Computational Epistemology

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28th of July 2021

The position we wish to put forward is that language (including natural language and mathematical language) should be formalised in some recursively axiomatised system P and that validity should be identified with provability and meaning with an actual proof. We call this thesis universal Hilbertian formalism (UHF).

There are many arguments that can be adduced in favour of UHF.

The first and most obvious one is that the same epistemological concerns which lead to the development of mathematical logic as a foundations of mathematics are equally applicable to all domains of science and philosophy as was expressed in the original concerns of Leibniz regarding his project of a formal and computable *universal characteristic*.

And there is the advantage of epistemic and ontological parsimony. However it may be remarked that UHF has just swept things under the rug and that all the original problems have been simply moved upstairs to the meta-logical and metal-mathematical level. This is indeed true. However it allows us to pose the question: what is the minimal cognitive-computational system required to justify the required metal-logical and meta-mathematical level ?(K1). Could the alleged parsimony be found precisely here ? We will return to this point which is the driving idea of this note.

We mention briefly some other strong arguments for UHF. Benaceraff has already remarked that there should not be an asymmetry between mathematical and natural language. That is, we should not have different semantic theories for each. This provides an argument against standard Tarskian semantics. This semantics, for all its merits in dealing with specifically first-order mathematical discourse, is manifestly insufficient for use alongside attempts to formalise natural language. One reason seems to be that it can be seen as *extensional* according to the Fregean framework wherein the denotation of a predicate is identified with the set of objects for which the predicate holds. But Frege himself agreed that in such utterances as *All men are mortal* we are certainly not referring to the relationship between the (obviously vague and unknown) extensions of the unary predicates Man and $Mortal$. It is clear that we are saying something about how our concept of Man includes the concept of $Mortal$. This is related to the general problem of induction in science, the justification of such propositions as *all living beings possess DNA*. We might phrase this fundamental problem as: how are universal valid statements about the world possible ? (K2). We will return to this later.

Tarskian semantics does not offer much of a theory of meaning. This brings us to another important point: that of the phenomenon of intensionality in natural language. It can be argued that Tarskian semantics (in a first-order context) is utterly insufficient to deal with intensionality, one reason being that it cannot deal with meaning in the framework of propositions and concepts being objects on the same level as individuals. For a general introduction to these problems see for instance Bealer's *Quality and Concept* or the introduction in our (hopefully) forthcoming article *Bealer's Intensional Logic* where we also argue both against nominalist or inscriptionalist approaches and the standard possible worlds semantics.

It might be added that the problems above have to do with the fact that the logical connectives and quantifiers[1] used in the standard foundations of mathematics based on mathematical logic are simply too different from their counterparts in natural language which are tied up to numerous constraints and spatial-temporal and other contextual factors, Mathematical logic is already intrinsically mathematical just as homotopy type theory foundations ends up working suprisingly well for...er...homotopy theory.

In our presentation of UHF above we equated meaning with proof. But we also equate proof with computation in the sense of an algorithmic procedure.

Consider how we learn the rules of chess. These are certainly not conceived as specified by subsets of the huge set of possible sequences of positions of pieces on the board. Rather they are learnt as algorithms. The game proceeds quite analogously to a formal proof. Suppose we knew that *the money is inside a book in the library.* Then such an existential quantifier is not equivalent to a finite disjunction regarding each individual book, for we may not know the extent of the library. Rather the proposition refers to the outcome of an algorithmic procedure of going through the books one by one.

In order to answer K1 we now introduce the thesis of computability (TC): *the human mind contains a Turing-complete computational system.* Note that this is not saying that the human mind is *only* a computational system or deny that it could contain non-computable capacities. Also this thesis is completely neutral with regards to any ontological or functionalist reductionism regarding consciousness and the physical brain. TC is the transcendental condition (in the Kantian sense) for the possibility of the metal-logic and meta-mathematics required for UHF. But in order for TC to fulfill this role Church's thesis must be invoked. We argue that the validity of metamathematics, which involves setting up recursive axiom systems and recursive deduction systems over finite strings, implies that the human mind[2] is at least a universal Turing machine and has all the elements and principles and categories necessary for consciousness to effect computations. *Church's thesis becomes a principle of transcendental logic* for it guarantees the universal agreement between the computations of all minds and between different syntactic embodiments of computability [3]. We are demanding a further analysis of the epistemic conditions for the possibility of finitary syntactic proofs (such as for instance the "syntactic" consistency proof of arithmetic based on the Hilbert-Ackerman theorem for open theories or the consistency proofs based on Gödel's system T.).

How are we to investigate the computational system of the mind ? We could proceed phenomenologically. What elements and principles in the human mind are required to play chess, what minimal finitary transcendental logic with its faculty of rules is in play ? It will help to analyse different data structures and functions used to implement a program for checking chess games, from a Turing machine, a counter automaton, a Boolean circuit, a cellular automaton to object-oriented code, to see which style or form corresponds best to the structure of the mind. The human mind is a kind of universal "machine" based on conscious perception and thought. Free exploration and the use analogy are no doubt important functions of cognition.

