImpElim 395 365
397. ¬¬((g'v) = (f'v))  ImpInt 396
398. ¬¬((g'v) = (f'v)) -> ((g'v) = (f'v))  PolySub 229
399. (g'v) = (f'v)  ImpElim 397 398
400. (f'v) = (g'v)  Symmetry 399
401. (g'u) = (g'v)  EqualitySub 308 400
402. 1-to-1(f) <-> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))  TheoremInt
403. (1-to-1(f) -> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))) & ((Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))) -> 1-to-1(f))  EquivExp 402
404. 1-to-1(f) -> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y))))  AndElimL 403
405. ∀f.(1-to-1(f) -> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y))) -> ¬((f'x) = (f'y)))))  ForallInt 404
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406. 1-to-1(g) -> (Function(g) & ∀x.∀y.(((x ε domain(g)) & ((y ε domain(g)) & ¬(x = y))) -> ¬((g'x) = (g'y)))) ForallElim 405
407. OrderPreserving(f,r,s) -> (1-to-1(f) & OrderPreserving((f)$^{-1}$,s,r)) TheoremInt
408. ∀f.(OrderPreserving(f,r,s) -> (1-to-1(f) & OrderPreserving((f)$^{-1}$,s,r))) ForallInt 407
409. OrderPreserving(g,r,s) -> (1-to-1(g) & OrderPreserving((g)$^{-1}$,s,r)) ForallElim 408
410. OrderPreserving(g,r,s) AndElimL 368
411. 1-to-1(g) & OrderPreserving((g)$^{-1}$,s,r) ImpElim 410 409
412. 1-to-1(g) AndElimL 411
413. Function(g) & ∀x.∀y.(((x ε domain(g)) & ((y ε domain(g)) & ¬(x = y))) -> ¬((g'x) = (g'y))) ImpElim 412 406
414. ∀x.∀y.(((x ε domain(g)) & ((y ε domain(g)) & ¬(x = y))) -> ¬((g'x) = (g'y))) AndElimR 413
415. ∀y.(((u ε domain(g)) & ((y ε domain(g)) & ¬(u = y))) -> ¬((g'u) = (g'y))) ForallElim 414
416. ((u ε domain(g)) & ((v ε domain(g)) & ¬(u = v))) -> ¬((g'u) = (g'v)) ForallElim 415
417. (u ε domain(f)) & (u ε domain(g)) AndInt 279 280
418. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x)) TheoremInt
419. Asymmetric(r,x) & TransIn(r,x) ImpElim 23 418
420. Asymmetric(r,x) AndElimL 419
421. ∀y.∀z.(((y ε x) & (z ε x)) -> (((y,z) ε r) -> ¬((z,y) ε r))) DefExp 420
422. ∀z.(((v ε x) & (z ε x)) -> (((v,z) ε r) -> ¬((z,v) ε r))) ForallElim 421
423. ((v ε x) & (u ε x)) -> (((v,u) ε r) -> ¬((u,v) ε r)) ForallElim 422
424. (u ε domain(f)) -> (u ε x) ForallElim 380
425. u ε domain(f) AndElimL 417
426. u ε x ImpElim 425 424
427. (v ε x) & (u ε x) AndInt 383 426
428. ((v,u) ε r) -> ¬((u,v) ε r) ImpElim 427 423
429. ¬((u,v) ε r) ImpElim 364 428
430. u = v Hyp
431. (v,v) ε r EqualitySub 364 430
432. ¬((v,v) ε r) EqualitySub 429 430
433. _|_ ImpElim 431 432
434. ¬(u = v) ImpInt 433
435. u ε domain(g) AndElimR 417
436. (v ε domain(g)) & ¬(u = v) AndInt 387 434
437. (u ε domain(g)) & ((v ε domain(g)) & ¬(u = v)) AndInt 384 436
438. ¬((g'u) = (g'v)) ImpElim 437 416
439. _|_ ImpElim 401 438
440. _|_ ExistsElim 299 300 439
441. _|_ OrElim 98 273 440 99 272
442. _|_ ExistsElim 41 42 441
443. ¬¬(class = 0) ImpInt 442
444. ¬¬(class = 0) -> (class = 0) PolySub 229
445. class = 0 ImpElim 443 444
446. {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} = 0 EqualitySub 445 16
447. (class = {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))}) -> ({z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} = 0) ImpInt 446
448. ∀class.((class = {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))}) -> ({z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} = 0)) ForallInt 447
449. ({z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} = {x_111: ((x_111 ε domain(f)) & ((x_111 ε domain(g)) & ¬((g'x_111) = (f'x_111))))}) -> ({x_111: ((x_111 ε domain(f)) & ((x_111 ε domain(g)) & ¬((g'x_111) = (f'x_111))))} = 0) ForallElim 448
450. {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} = {z: ((z ε domain(f)) & ((z ε domain(g)) & ¬((g'z) = (f'z))))} Identity
451. {x_111: ((x_111 ε domain(f)) & ((x_111 ε domain(g)) & ¬((g'x_111) = (f'x_111))))} = 0 ImpElim 450 449
452. z ε f Hyp
453. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}) TheoremInt
454. f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} ImpElim 67 453
455. z ε {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))} EqualitySub 452 454
456. Set(z) & ∃x.∃y.((z = (x,y)) & ((f'x) = y)) ClassElim 455
457. ∃x.∃y.((z = (x,y)) & ((f'x) = y)) AndElimR 456
458. ∃y.((z = (a,y)) & ((f'a) = y)) Hyp
459. (z = (a,b)) & ((f'a) = b) Hyp
460. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f))) TheoremInt
461. z = (a,b) AndElimL 459
462. (a,b) ε f EqualitySub 452 461
463. (a ε domain(f)) & (b ε range(f)) ImpElim 462 460
464. a ε domain(f) AndElimL 463

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465. ∀z.((z ε domain(f)) -> (z ε domain(g)))  DefExp 15
466. (a ε domain(f)) -> (a ε domain(g))  ForallElim 465
467. a ε domain(g)  ImpElim 464 466
468. ¬((g'a) = (f'a))  Hyp
469. (a ε domain(g)) & ¬((g'a) = (f'a))  AndInt 467 468
470. (a ε domain(f)) & ((a ε domain(g)) & ¬((g'a) = (f'a)))  AndInt 464 469
471. ∃w.(a ε w)  ExistsInt 464
472. Set(a)  DefSub 471
473. Set(a) & ((a ε domain(f)) & ((a ε domain(g)) & ¬((g'a) = (f'a))))  AndInt 472 470
474. a ε {w: ((w ε domain(f)) & ((w ε domain(g)) & ¬((g'w) = (f'w))))}  ClassInt 473
475. a ε 0  EqualitySub 474 451
476. 0 = {x: ¬(x = x)}  DefEqInt
477. a ε {x: ¬(x = x)}  EqualitySub 475 476
478. Set(a) & ¬(a = a)  ClassElim 477
479. ¬(a = a)  AndElimR 478
480. a = a  Identity
481. _|_  ImpElim 480 479
482. ¬¬((g'a) = (f'a))  ImpInt 481
483. ¬¬((g'a) = (f'a)) -> ((g'a) = (f'a))  PolySub 229
484. (g'a) = (f'a)  ImpElim 482 483
485. (f'a) = b  AndElimR 459
486. b = (f'a)  Symmetry 485
487. (f'a) = (g'a)  Symmetry 484
488. b = (g'a)  EqualitySub 486 487
489. z = (a,(g'a))  EqualitySub 461 488
490. (Function(f) & (x ε domain(f))) -> ((x,(f'x)) ε f)  TheoremInt
491. ∀f.((Function(f) & (x ε domain(f))) -> ((x,(f'x)) ε f))  ForallInt 490
492. (Function(g) & (x ε domain(g))) -> ((x,(g'x)) ε g)  ForallElim 491
493. ∀x.((Function(g) & (x ε domain(g))) -> ((x,(g'x)) ε g))  ForallInt 492
494. (Function(g) & (a ε domain(g))) -> ((a,(g'a)) ε g)  ForallElim 493
495. Function(g) & (a ε domain(g))  AndInt 79 467
496. (a,(g'a)) ε g  ImpElim 495 494
497. (a,(g'a)) = z  Symmetry 489
498. z ε g  EqualitySub 496 497
499. z ε g  ExistsElim 458 459 498
500. z ε g  ExistsElim 457 458 499
501. (z ε f) -> (z ε g)  ImpInt 500
502. ∀z.((z ε f) -> (z ε g))  ForallInt 501
503. f ⊂ g  DefSub 502
504. domain(g) ⊂ domain(f)  Hyp
505. z ε g  Hyp
506. ∀f.(Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))}))  ForallInt 453
507. Function(g) -> (g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))})  ForallElim 506
508. g = {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  ImpElim 79 507
509. z ε {w: ∃x.∃y.((w = (x,y)) & ((g'x) = y))}  EqualitySub 505 508
510. Set(z) & ∃x.∃y.((z = (x,y)) & ((g'x) = y))  ClassElim 509
511. ∃x.∃y.((z = (x,y)) & ((g'x) = y))  AndElimR 510
512. ∃y.((z = (a,y)) & ((g'a) = y))  Hyp
513. (z = (a,b)) & ((g'a) = b)  Hyp
514. z = (a,b)  AndElimL 513
515. (a,b) ε g  EqualitySub 505 514
516. ∀f.(((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f))))  ForallInt 460
517. ((a,b) ε g) -> ((a ε domain(g)) & (b ε range(g)))  ForallElim 516
518. (a ε domain(g)) & (b ε range(g))  ImpElim 515 517
519. ∀z.((z ε domain(g)) -> (z ε domain(f)))  DefExp 504
520. (a ε domain(g)) -> (a ε domain(f))  ForallElim 519
521. a ε domain(g)  AndElimL 518
522. a ε domain(f)  ImpElim 521 520
523. ¬((g'a) = (f'a))  Hyp
524. (a ε domain(g)) & ¬((g'a) = (f'a))  AndInt 521 523
525. (a ε domain(f)) & ((a ε domain(g)) & ¬((g'a) = (f'a)))  AndInt 522 524
526. ∃w.(a ε w)  ExistsInt 521
527. Set(a)  DefSub 526
528. Set(a) & ((a ε domain(f)) & ((a ε domain(g)) & ¬((g'a) = (f'a))))  AndInt 527 525
529. a ε {w: ((w ε domain(f)) & ((w ε domain(g)) & ¬((g'w) = (f'w))))}  ClassInt 528
530. a ε 0  EqualitySub 529 451
531. a ε {x: ¬(x = x)}  EqualitySub 530 476
532. Set(a) & ¬(a = a)  ClassElim 531
533. ¬(a = a)  AndElimR 532
534. a = a  Identity
535. _|_  ImpElim 534 533
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536. ¬¬((g'a) = (f'a))  ImpInt 535
537. (g'a) = (f'a)  ImpElim 536 483
538. (g'a) = b  AndElimR 513
539. b = (g'a)  Symmetry 538
540. b = (f'a)  EqualitySub 539 537
541. z = (a,(f'a))  EqualitySub 514 540
542. (Function(f) & (x ε domain(f))) -> ((x,(f'x)) ε f)  TheoremInt
543. ∀x.((Function(f) & (x ε domain(f))) -> ((x,(f'x)) ε f))  ForallInt 542
544. (Function(f) & (a ε domain(f))) -> ((a,(f'a)) ε f)  ForallElim 543
545. Function(f) & (a ε domain(f))  AndInt 67 522
546. (a,(f'a)) ε f  ImpElim 545 544
547. (a,(f'a)) = z  Symmetry 541
548. z ε f  EqualitySub 546 547
549. z ε f  ExistsElim 512 513 548
550. z ε f  ExistsElim 511 512 549
551. (z ε g) -> (z ε f)  ImpInt 550
552. ∀z.((z ε g) -> (z ε f))  ForallInt 551
553. g ⊂ f  DefSub 552
554. (f ⊂ g) v (g ⊂ f)  OrIntR 503
555. (f ⊂ g) v (g ⊂ f)  OrIntL 553
556. (f ⊂ g) v (g ⊂ f)  OrElim 14 15 554 504 555
557. (OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) &
(Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))))) -> ((f ⊂ g)
v (g ⊂ f))  ImpInt 556 Qed
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Used Theorems

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1. (Section(r,z,a) & Section(r,z,b)) -> ((a ⊂ b) v (b ⊂ a))
2. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))
3. (Function(f) & ((a,b) ε f)) -> ((f'a) = b)
4. OrderPreserving(f,r,s) -> (1-to-1(f) & OrderPreserving((f)⁻¹,s,r))
5. (Function(f) & (Function((f)⁻¹) & (a ε domain(f)))) -> (((f'a) ε domain((f)⁻¹)) &
(((f)⁻¹'(f'a)) = a))
6. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
8. (A -> B) -> (¬B -> ¬A)
9. D <-> ¬¬D
10. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
11. (Function(f) & (a ε domain(f))) -> ((f'a) ε range(f))
12. 1-to-1(f) <-> (Function(f) & ∀x.∀y.(((x ε domain(f)) & ((y ε domain(f)) & ¬(x = y)))
-> ¬((f'x) = (f'y))))
13. Function(f) -> (f = {w: ∃x.∃y.((w = (x,y)) & ((f'x) = y))})
14. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))
15. (Function(f) & (x ε domain(f))) -> ((x,(f'x)) ε f)
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PairEquals. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))

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0. Set((a,b)) & ((a,b) = (x,y))  Hyp
1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
2. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 1
3. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 2
4. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 3
5. Set((a,b))  AndElimL 0
6. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 4
7. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 6
8. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 7
9. Set((a,b)) -> (Set(a) & Set(b))  ForallElim 8
10. Set(a) & Set(b)  ImpElim 5 9
11. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))  TheoremInt
12. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)))  ForallInt 11
13. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))  ForallElim 12
14. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)))  ForallInt 13
15. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))  ForallElim 14
16. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)))  ForallInt 15
17. ((Set(a) & Set(b)) & ((a,b) = (x,v))) -> ((a = x) & (b = v))  ForallElim 16
18. ∀v.(((Set(a) & Set(b)) & ((a,b) = (x,v))) -> ((a = x) & (b = v)))  ForallInt 17
19. ((Set(a) & Set(b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  ForallElim 18
20. (a,b) = (x,y)  AndElimR 0
21. (Set(a) & Set(b)) & ((a,b) = (x,y))  AndInt 10 20
22. (a = x) & (b = y)  ImpElim 21 19
23. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  ImpInt 22 Qed
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Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))

WellOrdersSubset. (WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b)

0. WellOrders(r,a) & (b ⊂ a) Hyp
1. (x ε b) & (y ε b) Hyp
2. b ⊂ a AndElimR 0
3. ∀z.((z ε b) -> (z ε a)) DefExp 2
4. (x ε b) -> (x ε a) ForallElim 3
5. (y ε b) -> (y ε a) ForallElim 3
6. x ε b AndElimL 1
7. y ε b AndElimR 1
8. x ε a ImpElim 6 4
9. y ε a ImpElim 7 5
10. WellOrders(r,a) AndElimL 0
11. Connects(r,a) & ∀y.(((y ⊂ a) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 10
12. Connects(r,a) AndElimL 11
13. ∀y.∀z.(((y ε a) & (z ε a)) -> ((y = z) v (((y,z) ε r) v ((z,y) ε r)))) DefExp 12
14. ∀z.(((x ε a) & (z ε a)) -> ((x = z) v (((x,z) ε r) v ((z,x) ε r)))) ForallElim 13
15. ((x ε a) & (y ε a)) -> ((x = y) v (((x,y) ε r) v ((y,x) ε r))) ForallElim 14
16. (x ε a) & (y ε a) AndInt 8 9
17. (x = y) v (((x,y) ε r) v ((y,x) ε r)) ImpElim 16 15
18. ((x ε b) & (y ε b)) -> ((x = y) v (((x,y) ε r) v ((y,x) ε r))) ImpInt 17
19. ∀y.(((x ε b) & (y ε b)) -> ((x = y) v (((x,y) ε r) v ((y,x) ε r)))) ForallInt 18
20. ∀x.∀y.(((x ε b) & (y ε b)) -> ((x = y) v (((x,y) ε r) v ((y,x) ε r)))) ForallInt 19
21. Connects(r,b) DefSub 20
22. (y ⊂ b) & ¬(y = 0) Hyp
23. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z) TheoremInt
24. ∀y.(((y ⊂ a) & ¬(y = 0)) -> ∃z.First(r,y,z)) AndElimR 11
25. ((y ⊂ a) & ¬(y = 0)) -> ∃z.First(r,y,z) ForallElim 24
26. y ⊂ b AndElimL 22
27. ∀y.(((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)) ForallInt 23
28. ((x ⊂ b) & (b ⊂ z)) -> (x ⊂ z) ForallElim 27
29. ∀z.(((x ⊂ b) & (b ⊂ z)) -> (x ⊂ z)) ForallInt 28
30. ((x ⊂ b) & (b ⊂ a)) -> (x ⊂ a) ForallElim 29
31. ∀x.(((x ⊂ b) & (b ⊂ a)) -> (x ⊂ a)) ForallInt 30
32. ((y ⊂ b) & (b ⊂ a)) -> (y ⊂ a) ForallElim 31
33. (y ⊂ b) & (b ⊂ a) AndInt 26 2
34. y ⊂ a ImpElim 33 32
35. ¬(y = 0) AndElimR 22
36. (y ⊂ a) & ¬(y = 0) AndInt 34 35
37. ∃z.First(r,y,z) ImpElim 36 25
38. ((y ⊂ b) & ¬(y = 0)) -> ∃z.First(r,y,z) ImpInt 37
39. ∀y.(((y ⊂ b) & ¬(y = 0)) -> ∃z.First(r,y,z)) ForallInt 38
40. Connects(r,b) & ∀y.(((y ⊂ b) & ¬(y = 0)) -> ∃z.First(r,y,z)) AndInt 21 39
41. WellOrders(r,b) DefSub 40
42. (WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b) ImpInt 41 Qed

Used Theorems

1. ((x ⊂ y) & (y ⊂ z)) -> (x ⊂ z)

ContCompl. ((y ⊂ x) & ((x ~ y) = 0)) -> (x = y)

0. (y ⊂ x) & ((x ~ y) = 0) Hyp
1. a ε x Hyp
2. ¬(a ε y) Hyp
3. ∃x.(a ε x) ExistsInt 1
4. Set(a) DefSub 3
5. Set(a) & ¬(a ε y) AndInt 4 2
6. a ε {w: ¬(w ε y)} ClassInt 5
7. ~x = {y: ¬(y ε x)} DefEqInt
8. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 7
9. ~y = {i: ¬(i ε y)} ForallElim 8
10. {i: ¬(i ε y)} = ~y Symmetry 9
11. a ε ~y EqualitySub 6 10
12. (a ε x) & (a ε ~y) AndInt 1 11
13. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))

TheoremInt
14. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 13
15. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 14
16. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 15
17. ∀z.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 16
18. ((a ε x) & (a ε y)) -> (a ε (x ∩ y)) ForallElim 17
19. ∀y.(((a ε x) & (a ε y)) -> (a ε (x ∩ y))) ForallInt 18
20. ((a ε x) & (a ε ~y)) -> (a ε (x ∩ ~y)) ForallElim 19
21. a ε (x ∩ ~y) ImpElim 12 20
22. (x ~ y) = (x ∩ ~y) DefEqInt
23. (x ∩ ~y) = (x ~ y) Symmetry 22
24. a ε (x ~ y) EqualitySub 21 23
25. (x ~ y) = 0 AndElimR 0
26. a ε 0 EqualitySub 24 25
27. 0 = {x: ¬(x = x)} DefEqInt
28. a ε {x: ¬(x = x)} EqualitySub 26 27
29. Set(a) & ¬(a = a) ClassElim 28
30. ¬(a = a) AndElimR 29
31. a = a Identity
32. _|_ ImpElim 31 30
33. ¬¬(a ε y) ImpInt 32
34. D <-> ¬¬D TheoremInt
35. (D -> ¬¬D) & (¬¬D -> D) EquivExp 34
36. ¬¬D -> D AndElimR 35
37. ¬¬(a ε y) -> (a ε y) PolySub 36
38. a ε y ImpElim 33 37
39. (a ε x) -> (a ε y) ImpInt 38
40. ∀a.((a ε x) -> (a ε y)) ForallInt 39
41. x ⊂ y DefSub 40
42. y ⊂ x AndElimL 0
43. (x ⊂ y) & (y ⊂ x) AndInt 41 42
44. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
45. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y)) EquivExp 44
46. ((x ⊂ y) & (y ⊂ x)) -> (x = y) AndElimR 45
47. x = y ImpElim 43 46
48. ((y ⊂ x) & ((x ~ y) = 0)) -> (x = y) ImpInt 47 Qed

Used Theorems

1. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
2. D <-> ¬¬D
3. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

Th99. (WellOrders(r,x) & WellOrders(s,y)) -> ∃f.((OrderPreserving(f,r,s) &
(Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f))))

0. WellOrders(r,x) & WellOrders(s,y) Hyp
1. f = {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))}
Hyp
2. a ε f Hyp
3. a ε {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))}
EqualitySub 2 1
4. Set(a) & ∃u.∃v.((a = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))
ClassElim 3
5. ∃u.∃v.((a = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) &
(Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) AndElimR 4
6. ∃v.((a = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) &
(Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) Hyp
7. (a = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) &
(Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))) Hyp
8. a = (u,v) AndElimL 7
9. ∃v.(a = (u,v)) ExistsInt 8
10. ∃u.∃v.(a = (u,v)) ExistsInt 9
11. ∃u.∃v.(a = (u,v)) ExistsElim 6 7 10
12. ∃u.∃v.(a = (u,v)) ExistsElim 5 6 11
13. (a ε f) -> ∃u.∃v.(a = (u,v)) ImpInt 12
14. ∀a.((a ε f) -> ∃u.∃v.(a = (u,v))) ForallInt 13

