OFF-CRITICAL STATISTICAL MODELS: FACTORIZED SCATTERING THEORIES AND BOOTSTRAP PROGRAM

Giuseppe MUSSARDO

International School for Advanced Studies, Strada Costiera 12, 34014 Trieste, Italy

NORTH-HOLLAND

PHYSICS REPORTS (Review Section of Physics Letters) 218, Nos. 5 & 6 (1992) 215–379. North-Holland PHYSICS REPORTS

Off-critical statistical models: factorized scattering theories and bootstrap program

Giuseppe Mussardo

International School for Advanced Studies, Strada Costiera 12, 34014 Trieste, Italy

Received February 1992; editor: A. Schwimmer

Contents:

0370-1573/92/$15.00

Abstract:

We analyze those integrable statistical systems which originate from some relevant perturbations of the minimal models of conformal field theories. When only massive excitations are present, the systems can be efficiently characterized in terms of the relativistic scattering data. We review the general properties of the factorizable S-matrix in two dimensions with particular emphasis on the bootstrap principle. The classification program of the allowed spins of conserved currents and of the non-degenerate S-matrices is discussed and illustrated by means of some significant examples. The scattering theories of several massive perturbations of the minimal models are fully discussed. Among them are the Ising model, the tricritical Ising model, the Potts models, the series of the non-unitary minimal models $\mathcal{M}_{2,2n+3}$, the non-unitary model $\mathcal{M}_{3,5}$ and the scaling limit of the polymer system. The ultraviolet limit of these massive integrable theories can be exploited by the thermodynamics Bethe ansatz, in particular the central charge of the original conformal theories can be recovered from the scattering data. We also consider the numerical method based on the so-called conformal space truncated approach which confirms the theoretical results and allows a direct measurement of the scattering data, i.e. the masses and the S-matrix of the particles in bootstrap interaction. The problem of computing the off-critical correlation functions is discussed in terms of the form-factor approach.

1. Introduction

Close to a second-order phase transition point, the thermodynamic behaviour of a statistical model is dominated by large-scale fluctuations. The most important scale is given by the correlation length ξ, which is much larger than all other microscopic lengths. In particular, it diverges at the critical point and the system becomes scale invariant. In the vicinity of the critical point the fine details of the microscopic structure of many systems are observed to play an irrelevant role: statistical models with the same internal symmetry and the same dimensionality of the space share the same set of critical exponents and define the same class of universality [1–6]. Taking into account this universal behaviour, an efficient description of critical phenomena can be given in terms of Euclidean Quantum Field Theory (QFT) and its underlying framework, the Renormalization Group (RG) [1–10].

Besides its predictions of extraordinary accuracy, the renormalization group also provides a geometrical interpretation of the physical aspects of the critical phenomena. The scaling invariant theories are identified with *fixed points* of the RG transformations (in the infinite dimensional space of the local Hamiltonians $\mathscr{H}$) whereas the universality classes are in correspondence with the attraction domains of these fixed points.

Two-dimensional systems are ideal theoretical laboratories for testing the RG approach to statistical models. The reason is twofold. First, there exists a complete characterization of the fixed points of the RG. They correspond to Conformal Fields Theories (CFT), widely studied in the last years [11–28]. Secondly, the off-critical theories can be considered as CFT perturbed by appropriate relevant operators, i.e. they correspond to the RG trajectories which originate from the critical points [29, 30]. One can expect that a RG trajectory starting from a fixed point either goes to another critical point or develops a finite correlation length, in which case the corresponding field theory presents a purely massive spectrum.

An important progress in the investigation of non-scale invariant theories lies in a remarkable observation made by Zamolodchikov [29, 30] that, among all possible deformations of CFT, some of them possess an infinite set of commuting integrals of motion, i.e. they define integrable quantum field theories. These models are solvable even away from the critical point, and, if massive, they can be characterized in terms of a factorized scattering theory. In this case the integrals of motion restrict the possible bound state structure and mass ratios in the theory. Assuming further the bootstrap principle, i.e. all bound states belong to the same set of asymptotic particles, it is possible to construct the exact S-matrix with only a finite number of physical poles.

The aim of this paper is to present a review of the general properties of factorized S-matrices in two dimensions and their relevance in understanding the scaling region around the fixed points. The range of phenomena described by them is rich and instructive, as we will have ample opportunity to show in later chapters. In this introduction we will give a brief outline of the subject, starting with some features of scale invariant systems in two dimensions.

1.1. Minimal models of conformal field theories

There exist several excellent reviews on Conformal Field Theories (CFT), see e.g. refs. [15–18], which we advise the reader to consult. Our purpose here is to simply provide the most basic results and notations of CFT without attempting to justify the details.

Let us consider a two-dimensional statistical model with short-range interactions, invariant under translations and rotations. In the vicinity of a critical point, its properties are described by a Euclidean QFT. The basic quantities of the theory are the correlation functions of local fields (order parameters) $\phi_n(r)$,

$$\langle \phi_a(r_1)\phi_b(r_2)\cdots \phi_n(r_n)\rangle \equiv Z^{-1}\int \mathscr{D}\phi\ \phi_a(r_1)\phi_b(r_2)\cdots \phi_n(r_n)\,\mathrm{e}^{-S[\phi]}\,, \tag{1.1}$$

where S is the action of the theory and Z the corresponding partition function,

$$Z=\int \mathscr{D}\phi\ \mathrm{e}^{-S[\phi]}\,. \tag{1.2}$$

The variation of the correlation functions (1.1) under infinitesimal coordinate transformation,

$$r^\mu \to r^\mu + \varepsilon^\mu(r)\,, \tag{1.3}$$

is expressed by the following Ward identity:

$$\delta\langle \phi_\mathrm{a}(r_1)\phi_\mathrm{b}(r_2)\cdots\rangle = \left\langle \int \frac{\mathrm{d}^2 r}{2\pi}\,\partial^\mu\varepsilon^\nu(r)\,T_{\mu\nu}(r)\phi_a(r_1)\phi_b(r_2)\cdots\right\rangle . \tag{1.4}$$

This defines the stress-energy tensor $T_{\mu\nu}$. Rotational invariance implies

$$T_{\mu\nu}=T_{\nu\mu}\,, \tag{1.5}$$

whereas translation invariance leads to the conservation law

$$\partial^\mu T_{\mu\nu}=0\,. \tag{1.6}$$

At the critical fixed point, the trace of the stress-energy tensor vanishes

$$\Theta = T^\mu_\mu = 0\,, \tag{1.7}$$

and the system becomes invariant under dilatations

$$r^\mu \to \lambda r^\mu\,. \tag{1.8}$$

Actually, the condition (1.7), together with the locality of $T_{\mu\nu}$, automatically implies invariance under a large group of symmetry given by all transformations such that the traceless symmetric part of $\partial^\mu\varepsilon^\nu(r)$ vanishes [11]. Such transformations, acting as a combination of a rotation plus a dilatation, preserve all angles. This is the set of *conformal* transformations.

According to the scaling hypothesis [3], in the space of local fields ϕ_n we can choose a basis given by the scaling operators, which are eigenfunctions of the dilatation operator

$$\Phi_n(\lambda r) \to \lambda^{-x_n} \Phi_n(r) . \tag{1.9}$$

The constants x_n define the anomalous dimensions of the fields Φ_n. Their computation is the main problem of the theory of critical phenomena because they determine the singularity of thermodynamical functions in the vicinity of the critical point.

A successful approach for solving this problem has been proposed originally by Polyakov [11] and has been fully developed by Belavin, Polyakov and Zamolodchikov [12]. In this approach, known as *conformal bootstrap approach*, one assumes the existence of an associative Operator Product Expansion (OPE) algebra for the scaling fields Φ_n,

$$\Phi_n(r)\Phi_m(0) = \sum_k \frac{C_{nmk}}{r^{x_n+x_m-x_k}} \Phi_k(0) , \tag{1.10}$$

where C_{nmk} are the structure constants of the algebra. Requiring the compatibility of the operator algebra with the conformal symmetry, one gets a system of equations which fixes, in principle, the values of the anomalous dimensions and the structure constants. The far-reaching consequences of the conformal bootstrap approach for two-dimensional systems become much evident by adopting complex coordinates,

$$z \equiv x^1 + \mathrm{i}x^2 , \quad \bar{z} \equiv x^1 - \mathrm{i}x^2 . \tag{1.11}$$

In the z, $\bar{z}$ basis, the symmetric stress-energy tensor has components

$$T = T_{zz} , \quad \bar{T} = T_{\bar{z}\bar{z}} , \quad T_{z\bar{z}} = \tfrac{1}{4}\Theta . \tag{1.12}$$

At the critical point $\Theta = 0$ and the conservation law (1.6) gives the equations

$$\partial_{\bar{z}} T = 0 , \quad \partial_z \bar{T} = 0 , \tag{1.13}$$

so that $T = T(z)$ is a purely holomorphic field whereas $\bar{T} = \bar{T}(\bar{z})$ is purely anti-holomorphic. The OPE of these fields are given by [12]

$$\begin{aligned} T(z_1)T(z_2) &= \frac{c}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{z_1 - z_2} + \text{regular terms} , \\ T(z_1)\bar{T}(\bar{z}_2) &= \text{regular terms} , \\ \bar{T}(\bar{z}_1)\bar{T}(\bar{z}_2) &= \frac{c}{(\bar{z}_1 - \bar{z}_2)^4} + \frac{2\bar{T}(\bar{z}_2)}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{\partial \bar{T}(\bar{z}_2)}{\bar{z}_1 - \bar{z}_2} + \text{regular terms} . \end{aligned} \tag{1.14}$$

The parameter c is the conformal anomaly called *central charge*. It is one of the most important quantities of a conformal universality class.

In complex coordinates, the conformal transformations – defined by the vanishing of the traceless symmetric part of $\partial^\mu \varepsilon^\nu(r)$ – satisfy the conditions

$$\partial_{\bar z} \varepsilon^z = 0\,, \quad \partial_z \varepsilon^{\bar z} = 0\,. \tag{1.15}$$

Equations (1.13) together with (1.15) imply that in two dimensions the conformal group $\mathscr{G}$, generated by $T(z)$ and $\bar T(\bar z)$, factorizes into a direct product of local transformations of the form

$$z \to f(z)\,, \quad \bar z \to \bar f(\bar z)\,, \tag{1.16}$$

where $f(z)$ and $\bar f(\bar z)$ are arbitrary holomorphic and anti-holomorphic functions. The decoupling of the variables z and $\bar z$ is a noteworthy simplification of two-dimensional conformal systems which allows us to concentrate only on the holomorphic part, keeping in mind that similar properties also hold for the anti-holomorphic one. It is convenient to introduce the operators L_n, $n = 0, \pm 1, \pm 2, \ldots$ as coefficients of the Laurent expansion of the analytic component of the stress-energy tensor,

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \tag{1.17}$$

(similarly, we can define $\bar L_n$ as coefficients of the Laurent expansion for $\bar T(\bar z)$). It follows from (1.14) that the operators L_n satisfy the commutation relations

$$[L_n, L_m] = (n-m) L_{n+m} + \tfrac{1}{12} c n (n^2 - 1) \delta_{n+m,0} \tag{1.18}$$

(with a similar result for the $\bar L_n$). The above infinite-dimensional conformal algebra is called the Virasoro algebra. A subalgebra sl(2, C) of (1.18) consists of the generators L_{-1}, L_0, L_1, associated to the global conformal transformations

$$z \to w(z) = (az + b)/(cz + d)\,, \quad ad - bc = 1\,. \tag{1.19}$$

They are also known as Möbius transformations and are the only invertible mappings of the whole z-plane completed by a point at infinity.

Basic fields in two-dimensional Conformal Field Theories (CFT) are those operators $\Phi_n(z, \bar z)$ which transform under the conformal transformations (1.16) as

$$\Phi_n(z, \bar z) \to (\mathrm{d}f/\mathrm{d}z)^{\Delta_n} (\mathrm{d}\bar f/\mathrm{d}\bar z)^{\bar\Delta_n} \Phi_n(f(z), \bar f(\bar z))\,. \tag{1.20}$$

Such operators are called *primary* fields [12]. The pair $(\Delta_n, \bar\Delta_n)$ defines their conformal weights. The combinations $x_n = \Delta_n + \bar\Delta_n$ and $s_n = \Delta_n - \bar\Delta_n$ are the anomalous scale dimension and the spin of the field Φ_n, respectively. A primary field Φ_n satisfies (1.20) for *all* conformal transformations. This property distinguishes them from the *quasi-primary* fields $\mathscr{Q}_n$, which transform as a tensor of weights $(\Delta_n, \bar\Delta_n)$ only under the global conformal transformation (1.19). Obviously, any primary field is also quasi-primary but the contrary does not hold.

One of the most powerful results of two-dimensional conformal field theories is the classification of the operator content of a given critical model according to irreducible representations of the Virasoro algebra [12]. To label the states, we use their conformal weights Δ_n, i.e. the eigenvalues of L_0, and every scaling operator is either a primary field or a descendent thereof. Primary operators correspond to highest weight vectors. These are eigenvectors of L_0 which are annihilated by all L_n for $n>0$,

$$L_0|\Delta_i\rangle = \Delta|\Delta\rangle\ , \quad L_n|\Delta_i\rangle = 0\ , \quad n>0\ . \tag{1.21}$$

Among them, there is the vacuum $|0\rangle$. However, this state is also annihilated by L_{-1} because it is invariant under the sl$(2,\mathbb{C})$ global conformal group. Hence, it satisfies

$$L_n|0\rangle = 0\ , \quad n\geq -1\ . \tag{1.22}$$

The correspondence between the conformal primary fields and the highest weight vectors is given by

$$|\Delta_i\rangle = \Phi_{\Delta_i}(0)|0\rangle\ . \tag{1.23}$$

Descendant operators at level N with scaling dimension $\Delta + N$ correspond to linear combination of states of the form

$$L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\Delta_i\rangle\ , \quad n_1\geq n_2\geq n_3\cdots\geq n_k\ , \quad \sum n_i = N\ . \tag{1.24}$$

These fields, together with the primary fields Φ_{Δ_i} form the so-called *conformal families* $[\Phi_i]$, equivalently known as the Verma modules. Under a conformal transformation, each member of $[\Phi_i]$ is mapped into a representative of the same conformal family, i.e. $[\Phi_i]$ form a representation of the Virasoro algebra.

Exact solutions of two-dimensional conformal field theories have been studied extensively in the last years (see, e.g. ref. [14]). In this review we are mainly concerned with an infinite series of them, known as *minimal models* [12–14]. The minimal models $\mathcal{M}_{p,p'}$ are characterized by a pair $(p, p']$ of coprime positive integers*). The operator product expansion algebra of these models closes within a finite number of primary fields $\Phi_{r,s}$ with conformal weights given by the Kac formula

$$\Delta_{r,s} = \Delta_{p-r,p'-s} = \frac{(rp'-sp)^2-(p-p')^2}{4pp'}\ , \quad 1\leq r\leq p-1\ , \quad 1\leq s\leq p'-1\ . \tag{1.25}$$

The central charge of these models is

$$c = 1-6(p-p')^2/pp'\ . \tag{1.26}$$

It is also convenient to define an *effective central charge* $\tilde{c}$ given by

$$\tilde{c}\equiv c-24\,\Delta_{\min} = 1-6/\mathrm{pp}'\ , \tag{1.27}$$

where $\Delta_{\min}$ is the lowest conformal weight in (1.25). For the minimal models, $\tilde{c}$ is a positive quantity

*) An important subset of these minimal theories consists of the unitary conformal models, for which p and p' are two consecutive integers: $|p-p'|=1$, see ref. [13].

always less than 1. For the unitary minimal model, $\Delta_{\min} = 0$ and the effective central charge coincides with the central charge itself.

In the minimal models $\mathcal{M}_{p,p'}$, the descendant states (1.24) are not all independent because some linear combinations of them vanish. The corresponding states are called *null-vectors* [12] and their general expression has been explicitly found in ref. [21]. In order to obtain irreducible representations of the Virasoro algebra for the minimal models $\mathcal{M}_{p,p'}$, we have to project out all null-vectors from the Verma modules. The correlation functions of the fields $\Phi_{r,s}$ and their descendants then satisfy linear differential equations expressing these "null-vector" conditions [12]. All correlators can be obtained in terms of a Coulomb gas representation [20] and provide an explicit solution of the theory.

The structure of the Verma modules defined by the primary fields $\Phi_{r,s}$ is encoded in the corresponding characters,

$$\chi(q, c)_{\Delta_{r,s}} \equiv q^{-c/24} \operatorname{Tr} q^{L_0}|_{\Delta_{r,s}} = q^{-(c/24)+\Delta_{r,s}} \sum_{n=0}^{\infty} d_{r,s}(n) q^n . \tag{1.28}$$

Herein, $d_{r,s}(n)$ is the number of linearly independent states of the representation $\{\Phi_{r,s}\}$ at level n. The explicit formula for $\chi_{\Delta_{r,s}}(q, c)$ is given by [23]

$$\chi_{\Delta_{r,s}}(q, c) = \eta^{-1}(q) q^{-(c-1)/24+\Delta_{r,s}} \sum_{k=-\infty}^{\infty} q^{pp'k^2} (q^{k(rp'-sp)} - q^{k(rp'+sp)}) , \tag{1.29}$$

where $\eta(q)$ is the Dedekind η-function

$$\eta(q) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k) . \tag{1.30}$$

The physical meaning of the variable q becomes clear once we consider a CFT defined on a (complex) torus with modular parameter τ. The variable q is then identified with

$$q = e^{2\pi i \tau} , \tag{1.31}$$

and the characters (1.28) enter the expression of the partition function on the torus as

$$Z(q, \bar{q}) = \sum_{h,\bar{h}} N_{h,\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}) . \tag{1.32}$$

$N_{h,\bar{h}}$ are non-negative integers which characterize the operator content of the theory and therefore its class of universality. They are fixed by the requirement of modular invariance [24], i.e. the independence of the partition function (1.32) from the particular parameterization of the complex torus. The group of these transformations is called *modular group* and is generated by two fundamental transformations

$$\tau \to \tau + 1 , \quad \tau \to -1/\tau . \tag{1.33}$$

The problem of finding all such modular invariant partition functions in the case of the minimal models $\mathcal{M}_{p,p'}$ was solved in ref. [25]. The result is that these partition functions are classified by the same ADE series which classify the simply laced Lie algebras. In the case of minimal unitary CFT, they are the continuum limit of the corresponding partition functions for the lattice statistical models solved in refs. [31–33]. Among them, there are familiar systems of statistical mechanics: $\mathcal{M}_{3,4}$ describes the class of universality of the Ising model [12], $\mathcal{M}_{4,5}$ can be identified with the tricritical Ising model [13] and, in general, the models falling into the A series can be interpreted as multicritical points of statistical models with a Z_2 symmetry [34]. The first model of the D series has an operator content which is a subalgebra of $\mathcal{M}_{5,6}$ and is identified with the class of universality of the 3-state Potts model [35].

1.2. *Scaling region near the critical points*

The analysis of the classes of universality of statistical models must include the construction of the conformal field theories of the fixed points as well as the description of the scaling region around them. The linear neighbourhood of these fixed points is spanned by the *relevant scalar operators* Φ_{Δ_i,Δ_i} present in the characteristic CFT. These are the operators with anomalous dimensions $x_i = 2\Delta_i < 2$. Any RG trajectory flowing away from a such fixed point is obtained by any combination of the relevant fields. The corresponding off-critical action is given by

$$\mathcal{S} = \mathcal{S}^* + \sum_i \lambda_i \int \Phi_i(x)\, d^2x\,, \tag{1.34}$$

where $\mathcal{S}^*$ is the action corresponding to CFT. The relevant operators are of superrenormalizable type with respect to the ultraviolet divergences encountered in the perturbation series of (1.34). Hence they do not affect the behaviour of the system at short distances but they do change it at large distance scales. One can expect that the RG trajectories either reach another critical point (and, in this case, the large distance behaviour is governed by the CFT associated to it) or end at a non-critical fixed point, corresponding to a massive quantum field theory. For the unitary models, an important result concerning the RG flows is given by the c-theorem of Zamolodchikov [36].

1.2.1. *c-theorem and some applications*

Zamolodchikov [36] established the following theorem about the RG trajectories. For a class of quantum field theories which possesses rotational invariance, reflection positivity and conservation of the stress-energy tensor, there is a function $C(\{\lambda_i\})$ of the coupling constants λ_i which is non-increasing along the RG trajectories and is stationary only at the fixed points. At the fixed points it coincides with the central charge c of the corresponding CFT.

The proof is very simple [16, 36]. Let T, Θ and $\bar{T}$ be the spin $2, 0$ and -2 components of the stress-energy tensor. With the above assumptions, their correlators can be written as

$$\begin{aligned} &\langle T(z,\bar{z})T(0,0)\rangle = F(mz\bar{z})/z^4\,, \quad \langle T(z,\bar{z})\Theta(0,0)\rangle = G(mz\bar{z})/z^3\bar{z}\,, \\ &\langle \Theta(z,\bar{z})\Theta(0,0)\rangle = H(mz\bar{z})/z^2\bar{z}^2\,, \end{aligned} \tag{1.35}$$

where m is a mass scale. Using the conservation of the stress-energy tensor

$$\partial_{\bar{z}}T + \tfrac{1}{4}\,\partial_z\Theta = 0\,, \quad \partial_z\bar{T} + \tfrac{1}{4}\,\partial_{\bar{z}}\Theta = 0\,, \tag{1.36}$$

we deduce the following differential equations for the scalar functions F, G and H,

$$\dot{F} + \tfrac{1}{4}(\dot{G} - 3G) = 0 , \quad \dot{G} - G + \tfrac{1}{4}(\dot{H} - 2H) = 0 , \tag{1.37}$$

where

$$\dot{F} \equiv \mathrm{d}F(x)/\mathrm{d}\log x .$$

Defining

$$C \equiv 2F - G - \tfrac{3}{8}H , \tag{1.38}$$

we have finally

$$\dot{C} = -\tfrac{3}{4}H . \tag{1.39}$$

The positivity condition of QFT implies that H is a positive quantity and C is thus a non-increasing function. At the critical points, the trace of the stress-energy tensor vanishes, $\Theta = 0$. Therefore $G = H = 0$ and $F = \frac{1}{2}c$, so C reduces to the central charge of the corresponding CFT.

A reformulation of the c-theorem in terms of the spectral representation of the two-point function of the stress-energy tensor has been proposed in ref. [37]. Moreover, Cardy [38] has put forward an interesting link between the CFT data and the off-critical correlators by considering the integral version of eq. (1.39). Let us consider the simplest case of a deformation of CFT achieved by perturbing only with one relevant field Φ with anomalous dimension $x = 2\Delta$. The trace of the stress-energy tensor is given in this case by

$$\Theta(x) = 2\pi\lambda(2 - 2\Delta)\Phi(r) . \tag{1.40}$$

Using eq. (1.39), Cardy established the following sum rule for the total change in C from short to large distances:

$$\Delta c = 3\pi\lambda^2(2 - 2\Delta)^2 \int \mathrm{d}^2x\, |x|^2 \langle \Phi(x)\Phi(0)\rangle . \tag{1.41}$$

The above formula has been checked for the thermal perturbation of Ising model [38], i.e. for a massive free fermion theory. A more interesting example is to apply Cardy's sum rule (1.41) to a quantum field theory depending on a coupling constant. Let us consider for instance the sine–Gordon model with a Lagrangian given by

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \varphi)^2 + (m^2/\beta^2)(\cos \beta\varphi - 1): . \tag{1.42}$$

We restrict our attention to the phase region defined by $\beta^2 < 8\pi$. This model is massive (with $c = 0$) and we may consider it as a deformation of the free massless theory (with $c = 1$). In this case we have at lowest order in (m^2/β^2)

$$\lambda\varphi \equiv \varepsilon(x) = (m^2/\beta^2) :(\cos \beta\varphi - 1): \tag{1.43}$$

with anomalous dimension

$$2\Delta = \beta^2/4\pi\,. \tag{1.44}$$

For $\beta^2 < 8\pi$, the field $\varepsilon(x)$ is a relevant operator. Equation (1.41) becomes

$$\Delta c = 3\pi(2 - \beta^2/4\pi)^2 \int \mathrm{d}^2x\, |x|^2 < \varepsilon(x)\varepsilon(0)\rangle\,. \tag{1.45}$$

Since the left-hand side of (1.45) does not depend on β (in particular, it is equal to 1) the same must hold for the right-hand side at the order in (m^2/β^2) we are working. Notice that the sum rule is saturated just by the zero-order term in β, namely by the term corresponding to a free massive theory. In fact, at this order we have

$$\begin{aligned} &\varepsilon(x) = \tfrac{1}{2}m^2\varphi^2\,, \quad \Delta = 0\,, \\ &\langle \varepsilon(x)\varepsilon(0)\rangle = \tfrac{1}{2}m^4\langle \varphi(x)\varphi(0)\rangle^2 = (m^4/8\pi^2)K_0^2(m|x|) \end{aligned} \tag{1.46}$$

(K_0 is a Bessel function) and therefore

$$\Delta c_0 = 3\pi\,\frac{m^4}{2\pi^2}\int \mathrm{d}^2x\,|x|^2K_0^2(m|x|) = 3\int_0^\infty \mathrm{d}R\,R^3K_0^2(R) = 1\,. \tag{1.47}$$

This means that, expanding in series of β^2 the right-hand side of (1.45), the first coefficients in β^2 must vanish. Taking the first derivative with respect to β^2 and then putting $\beta^2 = 0$,

$$\frac{1}{3\pi}\,\frac{\mathrm{d}(\Delta c)}{\mathrm{d}\beta^2}\bigg|_{\beta^2=0} = -\frac{1}{\pi}\int \mathrm{d}^2x\,|x|^2\langle \varepsilon(x)\varepsilon(0)\rangle + 4\int \mathrm{d}^2x\,|x|^2\,\frac{\mathrm{d}}{\mathrm{d}\beta^2}\langle \varepsilon(x)\varepsilon(0)\rangle\,. \tag{1.48}$$

The first term is what we computed before for the free massive theory. In order to evaluate the second term, we expand the term $\cos(\beta\varphi)$ to the fourth order and use Wick's theorem. The result is

$$\frac{1}{3\pi}\,\frac{\mathrm{d}(\Delta c)}{\mathrm{d}\beta^2}\bigg|_{\beta^2=0} = -\frac{1}{12\pi^2} + \frac{m^6}{(2\pi)^4}\int \mathrm{d}^2x\,\mathrm{d}^2z\,|x|^2K_0^2(m|x-z|)K_0^2(m|z|)\,. \tag{1.49}$$

The last integral is easily computed: with the change of variable $|x - z| \to |t|$, it becomes the product of the integrals

$$2\pi\int_0^\infty R^3K_0^2(R)\,\mathrm{d}R = \tfrac{2}{3}\pi\,, \quad 2\pi\int_0^\infty RK_0^2(R)\,\mathrm{d}R = \pi \tag{1.50}$$

(the term in (1.49) which contains the scalar product $\boldsymbol{x}\cdot\boldsymbol{z}$ is zero after the angular integration). Inserting this into (1.49), we see that the variation of the central charge with respect to β^2 is zero, as it should be.

The c-theorem has been exploited in the analysis of quite a large number of physical systems, see e.g. refs. [39–42]. A rather remarkable example has been recently discussed by Zamolodchikov [43]: he found a remarkable pattern of "roaming" trajectories of the RG which flow near the succession of the $\mathcal{M}_{p,p+1}$ fixed points. We refer to the original literature for a detailed discussion of this point.

1.3. Minimal integrable models

An integrable model is characterized by an infinite set of conserved currents. In two-dimensional systems we can use holomorphic and anti-holomorphic indices and write the conservation laws for a current $(J_{z,z,\dots}, J_{\bar{z},\bar{z},\dots})$ in the form

$$\partial_{\bar{z}} J_{z,z,\dots} + \partial_z J_{\bar{z},\bar{z},\dots} = 0 . \tag{1.51}$$

As we noticed before, an important feature of the CFT is the decoupling of the z and $\bar{z}$ dependence. We may take as $J_{z,\dots}$ any independent operator in the conformal block of the stress-energy tensor. Since they are analytic functions, therefore eq. (1.51) is trivially satisfied by them, i.e. the conformal field theories of the fixed points possess an infinite number of conserved currents.

In the deformed theory defined by the action (1.34), this hierarchy of conserved currents – together with the decoupling of the z and $\bar{z}$ dependence – is generally destroyed. An analysis of the models which arise by an arbitrary deformation of the critical point action may be performed by the usual perturbative methods, but many aspects of the theory might remain hidden in this approach. Alternatively, the discovery by Zamolodchikov [29, 30] of the so-called integrable deformations of CFT has opened a completely new perspective in this investigation. The corresponding QFT possesses an infinite set of conserved charges $\mathcal{P}_s$ in involution which permits to solve the theory non-perturbatively. We call this class of theories *Minimal Integrable Models* (MIM).

In this review we are interested in those deformations of CFT which give rise to massive integrable models. The infinite number of integrals of motion present in these theories, precludes the possibility of inelastic scattering of the massive excitations and the n-particle S-matrix factorizes into a product of $n(n-1)/2$ elastic two-particle S-matrices. As a consequence of this factorization, the two-particle S-matrices satisfy, in addition to the usual requirements of unitarity and crossing, the star-triangle equations [120]. Moreover, they are linked among themselves by the bootstrap equations [30].

The knowledge of the exact on-shell S-matrix of a model is the starting point of further investigations of its properties. Once the scattering amplitudes are known, one can exploit bootstrap methods either to construct matrix elements of local fields (the so-called form factors), or to compute finite-size effects of such theories by means of the thermodynamical Bethe ansatz.

1.3.1. Bootstrap program for the minimal integrable models

A constructive approach to the MIM consists in the following program. Given the conformal data*) – central charge, dimensions, fusion algebras, etc. –

(i) find possible integrable perturbations, i.e. identify the off-critical conservation laws;

(ii) solve the bootstrap equations for the S-matrix consistent with the conservation laws, satisfying unitarity, crossing symmetry, analyticity and factorization;

*) In QFT this is equivalent to knowing the short distance behaviour, i.e. the ultraviolet properties of the theory; the infrared properties are the subject of our analysis.

(iii) find the field equations and the off mass-shell theories corresponding to these exact S-matrices.

Each of these points needs some comments.

The realization of point (i) of this program is far from being conclusive although there is a large and still growing number of examples of integrable deformations of CFT. Let T^i_{2s+2} be the quasi-primary descendents of the stress-energy tensor. Given a perturbation $\lambda_i \Phi_i$, the problem is to see which of the conformal conservation laws

$$\partial_{\bar{z}} T^i_{2s+2} = 0 \tag{1.52}$$

gives rise to the off-critical conservation laws, i.e. to find the spins s and the local fields Θ_{2s}, such that

$$\partial_{\bar{z}} T^i_{2s+2} = \partial_z \Theta_{2s} . \tag{1.53}$$

Whenever the initial system presents an invariance under an infinite algebra in addition to conformal symmetry [26] (this is the case, for instance, of the superconformal minimal models [27] or the parafermionic models [28]), the previous discussion can be generalized to the quasi-primary descendents of the corresponding currents [156, 179] (see chapter 2). Therefore, we may have conserved charges

$$P_{2s+1} = \int (T_{2s+2}\, \mathrm{d}z + \Theta_{2s}\, \mathrm{d}\bar{z}) \tag{1.54}$$

of integer, half-integer or fractional spins. Their existence is deeply related to the null-vector conditions of the primary field Φ_i [29, 30, 150, 189].

The bootstrap equations of point (ii) are the following functional equations for the two-particle scattering amplitude $S_{ab}(\theta)$ [30, 121]

$$S_{\mathrm{cd}}(\theta) = S_{\mathrm{bd}}(\theta - \mathrm{i}\bar{u}^a_{bc}) S_{ad}(\theta + i\bar{u}^b_{ac}) , \quad \bar{u} = \pi - u , \quad u^a_{bc} + u^b_{ac} + u^c_{ab} = 2\pi . \tag{1.55}$$

We have introduced the rapidity θ, related to the light-cone components of the momentum by $p = m\, \mathrm{e}^{\theta}$ and $\bar{p} = m\, \mathrm{e}^{-\theta}$. To solve these equations, it is often necessary to find an ansatz for the two-particle bootstrap fusions of some minimal subset of particles. These fusions should be consistent with the conservation laws and the symmetries of the model. The masses of the particles enter the analytic structure of the scattering amplitudes, i.e. they fix the location of the physical poles $\theta = \mathrm{i}u^c_{ab}$ *of* S_{ab}. For example, the fusions

$$a \times a \to \bar{a} + \bar{b} , \quad b \times b \to \bar{a} \tag{1.56}$$

($\bar{i}$ denote the antiparticle of $i = a, b$, with the same mass $m_i = m_{\bar{i}}$) are consistent with the conserved spins

$$s = 1, 4, 5, 7, 8, 11 \pmod{12} \tag{1.57}$$

iff

$$u^{\bar{b}}_{aa} = \tfrac{11}{12}\pi , \quad u^{\bar{a}}_{aa} = \tfrac{2}{3}\pi , \quad u^{\bar{a}}_{bb} = \tfrac{7}{12}\pi . \tag{1.58}$$

With this "initial condition", one can find a closed solution of eqs. (1.55) including only two new particles [156, 159] (see section 6.7).