We are "given" primitives of sensory input and perception and a kind of action-ouput. We are given the comparison function between two objects. We are given objects with attributes. We are given the idea of a rule with a condition. We can check the condition and if it obtains we apply the rule. The human mind functions by symbolic spatial-temporal representations and finitary rules. Unlike the linear address space in modern hardware, the mind's memory appears

[1] A moment's thought will convince one that there is really not such thing as a totally "unbounded" quantifier.

[2] By "mind" we mean the system of our conscious experience.

[3] The significance of Church's thesis for philosophy cannot be overestimated. Too little is made of the interesting result that total computable functions are not recursively enumerable. This is usually proven by a standard diagonal argument using Turing machines; but by Church's thesis it is readily seen to imply that no recursively axiomatisable system (such as Heyting Arithmetic in all finite types) can represent all total computable functions from the natural numbers to the natural numbers. This computational "incompleteness" seems to us just as significant as Gödel's proof-theoretic incompleteness.

to be a multidimensional array. The geometric, diagramatic aspect of Category Theory and the fact that it seems to employ a negation and disjunction free fragment of intuitionistic predicate logic is certainly worthy of note. We can also consider A. Wierzbicka's approach in *Semantic Primes and Universals* and find the minimal set of primitives sufficient to act as a universal Turing machine in the sense that it furnishes a minimal vocabulary and grammar capable of describing all algorithms. Linguistic activity certainly includes finitary symbolic spatial-temporal operations codified by grammar. The different functions of language, the different ways we use language can be likened to the different programing languages paradigms. It is important to stress the *essentially dynamic diagramatic nature* of most logical and computational systems.

We now introduce the Logic-Arithmetic-Computability thesis (LAG) which might be seen as an extension of Church's thesis. This states that there is a common ground in which computational systems, arithmetic and logic are different ways of looking at the same reality. We argue that the common ground of LAG is contained in the system of TC. LAG also justifies classical logicism for a suitable fragment of arithmetic.

If TC (in particular augmented by LAG) allows us to answer K1 it also show us a way to answer K2 as well. If the physical world[4], the laws of physics, are computable in the sense that nature is a computational system in the act of computing (as suggested by the astonishing success of methods of numerical approximation of the equations of mathematical physics) then again by Church' thesis we are lead to justify our *a priori* conditional knowledge concerning the world, which has the form: if the rules were such and such then we can say such and such.

Incompleteness is a fundamental property of formal systems when analysed within other (stronger) formal systems, perhaps tied to the fundamental reflective property of consciousness. We answer Gödel's arguments in *Is Mathematics the Syntax of Language ?* by observing that metamathematical proofs are always themselves formalisable and verifiable within some wider encompassing equally recursively axiomatisable system.

We still must consider the fundamental problem with Kant's approach: how can reason justify itself in its claim of a transcendental self-critique ? Also, the clarification of experience, intuition, introspection and the nature of concepts.

We end with some considerations about LAG. When we investigate the concept of computability we necessarily require arithmetic. When we investigate arithmetic from a logical point of view it is inevitable that we consider computability or are lead to it. Arithmetic and computability are inseparable notions and it is likely that the Turing-Church thesis is tied to the categoricity of the theory of natural numbers N. When one defines a natural number object in a topos the universal property gives us automatically the notion of a primitive recursive function. The fundamental concept in recursion theory is that of partial recursive function, which can be embodied or implemented in a variety of abstract machine models such as Turing machines. Partial recursive functions, which are partial functions (partial because the computation need not stop) from natural numbers to natural numbers can themselves be codified by natural numbers. This allows us to have a concept of constructive functional, extending the notion of recursivity to all finite types over the natural numbers. There are several possible constructions such as HRO and HEO. This entire process can be generalised to the algebraic setting of partial combinatoric algebras. Intuitionistic arithmetic in all finite types HA^ω, the terms of which form Gödel's system T, is an alternative to set theory, category theory and type theory (or HOL) for doing mathematics. It has a model in ZFC given by interpreting terms of a given finite type by means set-theoretic maps and the natural numbers. This model is extremely complex (it can represent the real numbers) and very little can be said about it in general. What is amazing is that models of HA^ω like HRO and HEO pack this entire structure

[4]Language talks about the world. So a "universal" formal logic cannot avoid the necessity of having at hand a general systems theory. Indeed, it is by considering a general systems theory that our universal logic might begin to take shape. This will be addressed in our paper *Towards a Theory of Consciousness*.

into the microcosm of N ! This gives us a vision of the mathematical universe in which, as for the ancient Pythagoreans, all things are number.