15. Relation(f) DefSub 14
16. ((a,b) ε f) & ((a,c) ε f) Hyp
17. (a,b) ε f AndElimL 16
18. (a,c) ε f AndElimR 16
19. (a,b) ε {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} EqualitySub 17 1
20. (a,c) ε {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} EqualitySub 18 1
21. Set((a,b)) & ∃u.∃v.(((a,b) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) ClassElim 19
22. Set((a,c)) & ∃u.∃v.(((a,c) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) ClassElim 20
23. ∃u.∃v.(((a,b) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) AndElimR 21
24. ∃u.∃v.(((a,c) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) AndElimR 22
25. ∃v.(((a,b) = (u1,v)) & ((u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v) ε g))))))) Hyp
26. ((a,b) = (u1,v1)) & ((u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g)))))) Hyp
27. ∃v.(((a,c) = (u2,v)) & ((u2 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u2 ε domain(g)) & ((u2,v) ε g))))))) Hyp
28. ((a,c) = (u2,v2)) & ((u2 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u2 ε domain(g)) & ((u2,v2) ε g)))))) Hyp
29. (u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g))))) AndElimR 26
30. (u2 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u2 ε domain(g)) & ((u2,v2) ε g))))) AndElimR 28
31. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g))))) AndElimR 29
32. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u2 ε domain(g)) & ((u2,v2) ε g))))) AndElimR 30
33. OrderPreserving(g1,r,s) & (Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1)))) Hyp
34. OrderPreserving(g2,r,s) & (Section(r,x,domain(g2)) & (Section(s,y,range(g2)) & ((u2 ε domain(g2)) & ((u2,v2) ε g2)))) Hyp
35. (OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))))) -> ((f ⊂ g) v (g ⊂ f)) TheoremInt
36. ∀f.((OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))))) -> ((f ⊂ g) v (g ⊂ f))) ForallInt 35
37. (OrderPreserving(g1,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(g1)) & (Section(r,x,domain(g)) & (Section(s,y,range(g1)) & Section(s,y,range(g))))))) -> ((g1 ⊂ g) v (g ⊂ g1)) ForallElim 36
38. ∀g.((OrderPreserving(g1,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(g1)) & (Section(r,x,domain(g)) & (Section(s,y,range(g1)) & Section(s,y,range(g))))))) -> ((g1 ⊂ g) v (g ⊂ g1))) ForallInt 37
39. (OrderPreserving(g1,r,s) & (OrderPreserving(g2,r,s) & (Section(r,x,domain(g1)) & (Section(r,x,domain(g2)) & (Section(s,y,range(g1)) & Section(s,y,range(g2))))))) -> ((g1 ⊂ g2) v (g2 ⊂ g1)) ForallElim 38
40. OrderPreserving(g1,r,s) AndElimL 33
41. Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1))) AndElimR 33
42. Section(r,x,domain(g1)) AndElimL 41
43. Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1)) AndElimR 41
44. Section(s,y,range(g1)) AndElimL 43
45. (u1 ε domain(g1)) & ((u1,v1) ε g1) AndElimR 43
46. OrderPreserving(g2,r,s) AndElimL 34
47. Section(r,x,domain(g2)) & (Section(s,y,range(g2)) & ((u2 ε domain(g2)) & ((u2,v2) ε g2))) AndElimR 34
48. Section(r,x,domain(g2)) AndElimL 47
49. Section(s,y,range(g2)) & ((u2 ε domain(g2)) & ((u2,v2) ε g2)) AndElimR 47

50. Section(s,y,range(g2)) AndElimL 49
51. (u2 ε domain(g2)) & ((u2,v2) ε g2) AndElimR 49
52. Section(s,y,range(g1)) & Section(s,y,range(g2)) AndInt 44 50
53. Section(r,x,domain(g2)) & (Section(s,y,range(g1)) & Section(s,y,range(g2))) AndInt 48 52
54. Section(r,x,domain(g1)) & (Section(r,x,domain(g2)) & (Section(s,y,range(g1)) & Section(s,y,range(g2)))) AndInt 42 53
55. OrderPreserving(g2,r,s) & (Section(r,x,domain(g1)) & (Section(r,x,domain(g2)) & (Section(s,y,range(g1)) & Section(s,y,range(g2))))) AndInt 46 54
56. OrderPreserving(g1,r,s) & (OrderPreserving(g2,r,s) & (Section(r,x,domain(g1)) & (Section(r,x,domain(g2)) & (Section(s,y,range(g1)) & Section(s,y,range(g2)))))) AndInt 40 55
57. (g1 ⊂ g2) v (g2 ⊂ g1) ImpElim 56 39
58. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
59. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 58
60. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 59
61. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 60
62. Set((a,b)) AndElimL 21
63. Set((a,c)) AndElimL 22
64. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 61
65. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 64
66. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 65
67. Set((a,b)) -> (Set(a) & Set(b)) ForallElim 66
68. Set(a) & Set(b) ImpElim 62 67
69. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 65
70. Set((a,c)) -> (Set(a) & Set(c)) ForallElim 69
71. Set(a) & Set(c) ImpElim 63 70
72. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v)) TheoremInt
73. (a,b) = (u1,v1) AndElimL 26
74. (a,c) = (u2,v2) AndElimL 28
75. (Set(a) & Set(b)) & ((a,b) = (u1,v1)) AndInt 68 73
76. (Set(a) & Set(c)) & ((a,c) = (u2,v2)) AndInt 71 74
77. ∀x.(((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))) ForallInt 72
78. ((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v)) ForallElim 77
79. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 78
80. ((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)) ForallElim 79
81. ∀u.(((Set(a) & Set(b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))) ForallInt 80
82. ((Set(a) & Set(b)) & ((a,b) = (u1,v))) -> ((a = u1) & (b = v)) ForallElim 81
83. ∀v.(((Set(a) & Set(b)) & ((a,b) = (u1,v))) -> ((a = u1) & (b = v))) ForallInt 82
84. ((Set(a) & Set(b)) & ((a,b) = (u1,v1))) -> ((a = u1) & (b = v1)) ForallElim 83
85. (a = u1) & (b = v1) ImpElim 75 84
86. ∀y.(((Set(a) & Set(y)) & ((a,y) = (u,v))) -> ((a = u) & (y = v))) ForallInt 78
87. ((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v)) ForallElim 86
88. ∀u.(((Set(a) & Set(c)) & ((a,c) = (u,v))) -> ((a = u) & (c = v))) ForallInt 87
89. ((Set(a) & Set(c)) & ((a,c) = (u2,v))) -> ((a = u2) & (c = v)) ForallElim 88
90. ∀v.(((Set(a) & Set(c)) & ((a,c) = (u2,v))) -> ((a = u2) & (c = v))) ForallInt 89
91. ((Set(a) & Set(c)) & ((a,c) = (u2,v2))) -> ((a = u2) & (c = v2)) ForallElim 90
92. (a = u2) & (c = v2) ImpElim 76 91
93. (u1,v1) ε g1 AndElimR 45
94. (u2,v2) ε g2 AndElimR 51
95. a = u1 AndElimL 85
96. b = v1 AndElimR 85
97. a = u2 AndElimL 92
98. c = v2 AndElimR 92
99. u1 = a Symmetry 95
100. v1 = b Symmetry 96
101. u2 = a Symmetry 97
102. v2 = c Symmetry 98
103. (a,v1) ε g1 EqualitySub 93 99
104. (a,b) ε g1 EqualitySub 103 100
105. (a,v2) ε g2 EqualitySub 94 101
106. (a,c) ε g2 EqualitySub 105 102
107. g1 ⊂ g2 Hyp
108. ∀z.((z ε g1) -> (z ε g2)) DefExp 107
109. ((a,b) ε g1) -> ((a,b) ε g2) ForallElim 108
110. (a,b) ε g2 ImpElim 104 109
111. OrderPreserving(g2,r,s) AndElimL 55
112. (Function(g2) & (WellOrders(r,domain(g2)) & WellOrders(s,range(g2)))) & ∀u.∀v.((((u ε domain(g2)) & (v ε domain(g2))) & ((u,v) ε r)) -> (((g2'u),(g2'v)) ε s)) DefExp 111
113. Function(g2) & (WellOrders(r,domain(g2)) & WellOrders(s,range(g2))) AndElimL 112
114. Function(g2) AndElimL 113

115. Relation(g2) & ∀x.∀y.∀z.((((x,y) ε g2) & ((x,z) ε g2)) -> (y = z)) DefExp 114
116. ∀x.∀y.∀z.((((x,y) ε g2) & ((x,z) ε g2)) -> (y = z)) AndElimR 115
117. ∀y.∀z.((((a,y) ε g2) & ((a,z) ε g2)) -> (y = z)) ForallElim 116
118. ∀z.((((a,b) ε g2) & ((a,z) ε g2)) -> (b = z)) ForallElim 117
119. (((a,b) ε g2) & ((a,c) ε g2)) -> (b = c) ForallElim 118
120. ((a,b) ε g2) & ((a,c) ε g2) AndInt 110 106
121. b = c ImpElim 120 119
122. g2 ⊂ g1 Hyp
123. ∀z.((z ε g2) -> (z ε g1)) DefExp 122
124. ((a,c) ε g2) -> ((a,c) ε g1) ForallElim 123
125. (a,c) ε g1 ImpElim 106 124
126. OrderPreserving(g1,r,s) AndElimL 56
127. (Function(g1) & (WellOrders(r,domain(g1)) & WellOrders(s,range(g1)))) & ∀u.∀v.((((u ε domain(g1)) & (v ε domain(g1))) & ((u,v) ε r)) -> (((g1'u),(g1'v)) ε s)) DefExp 126
128. Function(g1) & (WellOrders(r,domain(g1)) & WellOrders(s,range(g1))) AndElimL 127
129. Function(g1) AndElimL 128
130. Relation(g1) & ∀x.∀y.∀z.((((x,y) ε g1) & ((x,z) ε g1)) -> (y = z)) DefExp 129
131. ∀x.∀y.∀z.((((x,y) ε g1) & ((x,z) ε g1)) -> (y = z)) AndElimR 130
132. ∀y.∀z.((((a,y) ε g1) & ((a,z) ε g1)) -> (y = z)) ForallElim 131
133. ∀z.((((a,b) ε g1) & ((a,z) ε g1)) -> (b = z)) ForallElim 132
134. (((a,b) ε g1) & ((a,c) ε g1)) -> (b = c) ForallElim 133
135. ((a,b) ε g1) & ((a,c) ε g1) AndInt 104 125
136. b = c ImpElim 135 134
137. b = c OrElim 57 107 121 122 136
138. b = c ExistsElim 32 34 137
139. b = c ExistsElim 31 33 138
140. b = c ExistsElim 27 28 139
141. b = c ExistsElim 24 27 140
142. b = c ExistsElim 25 26 141
143. b = c ExistsElim 23 25 142
144. (((a,b) ε f) & ((a,c) ε f)) -> (b = c) ImpInt 143
145. ∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 144
146. ∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 145
147. ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) ForallInt 146
148. Relation(f) & ∀a.∀b.∀c.((((a,b) ε f) & ((a,c) ε f)) -> (b = c)) AndInt 15 147
149. Function(f) DefSub 148
150. ((a ε x) & (b ε domain(f))) & ((a,b) ε r) Hyp
151. (a ε x) & (b ε domain(f)) AndElimL 150
152. b ε domain(f) AndElimR 151
153. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
154. b ε {x: ∃y.((x,y) ε f)} EqualitySub 152 153
155. Set(b) & ∃y.((b,y) ε f) ClassElim 154
156. ∃y.((b,y) ε f) AndElimR 155
157. (b,j) ε f Hyp
158. (b,j) ε {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} EqualitySub 157 1
159. Set((b,j)) & ∃u.∃v.(((b,j) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) ClassElim 158
160. ∃u.∃v.(((b,j) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) AndElimR 159
161. ∃v.(((b,j) = (u1,v)) & ((u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v) ε g))))))) Hyp
162. ((b,j) = (u1,v1)) & ((u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g)))))) Hyp
163. (u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g))))) AndElimR 162
164. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g))))) AndElimR 163
165. OrderPreserving(g1,r,s) & (Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1)))) Hyp
166. Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1))) AndElimR 165
167. Section(r,x,domain(g1)) AndElimL 166
168. Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1)) AndElimR 166
169. (u1 ε domain(g1)) & ((u1,v1) ε g1) AndElimR 168
170. u1 ε domain(g1) AndElimL 169
171. (b,j) = (u1,v1) AndElimL 162

172. Set((b,j)) AndElimL 159
173. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
174. ∀b.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))) ForallInt 173
175. (Set((a,j)) & ((a,j) = (x,y))) -> ((a = x) & (j = y)) ForallElim 174
176. ∀a.((Set((a,j)) & ((a,j) = (x,y))) -> ((a = x) & (j = y))) ForallInt 175
177. (Set((b,j)) & ((b,j) = (x,y))) -> ((b = x) & (j = y)) ForallElim 176
178. ∀x.((Set((b,j)) & ((b,j) = (x,y))) -> ((b = x) & (j = y))) ForallInt 177
179. (Set((b,j)) & ((b,j) = (u1,y))) -> ((b = u1) & (j = y)) ForallElim 178
180. ∀y.((Set((b,j)) & ((b,j) = (u1,y))) -> ((b = u1) & (j = y))) ForallInt 179
181. (Set((b,j)) & ((b,j) = (u1,v1))) -> ((b = u1) & (j = v1)) ForallElim 180
182. Set((b,j)) & ((b,j) = (u1,v1)) AndInt 172 171
183. (b = u1) & (j = v1) ImpElim 182 181
184. b = u1 AndElimL 183
185. j = v1 AndElimR 183
186. u1 = b Symmetry 184
187. v1 = j Symmetry 185
188. b ε domain(g1) EqualitySub 170 186
189. (u1,v1) ε g1 AndElimR 169
190. (b,v1) ε g1 EqualitySub 189 186
191. (b,j) ε g1 EqualitySub 190 187
192. (a,b) ε r AndElimR 150
193. (a ε x) & (b ε domain(f)) AndElimL 150
194. a ε x AndElimL 193
195. ((domain(g1) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(g1))) & ((u,v) ε r)) -> (u ε domain(g1))) DefExp 167
196. ∀u.∀v.((((u ε x) & (v ε domain(g1))) & ((u,v) ε r)) -> (u ε domain(g1))) AndElimR 195
197. ∀v.((((a ε x) & (v ε domain(g1))) & ((a,v) ε r)) -> (a ε domain(g1))) ForallElim 196
198. (((a ε x) & (b ε domain(g1))) & ((a,b) ε r)) -> (a ε domain(g1)) ForallElim 197
199. (a ε x) & (b ε domain(g1)) AndInt 194 188
200. ((a ε x) & (b ε domain(g1))) & ((a,b) ε r) AndInt 199 192
201. a ε domain(g1) ImpElim 200 198
202. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
203. ∀f.(domain(f) = {x: ∃y.((x,y) ε f)}) ForallInt 202
204. domain(g1) = {x: ∃y.((x,y) ε g1)} ForallElim 203
205. a ε {x: ∃y.((x,y) ε g1)} EqualitySub 201 204
206. Set(a) & ∃y.((a,y) ε g1) ClassElim 205
207. ∃y.((a,y) ε g1) AndElimR 206
208. (a,d) ε g1 Hyp
209. w = (a,d) Hyp
210. (a ε domain(g1)) & ((a,d) ε g1) AndInt 201 208
211. Section(s,y,range(g1)) AndElimL 168
212. Section(s,y,range(g1)) & ((a ε domain(g1)) & ((a,d) ε g1)) AndInt 211 210
213. Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((a ε domain(g1)) & ((a,d) ε g1))) AndInt 167 212
214. OrderPreserving(g1,r,s) AndElimL 165
215. OrderPreserving(g1,r,s) & (Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((a ε domain(g1)) & ((a,d) ε g1)))) AndInt 214 213
216. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((a ε domain(g)) & ((a,d) ε g))))) ExistsInt 215
217. (a ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((a ε domain(g)) & ((a,d) ε g))))) AndInt 194 216
218. (w = (a,d)) & ((a ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((a ε domain(g)) & ((a,d) ε g)))))) AndInt 209 217
219. ∃d.((w = (a,d)) & ((a ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((a ε domain(g)) & ((a,d) ε g))))))) ExistsInt 218
220. ∃a.∃d.((w = (a,d)) & ((a ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((a ε domain(g)) & ((a,d) ε g))))))) ExistsInt 219
221. ∃w.((a,d) ε w) ExistsInt 208
222. Set((a,d)) DefSub 221
223. (a,d) = w Symmetry 209
224. Set(w) EqualitySub 222 223
225. Set(w) & ∃a.∃d.((w = (a,d)) & ((a ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((a ε domain(g)) & ((a,d) ε g))))))) AndInt 224 220
226. w ε {w: ∃a.∃d.((w = (a,d)) & ((a ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((a ε domain(g)) & ((a,d) ε g)))))))} ClassInt 225
227. {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))}

= f Symmetry 1
228. w ε f EqualitySub 226 227
229. (a,d) ε f EqualitySub 228 209
230. (w = (a,d)) -> ((a,d) ε f) ImpInt 229
231. ∀w.((w = (a,d)) -> ((a,d) ε f)) ForallInt 230
232. ((a,d) = (a,d)) -> ((a,d) ε f) ForallElim 231
233. (a,d) = (a,d) Identity
234. (a,d) ε f ImpElim 233 232
235. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f))) TheoremInt
236. ∀b.(((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))) ForallInt 235
237. ((a,d) ε f) -> ((a ε domain(f)) & (d ε range(f))) ForallElim 236
238. (a ε domain(f)) & (d ε range(f)) ImpElim 234 237
239. a ε domain(f) AndElimL 238
240. a ε domain(f) ExistsElim 207 208 239
241. a ε domain(f) ExistsElim 164 165 240
242. a ε domain(f) ExistsElim 161 162 241
243. a ε domain(f) ExistsElim 160 161 242
244. a ε domain(f) ExistsElim 156 157 243
245. (((a ε x) & (b ε domain(f))) & ((a,b) ε r)) -> (a ε domain(f)) ImpInt 244
246. WellOrders(r,x) AndElimL 0
247. w ε domain(f) Hyp
248. w ε {x: ∃y.((x,y) ε f)} EqualitySub 247 202
249. Set(w) & ∃y.((w,y) ε f) ClassElim 248
250. ∃y.((w,y) ε f) AndElimR 249
251. (w,y1) ε f Hyp
252. (w,y1) ε {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} EqualitySub 251 1
253. Set((w,y1)) & ∃u.∃v.(((w,y1) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) ClassElim 252
254. ∃u.∃v.(((w,y1) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) AndElimR 253
255. ∃v.(((w,y1) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) Hyp
256. ((w,y1) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))) Hyp
257. (u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))) AndElimR 256
258. u ε x AndElimL 257
259. (w,y1) = (u,v) AndElimL 256
260. Set((w,y1)) AndElimL 253
261. Set((w,y1)) & ((w,y1) = (u,v)) AndInt 260 259
262. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
263. ∀x.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))) ForallInt 262
264. (Set((a,b)) & ((a,b) = (u,y))) -> ((a = u) & (b = y)) ForallElim 263
265. ∀y.((Set((a,b)) & ((a,b) = (u,y))) -> ((a = u) & (b = y))) ForallInt 264
266. (Set((a,b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v)) ForallElim 265
267. ∀a.((Set((a,b)) & ((a,b) = (u,v))) -> ((a = u) & (b = v))) ForallInt 266
268. (Set((w,b)) & ((w,b) = (u,v))) -> ((w = u) & (b = v)) ForallElim 267
269. ∀b.((Set((w,b)) & ((w,b) = (u,v))) -> ((w = u) & (b = v))) ForallInt 268
270. (Set((w,y)) & ((w,y) = (u,v))) -> ((w = u) & (y = v)) ForallElim 269
271. ∀y.((Set((w,y)) & ((w,y) = (u,v))) -> ((w = u) & (y = v))) ForallInt 270
272. (Set((w,y1)) & ((w,y1) = (u,v))) -> ((w = u) & (y1 = v)) ForallElim 271
273. (w = u) & (y1 = v) ImpElim 261 272
274. w = u AndElimL 273
275. u = w Symmetry 274
276. w ε x EqualitySub 258 275
277. w ε x ExistsElim 255 256 276
278. w ε x ExistsElim 254 255 277
279. w ε x ExistsElim 250 251 278
280. (w ε domain(f)) -> (w ε x) ImpInt 279
281. ∀w.((w ε domain(f)) -> (w ε x)) ForallInt 280
282. domain(f) ⊂ x DefSub 281
283. (domain(f) ⊂ x) & WellOrders(r,x) AndInt 282 246
284. ∀b.((((a ε x) & (b ε domain(f))) & ((a,b) ε r)) -> (a ε domain(f))) ForallInt 245
285. ∀a.∀b.((((a ε x) & (b ε domain(f))) & ((a,b) ε r)) -> (a ε domain(f))) ForallInt 284
286. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀a.∀b.((((a ε x) & (b ε domain(f))) & ((a,b) ε

r)) -> (a ε domain(f))) AndInt 283 285
287. Section(r,x,domain(f)) DefSub 286
288. ((a ε y) & (b ε range(f))) & ((a,b) ε s) Hyp
289. (a ε y) & (b ε range(f)) AndElimL 288
290. b ε range(f) AndElimR 289
291. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
292. b ε {y: ∃x.((x,y) ε f)} EqualitySub 290 291
293. Set(b) & ∃x.((x,b) ε f) ClassElim 292
294. ∃x.((x,b) ε f) AndElimR 293
295. (i,b) ε f Hyp
296. (i,b) ε {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} EqualitySub 295 1
297. Set((i,b)) & ∃u.∃v.(((i,b) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) ClassElim 296
298. ∃u.∃v.(((i,b) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g))))))) AndElimR 297
299. ∃v.(((i,b) = (u1,v)) & ((u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v) ε g))))))) Hyp
300. ((i,b) = (u1,v1)) & ((u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g)))))) Hyp
301. (u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g))))) AndElimR 300
302. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g))))) AndElimR 301
303. OrderPreserving(g1,r,s) & (Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1)))) Hyp
304. Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1))) AndElimR 303
305. Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1)) AndElimR 304
306. Section(s,y,range(g1)) AndElimL 305
307. ((range(g1) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(g1))) & ((u,v) ε s)) -> (u ε range(g1))) DefExp 306
308. ∀u.∀v.((((u ε y) & (v ε range(g1))) & ((u,v) ε s)) -> (u ε range(g1))) AndElimR 307
309. (i,b) = (u1,v1) AndElimL 300
310. Set((i,b)) AndElimL 297
311. Set((i,b)) & ((i,b) = (u1,v1)) AndInt 310 309
312. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
313. ∀a.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))) ForallInt 312
314. (Set((i,b)) & ((i,b) = (x,y))) -> ((i = x) & (b = y)) ForallElim 313
315. ∀x.((Set((i,b)) & ((i,b) = (x,y))) -> ((i = x) & (b = y))) ForallInt 314
316. (Set((i,b)) & ((i,b) = (u1,y))) -> ((i = u1) & (b = y)) ForallElim 315
317. ∀y.((Set((i,b)) & ((i,b) = (u1,y))) -> ((i = u1) & (b = y))) ForallInt 316
318. (Set((i,b)) & ((i,b) = (u1,v1))) -> ((i = u1) & (b = v1)) ForallElim 317
319. (i = u1) & (b = v1) ImpElim 311 318
320. b = v1 AndElimR 319
321. i = u1 AndElimL 319
322. v1 = b Symmetry 320
323. u1 = i Symmetry 321
324. ∀v.((((a ε y) & (v ε range(g1))) & ((a,v) ε s)) -> (a ε range(g1))) ForallElim 308
325. (((a ε y) & (b ε range(g1))) & ((a,b) ε s)) -> (a ε range(g1)) ForallElim 324
326. Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1)) AndElimR 304
327. (u1 ε domain(g1)) & ((u1,v1) ε g1) AndElimR 326
328. (u1,v1) ε g1 AndElimR 327
329. (u1,b) ε g1 EqualitySub 328 322
330. (i,b) ε g1 EqualitySub 329 323
331. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f))) TheoremInt
332. ∀a.(((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))) ForallInt 331
333. ((i,b) ε f) -> ((i ε domain(f)) & (b ε range(f))) ForallElim 332
334. ∀f.(((i,b) ε f) -> ((i ε domain(f)) & (b ε range(f)))) ForallInt 333
335. ((i,b) ε g1) -> ((i ε domain(g1)) & (b ε range(g1))) ForallElim 334
336. (i ε domain(g1)) & (b ε range(g1)) ImpElim 330 335
337. b ε range(g1) AndElimR 336
338. (a ε y) & (b ε range(f)) AndElimL 288
339. (a,b) ε s AndElimR 288
340. a ε y AndElimL 338
341. (a ε y) & (b ε range(g1)) AndInt 340 337
342. ((a ε y) & (b ε range(g1))) & ((a,b) ε s) AndInt 341 339