The heuristic approach to integrable models discussed so far is complementary to the quantum group approach and both approaches enlighten the structure of these QFT. In fact, if we knew the symmetries of the perturbed models and the representations which are consistent with the conservation laws, then the bootstrap fusions would be nothing but the tensor product rules of these representations. Therefore, the bootstrap fusions encoded in the physical poles of the exact S-matrices are an important tool in searching for the hidden symmetries of the MIM. We present a general scheme to classify the N-particle bootstrap fusions and the corresponding S-matrices in section 3.5 [149, 156, 158, 160]. The question we address is: given a set of N massive particles $a_i(p_i)$, where p_i is the momentum, with the fusions

$$a_i(p_i) \times a_j(p_j) \rightarrow \sum_k C_{ijk} a_k(p_i + p_j) \tag{1.59}$$

find

- the resonance angles u_{ij}^k entering the on-shell mass condition

$$m_k^2 - m_i^2 - m_j^2 = 2m_i m_j \cos u_{ij}^k\,, \quad u_{ij}^k + u_{jk}^i + u_{ik}^j = 2\pi \tag{1.60}$$

the mass ratios m_j/m_i, and the solution of bootstrap equations (1.55);

- the infinite set of conserved charges P_s consistent with (1.59) and (1.60), such that*)

$$\begin{aligned} &P_s|a_i(p_i)\rangle = \gamma_s^i(p_i/m_i)^s|a_i(p_i)\rangle\,, \\ &P_s|a_1(p_i)\cdots a_n(p_n)\rangle = \sum_{k=1}^n \gamma_s^k(p_k/m_k)^s|a_1(p_1)\cdots a_n(p_n)\rangle\,, \end{aligned} \tag{1.61}$$

where γ_s^k are some constants. For instance, if $N = 1$ the only possible bootstrap fusion is

$$a(p_1) \times a(p_2) \rightarrow a(p_1 + p_2)\,. \tag{1.62}$$

The particle a appears as bound state of itself. The consistency condition reads

$$2\cos(s\pi/3) = 1\,. \tag{1.63}$$

Then the conserved spins are $s = 1, 5 \pmod 6$ and the minimal elastic S-matrix, satisfying the bootstrap fusion (1.62), has the form [148]

$$S_{aa}(\theta) = \tanh[\tfrac{1}{2}(\theta + \tfrac{2}{3}\mathrm{i}\pi)]/\tanh[\tfrac{1}{2}(\theta - \tfrac{2}{3}\mathrm{i}\pi)]\,. \tag{1.64}$$

Point (iii) is the most difficult one, so far. The existence of infinite-dimensional symmetries in these integrable models suggests that they are completely solvable, not only on mass-shell, but also in an

*) This form of the eigenvalues of P_s is a consequence of Lorentz invariance.

arbitrary domain of momentum space. However, a powerful scheme for dealing with such a problem (analogous to the conformal approach of Belavin, Polyakov and Zamolodchikov [12] for the QFT of the critical points) has not yet been accomplished. The computation of the correlation functions away from criticality remains a formidable task and exact results are known only for few cases [78–81]. However, a lot of progress has been achieved in the computation of the form factors of the integrable theories [69–71, 83–86, 147]. These are matrix elements of local quantum fields $\mathcal{O}_a$

$$F_n^a = {}_{\text{out}}\langle p_1, p_2, \ldots, p_m | \mathcal{O}_a(0) | p_{m+1}, \ldots, p_n \rangle_{\text{in}} . \tag{1.65}$$

The two-point functions of the operators $\mathcal{O}_a$ can be computed, in principle, as an infinite sum over the form factors exploiting the following spectral representation

$$\langle \mathcal{O}_a(p) \mathcal{O}_a(0) \rangle = G_a(p) = \int \frac{\rho_a(\kappa^2)\, \mathrm{d}\kappa^2}{p^2 - \kappa^2 + \mathrm{i}\varepsilon} , \tag{1.66}$$

with the spectral functions given by

$$\rho_a(\kappa^2) = \sum_n \frac{1}{n!} \int \prod_i \frac{\mathrm{d}^2 p_i}{(2\pi)^2}\, \delta(\kappa^0 - \Sigma_i\, p_i^0)\, \delta(\kappa^1 - \Sigma_i\, p_i^1) |F_n^a|^2 . \tag{1.67}$$

Although the computation of the form factors has been performed in several interesting cases, the difficult step to find a close expression for the spectral functions $\rho_a(\kappa)$ remains unsolved.

1.3.2. Quantum group symmetry

It has been recognized recently that the massive integrable quantum field theories are characterized by a quantum group symmetry [91–97]. A well-known example is provided by the relation between the $\Phi_{1,3}$ deformation of minimal models and the sine–Gordon theory at rational values of the coupling constant. The sine–Gordon theory possesses an $SU(2)_q$ invariance where the q-parameter is a function of the coupling constant β of the model. The soliton and the anti-soliton states of the sine–Gordon theory act like a doublet representation of this group with spin $j = 1/2$. The higher multisoliton states are obtained as tensor products of this fundamental representation. However, at special rational values of $\beta^2/8\pi$, q becomes a root of unity and the representation theory changes drastically, i.e. the model cannot sustain solitons exceeding a certain number and a reduction takes place in the system. This means that some degrees of freedom become frozen, whereas the others combine together and give rise to a new basis of the reduced Hilbert space. This mathematical analysis perfectly agrees with the observed physical realization of $\Phi_{1,3}$-perturbed CFT, which can be checked, for instance, by numerical methods [53, 134, 139]: after the perturbation, the original multiple degenerate vacuum of the fixed point splits into a finite number of different ground states which are connected only by a finite number of soliton states.

An analogous situation also occurs for the $\Phi_{1,2}$ and $\Phi_{2,1}$ deformations of the CFT. As shown by Smirnov [93], they are related to the quantum group reduction of the Zhiber–Mikhailov–Shabat model defined by the following Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + [\tfrac{1}{2} \exp(\mathrm{i}2\sqrt{2\pi}\phi) + \exp(\mathrm{i}\sqrt{2}\gamma\phi)] . \tag{1.68}$$

In this paper, we will summarize the results of this reduction procedure and discuss some significant examples of deformed CFT.

1.4. Affine Toda field theories

In chapters 7 and 8 of this review we will discuss a class of integrable Lagrangian theories known as Affine Toda Field Theories (ATFT). Previous investigations of these models have been carried out by several authors, see e.g. refs. [213–215, 222–224]. These theories have recently attracted much attention in relation to the deformations of CFT and the bootstrap approach.

The ATFT associated to a Lie algebra $\mathcal{G}$ of rank r is a theory of r bosonic fields ϕ^i with a Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{r} (\partial_\mu \phi^j)^2 - \frac{m_0^2}{\beta^2}\left[\sum_{i=1}^{r} n_i \exp\left(\beta \sum_{j=1}^{r} \alpha_i^j \phi^j\right) + \exp\left(-\beta \sum_{kj}^{r} n_k \alpha_k^j \phi^j\right)\right], \tag{1.69}$$

where m_0^2 is a mass scale, β the coupling constant and $\alpha_i (i = 1, 2, \dots, r)$ the simple roots of the algebra $\mathcal{G}$. The last term in (1.69) involves the maximal root ω

$$\omega = -\sum_{i=1}^{r} n_i \alpha_i . \tag{1.70}$$

The set of integers $\{n_i\}$ is specific for each algebra. The Coxeter number of the algebra is given by

$$h = \sum_{i=1}^{r} n_i + 1 . \tag{1.71}$$

The connection between ATFT and deformations of CFT has been suggested in refs. [189, 190] and has proved to be a rich subject. Significant examples are given by the ATFT constructed in terms of the Dynkin diagram of the exceptional algebras E_n. The minimal*) S-matrices of these ATFT are related to the first models of the minimal unitary series in CFT: E_8 is associated to the Ising model in a magnetic field [30, 159], E_7 to the tricritical Ising model (TIM) [157, 159] and E_6 to the tricritical Potts model (TPM) [156, 159] both in their high temperature phase.

The S-matrices corresponding to the conserved charges whose spins coincide with the Coxeter exponents of the affine (Kac–Moody) algebras have been related to the generalized Toda system in refs. [158, 160]. The same set of S-matrices has also been computed in ref. [171] and their relation with the root systems has been investigated in refs. [174, 177].

A perturbative analysis of the ATFT with real coupling constant has been pursued in refs. [157, 158, 171–173]. A distinction occurs between the ATFT constructed on the simply laced algebras and those constructed on the non-simply laced ones. For instance, the mass ratios of the simply laced ATFT are stable under quantum corrections. At one loop order, there is a universal formula for the mass shift Δm_i of the ith particle [158, 171]

$$\Delta m_i^2 / m_i^2 = (\beta^2/4h) \cot(\pi/h) . \tag{1.72}$$

*) The exact scattering amplitudes of these Lagrangian models can be written as $S(\theta, \beta) = S^{\min}(\theta) Z(\theta, \beta)$. The "minimal" S-matrix does not involve the coupling constant and contains poles only in the physical strip. $Z(\theta, \beta)$ is a coupling constant dependent solution which ensures $S(\theta)|_{\beta=0} = 1$. Both terms satisfy the requirements of unitarity, crossing symmetry and the bootstrap principle.

The rhs of (1.72) is nothing but the area of the regular planar polygon with h equal sides and perimeter β. On the contrary, the mass ratios of the non-simply laced ATFT are generally not robust under loop corrections. Moreover, the conjectured exact S-matrices for these models present some mysterious higher-order poles on the physical sheet that cannot be explained in terms of multiple scattering processes [158, 171]. For their quantum consistency, the introduction of some additional degrees of freedom is required [178].

1.5. Layout of the paper

In this introduction we have conveyed some basic ideas on the off-critical statistical systems. In the rest of this work we will discuss some of the major approaches to this large subject systematically. The paper is organized as follows. In chapter 2, the conformal approach to the conservation laws for off-critical systems is examined. In chapter 3 we discuss the general properties of the factorized S-matrix, including the restricted models originating from the sine–Gordon and from the Zhiber–Mikhailov–Shabat models. They provide the scattering theories for the massive deformations of CFT made by the operators $\Phi_{1,3}$, $\Phi_{1,2}$ and $\Phi_{2,1}$ respectively. In chapter 4 we review the thermodynamical Bethe ansatz, which has become an important tool in the analysis of the purely elastic scattering theories. Another efficient method of investigation is given by the numerical diagonalization of the off-critical Hamiltonian for systems defined on a domain with cylindrical geometry. We will discuss the basic ideas of this approach in chapter 5. Chapter 6 covers some significant examples of scattering theories for the minimal models away from criticality. Chapters 7 and 8 are devoted to affine Toda field theories with real coupling. Chapter 9 gives an introduction to the problem of computing the correlation functions away from the critical points. Our conclusions are in chapter 10. There are also several appendices where technical details are gathered in a concise way.

2. Conservation laws

An integrable quantum field theory is characterized by the existence of an infinite number of conservation laws. For integrable models originating from a perturbation of a CFT, these integrals of motion can be interpreted as deformations of the conformal conservation laws. A criterion for establishing their existence, at least for the lowest degrees of these conserved laws, is easily obtained in terms of the Operator Product Expansion (OPE) and the character formulas.

2.1. Deformations of the conformal conservation laws

Let us consider the conformal minimal models $\mathcal{M}_{p,p'}$ deformed by a relevant primary scalar field $\Phi_{lk}(z,\bar{z}) = \phi_{lk}(z)\bar{\phi}_{lk}(\bar{z})$, with anomalous dimension $x = 2\Delta < 2$. The off-critical action is given by

$$\mathcal{S} = \mathcal{S}_0 + \lambda \int \Phi_{lk}(z,\bar{z})\, \mathrm{d}^2 z\,. \tag{2.1}$$

Let $\mathcal{C}_s(z)$ be a conserved current of the model $\mathcal{M}_{p,p'}$ $(\partial_{\bar{z}}\mathcal{C}_s(z) = 0)$ with spin s (integer or fractional), local with respect to Φ_{lk}:

$$C_s(z)\Phi_{lk}(w,\bar{w}) = \sum_{n=2}^{m} \frac{d_{lk}^{(n)}}{(z-w)^n}\,\Phi_{lk}^{(n)}(w,\bar{w}) + \frac{1}{z-w}\,B_{lk}(w,\bar{w}) + \cdots \tag{2.2}$$

(n is an integer, $\Phi_{lk}^{(n)}$ and B_{lk} are descendents of Φ_{lk} and $d_{lk}^{(n)}$ some constants). The corresponding deformed Ward identities for $\mathscr{C}_s(z,\bar{z})$ can be written in terms of the conformal ones [16]:

$$\langle C_s(z,\bar{z})\cdots\rangle = \langle C_s(z)\cdots\rangle_0 + \lambda\int \mathrm{d}w\,\mathrm{d}\bar{w}\,\langle C_s(z)\Phi_{lk}(w,\bar{w})\cdots\rangle_0 + \mathrm{O}(\lambda^2)\,. \tag{2.3}$$

Equations (2.2) and (2.3), together with the identity

$$\partial_{\bar{z}}\,\frac{1}{z-w+\mathrm{i}\varepsilon} = \delta(z-w)\delta(\bar{z}-\bar{w})\,,$$

lead to the perturbed counterpart of the conformal conservation laws: to the first order in λ we have

$$\partial_{\bar{z}}C_s(z,\bar{z}) = \lambda[B_{lk}(z,\bar{z}) - d_{lk}^{(2)}\,\partial_z\Phi_{lk}^{(2)}]\,. \tag{2.4}$$

The existence of the off-critical conservation law depends on whether B_{lk} is a total derivative with respect to z. The simplest example is provided by energy–momentum conservation: if $\mathscr{C}_s = T$, where T is the stress-energy operator, we have

$$B_{lk} - d_{lk}^{(2)}\,\partial_z\Phi_{lk}^{(2)} = (1-\Delta)\,\partial_z\Phi_{lk}(z,\bar{z}) \tag{2.5}$$

and therefore

$$\partial_{\bar{z}}T(z,\bar{z}) = -\tfrac{1}{4}\,\partial_z\Theta\,,\quad \Theta = -4\lambda(1-\Delta)\Phi_{lk}(z,\bar{z})\,. \tag{2.6}$$

The corresponding conserved charge has the form

$$\mathscr{C}_1 \equiv P = \int (T\,\mathrm{d}z + \tfrac{1}{4}\Theta\,\mathrm{d}\bar{z})\,. \tag{2.7}$$

In order to understand the occurrence of higher integrals of motion, let us consider some significant examples:

(i) The minimal model $\mathscr{M}_{4,5}$ corresponds to the class of universality of the Tricritical Ising Model*) (TIM), known to be also the first superconformal model [13, 27] (see section 6.5). We choose as $\mathscr{C}_s$ the supercurrent $G_{3/2}$ with spin $s=\frac{3}{2}$ and as deformation field the vacancy density operator $\Phi_{13} = \Phi_{3/5,3/5}$. In the following we also use the notation $\Phi_{\Delta,\bar{\Delta}}$ to denote the conformal fields. The supersymmetric Ward identity

$$G(z_1)\Phi_{3/5,3/5}(z_2,\bar{z}_2) = \left(\frac{1}{5z_{12}^2} + \frac{1}{z_{12}}\,\partial_2\right)\Phi_{1/10,3/5}(z_2,\bar{z}_2) + \cdots \tag{2.8}$$

*) A microscopic realization of this model may be given by an Ising ferromagnet with vacancies, see section 6.5.

($z_{12} \equiv z_1 - z_2$) leads to the conservation law [179]

$$\partial_{\bar{z}} G(z, \bar{z}) = \partial_z \bar{\Psi}(z, \bar{z}) , \quad \bar{\Psi}(z, \bar{z}) = \tfrac{4}{5} \lambda \Phi_{1/10,3/5}(z, \bar{z}) . \tag{2.9}$$

The corresponding conserved charge

$$P_{1/2} \equiv Q = \int (G \, \mathrm{d}z + \bar{\Psi} \, \mathrm{d}\bar{z}) \tag{2.10}$$

has spin $s = \frac{1}{2}$. By using the conformal OPE's

$$G(z_1)G(z_2) = \frac{2}{z_{12}} T(z_2) + \cdots , \quad G(z_1)\Phi_{1/10,1/10}(z_2, \bar{z}_2) = \frac{1}{z_{12}} \Phi_{3/5,1/10}(z_2, \bar{z}_2) + \cdots , \tag{2.11}$$

it is easy to show that

$$\begin{aligned} Q^2 &= \int \mathrm{d}z_1 \, \mathrm{d}z_2 \, G(1)G(2) + \tfrac{4}{5}\lambda \int \mathrm{d}\bar{z}_1 \, \mathrm{d}_{\bar{2}} \{G(1), \Phi_{1/10,3/5}(2)\} \\ &= \int (2T \, \mathrm{d}z + \tfrac{4}{5}\lambda \Phi_{3/5,3/5} \, \mathrm{d}\bar{z}) = 2P . \end{aligned} \tag{2.12}$$

Similarly, $\bar{Q}^2 = 2\bar{P}$ and

$$Q\bar{Q} + \bar{Q}Q = \tfrac{4}{5}\lambda \int [(\partial_z \Phi_{1/10,1/10}) \, \mathrm{d}z + (\partial_{\bar{z}} \Phi_{1/10,1/10}) \, \mathrm{d}\bar{z}] . \tag{2.13}$$

The right-hand side of this equation is the topological charge of the TIM and therefore $\{Q, \bar{Q}, P, \bar{P}\}$ generate an off-critical global supersymmetry [179].

(ii) The class of universality of the 3-state Tricritical Potts Model (TPM) corresponds to a sub-algebra of the minimal model $\mathcal{M}_{6,7}$ [13]. We choose as $\mathscr{C}_s$ the chiral field W of spin $s = 5$ and as deformation the field $\Phi_{1,2}(z, \bar{z}) = \Phi_{1/7,1/7}$. The conformal OPE

$$\mathscr{W}(z_1)\Phi_{1/7,1/7}(z_2) = \left(\frac{w_0}{z_{12}^2} + \frac{1}{z_{12}} \partial_2 \right) \Phi_{22/7,1/7}(z_2, \bar{z}_2) + \cdots \tag{2.14}$$

(where w_0 is a constant) gives rise to the conserved charge of spin 4:

$$P_4 = \int (\mathscr{W} \, \mathrm{d}z + \Lambda \, \mathrm{d}\bar{z}) , \quad \Lambda = (w_0 - \tfrac{2}{7}) \Phi_{22/7,1/7} . \tag{2.15}$$

The field $\Phi_{1,2}$ is the scaling limit of the energy-density operator of the statistical model. Its insertion in the action thus shifts the temperature away from the critical value. This perturbation preserves the $S_3 = Z_2 \otimes Z_3$ permutation symmetry of the model, generated by C (charge conjugation operator) and by ϑ [156, 159, 179],

$$C^2 = \vartheta^3 = 1 .$$

P_4 is an odd operator under C, i.e. $CP_4 = -P_4 C$, while P_1 is an even operator, $CP_1 = P_1 C$.

(iii) Let us consider again the model $\mathcal{M}_{6,7}$, but this time we are interested in the deformation by means of the field $\Phi_{1,3}(z,\bar{z}) = \Phi_{5/7,5/7}$ and in the integral of motion originating from the operator $\mathcal{C}_s = \psi_{4/3}$ with spin $s=\frac{4}{3}$. The parafermionic OPE

$$\psi_{4/3}(z_1)\Phi_{5/7,5/7}(z_2) = \mathcal{A}\left(\frac{1}{z_{12}^2} + \frac{7}{z_{12}}\,\partial_z\right)\Phi_{1/21,5/7}(z_2) + \cdots \tag{2.16}$$

($\mathcal{A}$ is a constant) leads to the conservation law:

$$\partial_{\bar{z}}\psi_{4/3}(z,\bar{z}) = 6\mathcal{A}\lambda\,\partial_z\Phi_{1/21,5/7}(z,\bar{z}) . \tag{2.17}$$

The corresponding spin-$\frac{1}{3}$ charges $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ (associated to the current $\psi^\dagger_{4/3}$ of opposite Z_3 charge), together with $\bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}^\dagger$, realize an interesting off-critical algebra [179]

$$\begin{aligned}
&(\mathcal{Q})^3 = (\mathcal{Q}^\dagger)^3 = P\,, \quad (\bar{\mathcal{Q}})^3 = (\bar{\mathcal{Q}}^\dagger)^3 = \bar{P}\,,\\
&\mathcal{Q}\bar{\mathcal{Q}} - q\bar{\mathcal{Q}}\mathcal{Q} = t\,, \quad \mathcal{Q}^\dagger\bar{\mathcal{Q}}^\dagger - \bar{q}\bar{\mathcal{Q}}\bar{\mathcal{Q}}^\dagger = t^\dagger\,,\\
&\mathcal{Q}\bar{\mathcal{Q}}^\dagger + \bar{q}\bar{\mathcal{Q}}^\dagger\mathcal{Q} = \tilde{t}\,, \quad \mathcal{Q}^\dagger\bar{\mathcal{Q}} - q\bar{\mathcal{Q}}\mathcal{Q}^\dagger = \tilde{t}^\dagger\,,
\end{aligned} \tag{2.18}$$

where $q = \exp(2\pi i/3)$ and $t, t^\dagger$ etc. are the topological charges of the model. Their expressions, in terms of the order–disorder parameter fields σ and μ with dimensions $(\frac{1}{21}, \frac{1}{21})$, are given by

$$\begin{aligned}
&t = \kappa\int\!\!\int(\partial_z\sigma\,\mathrm{d}z + \partial_{\bar{z}}\sigma\,\mathrm{d}\bar{z})\,, \quad t^\dagger = \kappa\int(\partial_z\sigma^\dagger\,\mathrm{d}z + \partial_{\bar{z}}\sigma^\dagger\,\mathrm{d}\bar{z})\,,\\
&t = \kappa\int\!\!\int(\partial_z\mu\,\mathrm{d}z + \partial_{\bar{z}}\mu\,\mathrm{d}\bar{z})\,, \quad \tilde{t}^\dagger = \kappa\int(\partial_z\mu^\dagger\,\mathrm{d}z + \partial_{\bar{z}}\mu^\dagger\,\mathrm{d}\bar{z})\,,
\end{aligned} \tag{2.19}$$

where $\kappa = 140/9$.

(iv) As our last example, let us consider the QFT defined by a $\Phi_{1,3}$ deformation of the minimal unitary model $\mathcal{M}_{p,p+1}$. Let $\mathcal{C}_s =: T^2:$ be the quasi-primary field of spin 4 in the conformal family of the identity operator. Applying the stress-energy tensor's OPE twice, we have*)

$$B_{13} = \lambda(\Delta-1)[2L_{-1}L_{-2} - 2L_{-3} + \tfrac{1}{6}(\Delta-3)L^3_{-1}]\Phi_{13}(z,\bar{z}) . \tag{2.20}$$

The only term that may spoil the existence of conservation law is the second one. However, the field Φ_{13} satisfies the following null-vector condition at the level 3 [12]

$$L_{-3}\Phi_{13} = \left(\frac{2}{\Delta+2}\,L_{-1}L_{-2} - \frac{1}{(\Delta+1)(\Delta+2)}\,L^3_{-1}\right)\Phi_{13} . \tag{2.21}$$

Therefore, we can express the piece in L_{-3} in terms of a pure derivative expression and, consequently,

*) We use: $L_{-1}[\cdots] = \partial_z[\cdots]$.

there exists a conserved charge with spin 3 for any value of p [30]:

$$P_3 = \int (T_4 \, dz + \Theta_2 \, d\bar{z}) , \tag{2.22}$$

where

$$\Theta_2 = \lambda \frac{\Delta - 1}{\Delta + 2} \left(2\Delta L_{-2} + \frac{(\Delta - 2)(\Delta - 1)(\Delta + 3)}{\cancel{6(\Delta + 1)}} L_{-1}^2 \right) \Phi_{13} . \tag{2.23}$$

Using the corresponding OPEs, it is easy to prove that P_1 and P_3 commute: $[P_1, P_3] = 0$.

Eguchi and Yang [189] have related the deformations of the minimal models of CFT with the operator Φ_{13} to the sine–Gordon model: they have explicitly constructed the currents corresponding to conserved charges of spins $s = 3, 5, 7, 9$ and they have further conjectured the existence of an infinite number of them, with odd values of s.

All these examples demonstrate the usefulness of the conformal OPEs for constructing the lowest spin conserved currents. A powerful method, known as the *counting argument*, has been introduced by Zamolodchikov [30] in order to provide a sufficient criterion for the existence of such non-trivial conservation laws.

2.2. Counting argument

We illustrate the counting argument in the case of conserved currents which originate from the conformal family of the identity operator. A similar result is obtained by considering conserved currents which are local with respect to the perturbing field Φ.

Let $\hat{T}_{s+1}$ be the space of the quasi-primary descendents of the identity operator at levels $s + 1$, i.e. the factor space:

$$\hat{T}_{s+1} = T_{s+1} / \partial_z T_s .$$

Analogously, let $\hat{\Phi}_s$ be the factor space at level s of the conformal field Φ which perturbs the action of the fixed point,

$$\hat{\Phi}_s = \Phi_s / \partial_z \Phi_{s-1} .$$

The mapping

$$\partial_{\bar{z}} : \quad \hat{T}_{s+1} \to \lambda \hat{\Phi}_s$$

has a nonvanishing kernel when

$$\dim \hat{T}_{s+1} > \dim \hat{\Phi}_s . \tag{2.24}$$

If this condition is realized, there exist some fields $T_{s+1}(z, \bar{z}) \in \hat{T}_{s+1}$ and $\Phi_{s-1}(z, \bar{z}) \in \hat{\Phi}_{s-1}$ such that

$$\partial_z T_{s+1}(z, \bar{z}) = \lambda \, \partial_{\bar{z}} \Phi_{s-1}(z, \bar{z}) , \tag{2.25}$$

i.e. we have a conserved charge with spin s. The condition (2.24) is easily checked by computing the dimensions of the spaces by means of the character formulae:

$$\sum_{s=0}^{\infty} q^s \dim \hat{T}_n = (1-q)\tilde{\chi}_{1,1}(q) + q\,, \quad \sum_{s=0}^{\infty} q^{s+\Delta_{kl}} \dim(\hat{\Phi}_{k,l})_s = (1-q)\tilde{\chi}_{k,l}(q)\,,$$

where

$$\tilde{\chi}_{r,s}(q) = q^{(c-1)/24-\Delta_{r,s}} \chi_{r,s}(q)\,,$$

and $\chi_{r,s}(q)$ is the character of the field $\Phi_{r,s}$, given in eq. (1.29).

The counting argument usually provides a useful source of information on the structure of the conserved currents only for low values of s*). Using the counting argument, Zamolodchikov [30] has proved the existence of higher integrals of motion for the off-critical systems defined by the deformation of the minimal models of CFT by the operators $\Phi_{1,3}$, $\Phi_{1,2}$ and $\Phi_{2,1}$. Therefore these operators always define an integrable deformation of the minimal models of CFT.

2.2.1. Examples

Let us apply the counting argument to example (ii) of the previous section, i.e. to the thermal perturbation of the tricritical Potts model. There are two classes of conserved operators. The first class originates from the descendents of the stress-energy tensor T whereas the second set is given by the descendents of the operator $\mathcal{W}$. They are further characterized by their parity properties under the charge conjugation operator C. The result is [156]

$$\dim \hat{T}_{s+1} > \dim(\hat{\Phi}_{1/7,1/7})_s \quad \text{for} \quad s = 1, 5, 7, 11 \quad (C\text{-even})\,,$$

$$\dim \hat{W}_{s+1} > \dim(\hat{\Phi}_{22/7,1/7})_s \quad \text{for} \quad s = 4, 8 \quad (C\text{-odd})\,.$$

The natural conjecture is that the spins of the infinite set of conserved charges are given by the following sequence

$$s = 1, 4, 5, 7, 8, 11 \pmod{12}\,. \tag{2.26}$$

These spins coincide with the Coxeter exponents, modulo the Coxeter number, of E_6. The appearance of this Lie algebra structure is due to an additional symmetry of the conformal tricritical Potts model, which can be equivalently obtained as the first model of the WE_6-extended conformal algebra.

The analogous calculation for the tricritical Ising model ($\mathcal{M}_{4,5}$) perturbed by the energy operator $\Phi_{1,2} = \Phi_{1/10,1/10}$ gives [157]

$$\dim \hat{T}_{s+1} > \dim(\hat{\Phi}_{1/10,1/10})_s \quad \text{for } s = 1, 5, 7, 9, 11, 13\,, \tag{2.27}$$

which coincide with some Coxeter exponents of E_7. The natural conjecture is that the spins of the infinite set of conserved charges are given in this case by

$$s = 1, 5, 7, 9, 11, 13, 17 \pmod{18}\,, \tag{2.28}$$

*) This because the dimension of the conformal block of any operator $\Phi_{r,s}$ at level s grows asymptotically much faster than the corresponding dimension of the identity operator.

where 18 is the Coxeter number of E_7. The E_7 structure of the TIM is related to the equivalent realization of this model in terms of the coset $[(E_7)_1 \otimes (E_7)_1]/(E_7)_2$ (see appendix B).

For the Ising model ($\mathcal{M}_{3,4}$) perturbed by the magnetization operator $\Phi_{1,2} = \Phi_{1/16,1/16}$, we have [30]

$$\dim \hat{T}_{s+1} > \dim(\hat{\Phi}_{1/16,1/16})_s \quad \text{for} \quad s = 1, 7, 11, 13, 17, 19\,,$$

i.e. the first numbers of a conjectured infinite set equal to the Coxeter exponents of E_8, modulo the Coxeter number $h = 30$,

$$s = 1, 7, 11, 13, 17, 19, 23, 29 \quad (\mathrm{mod}\, 30)\,. \tag{2.29}$$

This is not a coincidence because the Ising model can also be realized in terms of the coset construction $[(E_8)_1 \otimes (E_8)_1]/(E_8)_2$ (see appendix C).

We will discuss in detail the off-critical theories of these models in chapter 6. $[(E_8)_1 \otimes (E_8)_1]/(E_8)_2$ (see appendix C).

2.3. Multi-coupling deformation of CFT

We have considered up to now some minimal models perturbed by only one relevant operator and we have discussed under which conditions the off-critical models are integrable. One might wonder whether the analysis made so far could be generalized to the case of multiple deformations of CFT. For instance, in the Ising model there are two individual integrable deformations corresponding to a thermal and a magnetic perturbation of the critical point action, respectively. One might ask whether there exist other integrable lines*) in the plane of the phase diagram defined by the axes of temperature and magnetic field. Similarly, for next minimal unitary model given by the tricritical Ising model, both $\Phi_{1,3}$ and $\Phi_{1,2}$ deformations are individually integrable: is it possible to find other integrable directions in the phase diagram of this model defined by the axes of the scaling fields $\Phi_{1,2}$ and $\Phi_{1,3}$? The same kind of question can be also addressed to other models as well.

A definite resolution about this problem has not been reached so far. However, we will show that

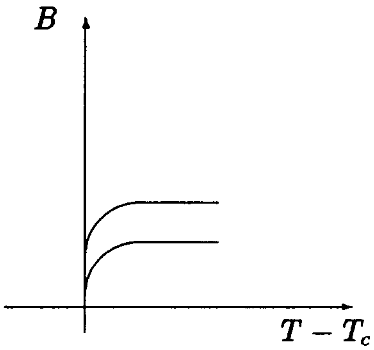


Fig. 1. Renormalization group trajectories of the Ising model.

*) If there exists an integrable point in the plane of the phase diagram, it necessarily belongs to one of the renormalization group trajectories (fig. 1), hence the model would be integrable along the full line.

conserved currents with low values of the spin s do not exist when multiple coupling deformations of minimal models of CFT are considered. The reasons are essentially the different null-vector conditions satisfied by the perturbing fields. Though our computations are restricted to the first non-trivial values of s, where such currents might exist, they reveal a geometrical structure that may be exploited in a more complete proof of the above statement for higher values of s.

To start with, we recall the derivation of the non critical conformal conservation law in presence of only one perturbation. We consider for simplicity the case of unitary theories and $\mathscr{C}_s \in \hat{T}_s$. Generally, up to derivative term, we have (see also eq. (2.4))

$$\partial_{\bar{z}} \mathscr{C}_s(z, \bar{z}) = \lambda B_{lk}^{(1)}(z, \bar{z}) + \cdots \lambda^n B_{lk}^{(n)}(z, \bar{z}) + \cdots \tag{2.30}$$

A dimensional analysis fixes the scaling dimensions of the operator $B_{lk}^{(n)}(z, \bar{z})$ to be

$$[s - n(1 - \Delta), 1 - n(1 - \Delta)] .$$

Since $\Delta < 1$, there exists an integer n_c such that for all $n > n_c$ the right dimensions of $B_{lk}^{(n)}(z, \bar{z})$ are negative. The absence of states with negative dimensions in the unitary minimal models forces the series (2.30) to stop and, actually, in most cases only the first term survives [29, 30]. If we now consider the case of two relevant perturbations (with scaling dimensions Δ_1, Δ_2 and coupling constants λ_1 and λ_2), the generalization of (2.30) becomes

$$\partial_{\bar{z}} \mathscr{C}_s(z, \bar{z}) = \sum_{n,m=1} \lambda_1^n \lambda_2^m B_{lk}^{(n,m)}(z, \bar{z}) . \tag{2.31}$$

The anomalous dimensions of the terms $B_{lk}^{(n,m)}$ are

$$[s - n(1 - \Delta_1) - m(1 - \Delta_2), 1 - n(1 - D_1) - m(1 - \Delta_2)] . \tag{2.32}$$

Again, we conclude that the series has to terminate. Moreover, at least for the Ising and the TIM models, the series splits into two independent series in λ_1 and λ_2*). The reason is very simple. In fact, the right dimension should coincide with one of the dimensions present in the conformal grid. For the Ising model in an external magnetic field and at a non-critical value of the temperature we have

$$1 - n \times \tfrac{1}{2} - m \times \tfrac{15}{16} = \Delta_r , \tag{2.33}$$

for some Δ_r of the model. As possible values of Δ_r we only have $\Delta_r = \{0, \frac{1}{2}, \frac{1}{16}\}$. Hence, it is impossible to have both n and m non-zero at the same time.

