A Logical Analysis of Plato's *Lysis*

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To formalise human knowledge let us fix a complete first-order theory T in a given first-order language $\mathcal{L}$. T represents the set of true sentences. It will include the set of all arithmetical truths. On the set $\mathcal{R}$ of binary relations in $\mathcal{L}$ there seems to be an operation of opposite: $\alpha : \mathcal{R} \to \mathcal{R}$ satisfying $\alpha \circ \alpha = id$ and the property

$$\text{(Op)} \qquad (S(x,y) \leftrightarrow R(x,y)) \to (S^\alpha(x,y) \leftrightarrow R^\alpha(x,y))$$

where R^α denotes $\alpha(R)$. Opposition need not coincide with negation. We have $R^\alpha(x,y) \to \sim R(x,y)$ but we do not necessarily have $\sim R(x,y) \to R^\alpha(x,y)$.

Let $C \subset T$ be the set of sentences of *common assent*. The Socratic goal is to determine whether a given ϕ belongs to T or not based on the assumptions C and a certain deductive system $\mathcal{D}$. In this paper we will take $\mathcal{D}$ to be natural deduction for classical first-order logic with equality. Among the constants, functions and predicates that occur in T some may be *eliminable*, but this may not be able to be determined based solely on C. In the Platonic dialogues there are a certain number of fundamental relations such as likeness, sameness, contrary, goal, good, evil, knowledge, virtue. Unary predicates express types and kind. We hold that the essence of Plato' thought is *logical analysis* with special moral-spiritual concerns as well as belief in the positive role of enlightened myth. The standards of Plato's logical analysis are no different than those of Leibniz, Frege or Russell. It is Plato the logician we wish to bring to light, so often overlooked in Aristotle's shadow. For Plato the problem of *definition* of a predicate $P(x_1, ..., x_n)$ is that of finding a formula $\phi(x_1, ..., x_n)$ in which P does not occur and in which the symbols occuring in ϕ are better known to us such that

$$\phi(x_1, ..., x_n) \leftrightarrow P(x_1, ..., x_n)$$

The *Lysis* concerns the definition of the binary relation $\Phi(x,y)$ (*y is x's friend*). We are given a relation $L(x,y)$(*x loves y*).

Socrates begins by inquiring concerning the logical relationship between Φ and L. Lysis assents to the first hypothesis for a definition

$$1.\ \ L(x,y)\ \&\ L(y,x) \leftrightarrow \Phi(x,y) \quad \text{Hyp}$$

Socrates then makes the deduction

$$2.\ \ \sim L(y,x) \to \sim \Phi(x,y) \quad 1$$

and brings out several sentences of C such as

$$3.\ \exists x,y.Man(x)\ \&\ Horse(y)\ \&\sim L(y,x)\ \&\ L(x,y)\ \&\ \Phi(x,y)$$

and $\exists x.Man(x)\ \&\ \Phi(x, wisdom)\ \&\sim L(wisdom, x)$ which refute 1 and thus we arrive at the conclusion

$$4.\ \sim (L(x,y)\ \&\ L(y,x) \leftrightarrow \Phi(x,y)) \quad 1,2,3$$

Then Socrates puts forward another hypothesis:

$$5.\ L(x,y) \leftrightarrow \Phi(x,y) \quad \text{Hyp}$$

Applying (Op) we get

$$6.\ L^{\alpha}(x,y) \leftrightarrow \Phi^{\alpha}(x,y)$$

But clearly $\exists x,y.L(x,y)\ \&\ L^{\alpha}(y,x)$. Hence we deduce that

$$7.\ \exists x,y.\Phi(x,y)\ \&\ \Phi^{\alpha}(y,x)$$

There will be people who are friends of their enemies and enemies of their friends. The negation of 7 is in C and so hypothesis 5 is also rejected and we have

$$8.\ \sim (L(x,y) \leftrightarrow \Phi(x,y)) \quad 5,6,7$$

Socrates then makes an appeal to the wisdom of the poets and brings forth the predicates $H(x,y)$(*likeness*) and $K(x)$ (*good*) alongside their opposites. For H opposition and negation seem to coincide.

Likeness can be extended to $\mathfrak{H}(x,y,z)$ likeness of x to y according to z and thereby resolve contradictions. So too for other relations. The following it put forth

$$9.\ H(x,y) \leftrightarrow \Phi(x,y) \quad \text{Hyp}$$

Then follows interesting propositions involving H:

$$10.\ \sim H(x,x) \rightarrow \forall y. \sim H(x,y)$$

$$11.\ K^{\alpha}(x) \rightarrow \sim H(x,x)\ \&\ K(x) \rightarrow H(x,x)$$

From these propositions (assumed to be part of C or at least T) Socrates then deduces (assuming $K(x)\ \&\ K(y) \rightarrow H(x,y)$) that

$$12.\ K(x)\ \&\ K(y) \rightarrow \Phi(x,y)\ \&\ K^{\alpha}(x)\ \&\ K^{\alpha}(y) \rightarrow \sim \Phi(x,y)$$

So bad people cannot friends.