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343. a ε range(g1)  ImpElim 342 325
344. range(f) = {y: ∃x.((x,y) ε f)}  DefEqInt
345. ∀f.(range(f) = {y: ∃x.((x,y) ε f)})  ForallInt 344
346. range(g1) = {y: ∃x.((x,y) ε g1)}  ForallElim 345
347. a ε {y: ∃x.((x,y) ε g1)}  EqualitySub 343 346
348. Set(a) & ∃x.((x,a) ε g1)  ClassElim 347
349. ∃x.((x,a) ε g1)  AndElimR 348
350. (k,a) ε g1  Hyp
351. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))  TheoremInt
352. ∀a.(((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f))))  ForallInt 351
353. ((k,b) ε f) -> ((k ε domain(f)) & (b ε range(f)))  ForallElim 352
354. ∀b.(((k,b) ε f) -> ((k ε domain(f)) & (b ε range(f))))  ForallInt 353
355. ((k,a) ε f) -> ((k ε domain(f)) & (a ε range(f)))  ForallElim 354
356. ∀f.(((k,a) ε f) -> ((k ε domain(f)) & (a ε range(f))))  ForallInt 355
357. ((k,a) ε g1) -> ((k ε domain(g1)) & (a ε range(g1)))  ForallElim 356
358. (k ε domain(g1)) & (a ε range(g1))  ImpElim 350 357
359. k ε domain(g1)  AndElimL 358
360. (k ε domain(g1)) & ((k,a) ε g1)  AndInt 359 350
361. Section(s,y,range(g1))  AndElimL 326
362. OrderPreserving(g1,r,s)  AndElimL 303
363. Section(r,x,domain(g1))  AndElimL 304
364. Section(s,y,range(g1)) & ((k ε domain(g1)) & ((k,a) ε g1))  AndInt 361 360
365. Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((k ε domain(g1)) & ((k,a) ε
g1)))  AndInt 363 364
366. OrderPreserving(g1,r,s) & (Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((k ε
domain(g1)) & ((k,a) ε g1))))  AndInt 362 365
367. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((k
ε domain(g)) & ((k,a) ε g)))))  ExistsInt 366
368. ((domain(g1) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(g1))) & ((u,v)
ε r)) -> (u ε domain(g1)))  DefExp 363
369. (domain(g1) ⊂ x) & WellOrders(r,x)  AndElimL 368
370. domain(g1) ⊂ x  AndElimL 369
371. ∀z.((z ε domain(g1)) -> (z ε x))  DefExp 370
372. (k ε domain(g1)) -> (k ε x)  ForallElim 371
373. k ε x  ImpElim 359 372
374. (k ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) &
(Section(s,y,range(g)) & ((k ε domain(g)) & ((k,a) ε g)))))  AndInt 373 367
375. v = (k,a)  Hyp
376. (v = (k,a)) & ((k ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) &
(Section(s,y,range(g)) & ((k ε domain(g)) & ((k,a) ε g))))))  AndInt 375 374
377. ∃a.((v = (k,a)) & ((k ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) &
(Section(s,y,range(g)) & ((k ε domain(g)) & ((k,a) ε g)))))))  ExistsInt 376
378. ∃k.∃a.((v = (k,a)) & ((k ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g))
& (Section(s,y,range(g)) & ((k ε domain(g)) & ((k,a) ε g)))))))  ExistsInt 377
379. ∃w.((k,a) ε w)  ExistsInt 350
380. Set((k,a))  DefSub 379
381. (k,a) = v  Symmetry 375
382. Set(v)  EqualitySub 380 381
383. Set(v) & ∃k.∃a.((v = (k,a)) & ((k ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((k ε domain(g)) & ((k,a) ε g)))))))
AndInt 382 378
384. v ε {w: ∃k.∃a.((w = (k,a)) & ((k ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((k ε domain(g)) & ((k,a) ε g)))))))}
ClassInt 383
385. {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))}
= f  Symmetry 1
386. v ε f  EqualitySub 384 385
387. (k,a) ε f  EqualitySub 386 375
388. (v = (k,a)) -> ((k,a) ε f)  ImpInt 387
389. ∀v.((v = (k,a)) -> ((k,a) ε f))  ForallInt 388
390. ((k,a) = (k,a)) -> ((k,a) ε f)  ForallElim 389
391. (k,a) = (k,a)  Identity
392. (k,a) ε f  ImpElim 391 390
393. ∃w.((w,a) ε f)  ExistsInt 392
394. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
395. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 394
396. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 395
397. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 396
398. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 397
399. Set((k,y)) -> (Set(k) & Set(y))  ForallElim 398
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400. ∀y.(Set((k,y)) -> (Set(k) & Set(y))) ForallInt 399
401. Set((k,a)) -> (Set(k) & Set(a)) ForallElim 400
402. Set(k) & Set(a) ImpElim 380 401
403. Set(a) AndElimR 402
404. Set(a) & ∃w.((w,a) ε f) AndInt 403 393
405. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
406. a ε {w: ∃x_66.((x_66,w) ε f)} ClassInt 404
407. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 405
408. a ε range(f) EqualitySub 406 407
409. a ε range(f) ExistsElim 349 350 408
410. a ε range(f) ExistsElim 302 303 409
411. a ε range(f) ExistsElim 299 300 410
412. a ε range(f) ExistsElim 298 299 411
413. a ε range(f) ExistsElim 294 295 412
414. (((a ε y) & (b ε range(f))) & ((a,b) ε s)) -> (a ε range(f)) ImpInt 413
415. j ε range(f) Hyp
416. j ε {y: ∃x.((x,y) ε f)} EqualitySub 415 405
417. Set(j) & ∃x.((x,j) ε f) ClassElim 416
418. ∃x.((x,j) ε f) AndElimR 417
419. (k,j) ε f Hyp
420. (k,j) ε {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))}
EqualitySub 419 1
421. Set((k,j)) & ∃u.∃v.(((k,j) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))
ClassElim 420
422. ∃u.∃v.(((k,j) = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))
AndElimR 421
423. ∃v.(((k,j) = (u1,v)) & ((u1 ε x) & ∃g.(OrderPreserving(g,r,s) &
(Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v) ε g)))))))
Hyp
424. ((k,j) = (u1,v1)) & ((u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g))
& (Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g)))))) Hyp
425. (u1 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) &
(Section(s,y,range(g)) & ((u1 ε domain(g)) & ((u1,v1) ε g))))) AndElimR 424
426. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u1
ε domain(g)) & ((u1,v1) ε g))))) AndElimR 425
427. OrderPreserving(g1,r,s) & (Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u1
ε domain(g1)) & ((u1,v1) ε g1)))) Hyp
428. Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε
g1))) AndElimR 427
429. Section(s,y,range(g1)) & ((u1 ε domain(g1)) & ((u1,v1) ε g1)) AndElimR 428
430. Section(s,y,range(g1)) AndElimL 429
431. (u1 ε domain(g1)) & ((u1,v1) ε g1) AndElimR 429
432. (u1,v1) ε g1 AndElimR 431
433. ((range(g1) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(g1))) & ((u,v) ε
s)) -> (u ε range(g1))) DefExp 430
434. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f))) TheoremInt
435. ∀a.(((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))) ForallInt 434
436. ((u1,b) ε f) -> ((u1 ε domain(f)) & (b ε range(f))) ForallElim 435
437. ∀b.(((u1,b) ε f) -> ((u1 ε domain(f)) & (b ε range(f)))) ForallInt 436
438. ((u1,v1) ε f) -> ((u1 ε domain(f)) & (v1 ε range(f))) ForallElim 437
439. ∀f.(((u1,v1) ε f) -> ((u1 ε domain(f)) & (v1 ε range(f)))) ForallInt 438
440. ((u1,v1) ε g1) -> ((u1 ε domain(g1)) & (v1 ε range(g1))) ForallElim 439
441. (u1 ε domain(g1)) & (v1 ε range(g1)) ImpElim 432 440
442. v1 ε range(g1) AndElimR 441
443. (range(g1) ⊂ y) & WellOrders(s,y) AndElimL 433
444. ∀z.((z ε range(g1)) -> (z ε y)) & WellOrders(s,y) DefExp 443
445. ∀z.((z ε range(g1)) -> (z ε y)) AndElimL 444
446. (v1 ε range(g1)) -> (v1 ε y) ForallElim 445
447. v1 ε y ImpElim 442 446
448. (k,j) = (u1,v1) AndElimL 424
449. Set((k,j)) AndElimL 421
450. Set((k,j)) & ((k,j) = (u1,v1)) AndInt 449 448
451. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
452. ∀a.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))) ForallInt 451
453. (Set((k,b)) & ((k,b) = (x,y))) -> ((k = x) & (b = y)) ForallElim 452
454. ∀b.((Set((k,b)) & ((k,b) = (x,y))) -> ((k = x) & (b = y))) ForallInt 453
455. (Set((k,j)) & ((k,j) = (x,y))) -> ((k = x) & (j = y)) ForallElim 454
456. ∀x.((Set((k,j)) & ((k,j) = (x,y))) -> ((k = x) & (j = y))) ForallInt 455

457. (Set((k,j)) & ((k,j) = (u1,y))) -> ((k = u1) & (j = y)) ForallElim 456
458. ∀y.((Set((k,j)) & ((k,j) = (u1,y))) -> ((k = u1) & (j = y))) ForallInt 457
459. (Set((k,j)) & ((k,j) = (u1,v1))) -> ((k = u1) & (j = v1)) ForallElim 458
460. (k = u1) & (j = v1) ImpElim 450 459
461. j = v1 AndElimR 460
462. k = u1 AndElimL 460
463. v1 = j Symmetry 461
464. j ε y EqualitySub 447 463
465. j ε y ExistsElim 426 427 464
466. j ε y ExistsElim 423 424 465
467. j ε y ExistsElim 422 423 466
468. j ε y ExistsElim 418 419 467
469. (j ε range(f)) -> (j ε y) ImpInt 468
470. ∀j.((j ε range(f)) -> (j ε y)) ForallInt 469
471. range(f) ⊂ y DefSub 470
472. ∀b.((((a ε y) & (b ε range(f))) & ((a,b) ε s)) -> (a ε range(f))) ForallInt 414
473. ∀a.∀b.((((a ε y) & (b ε range(f))) & ((a,b) ε s)) -> (a ε range(f))) ForallInt 472
474. WellOrders(s,y) AndElimR 0
475. (range(f) ⊂ y) & WellOrders(s,y) AndInt 471 474
476. ((range(f) ⊂ y) & WellOrders(s,y)) & ∀a.∀b.((((a ε y) & (b ε range(f))) & ((a,b) ε s)) -> (a ε range(f))) AndInt 475 473
477. Section(s,y,range(f)) DefSub 476
478. ((v ε domain(f)) & (u ε domain(f))) & ((v,u) ε r) Hyp
479. (v ε domain(f)) & (u ε domain(f)) AndElimL 478
480. u ε domain(f) AndElimR 479
481. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
482. u ε {x: ∃y.((x,y) ε f)} EqualitySub 480 481
483. Set(u) & ∃y.((u,y) ε f) ClassElim 482
484. ∃y.((u,y) ε f) AndElimR 483
485. (u,v1) ε f Hyp
486. (u,v1) ε {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} EqualitySub 485 1
487. Set((u,v1)) & ∃x_87.∃v.(((u,v1) = (x_87,v)) & ((x_87 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((x_87 ε domain(g)) & ((x_87,v) ε g))))))) ClassElim 486
488. ∃x_87.∃v.(((u,v1) = (x_87,v)) & ((x_87 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((x_87 ε domain(g)) & ((x_87,v) ε g))))))) AndElimR 487
489. ∃v.(((u,v1) = (u2,v)) & ((u2 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u2 ε domain(g)) & ((u2,v) ε g))))))) Hyp
490. ((u,v1) = (u2,v2)) & ((u2 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u2 ε domain(g)) & ((u2,v2) ε g)))))) Hyp
491. (u2 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u2 ε domain(g)) & ((u2,v2) ε g))))) AndElimR 490
492. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u2 ε domain(g)) & ((u2,v2) ε g))))) AndElimR 491
493. OrderPreserving(g1,r,s) & (Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u2 ε domain(g1)) & ((u2,v2) ε g1)))) Hyp
494. OrderPreserving(g1,r,s) AndElimL 493
495. (Function(g1) & (WellOrders(r,domain(g1)) & WellOrders(s,range(g1)))) & ∀u.∀v.((((u ε domain(g1)) & (v ε domain(g1))) & ((u,v) ε r)) -> (((g1'u),(g1'v)) ε s)) DefExp 494
496. Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((u2 ε domain(g1)) & ((u2,v2) ε g1))) AndElimR 493
497. Section(r,x,domain(g1)) AndElimL 496
498. ((domain(g1) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(g1))) & ((u,v) ε r)) -> (u ε domain(g1))) DefExp 497
499. ∀u.∀v.((((u ε x) & (v ε domain(g1))) & ((u,v) ε r)) -> (u ε domain(g1))) AndElimR 498
500. (v,u) ε r AndElimR 478
501. Section(s,y,range(g1)) & ((u2 ε domain(g1)) & ((u2,v2) ε g1)) AndElimR 496
502. (u2 ε domain(g1)) & ((u2,v2) ε g1) AndElimR 501
503. u2 ε domain(g1) AndElimL 502
504. Set((u,v1)) AndElimL 487
505. (u,v1) = (u2,v2) AndElimL 490
506. Set((u,v1)) & ((u,v1) = (u2,v2)) AndInt 504 505
507. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
508. ∀a.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))) ForallInt 507
509. (Set((u,b)) & ((u,b) = (x,y))) -> ((u = x) & (b = y)) ForallElim 508

510. ∀b.((Set((u,b)) & ((u,b) = (x,y))) -> ((u = x) & (b = y))) ForallInt 509
511. (Set((u,v1)) & ((u,v1) = (x,y))) -> ((u = x) & (v1 = y)) ForallElim 510
512. ∀x.((Set((u,v1)) & ((u,v1) = (x,y))) -> ((u = x) & (v1 = y))) ForallInt 511
513. (Set((u,v1)) & ((u,v1) = (u2,y))) -> ((u = u2) & (v1 = y)) ForallElim 512
514. ∀y.((Set((u,v1)) & ((u,v1) = (u2,y))) -> ((u = u2) & (v1 = y))) ForallInt 513
515. (Set((u,v1)) & ((u,v1) = (u2,v2))) -> ((u = u2) & (v1 = v2)) ForallElim 514
516. (u = u2) & (v1 = v2) ImpElim 506 515
517. u = u2 AndElimL 516
518. u2 = u Symmetry 517
519. u ε domain(g1) EqualitySub 503 518
520. ∀x_98.((((v ε x) & (x_98 ε domain(g1))) & ((v,x_98) ε r)) -> (v ε domain(g1))) ForallElim 499
521. (((v ε x) & (u ε domain(g1))) & ((v,u) ε r)) -> (v ε domain(g1)) ForallElim 520
522. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f))) DefExp 287
523. (domain(f) ⊂ x) & WellOrders(r,x) AndElimL 522
524. v ε domain(f) AndElimL 479
525. domain(f) ⊂ x AndElimL 523
526. ∀z.((z ε domain(f)) -> (z ε x)) DefExp 525
527. (v ε domain(f)) -> (v ε x) ForallElim 526
528. v ε x ImpElim 524 527
529. (v ε x) & (u ε domain(g1)) AndInt 528 519
530. ((v ε x) & (u ε domain(g1))) & ((v,u) ε r) AndInt 529 500
531. v ε domain(g1) ImpElim 530 521
532. ∀u.∀v.((((u ε domain(g1)) & (v ε domain(g1))) & ((u,v) ε r)) -> (((g1'u),(g1'v)) ε s)) AndElimR 495
533. ∀x_104.((((v ε domain(g1)) & (x_104 ε domain(g1))) & ((v,x_104) ε r)) -> (((g1'v),(g1'x_104)) ε s)) ForallElim 532
534. (((v ε domain(g1)) & (u ε domain(g1))) & ((v,u) ε r)) -> (((g1'v),(g1'u)) ε s) ForallElim 533
535. (v ε domain(g1)) & (u ε domain(g1)) AndInt 531 519
536. ((v ε domain(g1)) & (u ε domain(g1))) & ((v,u) ε r) AndInt 535 500
537. ((g1'v),(g1'u)) ε s ImpElim 536 534
538. Section(s,y,range(g1)) & ((u2 ε domain(g1)) & ((u2,v2) ε g1)) AndElimR 496
539. (u2 ε domain(g1)) & ((u2,v2) ε g1) AndElimR 538
540. (u2,v2) ε g1 AndElimR 539
541. (u,v2) ε g1 EqualitySub 540 518
542. v1 = v2 AndElimR 516
543. v2 = v1 Symmetry 542
544. (u,v1) ε g1 EqualitySub 541 543
545. Function(g1) & (WellOrders(r,domain(g1)) & WellOrders(s,range(g1))) AndElimL 495
546. Function(g1) AndElimL 545
547. (Function(f) & ((a,b) ε f)) -> ((f'a) = b) TheoremInt
548. ∀f.((Function(f) & ((a,b) ε f)) -> ((f'a) = b)) ForallInt 547
549. (Function(g1) & ((a,b) ε g1)) -> ((g1'a) = b) ForallElim 548
550. ∀a.((Function(g1) & ((a,b) ε g1)) -> ((g1'a) = b)) ForallInt 549
551. (Function(g1) & ((u,b) ε g1)) -> ((g1'u) = b) ForallElim 550
552. ∀b.((Function(g1) & ((u,b) ε g1)) -> ((g1'u) = b)) ForallInt 551
553. (Function(g1) & ((u,v1) ε g1)) -> ((g1'u) = v1) ForallElim 552
554. Function(g1) & ((u,v1) ε g1) AndInt 546 544
555. (g1'u) = v1 ImpElim 554 553
556. Function(f) & ((u,v1) ε f) AndInt 149 485
557. ∀a.((Function(f) & ((a,b) ε f)) -> ((f'a) = b)) ForallInt 547
558. (Function(f) & ((u,b) ε f)) -> ((f'u) = b) ForallElim 557
559. ∀b.((Function(f) & ((u,b) ε f)) -> ((f'u) = b)) ForallInt 558
560. (Function(f) & ((u,v1) ε f)) -> ((f'u) = v1) ForallElim 559
561. (f'u) = v1 ImpElim 556 560
562. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
563. ∀f.(domain(f) = {x: ∃y.((x,y) ε f)}) ForallInt 562
564. domain(g1) = {x: ∃y.((x,y) ε g1)} ForallElim 563
565. v ε {x: ∃y.((x,y) ε g1)} EqualitySub 531 564
566. Set(v) & ∃y.((v,y) ε g1) ClassElim 565
567. ∃y.((v,y) ε g1) AndElimR 566
568. (v,j) ε g1 Hyp
569. (v ε domain(g1)) & ((v,j) ε g1) AndInt 531 568
570. Section(s,y,range(g1)) AndElimL 538
571. Section(s,y,range(g1)) & ((v ε domain(g1)) & ((v,j) ε g1)) AndInt 570 569
572. Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((v ε domain(g1)) & ((v,j) ε g1))) AndInt 497 571
573. OrderPreserving(g1,r,s) & (Section(r,x,domain(g1)) & (Section(s,y,range(g1)) & ((v ε domain(g1)) & ((v,j) ε g1)))) AndInt 494 572

574. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((v ε domain(g)) & ((v,j) ε g))))) ExistsInt 573
575. Section(r,x,domain(g1)) AndElimL 572
576. ((domain(g1) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(g1))) & ((u,v) ε r)) -> (u ε domain(g1))) DefExp 575
577. (domain(g1) ⊂ x) & WellOrders(r,x) AndElimL 576
578. ∀z.((z ε domain(g1)) -> (z ε x)) & WellOrders(r,x) DefExp 577
579. ∀z.((z ε domain(g1)) -> (z ε x)) AndElimL 578
580. (v ε domain(g1)) -> (v ε x) ForallElim 579
581. v ε domain(g1) AndElimL 569
582. v ε x ImpElim 581 580
583. (v ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((v ε domain(g)) & ((v,j) ε g))))) AndInt 582 574
584. w = (v,j) Hyp
585. (w = (v,j)) & ((v ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((v ε domain(g)) & ((v,j) ε g)))))) AndInt 584 583
586. ∃j.((w = (v,j)) & ((v ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((v ε domain(g)) & ((v,j) ε g))))))) ExistsInt 585
587. ∃v.∃j.((w = (v,j)) & ((v ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((v ε domain(g)) & ((v,j) ε g))))))) ExistsInt 586
588. ∃w.((v,j) ε w) ExistsInt 568
589. Set((v,j)) DefSub 588
590. Set((v,j)) & ∃v.∃j.((w = (v,j)) & ((v ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((v ε domain(g)) & ((v,j) ε g))))))) AndInt 589 587
591. (v,j) = w Symmetry 584
592. Set(w) & ∃v.∃j.((w = (v,j)) & ((v ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((v ε domain(g)) & ((v,j) ε g))))))) EqualitySub 590 591
593. w ε {w: ∃v.∃j.((w = (v,j)) & ((v ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((v ε domain(g)) & ((v,j) ε g)))))))} ClassInt 592
594. {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} = f Symmetry 1
595. w ε f EqualitySub 593 594
596. (v,j) ε f EqualitySub 595 584
597. Function(f) & ((v,j) ε f) AndInt 149 596
598. Function(g1) & ((v,j) ε g1) AndInt 546 568
599. (Function(f) & ((a,b) ε f)) -> ((f'a) = b) TheoremInt
600. ∀a.((Function(f) & ((a,b) ε f)) -> ((f'a) = b)) ForallInt 599
601. (Function(f) & ((v,b) ε f)) -> ((f'v) = b) ForallElim 600
602. ∀b.((Function(f) & ((v,b) ε f)) -> ((f'v) = b)) ForallInt 601
603. (Function(f) & ((v,j) ε f)) -> ((f'v) = j) ForallElim 602
604. (f'v) = j ImpElim 597 603
605. ∀f.((Function(f) & ((a,b) ε f)) -> ((f'a) = b)) ForallInt 599
606. (Function(g1) & ((a,b) ε g1)) -> ((g1'a) = b) ForallElim 605
607. ∀a.((Function(g1) & ((a,b) ε g1)) -> ((g1'a) = b)) ForallInt 606
608. (Function(g1) & ((v,b) ε g1)) -> ((g1'v) = b) ForallElim 607
609. ∀b.((Function(g1) & ((v,b) ε g1)) -> ((g1'v) = b)) ForallInt 608
610. (Function(g1) & ((v,j) ε g1)) -> ((g1'v) = j) ForallElim 609
611. (g1'v) = j ImpElim 598 610
612. j = (f'v) Symmetry 604
613. (g1'v) = (f'v) EqualitySub 611 612
614. v1 = (f'u) Symmetry 561
615. (g1'u) = (f'u) EqualitySub 555 614
616. ((f'v),(g1'u)) ε s EqualitySub 537 613
617. ((f'v),(f'u)) ε s EqualitySub 616 615
618. (w = (v,j)) -> (((f'v),(f'u)) ε s) ImpInt 617
619. ∀w.((w = (v,j)) -> (((f'v),(f'u)) ε s)) ForallInt 618
620. ((v,j) = (v,j)) -> (((f'v),(f'u)) ε s) ForallElim 619
621. (v,j) = (v,j) Identity
622. ((f'v),(f'u)) ε s ImpElim 621 620
623. ((f'v),(f'u)) ε s ExistsElim 567 568 622
624. ((f'v),(f'u)) ε s ExistsElim 492 493 623
625. ((f'v),(f'u)) ε s ExistsElim 489 490 624
626. ((f'v),(f'u)) ε s ExistsElim 488 489 625
627. ((f'v),(f'u)) ε s ExistsElim 484 485 626
628. (((v ε domain(f)) & (u ε domain(f))) & ((v,u) ε r)) -> (((f'v),(f'u)) ε s) ImpInt 627
629. ∀v.((((v ε domain(f)) & (u ε domain(f))) & ((v,u) ε r)) -> (((f'v),(f'u)) ε s))