The same happens for the TIM, perturbed by the energy operator $\Phi_{1/10,1/10}$ and by the vacancy density operator $\Phi_{3/5,3/5}$.

In these two models, we thus have in (2.31) the direct sum of the contributions due to both terms. If a conserved current exists, it must occur at the common level of the conserved currents for each perturbation. For what concerns the fields $\Phi_{1/2,1/2}$ for Ising and $\Phi_{3/5,3/5}$ for TIM, they are both $\Phi_{1,3}$ operators. Therefore, from example (iv) of section 2.1, the conserved currents associated to these

*) We suspect that a similar conclusion holds for all minimal models $\mathscr{M}_{p,p'}$, although we do not have a general proof for that.

operators occur at the values of s

$$s = 1, 3, 5, 7, \ldots$$

i.e. at each odd level. On the contrary, for the Ising model the perturbation given by the other operator $\Phi_{1/16,1/16}$ gives rise to a set of conserved currents with spins equal to the Coxeter exponents of E_8

$$s = 1, 7, 11, 13, 19, 23, 29 \quad (\mathrm{mod}\, 30)\,.$$

For the tricritical Ising model, the perturbation given by the second operator $\Phi_{1/10,1/10}$ possesses a set of conserved currents with spins equal to the Coxeter components of E_7

$$s = 1, 5, 7, 9, 11, 13, 17 \quad (\mathrm{mod}\, 18)\,.$$

Therefore, in both models the common set of conserved spins coincides with the respective set of Coxeter exponents.

In the following, we will explicitly show that currents which are conserved under both deformations do not exist in these models, at least for the lowest values of s. This result seems to indicate the absence of integrability of the multi-coupling deformations of these models.

2.3.1. Tricritical Ising model

Let us start with the TIM because this model has a possible common conserved current at the value of the spin equal to $s = 5$, whereas for the Ising model we should consider at least the currents with spin $s = 7$ (see below). The explicit expression for the conserved current $\mathscr{C}_6^{(1)}$ of a Φ_{13}, perturbation of a minimal model $\mathscr{M}_{p,p+1}$ of CFT has been computed in refs. [30, 189],

$$C_6^{(1)} = (T(T^2)) + \tfrac{1}{12}(c+2)(T\,\partial^2 T)\,, \tag{2.34}$$

where c is the central charge of the model and (AB) denotes the normal ordered product of the operators A and B,

$$(AB)(z) \equiv \oint \frac{\mathrm{d}w}{w-z}\, A(w)B(z)\,.$$

Applying $\partial_{\bar z}$ to (2.34) and using eq. (2.3), we get

$$\partial_{\bar z} C_6 = \lambda_1(1+\Delta_{13})[(p+1)L_{-5} - pL_{-2}L_{-3}]\Phi_{13} + L_{-1}[\cdots]\,. \tag{2.35}$$

The first term on the right-hand side is indeed zero for the field Φ_{13} as a consequence of the level-3 null-vector condition (2.21). Therefore, $C_6^{(1)}$ is conserved under a $\Phi_{1,3}$ deformation. Plugging $c = 7/10$ into (2.34), we get the expression of the conserved current of the TIM deformed by the field $\Phi_{1,3} = \Phi_{3/5,3/5}$. We have now to see if $C_6^{(1)}$ is still conserved once we perturb the TIM with the second operator $\Phi_{1,2} = \Phi_{1/10,1/10}$. Repeating the computation, we obtain

$$\partial_{\bar z} C_6 = \lambda_2(1+\Delta_{12})\Big(\frac{3(11p^2+14p+8)}{4p(p+1)}\, L_{-5} - 6L_{-2}L_{-3}\Big)\Phi_{1,2} + L_{-1}[\cdots]\,. \tag{2.36}$$

We see, however, that the null-vector condition for $\Phi_{1,2}$,

$$\left(L_{-2} - \frac{3}{2(2\Delta_{12}+1)} L_{-1}^2\right)\Phi_{1,2} = 0\,, \tag{2.37}$$

does not imply now the vanishing of the first term on the right-hand side of (2.36)! And indeed, the current which is conserved under $\Phi_{1,2}$ is not that one given in (2.34) but is given by

$$C_6^{(2)} = (T(T^2)) + \tfrac{1}{16}(181 - 17c + 15\sqrt{25 - 26c + c^2})(T\,\partial^2 T)\,. \tag{2.38}$$

Therefore, a conserved current of spin $s = 5$ for this multi-coupling deformation of the TIM does not exist.

The TIM has also conserved currents of spin 7. One of them comes from the deformation under $\Phi_{1/10,1/10}$ (corresponding to the E_7 structure), the other one from the operator $\Phi_{1,3} = \Phi_{3/5,3/5}$. The explicit expression of the conserved current under a $\Phi_{1,3}$ deformation is given by [189]

$$C_8(T(T(T^2))) + \tfrac{1}{6}(c+8)(T(T\,\partial_z^2 T)) + \tfrac{1}{180}(c^2 + 4c - 101)(T\,\partial_z^4 T)\,, \tag{2.39}$$

where $c = \frac{7}{10}$ for the TIM. The question is whether or not this current is still conserved when we consider the deformation of the critical theory under the energy density operator $\Phi_{1/10,1/10}$. The explicit computation, which can be found in appendix D, shows that is not the case.

2.3.2. *Ising model*

For the Ising model*) in an external magnetic field and at a non-critical value of the temperature, the first common conserved charge may appear for the spin $s = 7$. The expression of the conserved current under the thermal perturbation is given by eq. (2.39) with $c = \frac{1}{2}$. In this specific example one can also take advantage of the null-vector condition of the identity operator at level 6 to simplify the computation. For technical reasons, the calculations are given in appendix C. The final result is that a conserved current under both perturbations does not exist.

These results point out the difficulties in obtaining conserved currents in the presence of two perturbations. Moreover, in section 8.8 we will see that the insertion of the energy operator $\Phi_{1/2,1/2}$ in the Ising model with an external magnetic field destroys the E_8 structure of the model. In fact, the particles with masses above threshold are no longer stable and they decay. The same occurs for the E_7 structure of the TIM at a non-critical temperature, if we additionally insert the vacancy density operator $\Phi_{3/5,3/5}$.

3. Factorized S-matrix

In this chapter we review the properties of two-dimensional elastic scattering theories. They provide useful on mass-shell information of the massive integrable deformations of CFT.

*) We are grateful to J.L. Cardy and S.K. Yang for discussion on this model. In particular, S.K. Yang has also done the computation, in agreement with our result [191].

3.1. General properties of purely elastic S-matrices

Consider a $(1+1)$-dimensional QFT with an infinite number of conserved charges $\mathcal{P}_s$ which transform according to higher representations of the Lorentz group. We assume that these charges are labelled by their "spin" s and are given by an integral of some current densities

$$\mathcal{P}_s = \int (T_{s+1}\, \mathrm{d}z + \Theta_{s-1}\, \mathrm{d}\bar{z}) . \tag{3.1}$$

T_{s+1} and Θ_{s-1} are local fields satisfying

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1} . \tag{3.2}$$

The S-matrix of such a QFT is restricted by several strong constraints. In fact, the existence of an infinite number of integrals of motion implies (i) absence of particle production; (ii) equality of the sets of initial and final momenta, $\{p_1, p_2, \dots p_n\} = \{p'_1, p'_2, \dots, p'_n\}$, such that the scattering processes which take place in these systems are purely elastic. It is also possible to show, that as a consequence of the existence of an infinite number of local currents, an arbitrary n-particle collision process becomes factorized into the product of $n(n-1)/2$ elastic pair collisions [108–110]. Hence, the scattering consists of a succession of individual elastic two-body collisions where possible exchanges of momenta and quantum numbers may occur only between particles of the same mass. The physical observables in such purely elastic scattering theories are the time delays (compared to the free case) of the outcoming particles.

The factorization property provides a drastic simplification of the scattering problem because in this case it reduces to an evaluation of the two-particle S-matrices. These two-body amplitudes satisfy the usual requirements of unitary and crossing symmetry. Furthermore, the equivalent ways of decomposing an n-particle amplitude into two-particle S-matrices give rise to cubic constraints (Yang–Baxter equations) for the two-particle scattering matrices and to the bootstrap equations.

3.2. Rapidity variable, unitary equations and crossing symmetry

We initially consider the scattering processes of particle-like excitations. The scattering amplitudes of kink-like excitations, i.e. of field configurations which interpolate between adjacent vacua, will be discussed at the end of this chapter and in some examples of deformed CFT (see chapter 6).

Let p_1 and p_2 be the initial momenta of the incoming particles A_i and A_j and p_3, p_4 the corresponding quantities of the outgoing states A_k and A_l (see fig. 2). Up to an overall energy–momentum δ function, Lorentz invariance fixes the two-body S-matrix to be a function of the momentum only through their Lorentz scalars. They are given by the Mandelstam variables s, t and u,

$$s = (p_1 + p_2)^2 , \quad t = (p_1 - p_3)^2 , \quad u = (p_1 - p_4)^2 , \quad s + t + u = \sum_{i=1}^{4} m_i^2 . \tag{3.3}$$

In $(1+1)$ dimensions and for elastic scattering, only one of them is independent. It is thus convenient to introduce a parameterization in terms of the rapidity variable θ. The momentum of the particle A_i is

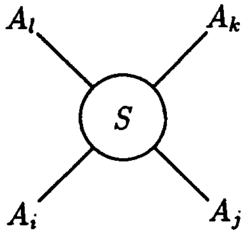


Fig. 2. Two-body S-matrix.

given by

$$p_i^0 = m_i \cosh \theta_i\ , \quad p_i^1 = m_i \sinh \theta\ . \tag{3.4}$$

For the invariant squared energy s of the collision process

$$A_i A_j \to A_k A_l\ , \tag{3.5}$$

we have

$$s(\theta) = (p_1 + p_2)^2 = m_i^2 + m_j^2 + 2m_i m_j \cosh \theta_{ij}\ , \quad \theta_{ij} = \theta_i - \theta_j\ . \tag{3.6}$$

The inverse transformation

$$\theta_{ij} = \ln\left(\frac{s - m_1^2 - m_2^2 + \sqrt{[(s-(m_1+m_2)^2)(s-(m_1-m_2)^2]}}{2m_1 m_2}\right) \tag{3.7}$$

maps the physical sheet of the s plane into the strip $0 \leq \operatorname{Im} \theta \leq \pi$. The second sheet is mapped into the strip $-\pi \leq \operatorname{Im} \theta \leq 0$ and this structure repeats with period $2\pi\mathrm{i}$. In terms of the θ variable, the two-particle S-matrix elements are defined by*)

$$|A_i(\theta_1) A_j(\theta_2)\rangle_{\text{in}} = S_{ij}^{kl}(\theta_{12}) |A_k(\theta_2) A_l(\theta_1)\rangle_{\text{out}}\ . \tag{3.8}$$

The functions $S_{ij}^{kl}(\theta)$ satisfy the unitarity equations

$$\sum_{n,m} S_{ij}^{nm}(\theta) S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l\ . \tag{3.9}$$

If the model is invariant under parity and time-reversal transformations, we have additional restrictions. Parity invariance implies

$$S_{ij}^{kl}(\theta) = S_{ji}^{kl}(\theta)\ , \tag{3.10}$$

*) For future purposes, it is also useful to introduce an expression of the S-matrix written in terms of the original Mandelstam variable s. We denote it by $\mathscr{S}$. Its relation with the S-matrix defined in (3.8) is established through the Jacobian of the transformation $s(\theta)$, i.e. $\mathscr{S}_{ij}^{kl}(s) = 4m_i m_j \sinh \theta_{ij}\, S_{ij}^{kl}(\theta_{ij})$.

while time-reversal symmetry leads to

$$S_{ij}^{kl}(\theta) = S_{kl}^{ij}(\theta) . \tag{3.11}$$

Furthermore, we can also exploit the crossing invariance of the scattering processes which provides a relation among the amplitudes of all possible channels in which the four particles A_a are involved. Namely, the amplitude for the cross-channel process

$$A_i A_k \rightarrow A_l A_j \tag{3.12}$$

is obtained from S_{ij}^{kl} by an analytic continuation from the "s-channel to the t-channel"*):

$$S_{ik}^{lj}(\theta) = S_{ij}^{kl}(i\pi - \theta) . \tag{3.13}$$

If the mass spectrum is not degenerate, the elasticity of the scattering processes implies the vanishing of the reflection amplitudes and, in this case, the S-matrix becomes completely diagonal and reduces to a set of phases. Vice versa, if the system presents multiplets of degenerate particles, the elasticity of the collisions induces only a redistribution of the momenta among the particles with the same masses.

3.3. Non-commutative algebra of the asymptotic states and Yang–Baxter equations

The general structure of the scattering processes can be described in terms of an associative non-commutative algebra [108]. Let $A_i(\theta)$ be a set of non-commutative operators which represent the corresponding particles. They are regarded to be the generators of the infinite-dimensional algebra given by all possible products of the form

$$A_{a_1}(\theta_1) A_{a_2}(\theta_2) \cdots A_{a_n}(\theta_n) . \tag{3.14}$$

The incoming and outgoing asymptotic states are in correspondence to a decreasing and increasing arrangement of the rapidities θ_i in the sequence (3.14), respectively. In this algebraic approach, the S-matrix plays the role of braiding operator for the non-commutative fields $A_i(\theta)$,

$$A_i(\theta_1) A_j(\theta_2) = S_{ij}^{kl}(\theta_{12}) A_l(\theta_2) A_k(\theta_1) . \tag{3.15}$$

These commutation relations should be compatible with the requirement of algebraic associativity. This translates into the following cubic relation (*star–triangle* or Yang–Baxter equations) for the two-body amplitudes S_{ij}^{kl} [104–108]

$$S_{i_1 i_2}^{k_1 k_2}(\theta_{12}) S_{k_1 k_3}^{j_1 j_3}(\theta_{13}) S_{k_2 i_3}^{j_2 k_3}(\theta_{23}) = S_{i_1 i_3}^{k_1 k_3}(\theta_{13}) S_{k_1 k_2}^{j_1 j_1}(\theta_{12}) S_{i_2 k_3}^{k_2 j_3}(\theta_{23}) \tag{3.16}$$

(where a sum on the intermediate indices is understood). This equation corresponds to the com-

*) The general expression of the crossing symmetry is given by $\hat{S}_{12}(i\pi - \theta) = C\hat{S}_{21}^{t_1}(\theta)C$, where the suffix t_1 means transposition with respect to the first space and C is the charge conjugation operator. The S-matrix can be considered as an operator $\hat{S}_{12}^{ij}$ that acts on the tensor product $V_1^i \otimes V_2^j$ of the isotopic subspaces V_1^i and V_2^j of the particles A_i and A_j.

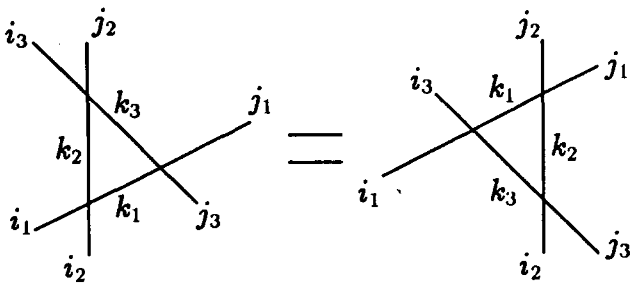


Fig. 3. Yang–Baxter equations.

mutativity of the processes shown in fig. 3 and can be justified by a particle-displacement argument [111]. The reasoning goes as follows: we can use the conserved charge $\mathcal{P}_s$ to define the series of operators

$$\mathcal{T}_s(a) = \exp(\mathrm{i}a\mathcal{P}_s)\,. \tag{3.17}$$

The first of them, $\mathcal{T}_1$, is constructed in terms of the momentum operator and, applied to any state of the system, only produces a uniform translation in space-time. But, applying any other operator $T_s(a)$ to the wave packets that describe the particles (localized both in coordinate and momentum space), we can move them by an amount which depends on their momentum. Hence, by means of a fine-tuning combination of the operators $\mathcal{T}_s(a)$, we can arbitrarily shift the points of interaction in any scattering process. Since the conserved charges commute with the Hamiltonian of the system, the different situations depicted in fig. 3 should correspond to the same amplitude – a requirement that leads to eqs. (3.16).

The conditions of unitarity, crossing symmetry and factorization give a system of equations for the two-particle S-matrix that, together with the knowledge of the symmetry of the system, is in many cases sufficient to determine a consistent solution*). The particle content of the theory is encoded into the analytic structure of the S-matrix.

3.4. *Analytic structure of the S-matrix and bootstrap principle*

The elastic S-matrices are analytic functions in the complex plane of the Mandelstam variable s with square branch cut singularities at $(m_a - m_b)^2$ and $(m_a + m_b)^2$ (fig. 4). Hence, the amplitudes $S_{ij}^{kl}(\theta)$ are meromorphic functions of the rapidity θ. The stable bound states are usually associated to the simple poles with positive residues which lie on the imaginary axis of the physical strip [211]. However, recent developments in the analysis of the deformations of CFT have shown how this assumption may be generalized both to the case of poles with negative residues**) [92, 148–150, 161, 163, 164] and to the case of odd higher order poles [157, 158, 161, 163, 171, 172]. Postponing the discussion of these topics to later sections, let us consider here for simplicity the case of an S-matrix with initial particle states A_i and A_j and with a simple pole in the s-channel at $\theta = \mathrm{i}u_{ij}^n$. In the vicinity of this singularity, we have

$$S_{ij}^{kl}(\theta) \sim \mathrm{i}R_{ij}^n/(\theta - \mathrm{i}u_{ij}^n)\,. \tag{3.18}$$

*) We remind that the general solution presents in any case an ambiguity related to the so-called CDD poles. The discussion of this point may be found in refs. [108, 216, 217].

**) This occurs in the S-matrices of the massive theories coming from the non-unitary CFT.

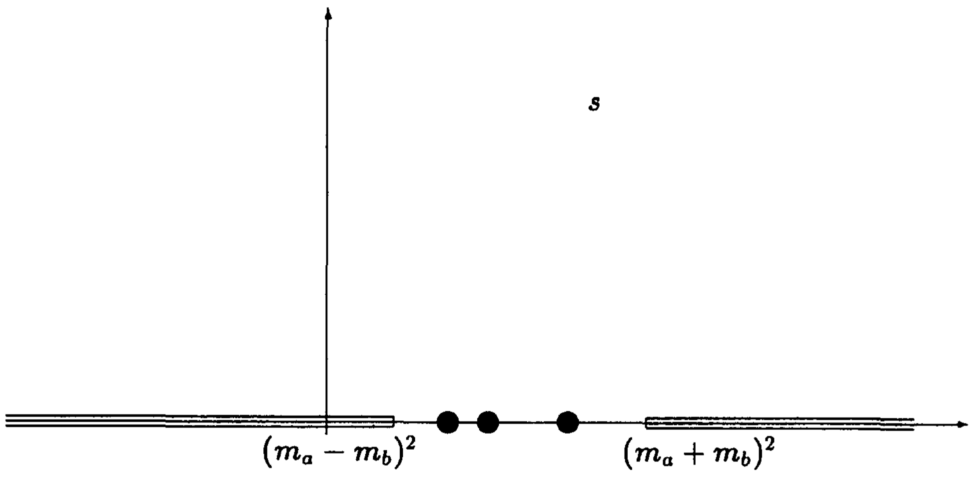


Fig. 4. Analytic structure of the S-matrix in the s plane. The dots are the bound states.

The residue R_{ij}^n in (3.18) is related to the on-mass shell coupling constants of the underlying quantum field theory. As shown in fig. 5, we have

$$R_{ij}^n = f_{ijn} f_{kln} . \tag{3.19}$$

The existence of a non-zero coupling constant f_{ijn} implies a pole singularity in the amplitudes S_{in} and S_{jn} as well, due to the intermediate bound states A_j and A_i, respectively [30]. In the bootstrap approach, the bound states are themselves identified with some of the particles appearing as asymptotic states. This leads to the following relation among the masses of the system: if $\theta = \mathrm{i} u_{ij}^n$ is a pole in the scattering process of the particle A_i and A_j, the mass of the "bound-state" A_n is given by

$$m_n^2 = m_i^2 + m_j^2 + 2 m_i m_j \cos u_{ij}^n . \tag{3.20}$$

From a geometrical point of view, we have a triangle*) with sides of lengths m_i, m_j and m_n (see fig. 6). The location of these three poles is thus restricted by the following identity (fig. 7),

$$u_{ij}^n + u_{in}^j + u_{jn}^i = 2\pi . \tag{3.21}$$

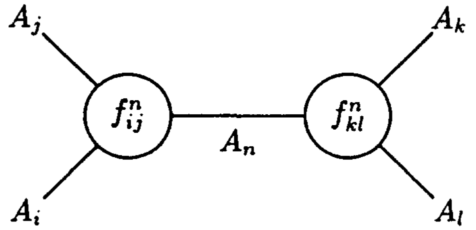


Fig. 5. Coupling constants versus residue.

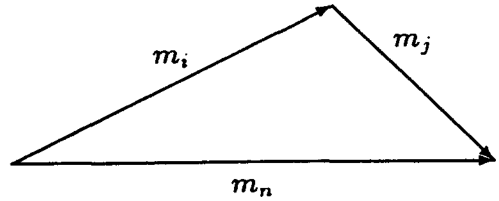


Fig. 6. Mass triangle.

*) This *mass triangle* will play an important role in the perturbative check of the exact S-matrices of affine Toda field theories.

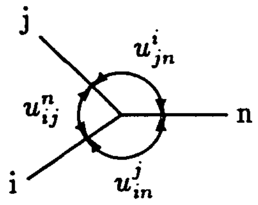


Fig. 7. Three-particle coupling.

The general discussion of the S-matrix carried out in the previous section, drastically simplifies in two cases. The first case is when the system under consideration has a non degenerate mass spectrum. The second one is when the system presents a spectrum of degenerate particles which can be unambiguously distinguished by their higher charge eigenvalues. Under these conditions, the S-matrix is diagonal and the *star–triangle* equations are trivially satisfied. Equations (3.9) and (3.13) become

$$S_{ab}(\theta)S_{ab}(-\theta)=1\,,\quad S_{ab}(\mathrm{i}\pi-\theta)=S_{\bar{a}b}(\theta) \tag{3.22}$$

($\bar{a}$ denotes the antiparticle). These equations imply that the $S_{ab}(\theta)$ are periodic functions of θ with period $2\pi\mathrm{i}$. Since the bootstrap principle gives the possibility to consider the asymptotic states on the same footing as the bound states, the S_{ij}s satisfy the following functional equations [30, 121]

$$S_{i\bar{l}}(\theta)=S_{ij}(\theta+\mathrm{i}\bar{u}^k_{jl})S_{ik}(\theta-\mathrm{i}\bar{u}^j_{lk})\,, \tag{3.23}$$

where

$$\bar{u}^c_{ab}\equiv\pi-u^c_{ab}\,. \tag{3.24}$$

A graphical representation of the bootstrap equations is illustrated in the fig. 8. The mathematical structure of these equations will be investigated in section 3.7. As a consequence of the bootstrap equations, scattering matrices of particles with heavier mass generally present higher-order poles.

The most general solution of (3.22) is expressed in terms of an arbitrary product of the following functions [217]

$$s_x(\theta)=\sinh[\tfrac{1}{2}(\theta+\mathrm{i}\pi x)]/\sinh[\tfrac{1}{2}(\theta-\mathrm{i}\pi x)]\,. \tag{3.25}$$

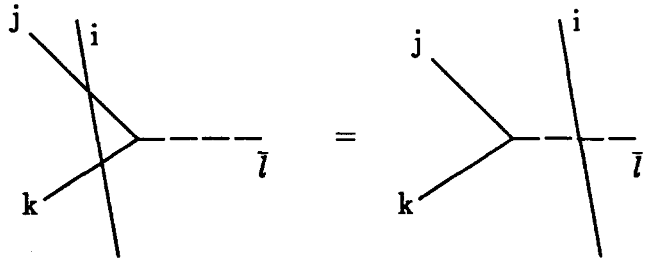


Fig. 8. Bootstrap equations.

It is not restrictive to choose x in the range $-1 \leq x \leq 1$. In the double covering of the s-plane, i.e. $-\pi \leq \operatorname{Im} \theta \leq \pi$, These functions have a simple pole at $\theta = \mathrm{i}x\pi$ and a simple zero at $\theta = -\mathrm{i}x\pi$. They satisfy the following properties:

$$
\begin{aligned}
& s_x(\theta)s_x(-\theta) = s_x(\theta)s_{-x}(\theta) = 1\,, \quad s_x(\theta) = s_{x+2}(\theta) = s_{-x}(-\theta)\,, \quad s_0(\theta) = -s_1(\theta) = 1\,, \\
& s_x(\mathrm{i}\pi - \theta) = -s_{1-x}(\theta)\,.
\end{aligned} \tag{3.26}
$$

In case all particles are self-conjugate, the functional space of the solutions of eq. (3.22) is spanned by products of functions $f_x(\theta)$ defined by

$$
f_x(\theta) = s_x(\theta)s_x(\mathrm{i}\pi - \theta) = \tanh[\tfrac{1}{2}(\theta + \mathrm{i}\pi x)]/\tanh[\tfrac{1}{2}(\theta - \mathrm{i}\pi x)]\,. \tag{3.27}
$$

The simple poles of these functions are located at the crossing symmetric points $\theta = \mathrm{i}\pi x$ and $\theta = \mathrm{i}(1-x)\pi$. There are zeros as well, located at $-\mathrm{i}x\pi$ and at $-\mathrm{i}(1-x)\pi$. Important properties of f_x are

$$
f_x(\theta) = f_x(\mathrm{i}\pi - \theta) = f_{1-x}(\theta)\,, \quad f_x(-\theta) = f_{-x}(\theta) = 1/f_x(\theta)\,. \tag{3.28}
$$

A simplification of the iterative procedure of the bootstrap equations (3.23) can be obtained by defining an operator $\mathcal{R}^y$ acting on a function $G(\theta)$ in the following way [157]:

$$
\mathcal{R}^y(G(\theta)) = G(\theta + \mathrm{i}\pi y)G(\theta - \mathrm{i}\pi y)\,. \tag{3.29}
$$

Applying $\mathcal{R}^y$ to the functions s_x and f_x we have

$$
\mathcal{R}^y(s_x(\theta)) = s_{x+y}(\theta)s_{x-y}(\theta)\,, \quad \mathcal{R}^y(f_x(\theta)) = f_{x+y}(\theta)f_{x-y}(\theta)\,. \tag{3.30}
$$

This operation is commutative and distributive

$$
\mathcal{R}^y(\mathcal{R}^z(G)) = \mathcal{R}^z(\mathcal{R}^y(G))\,, \quad \mathcal{R}^y(G_1)\mathcal{R}^y(G_2) = \mathcal{R}^y(G_1G_2)\,. \tag{3.31}
$$

Moreover, if a function $G(\theta)$ satisfies

$$
G(\theta) = G(\mathrm{i}\pi - \theta) = 1/G(-\theta)\,, \tag{3.32}
$$

it is easy to see that the operator $\mathcal{R}$ preserves these properties as well.

3.5. Bootstrap consistency equations

The integrals of motion are specific sets of data for each theory. If a Lagrangian of the system under consideration was known, their expressions in terms of the elementary fields could in principle be derived (see, e.g. refs. [122–125]). On the contrary, the knowledge of the S-matrix only gives a restriction on the possible values of the spin s of the conserved charges P_s. In this case, the computable quantities are just the ratios of the eigenvalues of $\mathcal{P}_s$. Zamolodchikov [30] has derived a system of consistency equations for the eigenvalues of the conserved charges on the basis of the bootstrap principle and locality alone.

Let P_s be the set of conserved charges. The asymptotic states $|A_a(\theta)\rangle$ are chosen to be eigenstates of P_s

$$P_s|A_a(\theta)\rangle = \omega_s^a(\theta)|A_a(\theta)\rangle\,. \tag{3.33}$$

The existence of a conserved current with spin s implies that at least one eigenvalue ω_s^a is different from zero. Lorentz invariance fixes the functional form of $\omega_s^a(\theta)$ to be

$$\omega_s^a(\theta) = \chi_s^a\, e^{s\theta}\,, \tag{3.34}$$

where χ_s^a are constants. χ_1^a are just the masses of the particles

$$\chi_1^a = m_a\,. \tag{3.35}$$

The locality requirement of the theory implies

$$P_s|A_{a_1}(\theta_1)\cdots A_{a_k}(\theta_k)\rangle = (\omega_s^{a_1}(\theta_1) + \cdots \omega_s^{a_k}(\theta_k))|A_{a_1}(\theta_1)\cdots A_{a_k}(\theta_k)\rangle\,. \tag{3.36}$$

For certain imaginary values of the rapidities, the two-particle states *fuse* together and give rise to the bound states. Hence, if $u_{ab}^{\bar{c}}$ is such a value for the asymptotic state $|A_a(\theta_1)A_b(\theta_2)\rangle$, we can identify the bound state $|A_{\bar{c}}\rangle$ as

$$\lim_{\varepsilon\to 0} \varepsilon|A_a(\theta + i\bar{u}_{ac}^b - \tfrac{1}{2}\varepsilon)A_b(\theta - i\bar{u}_{bc}^a + \tfrac{1}{2}\varepsilon)\rangle = |A_{\bar{c}}(\theta)\rangle\,. \tag{3.37}$$

Applying P_s to both sides of this equation and using (3.36) and (3.34) we get the following infinite system of linear equations for the χ_s^as,

$$\chi_s^a \exp(is\bar{u}_{ac}^b) + \chi_s^b \exp(-is\bar{u}_{bc}^a) = \chi_s^{\bar{c}}\,. \tag{3.38}$$

This set of homogeneous equations is always satisfied by $\chi_s^a = 0\ (\forall a, s)$. However this solution is not interesting because it implies the absence of any operator P_s and, in this case, we cannot any longer use the factorization property of the S-matrix to solve the scattering problem. Non-trivial solutions of (3.38) are obtained for special sets of resonance angles u_{ab}^c of the S-matrix. Since they are homogeneous equations, they provide significant limitations on the values of the spin s. Let us consider, for example, the case $a = b$ and $\chi_s^a \neq 0$. Equation (3.38) reduces to

$$2\cos(s\bar{u}_{ac}^a) = \chi_s^{\bar{c}}/\chi_s^a\,. \tag{3.39}$$

In the case $a = b = c$, the above equation has an unique solution

$$\bar{u}_{aa}^a = \tfrac{1}{3}\pi\,, \quad s = 1, 5 \pmod 6\,. \tag{3.40}$$

The corresponding S-matrix presents the so-called "Φ^3" property, i.e. the particle A_i appears as bound state of itself.

To proceed further in our analysis, let us also introduce the notion of *bootstrap fusions* [156, 157]. Let $\{A_i\}$ be a complete set of one-particle states in a theory with bootstrap interaction. The bootstrap structure of the bound states is encoded into the following non associative algebra [157, 156]

$$A_i \times A_j = \sum_k \mathcal{N}_{ij}^k A_k \,. \tag{3.41}$$

$\mathcal{N}_{ij}^k$ is a Boolean variable, different from zero only when A_k appears as a bound state in the scattering process of the particles A_i and A_j. It is worth noticing the analogy with the fusion rule algebra of the CFT [22], the key difference being the associativity condition.

3.5.1. Non-degenerate bootstrap systems

Let us assume the existence of non-trivial solutions for (3.38). We choose their normalization in such a way that the nonzero eigenvalues of the lightest particle A_1 is equal to 1. For self-conjugate particle systems, it is easy to show – by induction – that all remaining eigenvalues are real. Equations (3.38) thus split into two equations:

$$(\chi_s^c)^2 = (\chi_s^a)^2 + (\chi_s^b)^2 + 2\chi_s^a \chi_s^b \cos(s u_{ab}^c) \,, \tag{3.42}$$

$$\chi_s^a / \chi_s^b = \sin(s \bar{u}_{bc}^a) / \sin(s \bar{u}_{ac}^b) \,. \tag{3.43}$$

The first equation appears as a generalization of the *mass triangle* equation (3.20) whereas eq. (3.43) expresses a simple property of such a triangle. The latter equation is particularly useful from a computational point of view because, in order to have a non zero value for χ_s^a and χ_s^b, the above ratio of sines should be *independent* of any bound state $|A_c\rangle$ appearing in the channel $|A_a A_b\rangle$. Knowing the resonance angle of one bound state in this channel, we can use this equation either (a) to correctly identify the location of the other ones or (b) to prove that it is impossible to have higher-order conserved charges compatible with the bootstrap.