Let $P(x,y,z)$ mean x *can do* z *to* y. Socrates assumes

$$13.\ H(x,y) \rightarrow \forall z.(P(x,y,z) \rightarrow P(x,x,z))$$

Let $\mathfrak{K}(x,y)$ be defined as $\exists z. \sim P(x,x,z)\ \&\ P(y,x,z)$. The axiom of the self-sufficiency of the good is:

$$K(x) \rightarrow \sim \exists y.\mathfrak{K}(x,y) \tag{K}$$

Socrates accepts the *pragmatic* condition for friendship

$$14.\ \Phi(x,y) \rightarrow \mathfrak{K}(x,y)$$

from which we deduce

$$15.\ \Phi(x,y) \rightarrow \sim H(x,y)$$

and hence hypothesis 9 is refuted

$$16.\ \ \sim (H(x,y) \leftrightarrow \Phi(x,y)) \quad 9{,}15$$

Next, inspired by Hesiod, Socrates proposes:

$$17.\ \ \sim H(x,y) \leftrightarrow \Phi(x,y)$$

For each n-ary predicate symbol P occuring in T there is an associated constant $[P]$ which expresses the abstract concept as an object of discourse. Opposition α can be internalised. We have $\mathfrak{A}([P],[Q])$ if and only if $P = Q^\alpha$. Clearly we should assume that $\sim H([P],[P^\alpha])$. Socrates then draws the obvious conclusion from 17:

$$18.\ \Phi([\Phi],[\Phi^\alpha])\ \ \&\ \ \Phi([\Phi^\alpha],[\Phi])$$

Friendship is friends with enmity and likewise justice is friends with injustice and so forth. Since this must be rejected we get

$$19.\ \ \sim (\sim H(x,y) \leftrightarrow \Phi(x,y)) \quad 17{,}18$$

There can be a middle ground or mixture between opposites or a least a ground which belongs to neither of the oppposites. Socrates proposes things which are neither good nor evil $\sim K(x)\ \ \&\sim K^\alpha(x)$.

We denote this predicate by $M(x)$. There are thus three classes of beings

$$20.\ \forall x.K(x) \vee K^\alpha(x) \vee M(x)$$

We can use these classes to propose a definition of the form

$$P(x)\ \ \&\ \ Q(x) \rightarrow \Phi(x,y)$$

where P and Q are among K, K^α and M. Since x such that $K^\alpha(x)$ is ruled out from entering into Φ it follows from 15 and (K) that the only possibility is

$$21.\ M(x)\ \ \&\ \ K(y) \rightarrow \Phi(x,y) \quad \text{Hyp}$$

We consider that there is an *extension* relation $\prec$ defined set of predicate symbols of $\mathcal{L}$. If $P \prec Q$ and P is n-ary then Q must be $n+1$-ary and the following axiom is satisfied

$$\text{(E)} \qquad P(x_1,...,x_n) \leftrightarrow \exists w.Q(x_1,...,x_n,w)$$

Let $\Phi^+(x,y,z)$ be an extension of $\Phi(x,y)$ which we takes as *y is friend to x for the sake of z*. Socrates states that

$$22.\ \Phi^+(x,y,z) \rightarrow \Phi(x,z)$$

and that the primary instance of $\Phi(x,y)$ we are interested in is that in which $\Phi^+(x,y,y)$.

Apparently Socrates rules out an endless sequence $\Phi^+(x,y,a_1), \Phi^+(x,a_2,a_3), \Phi^+(x,a_3,a_4),...$ as well as a cycle such as $\Phi^+(x,y,z), \Phi^+(x,z,y)$. Let$\mathfrak{F}(x,y)$ be by definition $\Phi^+(x,y,x)$.

There is also an extension $\Phi^a(x,y,z)$ which expresses that z is a cause for $\Phi(x,y)$ in that way that sickness is the cause of a doctor being a friend to a man. Socrates states that

$$23.\ \Phi^a(x,y,z) \rightarrow K^\alpha(z)$$

But

$$24.\ \exists x, y, z. \sim K^{\alpha}(z)\ \&\ \Phi^{a}(x, y, z)$$

so 23 must be rejected.

There is the predicate $B(x, y)$, y *belongs to* x. Socrates proposes

$$25.\ B(x, y) \rightarrow \Phi(x, y)$$

but

$$26.\ K(x) \rightarrow (B(x, y) \rightarrow K(y))$$

$$27.\ M(x) \rightarrow (B(x, y) \rightarrow M(y))$$

$$28.\ K^{\alpha}(x) \rightarrow (B(x, y) \rightarrow K^{\alpha}(y))$$

Hence since there is a y such that $B(x, y)$ and $K^{\alpha}(x)$ then by 24 we will have $\Phi(x, y)$ which contradicts 12. Hence we reject 25.