ForallInt 628
630. ∀u.∀v.((((v ε domain(f)) & (u ε domain(f))) & ((v,u) ε r)) -> (((f'v),(f'u)) ε s))
ForallInt 629
631. (WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b) TheoremInt
632. WellOrders(r,x) AndElimL 0
633. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε
r)) -> (u ε domain(f))) DefExp 287
634. (domain(f) ⊂ x) & WellOrders(r,x) AndElimL 633
635. domain(f) ⊂ x AndElimL 634
636. ∀a.((WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b)) ForallInt 631
637. (WellOrders(r,x) & (b ⊂ x)) -> WellOrders(r,b) ForallElim 636
638. ∀b.((WellOrders(r,x) & (b ⊂ x)) -> WellOrders(r,b)) ForallInt 637
639. (WellOrders(r,x) & (domain(f) ⊂ x)) -> WellOrders(r,domain(f)) ForallElim 638
640. WellOrders(r,x) & (domain(f) ⊂ x) AndInt 632 635
641. WellOrders(r,domain(f)) ImpElim 640 639
642. WellOrders(s,y) AndElimR 0
643. ((range(f) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε
s)) -> (u ε range(f))) DefExp 477
644. (range(f) ⊂ y) & WellOrders(s,y) AndElimL 643
645. range(f) ⊂ y AndElimL 644
646. ∀r.((WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b)) ForallInt 631
647. (WellOrders(s,a) & (b ⊂ a)) -> WellOrders(s,b) ForallElim 646
648. ∀a.((WellOrders(s,a) & (b ⊂ a)) -> WellOrders(s,b)) ForallInt 647
649. (WellOrders(s,y) & (b ⊂ y)) -> WellOrders(s,b) ForallElim 648
650. ∀b.((WellOrders(s,y) & (b ⊂ y)) -> WellOrders(s,b)) ForallInt 649
651. (WellOrders(s,y) & (range(f) ⊂ y)) -> WellOrders(s,range(f)) ForallElim 650
652. WellOrders(s,y) & (range(f) ⊂ y) AndInt 642 645
653. WellOrders(s,range(f)) ImpElim 652 651
654. WellOrders(r,domain(f)) & WellOrders(s,range(f)) AndInt 641 653
655. Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f))) AndInt 149 654
656. ∀u.((((v ε domain(f)) & (u ε domain(f))) & ((v,u) ε r)) -> (((f'v),(f'u)) ε s))
ForallInt 628
657. ∀v.∀u.((((v ε domain(f)) & (u ε domain(f))) & ((v,u) ε r)) -> (((f'v),(f'u)) ε s))
ForallInt 656
658. (Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f)))) & ∀v.∀u.((((v ε
domain(f)) & (u ε domain(f))) & ((v,u) ε r)) -> (((f'v),(f'u)) ε s)) AndInt 655 657
659. OrderPreserving(f,r,s) DefSub 658
660. Section(r,x,domain(f)) & Section(s,y,range(f)) AndInt 287 477
661. OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f))) AndInt
659 660
662. ¬((x ~ domain(f)) = 0) & ¬((y ~ range(f)) = 0) Hyp
663. z ε (x ~ domain(f)) Hyp
664. (x ~ y) = (x ∩ ~y) DefEqInt
665. ∀y.((x ~ y) = (x ∩ ~y)) ForallInt 664
666. (x ~ domain(f)) = (x ∩ ~domain(f)) ForallElim 665
667. z ε (x ∩ ~domain(f)) EqualitySub 663 666
668. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
669. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 668
670. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y)))
EquivExp 669
671. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 670
672. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) ForallInt 671
673. (z ε (x ∩ ~domain(f))) -> ((z ε x) & (z ε ~domain(f))) ForallElim 672
674. (z ε x) & (z ε ~domain(f)) ImpElim 667 673
675. z ε x AndElimL 674
676. (z ε (x ~ domain(f))) -> (z ε x) ImpInt 675
677. ∀z.((z ε (x ~ domain(f))) -> (z ε x)) ForallInt 676
678. (x ~ domain(f)) ⊂ x DefSub 677
679. z ε (y ~ range(f)) Hyp
680. ∀y.((x ~ y) = (x ∩ ~y)) ForallInt 664
681. (x ~ range(f)) = (x ∩ ~range(f)) ForallElim 680
682. ∀x.((x ~ range(f)) = (x ∩ ~range(f))) ForallInt 681
683. (y ~ range(f)) = (y ∩ ~range(f)) ForallElim 682
684. z ε (y ∩ ~range(f)) EqualitySub 679 683
685. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) ForallInt 671
686. (z ε (x ∩ ~range(f))) -> ((z ε x) & (z ε ~range(f))) ForallElim 685
687. ∀x.((z ε (x ∩ ~range(f))) -> ((z ε x) & (z ε ~range(f)))) ForallInt 686
688. (z ε (y ∩ ~range(f))) -> ((z ε y) & (z ε ~range(f))) ForallElim 687
689. (z ε y) & (z ε ~range(f)) ImpElim 684 688
690. z ε y AndElimL 689

691. (z ε (y ~ range(f))) -> (z ε y) ImpInt 690
692. ∀z.((z ε (y ~ range(f))) -> (z ε y)) ForallInt 691
693. (y ~ range(f)) ⊂ y DefSub 692
694. WellOrders(r,x) AndElimL 0
695. Connects(r,x) & ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z)) DefExp 694
696. ∀y.(((y ⊂ x) & ¬(y = 0)) -> ∃z.First(r,y,z)) AndElimR 695
697. (((x ~ domain(f)) ⊂ x) & ¬((x ~ domain(f)) = 0)) -> ∃z.First(r,(x ~ domain(f)),z) ForallElim 696
698. ¬((x ~ domain(f)) = 0) AndElimL 662
699. ((x ~ domain(f)) ⊂ x) & ¬((x ~ domain(f)) = 0) AndInt 678 698
700. ∃z.First(r,(x ~ domain(f)),z) ImpElim 699 697
701. WellOrders(s,y) AndElimR 0
702. Connects(s,y) & ∀x_128.(((x_128 ⊂ y) & ¬(x_128 = 0)) -> ∃z.First(s,x_128,z)) DefExp 701
703. ∀x_128.(((x_128 ⊂ y) & ¬(x_128 = 0)) -> ∃z.First(s,x_128,z)) AndElimR 702
704. (((y ~ range(f)) ⊂ y) & ¬((y ~ range(f)) = 0)) -> ∃z.First(s,(y ~ range(f)),z) ForallElim 703
705. ¬((y ~ range(f)) = 0) AndElimR 662
706. ((y ~ range(f)) ⊂ y) & ¬((y ~ range(f)) = 0) AndInt 693 705
707. ∃z.First(s,(y ~ range(f)),z) ImpElim 706 704
708. First(r,(x ~ domain(f)),m) Hyp
709. First(s,(y ~ range(f)),n) Hyp
710. (a ε domain(f)) & ((m,a) ε r) Hyp
711. Section(r,x,domain(f)) AndElimL 660
712. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f))) DefExp 711
713. ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f))) AndElimR 712
714. ∀v.((((m ε x) & (v ε domain(f))) & ((m,v) ε r)) -> (m ε domain(f))) ForallElim 713
715. (((m ε x) & (a ε domain(f))) & ((m,a) ε r)) -> (m ε domain(f)) ForallElim 714
716. (m ε (x ~ domain(f))) & ∀y.((y ε (x ~ domain(f))) -> ¬((y,m) ε r)) DefExp 708
717. m ε (x ~ domain(f)) AndElimL 716
718. ∀z.((z ε (x ~ domain(f))) -> (z ε x)) DefExp 678
719. (m ε (x ~ domain(f))) -> (m ε x) ForallElim 718
720. m ε x ImpElim 717 719
721. (m ε x) & (m ε (x ~ domain(f))) AndInt 720 717
722. (m,a) ε r AndElimR 710
723. a ε domain(f) AndElimL 710
724. (m ε x) & (a ε domain(f)) AndInt 720 723
725. (m,a) ε r AndElimR 710
726. ((m ε x) & (a ε domain(f))) & ((m,a) ε r) AndInt 724 725
727. m ε domain(f) ImpElim 726 715
728. (m ε (x ~ domain(f))) & ∀y.((y ε (x ~ domain(f))) -> ¬((y,m) ε r)) DefExp 708
729. m ε (x ~ domain(f)) AndElimL 728
730. (x ~ y) = (x ∩ ~y) DefEqInt
731. ∀y.((x ~ y) = (x ∩ ~y)) ForallInt 730
732. (x ~ domain(f)) = (x ∩ ~domain(f)) ForallElim 731
733. m ε (x ∩ ~domain(f)) EqualitySub 729 732
734. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
735. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 734
736. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 735
737. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 736
738. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) ForallInt 737
739. (z ε (x ∩ ~domain(f))) -> ((z ε x) & (z ε ~domain(f))) ForallElim 738
740. ∀z.((z ε (x ∩ ~domain(f))) -> ((z ε x) & (z ε ~domain(f)))) ForallInt 739
741. (m ε (x ∩ ~domain(f))) -> ((m ε x) & (m ε ~domain(f))) ForallElim 740
742. (m ε x) & (m ε ~domain(f)) ImpElim 733 741
743. m ε ~domain(f) AndElimR 742
744. ~x = {y: ¬(y ε x)} DefEqInt
745. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 744
746. ~domain(f) = {y: ¬(y ε domain(f))} ForallElim 745
747. m ε {y: ¬(y ε domain(f))} EqualitySub 743 746
748. Set(m) & ¬(m ε domain(f)) ClassElim 747
749. ¬(m ε domain(f)) AndElimR 748
750. _|_ ImpElim 727 749
751. ¬((a ε domain(f)) & ((m,a) ε r)) ImpInt 750
752. (a ε range(f)) & ((n,a) ε s) Hyp
753. Section(s,y,range(f)) AndElimR 660
754. ((range(f) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε s)) -> (u ε range(f))) DefExp 753

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755. ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε s)) -> (u ε range(f)))  AndElimR 754
756. ∀v.((((n ε y) & (v ε range(f))) & ((n,v) ε s)) -> (n ε range(f)))  ForallElim 755
757. (((n ε y) & (a ε range(f))) & ((n,a) ε s)) -> (n ε range(f))  ForallElim 756
758. ∀z.((z ε (y ~ range(f))) -> (z ε y))  DefExp 693
759. (n ε (y ~ range(f))) -> (n ε y)  ForallElim 758
760. (n ε (y ~ range(f))) & ∀x_148.((x_148 ε (y ~ range(f))) -> ¬((x_148,n) ε s))  DefExp 709
761. n ε (y ~ range(f))  AndElimL 760
762. n ε y  ImpElim 761 759
763. a ε range(f)  AndElimL 752
764. (n ε y) & (a ε range(f))  AndInt 762 763
765. (n,a) ε s  AndElimR 752
766. ((n ε y) & (a ε range(f))) & ((n,a) ε s)  AndInt 764 765
767. n ε range(f)  ImpElim 766 757
768. ∀y.((x ~ y) = (x ∩ ~y))  ForallInt 730
769. (x ~ range(f)) = (x ∩ ~range(f))  ForallElim 768
770. ∀x.((x ~ range(f)) = (x ∩ ~range(f)))  ForallInt 769
771. (y ~ range(f)) = (y ∩ ~range(f))  ForallElim 770
772. n ε (y ∩ ~range(f))  EqualitySub 761 771
773. ∀y.((z ε (x ∩ y)) -> ((z ε x) & (z ε y)))  ForallInt 737
774. (z ε (x ∩ ~range(f))) -> ((z ε x) & (z ε ~range(f)))  ForallElim 773
775. ∀x.((z ε (x ∩ ~range(f))) -> ((z ε x) & (z ε ~range(f))))  ForallInt 774
776. (z ε (y ∩ ~range(f))) -> ((z ε y) & (z ε ~range(f)))  ForallElim 775
777. ∀z.((z ε (y ∩ ~range(f))) -> ((z ε y) & (z ε ~range(f))))  ForallInt 776
778. (n ε (y ∩ ~range(f))) -> ((n ε y) & (n ε ~range(f)))  ForallElim 777
779. (n ε y) & (n ε ~range(f))  ImpElim 772 778
780. n ε ~range(f)  AndElimR 779
781. ∀x.(~x = {y: ¬(y ε x)})  ForallInt 744
782. ~range(f) = {y: ¬(y ε range(f))}  ForallElim 781
783. n ε {y: ¬(y ε range(f))}  EqualitySub 780 782
784. Set(n) & ¬(n ε range(f))  ClassElim 783
785. ¬(n ε range(f))  AndElimR 784
786. _|_  ImpElim 767 785
787. ¬((a ε range(f)) & ((n,a) ε s))  ImpInt 786
788. ¬((a ε domain(f)) & ((m,a) ε r)) & ¬((a ε range(f)) & ((n,a) ε s))  AndInt 751 787
789. g = (f ∪ {(m,n)})  Hyp
790. z ε g  Hyp
791. z ε (f ∪ {(m,n)})  EqualitySub 790 789
792. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y))  AndElimL 734
793. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 792
794. (z ε (x ∪ y)) -> ((z ε x) v (z ε y))  AndElimL 793
795. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y)))  ForallInt 794
796. (z ε (f ∪ y)) -> ((z ε f) v (z ε y))  ForallElim 795
797. ∀y.((z ε (f ∪ y)) -> ((z ε f) v (z ε y)))  ForallInt 796
798. (z ε (f ∪ {(m,n)})) -> ((z ε f) v (z ε {(m,n)}))  ForallElim 797
799. (z ε f) v (z ε {(m,n)})  ImpElim 791 798
800. z ε f  Hyp
801. Relation(f) & ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  DefExp 149
802. Relation(f)  AndElimL 801
803. ∀z.((z ε f) -> ∃x.∃y.(z = (x,y)))  DefExp 802
804. (z ε f) -> ∃x.∃y.(z = (x,y))  ForallElim 803
805. ∃x.∃y.(z = (x,y))  ImpElim 800 804
806. z ε {(m,n)}  Hyp
807. ∃w.(m ε w)  ExistsInt 720
808. Set(m)  DefSub 807
809. ∃w.(n ε w)  ExistsInt 762
810. Set(n)  DefSub 809
811. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
812. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 811
813. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 812
814. (Set(x) & Set(y)) -> Set((x,y))  AndElimL 813
815. ∀x.((Set(x) & Set(y)) -> Set((x,y)))  ForallInt 814
816. (Set(m) & Set(y)) -> Set((m,y))  ForallElim 815
817. ∀y.((Set(m) & Set(y)) -> Set((m,y)))  ForallInt 816
818. (Set(m) & Set(n)) -> Set((m,n))  ForallElim 817
819. Set(m) & Set(n)  AndInt 808 810
820. Set((m,n))  ImpElim 819 818
821. Set(x) -> ((y ε {x}) <-> (y = x))  TheoremInt
822. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x)))  ForallInt 821
823. Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n)))  ForallElim 822
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824. ∀y.(Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n))))  ForallInt 823
825. Set((m,n)) -> ((z ε {(m,n)}) <-> (z = (m,n)))  ForallElim 824
826. (z ε {(m,n)}) <-> (z = (m,n))  ImpElim 820 825
827. ((z ε {(m,n)}) -> (z = (m,n))) & ((z = (m,n)) -> (z ε {(m,n)}))  EquivExp 826
828. (z ε {(m,n)}) -> (z = (m,n))  AndElimL 827
829. z = (m,n)  ImpElim 806 828
830. ∃y.(z = (m,y))  ExistsInt 829
831. ∃x.∃y.(z = (x,y))  ExistsInt 830
832. ∃x.∃y.(z = (x,y))  OrElim 799 800 805 806 831
833. (z ε g) -> ∃x.∃y.(z = (x,y))  ImpInt 832
834. ∀z.((z ε g) -> ∃x.∃y.(z = (x,y)))  ForallInt 833
835. Relation(g)  DefSub 834
836. ((a,b) ε g) & ((a,c) ε g)  Hyp
837. (a,b) ε g  AndElimL 836
838. (a,b) ε (f ∪ {(m,n)})  EqualitySub 837 789
839. ∀z.((z ε (f ∪ {(m,n)})) -> ((z ε f) v (z ε {(m,n)})))  ForallInt 798
840. ((a,b) ε (f ∪ {(m,n)})) -> (((a,b) ε f) v ((a,b) ε {(m,n)}))  ForallElim 839
841. ((a,b) ε f) v ((a,b) ε {(m,n)})  ImpElim 838 840
842. (a,b) ε f  Hyp
843. (a,c) ε g  AndElimR 836
844. ∀z.((z ε (f ∪ {(m,n)})) -> ((z ε f) v (z ε {(m,n)})))  ForallInt 798
845. ((a,c) ε (f ∪ {(m,n)})) -> (((a,c) ε f) v ((a,c) ε {(m,n)}))  ForallElim 844
846. (a,c) ε (f ∪ {(m,n)})  EqualitySub 843 789
847. ((a,c) ε f) v ((a,c) ε {(m,n)})  ImpElim 846 845
848. (a,c) ε f  Hyp
849. ∀x.∀y.∀z.((((x,y) ε f) & ((x,z) ε f)) -> (y = z))  AndElimR 801
850. ∀y.∀z.((((a,y) ε f) & ((a,z) ε f)) -> (y = z))  ForallElim 849
851. ∀z.((((a,b) ε f) & ((a,z) ε f)) -> (b = z))  ForallElim 850
852. (((a,b) ε f) & ((a,c) ε f)) -> (b = c)  ForallElim 851
853. ((a,b) ε f) & ((a,c) ε f)  AndInt 842 848
854. b = c  ImpElim 853 852
855. (a,c) ε {(m,n)}  Hyp
856. ∀z.((z ε {(m,n)}) -> (z = (m,n)))  ForallInt 828
857. ∀z.((z ε {(m,n)}) -> (z = (m,n)))  ForallInt 828
858. ((a,c) ε {(m,n)}) -> ((a,c) = (m,n))  ForallElim 857
859. (a,c) = (m,n)  ImpElim 855 858
860. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))  TheoremInt
861. (m,n) = (a,c)  Symmetry 859
862. Set((m,n)) & ((m,n) = (a,c))  AndInt 820 861
863. ∀a.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)))  ForallInt 860
864. (Set((m,b)) & ((m,b) = (x,y))) -> ((m = x) & (b = y))  ForallElim 863
865. ∀b.((Set((m,b)) & ((m,b) = (x,y))) -> ((m = x) & (b = y)))  ForallInt 864
866. (Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y))  ForallElim 865
867. ∀x.((Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y)))  ForallInt 866
868. (Set((m,n)) & ((m,n) = (a,y))) -> ((m = a) & (n = y))  ForallElim 867
869. ∀y.((Set((m,n)) & ((m,n) = (a,y))) -> ((m = a) & (n = y)))  ForallInt 868
870. (Set((m,n)) & ((m,n) = (a,c))) -> ((m = a) & (n = c))  ForallElim 869
871. (m = a) & (n = c)  ImpElim 862 870
872. ∃w.((a,w) ε f)  ExistsInt 848
873. ∃w.((a,c) ε w)  ExistsInt 848
874. Set((a,c))  DefSub 873
875. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))  TheoremInt
876. (Set(x) & Set(y)) <-> Set((x,y))  AndElimL 875
877. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y)))  EquivExp 876
878. Set((x,y)) -> (Set(x) & Set(y))  AndElimR 877
879. ∀x.(Set((x,y)) -> (Set(x) & Set(y)))  ForallInt 878
880. Set((a,y)) -> (Set(a) & Set(y))  ForallElim 879
881. ∀y.(Set((a,y)) -> (Set(a) & Set(y)))  ForallInt 880
882. Set((a,c)) -> (Set(a) & Set(c))  ForallElim 881
883. Set(a) & Set(c)  ImpElim 874 882
884. Set(a)  AndElimL 883
885. Set(a) & ∃w.((a,w) ε f)  AndInt 884 872
886. a ε {w: ∃x_155.((w,x_155) ε f)}  ClassInt 885
887. domain(f) = {x: ∃y.((x,y) ε f)}  DefEqInt
888. {x: ∃y.((x,y) ε f)} = domain(f)  Symmetry 887
889. a ε domain(f)  EqualitySub 886 888
890. m = a  AndElimL 871
891. a = m  Symmetry 890
892. m ε domain(f)  EqualitySub 889 891
893. _|_  ImpElim 892 749
894. b = c  AbsI 893
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895. b = c OrElim 847 848 854 855 894
896. (a,b) ε {(m,n)} Hyp
897. (a,c) ε f Hyp
898. ((a,b) ε {(m,n)}) -> ((a,b) = (m,n)) ForallElim 857
899. (a,b) = (m,n) ImpElim 896 898
900. (m,n) = (a,b) Symmetry 899
901. ∀y.((Set((m,n)) & ((m,n) = (a,y))) -> ((m = a) & (n = y))) ForallInt 868
902. (Set((m,n)) & ((m,n) = (a,b))) -> ((m = a) & (n = b)) ForallElim 901
903. Set((m,n)) & ((m,n) = (a,b)) AndInt 820 900
904. (m = a) & (n = b) ImpElim 903 902
905. m = a AndElimL 904
906. ∃w.((a,c) ε w) ExistsInt 897
907. Set((a,c)) DefSub 906
908. Set(a) & Set(c) ImpElim 907 882
909. Set(a) AndElimL 908
910. ∃w.((a,w) ε f) ExistsInt 897
911. Set(a) & ∃w.((a,w) ε f) AndInt 909 910
912. a ε {w: ∃x_157.((w,x_157) ε f)} ClassInt 911
913. a ε domain(f) EqualitySub 912 888
914. a = m Symmetry 905
915. m ε domain(f) EqualitySub 913 914
916. _|_ ImpElim 915 749
917. b = c AbsI 916
918. (a,c) ε {(m,n)} Hyp
919. (a,c) = (m,n) ImpElim 918 858
920. (m,n) = (a,c) Symmetry 919
921. Set((m,n)) & ((m,n) = (a,c)) AndInt 820 920
922. (m = a) & (n = c) ImpElim 921 870
923. n = b AndElimR 904
924. n = c AndElimR 922
925. b = n Symmetry 923
926. b = c EqualitySub 925 924
927. b = c OrElim 847 897 917 918 926
928. b = c OrElim 841 842 895 896 927
929. (((a,b) ε g) & ((a,c) ε g)) -> (b = c) ImpInt 928
930. ∀c.((((a,b) ε g) & ((a,c) ε g)) -> (b = c)) ForallInt 929
931. ∀b.∀c.((((a,b) ε g) & ((a,c) ε g)) -> (b = c)) ForallInt 930
932. ∀a.∀b.∀c.((((a,b) ε g) & ((a,c) ε g)) -> (b = c)) ForallInt 931
933. Relation(g) & ∀a.∀b.∀c.((((a,b) ε g) & ((a,c) ε g)) -> (b = c)) AndInt 835 932
934. Function(g) DefSub 933
935. (a ε domain(g)) & ((b ε domain(g)) & ((a,b) ε r)) Hyp
936. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
937. ∀g.(domain(f) = {x: ∃y.((x,y) ε f)}) ForallInt 936
938. ∀f.(domain(f) = {x: ∃y.((x,y) ε f)}) ForallInt 936
939. domain(g) = {x: ∃y.((x,y) ε g)} ForallElim 938
940. (a ε {x: ∃y.((x,y) ε g)}) & ((b ε {x: ∃y.((x,y) ε g)}) & ((a,b) ε r)) EqualitySub 935 939
941. a ε {x: ∃y.((x,y) ε g)} AndElimL 940
942. (b ε {x: ∃y.((x,y) ε g)}) & ((a,b) ε r) AndElimR 940
943. b ε {x: ∃y.((x,y) ε g)} AndElimL 942
944. Set(a) & ∃y.((a,y) ε g) ClassElim 941
945. Set(b) & ∃y.((b,y) ε g) ClassElim 943
946. ∃y.((a,y) ε g) AndElimR 944
947. ∃y.((b,y) ε g) AndElimR 945
948. (a,p) ε g Hyp
949. (b,q) ε g Hyp
950. (a,p) ε (f ∪ {(m,n)}) EqualitySub 948 789
951. (b,q) ε (f ∪ {(m,n)}) EqualitySub 949 789
952. ((a,p) ε (f ∪ {(m,n)})) -> (((a,p) ε f) v ((a,p) ε {(m,n)})) ForallElim 844
953. ((a,p) ε f) v ((a,p) ε {(m,n)}) ImpElim 950 952
954. (a,p) ε f Hyp
955. ((b,q) ε (f ∪ {(m,n)})) -> (((b,q) ε f) v ((b,q) ε {(m,n)})) ForallElim 844
956. ((b,q) ε f) v ((b,q) ε {(m,n)}) ImpElim 951 955
957. (b,q) ε f Hyp
958. ∃w.((a,p) ε w) ExistsInt 954
959. Set((a,p)) DefSub 958
960. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 878
961. Set((a,y)) -> (Set(a) & Set(y)) ForallElim 960
962. ∀y.(Set((a,y)) -> (Set(a) & Set(y))) ForallInt 961
963. Set((a,p)) -> (Set(a) & Set(p)) ForallElim 962
964. Set(a) & Set(p) ImpElim 959 963