Some examples may clarify the above observation [160]. Let us consider two different bootstrap systems defined by the following S-matrix for the lowest particle, S_{11}:

$$S_{11} = -f_{1/9} f_{5/9} \,, \tag{3.44}$$

for the first one and

$$S_{11} = -f_{1/5} f_{1/7} \,, \tag{3.45}$$

for the second one. We will use the criterion of the independence of the ratio (3.43) from the index c in order to select which of the two models may give rise to a consistent bootstrap system.

(i) In the case of the bootstrap model defined by the S-matrix (3.44), we identify the poles at $\theta = \mathrm{i}\pi/9$ and $\mathrm{i}5\pi/9$ with two new bound states m_2 and m_3. Applying (3.23) we can compute

$$S_{12} = f_{1/6} f_{17/18} f_{11/18} f_{1/2} \,. \tag{3.46}$$

In this amplitude the pole at $\theta = \mathrm{i}17\pi/18$ corresponds to the particle A_1. This angle is u_{12}^1 and therefore fixes the ratios (3.43). From table 1 we see that we can identify the poles at $\theta = \mathrm{i}\pi/6$ and $\theta = \mathrm{i}11\pi/18$ as due to additional bound states. The conserved spins are $s = 1, 5, 7, 9, 11, 13, 17 \pmod{18}$.

(ii) For the model defined by the S-matrix (3.45), let us suppose we identify the pole at $\theta = \mathrm{i}\pi/7$ as a singularity due to a particle A_2, i.e. $u_{11}^2 = \pi/7$. We can compute the amplitude S_{12} by applying the bootstrap equation

$$S_{12} = f_{3/14} f_{1/14} f_{19/70} f_{9/70} \,. \tag{3.47}$$

In this amplitude, the pole at $\theta = \mathrm{i}13\pi/14$ is due to the particle A_1. Hence, this angle fixes the ratios χ_1^s/χ_2^s. However, from table 2 we see that there is no other pole in this amplitude which gives the same value of these ratios for $s = 1, 3, \ldots, 35$. Therefore the bootstrap system defined by (3.45) is not supported by the existence of higher additional charges and, consequently, is not a consistent bootstrap model. The same conclusion is reached starting with any other possible u_{11}^k. In order to classify the off-critical conservation laws, we will assume the knowledge of the total number N of massive particles

Table 1
Ratios γ_s^2/γ_s^1, $s = 1, \ldots, 17$, calculated for the poles occurring in the amplitude S_{12} calculated starting with $S_{11} = -f_{1/9} f_{5/9}$. The first column contains the ratios for the identified pole u_{12}^1.

s	γ^2/γ^1	$\frac{1}{6}\pi$	$\frac{5}{6}\pi$	$\frac{11}{18}\pi$	$\frac{7}{18}\pi$
1	1.970	1.970	1.970	1.970	1.970
3	1.732	1.732	0.303	−0.866	0.647
5	1.286	1.286	−1.177	1.286	−0.920
7	0.684	0.684	2.159	0.684	−0.989
9	0	0	−1.219	0	−2.888
11	−0.684	−0.684	0.850	−0.684	−0.258
13	−1.286	−1.286	0.333	−1.286	1.040
15	−1.732	−1.732	10.190	0.866	0.068
17	−1.970	−1.970	−0.507	−1.970	3.603

Table 2
Ratios γ_s^2/γ_s^1, $s = 1, \ldots, 35$, calculated for the poles occurring in the amplitude S_{12} calculated starting with $S_{11} = -f_{1/5} f_{1/7}$. The first column contains the ratios for the identified pole u_{12}^1.

s	γ^2/γ^1	$\frac{3}{14}\pi$	$\frac{11}{14}\pi$	$\frac{19}{70}\pi$	$\frac{51}{70}\pi$	$\frac{9}{70}\pi$	$\frac{61}{70}\pi$
1	1.950	1.950	1.950	1.950	1.950	1.950	1.950
3	1.564	1.564	−0.344	1.325	−0.841	1.812	0.836
5	0.868	0.868	−0.705	0.284	1.204	1.547	−0.66
7	0	0	−3.343	−0.821	0.450	1.171	−1.41
9	−0.868	−0.868	0.906	−1.582	0.142	0.710	−4.28
11	−1.564	−1.564	−0.296	0.087	−1.086	0.195	−1.08
13	−1.950	−1.950	6.053	−1.857	−1.438	−0.340	0.534
15	−1.950	−1.950	−1.122	−0.855	0.961	−0.862	1.184
17	−1.564	−1.564	0.905	0.289	−2.275	−1.340	−5.405
19	−0.868	−0.868	−1.064	1.225	−1.569	−1.765	1.498
21	0	0	0.840	1.441	0.659	−2.194	−0.423
23	0.868	0.868	−1.039	3.016	−0.792	−11.529	−1.069
25	1.564	1.564	2.395	1.458	0.078	−1.548	1.136
27	1.950	1.950	−0.002	0.281	−0.416	−1.596	−2.353
29	1.950	1.950	0.731	−0.798	1.841	−1.381	0.315
31	1.564	1.564	−2.052	−1.388	1.527	−1.023	1.013
33	0.868	0.868	−1.410	0.088	−1.009	−0.568	−0.429
35	0	0	−0.118	−2.271	2.518	−0.052	5.248

which close the bootstrap and we will exploit the consistency of the fusions (3.41) [149, 156, 158]. We concentrate on some significant examples of this classification program of the bootstrap models.

$N = 1$ bootstrap systems. The bootstrap fusion is given by

$$A \times A \rightarrow A\,. \tag{3.48}$$

In this case, the model satisfies the "Φ^3" property and the possible values of conserved spins are

$$s = 1, 5 \quad (\mathrm{mod}\, 6)\,. \tag{3.49}$$

A physical realization of such a system is given by the off-critical Yang–Lee Model, i.e. the simplest minimal (non-unitary) CFT perturbed by an imaginary magnetic field [148].

$N = 2$ bootstrap systems. For the system with $N = 2$ particles in bootstrap interaction, neglecting the reducible fusions $A_a \times A_a \rightarrow A_a$, $A_b \times A_b \rightarrow A_b$, we consider the following examples*):

$$\text{(i)} \quad A_a \times A_a \rightarrow A_b\,, \quad A_b \times A_b \rightarrow A_b\,,$$

$$\text{(ii)} \quad A_a \times A_a \rightarrow A_a + A_b\,, \quad A_b \times A_b \rightarrow A_a\,.$$

The consistency conditions for the processes (i) are

$$2\chi_s^a \cos(s\bar{u}_{ab}^a) = \chi_s^b\,, \quad 2\chi_s^b \cos(s\bar{u}_{ab}^b) = \chi_s^a\,. \tag{3.50}$$

For $\chi_s^{a,b} \neq 0$, they reduce to

$$\cos(s\bar{u}_{ab}^a)\cos(s\bar{u}_{ab}^b) = \tfrac{1}{4}\,. \tag{3.51}$$

This equation admits two solutions [156]:

$$\bar{u}_{ab}^a = \tfrac{1}{12}\pi\,, \quad \bar{u}_{ab}^b = \tfrac{5}{12}\pi\,, \quad s = 1, 4, 5, 7, 8, 11 \quad (\mathrm{mod}\, 12)\,, \tag{3.52}$$

$$\bar{u}_{ab}^a = \tfrac{1}{5}\pi\,, \quad \bar{u}_{ab}^b = \tfrac{2}{5}\pi\,, \quad s = 1, 3, 7, 9 \quad (\mathrm{mod}\, 10)\,. \tag{3.53}$$

For self-conjugate particles the even spins do not exist. Then the allowed set of conserved spins in (3.52) are only

$$s = 1, 5, 7, 11 \quad (\mathrm{mod}\, 12)\,. \tag{3.54}$$

For the process (ii), we have to take for the spin s the common solution of eq. (3.40) and of eqs. (3.53) and (3.54). For instance, in the case of the solution (3.53), we have

$$s = 1, 7, 11, 13, 17, 19, 23, 29 \quad (\mathrm{mod}\, 30)\,. \tag{3.55}$$

*) The complete analysis of the $N = 2$ bootstrap system was carried out in refs. [149, 158].

N = 3 bootstrap systems. We consider the simplest $N=3$ system, defined by the bootstrap fusion

$$A_a \times A_a \to A_b\,, \quad A_b \times A_b \to A_c\,, \quad A_c \times A_c \to A_a\,. \tag{3.56}$$

The consistency condition is

$$\cos(sx_1)\cos(sx_2)\cos(sx_3) = \tfrac{1}{8}\,. \tag{3.57}$$

The four solutions of this equation are [156]

$$\begin{aligned} &x_1 = \tfrac{1}{9}\pi\,, \quad x_2 = \tfrac{2}{9}\pi\,, \quad x_3 = \tfrac{4}{9}\pi\,, \quad s = 1,3,5,7,11,13,15,17 \pmod{18}\,,\\ &x_1 = \tfrac{1}{7}\pi\,, \quad x_2 = \tfrac{2}{7}\pi\,, \quad x_3 = \tfrac{3}{7}\pi\,, \quad s = 1,2,3,4,5,6 \pmod{7}\,,\\ &x_1 = \tfrac{1}{20}\pi\,, \quad x_2 = \tfrac{4}{20}\pi\,, \quad x_3 = \tfrac{9}{20}\pi\,, \quad s = 1,3,7,9,11,13,17 \pmod{20}\,,\\ &x_1 = \tfrac{3}{20}\pi\,, \quad x_2 = \tfrac{8}{20}\,, \quad x_3 = \tfrac{7}{20}\pi\,, \quad s = 1,3,7,9,11,13,17 \pmod{20}\,. \end{aligned} \tag{3.58}$$

If we also add to (3.56) the process $A_a \times A_a \to A_a$, the admissible spins change as follows: those of the first equation in (3.58) become equal to the $E_6^{(1)}$ exponents and those of the last equation in (3.58) become equal to the $E_8^{(1)}$ exponents (see table 15 in chapter 7).

One can further complicate the 3-particles processes and consider, for instance, the bootstrap fusions

$$A_a \times A_a \to A_b + A_c\,, \quad A_b \times A_b \to A_a\,, \quad A_c \times A_c \to A_a\,; \tag{3.59}$$

$$A_a \times A_a \to A_b + A_c\,, \quad A_b \times A_b \to A_c\,, \quad A_c \times A_c \to A_a\,. \tag{3.60}$$

The consistency equations for (3.59) are:

$$\cos(sx_1)\cos(sx_3) = \tfrac{1}{4}\,, \quad \cos(sx_2)\cos(sx_4) = \tfrac{1}{4}\,. \tag{3.61}$$

They are satisfied by (3.52) and (3.53) with the $E_8^{(1)}$ exponents as the admissible spins. In the case (3.60) we have to add to (3.57) the condition

$$\cos(sx_3)\cos(sx_4) = \tfrac{1}{4}\,. \tag{3.62}$$

The solution is given by the last equation in (3.58) with $x_4 = \frac{1}{5}\pi$ (or $\frac{2}{5}\pi$).

Chain bootstrap fusions. For a generic system with N self-conjugate particles, it is easy to analyze the so called *pure chain bootstrap processes* [149, 156, 158]:

$$A_K \times A_k \to A_{k+1} \quad i = 1,2,\ldots,N\,, \quad A_{N+1} \equiv A_1\,. \tag{3.63}$$

The consistency condition is

$$\prod_{k=1}^{N} 2\cos(s\bar{u}^{i}_{k,k+1}) = 1\,. \tag{3.64}$$

The solution is given, up to a permutation, by the resonance angles

$$\bar{u}^{k}_{k,k+1} = k\pi/(2N+1)\,, \tag{3.65}$$

and by the $A^{(2)}_{2N}$ Coxeter exponents

$$s = 1, 3, \ldots, 2N-1, 2N+3, \ldots, 4N+1 \quad (\mathrm{mod}\, 4N+2)\,. \tag{3.66}$$

A more detailed analysis of the bootstrap fusions is necessary when some of the χ^{i}_{s}s vanish for particular values of the indices i and s. This happens, for instance, for the current $\mathcal{P}_9$ in the tricritical Ising model with a thermal perturbation and in the $E^{(1)}_7$ affine Toda field theory.

3.5.2. Example of degenerate multiplets in bootstrap interaction

The previous analysis can be extended to the bootstrap fusions of non self-conjugate particles. This situation occurs, for instance, in those QFT which are invariant under a group of symmetry as U(1), Z_p, S_3 etc. We choose the symmetric group S_3 for our discussion. Physical realizations of bootstrap systems with S_3 invariance are given by the off-critical 3-state Potts model [146] and its tricritical version [156, 159]. The group S_3 is a direct product of two Abelian subgroup Z_3 and Z_2. The former is generated by ϑ, with $\vartheta^3 = 1$ whereas the latter is generated by C, with $C^2 = 1$. We can interpret C as the *charge conjugation* operator. The irreducible representations of S_3 are given by two one-dimensional representations and one bidimensional representation. Let us consider the bidimensional representation $(A_a, A_{\bar{a}})$ of this group. The generators C and ϑ of S_3 act on this doublet as follows:

$$\vartheta A_a = q A_a\,, \quad \vartheta A_{\bar{a}} = \bar{q} A_{\bar{a}}\,; \quad C A_a = A_{\bar{a}}\,. \tag{3.67}$$

Herein, $q = \exp(\frac{2}{3}\pi \mathrm{i})$. Let us consider the following bootstrap fusions of two doublets $(A_a, A_{\bar{a}})$ and $(A_b, A_{\bar{b}})$:

$$A_a \times A_{\bar{a}} \to A_{\bar{a}} + A_{\bar{b}}\,, \quad A_b \times A_b \to A_{\bar{a}}\,. \tag{3.68}$$

It is possible to prove that in such a model conserved currents with even spins are C-odd

$$C P_{2s} = -P_{2s} C\,, \quad \text{i.e.} \quad P_{2s} A_{\bar{a}} = -\chi_{2s} (p/m)^{2s} A_{\bar{a}}\,, \tag{3.69}$$

whereas conserved currents P_{2s+1} with odd spins are C-even [156, 159]. The consistency conditions can be obtained using eq. (3.38), taking into account the C-parity of the currents

$$2\chi^{a}_{s} \cos(s x_a) = \pm \chi^{b}_{s}\,, \quad 2\chi^{b}_{s} \cos(s x_b) = \pm \chi^{a}_{s}\,, \quad 2\cos(\tfrac{1}{3} s\pi) = \pm 1\,, \tag{3.70}$$

where the plus sign is for the s odd and the minus sign for the s even. The solution of this system is again (3.52). Taking into account this sign modification, one can easily repeat the previous analysis for the case of S_3 charged particles.

3.5.3. General remarks

The analysis of more complicated sets of fusions, for example

$$A_a \times A_a \rightarrow A_b\,,\quad A_b \times A_b \rightarrow (\text{no bound state})\,, \tag{3.71}$$

or

$$\begin{aligned} &A_a \times A_a \rightarrow A_b\,,\quad A_a \times A_b \rightarrow A_a + A_c\,; \\ &A_b \times A_b \rightarrow (\text{no bound state})\,,\quad A_c \times A_c \rightarrow A_b\,, \end{aligned} \tag{3.72}$$

requires an explicit construction of the solutions of the bootstrap equations (3.23). We leave this problem to the next sections. Here, our purpose was to show how the bootstrap fusions can be used for the analysis of possible conservation laws. As a byproduct of each example, we have found a set of resonance angles u_{ij}^k which are the necessary extra data we need to solve the bootstrap equations.

The discussion made so far naturally suggests that there is a deep relationship between the bootstrap equations and the consistency equations for the conserved charges. A key observation in this respect was put forward by Braden et al. [171]. Their argument was as follows. Suppose we consider the logarithmic derivative of the S-matrix elements

$$\varphi_{ab}(\theta) = -\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\theta}\ln S_{ab}(\theta)\,. \tag{3.73}$$

In terms of φ_{ab}, the bootstrap equations (3.23) can be written as

$$\varphi_{i\bar{l}}(\theta) = \varphi_{ij}(\theta + \mathrm{i}\bar{u}_{jl}^k) + \varphi_{ik}(\theta - \bar{u}_{lk}^j)\,. \tag{3.74}$$

We can expand $\varphi_{ab}(\theta)$ as a Fourier series in θ because the non-degenerate S-matrices are $2\pi\mathrm{i}$ periodic functions

$$\varphi_{ab}(\theta) = \sum_{-\infty}^{\infty} \varphi_{ab}^{(s)}\,\mathrm{e}^{s\theta}\,. \tag{3.75}$$

Plugging this expansion into (3.74), we get the following constraints on the coefficients $\varphi_{ab}^{(s)}$

$$\varphi_{i\bar{l}}^{(s)} = \varphi_{ij}^{(s)}\exp(\mathrm{i}s\bar{u}_{jl}^k) + \varphi_{ik}^{(s)}\exp(-\mathrm{i}s\bar{u}_{lk}^j)\,, \tag{3.76}$$

which are very similar to eq. (3.38). Additional restrictions on the coefficients in (3.75) come from the unitary conditions (3.9), namely

$$\varphi_{ab}(\theta) = \varphi_{ab}(-\theta) \rightarrow \varphi_{ab}^{(s)} = \varphi_{ab}^{(-s)}\,, \tag{3.77}$$

as well as from the crossing relation (3.13),

$$\varphi_{ab}(\mathrm{i}\pi - \theta) = -\varphi_{\bar{a}b}(\theta) \rightarrow \varphi_{ab}^{(s)} = (-1)^{s+1}\varphi_{\bar{a}b}^{(-s)}\,. \tag{3.78}$$

Let us work out the coefficients $\varphi_{ab}^{(s)}$ for an S-matrix of the form

$$S_{ab}(\theta) = \prod_{i=1}^{n} s_{x_i}(\theta)\,, \tag{3.79}$$

where

$$s_{x_i}(\theta) = \sinh[\tfrac{1}{2}(\theta + \mathrm{i}\pi x_i)]/\sinh[\tfrac{1}{2}(\theta - \mathrm{i}\pi x_i)] .$$

With the position $\varphi_{x_i} = -\mathrm{i}\, \mathrm{d}s_{x_i}/\mathrm{d}\theta$, we have

$$\varphi_{x_i}(\theta) = -(\sin \pi x_i)/(\cosh \theta - \cos \pi x_i) . \tag{3.80}$$

The coefficients of the corresponding Fourier series,

$$\varphi_{ab}(\theta) = -\sum_{s=1}^{\infty} \varphi_{ab}^{s} \exp(-s|\theta|) , \tag{3.81}$$

are given by

$$\varphi_{ab}^{(s)} = 2 \sum_{x_i} \sin(s\pi x_i) . \tag{3.82}$$

As a consequence of the bootstrap consistency equations (3.42) and (3.43), the mass ratios satisfy

$$m_a/m_b = \varphi_{ad}^{(1)}/\varphi_{bd}^{(1)} , \tag{3.83}$$

for any d. Therefore, any $\varphi_{ab}^{(1)}$ can be expressed in terms of the corresponding quantity of the fundamental particle as

$$\varphi_{ab}^{(1)} = \varphi_{11}^{(1)}(m_a/m_1)(m_b/m_1) . \tag{3.84}$$

For an S-matrix of the form

$$S_{ab}(\theta) = \prod_{i=1}^{n} f_{x_i}(\theta) , \tag{3.85}$$

where

$$f_{x_i}(\theta) = \tanh[\tfrac{1}{2}(\theta + \mathrm{i}\pi x_i)]/\tanh[\tfrac{1}{2}(\theta - \mathrm{i}\pi x_i)] ,$$

we have a crossing symmetric combination of the previous case. Taking the logarithmic derivative, we have

$$\varphi_{ab}(\theta) = \sum_{i=1}^{n} \left(\frac{1}{\cancel{\sinh(\theta + }\mathrm{i}\pi x_i)} - \frac{1}{\cancel{\sinh(\theta} - \mathrm{i}\pi x_i)} \right) \equiv \sum_{i=1}^{n} \mathcal{T}_{x_i}(\theta) , \tag{3.86}$$

where

$$\mathcal{T}_{x_i} = -\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\theta} f_{x_i}(\theta) .$$

The Fourier series of each individual term $\mathcal{T}_{x_i}$ is given by

$$\mathcal{T}_{x_i}(\theta) = -2 \sum_{n=0}^{\infty} \mathrm{e}^{(2n+1)\theta}\left(\frac{A_i^{4n+2}-1}{A_i^{2n+1}}\right), \quad A_i = \mathrm{e}^{\mathrm{i}\pi x_i}. \tag{3.87}$$

The previously mentioned connection with the conserved spins is clarified by some examples. Let us consider the S-matrix of the Yang–Lee model [148] (see chapter 6)

$$S(\theta) = F_{2/3}(\theta).$$

In this case, we have

$$A^6 = 1,$$

and correspondingly all terms $\varphi^{(s)}$ with $s = 3n$ are absent from the expansion (3.75). This means that currents corresponding to these spins do not appear in the set of conserved quantities. Repeating the same analysis for the S-matrix for the lightest particle of the Ising model in a magnetic field [30] (see chapter 6)

$$S(\theta) = f_{2/3}(\theta) f_{2/5}(\theta) f_{1/15}(\theta),$$

the only non-zero coefficients $\varphi^{(s)}$ in (3.75) are those with

$$s = 1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}.$$

As a last remark, it is worth noticing that in the bootstrap approach all particles are on the same footing and no distinction occurs between asymptotic and bound states*). The corresponding functional equations for the scattering matrices S_{ij} respect this equivalence but the solutions do not. In fact, in order to solve the bootstrap equations (3.23), we have to choose one particle as a fundamental one to start with. Usually this is the particle with the lightest mass. The S-matrices of this particle will have a minimal pole structure, whereas the S-matrices of the other ones will present a higher pole structure describing the multi-scattering processes [148, 158, 171, 172, 193]. These higher singularities are produced by an iterative application of eqs. (3.23). It is suggestive that, considering particles with heavier mass, the analytic structure of their S-matrix elements gets richer and more complicated patterns, supporting the interpretation of the mass of a particle as a dynamical parameter related to the complexity of its interactions.

3.6. Multiple scattering processes and higher-order poles

One of the interesting features of the exact S-matrices is their multiple pole structure – an unavoidable consequence of the iterative application of the bootstrap equations. A consistent interpretation of the scattering theory demands its explanation in terms of elementary collision processes which take place in the system.

*) In the past, such an approach was pursued for describing the strong interaction of the hadronic particles, see refs. [126, 127].

The simple poles correspond to the bound states which appear in the intermediate channels (see fig. 9). This identification is true in any dimension of the space-time in which the scattering processes occur and is one of the main points of the analytic theory of the S-matrix [211]. Concerning the higher-order poles, their appearance is instead a peculiar feature of $(1+1)$-dimensional systems. In more familiar four-dimensional theories, the singularities which may appear in perturbation theory are just branch cuts. It is only the dimensionality of phase space that makes these anomalous threshold singularities (often called Landau singularities) double or higher-order poles in two dimensions and branch cuts in four dimensions. This observation is the basis of the Källen–Toll theorem [195].

Historically, the problem of the second-order poles which populate the physical sheets of the $(1+1)$ elastic S-matrices was discussed in the context of sine–Gordon model by Coleman and Thun [193] and, independently, by Goebel [194]. Their results were recently generalized to the higher-order poles of the S-matrices of the affine Toda field theories by Christe and Mussardo [157, 158] and by Braden et al. [171, 172]. In particular, a systematic analysis of this problem was pursued in ref. [172] and we refer the reader to it for details. In this section, following ref. [172], we will briefly review the procedure for isolating the leading-order from Feynman diagrams.

Let us assume the existence of a set of Feynman rules which define the dynamics of a theory. The scattering processes are given by a sum of Feynman diagrams and the problem is to single out those Feynman diagrams which become singular at a particular value of θ, say $\theta \to \mathrm{i}\theta_0$. The leading-order singularities arise when all the propagators of the intermediate particles are simultaneously on shell [196, 211]. It is possible to prove that to isolate the most singular term in a scattering amplitude it is enough to consider only the vertices which involve three particles [172]. Therefore, for the sake of simplicity, we consider a bosonic theory with $g\phi^3$ interaction*⁾. The singularities in the S-matrix have the form

$$S_{ab}(\theta) \sim g^{2p} R_p / (\theta - \theta_0)^p , \tag{3.88}$$

and they originate from the Feynman diagrams with P propagators and L loops, with the condition $p = P - 2L$. The proof requires some preliminary discussion about the choice of variables.

3.6.1. Choice of variables

Let us consider a Feynman diagram $\mathscr{G}$ with four external legs, P internal propagators and L loops. Since each vertex is of three-particle type, there will be $2p$ vertices. Hence, such a graph is of order g^{2p}. Let p_a and p_b be the momenta of the external incoming particles, with $p_a^2 = m_a^2$, $p_b^2 = m_b^2$ and $s = (p_a + p_b)^2$. In addition, let $p_a^{(0)}$ and $p_b^{(0)}$ be the values of the momenta such that $s = s_0$, i.e. the pole position. When the external momenta and all P internal momenta are on-shell simultaneously, we can

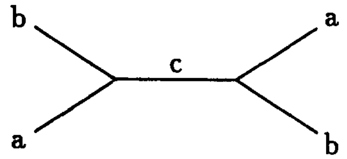


Fig. 9. Pole singularity.

*⁾ ϕ may stay for a multiplet of fields.

write the internal on mass shell momenta $p_i^{(0)}$ ($[p_i^{(0)}]^2 = m_i^2$) as linear combination of the momenta $p_a^{(0)}$ and $p_b^{(0)}$,

$$p_i^{(0)} = a_i p_a^{(0)} + b_i p_b^{(0)}\,, \quad i = 1, 2, \ldots, P\,. \tag{3.89}$$

A further definition is necessary if we shift p_a and p_b away from their singular values. We define

$$p_i \equiv a_i p_a + b_i p_b\,. \tag{3.90}$$

Then

$$p_i^2 - m_i^2 = a_i b_i (s - s_0) \equiv \varepsilon_i (s - s_0)\,. \tag{3.91}$$

Each propagator momentum P_i at a generic non-singular position can be expressed as

$$P_i = p_i + \sum_{j=1}^{L} \lambda_{ij} l_j \equiv p_i + k_i\,, \tag{3.92}$$

where l_j is the jth loop momentum and λ_{ij} some constant coefficients. The appropriate linear combination is fixed by the requirement of momentum conservation at each vertex of the graph. We have

$$P_i^2 - m_i^2 = \varepsilon_i (s - s_0) + 2 p_i \cdot k_i + k_i \cdot k_i\,. \tag{3.93}$$

With the shift $l_i \to (s - s_0) l_i$, finally we get

$$P_i^2 - m_i^2 = (s - s_0)[\varepsilon_i + 2 p_i \cdot k_i + (s - s_0) k_i \cdot k_i]\,. \tag{3.94}$$

Hence, the final expression of a Feynman diagram $\mathcal{G}$ with P propagators and L loops becomes

$$\begin{aligned}\mathcal{G} &\sim g^{2p} \int \prod_{i=1}^{L} \mathrm{d}^2 l_i \, \frac{1}{P_1^2 - m_1^2 + \mathrm{i}\varepsilon} \, \frac{1}{P_2^2 - m_2^2 + \mathrm{i}\varepsilon} \cdots \frac{1}{P_P^2 - m_P^2 + \mathrm{i}\varepsilon} \\ &\sim \frac{g^{2p}}{(s - s_0)^{P-2L}} \int \prod_{i=1}^{L} \mathrm{d}^2 l_i \, \frac{1}{2 p_i^{(0)} \cdot k_i + \varepsilon_i + \mathrm{i}\varepsilon} \cdots \frac{1}{2 p_P^{(0)} \cdot k_P + \varepsilon_P + \mathrm{i}\varepsilon}\,.\end{aligned} \tag{3.95}$$

The order of pole matches with the singularity (3.88) of the S-matrix and the residue can be computed explicitly via the multiple integral with the propagators replaced by linear functions of the loop momenta.

3.6.2. Double poles

A simple application of eq. (3.95) is the explanation of the double pole singularities. This problem was firstly analyzed by Coleman and Thun for the S-matrices of the sine–Gordon model [193]. Let us consider the graph depicted in fig. 10. This Feynman diagram produces a singularity in the forward scattering of two particles A and B. Once all external and internal momenta of the particles involved in

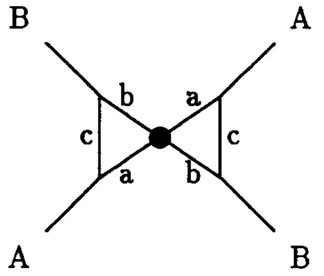


Fig. 10. Multiscattering process responsible for higher-order pole singularities in the S-matrix.

Table 3
Resonance angles of the $A_4^{(2)}$ model.

$u_{11}^{2} = \frac{2}{5}\pi$	$u_{12}^{1} = \frac{4}{5}\pi$
	$u_{12}^{2} = \frac{3}{5}\pi$
	$u_{22}^{3} = \frac{4}{5}\pi$

this multi-scattering process are put on shell, the problem of locating the singularity becomes a problem of plane geometry. In fact, the angles drawn in fig. 10 are exactly the resonance angles u_{ij}^{k} of the elementary scattering collision. The dot in the middle of the graph represents the S-matrix S_{ab} of the intermediate particles. From the previous analysis, we obtain that each loop gives rise to a simple pole and the consequent double pole is located at

$$\theta_{AB} = 2\pi - u_{Ac}^{a} - u_{Bc}^{b} . \tag{3.96}$$

Of course, such a double pole arises only if the graph can actually be drawn. This gives the following bound for the resonance angles:

$$\theta_{ab} = u_{ac}^{A} + u_{bc}^{B} < \pi . \tag{3.97}$$

This is a dynamical condition on the set of resonance angles that a bootstrap system may possess. As an example, consider the two-particle bootstrap chain defined by

$$A_1 \times A_1 \to A_2 , \quad A_2 \times A_2 \to A_1 . \tag{3.98}$$

The S-matrix was computed in ref. [149],

$$S_{11}(\theta) = f_{2/5}(\theta) , \quad S_{12}(\theta) = f_{3/5}(\theta) f_{4/5}(\theta) , \quad S_{22}(\theta) = f_{4/5}(\theta)[f_{2/5}(\theta)]^2 . \tag{3.99}$$

The poles corresponding to the bound states are given in table 3. The mass spectrum is

$$m_1 = M , \quad m_2 = 2M \cos \tfrac{1}{5}\pi . \tag{3.100}$$

The amplitude S_{22} of the heaviest particle A_2 presents a double pole at $\theta = \frac{2}{5}\pi i$, which corresponds to the double scattering process of fig. 10, all internal particles being A_1.

3.6.3. Third- and higher-order poles

In our previous example we assumed that the amplitude S_{ab} in the middle of the diagram was a regular function at $\theta = \theta_{ab}$. A new singularity occurs when θ_{ab} is itself a pole of S_{ab}. In this case, the effective singularity of such a graph is of the kind of a higher-order pole. For instance, if θ_{ab} is a location of a simple pole for S_{ab} we can further stretch the dot in the middle of fig. 10 to replace it by means of an intermediate propagator. Altogether, we have the situation depicted in fig. 11 and a third-order pole occurs. Depending on whether the system has a degenerate spectrum or not, we will have a term f_x^3 or s_x^3 in the S-matrix. As noticed in refs. [157, 172], the odd-order poles provide a new mechanism to produce bound states and they should be taken in account for computing consistently the bootstrap fusion algebra.

Concerning the even-order poles, let us consider the case when S_{ab} has a double pole at θ_{ab}. Stretching S_{ab} as before, we have the situation shown in fig. 12. There is not a propagator in between which connects the left and right loops and therefore this graph describes a purely multi-scattering process without the creation of intermediate bound states. This conclusion seems to apply to any even-order pole singularity.

The actual analysis of higher-order pole singularities is quite involved and will not be pursued here. The interested reader is encouraged to consult the original literature, in particular refs. [157, 171, 172]. A simple consequence of the previous discussion is that the scattering amplitude S_{11} of the lightest particle cannot have higher-order poles, because the resonance angle of two heavy particles with the lightest one is greater than $2\pi/3$ and therefore it is impossible to draw a figure like fig. 10 with the particle A_1 on all four external legs and the internal ones on-shell [172].

The results so far obtained on higher-order singularities are particularly useful in the discussion of the mathematical structure of the bootstrap equations and their classification, which we will discuss in the next section.