965. Set(a) AndElimL 964
966. ∃w.((a,w) ε f) ExistsInt 954
967. Set(a) & ∃w.((a,w) ε f) AndInt 965 966
968. a ε {w: ∃x_160.((w,x_160) ε f)} ClassInt 967
969. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
970. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 969
971. a ε domain(f) EqualitySub 968 970
972. ∃w.((b,q) ε w) ExistsInt 957
973. Set((b,q)) DefSub 972
974. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 878
975. Set((b,y)) -> (Set(b) & Set(y)) ForallElim 974
976. ∀y.(Set((b,y)) -> (Set(b) & Set(y))) ForallInt 975
977. Set((b,q)) -> (Set(b) & Set(q)) ForallElim 976
978. Set(b) & Set(q) ImpElim 973 977
979. Set(b) AndElimL 978
980. ∃w.((b,w) ε f) ExistsInt 957
981. Set(b) & ∃w.((b,w) ε f) AndInt 979 980
982. b ε {w: ∃x_162.((w,x_162) ε f)} ClassInt 981
983. b ε domain(f) EqualitySub 982 970
984. (Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) DefExp 659
985. ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) AndElimR 984
986. ∀v.((((a ε domain(f)) & (v ε domain(f))) & ((a,v) ε r)) -> (((f'a),(f'v)) ε s)) ForallElim 985
987. (((a ε domain(f)) & (b ε domain(f))) & ((a,b) ε r)) -> (((f'a),(f'b)) ε s) ForallElim 986
988. (a ε domain(f)) & (b ε domain(f)) AndInt 971 983
989. (b ε domain(g)) & ((a,b) ε r) AndElimR 935
990. (a,b) ε r AndElimR 989
991. ((a ε domain(f)) & (b ε domain(f))) & ((a,b) ε r) AndInt 988 990
992. ((f'a),(f'b)) ε s ImpElim 991 987
993. (Function(f) & ((a,b) ε f)) -> ((f'a) = b) TheoremInt
994. Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f))) AndElimL 984
995. ∀b.((Function(f) & ((a,b) ε f)) -> ((f'a) = b)) ForallInt 993
996. (Function(f) & ((a,p) ε f)) -> ((f'a) = p) ForallElim 995
997. ∀f.((Function(f) & ((a,p) ε f)) -> ((f'a) = p)) ForallInt 996
998. (Function(g) & ((a,p) ε g)) -> ((g'a) = p) ForallElim 997
999. Function(g) & ((a,p) ε g) AndInt 934 948
1000. (g'a) = p ImpElim 999 998
1001. Function(f) AndElimL 994
1002. Function(f) & ((a,p) ε f) AndInt 1001 954
1003. (f'a) = p ImpElim 1002 996
1004. ∀b.((Function(f) & ((a,b) ε f)) -> ((f'a) = b)) ForallInt 993
1005. (Function(f) & ((a,q) ε f)) -> ((f'a) = q) ForallElim 1004
1006. ∀a.((Function(f) & ((a,q) ε f)) -> ((f'a) = q)) ForallInt 1005
1007. (Function(f) & ((b,q) ε f)) -> ((f'b) = q) ForallElim 1006
1008. Function(f) & ((b,q) ε f) AndInt 1001 957
1009. (f'b) = q ImpElim 1008 1007
1010. ∀f.((Function(f) & ((b,q) ε f)) -> ((f'b) = q)) ForallInt 1007
1011. (Function(g) & ((b,q) ε g)) -> ((g'b) = q) ForallElim 1010
1012. Function(g) & ((b,q) ε g) AndInt 934 949
1013. (g'b) = q ImpElim 1012 1011
1014. p = (g'a) Symmetry 1000
1015. q = (g'b) Symmetry 1013
1016. (f'a) = (g'a) EqualitySub 1003 1014
1017. (f'b) = (g'b) EqualitySub 1009 1015
1018. ((g'a),(f'b)) ε s EqualitySub 992 1016
1019. ((g'a),(g'b)) ε s EqualitySub 1018 1017
1020. (b,q) ε {(m,n)} Hyp
1021. Set((m,n)) & ((b,q) ε {(m,n)}) AndInt 820 1020
1022. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1023. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1022
1024. Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n))) ForallElim 1023
1025. ∀y.(Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n)))) ForallInt 1024
1026. Set((m,n)) -> (((b,q) ε {(m,n)}) <-> ((b,q) = (m,n))) ForallElim 1025
1027. ((b,q) ε {(m,n)}) <-> ((b,q) = (m,n)) ImpElim 820 1026
1028. (((b,q) ε {(m,n)}) -> ((b,q) = (m,n))) & (((b,q) = (m,n)) -> ((b,q) ε {(m,n)})) EquivExp 1027
1029. ((b,q) ε {(m,n)}) -> ((b,q) = (m,n)) AndElimL 1028
1030. (b,q) = (m,n) ImpElim 1020 1029

1031. (m,n) = (b,q) Symmetry 1030
1032. Set((m,n)) & ((m,n) = (b,q)) AndInt 820 1031
1033. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
1034. ∀a.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))) ForallInt 1033
1035. (Set((m,b)) & ((m,b) = (x,y))) -> ((m = x) & (b = y)) ForallElim 1034
1036. ∀b.((Set((m,b)) & ((m,b) = (x,y))) -> ((m = x) & (b = y))) ForallInt 1035
1037. (Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y)) ForallElim 1036
1038. ∀x.((Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y))) ForallInt 1037
1039. (Set((m,n)) & ((m,n) = (b,y))) -> ((m = b) & (n = y)) ForallElim 1038
1040. ∀y.((Set((m,n)) & ((m,n) = (b,y))) -> ((m = b) & (n = y))) ForallInt 1039
1041. (Set((m,n)) & ((m,n) = (b,q))) -> ((m = b) & (n = q)) ForallElim 1040
1042. (m = b) & (n = q) ImpElim 1032 1041
1043. m = b AndElimL 1042
1044. n = q AndElimR 1042
1045. b = m Symmetry 1043
1046. q = n Symmetry 1044
1047. (m,q) ε g EqualitySub 949 1045
1048. (m,n) ε g EqualitySub 1047 1046
1049. (Function(f) & ((a,b) ε f)) -> ((f'a) = b) TheoremInt
1050. ∀f.((Function(f) & ((a,b) ε f)) -> ((f'a) = b)) ForallInt 1049
1051. (Function(g) & ((a,b) ε g)) -> ((g'a) = b) ForallElim 1050
1052. ∀a.((Function(g) & ((a,b) ε g)) -> ((g'a) = b)) ForallInt 1051
1053. (Function(g) & ((m,b) ε g)) -> ((g'm) = b) ForallElim 1052
1054. ∀b.((Function(g) & ((m,b) ε g)) -> ((g'm) = b)) ForallInt 1053
1055. (Function(g) & ((m,n) ε g)) -> ((g'm) = n) ForallElim 1054
1056. Function(g) & ((m,n) ε g) AndInt 934 1048
1057. (g'm) = n ImpElim 1056 1055
1058. (g'b) = n EqualitySub 1057 1043
1059. ∃w.((w,p) ε f) ExistsInt 954
1060. Set(p) AndElimR 964
1061. Set(p) & ∃w.((w,p) ε f) AndInt 1060 1059
1062. p ε {w: ∃x_166.((x_166,w) ε f)} ClassInt 1061
1063. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
1064. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 1063
1065. p ε range(f) EqualitySub 1062 1064
1066. ∀a.¬((a ε range(f)) & ((n,a) ε s)) ForallInt 787
1067. ¬((p ε range(f)) & ((n,p) ε s)) ForallElim 1066
1068. (n,p) ε s Hyp
1069. (p ε range(f)) & ((n,p) ε s) AndInt 1065 1068
1070. _|_ ImpElim 1069 1067
1071. ¬((n,p) ε s) ImpInt 1070
1072. n = p Hyp
1073. p = n Symmetry 1072
1074. n ε range(f) EqualitySub 1065 1073
1075. _|_ ImpElim 1074 785
1076. ¬(n = p) ImpInt 1075
1077. WellOrders(s,y) AndElimR 0
1078. Connects(s,y) & ∀x_169.(((x_169 ⊂ y) & ¬(x_169 = 0)) -> ∃z.First(s,x_169,z))
DefExp 1077
1079. Connects(s,y) AndElimL 1078
1080. ∀x_172.∀z.(((x_172 ε y) & (z ε y)) -> ((x_172 = z) v (((x_172,z) ε s) v ((z,x_172)
ε s)))) DefExp 1079
1081. ∀z.(((n ε y) & (z ε y)) -> ((n = z) v (((n,z) ε s) v ((z,n) ε s)))) ForallElim
1080
1082. ((n ε y) & (p ε y)) -> ((n = p) v (((n,p) ε s) v ((p,n) ε s))) ForallElim 1081
1083. (p ε range(f)) -> (p ε y) ForallElim 470
1084. p ε y ImpElim 1065 1083
1085. (n ε y) & (p ε y) AndInt 762 1084
1086. (n = p) v (((n,p) ε s) v ((p,n) ε s)) ImpElim 1085 1082
1087. n = p Hyp
1088. _|_ ImpElim 1087 1076
1089. (p,n) ε s AbsI 1088
1090. ((n,p) ε s) v ((p,n) ε s) Hyp
1091. (n,p) ε s Hyp
1092. _|_ ImpElim 1091 1071
1093. (p,n) ε s AbsI 1092
1094. (p,n) ε s Hyp
1095. (p,n) ε s OrElim 1090 1091 1093 1094 1094
1096. (p,n) ε s OrElim 1086 1087 1089 1090 1095
1097. n = (g'b) Symmetry 1058
1098. (p,(g'b)) ε s EqualitySub 1096 1097

1099. p = (g'a) Symmetry 1000
1100. ((g'a),(g'b)) ε s EqualitySub 1098 1099
1101. ((g'a),(g'b)) ε s OrElim 956 957 1019 1020 1100
1102. (a,p) ε {(m,n)} Hyp
1103. Set((m,n)) -> (((a,p) ε {(m,n)}) <-> ((a,p) = (m,n))) ForallElim 1025
1104. ((a,p) ε {(m,n)}) <-> ((a,p) = (m,n)) ImpElim 820 1103
1105. (((a,p) ε {(m,n)}) -> ((a,p) = (m,n))) & (((a,p) = (m,n)) -> ((a,p) ε {(m,n)}))
EquivExp 1104
1106. ((a,p) ε {(m,n)}) -> ((a,p) = (m,n)) AndElimL 1105
1107. (a,p) = (m,n) ImpElim 1102 1106
1108. (m,n) = (a,p) Symmetry 1107
1109. Set((m,n)) & ((m,n) = (a,p)) AndInt 820 1108
1110. ∀x.((Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y))) ForallInt 1037
1111. (Set((m,n)) & ((m,n) = (a,y))) -> ((m = a) & (n = y)) ForallElim 1110
1112. ∀y.((Set((m,n)) & ((m,n) = (a,y))) -> ((m = a) & (n = y))) ForallInt 1111
1113. (Set((m,n)) & ((m,n) = (a,p))) -> ((m = a) & (n = p)) ForallElim 1112
1114. (m = a) & (n = p) ImpElim 1109 1113
1115. m = a AndElimL 1114
1116. a = m Symmetry 1115
1117. (b ε domain(g)) & ((a,b) ε r) AndElimR 935
1118. b ε domain(g) AndElimL 1117
1119. (a,b) ε r AndElimR 1117
1120. ¬((a ε domain(f)) & ((m,a) ε r)) AndElimL 788
1121. ∀a.¬((a ε domain(f)) & ((m,a) ε r)) ForallInt 1120
1122. ¬((b ε domain(f)) & ((m,b) ε r)) ForallElim 1121
1123. (b,q) ε f Hyp
1124. ∃w.((b,q) ε w) ExistsInt 1123
1125. Set((b,q)) DefSub 1124
1126. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
1127. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 1126
1128. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp
1127
1129. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 1128
1130. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 1129
1131. Set((b,y)) -> (Set(b) & Set(y)) ForallElim 1130
1132. ∀y.(Set((b,y)) -> (Set(b) & Set(y))) ForallInt 1131
1133. Set((b,q)) -> (Set(b) & Set(q)) ForallElim 1132
1134. Set(b) & Set(q) ImpElim 1125 1133
1135. Set(b) AndElimL 1134
1136. ∃w.((b,w) ε f) ExistsInt 1123
1137. Set(b) & ∃w.((b,w) ε f) AndInt 1135 1136
1138. b ε {w: ∃x_174.((w,x_174) ε f)} ClassInt 1137
1139. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
1140. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 1139
1141. b ε domain(f) EqualitySub 1138 1140
1142. (m,b) ε r EqualitySub 1119 1116
1143. (b ε domain(f)) & ((m,b) ε r) AndInt 1141 1142
1144. _|_ ImpElim 1143 1122
1145. ((g'a),(g'b)) ε s AbsI 1144
1146. (b,q) ε {(m,n)} Hyp
1147. (b,q) = (m,n) ImpElim 1146 1029
1148. (m,n) = (b,q) Symmetry 1147
1149. Set((m,n)) & ((m,n) = (b,q)) AndInt 820 1148
1150. (m = b) & (n = q) ImpElim 1149 1041
1151. m = b AndElimL 1150
1152. (m,b) ε r EqualitySub 1119 1116
1153. b = m Symmetry 1151
1154. (m,m) ε r EqualitySub 1152 1153
1155. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x)) TheoremInt
1156. WellOrders(r,x) AndElimL 0
1157. Asymmetric(r,x) & TransIn(r,x) ImpElim 1156 1155
1158. Asymmetric(r,x) AndElimL 1157
1159. ∀y.∀z.(((y ε x) & (z ε x)) -> (((y,z) ε r) -> ¬((z,y) ε r))) DefExp 1158
1160. ∀z.(((m ε x) & (z ε x)) -> (((m,z) ε r) -> ¬((z,m) ε r))) ForallElim 1159
1161. ((m ε x) & (m ε x)) -> (((m,m) ε r) -> ¬((m,m) ε r)) ForallElim 1160
1162. m ε x AndElimL 742
1163. (m ε x) & (m ε x) AndInt 1162 1162
1164. ((m,m) ε r) -> ¬((m,m) ε r) ImpElim 1163 1161
1165. ¬((m,m) ε r) ImpElim 1154 1164
1166. _|_ ImpElim 1154 1165
1167. ((g'a),(g'b)) ε s AbsI 1166

1168. ((g'a),(g'b)) ε s OrElim 956 1123 1145 1146 1167
1169. ((g'a),(g'b)) ε s OrElim 953 954 1101 1102 1168
1170. ((g'a),(g'b)) ε s ExistsElim 947 949 1169
1171. ((g'a),(g'b)) ε s ExistsElim 946 948 1170
1172. ((a ε domain(g)) & ((b ε domain(g)) & ((a,b) ε r))) -> (((g'a),(g'b)) ε s) ImpInt 1171
1173. ∀b.(((a ε domain(g)) & ((b ε domain(g)) & ((a,b) ε r))) -> (((g'a),(g'b)) ε s)) ForallInt 1172
1174. ∀a.∀b.(((a ε domain(g)) & ((b ε domain(g)) & ((a,b) ε r))) -> (((g'a),(g'b)) ε s)) ForallInt 1173
1175. a ε domain(g) Hyp
1176. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
1177. ∀f.(domain(f) = {x: ∃y.((x,y) ε f)}) ForallInt 1176
1178. domain(g) = {x: ∃y.((x,y) ε g)} ForallElim 1177
1179. a ε {x: ∃y.((x,y) ε g)} EqualitySub 1175 1178
1180. Set(a) & ∃y.((a,y) ε g) ClassElim 1179
1181. ∃y.((a,y) ε g) AndElimR 1180
1182. (a,b) ε g Hyp
1183. (a,b) ε (f ∪ {(m,n)}) EqualitySub 1182 789
1184. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
1185. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 1184
1186. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 1185
1187. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 1186
1188. ∀x.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 1187
1189. (z ε (f ∪ y)) -> ((z ε f) v (z ε y)) ForallElim 1188
1190. ∀y.((z ε (f ∪ y)) -> ((z ε f) v (z ε y))) ForallInt 1189
1191. (z ε (f ∪ {(m,n)})) -> ((z ε f) v (z ε {(m,n)})) ForallElim 1190
1192. ∀z.((z ε (f ∪ {(m,n)})) -> ((z ε f) v (z ε {(m,n)}))) ForallInt 1191
1193. ((a,b) ε (f ∪ {(m,n)})) -> (((a,b) ε f) v ((a,b) ε {(m,n)})) ForallElim 1192
1194. ((a,b) ε f) v ((a,b) ε {(m,n)}) ImpElim 1183 1193
1195. (a,b) ε f Hyp
1196. ∃b.((a,b) ε f) ExistsInt 1195
1197. Set(a) AndElimL 1180
1198. Set(a) & ∃b.((a,b) ε f) AndInt 1197 1196
1199. a ε {w: ∃b.((w,b) ε f)} ClassInt 1198
1200. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 1176
1201. a ε domain(f) EqualitySub 1199 1200
1202. (a ε domain(f)) v (a ε {m}) OrIntR 1201
1203. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 1185
1204. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 1203
1205. ∀x.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 1204
1206. ((z ε domain(f)) v (z ε y)) -> (z ε (domain(f) ∪ y)) ForallElim 1205
1207. ∀y.(((z ε domain(f)) v (z ε y)) -> (z ε (domain(f) ∪ y))) ForallInt 1206
1208. ((z ε domain(f)) v (z ε {m})) -> (z ε (domain(f) ∪ {m})) ForallElim 1207
1209. ∀z.(((z ε domain(f)) v (z ε {m})) -> (z ε (domain(f) ∪ {m}))) ForallInt 1208
1210. ((a ε domain(f)) v (a ε {m})) -> (a ε (domain(f) ∪ {m})) ForallElim 1209
1211. a ε (domain(f) ∪ {m}) ImpElim 1202 1210
1212. (a,b) ε {(m,n)} Hyp
1213. Set((m,n)) & ((a,b) ε {(m,n)}) AndInt 820 1212
1214. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1215. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1214
1216. Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n))) ForallElim 1215
1217. ∀y.(Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n)))) ForallInt 1216
1218. Set((m,n)) -> (((a,b) ε {(m,n)}) <-> ((a,b) = (m,n))) ForallElim 1217
1219. Set((m,n)) AndElimL 1213
1220. ((a,b) ε {(m,n)}) <-> ((a,b) = (m,n)) ImpElim 1219 1218
1221. (((a,b) ε {(m,n)}) -> ((a,b) = (m,n))) & (((a,b) = (m,n)) -> ((a,b) ε {(m,n)})) EquivExp 1220
1222. ((a,b) ε {(m,n)}) -> ((a,b) = (m,n)) AndElimL 1221
1223. (a,b) = (m,n) ImpElim 1212 1222
1224. (m,n) = (a,b) Symmetry 1223
1225. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
1226. ∀a.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))) ForallInt 1225
1227. (Set((m,b)) & ((m,b) = (x,y))) -> ((m = x) & (b = y)) ForallElim 1226
1228. ∀b.((Set((m,b)) & ((m,b) = (x,y))) -> ((m = x) & (b = y))) ForallInt 1227
1229. (Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y)) ForallElim 1228
1230. ∀x.((Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y))) ForallInt 1229
1231. (Set((m,n)) & ((m,n) = (a,y))) -> ((m = a) & (n = y)) ForallElim 1230