3.7. Classification program of the bootstrap systems

The classification of all massive integrable systems with bootstrap interaction is quite a vast problem and still far from being completed. In the context of Bethe ansatz approach, there are remarkable papers on the classification of the factorized S-matrix of particles which transform according to the representations of the simple laced Lie groups [180–182]. In the present approach, we will not assume any a priori underlying algebraic structure of the models but, on the contrary, we will take the point of view of considering the bootstrap equations as our basic entities, restricted only by the consistency equations of the conserved spins and by a multi-scattering interpretation of higher-order poles. Interesting results have been obtained in the case of non-degenerate systems [158, 160]. As we have already discussed, for these models the general solution of the unitary and crossing-symmetry equations is given by the functions $f_x(\theta)$, i.e. any S-matrix element can be written as

$$S_{ab}(\theta) = \prod_{x \in X_{ab}} f_x(\theta) . \tag{3.101}$$

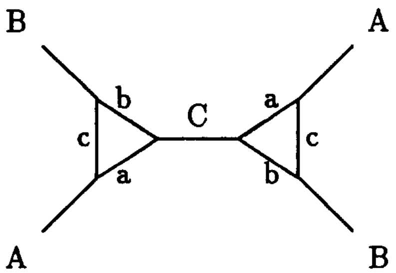


Fig. 11. Third-order pole.

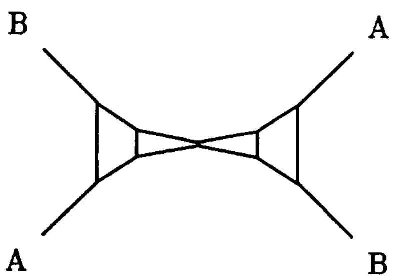


Fig. 12. Fourth-order pole.

We denote by $\mathcal{F}$ the space of the functions f_x and their products. The sets of positive $\{x\}$s or $\{1-x\}$s in (3.101) are related to the position of the poles in the scattering amplitudes S_{ab}, i.e. they signal possible bound states in this channel. A priori, we will not put any requirement on the sign of their residues but we will demand that their values must be consistently determined by the dynamical principle of bootstrap.

In the *strong* version of the bootstrap program, any odd-order pole is assumed to be in correspondence with a bound state (appearing or in the s-channel or in the t-channel) and the whole set of bound states should form a complete set of states which describes the asymptotic particles as well. This translates into the bootstrap equations (3.23). The resonance angles fix the values of the masses by means of eq. (3.20). Hence, starting from the scattering amplitude S_{11} of the lightest particle, one can compute the S-matrices of the bound states with higher mass using iteratively eq. (3.23). However, not all initial S_{11} give rise to a closed bootstrap process. Furthermore, since in principle we have the possibility to choose for each function f_x the pole at $\theta = \mathrm{i}\pi x$ or at $\theta = \mathrm{i}\pi(1-x)$ in order to continue the bootstrap, at each step of the process we can have ramification points. The natural structure associated to eqs. (3.23) is that of a schematic tree with the node of each set of branches representing an S-matrix reached at that stage of iteration and the branches originating from each node, the possible new singularities can be used to continue the process (fig. 13). The problem is thus to select S_{11} and then, out of all possible trees arising from it, only those ones, which give rise to a consistent set of S-matrices. The consistency requirements are those of existence of higher conserved charges and, at the same time, the explanation of all singularities in terms of the basic principles of the analytic S-matrix. It is easy to see that many initial choices for S_{11} can be immediately discarded because the result of applying eqs. (3.23) will lie outside the space $\mathcal{F}$. But it remains to investigate those ones which give rise to a set of functions which belong to the functional space $\mathcal{F}$.

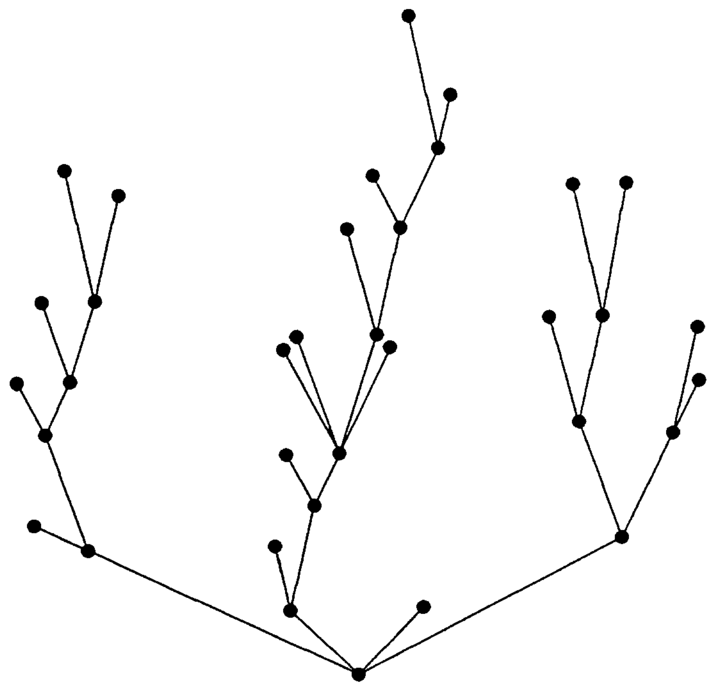


Fig. 13. Bootstrap tree.

It was observed in ref. [160] that such a formulation of the bootstrap classification of the integrable systems is closely related to basic questions which arise in the analysis of algorithms and their complexity, as discussed e.g. in refs. [128–131]. Fundamental developments in mathematical logics show the existence of *insolvable* problems: No algorithm can possibly exist for the solution of some problems. A typical one is the so-called *halting problem*: Given a computer algorithm*) (as, for instance, that one defined by eqs. (3.23) plus an initial condition) is it possible to decide whether or not it will ever halt for any initial input? The answer is negative, i.e. there is no algorithm that solves correctly all instances of the problem [132].

In our case the situation is not so extreme because the bootstrap systems defined by eqs. (3.23) are severely constrained by the consistency equations and, furthermore, by a coherent explanation of the higher-order singularities produced by the iterative application of (3.23). Because of that, in general the construction of a consistent set of S-matrices is an overdetermined problem and one could hope to find an answer to the following questions: Is it possible that the same S_{11} gives rise to different bootstrap systems? Is there a way to know a priori whether or not the bootstrap closes within a finite number of particles and how many there are? Starting with S_{11} with a finite number of f_x terms could we end up with an arbitrarily long path in the bootstrap tree?

For instance, it seems easy to generate an infinite path in the following way. Let us take $S_{11} = f_{1/3-\varepsilon}$ ($\varepsilon > 0$) and choose the singularity at $\theta = \mathrm{i}\pi(\frac{1}{3} - \varepsilon)$ to start with. This gives a bound state A_2 with mass equal to $m_2 = 2\cos\pi(\frac{1}{6} - \frac{1}{6} - \frac{1}{2}\varepsilon)]m_1$ and, using eq. (3.23), $S_{22} = f_{1/3+2\varepsilon}(f_{1/3-\varepsilon})^2$. Then, choosing as a new singularity the pole at $\theta = \mathrm{i}\pi(\frac{1}{3} + 2\varepsilon)$, we find a new bound state with mass $m_3 = 2\cos\pi(\frac{1}{6} + \varepsilon)m_2$. Proceeding in this way and taking ε arbitrarily small, we will generate a bootstrap with infinitely many particles, a subset of those with the unbounded masses $m_n \sim 3^{(n-1)/2} m_1$. Actually this is not a consistent system, in view of a theorem which we will prove in the next section.

3.7.1. *Bootstrap trees generated by an initial S-matrix with one singularity*

Let us consider a bootstrap system with

$$S_{11} = f_x(\theta)\,. \tag{3.102}$$

Our approach consists in the application of eqs. (3.23) as far as there are singularities in the functions S_{ab} identifiable as bound states. At this stage, we will not make a distinction between *real* and *virtual* states which is made, for instance, in the discussion of sine–Gordon model, i.e. those related to poles with positive and negative residue, but we try to find a self-consistent solution of the bootstrap equations. A simple theorem was established in ref. [160] for dealing with bootstrap trees generated by an amplitude S_{11} with only one singularity. The first part of the theorem says that there is only one possible way to implement the bootstrap which satisfies the consistency equations (3.38). The resulting spectrum is given by

$$m_k = 2m\sin(kx/2)\,. \tag{3.103}$$

The second part of the theorem deals with the implementation of the second constraint that an S-matrix has to satisfy, i.e. a consistent explanation of the higher-order poles. To fulfill this requirement, it is necessary to decouple the lightest particle A_{2n+1} produced by the bootstrap from the massive sector of

*) This notion is usually formalized in terms of Turing machines. See refs. [128–131] for details.

the theory. This is achieved by putting its mass equal to zero. Hence, we have the following condition for x

$$m_{2n+1}=0\,, \quad \mapsto x=2\pi/(2n+1)\,. \tag{3.104}$$

Before attacking the proof of this theorem, it is instructive to study first the cases when x is close to $\frac{2}{3}\pi$.

(a) $x>\frac{2}{3}\pi$. In this case, the singularity at $\theta=\mathrm{i}\pi x$ corresponds to a bound state A_2 with mass m_2 less than m_1,

$$m_2/m_1=2\cos(x/2) \tag{3.105}$$

and therefore contradicts the assumption that A_1 was the lightest particle. This fact alone is not necessarily a drawback since the bootstrap allows us to compute the scattering amplitudes choosing any arbitrary particle as starting point. Therefore it could only mean that our initial identification of the lightest particle was wrong. But the real difficulty comes when we compute

$$S_{22}=f_{2x}(f_x)^2\,, \tag{3.106}$$

because we see that in this amplitude (which is now the amplitude of the particle with lightest mass) appears a double pole which cannot occur (see section 3.6.3). Hence, there is no consistent set of S-matrices starting from $S_{11}=f_x$ when $x>\frac{2}{3}\pi$.

(b) x slightly less than $\frac{2}{3}\pi$, $x=(\frac{2}{3}-\varepsilon)\pi$ $(\varepsilon\to 0)$. In this case, the bootstrap produces three bound states with masses (fig. 14)

$$m_k=2m\sin(kx/2)\,, \quad k=1,2,3\,, \tag{3.107}$$

and S-matrices

$$\begin{aligned} &S_{11}=f_x\,, \quad S_{12}=f_{3x/2}f_{x/2}\,, \quad S_{13}=f_xf_{2x}\,, \\ &S_{22}=f_{2x}(f_x)^2\,, \quad S_{23}=f_{5x/2}f_{x/2}(f_{3x/2})^2\,, \quad S_{33}=f_{3x}(f_xf_{2x})^2\,. \end{aligned} \tag{3.108}$$

As in case (a), we obtain a particle A_3 with mass m_3 less than m_1, which is the one we started with. S_{33} contains as well unwanted double poles. The only way to make this system consistent is to push $m_3\to 0$ and, correspondingly, decouple A_3 from the rest of the theory. In this limit all S-matrices involving A_3 go to the identity and the particle state A_2 becomes identical to A_1. The initial three-particle system collapses to that one with only one particle state and S-matrix

$$S_{11}=f_{2/3}(\theta)\,. \tag{3.109}$$

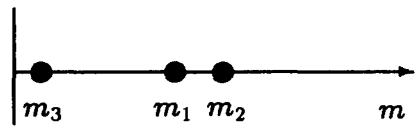


Fig. 14. Mass spectrum generated by $S_{11}=f_x$ with x close to $2\pi/3$.

This corresponding to the S-matrix of the Yang–Lee model [148].

Let us consider now the general case and prove that there exists only one path in the bootstrap tree which respects the consistency equations. The proof is given by induction. Starting with $S_{11} = f_x$, we obtain a new bound state whose mass can be written as

$$m_2/m_1 = 2\cos(x/2) = 2m\sin(x)/2m\sin(x/2)\,, \tag{3.110}$$

where m is an arbitrary mass scale. We can compute S_{12} by applying eq. (3.23)

$$S_{12} = S_{11}(\theta - \mathrm{i}x/2)S_{11}(\theta + \mathrm{i}x/2) = f_{3x/2}f_{x/2}\,. \tag{3.111}$$

We get a function with four singularities: those at $\theta = \mathrm{i}\pi x/2$ and $\theta = (1 - x/2)\pi$ (from the $f_{x/2}$ term) and those at $\theta = \mathrm{i}\frac{3}{2}\pi x$ and $\theta = \mathrm{i}(1 - \frac{3}{2}x)\pi$ (from the $f_{3x/2}$ term). Among these, the one at $\theta = \mathrm{i}(1 - x/2)\pi$ corresponds to the bound state A_1. Therefore we have correctly identified this angle as the resonance angle due to a bound state. We can now apply eq. (3.43) in order to decide which of the two poles in $f_{3x/2}$ corresponds to a new bound state A_3. The answer turns out to be that one at $\theta = \mathrm{i}\frac{3}{2}\pi x$. This means that we cannot use the other singularity at $\theta = (1 - \frac{3}{2}x)\pi$ in order to implement further the bootstrap if we require a non zero solution of the consistency equations. On the contrary, we are obliged to follow the path defined by the resonance angle $u_{12}^{3} = \frac{3}{2}\pi x$. The mass of the new particle is

$$m_3/m_1 = 2m\sin(\tfrac{3}{2}\pi x)/2m\sin(\tfrac{1}{2}\pi x)\,. \tag{3.112}$$

We can compute

$$S_{13} = S_{12}(\theta - \mathrm{i}\bar{u}_{23}^{1})S_{11}(\theta + \mathrm{i}\bar{u}_{13}^{2}) = f_x f_{2x}\,. \tag{3.113}$$

Repeating the same reasoning of before, we identity the pole at $\theta = \mathrm{i}(1 - x)\pi$ as u_{13} and in this way we fix the ratio of the conserved quantities γ_1/γ_3 in (3.43). The singularity due to a new bound state A_4 is that at $\theta = \mathrm{i}u_{13}^{4} = \mathrm{i}2\pi x$. The mass of this new bound state is

$$m_4/m_1 = 2m\sin(2\pi x)/2m\sin(\tfrac{1}{2}\pi x)\,. \tag{3.114}$$

The process can be continued up to the particle A_{2n+1} where n is defined by

$$2\pi/(2n+3) < x \leq 2\pi/(2n+1) \tag{3.115}$$

and has to be completed by the computation of the remaining S-matrices. The mass spectrum is given by

$$m_k = 2m\sin(\tfrac{1}{2}kx)\,, \quad k = 1, 2, \ldots, 2n+1\,. \tag{3.116}$$

The particle $A_{2n+1} \equiv A_l$ is the lightest one and its S_{ll}-matrix has a plethora of double poles. We can get a consistent set of S-matrices only if we put $m_{2n+1} = 0$ and decouple this particle from the theory. In this limit the remaining $2n$ particles become pairwise identical and we end up with an n-particle system with

S-matrix

$$S_{ab} = f_{|a-b|/(2n+1)} f_{(a+b)/(2n+1)} \prod_{k=1}^{\min(a,b)-1} (f_{(|a-b|+2k)/(2n+1)})^2 \tag{3.117}$$

$(a, b = 1, 2, \ldots, n)$. All double poles have now explanation in terms of multi-scattering processes and the conserved spins are all odd numbers but multiples of $2n+1$

$$s = 1, 3, \ldots, 2n-1, 2n+3, \ldots, 4n+1 \quad (\mathrm{mod}\, 4n+2)\,. \tag{3.118}$$

The price to be paid is that these S-matrices are not *one-particle unitary* and indeed correspond to the $\Phi_{1,3}$ deformation of the non-unitary minimal models $\mathcal{M}_{2,2n+3}$ [92, 148, 150] (see section 6.2.1).

3.7.2. Bootstrap trees generated by an S_{11} with multiple poles

The problem to find consistent sets of S-matrices starting with a scattering amplitude S_{11} with more than one singularity becomes more complicated. In order to investigate the bootstrap tree generated by a generic S_{11}, a computer algorithm was designed to span all possible paths and to select only those ones which satisfy our requirements of consistency [160]. The algorithm was based on the following items:

(i) m_1 is the lowest particle,
(ii) the theory is one-particle unitary,
(iii) the singularities in S_{11} are simple poles with rational values.

As input data, it is necessary to specify the number of functions f_x in S_{11} and their common denominator D. The value of D depends on the memory capacity of the computer. The analysis was performed for the cases $D \leq 128$ (which is equivalent to analyze $\sim 10^9$ initial S_{11}). The results can be summarized as follows.

$$\text{(a)} \quad S_{11} = \prod_{i=1}^{2} f_{x_i}\,. \tag{3.119}$$

The only consistent set of S-matrices is obtained when

$$S_{11} = -f_{1/9} f_{5/9} \tag{3.120}$$

and this corresponds to the minimal S-matrix of the E_7 system [157, 159].

$$\text{(b)} \quad S_{11} = \prod_{i=1}^{3} f_{x_i}\,. \tag{3.121}$$

The only consistent set of S-matrices is obtained when

$$S_{11} = f_{1/3} f_{2/5} f_{1/15}\,, \tag{3.122}$$

which corresponds to the minimal S-matrix of the E_8 system [30, 159].

$$\text{(c)} \quad S_{11} = \prod_{i=1}^{N} f_{x_i}\,, \quad N = 4, 5\,. \tag{3.123}$$

In the range of D previously discussed, the bootstrap does not give rise to any consistent system.

3.8. Factorized S-matrix of kink states in deformed CFT

Many perturbed CFT are described by integrable massive field theories with soliton behaviour. This means that the deformation of the critical point action gives rise to an effective potential with a finite number of degenerate vacua. It is therefore interesting to extent our analysis to these cases. The soliton configurations which interpolate between different vacua are the basic massive excitations of the spectrum. A classical example of such a theory is given by the thermal deformation of the Ising model. At the critical point ($T = T_c$), this model can be described in terms of a scalar field Φ with a quartic interaction

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \Phi)^2 + g\Phi^4 . \tag{3.124}$$

A thermal perturbation is realized by adding a quadratic term to the previous Lagrangian and changing the effective potential to

$$V(\Phi) = (T - T_c)\Phi^2 + g\Phi^4 . \tag{3.125}$$

If $T > T_c$, we have a unique vacuum. On the contrary, when $T < T_c$, the model has two degenerate vacua and the one-particle excitations of the spectrum are given by the kinks connecting the two ground states.

A general approach for dealing with the massive integrable QFT with soliton behaviour obtained as $\Phi_{1,3}$, $\Phi_{1,2}$ and $\Phi_{2,1}$ deformations of minimal models of CFT has been developed in recent years. The main idea of this approach is based on the well-known relation between the S-matrices and the R-matrices of integrable models. Several authors have pointed out that the above deformations of CFT are obtained by restricting the Hilbert space of the integrable sine–Gordon (SG) and Zhiber–Mikhailov–Shabat (ZMS) models while preserving their integrability [91–96]. A heuristic argument for understanding the relevance of the SG and the ZMS models in the description of these integrable deformations of CFT is based on the realization of the minimal models $\mathscr{M}_{p,p'}$ in terms of a conformal quantization of a Liouville theory [208–210]

$$\mathscr{L} = (\partial_\mu \varphi)^2 + e^{i\beta\varphi} . \tag{3.126}$$

The primary fields $\Phi_{1,n}$ are identified with the vertex operators $\exp[-\frac{1}{2}i(n-1)\beta\varphi]$. A deformation of the conformal theory by the operator $\Phi_{1,3}$ is thus obtained by inserting its vertex representation $e^{-i\beta\varphi}$ into the Lagrangian (3.126). The resulting theory is the sine–Gordon model. Analogously, the deformation of CFT by means of the operator $\Phi_{1,2} \equiv \exp(-\frac{1}{2}\beta\varphi)$ gives rise to the Zhiber–Mikhailov–Shabat model.

The heuristic argument given above is supported by a more detailed analysis of the Hilbert space of the SG and ZMS models. An important feature of both models is the invariance of their R-matrix under the quantum group*) $SL(2)_q$. The q-parameter is a function of the coupling constant β. When q is a root of unity, we can restrict the Hilbert space of the original model preserving both the integrability and the locality of an invariant set of operators. This happens when $\beta^2/8\pi$ is a rational number. The restricted models obtained in this way renormalize in the ultraviolet limit to a minimal model of CFT

*) An introduction to the theory of quantum groups can be found in refs. [218, 204–206].

with central charge given by

$$c = 1 - 6(\beta^2/8\pi + 8\pi/\beta^2 - 2)\,. \tag{3.127}$$

In this section we will focus only on the results of this analysis referring to the original literature for their derivative and justification. Interesting applications of them will be presented in chapter 6.

3.8.1. $\Phi_{1,3}$ deformation of minimal models of CFT as restricted sine–Gordon model

The sine–Gordon (SG) model is described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \varphi)^2 + (m^2/\beta^2)(\cos \beta\varphi - 1)\,. \tag{3.128}$$

The theory has an infinite number of degenerate vacua. The soliton solutions of the equation of motion are given by those field configurations which interpolate between two vacua. They can be characterized by their topological charge

$$t = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\, \mathrm{d}x = (\beta/2\pi)[\varphi(+\infty) - \varphi(-\infty)]\,, \tag{3.129}$$

which takes values $t = \pm n$, $n = 0, 1, 2, \ldots$

The sine–Gordon model is integrable both at the classical and quantum level [108, 124, 192–194, 197–200]. In the quantum version of the theory, the solitons $(A, \bar{A})$ with $t = \pm 1$ play the role of fundamental particles whereas the solitons with higher topological charges form the multi-soliton configurations. The spectrum also contains neutral particles B_n (the so-called breathers), which are bound states of the soliton–antisoliton states. The total number of breathers is $[\pi/\gamma]$, where γ is the renormalized coupling constant

$$\gamma = \pi\beta^2/(8\pi - \beta^2)\,, \tag{3.130}$$

and $[x]$ stays for the integer part of the real number x. The S-matrix of this model for $\beta^2 < 8\pi$ has been derived in ref. [108] by solving the Yang–Baxter equations and the usual requirements of unitarity and crossing symmetry. The SG model possesses a hidden $\mathrm{SL}(2)_q$ quantum symmetry which permits to assemble together the scattering amplitudes of the soliton sector into the structure of the R-matrix corresponding to $\mathrm{A}_1^{(1)}$ which has, in addition to $\mathrm{SL}(2)_q$, a spectral parameter [91, 92, 94–96]. The deformation parameter is given in terms of the coupling constant by

$$q = \exp(2\pi^2 \mathrm{i}/\gamma)\,. \tag{3.131}$$

The quantum group $\mathrm{SL}(2)_q$ is defined as the universal enveloping algebra $\mathcal{U}_q[\mathrm{sl}(2)]$ with the commutation relations [206]

$$[J_+, J_-] = (q^H - q^{-H})/(q - q^{-1})\,, \quad [H, J_\pm] = \pm 2J_\pm\,. \tag{3.132}$$

A "comultiplication" Δ_q is defined by

$$\Delta_q(H) = 1 \otimes H + H \otimes 1\,, \quad \Delta_q(J_\pm) = q^{H/2} \otimes J_\pm + J_\pm \otimes q^{-H/2}\,. \tag{3.133}$$

The irreducible representations of $SL(2)_q$ are obtained by means of the comultiplication Δ_q [206].

The doublet $(A, \bar{A})$ forms the fundamental spin 1/2 representation of the quantum group $SL(2)_q$ whereas B_ns are singlets. We can generate all the irreducible representations with higher spins of $SL(2)_q$ using the q-analog of the Clebsch–Gordan coefficients,

$$|J, M\rangle = \sum_{m_1, m_2} \begin{bmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{bmatrix} |j_1, m_1\rangle \otimes |j_2, m_2\rangle . \tag{3.134}$$

The explicit expressions of the q-analogs of the Clebsch–Gordan coefficients and of the 6-j symbols can be found in ref. [206]. They are expressed in terms of the q-numbers $[n]_q$, defined by

$$[n]_q = (q^n - q^{-n})/(q - q^{-1}) . \tag{3.135}$$

For q not a root of unity, the representations of $SL(2)_q$ are thus obtained as deformations of the representations of the classical algebra SL(2). However, when q is a root of unity ($q^N = \pm 1$), some of the Clebsch–Gordan and the 6-j coefficients become singular and therefore the usual representation theory is not well-defined. In this case, the sensible representations are those with an upper bound for the value of the spins $0 \leq J \leq J_{\max}$, where $J_{\max}$ is fixed by the condition $[2J_{\max} + 1]_q = 0$, i.e.

$$J_{\max} = \tfrac{1}{2}N - 1 . \tag{3.136}$$

For the SG model, this happens when $\beta^2/8\pi$ is a rational number,

$$\beta^2/8\pi = p/p' , \tag{3.137}$$

with p and p' two coprime integers ($p' > p$). Correspondingly,

$$\gamma/\pi = p/(p' - p) , \tag{3.138}$$

and the resulting Hilbert space of the SG model decomposes into a finite number of subspaces with spin J bounded by

$$J_{\max} = \tfrac{1}{2}p - 1 . \tag{3.139}$$

Performing a change of basis from vertex to IRF basis [206], the solitons and the antisolitons are replaced by kinks K_{ab} with $|a - b| = 1/2$ and $a, b = 0, \tfrac{1}{2}, \ldots, J_{\max}$. This means that the restricted SG model does not possess an infinite number of vacuum states but only a finite number of them, connected by the kink states K_{ab}. The kink-kink S-matrices $S^{ab}_{dc}(\theta)$ which describe the scattering $K_{da} + K_{ab} \rightarrow K_{dc} + K_{cd}$ are given by a RSOS restriction of the original soliton S-matrices. Its explicit form is given by [91, 92, 94–96]

$$\begin{aligned} S^{ab}_{dc}(\theta) = {} & \frac{U(\theta)}{2\pi \mathrm{i}} \left(\frac{[2a+1]_q [2c+1]_q}{[2d+1]_q [2b+1]_q} \right)^{-\theta/2\pi \mathrm{i}} \\ & \times \left[\delta_{db} \sinh\left(\frac{\theta}{\gamma}\right) \left(\frac{[2a+1]_q [2c+1]_q}{[2d+1]_q [2b+1]_q} \right)^{1/2} + \delta_{ac} \sinh\left(\frac{\mathrm{i}\pi - \theta}{\gamma} \right) \right] , \end{aligned} \tag{3.140}$$

where

$$U(\theta)=\Gamma(1/\gamma)\Gamma(1+\mathrm{i}\theta/\gamma)\Gamma(1-1/\gamma-\mathrm{i}\theta/\gamma)\prod_{n=1}^{\infty}\frac{R_n(\theta)R_n(\mathrm{i}\pi-\theta)}{R_n(0)R_n(i\pi)}\,, \tag{3.141}$$

$$R_n(\theta)=\frac{\Gamma(2n/\gamma+\mathrm{i}\theta/\gamma)\Gamma(1+2n/\gamma+\mathrm{i}\theta/\gamma)}{\Gamma((2n+1)/\gamma+\mathrm{i}\theta/\gamma)\Gamma(1+(2n-1)/\gamma+\mathrm{i}\theta/\gamma)}\,. \tag{3.142}$$

The breather sector remains untouched in the reduction procedure since it is a singlet subspace of $SL(2)_q$. The S-matrices $S^{(n)}(\theta)$ of the kink–breather scattering $K_{ab}+B_n\to B_n+K_{ab}$ are given by the corresponding scattering amplitudes of the original SG theory [108],

$$S^{(n)}(\theta)=\frac{\sinh\theta+\mathrm{i}\cos(\frac{1}{2}n\gamma)}{\sinh\theta-\mathrm{i}\cos(\frac{1}{2}n\gamma)}\prod_{l=1}^{n-1}\frac{\sin^2[\frac{1}{4}(n-2l)\gamma-\frac{1}{4}\pi+\frac{1}{2}\mathrm{i}\theta]}{\sin^2[\frac{1}{4}(n-2l)\gamma-\frac{1}{4}\pi-\frac{1}{2}\mathrm{i}\theta]}\,. \tag{3.143}$$

An analogous result holds for the scattering of two breathers $B_n+B_m\to B_m+B_n$. The S-matrix $S^{(n,m)}(\theta)$ $(n\geq m)$ is given by [108]

$$S^{(n,m)}(\theta)=\frac{\sinh\theta+\mathrm{i}\sin[\frac{1}{2}(n+m)\gamma]\sinh\theta+\mathrm{i}\sin[\frac{1}{2}(n-m)\gamma]}{\sinh\theta-\mathrm{i}\sin[\frac{1}{2}(n+m)\gamma]\sinh\theta-\mathrm{i}\sin[\frac{1}{2}(n-m)\gamma]}$$

$$\times\prod_{l=1}^{m-1}\frac{\sin^2[\frac{1}{4}(m-n-2l)\gamma+\frac{1}{2}\mathrm{i}\theta]\cos^2[\frac{1}{4}(m+n-2l)\gamma+\frac{1}{2}\mathrm{i}\theta]}{\sin^2[\frac{1}{4}(m-n-2l)\gamma-\frac{1}{2}\mathrm{i}\theta]\cos^2[\frac{1}{4}(m+n-2l)\gamma-\frac{1}{2}\mathrm{i}\theta]}\,. \tag{3.144}$$

The S-matrices of the kink states satisfy the relation

$$S_{dc}^{ab}(\theta)=S_{ad}^{bc}(\mathrm{i}\pi-\theta)\,, \tag{3.145}$$

which expresses the crossing symmetry. An important constraint comes from the condition of unitarity. In fact, the S-matrices of the kink states satisfy the equation

$$\sum_l S_{ld}^{ab}(\theta)S_{cd}^{lb}(-\theta)=\delta_{ac}\,. \tag{3.146}$$

Hence, as far as the condition $S^{\dagger}(\theta)=S(-\theta)$ is satisfied, the scattering theory is unitary. This occurs for the following values of γ/π [91]

$$\gamma/\pi=p/(p'-p)=N/(Nk+1)\quad N\geq 2,\,k\geq 0\,,$$

$$=3/(3k+2)\qquad k\geq 0\,. \tag{3.147}$$

It has been argued that the ultraviolet limit of the integrable models constructed that way is controlled by the minimal conformal model $\mathcal{M}_{p,p'}$ deformed by the operator $\Phi_{1,3}$ [91, 92, 94–96]. A non trivial check of this conjecture has been obtained by means of the thermodynamical Bethe ansatz [53, 54].

3.8.2. $\Phi_{1,2}$ and $\Phi_{2,1}$ deformations of minimal models of CFT

It has been pointed out by Smirnov [93] that the deformation of minimal model of CFT by the operator $\Phi_{1,2}$ can be considered as reduced systems of the Zhibner–Mikhailov–Shabat (ZMS) model with imaginary coupling constant. The Lagrangian is formally given by

$$\mathscr{L} = \int \mathrm{d}x\, [\tfrac{1}{2}(\partial_\mu \varphi)^2 + (\tfrac{1}{2}\, \mathrm{e}^{\mathrm{i}\beta\varphi} + \mathrm{e}^{-\mathrm{i}\beta\varphi/2})] . \tag{3.148}$$

An important difference of this model with respect to the SG model is that the Lagrangian (3.148) is not a Hermitian operator and, therefore, the definition itself of the theory seems to be problematic. The solution of this problem and the resulting S-matrices for the massive deformations of CFT is one of the most beautiful results in the quantum group approach to off-critical models. In fact, as shown by Smirnov [93], in this case only the restricted theories have a physical meaning.

The first step of Smirnov's analysis is to consider the R-matrix of the theory. It is intrinsically related to the algebra $A_2^{(2)}$, which similarly to the case of sine–Gordon contains a spectral parameter but is constructed starting from the spin 1 representation of $SL(2)_q$ [99]. Its expression is given by

$$R_{12}(\lambda, q) = (\lambda^{-1} - 1) q^{3/2} R_{1,2}(q) + (1 - \lambda) q^{-3/2} R_{21}^{-1}(q) + q^{-5/2}(q^2 - 1)(q^3 + 1) P_{12} . \tag{3.149}$$

$R_{12}(\lambda, q)$ is an operator acting on the vector space $\mathbb{C}^3 \otimes \mathbb{C}^3$, λ is the spectral operator, q is the parameter of the quantum group and P_{12} is the permutation operator. The matrix $R_{12}(q)$ is the constant solution of the Yang–Baxter equation for spin 1 representations of the quantum group $SL(2)_q$, given by

$$R_{12}(q) = \exp(\tfrac{1}{4} H \otimes H)[1 + (q^2 - 1) E \otimes F + (q-1)^2 (q+1) E^2 \otimes F^2] , \tag{3.150}$$

where

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & q^{-1/2} \\ 0 & 0 & 0 \end{pmatrix}, \quad F = q^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q^{1/2} & 0 \end{pmatrix} . \tag{3.151}$$

In order to interpret the matrix $R(\lambda, q)$ as an S-matrix, one needs to relate λ to the rapidity variable θ and q to the coupling constant β of the model. With the following identifications

$$q = \exp(16\pi^2 \mathrm{i}/\beta^2) , \quad \lambda = \exp(2\pi\theta/\xi) , \quad \xi = \tfrac{2}{3}[\pi\beta^2/(16\pi - \beta^2)] , \tag{3.152}$$

the hypothetical S-matrix of the three-component kink (which is the fundamental particle of the ZMS model) is given by

$$S_{12}(\theta) = S_0(\theta) R_{12}(\exp(2\pi\theta/\xi), \exp(16\pi^2 \mathrm{i}/\beta^2)) . \tag{3.153}$$

The prefactor $S_0(\theta)$ ensures the validity of the "unitary" relation

$$S_{12}(\theta) S_{21}(-\theta) = 1 , \tag{3.154}$$

and fixes the pole structure of the scattering amplitude. It reads

$$S_0(\theta) = \{\sinh[(\pi/\xi)(\theta - \mathrm{i}\pi)]\,\sinh[(\pi/\xi)(\theta - \tfrac{2}{3}\pi\mathrm{i})]\}^{-1}$$

$$\times \exp\left(-2\mathrm{i}\int_0^\infty \frac{\mathrm{d}x}{x}\, \frac{\sin(\theta x)\sinh(\frac{1}{3}\pi x)\cosh[(\frac{1}{6}\pi - \frac{1}{2}\xi)x]}{\cosh(\frac{1}{2}\pi x)\sinh(\frac{1}{2}\xi x)}\right). \tag{3.155}$$

In order to discuss the analytic structure of $S_0(\beta)$, it is more convenient to write it as

$$S_0(\beta) = \pi^{-2}\Gamma((\pi + \mathrm{i}\beta)/\xi)\Gamma((\xi - \pi - \mathrm{i}\beta)/\xi)\Gamma((\tfrac{2}{3}\pi + \mathrm{i}\beta)/\xi)\Gamma((\xi - \tfrac{2}{3}\pi - \mathrm{i}\beta)/\xi)\Xi(\beta)\,, \tag{3.156}$$

where $\Xi(\beta)$ is the following infinite product

$$\Xi(\beta) = \prod_{k=0}^{\infty} \frac{\Gamma(\pi/\xi + (2k\pi - \mathrm{i}\beta)/\xi)\Gamma(2\pi/\xi + (2k\pi + \mathrm{i}\beta)/\xi)}{\Gamma(\pi/\xi + (2k\pi + \mathrm{i}\beta)/\xi)\Gamma(2\pi/\xi + (2k\pi - \mathrm{i}\beta)/\xi)}$$

$$\times \frac{\Gamma(1 + (2k\pi + \mathrm{i}\beta)/\xi)\Gamma((\xi + \pi)/\xi + (2k\pi - \mathrm{i}\beta)/\xi)}{\Gamma(1 + (2k\pi - \mathrm{i}\beta)/\xi)\Gamma((\xi + \pi)/\xi + (2k\pi + \mathrm{i}\beta)/\xi)}$$

$$\times \frac{\Gamma(\pi/3\xi + (2k\pi + \mathrm{i}\beta)/\xi)\Gamma(4\pi/3\xi + (2k\pi - \mathrm{i}\beta)/\xi)}{\Gamma(\pi/3\xi + (2k\pi - \mathrm{i}\beta)/\xi)\Gamma(4\pi/3\xi + (2k\pi + \mathrm{i}\beta)/\xi)}$$

$$\times \frac{\Gamma((2\pi + 3\xi)/3\xi + (2k\pi - \mathrm{i}\beta)/\xi)\Gamma((5\pi + 3\xi)/3\xi + (2k\pi + \mathrm{i}\beta)/\xi)}{\Gamma((2\pi + 3\xi)/3\xi + (2k\pi + \mathrm{i}\beta)/\xi)\Gamma((5\pi + 3\xi)/3\xi + (2k\pi - \mathrm{i}\beta)/\xi)}\,. \tag{3.157}$$

It is easy to check that $S_0(\beta)$ satisfies the equation

$$S_0(\beta) = S_0(\mathrm{i}\pi - \beta)\,. \tag{3.158}$$

For generic ξ, the simple poles which lie on the physical sheet are at the crossing symmetric locations

$$\theta = \mathrm{i}\pi - \mathrm{i}\xi m\,, \qquad \mathrm{i}\xi m\,, \qquad\qquad m > 0\,, \tag{3.155a}$$

$$= \tfrac{2}{3}\pi\mathrm{i} - \mathrm{i}\xi m\,, \quad \tfrac{1}{3}\pi\mathrm{i} + \mathrm{i}\xi m\,, \quad m \geq 0\,. \tag{3.155b}$$

In both sets, the first poles are the singularities in the direct channel and the second ones are the crossing poles.

For the first set (3.155a), the R-matrix degenerates into a one-dimensional projector. Smirnov has identified these singularities as those of the breather bound states. Using the bootstrap equation, the S-matrix of the fundamental breather (corresponding to the pole $\theta = \mathrm{i}\pi - \mathrm{i}\xi$) is given by

$$S_{b_1,b_1}(\theta) = f_{2/3}(\theta)f_{\xi/\pi}(\theta)f_{\xi/\pi - 1/3}(\theta)\,. \tag{3.59}$$

On the other hand, considering the second set (3.155b), the R-matrix degenerates into a three-dimensional projector. Hence, these poles can be interpreted as those corresponding to the creation of higher kinks. However, from a physical point of view, the definiton itself of S_{12} has some drawbacks which forbid to interpret it as the correct scattering amplitude of the model [93, 207]. One of them is that the matrix R_{12}, for the important cases when $|q| = 1$, does not satisfy the relation

$$R^*_{12}(\lambda) = R_{21}(\lambda^{-1})\,, \tag{3.160}$$

which is crucial in order to correctly implement the unitarity requirement of the scattering amplitudes. Therefore, the S-matrix (3.153) as it is, cannot be interpreted as the scattering amplitude of the ZMS model. Though, as Smirnov observed, the RSOS restriction of the R-matrix yields S-matrices which have a sensible physical interpretation. This happens when $q^r = 1$. The RSOS states which appear in the reduced model,

$$|\theta_1, j_1, k_1, |a_1|\theta_2, j_2, k_2, \dots |a_{n-1}|\theta_n, j_n, k_n\rangle , \tag{3.161}$$

are characterized by their rapidity θ_i, by their type k (which distinguishes the kinks from the breathers), by their $SL(2)_q$ spin j and by a string of numbers a_i, constrained by the following limitations:

$$a_i \leq \tfrac{1}{2}(r-2) , \quad |a_k - 1| \leq a_{k+1} \leq \min(a_k + 1, r - 3 - a_k) . \tag{3.162}$$

The S-matrix of these RSOS states is given by

$$\begin{aligned} S\left(\theta_k - \theta_{k+1}\left|\begin{matrix} a_{k-1} & a_k \\ a_{k+1} & a'_k \end{matrix}\right|\right) &= \frac{1}{4\mathrm{i}}\, S_0(\theta_k - \theta_{k+1})\left[\begin{Bmatrix} 1 & a_{k-1} & a_k \\ 1 & a_{k+1} & a'_k \end{Bmatrix}_q\right. \\ &\times \{[\exp((2\pi/\xi)(\theta_{k+1} - \theta_k)) - 1]\, q^{(c_{a_{k+1}} + c_{a_{k-1}} - c_{a_k} - c_{a'_k} + 3)/2} \\ &- [\exp(-(2\pi/\xi)(\theta_{k+1} - \theta_k)) - 1]\, q^{-(c_{a_{k+1}} + c_{a_{k-1}} - c_{a_k} - c_{a'_k} + 3)/2}\} \\ &\left. + q^{-5/2}(q^3 + 1)(q^2 - 1)\delta_{a_k, a'_k}\right] . \end{aligned} \tag{3.163}$$

Herein, c_a are given by $c_a = a(a+1)$. The expression of the 6-j symbols can be found in ref. [93]. The above S-matrix is unitary if and only if the 6-j symbols are real and this happens for the following values of $\beta^2/8\pi$: (i)

$$\beta^2/8\pi = r/(r+1) , \tag{3.164a}$$

which correspond to the $\Phi_{1,2}$ deformation of the minimal unitary models $\mathcal{M}_{r,r+1}$; (ii)

$$\beta^2/8\pi = 2/(2n+1) , \quad \beta^2/8\pi = 3\pi/(3n+1) , \tag{3.164b}$$

which are related to the $\Phi_{1,2}$ deformation of the nonunitary minimal models $\mathcal{M}_{2,2n+1}$ and $\mathcal{M}_{3,3n\pm1}$. For these values of $\beta^2/8\pi$, the maximal allowed spin is 0 and $\frac{1}{2}$. Hence, the kinks disappear from the reduced space and only breathers remain as asymptotic states in the spectrum; (iii)

$$\beta^2/8\pi = 4\pi/(4n \pm 1) , \tag{3.164c}$$

which correspond to the $\Phi_{1,2}$ deformation of the nonunitary minimal model $\mathcal{M}_{4,4n\pm1}$. For this series the maximal allowed spin is equal to 1 and, according to the RSOS restriction, the kinks behave as scalar particles.

The discussion of the S-matrices of the massive states originating from the deformation of minimal model of CFT by the operator $\Phi_{2,1}$ is similar to the one above, the only difference being in the definition of the q-parameter and the spectral parameter λ. In this case, the correct identifications are as follows

$$q=\exp(\mathrm{i}\pi^2\beta^2/4)\,,\quad \lambda=\exp(2\pi\theta/\tilde{\xi})\,,\quad \tilde{\xi}=\tfrac{8}{3}\pi^2/(\beta^2-4\pi)\,,\quad \beta^2/8\pi>\tfrac{1}{2}\,. \tag{3.165}$$

Correspondingly, in all previous formulas ξ has to be changed to $\tilde{\xi}$. The condition $\beta^2/8\pi>\frac{1}{2}$ selects the range of β for which the field $\Phi_{2,1}$ is a relevant operator, since its anomalous dimension is given by

$$\Delta_{2,1}=-\tfrac{1}{2}+6\pi/\beta^2\,. \tag{3.166}$$

3.9. Supersymmetric S-matrices

Our previous analysis can be also generalized to the supersymmetric case and gives rise to interesting models of fermion–boson interaction. Integrable and supersymmetric deformations of the superconformal models have been discussed by several authors [115–119, 179]. Factorized S-matrices that have both O(N) symmetry and supersymmetry have been analyzed by Shankar and Witten [111]. They correspond to the scattering amplitudes of the Lagrangian model of the O(N) supersymmetric non-linear sigma model [112–114]. We will present here their basic findings.

The Lagrangian of the O(N) supersymmetric non-linear sigma model is given by

$$\mathscr{L}=\frac{1}{g^2}\int \mathrm{d}^2x\,[\tfrac{1}{2}(\partial_\mu n^a)^2+\tfrac{1}{2}\bar{\psi}^a\,\mathrm{i}\,\partial_\mu\gamma^\mu\psi^a+\tfrac{1}{8}(\bar{\psi}^a\psi^a)^2]\,, \tag{3.167}$$

with the constraints

$$\sum_{a=1}^{N} n^a n^a=1\,,\quad \sum_{a=1}^{N} n^a\psi^a_\mu=0\,.$$

Here n^a is an N-component real scalar field and ψ^a_μ is an N-component Majorana fermion. This model may be seen as a hybridization of the non-linear σ model and the Majorana version of the Gross–Neveu model.

In the large-N limit, the model (3.167) presents asymptotic freedom and dynamical mass generation [114]. This means that the spectrum consists of a degenerate supermultiplet of N massive boson states $|b^a\rangle$ and N massive fermion states $|f^a\rangle$. Concerning the bound states, the large-N expansion shows the existence of a fermion–boson bound state in addition to the fermion–fermion bound state. These are the only extra informations we need in order to fix completely the exact S-matrix of the model.

Among the conserved charges of the model, we have the chiral components of the supercharge $\mathscr{Q}$. They satisfy the supersymmetry algebra

$$\mathscr{Q}_+^2=P_0+P_1\,,\quad \mathscr{Q}_-^2=P_0-P_1\,,\quad \mathscr{Q}_+\mathscr{Q}_-+\mathscr{Q}_-\mathscr{Q}_+=0\,,\quad \mathscr{Q}_\pm P_\mu-P_\mu\mathscr{Q}_\pm=0\,. \tag{3.168}$$

Their action on the one-particle states is

$$\begin{aligned}&\mathscr{Q}_+|b^a(\theta)=\mathrm{e}^{\theta/2}|f^a(\theta)\rangle\,,\quad \mathscr{Q}_+|f^a(\theta)=\mathrm{e}^{\theta/2}|b^a(\theta)\rangle\,,\\ &\mathscr{Q}_-|b^a(\theta)=\mathrm{i}\,\mathrm{e}^{-\theta/2}|f^a(0)\rangle\,,\quad \mathscr{Q}_-|f^a(\theta)=-\mathrm{i}\,\mathrm{e}^{-\theta/2}|b^a(\theta)\rangle\,.\end{aligned} \tag{3.169}$$

A basis of the two-particle asymptotic states is given by

$$|b^a(\theta_1)b^b(\theta_2)\rangle\,,\quad |f^a(\theta_1)f^b(\theta_2)\rangle\,,\quad |f^a(\theta_1)b^b(\theta_2)\rangle\,,\quad |b^a(\theta_1)f^b(\theta_2)\rangle\,. \tag{3.170}$$

The conservation of the fermionic number implies that the first two states scatter into each other and so do the last two. Therefore the 4×4 S-matrix splits into two 2×2 block matrices. Their form is further restricted by supersymmetry. In fact, the S-matrix must commute with the bosonic operator $\mathcal{Q}_+\mathcal{Q}_-$, which in our basis is

$$\mathcal{Q}_+\mathcal{Q}_- = 2\mathrm{i}\begin{pmatrix} 1 & \sinh\frac{1}{2}\theta & 0 & 0 \\ \sinh\frac{1}{2}\theta & -1 & 0 & 0 \\ 0 & 0 & 0 & -\cosh\frac{1}{2}\theta \\ 0 & 0 & -\cosh\frac{1}{2}\theta & 0 \end{pmatrix} \tag{3.171}$$

$(\theta=\theta_1-\theta_2)$. A set of common eigenstates is given by

$$\begin{aligned}
|S^{ab}\rangle &= (\cosh\tfrac{1}{2}\theta)^{-1/2}[\cosh\tfrac{1}{4}\theta\,|b^a(\theta_1)b^b(\theta_2)\rangle + \sinh\tfrac{1}{4}\theta\,|f^a(\theta_1)f^b(\theta_2)\rangle]\,,\\
|T^{ab}\rangle &= (\cosh\tfrac{1}{2}\theta)^{-1/2}[-\sinh\tfrac{1}{4}\theta\,|b^a(\theta_1)b^b(\theta_2)\rangle + \cosh\tfrac{1}{4}\theta\,|f^a(\theta_1)f^b(\theta_2)\rangle]\,,\\
|U^{ab}\rangle &= [|b^a(\theta_1)f^b(\theta_2)\rangle + f^a(\theta_1)b^b(\theta_2)\rangle]/\sqrt{2}\,,\\
|V^{ab}\rangle &= [|b^a(\theta_1)f^b(\theta_2)\rangle - |f^a(\theta_1)b^b(\theta_2)\rangle]/\sqrt{2}\,.
\end{aligned} \tag{3.172}$$

In the new basis, the S-matrix assumes a diagonal form. It is also easy to see that supersymmetry implies

$$\langle S^{cd}|S|S^{ab}\rangle = \langle U^{cd}|S|U^{ab}\rangle\,,\quad \langle T^{cd}|S|T^{ab}\rangle = \langle V^{cd}|S|V^{ab}\rangle\,. \tag{3.173}$$

Hence, the general form of the S-matrix compatible with $\mathrm{O}(N)$ symmetry and supersymmetry can be parameterized as follows

$$\begin{aligned}
\langle S^{cd}|S|S^{ab}\rangle &= \langle U^{cd}|S|U^{ab}\rangle = S_1(\theta)\delta^{ac}\delta^{bd} + S_2(\theta)\delta^{ab}\delta^{cd} + S_3(\theta)\delta^{ad}\delta^{bc}\,,\\
\langle T^{cd}|S|T^{ab}\rangle &= \langle V^{cd}|S|V^{ab}\rangle = T_1(\theta)\delta^{ac}\delta^{bd} + T_2(\theta)\delta^{ab}\delta^{cd} + T_3(\theta)\delta^{ad}\delta^{bc}\,.
\end{aligned} \tag{3.174}$$

The functions S_1 and S_3 (T_1 and T_3) are the transition and reflection amplitudes respectively, while S_2 (T_2) is the amplitude of the annihilation processes. They must satisfy the unitary equations

$$\begin{aligned}
&[S_1(\theta)+S_3(\theta)][S_1(-\theta)+S_3(-\theta)]=1\,,\quad [S_1(\theta)-S_3(\theta)][S_1(-\theta)-S_3(-\theta)]=1\,,\\
&[S_1(\theta)+S_3(\theta)+NS_2(\theta)][S_1(-\theta)+S_3(-\theta)+NS_2(-\theta)]=1\,,
\end{aligned} \tag{3.175}$$

(the same for the T_is), corresponding respectively to the three isospin channels which are the symmetric and traceless channel, the antisymmetric channel and the isosinglet one. The crossing relations are

$$\begin{aligned}
&S_1(\mathrm{i}\pi-\theta)+T_1(\mathrm{i}\pi-\theta)=S_1(\theta)+T_1(\theta)\,,\\
&S_1(\mathrm{i}\pi-\theta)-T_1(\mathrm{i}\pi-\theta)=-\mathrm{i}\tanh\tfrac{1}{2}\theta\,[S_1(\theta)-T_1(\theta)]\,,\\
&S_2(\mathrm{i}\pi-\theta)+T_2(\mathrm{i}\pi-\theta)=S_3(\theta)+T_3(\theta)\,,\\
&S_2(\mathrm{i}\pi-\theta)-T_2(\mathrm{i}\pi-\theta)=-\mathrm{i}\tanh\tfrac{1}{2}\theta\,[S_3(\theta)-T_3(\theta)]\,.
\end{aligned} \tag{3.176}$$

Additional constraints come from the Yang–Baxter equations. Solving the cubic constraints of the Yang–Baxter equations, the S-matrix can be put in the form

$$\begin{aligned}
&S_1(\theta)=\left(1-\frac{\mathrm{i}f}{\sinh\frac{1}{2}\theta}\right)\Xi(\theta)\,,\quad S_2(\theta)=-\frac{2\pi\mathrm{i}}{N-2}\,\frac{S_1(\theta)}{(\mathrm{i}\pi-\theta)}\,,\\
&S_3(\theta)=-\frac{2\pi\mathrm{i}}{N-2}\,\frac{S_1(\theta)}{\theta}\,,\quad T_1(\theta)=\left(1+\frac{\mathrm{i}f}{\sinh\frac{1}{2}\theta}\right)\Xi(\theta)\,,\\
&T_2(\theta)=-\frac{2\pi\mathrm{i}}{N-2}\,\frac{T_1(\theta)}{(\mathrm{i}\pi-\theta)}\,,\quad T_3(\theta)=-\frac{2\pi\mathrm{i}}{N-2}\,\frac{T_1(\theta)}{\theta}\,,
\end{aligned} \tag{3.177}$$

where f is an unknown constant and $\Xi(\theta)$ is a meromorphic function which satisfies the following equations:

$$\Xi(\theta)\Xi(-\theta)=\frac{\theta^2}{\theta^2+\Delta^2}\,\frac{\sinh^2\frac{1}{2}\theta}{\sinh^2\frac{1}{2}\theta+f^2}\,,\quad \Xi(\theta)=\Xi(\mathrm{i}\pi-\theta)\,, \tag{3.178}$$

with $\Delta=2\pi/(N-2)$. We may fix f using the results of the large-N limit on the bound-state structure of the theory. The large-N expansion indicates that the interaction is repulsive in the channels corresponding to S_1, S_2 and S_3 but attractive in the channels T_1, T_2 and T_3 [114]. Hence, the possible bound states may occur only in the T_i channels but not in the S_i ones. To implement this condition, we impose that the prefactor $(1-\mathrm{i}f/\sinh\theta/2)$ appearing in S_i vanishes at the values θ_0 where $\Xi(\theta)$ presents a pole, i.e.

$$f=-\mathrm{i}\sinh\tfrac{1}{2}\theta_0\,. \tag{3.179}$$

In addition, the large-N expansion selects as possible channels where the bound state poles appear those ones of T_2 and T_1-T_3 but not that of T_1+T_3. This gives the following condition on the resonance pole θ_0:

$$1-2\pi\mathrm{i}/(N-2)\theta_0=0\,, \tag{3.180}$$

$$\theta_0=\mathrm{i}\Delta\,. \tag{3.181}$$

Combining with eq. (3.179), we obtain

$$f=\sin[\pi/(N-2)]\,. \tag{3.182}$$

It remains to find $\Xi(\theta)$ by solving the functional equations (3.178), with the requirement that it has a simple pole at $\theta = \mathrm{i}\Delta$. It is convenient to factorize $\Xi(\theta)$ in terms of two crossing symmetric functions $Q(\theta)$ and $Y(\theta)$, i.e. $\Xi(\theta) = Q(\theta)Y(\theta)$, which satisfy

$$Q(\theta)Q(-\theta) = \theta^2/(\theta^2 + \Delta^2)\,, \quad Q(\theta) = Q(\mathrm{i}\pi - \theta)\,, \tag{3.183}$$

$$Y(\theta)Y(-\theta) = \frac{\sinh^2 \frac{1}{2}\theta}{\sinh^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\Delta}\,, \quad Y(\theta) = Y(\mathrm{i}\pi - \theta)\,. \tag{3.184}$$

Let us discuss first the function $Q(\theta)$. The simplest solution of the first equation in (3.183) is given by*)

$$Q^{(0)}(\theta) = \theta/(\theta + \mathrm{i}\Delta)\,. \tag{3.185}$$

However, this does not satisfy the crossing symmetry relation. We may repair that by using

$$Q^{(1)}(\theta) = [\theta/(\theta + \mathrm{i}\Delta)]\,(\mathrm{i}\pi - \theta)/(\mathrm{i}\pi - \theta + \mathrm{i}\Delta)\,, \tag{3.186}$$

which is crossing symmetric but no longer satisfies unitarity. Unitarity is restored by multiplying by another factor

$$Q^{(2)}(\theta) = [\theta/(\theta + \mathrm{i}\Delta)[(\mathrm{i}\pi - \theta)/(\mathrm{i}\pi - \theta + \mathrm{i}\Delta)][(\mathrm{i}\pi + \theta + \mathrm{i}\Delta)/(\mathrm{i}\pi + \theta)]\,. \tag{3.187}$$

But we have spoiled crossing symmetry again. This can be adjusted by including an extra factor. It is therefore clear that only an infinite product of factors can simultaneously satisfy unitarity and the crossing condition. This infinite product can be rearranged in terms of Γ functions. The final result is

$$Q(\theta, \Delta) = R(\theta, \Delta)R(\mathrm{i}\pi - \theta, \Delta)\,, \tag{3.188}$$

where

$$R(\theta, \Delta) = \frac{\Gamma(\Delta/2\pi - \mathrm{i}\,\theta/2\pi)\Gamma(\frac{1}{2} - \mathrm{i}\,\theta/2\pi)}{\Gamma(-\mathrm{i}\theta/2\pi)\Gamma(\frac{1}{2} + \Delta/2\pi - \mathrm{i}\,\theta/2\pi)}\,. \tag{3.189}$$

An equivalent integral representation is by

$$Q(\theta, \Delta) = [\theta/(\theta + \mathrm{i}\Delta]\exp\left(-2\mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x}\,\frac{\sin(\theta x/2\pi)\sinh(\Delta x/4\pi)\sinh[(\frac{1}{4} + \Delta/4\pi)x]}{\cosh\frac{1}{4}x}\right). \tag{3.190}$$

We proceed similarly for the $Y(\theta)$ function, starting from**)

$$Y^{(0)} = \frac{\sinh\frac{1}{2}\theta}{\sinh\frac{1}{2}\theta - \mathrm{i}\sin\frac{1}{2}\Delta}\,. \tag{3.191}$$

*) The other solution, $Q^{(0)} = \theta/(\theta - \mathrm{i}\Delta)$ will be discussed later on.

**) The second minimal solution obtained started from (3.191) will be discussed later on.

The final result is

$$Y(\theta, \Delta) = \mathscr{R}(\theta, \Delta)\mathscr{R}_a(\mathrm{i}\pi - \theta, \Delta) , \tag{3.192}$$

where

$$\mathscr{R}(\theta, \Delta) = \frac{\Gamma(-\mathrm{i}\theta/2\pi)}{\Gamma(\frac{1}{2}-\mathrm{i}\theta/2\pi)} \prod_{l=1}^{\infty} \frac{\Gamma(\Delta/2\pi - \mathrm{i}\theta/2\pi + l)}{\Gamma(\Delta/2\pi - \mathrm{i}\theta/2\pi + l + \frac{1}{2})}$$
$$\times \frac{\Gamma(-\Delta/2\pi - \mathrm{i}\theta/2\pi + l - 1)\Gamma^2(-\mathrm{i}\theta/2\pi + l - \frac{1}{2})}{\Gamma(-\Delta/2\pi - \mathrm{i}\theta/2\pi + l - \frac{1}{2})\Gamma(-\mathrm{i}\theta/2\pi + l - 1)} . \tag{3.193}$$

The above factorization of $\Xi(\theta)$ in terms of the functions $Q(\theta)$ and $Y(\theta)$ reflects an important property of the $\mathrm{O}(N)$ supersymmetric non-linear sigma model, namely its close relation both to supersymmetric sine–Gordon and to the Gross–Neveu model. In fact, $Q(\theta)$ is the same function which enters the S-matrix of the bosonic $\mathrm{O}(N)$ non-linear sigma model, solved by Zamolodchikov and Zamolodchikov [108]. On the other hand, the function $Y(\theta)$ appears in the scattering amplitude of the supersymmetric sine–Gordon. The Lagrangian of this model is

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}\bar{\psi}\, \mathrm{i}\partial_\mu \gamma^\mu \psi + (1/4\beta^2)\cos^2\beta\phi - \tfrac{1}{2}(\cos\beta\phi)\bar{\psi}\psi . \tag{3.194}$$

The invariant amplitudes for the elementary boson and fermion particles of the supersymmetric sine–Gordon which satisfy the constraints of supersymmetry, the Yang–Baxter equations, the unitarity and crossing symmetry relations are given by*) [111]

$$S_{\mathrm{b,f}}(\theta) = [1 \pm \mathrm{i}(\sin\tfrac{1}{2}\Delta)/\sinh\tfrac{1}{2}\theta)]Y(\theta) , \tag{3.195}$$

where the expression with the plus sign holds for the bosonic case and the other one for the fermionic case.

Hence, the S-matrix of the elementary excitations of the supersymmetric $\mathrm{O}(N)$ non-linear sigma model is a product of two S-matrices, one of them related to the ordinary bosonic non-linear σ model, the other one related to the supersymmetric sine–Gordon. The scattering amplitudes of the fermionic bound states of the supersymmetric σ model, with mass ratios

$$m_k/m_1 = \sin[k\pi/(N-2)]/\sin[\pi/(N-2)] , \quad k = 1, 2, \ldots, \tfrac{1}{2}N - 1 , \tag{3.196}$$

can be computed via the usual bootstrap equations.

As a last remark, we come back to the ambiguity inherent in solving the functional equations (3.183) and (3.184). Equation (3.183) admits two possible minimal solutions: the first is given in terms of $R(\theta, \Delta)$, eq. (3.189), the second one is constructed by means of $R(\theta, -\Delta)$. Analogously, eq. (3.184) has two minimal solutions: the first one is given in terms of $\mathscr{R}(\theta, \Delta)$, eqs. (3.193), the second one is obtained changing $\Delta \to -\Delta$ into $\mathscr{R}(\theta, \Delta)$. Therefore, for $\Xi(\theta)$ we can have the following four solutions:

*) The complete S-matrix of the supersymmetric sine–Gordon has been discussed in ref. [119].

$$
\begin{aligned}
&\Xi_1(\theta) = Q(\theta, \Delta)\, Y(\theta, \Delta)\,, \quad \Xi_2(\theta) = Q(\theta, \Delta)\, Y(\theta, -\Delta)\,, \\
&\Xi_3(\theta) = Q(\theta, -\Delta)\, Y(\theta, \Delta)\,, \quad \Xi_4(\theta) = Q(\theta, -\Delta)\, Y(\theta, -\Delta)\,.
\end{aligned}
\tag{3.197}
$$

However, the physical requirement is that $\Xi(\theta)$ contains a single pole at $\theta = \mathrm{i}\Delta$. Therefore neither $\Xi_2(\theta)$ nor $\Xi_3(\theta)$ satisfy this condition because the former does not have any pole and the latter, on the contrary, has a double pole at $\theta = \mathrm{i}\Delta$. About $\Xi_1(\theta)$ and $\Xi_4(\theta)$, they actually coincide. The interpretation of this fact is the aforementioned relation of the $\mathrm{O}(N)$ supersymmetric sigma model with the non-linear sigma model or with the Gross–Neveu model. Choosing $\Xi_1(\theta)$, we assign the origin of the pole at $\theta = \mathrm{i}\Delta$ to the supersymmetric sine–Gordon model, whereas the original $\mathrm{O}(N)$ symmetry is realized in terms of the usual bosonic non-linear sigma model, which has no bound states. On the other hand, choosing for $\Xi(\theta)$ the solution $\Xi_4(\theta)$, the $\mathrm{O}(N)$ symmetry is realized in terms of the fermions of the Gross–Neveu model, which has instead bound states. Hence, the supersymmetric $\mathrm{O}(N)$ non-linear sigma model can be considered equivalently as the supersymmetric version of both the Gross–Neveu model and the non-linear sigma model.

The scattering theory so far discussed is believed to describe correctly the dynamics of the model for $N > 4$. For $N \leq 4$, the S-matrix presents some drawbacks which spoils its interpretation. For instance, for $N \to 4$, the mass of one of the bound states goes to zero and for $N \to 3$ the theory does not have an enlarged supersymmetry which is known, on the contrary, to exist [112].

4. Thermodynamical Bethe ansatz

The infinite-volume thermodynamics of a massive QFT can be computed in terms of its S-matrix. This idea, originally proposed in ref. [45], has recently found wide applications for the $(1+1)$-dimensional integrable theories essentially for two reasons: (a) the special properties of the S-matrices for these theories; (b) the possibility to extend the techniques of the Thermodynamical Bethe Ansatz (TBA) proposed for the non-relativistic models by Yang and Yang [46, 47] to the relativistic cases. By means of the TBA, the derivation of the thermodynamical quantities of a purely elastic scattering theory reduces to the solution of a set of coupled nonlinear integral equations for the one-particle excitation energies and the rapidity distributions of the particles of the theory.

Originally, the TBA equations for relativistic models with diagonal S-matrices has been proposed by Zamolodchikov [48] and several applications have been discussed by him and other authors [49–52]. Recent developments consist in the generalization of the TBA equations to the non-diagonal S-matrices [53–55] and to the excited states [56–58].

4.1. Casimir energy

The basic geometry we consider is that of a cylinder with periodic boundary condition both in the R and L direction (see fig. 15). There are two alternative ways to define a Euclidean QFT on such a geometry [48]: since the R-direction and the L-direction play a symmetric role, we can choose equivalently one of them as quantization axis, the other one playing the role of Euclidean time*⁾.

*⁾ In the context of CFT, this is nothing but the invariance under the modular group [24].

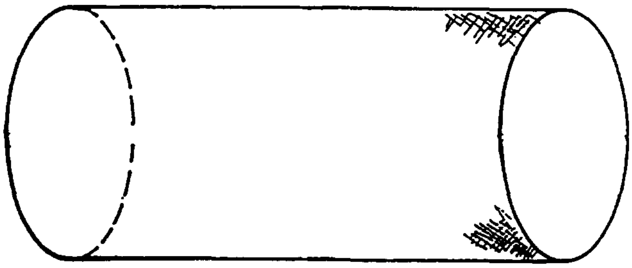


Fig. 15. Toroidal geometry with orthogonal circles of circumference R and L.

Hence, the partition function assumes the two alternative expressions:

$$Z(R, L) = \mathrm{Tr}\,\exp(-L\mathcal{H}_R)\,, \tag{4.1}$$

or

$$Z(R, L) = \mathrm{Tr}\,\exp(-R\mathcal{H}_L)\,. \tag{4.2}$$

$\mathcal{H}_R$ and $\mathcal{H}_L$ are the Hamiltonians for the system quantized along the R-axes and the L-axes, respectively. In the limit $L \to \infty$, the partition function (4.1) is dominated by the ground state energy $E_0(R)$ of $\mathcal{H}_R$ and we have

$$Z(R, L) \simeq \exp[-LE_0(R)]\,. \tag{4.3}$$

On the other hand, looking at the second expression (4.2) of the partition function, $L \to \infty$ results in the thermodynamical limit of the one-dimensional quantum system defined on the L-axes at temperature $T \equiv 1/R$. In this limit, the partition function becomes

$$Z(R, L) \simeq \exp[-LRf(R)]\,, \tag{4.4}$$

where $f(R)$ is the bulk free energy of the system at temperature $1/R$. Comparing the two expressions, the relationship between the Casimir energy $E_0(R)$ of the finite volume and the free energy $f(R)$ of the infinite one-dimensional system is given by

$$E_0(R) = Rf(R)\,. \tag{4.5}$$

A useful parameterization of $E_0(R)$ is given by

$$E_0(R) = -\pi\tilde{c}(r)/6R\,, \tag{4.6}$$

where $r = m_1 R$ and m_1 is the lowest mass in the theory. As we will show later on, the scaling function $\tilde{c}(r)$ can be determined from the scattering data using the thermodynamical Bethe ansatz equations. In the limit $r \to 0$, conformal invariance predicts for the ground state energy [24]

$$E_0(R) = (2\pi/R)(\Delta_{\min} + \bar{\Delta}_{\min} - \tfrac{1}{12}c)\,, \tag{4.7}$$

and $\tilde{c}(r)$ should thus reduce to the effective central charge

$$\lim_{r\to 0} \tilde{c}(r) = c - 24\Delta_{\min} . \tag{4.8}$$

This limit establishes a remarkable link between the scattering data of a given massive theory and CFT governing its ultraviolet behaviour. Several deformations of minimal models of CFT have been checked in this way (see chapter 6). In this chapter we will derive the TBA equations following the original works of Yang and Yang [47] and Zamolodchikov [48].