1232. ∀y.((Set((m,n)) & ((m,n) = (a,y))) -> ((m = a) & (n = y))) ForallInt 1231
1233. (Set((m,n)) & ((m,n) = (a,b))) -> ((m = a) & (n = b)) ForallElim 1232
1234. Set((m,n)) & ((m,n) = (a,b)) AndInt 820 1224
1235. (m = a) & (n = b) ImpElim 1234 1233
1236. m = a AndElimL 1235
1237. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
1238. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 1237
1239. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp
1238
1240. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 1239
1241. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 1240
1242. Set((m,y)) -> (Set(m) & Set(y)) ForallElim 1241
1243. ∀y.(Set((m,y)) -> (Set(m) & Set(y))) ForallInt 1242
1244. Set((m,n)) -> (Set(m) & Set(n)) ForallElim 1243
1245. Set(m) & Set(n) ImpElim 1219 1244
1246. Set(m) AndElimL 1245
1247. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1248. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1247
1249. Set(m) -> ((y ε {m}) <-> (y = m)) ForallElim 1248
1250. ∀y.(Set(m) -> ((y ε {m}) <-> (y = m))) ForallInt 1249
1251. Set(m) -> ((a ε {m}) <-> (a = m)) ForallElim 1250
1252. (a ε {m}) <-> (a = m) ImpElim 1246 1251
1253. ((a ε {m}) -> (a = m)) & ((a = m) -> (a ε {m})) EquivExp 1252
1254. (a = m) -> (a ε {m}) AndElimR 1253
1255. a = m Symmetry 1236
1256. a ε {m} ImpElim 1255 1254
1257. (a ε domain(f)) v (a ε {m}) OrIntL 1256
1258. a ε (domain(f) ∪ {m}) ImpElim 1257 1210
1259. a ε (domain(f) ∪ {m}) OrElim 1194 1195 1211 1212 1258
1260. a ε (domain(f) ∪ {m}) ExistsElim 1181 1182 1259
1261. (a ε domain(g)) -> (a ε (domain(f) ∪ {m})) ImpInt 1260
1262. ∀a.((a ε domain(g)) -> (a ε (domain(f) ∪ {m}))) ForallInt 1261
1263. domain(g) ⊂ (domain(f) ∪ {m}) DefSub 1262
1264. a ε (domain(f) ∪ {m}) Hyp
1265. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
TheoremInt
1266. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 1265
1267. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y)))
EquivExp 1266
1268. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 1267
1269. ∀z.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 1268
1270. (a ε (x ∪ y)) -> ((a ε x) v (a ε y)) ForallElim 1269
1271. ∀x.((a ε (x ∪ y)) -> ((a ε x) v (a ε y))) ForallInt 1270
1272. (a ε (domain(f) ∪ y)) -> ((a ε domain(f)) v (a ε y)) ForallElim 1271
1273. ∀y.((a ε (domain(f) ∪ y)) -> ((a ε domain(f)) v (a ε y))) ForallInt 1272
1274. (a ε (domain(f) ∪ {m})) -> ((a ε domain(f)) v (a ε {m})) ForallElim 1273
1275. (a ε domain(f)) v (a ε {m}) ImpElim 1264 1274
1276. a ε domain(f) Hyp
1277. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
1278. a ε {x: ∃y.((x,y) ε f)} EqualitySub 1276 1277
1279. Set(a) & ∃y.((a,y) ε f) ClassElim 1278
1280. ∃y.((a,y) ε f) AndElimR 1279
1281. (a,b) ε f Hyp
1282. ((a,b) ε f) v ((a,b) ε {(m,n)}) OrIntR 1281
1283. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 1267
1284. ∀z.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 1283
1285. (((a,b) ε x) v ((a,b) ε y)) -> ((a,b) ε (x ∪ y)) ForallElim 1284
1286. ∀x.((((a,b) ε x) v ((a,b) ε y)) -> ((a,b) ε (x ∪ y))) ForallInt 1285
1287. (((a,b) ε f) v ((a,b) ε y)) -> ((a,b) ε (f ∪ y)) ForallElim 1286
1288. ∀y.((((a,b) ε f) v ((a,b) ε y)) -> ((a,b) ε (f ∪ y))) ForallInt 1287
1289. (((a,b) ε f) v ((a,b) ε {(m,n)})) -> ((a,b) ε (f ∪ {(m,n)})) ForallElim 1288
1290. (a,b) ε (f ∪ {(m,n)}) ImpElim 1282 1289
1291. (f ∪ {(m,n)}) = g Symmetry 789
1292. (a,b) ε g EqualitySub 1290 1291
1293. ∃b.((a,b) ε g) ExistsInt 1292
1294. Set(a) AndElimL 1279
1295. Set(a) & ∃b.((a,b) ε g) AndInt 1294 1293
1296. a ε {w: ∃b.((w,b) ε g)} ClassInt 1295
1297. ∀f.(domain(f) = {x: ∃y.((x,y) ε f)}) ForallInt 1277
1298. domain(g) = {x: ∃y.((x,y) ε g)} ForallElim 1297
1299. {x: ∃y.((x,y) ε g)} = domain(g) Symmetry 1298

1300. a ε domain(g) EqualitySub 1296 1299
1301. a ε domain(g) ExistsElim 1280 1281 1300
1302. a ε {m} Hyp
1303. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1304. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1303
1305. Set(m) -> ((y ε {m}) <-> (y = m)) ForallElim 1304
1306. ∀y.(Set(m) -> ((y ε {m}) <-> (y = m))) ForallInt 1305
1307. Set(m) -> ((a ε {m}) <-> (a = m)) ForallElim 1306
1308. (a ε {m}) <-> (a = m) ImpElim 808 1307
1309. ((a ε {m}) -> (a = m)) & ((a = m) -> (a ε {m})) EquivExp 1308
1310. (a ε {m}) -> (a = m) AndElimL 1309
1311. a = m ImpElim 1302 1310
1312. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1303
1313. Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n))) ForallElim 1312
1314. ∀y.(Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n)))) ForallInt 1313
1315. Set((m,n)) -> (((m,n) ε {(m,n)}) <-> ((m,n) = (m,n))) ForallElim 1314
1316. ((m,n) ε {(m,n)}) <-> ((m,n) = (m,n)) ImpElim 820 1315
1317. (((m,n) ε {(m,n)}) -> ((m,n) = (m,n))) & (((m,n) = (m,n)) -> ((m,n) ε {(m,n)}))
EquivExp 1316
1318. ((m,n) = (m,n)) -> ((m,n) ε {(m,n)}) AndElimR 1317
1319. (m,n) = (m,n) Identity
1320. (m,n) ε {(m,n)} ImpElim 1319 1318
1321. ((m,n) ε f) v ((m,n) ε {(m,n)}) OrIntL 1320
1322. ∀z.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 1283
1323. (((m,n) ε x) v ((m,n) ε y)) -> ((m,n) ε (x ∪ y)) ForallElim 1322
1324. ∀x.((((m,n) ε x) v ((m,n) ε y)) -> ((m,n) ε (x ∪ y))) ForallInt 1323
1325. (((m,n) ε f) v ((m,n) ε y)) -> ((m,n) ε (f ∪ y)) ForallElim 1324
1326. ∀y.((((m,n) ε f) v ((m,n) ε y)) -> ((m,n) ε (f ∪ y))) ForallInt 1325
1327. (((m,n) ε f) v ((m,n) ε {(m,n)})) -> ((m,n) ε (f ∪ {(m,n)})) ForallElim 1326
1328. (m,n) ε (f ∪ {(m,n)}) ImpElim 1321 1327
1329. (m,n) ε g EqualitySub 1328 1291
1330. ∃n.((m,n) ε g) ExistsInt 1329
1331. Set(m) & ∃n.((m,n) ε g) AndInt 808 1330
1332. m ε {w: ∃n.((w,n) ε g)} ClassInt 1331
1333. m ε domain(g) EqualitySub 1332 1299
1334. m = a Symmetry 1311
1335. a ε domain(g) EqualitySub 1333 1334
1336. a ε domain(g) OrElim 1275 1276 1301 1302 1335
1337. (a ε (domain(f) ∪ {m})) -> (a ε domain(g)) ImpInt 1336
1338. ∀a.((a ε (domain(f) ∪ {m})) -> (a ε domain(g))) ForallInt 1337
1339. (domain(f) ∪ {m}) ⊂ domain(g) DefSub 1338
1340. (domain(g) ⊂ (domain(f) ∪ {m})) & ((domain(f) ∪ {m}) ⊂ domain(g)) AndInt 1263 1339
1341. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
1342. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y)) EquivExp 1341
1343. ((x ⊂ y) & (y ⊂ x)) -> (x = y) AndElimR 1342
1344. ∀x.(((x ⊂ y) & (y ⊂ x)) -> (x = y)) ForallInt 1343
1345. ((domain(g) ⊂ y) & (y ⊂ domain(g))) -> (domain(g) = y) ForallElim 1344
1346. ∀y.(((domain(g) ⊂ y) & (y ⊂ domain(g))) -> (domain(g) = y)) ForallInt 1345
1347. ((domain(g) ⊂ (domain(f) ∪ {m})) & ((domain(f) ∪ {m}) ⊂ domain(g))) -> (domain(g) =
(domain(f) ∪ {m})) ForallElim 1346
1348. domain(g) = (domain(f) ∪ {m}) ImpElim 1340 1347
1349. a ε range(g) Hyp
1350. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
1351. ∀f.(range(f) = {y: ∃x.((x,y) ε f)}) ForallInt 1350
1352. range(g) = {y: ∃x.((x,y) ε g)} ForallElim 1351
1353. a ε {y: ∃x.((x,y) ε g)} EqualitySub 1349 1352
1354. Set(a) & ∃x.((x,a) ε g) ClassElim 1353
1355. ∃x.((x,a) ε g) AndElimR 1354
1356. (b,a) ε g Hyp
1357. (b,a) ε (f ∪ {(m,n)}) EqualitySub 1356 789
1358. ∀z.((z ε (f ∪ {(m,n)})) -> ((z ε f) v (z ε {(m,n)}))) ForallInt 1191
1359. ((b,a) ε (f ∪ {(m,n)})) -> (((b,a) ε f) v ((b,a) ε {(m,n)})) ForallElim 1358
1360. ((b,a) ε f) v ((b,a) ε {(m,n)}) ImpElim 1357 1359
1361. (b,a) ε f Hyp
1362. ∃b.((b,a) ε f) ExistsInt 1361
1363. Set(a) AndElimL 1354
1364. Set(a) & ∃b.((b,a) ε f) AndInt 1363 1362
1365. a ε {w: ∃b.((b,w) ε f)} ClassInt 1364
1366. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
1367. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 1366
1368. a ε range(f) EqualitySub 1365 1367

1369. (a ε range(f)) v (a ε {n}) OrIntR 1368
1370. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
1371. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 1370
1372. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 1371
1373. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 1372
1374. ∀z.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 1373
1375. ((a ε x) v (a ε y)) -> (a ε (x ∪ y)) ForallElim 1374
1376. ∀x.(((a ε x) v (a ε y)) -> (a ε (x ∪ y))) ForallInt 1375
1377. ((a ε range(f)) v (a ε y)) -> (a ε (range(f) ∪ y)) ForallElim 1376
1378. ∀y.(((a ε range(f)) v (a ε y)) -> (a ε (range(f) ∪ y))) ForallInt 1377
1379. ((a ε range(f)) v (a ε {n})) -> (a ε (range(f) ∪ {n})) ForallElim 1378
1380. a ε (range(f) ∪ {n}) ImpElim 1369 1379
1381. (b,a) ε {(m,n)} Hyp
1382. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1383. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1382
1384. Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n))) ForallElim 1383
1385. ∀y.(Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n)))) ForallInt 1384
1386. Set((m,n)) -> (((b,a) ε {(m,n)}) <-> ((b,a) = (m,n))) ForallElim 1385
1387. ((b,a) ε {(m,n)}) <-> ((b,a) = (m,n)) ImpElim 820 1386
1388. (((b,a) ε {(m,n)}) -> ((b,a) = (m,n))) & (((b,a) = (m,n)) -> ((b,a) ε {(m,n)})) EquivExp 1387
1389. ((b,a) ε {(m,n)}) -> ((b,a) = (m,n)) AndElimL 1388
1390. (b,a) = (m,n) ImpElim 1381 1389
1391. (m,n) = (b,a) Symmetry 1390
1392. Set((m,n)) & ((m,n) = (b,a)) AndInt 820 1391
1393. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y)) TheoremInt
1394. ∀a.((Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))) ForallInt 1393
1395. (Set((m,b)) & ((m,b) = (x,y))) -> ((m = x) & (b = y)) ForallElim 1394
1396. ∀b.((Set((m,b)) & ((m,b) = (x,y))) -> ((m = x) & (b = y))) ForallInt 1395
1397. (Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y)) ForallElim 1396
1398. ∀x.((Set((m,n)) & ((m,n) = (x,y))) -> ((m = x) & (n = y))) ForallInt 1397
1399. (Set((m,n)) & ((m,n) = (b,y))) -> ((m = b) & (n = y)) ForallElim 1398
1400. ∀y.((Set((m,n)) & ((m,n) = (b,y))) -> ((m = b) & (n = y))) ForallInt 1399
1401. (Set((m,n)) & ((m,n) = (b,a))) -> ((m = b) & (n = a)) ForallElim 1400
1402. (m = b) & (n = a) ImpElim 1392 1401
1403. n = a AndElimR 1402
1404. a = n Symmetry 1403
1405. Set(m) & Set(n) ImpElim 820 1244
1406. Set(m) AndElimL 1405
1407. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1382
1408. Set(n) -> ((y ε {n}) <-> (y = n)) ForallElim 1407
1409. ∀y.(Set(n) -> ((y ε {n}) <-> (y = n))) ForallInt 1408
1410. Set(n) -> ((a ε {n}) <-> (a = n)) ForallElim 1409
1411. Set(n) AndElimR 1405
1412. (a ε {n}) <-> (a = n) ImpElim 1411 1410
1413. ((a ε {n}) -> (a = n)) & ((a = n) -> (a ε {n})) EquivExp 1412
1414. (a = n) -> (a ε {n}) AndElimR 1413
1415. a ε {n} ImpElim 1404 1414
1416. (a ε range(f)) v (a ε {n}) OrIntL 1415
1417. a ε (range(f) ∪ {n}) ImpElim 1416 1379
1418. a ε (range(f) ∪ {n}) OrElim 1360 1361 1380 1381 1417
1419. a ε (range(f) ∪ {n}) ExistsElim 1355 1356 1418
1420. (a ε range(g)) -> (a ε (range(f) ∪ {n})) ImpInt 1419
1421. ∀a.((a ε range(g)) -> (a ε (range(f) ∪ {n}))) ForallInt 1420
1422. range(g) ⊂ (range(f) ∪ {n}) DefSub 1421
1423. a ε domain(g) Hyp
1424. a ε (domain(f) ∪ {m}) EqualitySub 1423 1348
1425. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
1426. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 1425
1427. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 1426
1428. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 1427
1429. ∀z.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 1428
1430. (a ε (x ∪ y)) -> ((a ε x) v (a ε y)) ForallElim 1429
1431. ∀x.((a ε (x ∪ y)) -> ((a ε x) v (a ε y))) ForallInt 1430
1432. (a ε (domain(f) ∪ y)) -> ((a ε domain(f)) v (a ε y)) ForallElim 1431
1433. ∀y.((a ε (domain(f) ∪ y)) -> ((a ε domain(f)) v (a ε y))) ForallInt 1432
1434. (a ε (domain(f) ∪ {m})) -> ((a ε domain(f)) v (a ε {m})) ForallElim 1433

1435. (a ε domain(f)) v (a ε {m}) ImpElim 1424 1434
1436. a ε domain(f) Hyp
1437. (a ε domain(f)) -> (a ε x) ForallElim 281
1438. a ε x ImpElim 1436 1437
1439. a ε {m} Hyp
1440. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1441. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1440
1442. Set(m) -> ((y ε {m}) <-> (y = m)) ForallElim 1441
1443. ∀y.(Set(m) -> ((y ε {m}) <-> (y = m))) ForallInt 1442
1444. Set(m) -> ((a ε {m}) <-> (a = m)) ForallElim 1443
1445. (a ε {m}) <-> (a = m) ImpElim 1406 1444
1446. ((a ε {m}) -> (a = m)) & ((a = m) -> (a ε {m})) EquivExp 1445
1447. (a ε {m}) -> (a = m) AndElimL 1446
1448. a = m ImpElim 1439 1447
1449. m = a Symmetry 1448
1450. a ε x EqualitySub 720 1449
1451. a ε x OrElim 1435 1436 1438 1439 1450
1452. (a ε domain(g)) -> (a ε x) ImpInt 1451
1453. ∀a.((a ε domain(g)) -> (a ε x)) ForallInt 1452
1454. domain(g) ⊂ x DefSub 1453
1455. a ε range(g) Hyp
1456. (a ε range(g)) -> (a ε (range(f) ∪ {n})) ForallElim 1421
1457. a ε (range(f) ∪ {n}) ImpElim 1455 1456
1458. ∀x.((a ε (x ∪ y)) -> ((a ε x) v (a ε y))) ForallInt 1430
1459. (a ε (range(f) ∪ y)) -> ((a ε range(f)) v (a ε y)) ForallElim 1458
1460. ∀y.((a ε (range(f) ∪ y)) -> ((a ε range(f)) v (a ε y))) ForallInt 1459
1461. (a ε (range(f) ∪ {n})) -> ((a ε range(f)) v (a ε {n})) ForallElim 1460
1462. (a ε range(f)) v (a ε {n}) ImpElim 1457 1461
1463. a ε range(f) Hyp
1464. (a ε range(f)) -> (a ε y) ForallElim 470
1465. a ε y ImpElim 1463 1464
1466. a ε {n} Hyp
1467. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1440
1468. Set(n) -> ((y ε {n}) <-> (y = n)) ForallElim 1467
1469. Set(n) AndElimR 1405
1470. ∀y.(Set(n) -> ((y ε {n}) <-> (y = n))) ForallInt 1468
1471. Set(n) -> ((a ε {n}) <-> (a = n)) ForallElim 1470
1472. (a ε {n}) <-> (a = n) ImpElim 1469 1471
1473. ((a ε {n}) -> (a = n)) & ((a = n) -> (a ε {n})) EquivExp 1472
1474. (a ε {n}) -> (a = n) AndElimL 1473
1475. a = n ImpElim 1466 1474
1476. n = a Symmetry 1475
1477. a ε y EqualitySub 762 1476
1478. a ε y OrElim 1462 1463 1465 1466 1477
1479. (a ε range(g)) -> (a ε y) ImpInt 1478
1480. ∀a.((a ε range(g)) -> (a ε y)) ForallInt 1479
1481. range(g) ⊂ y DefSub 1480
1482. (WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b) TheoremInt
1483. ∀a.((WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b)) ForallInt 1482
1484. (WellOrders(r,x) & (b ⊂ x)) -> WellOrders(r,b) ForallElim 1483
1485. ∀b.((WellOrders(r,x) & (b ⊂ x)) -> WellOrders(r,b)) ForallInt 1484
1486. (WellOrders(r,x) & (domain(g) ⊂ x)) -> WellOrders(r,domain(g)) ForallElim 1485
1487. WellOrders(r,x) AndElimL 0
1488. WellOrders(r,x) & (domain(g) ⊂ x) AndInt 1487 1454
1489. WellOrders(r,domain(g)) ImpElim 1488 1486
1490. WellOrders(s,y) AndElimR 0
1491. ∀r.((WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b)) ForallInt 1482
1492. (WellOrders(s,a) & (b ⊂ a)) -> WellOrders(s,b) ForallElim 1491
1493. ∀a.((WellOrders(s,a) & (b ⊂ a)) -> WellOrders(s,b)) ForallInt 1492
1494. (WellOrders(s,y) & (b ⊂ y)) -> WellOrders(s,b) ForallElim 1493
1495. ∀b.((WellOrders(s,y) & (b ⊂ y)) -> WellOrders(s,b)) ForallInt 1494
1496. (WellOrders(s,y) & (range(g) ⊂ y)) -> WellOrders(s,range(g)) ForallElim 1495
1497. WellOrders(s,y) & (range(g) ⊂ y) AndInt 1490 1481
1498. WellOrders(s,range(g)) ImpElim 1497 1496
1499. WellOrders(r,domain(g)) & WellOrders(s,range(g)) AndInt 1489 1498
1500. Function(g) & (WellOrders(r,domain(g)) & WellOrders(s,range(g))) AndInt 934 1499
1501. ((a ε domain(g)) & (b ε domain(g))) & ((a,b) ε r) Hyp
1502. (a ε domain(g)) & (b ε domain(g)) AndElimL 1501
1503. (a,b) ε r AndElimR 1501
1504. a ε domain(g) AndElimL 1502
1505. b ε domain(g) AndElimR 1502