4.2. The relativistic Bethe wave function

Let us consider an integrable QFT on a circle of length L. We assume that the spectrum consists of a set of particles A_a ($a = 1, 2, \ldots, n$) of mass m_a and that the scattering amplitudes are diagonal and characterized by their phase-shifts, $S_{ab}(\beta) = \exp[\mathrm{i}\delta_{ab}(\beta)]$. The lowest mass fixes the correlation length of the system, i.e. $\xi = 1/m_1$.

The Hilbert space of such a theory is quite simple. Given any N-particle state, the integrability of the model ensures that the identities of the particles and their individual momenta are preserved all over its time evolution. Hence, it is meaningful to associate to any state of the system a wave function $\Psi(x_1, x_2, \ldots, x_N)$. In the configuration space of an N-particle state we select $N!$ regions where all the particles are far apart, i.e. $|x_i - x_{i+1}| \gg \xi$, and consequently the relativistic effects due to off-mass shell processes can be neglected. Each such domain is labelled by an ordering $x_{i_1} \ll x_{i_2} \ll x_{i_3} \cdots \ll x_{i_N}$ of the coordinates of the particles and here the wave function is simply

$$\Psi(x_{i_1}, x_{i_2}, \ldots, x_{i_N}) = \prod_{k=1}^{N} \exp(\mathrm{i}p_{i_k} x_{i_k}) . \tag{4.9}$$

The interchanging of two particle positions maps one domain into another and every transition results in a multiplication of the wave function by the corresponding scattering amplitude. Imposing periodic (anti-periodic) boundary conditions for the total wave function of bosons (fermions), we arrive at the following quantization equations*) for the momenta p_i:

$$\exp(\mathrm{i}p_i L) \prod_{j\neq i}^{N} S(\theta_i - \theta_j) = \pm 1 , \quad i = 1, 2, \ldots, N , \tag{4.10}$$

or

$$m_i L \sinh \theta_i + \sum_{j\neq i}^{N} \delta_{ij}(\theta_i - \theta_j) = 2\pi n_i , \tag{4.11}$$

where $\delta_{ij}(\theta) = -\mathrm{i} \ln S_{ij}(\theta)$. The numbers $\{n_i\}$ are integers for bosons and semi-integers for fermions. Together with the set of rapidities satisfying eq. (4.11), they label the Bethe ansatz states $|n_1, \theta_1; n_2, \theta_2; \ldots; n_N, \theta_N\rangle$. The corresponding energy and momentum are given by

$$E = \sum_{i=1}^{N} m_i \cosh \theta_i , \quad p = \sum_{i=1}^{N} m_i \sinh \theta_i . \tag{4.12}$$

*) In absence of any interaction, we recover the usual quantization condition $p_i = 2\pi n_i / L$.

4.3. Selection rules

The Bethe wave function must be symmetric (antisymmetric) under the interchange of two identical bosons (fermions) of the same rapidity. Therefore, we have to take into account some selection rules arising from the identity of the particles. For the diagonal S-matrices, the unitarity condition implies

$$S^2_{aa}(0) = 1 , \qquad (4.13)$$

and two different cases may occur:

(i) The first case,

$$S_{aa}(0) = -1 , \qquad (4.14)$$

results in a wave function which is antisymmetric under the exchange of the two particles with the same rapidity. If they are bosons, this is clearly incompatible with the Bose statistics. Therefore, two bosons A_a cannot have the same rapidity and each value of rapidity can be occupied by at most one particle. This means that all the integers $n_i^{(a)}$ of the species a, appearing in (4.11), must be different. On the other hand, if the identical particles are fermions no such restriction arises.

(ii) In the second case, we have

$$S_{aa}(0) = 1 . \qquad (4.15)$$

Here the situation is reversed with respect to the previous one. Two identical bosons can take the same value of the rapidity and there is no restriction on $n_i^{(a)}$. On the contrary, if the two identical particles are fermions, each value of the rapidity can be occupied by only one particle, i.e. all the integers $n_i^{(a)}$ of the species a are different.

For the sake of simplicity, in the following we restrict our attention only to the case of S-matrices of bosonic particles with $S_{aa} = -1$. This is the situation that generally occurs in the context of perturbed CFT. The discussion of the general case can be found in ref. [48].

4.4. Derivation of thermodynamics

In the thermodynamic limit, both L and all N_a go to infinity but the densities N_a/L remain finite. It is thus convenient to introduce continuous rapidity densities of particles $\rho_a^{(r)}$. They are defined as the number of particles A_a with rapidity between θ and $\theta + \Delta\theta$ divided by $L\,\Delta\theta$. In terms of them, the energy per unit length of the system becomes

$$E[\rho^{(r)}] = \sum_{a=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}\theta\, \rho_a^{(r)} m_a \cosh\theta . \qquad (4.16)$$

The Bethe ansatz equations can be written in the form

$$m_a \sinh\theta_i^{(a)} + \sum_{b=1}^{n} (\delta_{ab} * \rho_b^{(r)})(\theta) = 2\pi n_i^{(a)}/L , \qquad (4.17)$$

where $*$ denotes the convolution,

$$(f*g)(\theta)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi}\,f(\theta-\theta')g(\theta')\,. \tag{4.18}$$

Whenever $n_i^{(a)}$ is a set of admissible quantum numbers, the corresponding solutions $\theta_i^{(a)}$ will be referred to as *roots* of species a and their densities are given by $\rho_a^{(\mathrm{r})}$. However, these equations have also solutions for those values of $n_i^{(a)}$ which do not correspond to actual states. Such values of θ are called *holes* of species a and the corresponding density is denoted by $\rho_a^{(\mathrm{h})}$. The absence of some integers in the sequence of the actual quantum numbers of the physical states, i.e. the existence of hole solutions for the Bethe ansatz equations, follows from the discussion we made on the selection rules: for instance, in the bosonic case with $S(0)=-1$, choosing an ordering for the $\theta_i^{(a)}$ variables, the $n_i^{(a)}$ corresponding to the physical states must form a progression of strictly increasing numbers and, therefore, some integers may be skipped in this sequence.

In the thermodynamic limit, there is thus a density distribution of roots as well as of holes. The total density ρ_a of occupied and empty levels of the particle A_a is equal to the derivative of the left-hand side of eq. (4.17),

$$\rho^{(a)}(\theta)=\rho_a^{(\mathrm{r})}+\rho_a^{(\mathrm{h})}=(m_a/2\pi)\cosh\theta+\sum_{b=1}^{n}(\varphi_{ab}*\rho_a^{(\mathrm{r})})(\theta)\,, \tag{4.19}$$

where

$$\varphi_{ab}(\theta)=\mathrm{d}\delta_{ab}(\theta)/\mathrm{d}\theta\,. \tag{4.20}$$

By the unitarity condition of the S-matrix, these functions satisfy $\varphi_{ab}(-\theta)=\varphi_{ab}(\theta)$. For an S-matrix built up in terms of the functions $s_{x_i}(\theta)$,

$$S_{ab}(\theta)=\prod_{x_i}s_{x_i}(\theta)\,,$$

their explicit expression was worked out in chapter 3. We recall that they are $2\pi\mathrm{i}$ periodic functions with a Fourier expansion given by

$$\varphi_{ab}(\theta)=-\sum_{s=1}^{\infty}\varphi_{ab}^{(s)}\exp(-s|\theta|)\,,\quad \varphi_{ab}^{(s)}=2\sum_{x_i}\sin(s\pi x_i)\,. \tag{4.21}$$

Because of the existence of holes, many microscopic states of approximately the same energy can be described by the same densities ρ_a and $\rho_a^{(\mathrm{r})}$. Their number is given by

$$\Omega_a=[L(\rho_a(\theta)\Delta\theta]!/[L\rho_a^{(\mathrm{r})}(\theta)\Delta\theta]![L\rho_a^{(\mathrm{h})}(\theta)\Delta\theta]!\,. \tag{4.22}$$

Correspondingly, the entropy $\mathscr{S}=\ln(\Pi_a\,\Omega_a)$ per unit length is equal to

$$\mathscr{S}[\rho,\rho^{(r)}]=\sum_{a=1}^{n}\int_{-\infty}^{\infty}\mathrm{d}\theta\,[\rho_a\ln\rho_a-\rho_a^{(\mathrm{r})}\ln\rho_a^{(\mathrm{r})}-(\rho_a-\rho_a^{(\mathrm{r})})\ln(\rho_a-\rho^{(\mathrm{r})}a)]\,. \tag{4.23}$$

The free energy per unit length is thus given by

$$f[\rho, \rho^{(r)}] = E[\rho^{(r)}] - R^{-1}\mathcal{S}[\rho, \rho^{(r)}] . \tag{4.24}$$

In order to obtain the thermodynamical properties of the system in thermal equilibrium at temperature $T = 1/R$, we have to minimize the free energy with respect to ρ_a and $\rho_a^{(r)}$ subject to the constraint (4.19). The extremum condition is given by

$$m_a R \cosh\theta = \ln[(\rho_a - \rho_a^{(r)})/\rho_a^{(r)}] + \sum_{b=1}^{n} (\varphi_{ab} * \ln[\rho_b/(\rho_b - \rho_b^{(r)})])(\theta) . \tag{4.25}$$

Defining the functions $\varepsilon_a(\theta)$ and $L_a(\theta)$ as

$$\rho_a(\theta)/\rho_a^{(r)}(\theta) = 1 + \exp(\varepsilon_a(\theta)) , \quad L_a(\theta) = \ln[1 + \exp(-\varepsilon_a(\theta))] , \tag{4.26}$$

eqs. (4.25) becomes

$$\hat{m}_a r \cosh\theta = \varepsilon_a(\theta) + \sum_{b=1}^{n} (\varphi_{ab} * L_b)(\theta) , \tag{4.27}$$

where $\hat{m}_a = m_a/m_1$. The equilibrium free energy is given by

$$f(R) = -\frac{1}{2\pi R} \sum_{a=1}^{n} m_a \int_{-\infty}^{\infty} L_a(\theta) \cosh\theta \, d\theta . \tag{4.28}$$

Using eqs. (4.5) and (4.6), we get the following expression for the scaling function $\tilde{c}(r)$,

$$\tilde{c}(r) = \frac{3}{\pi^2} \sum_{a=1}^{n} \hat{m}_a r \int_{-\infty}^{\infty} L_a(\theta) \cosh\theta \, d\theta . \tag{4.29}$$

4.5. Conformal limit of purely elastic scattering theories

The conformal limit of the massive field theory is reached when $r \to 0$. To evaluate the scaling function $\tilde{c}(r)$ in this limit, we need to analyze some properties of the integral equations (4.27). The solutions $\varepsilon_a(\theta)$ are even functions of θ and, when $r \to 0$, they become constant in the region $-\ln(2/r) \ll \theta \ll \ln(2/r)$. Their limiting constant values ε_a satisfy the transcendental equations

$$\varepsilon_a = \sum_{b=1}^{n} N_{ab} \ln[1 + \exp(-\varepsilon_b)] , \tag{4.30}$$

where N_{ab} is a symmetric matrix given by

$$N_{ab} = -\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \varphi_{ab}(\theta) = -\frac{1}{2\pi} [\delta_{ab}(+\infty) - \delta_{ab}(-\infty)] . \tag{4.31}$$

For large values of θ, the right-hand side of eq. (4.27) can be written as

$$\hat{m}_a r \cosh\theta \sim \tfrac{1}{2}\hat{m}_a\, e^{\theta} = \hat{m}_a \exp[\theta - \ln(2/r)]\,, \tag{4.32}$$

and therefore the r-dependence of the functions $\varepsilon_a(\theta)$ reduces to a simple shift*)

$$\theta \to \theta - 2/r\,. \tag{4.33}$$

For $r \to 0$, their behaviour near the edge of the interval is universal and dictated by the equation

$$\hat{m}_a\, e^{\theta} = \tilde{u}_a(\theta) + \sum_{b=1}^{n} (\varphi_{ab} * \tilde{L}_b)(\theta)\,. \tag{4.34}$$

The $\tilde{\varepsilon}_a(\theta)$ assume the constant values ε_a for $\theta \ll 2/r$ and go exponentially to infinity when $\theta \to \infty$. The corresponding $\tilde{L}_a(\theta)$ interpolate between zero and the limit values given by eq. (4.30). For this reason the universal functions $\tilde{\varepsilon}_a$ are called the *kink* solutions [48]. In terms of them, the value of the scaling function $\tilde{c}(r)$ at zero assumes the form

$$\tilde{c}(0) = \frac{6}{\pi^2} \sum_{a=1}^{n} \int_0^{\infty} d\theta\, \tilde{L}_a(\theta) \tilde{m}_a\, e^{\theta}\,. \tag{4.35}$$

Substituting for $\tilde{m}_a\, e^{\theta}$ the derivative of the left-hand side of eq. (4.34),

$$e^{\theta} = \frac{d\tilde{\varepsilon}_a(\theta)}{d\theta} - \sum_{b=1}^{n} \left(\varphi_{ab} * \frac{\exp(-\tilde{\varepsilon}_b)}{1+\exp(-\tilde{\varepsilon}_b)}\, \frac{d\tilde{\varepsilon}_b}{d\theta}\right)(\theta)\,, \tag{4.36}$$

we have

$$\tilde{c}(0) = \frac{6}{\pi^2} \sum_{a=1}^{n} \int_0^{\infty} d\theta\, \tilde{L}_a(\theta) \left[\frac{d\tilde{\varepsilon}_a(\theta)}{d\theta} - \sum_{b=1}^{n} \left(\varphi_{ab} * \frac{\exp(-\tilde{\varepsilon}_b)}{1+\exp(\tilde{\varepsilon}_b)}\, \frac{d\tilde{\varepsilon}_b}{d\theta}\right)(\theta)\right]. \tag{4.37}$$

Since the $\tilde{\varepsilon}_a$ are monotonic increasing functions, the first term on the right-hand side becomes simply

$$\int_0^{\infty} d\theta\, \tilde{L}(\theta)\, \frac{d\tilde{\varepsilon}_a(\theta)}{d\theta} = \int_{\varepsilon_a}^{\infty} d\varepsilon\, \tilde{L}(\varepsilon)\,. \tag{4.38}$$

The convolution term in (4.37) can be replaced using the same equation (4.34). After an integration by part, the final result is given by [48, 51]

$$\tilde{c}(0) = \sum_{a=1}^{n} \tilde{c}_a(\varepsilon_a)\,, \tag{4.39}$$

*) We discuss the behaviour of the $\varepsilon_a(\theta)$ functions for positive values of θ, the corresponding pattern for negative values of θ can be obtained by their parity properties.

where

$$\tilde{c}_a(\varepsilon_a) = \frac{6}{\pi^2}\left(\int_{\varepsilon_a}^{\infty} \mathrm{d}x \ln(1+\mathrm{e}^{-x}) + \tfrac{1}{2}\varepsilon_a \ln(1+\mathrm{e}^{-\varepsilon_a})\right) = \frac{6}{\pi^2} L(1/(1+\mathrm{e}^{\varepsilon a})) , \tag{4.40}$$

and $L(x)$ is the Rogers dilogarithm function [59]

$$L(x) = -\frac{1}{2}\int_0^x \mathrm{d}t \left(\frac{\ln t}{1-t} + \frac{\ln(1-t)}{t}\right). \tag{4.41}$$

Hence, the computation of the central charge of the underlying CFT of a massive field theory with a purely elastic diagonal S-matrix reduces to solve the transcendental equations (4.30) and to plug their solutions ε_a into eq. (4.39).

4.6. Universal bulk energy term

In a theory with a massive scale, additivity of energy predicts a linear growing of the ground state energy with respect to the dimension of the system

$$E_0 \sim \mathscr{E}_0 R . \tag{4.42}$$

$\mathscr{E}_0$ is interpreted as the singular part of the infinite bulk energy which arises because long-range fluctuations are present in the system. Usually its value is not universal, being related to the ultraviolet regularization of the theory. However, in a perturbed CFT the scheme of regularization is fixed by the requirement that the off-critical quantities reduce in the ultraviolet limit to the conformal data and a universal $\mathscr{E}_0$, depending only on the scattering data, can be extracted. Indeed, since E_0 is related to the scaling function $\tilde{c}(r)$, eq. (4.6), the bulk energy $\mathscr{E}_0$ is given by

$$\mathscr{E}_0 = -\frac{\pi}{12} m_1^2 \frac{1}{r}\frac{\mathrm{d}\tilde{c}}{\mathrm{d}r}\bigg|_{r=0} . \tag{4.43}$$

For the evaluation of this limit, let us introduce the functions [48]

$$\psi_a(\theta) = (\partial_r + r^{-1}\partial_\theta)\varepsilon_a(\theta) . \tag{4.44}$$

$\varepsilon_a(\theta)$ depends on r through eq. (4.27). It is easy to see that ψ_a satisfies the integral equations

$$\psi_a(\theta) = \hat{m}_a \mathrm{e}^{\theta} + \sum_{b=1}^{n} (\varphi_{ab} * \psi_b/(\mathrm{e}^{\varepsilon_b}+1))(\theta) . \tag{4.45}$$

Using eq. (4.29), we have

$$\frac{1}{r}\frac{\mathrm{d}\tilde{c}(r)}{\mathrm{d}r} = -\frac{3}{\pi^2}\sum_{a=1}^{n}\int_{-\infty}^{\infty} \mathrm{d}\theta\, \hat{m}_a \mathrm{e}^{-\theta} \frac{\psi_a(\theta)}{\mathrm{e}^{\varepsilon(\theta)}+1} . \tag{4.46}$$

In the limit $r \to 0$, the integrand is localized near the edge of the flat region and its behaviour is fixed by the kink functions $\tilde{L}_a(\theta)$. Therefore we obtain

$$\frac{1}{r}\frac{\mathrm{d}\tilde{c}(r)}{\mathrm{d}r}\bigg|_{r=0} = \frac{3}{\pi^2}\sum_{a=1}^{n} \tilde{m}_a \int_{-\infty}^{\infty} \mathrm{d}\theta\, \mathrm{e}^{-\theta}\, \partial_\theta \tilde{L}_a(\theta) \equiv -\frac{3}{\pi^2}\sum_{a=1}^{n} \hat{m}_a T_a\,. \tag{4.47}$$

In order to compute the right-hand side of this equation, we proceed as follows. Firstly, looking at the asymptotic expansion for $\theta \to -\infty$ of the convolution term, we get

$$\sum_{b=1}^{n} (\varphi_{ab} * \tilde{L}_b)(\theta) = -\varepsilon_a + \frac{\mathrm{e}^{\theta}}{2\pi}\sum_{b=1}^{n} \varphi_{ab}^{(1)} T_b + \cdots, \tag{4.48}$$

where $\varphi_{ab}^{(1)}$ is the first mode in the Fourier expansion of these functions, given in (4.21). Matching with the exponential term in eq. (4.36), we obtain

$$\sum_{b=1}^{n} \varphi_{ab}^{(1)} T_b = 2\pi\,. \tag{4.49}$$

Secondly, using eq. (3.84), we have

$$\sum_{b=1}^{n} \varphi_{ab}^{(1)} T_b = \varphi_{11}^{(1)} \hat{m}_a \sum_{b=1}^{n} \hat{m}_b T_b\,, \tag{4.50}$$

where $\varphi_{11}^{(1)}$ is the corresponding quantity for the lightest particle. Therefore, the universal bulk energy term is fixed by the S-matrix of the lightest particle via

$$\mathscr{E}_0 = m_1^2 / 2\varphi_{11}^{(1)}\,. \tag{4.51}$$

A direct measurement of this quantity can be obtained by means of a numerical algorithm which is discussed in the next chapter.

5. The conformal space truncation approach

The scaling region around the fixed points of the minimal models of CFT can be efficiently investigated by means of a numerical approach. The method, known as Conformal Space Truncation Approach (CSTA), has been suggested by Yurov and Zamolodchikov [133] and later on developed and applied by other authors [134–137, 139, 162]. It consists in studying the numerical spectrum of the off-critical Hamiltonian in a finite volume (a circle). The perturbed Hamiltonian acts on the Hilbert space defined by the fixed point action and its matrix elements can be extracted from the conformal field theory. Truncation of the space at a suitable level reduces the problem to a numerical diagonalization of a finite dimensional Hamiltonian.

The truncation method is particularly useful for the analysis of the massive deformations of the CFT action and in the following we will restrict our considerations only to these situations. Several non perturbative parameters of the theories can be extracted in this way, among them the mass gaps and the bulk energy. A direct numerical measurement of the exact S-matrix of the integrable models becomes also available by analyzing the threshold lines of the massive models.

5.1. Truncated off-critical Hamiltonian

Let us consider a CFT which is perturbed by a relevant scaling field Φ with angular momentum $\Delta - \bar{\Delta} = 0$ and scaling dimension $\Delta + \bar{\Delta} = x$. A very convenient approach to analyze such a theory is to use a Hamiltonian formalism. To this aim, let us consider the model defined on an infinitely long cylinder, i.e. the strip $-\infty < u < \infty$, $0 \leq v \leq R$ with periodic boundary conditions. The conformal transformation

$$\omega \equiv u + \mathrm{i}v = (R/2\pi) \ln z , \tag{5.1}$$

maps the plane onto such a strip. The whole information about a statistical theory is encoded in its Hamiltonian (the logarithm of the transfer matrix) which can be written as

$$H_\lambda = H_0 + \lambda V . \tag{5.2}$$

The Hamiltonian of the fixed point can be expressed in terms of the Virasoro generators L_0, $\bar{L}_0$ and the central charge c [24],

$$H_0 = (2\pi/R)(L_0 + \bar{L}_0 - \tfrac{1}{12}c) . \tag{5.3}$$

The interaction V is formally given by

$$V = \int_0^R \Phi(u, v)\, \mathrm{d}v . \tag{5.4}$$

By a scaling argument, the spectrum of H_λ depends on only the dimensionless variable λR^{2-x}. Therefore, we can set $\lambda = 1$ and study the spectrum as a function of R.

Since Φ is a scalar operator, both H_0 and V commute with the momentum operator on the strip,

$$K = (2\pi/R)(L_0 - \bar{L}_0) . \tag{5.5}$$

The eigenstates of H_0, which are the conformal states of the fixed point, are assumed to form a basis of the Hilbert space. They are labelled by their conformal dimensions, their momentum and additional quantum numbers which distinguish degenerate states at the conformal point. The matrix elements of V in this basis are easily computed. In fact, the space integration in (5.4) ensures the momentum conservation,

$$\langle \phi_i | V | \phi_j \rangle = (R/2\pi) \langle \phi_i | \Phi(0, 0) | \phi_j \rangle \delta_{K_i, K_j} , \tag{5.6}$$

and it remains to compute conformal three-point functions in the plane. By virtue of the infinite-dimensional conformal symmetry, these functions in turn can be expressed in terms of a finite number of dimensionless structure constants of the primary fields which characterize the short distance behaviour of CFT [12]. An algorithm that performs all these computations has been written by Lässig and Mussardo [135].

The analysis of the spectrum of the infinite-dimensional Hamiltonian (5.3) is impracticable. However, we can construct a sequence of truncated Hamiltonians H_Λ with a finite number of elements. The truncation can be performed according to the conformal dimensions of the states, i.e. putting an upper bound Λ in the energies. The sequence H_Λ of Hamiltonians approximates the infinite-dimensional Hamiltonian (5.3) and converges to it in the limit $\Lambda \to \infty$. Therefore, after a suitable truncation of the Hilbert space, it remains to numerically diagonalize the Hamiltonian H_Λ. A typical spectrum obtained by the diagonalization of a truncated Hamiltonian is shown in fig. 16. In order to extract sensible physical quantities and to control the truncation effects of the Hilbert space, we need to know some general features of the energy levels.

Without truncation, the energy levels E_i on a strip of width R satisfy the scaling law

$$E_i(R, \lambda) = (2\pi/R) f_i(R/\xi) , \tag{5.7}$$

i.e. the dependence on the coupling constant λ is contained only in the correlation length $\xi(\lambda)$. From a quantum field theory point of view, ξ is the inverse of the lowest mass gap m_1 and is a non perturbative parameter. Simple expressions for the energy level are obtained in two different asymptotic regimes, $R \ll \xi$ and $R \gg \xi$.

In the ultraviolet regime, $R \ll \xi$, the system is conformal invariant and the spectrum of (5.3) coincides with that of H_0, hence f_i are directly related to the conformal data,

$$E_i \sim (\pi/R)(\Delta_i + \bar{\Delta}_i - \tfrac{1}{12}c) \quad (R \ll \xi) . \tag{5.8}$$

In the infrared regime, $R \gg \xi$, we are essentially dealing with R/ξ copies of independent systems. The

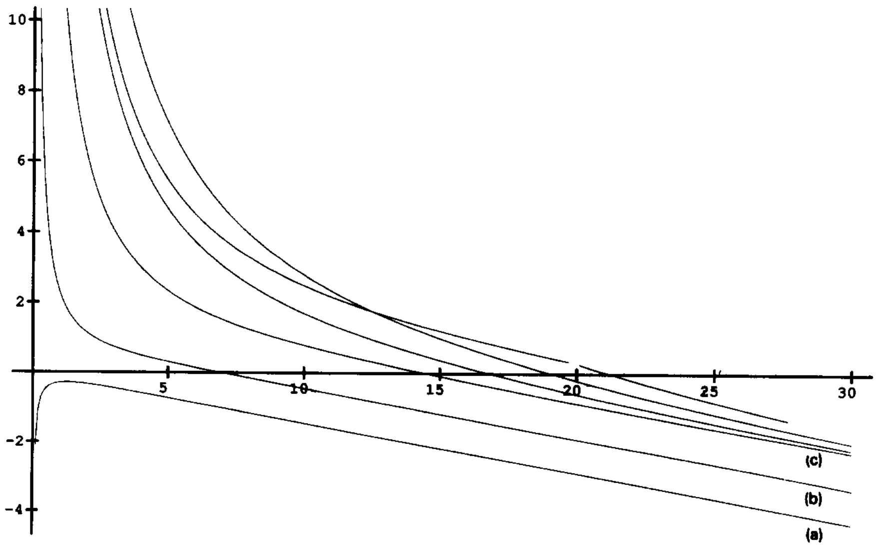


Fig. 16. A typical spectrum $E_i(R)$ obtained from TCSA. (a) Ground state energy, (b) one-particle line, and (c) threshold line.

additivity property of the energy leads to a linear behaviour of the energy levels and their general form is given by

$$E_i(R) = (\varepsilon_0/\xi^2)R + M_i \,. \tag{5.9}$$

The dimensionless constant ε_0 is the anti-bulk vacuum energy which, for the integrable massive field theories, can be extracted from the thermodynamical Bethe ansatz as

$$\varepsilon_0 = -\mathscr{E}_0 \,, \tag{5.10}$$

where $\mathscr{E}_0$ is given in terms of the scattering data by eq. (4.51). M_i is the (multi)particle mass term of the ith level.

The previous discussion is slightly modified in the presence of a truncation of the Hilbert space. The truncation results in the introduction of an additional scale parameter ρ related to the upper eigenvalues of the unperturbed Hamiltonian H_0 and the scaling form for the energies accordingly becomes

$$E_i^{\mathrm{tr}}(R, \lambda, \rho) = (2\pi/R) f_i^{\mathrm{tr}}(R/\xi, R/\rho) \,. \tag{5.11}$$

The correlation length ξ rules the crossover from the ultraviolet regime to the infrared regime, while ρ characterizes the onset of truncation effects. H_0 is now a bounded operator. This implies that for R larger than ρ, the eigenvalues we extract from the diagonalization of (5.3) are essentially those of V. In this unphysical regime, instead of the behaviour predicted by eq. (5.9), the energies scale like

$$E_i^{\mathrm{tr}} \sim R^{1-x} \quad (R \gg \rho) \,. \tag{5.12}$$

To extract reliable information about the infrared region, a sufficient number of states has to be included such that $\rho \gg \xi$. Consequently, in the spectrum of the truncated levels we have to distinguish three different scaling regimes (5.8), (5.9) and (5.12), separated by two crossover regions at $R \sim \xi$ and $R \sim \rho$. These three regimes can be easily identified by means of an effective scaling exponent [134]

$$a_i(R) = R \,\mathrm{d} \ln E_i/\mathrm{d}R \,. \tag{5.13}$$

Measurements of non perturbative parameters in the infrared region must be performed in the region where a_i reaches a plateau at the value $a_i = 1$. In the unphysical regime, we have instead $a_i = 1 - x$.

In virtue of its formulation, the CSTA is not restricted to the analysis of integrable deformations but can be applied as well to the investigation of a generic deformation of CFT. However, an interesting situation occurs when the deformation of the conformal theory happens to be integrable. In this case, several intersections between different eigenvalues are observed at finite values of R [133–136]. It is a general result of quantum mechanics that for a Hamiltonian which depends on a parameter (in our case R), the condition for possible crossings of two energy levels is that they belong to different irreducible representations of the symmetry group of the Hamiltonian. In other words, generic eigenvalue lines (in the absence of any symmetry) do not cross [140]. Therefore, the energy level intersections observed in several integrable deformations of CFT were interpreted as the signal of the infinite dimensional symmetry underlying these models.

5.2. Measurement of the S-matrix

For the integrable deformations of CFT, the truncation approach together with the Bethe ansatz allows a direct numerical measurement of the elastic S-matrix. The basic idea is due to Lüscher [141] and further developed by Lüscher and Wolff [142]. It consists in applying the Bethe ansatz equations to the case of the two-particle states $|A_a A_b\rangle$. For any observed level with zero total momentum, relative momentum k and energy E,

$$E = \sqrt{m_a^2 + k^2} + \sqrt{m_m^2 + k^2}\,, \tag{5.14}$$

the phase shift $\delta_{ab}(k)$ is given by

$$\exp[2\mathrm{i}\,\delta_{ab}(k)] = \exp(-\mathrm{i}kR)\,. \tag{5.15}$$

The energy E and the mass gaps m_a and m_b can be determined by the CSTA and correspondingly the momentum k extracted from eq. (5.14). Hence, the scattering phase $\delta_{ab}(k)$ can be measured directly from eq. (5.15).

Lässig and Martins [136] have applied this method to several integrable deformations of minimal conformal models and confirmed the conjectured scattering theories of these quantum field theories.

6. Elastic S-matrices of minimal models away from criticality

In this chapter we present the S-matrices proposed for the description of massive integrable perturbations of some minimal models of conformal field theories. The starting point of almost all examples is the knowledge of the first degrees (spins) of the additional conserved currents away from the critical point. The minimal form of the S-matrix may be fixed on the basis of symmetry and analyticity arguments alone.

Initially we will consider simple bootstrap models associated with integrable deformations of the non-unitary minimal models $\mathcal{M}_{2,2n+3}$ and $\mathcal{M}_{3,5}$. The corresponding S-matrices satisfy the unitarity condition $S(\theta)S(-\theta) = 1$ but they are not *one-particle unitary*. This means they may have negative residues at some of the poles.

Then we will analyze the integrable massive deformations of the Ising model. Although the Ising model is the simplest critical system, its perturbation by a magnetic field gives rise to a bootstrap model which closes with eight particles and exhibits a very rich structure of higher order poles. This is related to a hidden E_8 symmetry of the model. The thermal perturbation of the Ising model is much simpler because it reduces to free massive Majorana fermions. However, in the spin sector of the model, the theory is not free and the S-matrix is equal to $S = -1$.

Next to the Ising model, its tricritical version also presents an interesting pattern away from criticality. There are three different integrable deformations of the action of the critical point. The first is a thermal perturbation which gives rise to a bootstrap system with seven particles. The second one is a sub-leading energy deformation which preserves the supersymmetry of the conformal model. The last one is the sub-leading magnetic deformation which presents an asymmetric scattering of the kinks K_+ and K_-, with one bound state in the channel $|K_- K_+\rangle$ but none in the channel $|K_+ K_-\rangle$.

We also discuss the thermal perturbations of the three-state Potts model and of its tricritical version.

Our last example is the exact factorized S-matrix of the quantum field theory corresponding to the scaling limit of the $O(n)$ vector model with $-2 \leq n \leq 2$ away from the critical point. In the limit $n \to 0$, this model describes the scaling behaviour of self-avoiding polymer chains.

We point out that the S-matrices so far proposed to describe the on-mass shell data of the off-critical theories are based on some conjectures and therefore they need additional checks. Their correct identification may be supported by means of the Thermodynamical Bethe Ansatz (TBA) and the Conformal Space Truncation Approach (CSTA).

6.1. *S-matrix of the Yang–Lee edge singularity*

The simplest S-matrix of a system with bootstrap interaction was found by Cardy and Mussardo [148] and identified with the scattering amplitude of the massive excitation of the Yang–Lee model away from criticality.