1506. (b ε domain(g)) & ((a,b) ε r) AndInt 1505 1503
1507. (a ε domain(g)) & ((b ε domain(g)) & ((a,b) ε r)) AndInt 1504 1506
1508. ∀b.(((a ε domain(g)) & ((b ε domain(g)) & ((a,b) ε r))) -> (((g'a),(g'b)) ε s)) ForallElim 1174
1509. ((a ε domain(g)) & ((b ε domain(g)) & ((a,b) ε r))) -> (((g'a),(g'b)) ε s) ForallElim 1508
1510. ((g'a),(g'b)) ε s ImpElim 1507 1509
1511. (((a ε domain(g)) & (b ε domain(g))) & ((a,b) ε r)) -> (((g'a),(g'b)) ε s) ImpInt 1510
1512. ∀b.((((a ε domain(g)) & (b ε domain(g))) & ((a,b) ε r)) -> (((g'a),(g'b)) ε s)) ForallInt 1511
1513. ∀a.∀b.((((a ε domain(g)) & (b ε domain(g))) & ((a,b) ε r)) -> (((g'a),(g'b)) ε s)) ForallInt 1512
1514. (Function(g) & (WellOrders(r,domain(g)) & WellOrders(s,range(g)))) & ∀a.∀b.((((a ε domain(g)) & (b ε domain(g))) & ((a,b) ε r)) -> (((g'a),(g'b)) ε s)) AndInt 1500 1513
1515. OrderPreserving(g,r,s) DefSub 1514
1516. ((a ε x) & (b ε domain(g))) & ((a,b) ε r) Hyp
1517. (a ε x) & (b ε domain(g)) AndElimL 1516
1518. b ε domain(g) AndElimR 1517
1519. (b ε domain(g)) -> (b ε (domain(f) ∪ {m})) ForallElim 1262
1520. b ε (domain(f) ∪ {m}) ImpElim 1518 1519
1521. ∀z.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 1268
1522. (b ε (x ∪ y)) -> ((b ε x) v (b ε y)) ForallElim 1521
1523. ∀x.((b ε (x ∪ y)) -> ((b ε x) v (b ε y))) ForallInt 1522
1524. (b ε (domain(f) ∪ y)) -> ((b ε domain(f)) v (b ε y)) ForallElim 1523
1525. ∀y.((b ε (domain(f) ∪ y)) -> ((b ε domain(f)) v (b ε y))) ForallInt 1524
1526. (b ε (domain(f) ∪ {m})) -> ((b ε domain(f)) v (b ε {m})) ForallElim 1525
1527. (b ε domain(f)) v (b ε {m}) ImpElim 1520 1526
1528. b ε domain(f) Hyp
1529. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f))) DefExp 287
1530. ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f))) AndElimR 1529
1531. ∀v.((((a ε x) & (v ε domain(f))) & ((a,v) ε r)) -> (a ε domain(f))) ForallElim 1530
1532. (((a ε x) & (b ε domain(f))) & ((a,b) ε r)) -> (a ε domain(f)) ForallElim 1531
1533. a ε x AndElimL 1517
1534. (a ε x) & (b ε domain(f)) AndInt 1533 1528
1535. (a,b) ε r AndElimR 1516
1536. ((a ε x) & (b ε domain(f))) & ((a,b) ε r) AndInt 1534 1535
1537. a ε domain(f) ImpElim 1536 1532
1538. (a ε domain(f)) v (a ε {m}) OrIntR 1537
1539. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 1267
1540. ∀z.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 1539
1541. ((a ε x) v (a ε y)) -> (a ε (x ∪ y)) ForallElim 1540
1542. ∀x.(((a ε x) v (a ε y)) -> (a ε (x ∪ y))) ForallInt 1541
1543. ((a ε domain(f)) v (a ε y)) -> (a ε (domain(f) ∪ y)) ForallElim 1542
1544. ∀y.(((a ε domain(f)) v (a ε y)) -> (a ε (domain(f) ∪ y))) ForallInt 1543
1545. ((a ε domain(f)) v (a ε {m})) -> (a ε (domain(f) ∪ {m})) ForallElim 1544
1546. a ε (domain(f) ∪ {m}) ImpElim 1538 1545
1547. b ε {m} Hyp
1548. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1549. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1548
1550. Set(m) -> ((y ε {m}) <-> (y = m)) ForallElim 1549
1551. ∀y.(Set(m) -> ((y ε {m}) <-> (y = m))) ForallInt 1550
1552. Set(m) -> ((b ε {m}) <-> (b = m)) ForallElim 1551
1553. (b ε {m}) <-> (b = m) ImpElim 1406 1552
1554. ((b ε {m}) -> (b = m)) & ((b = m) -> (b ε {m})) EquivExp 1553
1555. (b ε {m}) -> (b = m) AndElimL 1554
1556. b = m ImpElim 1547 1555
1557. (a,b) ε r AndElimR 1516
1558. (a,m) ε r EqualitySub 1557 1556
1559. (m ε (x ~ domain(f))) & ∀y.((y ε (x ~ domain(f))) -> ¬((y,m) ε r)) DefExp 708
1560. ∀y.((y ε (x ~ domain(f))) -> ¬((y,m) ε r)) AndElimR 1559
1561. (a ε (x ~ domain(f))) -> ¬((a,m) ε r) ForallElim 1560
1562. ¬(a ε domain(f)) Hyp
1563. ∃w.(a ε w) ExistsInt 1533
1564. Set(a) DefSub 1563
1565. Set(a) & ¬(a ε domain(f)) AndInt 1564 1562
1566. a ε {w: ¬(w ε domain(f))} ClassInt 1565
1567. ~x = {y: ¬(y ε x)} DefEqInt

1568. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1567
1569. ~domain(f) = {y: ¬(y ε domain(f))} ForallElim 1568
1570. {y: ¬(y ε domain(f))} = ~domain(f) Symmetry 1569
1571. a ε ~domain(f) EqualitySub 1566 1570
1572. (a ε x) & (a ε ~domain(f)) AndInt 1533 1571
1573. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
1574. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 1573
1575. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 1574
1576. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 1575
1577. ∀z.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 1576
1578. ((a ε x) & (a ε y)) -> (a ε (x ∩ y)) ForallElim 1577
1579. ∀y.(((a ε x) & (a ε y)) -> (a ε (x ∩ y))) ForallInt 1578
1580. ((a ε x) & (a ε ~domain(f))) -> (a ε (x ∩ ~domain(f))) ForallElim 1579
1581. a ε (x ∩ ~domain(f)) ImpElim 1572 1580
1582. (x ~ y) = (x ∩ ~y) DefEqInt
1583. ∀y.((x ~ y) = (x ∩ ~y)) ForallInt 1582
1584. (x ~ domain(f)) = (x ∩ ~domain(f)) ForallElim 1583
1585. (x ∩ ~domain(f)) = (x ~ domain(f)) Symmetry 1584
1586. a ε (x ~ domain(f)) EqualitySub 1581 1585
1587. ¬((a,m) ε r) ImpElim 1586 1561
1588. _|_ ImpElim 1558 1587
1589. ¬¬(a ε domain(f)) ImpInt 1588
1590. D <-> ¬¬D TheoremInt
1591. (D -> ¬¬D) & (¬¬D -> D) EquivExp 1590
1592. ¬¬D -> D AndElimR 1591
1593. ¬¬(a ε domain(f)) -> (a ε domain(f)) PolySub 1592
1594. a ε domain(f) ImpElim 1589 1593
1595. (a ε domain(f)) v (a ε {m}) OrIntR 1594
1596. a ε (domain(f) ∪ {m}) ImpElim 1595 1545
1597. a ε (domain(f) ∪ {m}) OrElim 1527 1528 1546 1547 1596
1598. (domain(f) ∪ {m}) = domain(g) Symmetry 1348
1599. a ε domain(g) EqualitySub 1597 1598
1600. (((a ε x) & (b ε domain(g))) & ((a,b) ε r)) -> (a ε domain(g)) ImpInt 1599
1601. ∀b.((((a ε x) & (b ε domain(g))) & ((a,b) ε r)) -> (a ε domain(g))) ForallInt 1600
1602. ∀a.∀b.((((a ε x) & (b ε domain(g))) & ((a,b) ε r)) -> (a ε domain(g))) ForallInt 1601
1603. WellOrders(r,x) AndElimL 0
1604. (domain(g) ⊂ x) & WellOrders(r,x) AndInt 1454 1603
1605. ((domain(g) ⊂ x) & WellOrders(r,x)) & ∀a.∀b.((((a ε x) & (b ε domain(g))) & ((a,b) ε r)) -> (a ε domain(g))) AndInt 1604 1602
1606. Section(r,x,domain(g)) DefSub 1605
1607. ((a ε y) & (b ε range(g))) & ((a,b) ε s) Hyp
1608. (a ε y) & (b ε range(g)) AndElimL 1607
1609. b ε range(g) AndElimR 1608
1610. (b ε range(g)) -> (b ε (range(f) ∪ {n})) ForallElim 1421
1611. b ε (range(f) ∪ {n}) ImpElim 1609 1610
1612. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
1613. (z ε (x ∪ y)) <-> ((z ε x) v (z ε y)) AndElimL 1612
1614. ((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) & (((z ε x) v (z ε y)) -> (z ε (x ∪ y))) EquivExp 1613
1615. (z ε (x ∪ y)) -> ((z ε x) v (z ε y)) AndElimL 1614
1616. ∀z.((z ε (x ∪ y)) -> ((z ε x) v (z ε y))) ForallInt 1615
1617. (b ε (x ∪ y)) -> ((b ε x) v (b ε y)) ForallElim 1616
1618. ∀x.((b ε (x ∪ y)) -> ((b ε x) v (b ε y))) ForallInt 1617
1619. (b ε (range(f) ∪ y)) -> ((b ε range(f)) v (b ε y)) ForallElim 1618
1620. ∀y.((b ε (range(f) ∪ y)) -> ((b ε range(f)) v (b ε y))) ForallInt 1619
1621. (b ε (range(f) ∪ {n})) -> ((b ε range(f)) v (b ε {n})) ForallElim 1620
1622. (b ε range(f)) v (b ε {n}) ImpElim 1611 1621
1623. b ε range(f) Hyp
1624. ((range(f) ⊂ y) & WellOrders(s,y)) & ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε s)) -> (u ε range(f))) DefExp 477
1625. ∀u.∀v.((((u ε y) & (v ε range(f))) & ((u,v) ε s)) -> (u ε range(f))) AndElimR 1624
1626. ∀v.((((a ε y) & (v ε range(f))) & ((a,v) ε s)) -> (a ε range(f))) ForallElim 1625
1627. (((a ε y) & (b ε range(f))) & ((a,b) ε s)) -> (a ε range(f)) ForallElim 1626
1628. a ε y AndElimL 1608
1629. (a ε y) & (b ε range(f)) AndInt 1628 1623
1630. (a,b) ε s AndElimR 1607
1631. ((a ε y) & (b ε range(f))) & ((a,b) ε s) AndInt 1629 1630

1632. a ε range(f) ImpElim 1631 1627
1633. b ε {n} Hyp
1634. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1635. Set(n) AndElimR 1405
1636. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1634
1637. Set(n) -> ((y ε {n}) <-> (y = n)) ForallElim 1636
1638. ∀y.(Set(n) -> ((y ε {n}) <-> (y = n))) ForallInt 1637
1639. Set(n) -> ((b ε {n}) <-> (b = n)) ForallElim 1638
1640. (b ε {n}) <-> (b = n) ImpElim 1635 1639
1641. ((b ε {n}) -> (b = n)) & ((b = n) -> (b ε {n})) EquivExp 1640
1642. (b ε {n}) -> (b = n) AndElimL 1641
1643. b = n ImpElim 1633 1642
1644. n = b Symmetry 1643
1645. (n ε (y ~ range(f))) & ∀x_206.((x_206 ε (y ~ range(f))) -> ¬((x_206,n) ε s)) DefExp 709
1646. ∀x_206.((x_206 ε (y ~ range(f))) -> ¬((x_206,n) ε s)) AndElimR 1645
1647. (a ε (y ~ range(f))) -> ¬((a,n) ε s) ForallElim 1646
1648. (a,n) ε s EqualitySub 1630 1643
1649. ¬(a ε range(f)) Hyp
1650. ∃w.(a ε w) ExistsInt 1628
1651. Set(a) DefSub 1650
1652. Set(a) & ¬(a ε range(f)) AndInt 1651 1649
1653. a ε {w: ¬(w ε range(f))} ClassInt 1652
1654. ~x = {y: ¬(y ε x)} DefEqInt
1655. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1654
1656. ~range(f) = {y: ¬(y ε range(f))} ForallElim 1655
1657. {y: ¬(y ε range(f))} = ~range(f) Symmetry 1656
1658. a ε ~range(f) EqualitySub 1653 1657
1659. (a ε y) & (a ε ~range(f)) AndInt 1628 1658
1660. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 1612
1661. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 1660
1662. ((z ε x) & (z ε y)) -> (z ε (x ∩ y)) AndElimR 1661
1663. ∀z.(((z ε x) & (z ε y)) -> (z ε (x ∩ y))) ForallInt 1662
1664. ((a ε x) & (a ε y)) -> (a ε (x ∩ y)) ForallElim 1663
1665. ∀y.(((a ε x) & (a ε y)) -> (a ε (x ∩ y))) ForallInt 1664
1666. ((a ε x) & (a ε ~range(f))) -> (a ε (x ∩ ~range(f))) ForallElim 1665
1667. ∀x.(((a ε x) & (a ε ~range(f))) -> (a ε (x ∩ ~range(f)))) ForallInt 1666
1668. ((a ε y) & (a ε ~range(f))) -> (a ε (y ∩ ~range(f))) ForallElim 1667
1669. a ε (y ∩ ~range(f)) ImpElim 1659 1668
1670. (x ~ y) = (x ∩ ~y) DefEqInt
1671. ∀y.((x ~ y) = (x ∩ ~y)) ForallInt 1670
1672. (x ~ range(f)) = (x ∩ ~range(f)) ForallElim 1671
1673. ∀x.((x ~ range(f)) = (x ∩ ~range(f))) ForallInt 1672
1674. (y ~ range(f)) = (y ∩ ~range(f)) ForallElim 1673
1675. (y ∩ ~range(f)) = (y ~ range(f)) Symmetry 1674
1676. a ε (y ~ range(f)) EqualitySub 1669 1675
1677. ¬((a,n) ε s) ImpElim 1676 1647
1678. _|_ ImpElim 1648 1677
1679. ¬¬(a ε range(f)) ImpInt 1678
1680. ¬¬(a ε range(f)) -> (a ε range(f)) PolySub 1592
1681. a ε range(f) ImpElim 1679 1680
1682. a ε range(f) OrElim 1622 1623 1632 1633 1681
1683. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
1684. a ε {y: ∃x.((x,y) ε f)} EqualitySub 1682 1683
1685. Set(a) & ∃x.((x,a) ε f) ClassElim 1684
1686. ∃x.((x,a) ε f) AndElimR 1685
1687. (b,a) ε f Hyp
1688. ((b,a) ε f) v ((b,a) ε {(m,n)}) OrIntR 1687
1689. ((z ε x) v (z ε y)) -> (z ε (x ∪ y)) AndElimR 1614
1690. ∀z.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 1689
1691. (((b,a) ε x) v ((b,a) ε y)) -> ((b,a) ε (x ∪ y)) ForallElim 1690
1692. ∀x.((((b,a) ε x) v ((b,a) ε y)) -> ((b,a) ε (x ∪ y))) ForallInt 1691
1693. (((b,a) ε f) v ((b,a) ε y)) -> ((b,a) ε (f ∪ y)) ForallElim 1692
1694. ∀y.((((b,a) ε f) v ((b,a) ε y)) -> ((b,a) ε (f ∪ y))) ForallInt 1693
1695. (((b,a) ε f) v ((b,a) ε {(m,n)})) -> ((b,a) ε (f ∪ {(m,n)})) ForallElim 1694
1696. (b,a) ε (f ∪ {(m,n)}) ImpElim 1688 1695
1697. (f ∪ {(m,n)}) = g Symmetry 789
1698. (b,a) ε g EqualitySub 1696 1697
1699. ∃b.((b,a) ε g) ExistsInt 1698
1700. ∃b.((b,a) ε g) ExistsElim 1686 1687 1699

1701. Set(a) AndElimL 1685
1702. Set(a) & ∃b.((b,a) ε g) AndInt 1701 1700
1703. a ε {w: ∃b.((b,w) ε g)} ClassInt 1702
1704. range(f) = {y: ∃x.((x,y) ε f)} DefEqInt
1705. {y: ∃x.((x,y) ε f)} = range(f) Symmetry 1704
1706. ∀f.({y: ∃x.((x,y) ε f)} = range(f)) ForallInt 1705
1707. {y: ∃x.((x,y) ε g)} = range(g) ForallElim 1706
1708. a ε range(g) EqualitySub 1703 1707
1709. (((a ε y) & (b ε range(g))) & ((a,b) ε s)) -> (a ε range(g)) ImpInt 1708
1710. ∀b.((((a ε y) & (b ε range(g))) & ((a,b) ε s)) -> (a ε range(g))) ForallInt 1709
1711. ∀a.∀b.((((a ε y) & (b ε range(g))) & ((a,b) ε s)) -> (a ε range(g))) ForallInt 1710
1712. WellOrders(s,y) AndElimR 0
1713. WellOrders(s,y) & (range(g) ⊂ y) AndInt 1712 1481
1714. (range(g) ⊂ y) & WellOrders(s,y) AndInt 1481 1712
1715. ((range(g) ⊂ y) & WellOrders(s,y)) & ∀a.∀b.((((a ε y) & (b ε range(g))) & ((a,b) ε s)) -> (a ε range(g))) AndInt 1714 1711
1716. Section(s,y,range(g)) DefSub 1715
1717. Set(x) -> ((y ε {x}) <-> (y = x)) TheoremInt
1718. ∀x.(Set(x) -> ((y ε {x}) <-> (y = x))) ForallInt 1717
1719. Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n))) ForallElim 1718
1720. ∀y.(Set((m,n)) -> ((y ε {(m,n)}) <-> (y = (m,n)))) ForallInt 1719
1721. Set((m,n)) -> (((m,n) ε {(m,n)}) <-> ((m,n) = (m,n))) ForallElim 1720
1722. Set((m,n)) AndElimL 921
1723. ((m,n) ε {(m,n)}) <-> ((m,n) = (m,n)) ImpElim 820 1721
1724. (((m,n) ε {(m,n)}) -> ((m,n) = (m,n))) & (((m,n) = (m,n)) -> ((m,n) ε {(m,n)})) EquivExp 1723
1725. ((m,n) = (m,n)) -> ((m,n) ε {(m,n)}) AndElimR 1724
1726. (m,n) = (m,n) Identity
1727. (m,n) ε {(m,n)} ImpElim 1726 1725
1728. ((m,n) ε f) v ((m,n) ε {(m,n)}) OrIntL 1727
1729. ∀z.(((z ε x) v (z ε y)) -> (z ε (x ∪ y))) ForallInt 1689
1730. (((m,n) ε x) v ((m,n) ε y)) -> ((m,n) ε (x ∪ y)) ForallElim 1729
1731. ∀x.((((m,n) ε x) v ((m,n) ε y)) -> ((m,n) ε (x ∪ y))) ForallInt 1730
1732. (((m,n) ε f) v ((m,n) ε y)) -> ((m,n) ε (f ∪ y)) ForallElim 1731
1733. ∀y.((((m,n) ε f) v ((m,n) ε y)) -> ((m,n) ε (f ∪ y))) ForallInt 1732
1734. (((m,n) ε f) v ((m,n) ε {(m,n)})) -> ((m,n) ε (f ∪ {(m,n)})) ForallElim 1733
1735. (m,n) ε (f ∪ {(m,n)}) ImpElim 1728 1734
1736. (f ∪ {(m,n)}) = g Symmetry 789
1737. (m,n) ε g EqualitySub 1735 1736
1738. ∃n.((m,n) ε g) ExistsInt 1737
1739. Set(m) & ∃n.((m,n) ε g) AndInt 808 1738
1740. m ε {w: ∃n.((w,n) ε g)} ClassInt 1739
1741. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
1742. ∀f.(domain(f) = {x: ∃y.((x,y) ε f)}) ForallInt 1741
1743. domain(g) = {x: ∃y.((x,y) ε g)} ForallElim 1742
1744. {x: ∃y.((x,y) ε g)} = domain(g) Symmetry 1743
1745. m ε domain(g) EqualitySub 1740 1744
1746. (m ε domain(g)) & ((m,n) ε g) AndInt 1745 1737
1747. Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g)) AndInt 1716 1746
1748. Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g))) AndInt 1606 1747
1749. OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g)))) AndInt 1515 1748
1750. ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g))))) ExistsInt 1749
1751. (m ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g))))) AndInt 1162 1750
1752. w = (m,n) Hyp
1753. (w = (m,n)) & ((m ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g)))))) AndInt 1752 1751
1754. ∃n.((w = (m,n)) & ((m ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g))))))) ExistsInt 1753
1755. ∃m.∃n.((w = (m,n)) & ((m ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g))))))) ExistsInt 1754
1756. (m,n) = w Symmetry 1752
1757. Set(w) EqualitySub 820 1756
1758. Set(w) & ∃m.∃n.((w = (m,n)) & ((m ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g))))))) AndInt 1757 1755

1759. w ε {w: ∃m.∃n.((w = (m,n)) & ((m ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((m ε domain(g)) & ((m,n) ε g)))))))} ClassInt 1758
1760. (m,n) ε {w: ∃x_211.∃x_212.((w = (x_211,x_212)) & ((x_211 ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((x_211 ε domain(g)) & ((x_211,x_212) ε g)))))))} EqualitySub 1759 1752
1761. {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} = f Symmetry 1
1762. (m,n) ε f EqualitySub 1760 1761
1763. (w = (m,n)) -> ((m,n) ε f) ImpInt 1762
1764. ∀w.((w = (m,n)) -> ((m,n) ε f)) ForallInt 1763
1765. ((m,n) = (m,n)) -> ((m,n) ε f) ForallElim 1764
1766. (m,n) = (m,n) Identity
1767. (m,n) ε f ImpElim 1766 1765
1768. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f))) TheoremInt
1769. ∀a.(((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))) ForallInt 1768
1770. ((m,b) ε f) -> ((m ε domain(f)) & (b ε range(f))) ForallElim 1769
1771. ∀b.(((m,b) ε f) -> ((m ε domain(f)) & (b ε range(f)))) ForallInt 1770
1772. ((m,n) ε f) -> ((m ε domain(f)) & (n ε range(f))) ForallElim 1771
1773. (m ε domain(f)) & (n ε range(f)) ImpElim 1767 1772
1774. m ε domain(f) AndElimL 1773
1775. (g = (f ∪ {(m,n)})) -> (m ε domain(f)) ImpInt 1774
1776. ∀g.((g = (f ∪ {(m,n)})) -> (m ε domain(f))) ForallInt 1775
1777. ((f ∪ {(m,n)}) = (f ∪ {(m,n)})) -> (m ε domain(f)) ForallElim 1776
1778. (f ∪ {(m,n)}) = (f ∪ {(m,n)}) Identity
1779. m ε domain(f) ImpElim 1778 1777
1780. m ε domain(f) ExistsElim 707 709 1779
1781. (m ε (x ~ domain(f))) & ∀y.((y ε (x ~ domain(f))) -> ¬((y,m) ε r)) DefExp 708
1782. m ε (x ~ domain(f)) AndElimL 1781
1783. (x ~ y) = (x ∩ ~y) DefEqInt
1784. ∀y.((x ~ y) = (x ∩ ~y)) ForallInt 1783
1785. (x ~ domain(f)) = (x ∩ ~domain(f)) ForallElim 1784
1786. m ε (x ∩ ~domain(f)) EqualitySub 1782 1785
1787. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y))) TheoremInt
1788. (z ε (x ∩ y)) <-> ((z ε x) & (z ε y)) AndElimR 1787
1789. ((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) & (((z ε x) & (z ε y)) -> (z ε (x ∩ y))) EquivExp 1788
1790. (z ε (x ∩ y)) -> ((z ε x) & (z ε y)) AndElimL 1789
1791. ∀z.((z ε (x ∩ y)) -> ((z ε x) & (z ε y))) ForallInt 1790
1792. (m ε (x ∩ y)) -> ((m ε x) & (m ε y)) ForallElim 1791
1793. ∀y.((m ε (x ∩ y)) -> ((m ε x) & (m ε y))) ForallInt 1792
1794. (m ε (x ∩ ~domain(f))) -> ((m ε x) & (m ε ~domain(f))) ForallElim 1793
1795. (m ε x) & (m ε ~domain(f)) ImpElim 1786 1794
1796. m ε ~domain(f) AndElimR 1795
1797. ~x = {y: ¬(y ε x)} DefEqInt
1798. ∀x.(~x = {y: ¬(y ε x)}) ForallInt 1797
1799. ~domain(f) = {y: ¬(y ε domain(f))} ForallElim 1798
1800. m ε {y: ¬(y ε domain(f))} EqualitySub 1796 1799
1801. Set(m) & ¬(m ε domain(f)) ClassElim 1800
1802. ¬(m ε domain(f)) AndElimR 1801
1803. _|_ ImpElim 1780 1802
1804. _|_ ExistsElim 700 708 1803
1805. ¬(¬((x ~ domain(f)) = 0) & ¬((y ~ range(f)) = 0)) ImpInt 1804
1806. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B)) TheoremInt
1807. ¬(A & B) <-> (¬A v ¬B) AndElimR 1806
1808. ¬(¬((x ~ domain(f)) = 0) & B) <-> (¬¬((x ~ domain(f)) = 0) v ¬B) PolySub 1807
1809. ¬(¬((x ~ domain(f)) = 0) & ¬((y ~ range(f)) = 0)) <-> (¬¬((x ~ domain(f)) = 0) v ¬¬((y ~ range(f)) = 0)) PolySub 1808
1810. (¬(¬((x ~ domain(f)) = 0) & ¬((y ~ range(f)) = 0)) -> (¬¬((x ~ domain(f)) = 0) v ¬¬((y ~ range(f)) = 0))) & ((¬¬((x ~ domain(f)) = 0) v ¬¬((y ~ range(f)) = 0)) -> ¬(¬((x ~ domain(f)) = 0) & ¬((y ~ range(f)) = 0))) EquivExp 1809
1811. ¬(¬((x ~ domain(f)) = 0) & ¬((y ~ range(f)) = 0)) -> (¬¬((x ~ domain(f)) = 0) v ¬¬((y ~ range(f)) = 0)) AndElimL 1810
1812. ¬¬((x ~ domain(f)) = 0) v ¬¬((y ~ range(f)) = 0) ImpElim 1805 1811
1813. ¬¬((y ~ range(f)) = 0) Hyp
1814. ¬¬((y ~ range(f)) = 0) -> ((y ~ range(f)) = 0) PolySub 1592
1815. (y ~ range(f)) = 0 ImpElim 1813 1814
1816. ((x ~ domain(f)) = 0) v ((y ~ range(f)) = 0) OrIntL 1815
1817. ¬¬((x ~ domain(f)) = 0) Hyp

1818. ¬¬((x ~ domain(f)) = 0) -> ((x ~ domain(f)) = 0) PolySub 1592
1819. (x ~ domain(f)) = 0 ImpElim 1817 1818
1820. ((x ~ domain(f)) = 0) v ((y ~ range(f)) = 0) OrIntR 1819
1821. ((x ~ domain(f)) = 0) v ((y ~ range(f)) = 0) OrElim 1812 1817 1820 1813 1816
1822. ((y ⊂ x) & ((x ~ y) = 0)) -> (x = y) TheoremInt
1823. ∀y.(((y ⊂ x) & ((x ~ y) = 0)) -> (x = y)) ForallInt 1822
1824. ((domain(f) ⊂ x) & ((x ~ domain(f)) = 0)) -> (x = domain(f)) ForallElim 1823
1825. ∀y.(((y ⊂ x) & ((x ~ y) = 0)) -> (x = y)) ForallInt 1822
1826. ((range(f) ⊂ x) & ((x ~ range(f)) = 0)) -> (x = range(f)) ForallElim 1825
1827. ∀x.(((range(f) ⊂ x) & ((x ~ range(f)) = 0)) -> (x = range(f))) ForallInt 1826
1828. ((range(f) ⊂ y) & ((y ~ range(f)) = 0)) -> (y = range(f)) ForallElim 1827
1829. (domain(f) ⊂ x) & (range(f) ⊂ y) AndInt 282 471
1830. (x ~ domain(f)) = 0 Hyp
1831. domain(f) ⊂ x AndElimL 1829
1832. (domain(f) ⊂ x) & ((x ~ domain(f)) = 0) AndInt 1831 1830
1833. x = domain(f) ImpElim 1832 1824
1834. (x = domain(f)) v (y = range(f)) OrIntR 1833
1835. (y ~ range(f)) = 0 Hyp
1836. range(f) ⊂ y AndElimR 1829
1837. (range(f) ⊂ y) & ((y ~ range(f)) = 0) AndInt 1836 1835
1838. y = range(f) ImpElim 1837 1828
1839. (x = domain(f)) v (y = range(f)) OrIntL 1838
1840. (x = domain(f)) v (y = range(f)) OrElim 1821 1830 1834 1835 1839
1841. (OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f))) AndInt 661 1840
1842. ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f)))) ExistsInt 1841
1843. (f = {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))}) -> ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f)))) ImpInt 1842
1844. ∀f.((f = {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))}) -> ∃x_216.((OrderPreserving(x_216,r,s) & (Section(r,x,domain(x_216)) & Section(s,y,range(x_216)))) & ((x = domain(x_216)) v (y = range(x_216))))) ForallInt 1843
1845. ({w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} = {x_217: ∃x_218.∃x_219.((x_217 = (x_218,x_219)) & ((x_218 ε x) & ∃x_220.(OrderPreserving(x_220,r,s) & (Section(r,x,domain(x_220)) & (Section(s,y,range(x_220)) & ((x_218 ε domain(x_220)) & ((x_218,x_219) ε x_220)))))))}) -> ∃x_216.((OrderPreserving(x_216,r,s) & (Section(r,x,domain(x_216)) & Section(s,y,range(x_216)))) & ((x = domain(x_216)) v (y = range(x_216)))) ForallElim 1844
1846. {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} = {w: ∃u.∃v.((w = (u,v)) & ((u ε x) & ∃g.(OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & (Section(s,y,range(g)) & ((u ε domain(g)) & ((u,v) ε g)))))))} Identity
1847. ∃x_216.((OrderPreserving(x_216,r,s) & (Section(r,x,domain(x_216)) & Section(s,y,range(x_216)))) & ((x = domain(x_216)) v (y = range(x_216)))) ImpElim 1846 1845
1848. (OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f))) Hyp
1849. ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f)))) ExistsInt 1848
1850. ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f)))) ExistsElim 1847 1848 1849
1851. (WellOrders(r,x) & WellOrders(s,y)) -> ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f)))) ImpInt 1850 Qed

Used Theorems

1. (OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))))) -> ((f ⊂ g) v (g ⊂ f))
2. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
3. ((Set(x) & Set(y)) & ((x,y) = (u,v))) -> ((x = u) & (y = v))
4. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))
5. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))
6. (Function(f) & ((a,b) ε f)) -> ((f'a) = b)

7. (WellOrders(r,a) & (b ⊂ a)) -> WellOrders(r,b)
8. ((z ε (x ∪ y)) <-> ((z ε x) v (z ε y))) & ((z ε (x ∩ y)) <-> ((z ε x) & (z ε y)))
9. Set(x) -> ((y ε {x}) <-> (y = x))
10. (Set((a,b)) & ((a,b) = (x,y))) -> ((a = x) & (b = y))
11. (Function(f) & ((a,b) ε f)) -> ((f'a) = b)
12. WellOrders(r,x) -> (Asymmetric(r,x) & TransIn(r,x))
13. (x = y) <-> ((x ⊂ y) & (y ⊂ x))
14. D <-> ¬¬D
15. ((a,b) ε f) -> ((a ε domain(f)) & (b ε range(f)))
16. (¬(A v B) <-> (¬A & ¬B)) & (¬(A & B) <-> (¬A v ¬B))
17. ((y ⊂ x) & ((x ~ y) = 0)) -> (x = y)

Th100aux. (Function(f) & (Function(g) & ((domain(f) = domain(g)) & (f ⊂ g)))) -> (f = g)

0. Function(f) & (Function(g) & ((domain(f) = domain(g)) & (f ⊂ g))) Hyp
1. x ε g Hyp
2. Function(g) & ((domain(f) = domain(g)) & (f ⊂ g)) AndElimR 0
3. Function(g) AndElimL 2
4. Relation(g) & ∀x.∀y.∀z.((((x,y) ε g) & ((x,z) ε g)) -> (y = z)) DefExp 3
5. Relation(g) AndElimL 4
6. ∀z.((z ε g) -> ∃x.∃y.(z = (x,y))) DefExp 5
7. (x ε g) -> ∃x_3.∃y.(x = (x_3,y)) ForallElim 6
8. ∃x_3.∃y.(x = (x_3,y)) ImpElim 1 7
9. ∃y.(x = (n,y)) Hyp
10. x = (n,y) Hyp
11. (n,y) ε g EqualitySub 1 10
12. ∃b.((n,b) ε g) ExistsInt 11
13. ∃c.((n,y) ε c) ExistsInt 11
14. Set((n,y)) DefSub 13
15. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U)) TheoremInt
16. (Set(x) & Set(y)) <-> Set((x,y)) AndElimL 15
17. ((Set(x) & Set(y)) -> Set((x,y))) & (Set((x,y)) -> (Set(x) & Set(y))) EquivExp 16
18. Set((x,y)) -> (Set(x) & Set(y)) AndElimR 17
19. ∀x.(Set((x,y)) -> (Set(x) & Set(y))) ForallInt 18
20. Set((n,y)) -> (Set(n) & Set(y)) ForallElim 19
21. Set(n) & Set(y) ImpElim 14 20
22. Set(n) AndElimL 21
23. Set(n) & ∃b.((n,b) ε g) AndInt 22 12
24. n ε {m: ∃b.((m,b) ε g)} ClassInt 23
25. domain(f) = {x: ∃y.((x,y) ε f)} DefEqInt
26. {x: ∃y.((x,y) ε f)} = domain(f) Symmetry 25
27. ∀f.({x: ∃y.((x,y) ε f)} = domain(f)) ForallInt 26
28. {x: ∃y.((x,y) ε g)} = domain(g) ForallElim 27
29. n ε domain(g) EqualitySub 24 28
30. (domain(f) = domain(g)) & (f ⊂ g) AndElimR 2
31. domain(f) = domain(g) AndElimL 30
32. domain(g) = domain(f) Symmetry 31
33. n ε domain(f) EqualitySub 29 32
34. n ε {x: ∃y.((x,y) ε f)} EqualitySub 33 25
35. Set(n) & ∃y.((n,y) ε f) ClassElim 34
36. ∃y.((n,y) ε f) AndElimR 35
37. (n,z) ε f Hyp
38. (domain(f) = domain(g)) & (f ⊂ g) AndElimR 2
39. f ⊂ g AndElimR 38
40. ∀z.((z ε f) -> (z ε g)) DefExp 39
41. ((n,z) ε f) -> ((n,z) ε g) ForallElim 40
42. (n,z) ε g ImpElim 37 41
43. ∀x.∀y.∀z.((((x,y) ε g) & ((x,z) ε g)) -> (y = z)) AndElimR 4
44. ∀y.∀z.((((n,y) ε g) & ((n,z) ε g)) -> (y = z)) ForallElim 43
45. ∀z.((((n,y) ε g) & ((n,z) ε g)) -> (y = z)) ForallElim 44
46. (((n,y) ε g) & ((n,z) ε g)) -> (y = z) ForallElim 45
47. ((n,y) ε g) & ((n,z) ε g) AndInt 11 42
48. y = z ImpElim 47 46
49. x = (n,z) EqualitySub 10 48
50. (n,z) = x Symmetry 49
51. x ε f EqualitySub 37 50
52. x ε f ExistsElim 9 10 51
53. x ε f ExistsElim 9 10 52
54. x ε f ExistsElim 36 37 52
55. x ε f ExistsElim 9 10 54
56. x ε f ExistsElim 8 9 55

57. (x ε g) -> (x ε f) ImpInt 56
58. ∀x.((x ε g) -> (x ε f)) ForallInt 57
59. g ⊂ f DefSub 58
60. (f ⊂ g) & (g ⊂ f) AndInt 39 59
61. (x = y) <-> ((x ⊂ y) & (y ⊂ x)) TheoremInt
62. ((x = y) -> ((x ⊂ y) & (y ⊂ x))) & (((x ⊂ y) & (y ⊂ x)) -> (x = y)) EquivExp 61
63. ((x ⊂ y) & (y ⊂ x)) -> (x = y) AndElimR 62
64. ∀x.(((x ⊂ y) & (y ⊂ x)) -> (x = y)) ForallInt 63
65. ((f ⊂ y) & (y ⊂ f)) -> (f = y) ForallElim 64
66. ∀y.(((f ⊂ y) & (y ⊂ f)) -> (f = y)) ForallInt 65
67. ((f ⊂ g) & (g ⊂ f)) -> (f = g) ForallElim 66
68. f = g ImpElim 60 67
69. (Function(f) & (Function(g) & ((domain(f) = domain(g)) & (f ⊂ g)))) -> (f = g) ImpInt 68 Qed

Used Theorems

1. ((Set(x) & Set(y)) <-> Set((x,y))) & (¬Set((x,y)) -> ((x,y) = U))
2. (x = y) <-> ((x ⊂ y) & (y ⊂ x))

Th100. ((WellOrders(r,x) & (WellOrders(s,y) & (Set(x) & ¬Set(y)))) -> ∃f. ((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & (x = domain(f)))) & ((((OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & Section(s,y,range(g)))) & (x = domain(g))) & ((OrderPreserving(h,r,s) & (Section(r,x,domain(h)) & Section(s,y,range(h)))) & (x = domain(h)))) -> (g = h))

0. WellOrders(r,x) & (WellOrders(s,y) & (Set(x) & ¬Set(y))) Hyp
1. WellOrders(r,x) AndElimL 0
2. WellOrders(s,y) & (Set(x) & ¬Set(y)) AndElimR 0
3. WellOrders(s,y) AndElimL 2
4. WellOrders(r,x) & WellOrders(s,y) AndInt 1 3
5. (WellOrders(r,x) & WellOrders(s,y)) -> ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f)))) TheoremInt
6. ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f)))) ImpElim 4 5
7. (OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f))) Hyp
8. OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f))) AndElimL 7
9. OrderPreserving(f,r,s) AndElimL 8
10. (Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f)))) & ∀u.∀v.((((u ε domain(f)) & (v ε domain(f))) & ((u,v) ε r)) -> (((f'u),(f'v)) ε s)) DefExp 9
11. Function(f) & (WellOrders(r,domain(f)) & WellOrders(s,range(f))) AndElimL 10
12. Function(f) AndElimL 11
13. (Function(f) & Set(domain(f))) -> Set(range(f)) AxInt
14. (x = domain(f)) v (y = range(f)) AndElimR 7
15. OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f))) AndElimL 7
16. Section(r,x,domain(f)) & Section(s,y,range(f)) AndElimR 15
17. Section(r,x,domain(f)) AndElimL 16
18. ((domain(f) ⊂ x) & WellOrders(r,x)) & ∀u.∀v.((((u ε x) & (v ε domain(f))) & ((u,v) ε r)) -> (u ε domain(f))) DefExp 17
19. (domain(f) ⊂ x) & WellOrders(r,x) AndElimL 18
20. domain(f) ⊂ x AndElimL 19
21. (Set(x) & (y ⊂ x)) -> Set(y) TheoremInt
22. WellOrders(s,y) & (Set(x) & ¬Set(y)) AndElimR 0
23. Set(x) & ¬Set(y) AndElimR 22
24. Set(x) AndElimL 23
25. ∀y.((Set(x) & (y ⊂ x)) -> Set(y)) ForallInt 21
26. (Set(x) & (domain(f) ⊂ x)) -> Set(domain(f)) ForallElim 25
27. (Function(f) & Set(domain(f))) -> Set(range(f)) AxInt
28. Set(x) & (domain(f) ⊂ x) AndInt 24 20
29. Set(domain(f)) ImpElim 28 26
30. Function(f) & Set(domain(f)) AndInt 12 29
31. Set(range(f)) ImpElim 30 27
32. x = domain(f) Hyp
33. y = range(f) Hyp
34. range(f) = y Symmetry 33
35. Set(y) EqualitySub 31 34
36. ¬Set(y) AndElimR 23
37. _|_ ImpElim 35 36
38. x = domain(f) AbsI 37

39. x = domain(f) OrElim 14 32 32 33 38
40. (OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & (x = domain(f)) AndInt 8 39
41. ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & (x = domain(f))) ExistsInt 40
42. ∃f.((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & (x = domain(f))) ExistsElim 6 7 41
43. ((OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & Section(s,y,range(g)))) & (x = domain(g))) & ((OrderPreserving(h,r,s) & (Section(r,x,domain(h)) & Section(s,y,range(h)))) & (x = domain(h))) Hyp
44. (OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & Section(s,y,range(g)))) & (x = domain(g)) AndElimL 43
45. (OrderPreserving(h,r,s) & (Section(r,x,domain(h)) & Section(s,y,range(h)))) & (x = domain(h)) AndElimR 43
46. OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & Section(s,y,range(g))) AndElimL 44
47. OrderPreserving(g,r,s) AndElimL 46
48. Section(r,x,domain(g)) & Section(s,y,range(g)) AndElimR 46
49. Section(s,y,range(g)) AndElimR 48
50. Section(r,x,domain(g)) AndElimL 48
51. OrderPreserving(h,r,s) & (Section(r,x,domain(h)) & Section(s,y,range(h))) AndElimL 45
52. OrderPreserving(h,r,s) AndElimL 51
53. Section(r,x,domain(h)) & Section(s,y,range(h)) AndElimR 51
54. Section(r,x,domain(h)) AndElimL 53
55. Section(s,y,range(h)) AndElimR 53
56. Section(s,y,range(g)) & Section(s,y,range(h)) AndInt 49 55
57. Section(r,x,domain(h)) & (Section(s,y,range(g)) & Section(s,y,range(h))) AndInt 54 56
58. Section(r,x,domain(g)) & (Section(r,x,domain(h)) & (Section(s,y,range(g)) & Section(s,y,range(h)))) AndInt 50 57
59. OrderPreserving(h,r,s) & (Section(r,x,domain(g)) & (Section(r,x,domain(h)) & (Section(s,y,range(g)) & Section(s,y,range(h))))) AndInt 52 58
60. OrderPreserving(g,r,s) & (OrderPreserving(h,r,s) & (Section(r,x,domain(g)) & (Section(r,x,domain(h)) & (Section(s,y,range(g)) & Section(s,y,range(h)))))) AndInt 47 59
61. (OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))))) -> ((f ⊂ g) v (g ⊂ f)) TheoremInt
62. ∀g.((OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))))) -> ((f ⊂ g) v (g ⊂ f))) ForallInt 61
63. (OrderPreserving(f,r,s) & (OrderPreserving(h,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(h)) & (Section(s,y,range(f)) & Section(s,y,range(h))))))) -> ((f ⊂ h) v (h ⊂ f)) ForallElim 62
64. ∀f.((OrderPreserving(f,r,s) & (OrderPreserving(h,r,s) & (Section(r,x,domain(f)) & (Section(r,x,domain(h)) & (Section(s,y,range(f)) & Section(s,y,range(h))))))) -> ((f ⊂ h) v (h ⊂ f))) ForallInt 63
65. (OrderPreserving(g,r,s) & (OrderPreserving(h,r,s) & (Section(r,x,domain(g)) & (Section(r,x,domain(h)) & (Section(s,y,range(g)) & Section(s,y,range(h))))))) -> ((g ⊂ h) v (h ⊂ g)) ForallElim 64
66. (g ⊂ h) v (h ⊂ g) ImpElim 60 65
67. x = domain(g) AndElimR 44
68. x = domain(h) AndElimR 45
69. domain(g) = x Symmetry 67
70. domain(g) = domain(h) EqualitySub 69 68
71. (Function(g) & (WellOrders(r,domain(g)) & WellOrders(s,range(g)))) & ∀u.∀v.((((u ε domain(g)) & (v ε domain(g))) & ((u,v) ε r)) -> (((g'u),(g'v)) ε s)) DefExp 47
72. (Function(h) & (WellOrders(r,domain(h)) & WellOrders(s,range(h)))) & ∀u.∀v.((((u ε domain(h)) & (v ε domain(h))) & ((u,v) ε r)) -> (((h'u),(h'v)) ε s)) DefExp 52
73. Function(g) & (WellOrders(r,domain(g)) & WellOrders(s,range(g))) AndElimL 71
74. Function(g) AndElimL 73
75. Function(h) & (WellOrders(r,domain(h)) & WellOrders(s,range(h))) AndElimL 72
76. Function(h) AndElimL 75
77. (Function(f) & (Function(g) & ((domain(f) = domain(g)) & (f ⊂ g)))) -> (f = g) TheoremInt
78. ∀g.((Function(f) & (Function(g) & ((domain(f) = domain(g)) & (f ⊂ g)))) -> (f = g)) ForallInt 77
79. (Function(f) & (Function(h) & ((domain(f) = domain(h)) & (f ⊂ h)))) -> (f = h) ForallElim 78
80. ∀f.((Function(f) & (Function(h) & ((domain(f) = domain(h)) & (f ⊂ h)))) -> (f = h))

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ForallInt 79
81. (Function(g) & (Function(h) & ((domain(g) = domain(h)) & (g ⊂ h)))) -> (g = h)
ForallElim 80
82. g ⊂ h  Hyp
83. (domain(g) = domain(h)) & (g ⊂ h)  AndInt 70 82
84. Function(h) & ((domain(g) = domain(h)) & (g ⊂ h))  AndInt 76 83
85. Function(g) & (Function(h) & ((domain(g) = domain(h)) & (g ⊂ h)))  AndInt 74 84
86. g = h  ImpElim 85 81
87. h ⊂ g  Hyp
88. ∀f.((Function(f) & (Function(g) & ((domain(f) = domain(g)) & (f ⊂ g)))) -> (f = g))
ForallInt 77
89. (Function(h) & (Function(g) & ((domain(h) = domain(g)) & (h ⊂ g)))) -> (h = g)
ForallElim 88
90. domain(h) = domain(g)  Symmetry 70
91. (domain(h) = domain(g)) & (h ⊂ g)  AndInt 90 87
92. Function(g) & ((domain(h) = domain(g)) & (h ⊂ g))  AndInt 74 91
93. Function(h) & (Function(g) & ((domain(h) = domain(g)) & (h ⊂ g)))  AndInt 76 92
94. h = g  ImpElim 93 89
95. g = h  Symmetry 94
96. g = h  OrElim 66 82 86 87 95
97. (((OrderPreserving(g,r,s) & (Section(r,x,domain(g)) & Section(s,y,range(g)))) & (x =
domain(g))) & ((OrderPreserving(h,r,s) & (Section(r,x,domain(h)) &
Section(s,y,range(h)))) & (x = domain(h)))) -> (g = h)  ImpInt 96
98. (WellOrders(r,x) & (WellOrders(s,y) & (Set(x) & ¬Set(y)))) -> ∃f.
((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & (x =
domain(f)))  ImpInt 42
99. ((WellOrders(r,x) & (WellOrders(s,y) & (Set(x) & ¬Set(y)))) -> ∃f.
((OrderPreserving(f,r,s) & (Section(r,x,domain(f)) & Section(s,y,range(f)))) & (x =
domain(f)))) & ((((OrderPreserving(g,r,s) & (Section(r,x,domain(g)) &
Section(s,y,range(g)))) & (x = domain(g))) & ((OrderPreserving(h,r,s) &
(Section(r,x,domain(h)) & Section(s,y,range(h)))) & (x = domain(h)))) -> (g = h))  AndInt
98 97 Qed
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Used Theorems

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1. (WellOrders(r,x) & WellOrders(s,y)) -> ∃f.((OrderPreserving(f,r,s) &
(Section(r,x,domain(f)) & Section(s,y,range(f)))) & ((x = domain(f)) v (y = range(f))))
3. (Set(x) & (y ⊂ x)) -> Set(y)
2. (OrderPreserving(f,r,s) & (OrderPreserving(g,r,s) & (Section(r,x,domain(f)) &
(Section(r,x,domain(g)) & (Section(s,y,range(f)) & Section(s,y,range(g))))))) -> ((f ⊂ g)
v (g ⊂ f))
4. (Function(f) & (Function(g) & ((domain(f) = domain(g)) & (f ⊂ g)))) -> (f = g)
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Succesfully checked 71 theorems with a total of 10119 lines in 55 seconds.