The Yang–Lee (YL) singularity describes the critical behaviour of an Ising model in a pure imaginary field $\mathrm{i}h$ [151, 152]. For $h > h_c$, the zeros of the partition function are dense on the imaginary h-axis. The Landau–Ginzburg Lagrangian of this model is given by [153]

$$\mathscr{L} = \int [\tfrac{1}{2}(\partial\phi)^2 - \mathrm{i}(h - h_c)\phi - \mathrm{i}g^3]\, \mathrm{d}^d x \,. \tag{6.1}$$

This theory at criticality ($h = h_c$) has only one relevant operator, namely the field ϕ itself. Cardy [154] showed that in two dimensions this property is satisfied by the minimal model $\mathscr{M}_{2,5}$ with central charge $c = -22/5$ and one relevant operator with scaling dimension equal to $2\Delta = -2/5$. The effective central charge of this model is thus $\tilde{c} = 2/5$. This theory is not unitary, i.e. the Hilbert space contains states with negative norm.

The scaling region around the fixed point of the YL model is a one-dimensional (complex) space spanned by the coupling constant of the relevant field ϕ. Taking this coupling constant purely imaginary, the corresponding Landau–Ginzburg Lagrangian is given by eq. (6.1). Using the counting argument, one can explicitly establish for the off-critical system the existence of conserved currents with spins

$$s = 1, 5, 7, 11, 13, 17, 19, 23 \,. \tag{6.2}$$

Cardy and Mussardo [148] have conjectured the existence of similar currents for each value of s not dividable by 2 or 3. This pattern of the conserved spins allows the existence of a particle A which appears as a bound state of itself, i.e. the S-matrix S_{AA} has a pole at $s = m_A^2$. In terms of the rapidity variable, the s-channel pole is located at $\theta = \frac{2}{3}\,\mathrm{i}\pi$. Crossing symmetry fixes the pole in the t-channel to be at $\theta = \frac{1}{3}\,\mathrm{i}\pi$. Assuming no further poles*), the unique solution of the bootstrap equation

$$S_{AA}(\theta) = S_{AA}(\theta - \tfrac{1}{3}\mathrm{i}\pi) S_{AA}(\theta + \tfrac{1}{3}\mathrm{i}\pi) \tag{6.3}$$

*) This assumption can be supported by the analysis of the non-relativistic Born approximation of the Lagrangian (6.1), which shows that the exchange of the A particle in the t-channel leads to a repulsive potential.

is given by

$$S_{AA} = \tanh[\tfrac{1}{2}(\theta + \tfrac{2}{3}i\pi)]/\tanh[\tfrac{1}{2}(\theta - \tfrac{2}{3}i\pi)] = f_{2/3} . \tag{6.4}$$

Comparing with eq. (6.1), one can easily extract the value of the renormalized coupling constant*)

$$-ig^2 = 3m^4 \sinh(\tfrac{2}{3}i\pi) = i\tfrac{3}{2}\sqrt{3}\, m^4 . \tag{6.5}$$

An intriguing observation is that the residue at the pole has the *opposite sign* to the one expected in a unitary theory. On the other hand, the S-matrix (6.4) satisfies by construction the unitarity condition $S(\theta)S(-\theta) = 1$. The solution of this paradox and, consequently, the compatibility of the above two different formulations of unitarity is the following [148]. Let us define an operator C ($C^2 = 1$) by

$$C\phi C = -\phi . \tag{6.6}$$

The Hamiltonian obtained by (6.1) is not Hermitian but rather satisfies the equation $H^\dagger = CHC$. The Fock space states are created by the repetitive action of the field ϕ on the vacuum, hence they are all eigenstates of C with eigenvalues $(-1)^N$, where N is the particle number. Since H is not hermitian, its left eigenstates $\langle n_\ell|$ are not the adjoints of the right eigenstates, but they are given by $\langle n_\ell| = \langle n_\mathrm{r}|C$. The completeness equation reads

$$\sum_n |n_\mathrm{r}\rangle\langle n_\ell| = \sum_n |n_\mathrm{r}\rangle\langle n_\mathrm{r}|C . \tag{6.7}$$

The unitarity of the S-matrix, namely

$$SS^\dagger = 1 , \tag{6.8}$$

deals only with the fact that the in-states and the out-states form a basis in the Hilbert space and does not concern whether or not the Hamiltonian is hermitian. However, when we insert the completeness equation into (6.8), each term will be weighted by $(-1)^N$. This is the reason of the wrong sign for the residue at the pole. Therefore the S-matrix is unitary in the sense that it preserves the probability but is not *one-particle unitary*.

The simplicity of this model selects it as an ideal theoretical "laboratory" for explicit checks on off-critical statistical systems. The ultraviolet limit of the massive field theory defined by the above S-matrix has been studied by Zamolodchikov [48] by means of the thermodynamical Bethe ansatz. In this case, we have only one pseudo-energy ε_1 which satisfies

$$\varepsilon_1 = \ln(1 + e^{-\varepsilon_1}) . \tag{6.9}$$

The positive solution of this equation is

$$\varepsilon_1 = \ln[\tfrac{1}{2}(\sqrt{5} + 1)] . \tag{6.10}$$

*) We renormalize the theory on the mass-shell, i.e. the renormalized coupling constant is defined by i times the residue of the pole.

Inserting into eq. (4.39), we get for the effective central charge the value

$$\tilde{c} = (6/\pi^2)L(1/(1+e^{\varepsilon_1})) = \tfrac{2}{5}\,, \tag{6.11}$$

in agreement with the CFT prediction. Using eq. (4.51), the universal bulk energy term is predicted to be

$$\varepsilon_0 = -\tfrac{1}{12}\sqrt{3}m^2\,. \tag{6.12}$$

A numerical analysis of the off-critical Yang–Lee model and a detailed comparison with the scattering theory of the model has been performed in ref. [133].

6.2. Integrable deformations of non-unitary models $\mathcal{M}_{2,2n+3}$

The Yang–Lee model belongs to the series of non-unitary minimal models $\mathcal{M}_{2,2n+3}$. They have the most asymmetrical Kac table consisting of only one column. There are n conformal fields in addition to the identity operator, all of them with negative conformal weights,

$$\Delta_{1,r} = \Delta_{1,2n+3-r} = -(r-1)(2n+2-r)/2(2n+3)\,, \quad r = 0, 1, \ldots, n\,. \tag{6.13}$$

The central charge c and the effective central charge $\tilde{c}$ are given by

$$c = -2n(6n+5)/(2n+3)\,, \quad \tilde{c} = 2n/(2n+3)\,. \tag{6.14}$$

The fusion rules of the conformal fields are particularly simple,

$$[\phi_{1,1+a}] \times [\phi_{1,1+b}] = \sum_{c=|a-b|}^{a+b} [\phi_{1,1+c}]\,, \tag{6.15}$$

where the sum over c is in steps of 2.

6.2.1. $\Phi_{1,3}$ deformation

The scattering theory originating from this deformation has been discussed in ref. [150]. The starting point is the knowledge of the conserved spins. An interesting result is obtained from the analysis of the Verma modules. In fact, in all these models, the second independent null-state of the identity field occurs at level $2n+2$. This means that the dimension of the space $\hat{\Lambda}_{2n+2}$ is lowered by 1 with respect to that of a non-minimal model. However, the second independent null vector of the field $\Phi_{1,3}$ exists at level $2n$, hence its descendent at level $2n+1$ is a total derivative and does not affect the dimension of the space $\hat{\Phi}_{2n+1}$. It is therefore plausible that the holomorphic part

$$\oint T_{s+1}\,\mathrm{d}z \tag{6.16}$$

($s+1 = 2n+2$) of the conserved charge Q_{2n+1} vanishes due to the second null-vector state in the Verma module of the identity field. An explicit check of this result in the cases $n = 1, \ldots, 4$ was done in ref. [150], where it was shown that the above component (6.16) is proportional, up to total derivative field, to the null-vector state. The conjectured set of conserved spins is thus given by

$$s = 1, 3, \ldots, 2n-1, 2n+3, \ldots, 4n+1 \quad (\mathrm{mod}\, 4n+2)\,. \tag{6.17}$$

Based on the above analysis of the conserved spins, Freund et al. [150] proposed that the massive theory associated to the $\Phi_{1,3}$ deformation of $\mathcal{M}_{2,2n+3}$ involves n massive particles with chain-like bootstrap fusions,

$$A_1 \times A_1 \to A_2\,, \quad A_2 \times A_2 \to A_3\,, \quad \ldots, \quad A_n \times A_n \to A_1\,. \tag{6.18}$$

Using the result of section 3.5.1, a solution of the consistency equations is given by

$$\bar{u}^{k}_{k,k+1} = k\pi/(2n+1)\,, \quad k = 1, 2, \ldots, n\,. \tag{6.19}$$

The mass spectrum then reads

$$m_a = \sin[a\pi/(2n+1)]\,, \quad a = 1, 2, \ldots, n\,. \tag{6.20}$$

The scattering amplitude of the lightest particle A_1 is

$$S_{11}(\theta) = f_{2/(2n+1)}(\theta)\,, \tag{6.21}$$

and the remaining amplitudes are obtained by induction

$$S_{ab}(\theta) = f_{|a-b|/(2n+1)} f_{(a+b)/(2n+1)} \prod_{k=1}^{\min(a,b)-1} (f_{(|a-b|+2k)/(2n+1)})^2\,. \tag{6.22}$$

The simple pole of the first factor (for $a \neq b$),

$$\theta = \mathrm{i}u^{|a-b|}_{ab} = \mathrm{i}[1 - |a-b|/(2n+1)]\pi\,, \tag{6.23}$$

is related to the particle $A_{|a-b|}$ appearing as a bound state in this scattering process. The simple pole of the second factor,

$$\theta = \mathrm{i}u^{n(a,b)}_{ab} = \mathrm{i}(a+b)\pi/(2n+1)\,, \tag{6.24}$$

is due to the particle of type $n(a, b) = \min(a + b, 2n + 1 - a - b)$. The double poles of the remaining functions are required by the closure of the bootstrap. They can be correctly identified as multiple re-scattering processes along the way described in section 3.6.2.

The TBA analysis permits us to obtain the correct effective central charge of these models [51]. The matrix N_{ab} entering the transcendental equations (4.30) has the form

$$N_{ab} = 2\min(a, b) - \delta_{ab}\,, \tag{6.25}$$

and the pseudo-energies ε_a, solution of the equations

$$\varepsilon_a = \sum_{b=1}^{n} N_{ab} \ln(1 + e^{-\varepsilon_b}) , \tag{6.26}$$

are given by

$$e^{\varepsilon_a} = \frac{\sin[a\pi/(2n+3)] \sin[(a+2)\pi/(2n+3)]}{\sin^2[\pi/(2n+3)]} . \tag{6.27}$$

The resulting value for $\tilde{c}$ follows from a sum rule of the Rogers dilogarithm functions,

$$\tilde{c} = \frac{6}{\pi^2} \sum_{a=1}^{n} L(1/(1 + e^{\varepsilon_a})) = 2n/(2n+3) . \tag{6.28}$$

6.2.2. $\Phi_{1,2}$ deformation

The $\Phi_{1,2}$ deformation of the non-unitary minimal models $\mathcal{M}_{2,2n+1}$ gives rise to an interesting situation where the underlying soliton structure only appears through the bound states, called breathers. This means that the solitons play the role of quarks, which do not appear as asymptotic states, whereas the breathers are the "mesonic" particles of the theory, sensible to measurement. This picture deduces from the RSOS reduction of the Zhiber–Mikhailov–Shabat model with respect to the quantum group $SL(2)_q$, found by Smirnov [93] (see section 3.8.2). This integral deformation of $\mathcal{M}_{2,2n+1}$ has been analysed in full detail in refs. [161, 163].

The spectrum consists in $n-1$ breathers b_i. The S-matrix of the fundamental particle b_1 is given by

$$S_{b_1 b_1}(\theta) = f_{1/3n}(\theta) f_{2/3}(\theta) f_{-(n-1)/3n(\theta)} . \tag{6.29}$$

From that, we extract the poles $u_{11}^1 = \frac{2}{3}\pi$ and $u_{11}^2 = \pi/3n$. This first pole is interpreted as a bound state corresponding to the fusion $b_1 b_1 \to b_1 \to b_1 b_1$. This S-matrix has the "Φ^3" property and therefore a current with spin $s = 3$ is absent from the set of conserved quantities (see section 3.5). The second pole corresponds to the breather b_2. Its mass is given by

$$m_2/m_1 = \sin(2\pi/6n)/\sin(\pi/6n) . \tag{6.30}$$

Using the bootstrap equations [30, 108], we can compute the amplitude $S_{b_1 b_2}$,

$$S_{b_1 b_2}(\theta) = f_{3/6n}(\theta) f_{1/6n}(\theta) f_{(2n+1)/6n}(\theta) f_{(-2n+3)/6n}(\theta) . \tag{6.31}$$

If n is larger than 3, we get a new particle b_3 at the pole $u_{12}^3 = \pi/2n$. Treating similarly the scattering of b_1 and b_3, a new bound state b_4 appears and so on. By induction, we obtain the whole sequence S_{b_1, b_k} $(k = 1, 2, \ldots \leq n-1)$

$$S_{b_1 b_k} = f_{(k+1)/6n} f_{(k-1)/6n} f_{(2n+k-1)/6n} f_{(-2n+k+1)/6n} . \tag{6.32}$$

The remaining scattering amplitudes are obtained by successive application of the bootstrap equations.

A compact form for the general S-matrix S_{b_p,b_k} (for $p, k = 1, 2, \ldots, n-1$) is given by

$$S_{b_p k_k} = f_{(k+p)/6n}(f_{(k+p-2)/6n} \cdots f_{(k-p+2)/6n})^2 f_{(k-p)/6n}$$
$$\times f_{(2n+k+p-2)/6n} \cdots f_{(2n+k-p+2)/6n} f_{(2n+k-p)/6n}$$
$$\times f_{(-2n+k-p+2)/6n} \cdots f_{(-2n+k+p-2)/6n} f_{(-2n+k+p)/6n} . \tag{6.33}$$

The mass spectrum is given by

$$m_k = \sin(k\pi/6n) , \quad k = 1, 2, \ldots, n-1 . \tag{6.34}$$

Notice that the first line in (6.33) corresponds exactly to the structure of the S-matrices found for the $\Phi_{1,3}$ deformation of these models [148, 150]. Moreover, the number of poles in the physical sheet given by the functions in the second line of (6.33), coincides with the number of zeros given by the functions of the third line. The matrix N_{ab} which enters the thermodynamical Bethe ansatz (TBA) is thus given by

$$N_{ab} = 2 \min(a, b) - \delta_{ab} . \tag{6.35}$$

The solutions of the equations for the pseudo-energies

$$\varepsilon_a = \sum_{b=1}^{n-1} N_{ab} \log(1 + \mathrm{e}^{-\varepsilon_b}) \tag{6.36}$$

are given by

$$\varepsilon^{\varepsilon_a} = \frac{\sin[a\pi/(2n+1)] \sin[(a+2)\pi/(2n+1)]}{\sin^2[\pi/(2n+1)]} , \quad a = 1, \ldots, n-1 . \tag{6.37}$$

Inserting them into the expression of the partial central charge contributions and using the sum rules of the Rogers dilogarithm function $L(x)$, we get

$$\tilde{c} = \frac{6}{\pi^2} \sum_{a=1}^{n-1} L(1/(1 + \mathrm{e}^{\varepsilon_a})) = 2(n-1)/(2n+1) . \tag{6.38}$$

This coincides with the effective central charge $\tilde{c} \equiv c - 24\Delta_{\min}$ of the models $\mathcal{M}_{2,2n+1}$. A more detailed analysis of the TBA for finite temperature and a comparison with the numerical spectrum extracted from the CSTA were carried out in ref. [163]. In figs. 17a–17c the first 20 lines of the scaling functions $RE(R)$ for the models $\mathcal{M}_{2,2n+1}$ for $n = 3, 4, 5$ are presented. The long dashed lines correspond to the one-particle states, with normalization $m_1 = 1$. The corresponding mass ratios are in agreement with eq. (6.34). The short dashed lines characterize the lowest two-particle state (threshold line). From fig. 17 one can see that only the two lowest one-particle states (m_1, m_2) are below the threshold line. The threshold line, in the ultraviolet limit, originates from the conformal family of most relevant fields $\Phi_{1,n}$ [136] and for $n \geq 3$ this produces non-trivial crossings with the one-particle states m_i, $i \geq 3$ (open circles in fig. 17).

For large R, the energy levels $E_i(R)$ tend to straight lines with the same slope, $E_i(R) \sim -\varepsilon_0 R$, where ε_0 is the vacuum bulk energy. The numerical results found in ref. [163] for ε_0 are in perfect agreement with the theoretical one predicted by eq. (4.51), namely

$$(8\varepsilon_0)^{-1} = \sin(\pi/3n) + \sin(\tfrac{1}{3}\pi) + \sin(\pi/3n - \tfrac{1}{3}\pi)\,. \tag{6.39}$$

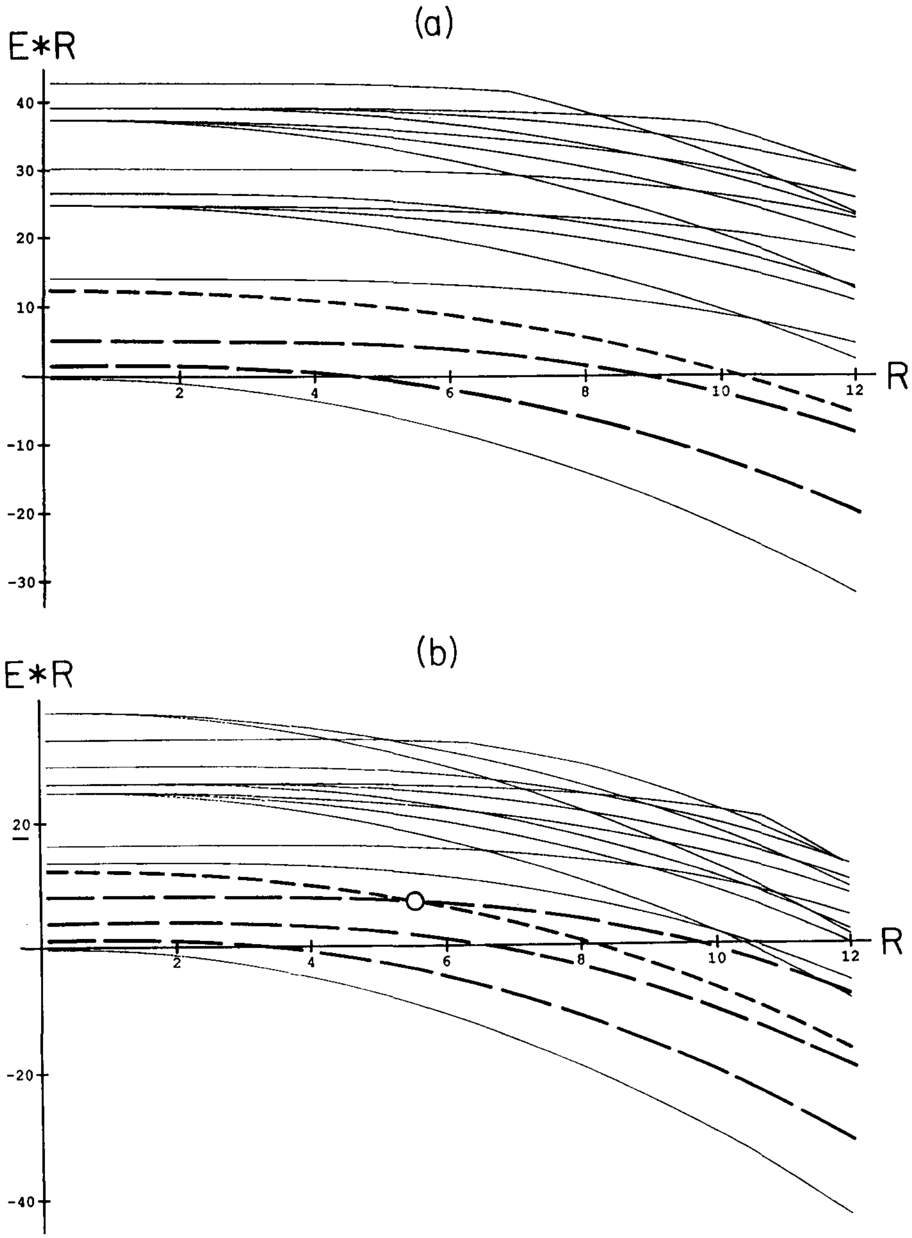


Fig. 17. First 20 levels of the scaling function $R * E_i(R)$ for the models (a) $\mathcal{M}_{2,7}$, (b) $\mathcal{M}_{2,9}$ and (c) $\mathcal{M}_{2,11}$. The short dashed lines correspond to the threshold line. The long dashed lines correspond to the masses m_i of the particles present in the spectrum. The open circles characterize level crossings between the threshold line and the one-particle state m_i, $i \geq 3$.

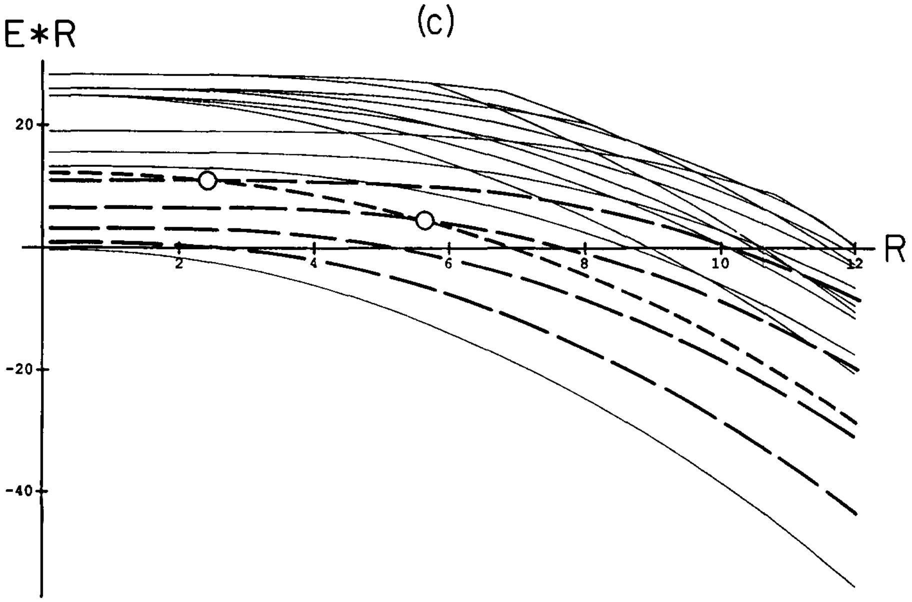


Fig. 17. (cont.).

The TBA equations for the ground state energy at temperature $T = 1/R$ are

$$\varepsilon_i(\theta) + \frac{1}{2\pi}\sum_{j=1}^{N}\int_{-\infty}^{\infty} d\theta'\, \varphi_{ij}(\theta-\theta')L_j(\varepsilon_j) = m_i R\cosh\theta\,, \quad i=1,2,\ldots,N\,,$$
$$E_0(R) = R^{-1}F(R)\,, \quad F(R) = -\frac{1}{2\pi}\sum_{i=1}^{N} m_i R\int_{-\infty}^{\infty} d\theta \cosh\theta\, L_i(\varepsilon_i)\,. \tag{6.40}$$

$F(r)$ is a function of the variable $r = m_1 R$ (m_1 is the lightest mass) and its general expression reads

$$F(r) = -\tfrac{1}{12}\tilde{c} + (\varepsilon_0/2\pi)r^2 + \sum_{k=1}^{\infty} f_i(r^{y(n)})^k\,, \tag{6.41}$$

where $y(n) = 2(1-\Delta_{1,2}) = 6n/(2n+1)$. In eq. (6.41) the first coefficient f_1 is proportional to the scale between the lowest mass m_1 and the perturbing coupling constant λ, namely

$$\lambda = [(2\pi)^{1-y}/C_{\Phi_{1,n}\Phi_{1,2}\Phi_{1,n}}]f_1 m_1^y\,, \tag{6.42}$$

where $C_{\Phi_{1,n}\Phi_{1,2}\Phi_{1,n}}$ are the structure constants of the $\mathcal{M}_{2,2n+1}$ models. Solving numerically eq. (6.40) for small r, the authors of ref. [163] computed the exponent y and found a good agreement with the prediction of conformal perturbation theory.

This analysis has confirmed that the ultraviolet limit of the massive theories defined by the scattering amplitudes (6.33) is controlled by the fixed points of the series $\mathcal{M}_{2,2n+1}$ and therefore the S-matrix (6.33) defines these theories away from criticality.

6.3. Integrable deformations of the non-unitary model $\mathcal{M}_{3,5}$

The Kac table of the non-unitary minimal model $\mathcal{M}_{3,5}$ contains four primary operators with anomalous dimensions given in table 4. With the identification

$$1 = \Phi_{0,0}\,, \quad \sigma = \Phi_{-1/20,-1/20}\,, \quad \varphi = \Phi_{1/5,1/5}\,, \quad \psi = \Phi_{3/4,3/4}\,, \tag{6.43}$$

the fusion algebra can be summarized as

$$\begin{aligned} &\psi \times \psi = 1\,, \quad \sigma \times \sigma = 1 + \varphi\,, \quad \varphi \times \varphi = 1 + \varphi\,, \\ &\psi \times \sigma = \varphi\,, \quad \psi \times \varphi = \sigma\,, \quad \varphi \times \sigma = \sigma + \psi\,. \end{aligned} \tag{6.44}$$

We can consistently introduce a Z_2 parity, identifying the fields 1 and φ as even operators while σ and ψ as odd ones. The central charge and effective central charge of the model are equal to

$$c = -\tfrac{3}{5}\,, \quad \tilde{c} = c - 24\Delta_{\text{min}} = \tfrac{3}{5}\,. \tag{6.45}$$

The counting argument predicts conserved currents with higher spins for any deformation of the fixed-point action given by the fields σ, φ and ψ

$$\sigma \to s = 1, 5, 11\,, \quad \varphi \to s = 1, 3, 5, 7, 9, 11\,, \quad \psi \to s = 1, 5, 7, 9, 11\,. \tag{6.46}$$

Therefore, the scaling region is described by three different integrable field theories.

6.3.1. σ deformation

From the analysis carried out by Smirnov [93] (see section 3.8.2), in this case the kinks disappear after the restriction and the spectrum consists only of breather particles. The S-matrix of the lowest-mass particle is given by

$$S_{11}(\theta) = f_{2/7}(\theta) f_{2/3}(\theta) f_{-1/21}(\theta)\,. \tag{6.47}$$

Notice the appearance of the function $f_{2/3}(\theta)$ which indicates the Φ^3 property of the model, in agreement with the set of conserved spins for this deformation. The bootstrap program for this model has not been carried out and therefore the complete spectrum of the theory is presently unknown.

Table 4
Kac table of the model $\mathcal{M}_{3,5}$ and Z_2 parity for the operators in the $\mathcal{M}_{3,5}$ model.

Kac table		ϕ	Z_2
$\frac{3}{4}$	0	1	$+$
$\frac{1}{5}$	$-\frac{1}{20}$	σ	$-$
$-\frac{1}{20}$	$\frac{1}{5}$	φ	$+$
0	$\frac{3}{4}$	ψ	$-$

6.3.2. φ deformation

The corresponding massive theory – proposed in ref. [91] – is related to a reduction of the sine–Gordon model. In this reduction, certain degrees of freedom of the original sine–Gordon model become frozen and the solitons combine into a single scalar particle a, with an S-matrix given by

$$S_{aa}(\theta) = \tanh[\tfrac{1}{2}(\theta - \mathrm{i}\tfrac{1}{2}\pi)]\,. \tag{6.48}$$

There is no additional bound state since the above S-matrix has only an unphysical pole at $\theta = -\mathrm{i}\frac{1}{2}\pi$. The thermodynamical Bethe ansatz computations for the scattering theory defined by (6.48) were performed in ref. [52]. The analysis confirms that the ultraviolet limit of this massive field theory is governed by the conformal model $\mathcal{M}_{3,5}$. The same conclusion is reached by studying this deformation by means of the conformal space truncation approach [164]. The first eigenvalues of the spectrum, as functions of R, are shown in fig. 18. The CSTA data present two degenerate ground states, which exponentially approach each other

$$E_1(R) - E_0(R) \sim \mathrm{e}^{-mR} \tag{6.49}$$

(m is the mass of the fundamental particle a) and no additional bound states. The slope of the massive levels in this case is positive [52]

$$\varepsilon_0 = \tfrac{1}{4}m^2\,. \tag{6.50}$$

The thermodynamical consequence of the positivity of ε has been investigated in ref. [164].

6.3.3. ψ deformation

The operator ψ appears in two places in the Kac table: at the position $(1,4)$ and at $(2,1)$. As field $\Phi_{2,1}$, ψ belongs to the class of integrable deformations of a conformal minimal model discussed by Smirnov [93] (see section 3.8.2). The complete solution and the novelty of this deformation has been

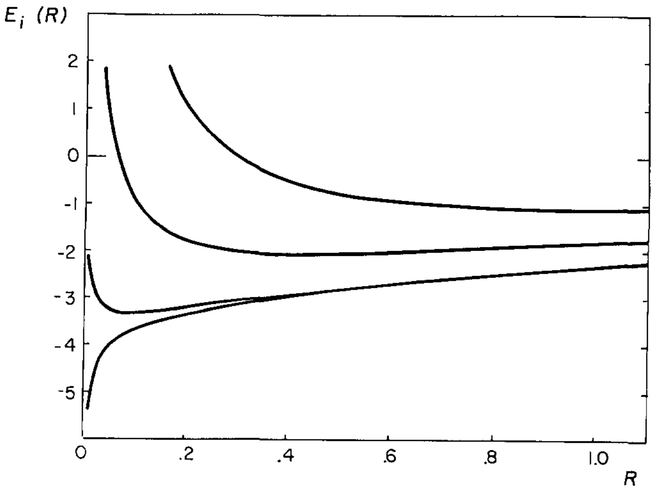


Fig. 18. First energy levels of the Hamiltonian associated to the ψ deformation of the model $\mathcal{M}_{3,5}$.

discussed in ref. [164]. The massive excitations of the models are kink-like and there are no additional breather bound states. One important feature of this scattering theory is the presence of poles with real residues and with "wrong" signs. This implies that the corresponding underlying field theory has some of the coupling constants purely imaginary.

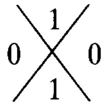
The RSOS reduction of the model related to its quantum group structure, selects out as possible one-particle states the vectors: $|K_{01}\rangle$, $|K_{10}\rangle$ and $|K_{11}\rangle$. All of them have the same mass m. The RSOS restriction projects out the state $|K_{00}\rangle$, which cannot appear neither as an asymptotic state nor as an intermediate one. Therefore a basis for the two-particle asymptotic states is given by

$$|K_{01}K_{10}\rangle\,,\quad |K_{01}K_{11}\rangle\,,\quad |K_{11}K_{11}\rangle\,,\quad |K_{11}K_{10}\rangle\,,\quad |K_{10}K_{01}\rangle\,. \tag{6.51}$$

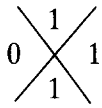
The scattering processes are

$$\begin{aligned}
|K_{01}(\theta_1)K_{10}(\theta_2)\rangle &= S_{00}^{11}(\theta_1-\theta_2)|K_{01}(\theta_2)K_{10}(\theta_1)\rangle\,,\\
|K_{01}(\theta_1)K_{11}(\theta_2)\rangle &= S_{01}^{11}(\theta_1-\theta_2)|K_{01}(\theta_2)K_{11}(\theta_1)\rangle\,,\\
|K_{11}(\theta_1)K_{10}(\theta_2)\rangle &= S_{01}^{11}(\theta_1-\theta_2)|K_{11}(\theta_2)K_{10}(\theta_1)\rangle\,,\\
|K_{11}(\theta_1)K_{11}(\theta_2)\rangle &= S_{11}^{11}(\theta_1-\theta_2)|K_{11}(\theta_2)K_{11}(\theta_1)\rangle + S_{11}^{10}(\theta_1-\theta_2)|K_{10}(\theta_2)K_{01}(\theta_1)\rangle\,,\\
|K_{10}(\theta_1)K_{01}(\theta_2)\rangle &= S_{11}^{00}(\theta_1-\theta_2)|K_{10}(\theta_2)K_{01}(\theta_1)\rangle + S_{11}^{10}(\theta_1-\theta_2)|K_{11}(\theta_2)K_{11}(\theta_1)\rangle\,.
\end{aligned} \tag{6.52}$$

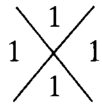
Explicitly, the above amplitude are given by



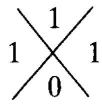
$$= S_{00}^{11}(\theta) = -\mathrm{i}\tfrac{1}{2}S_0(\theta)\sinh(\tfrac{3}{5}\theta + \mathrm{i}\tfrac{3}{5}\pi)\,, \tag{6.53a}$$



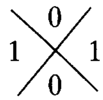
$$= S_{01}^{11}(\theta) = \mathrm{i}\tfrac{1}{2}S_0(\theta)\sinh(\tfrac{3}{5}\theta - \mathrm{i}\tfrac{3}{5}\pi)\,, \tag{6.53b}$$



$$= S_{11}^{11}(\theta) = -\mathrm{i}\tfrac{1}{2}S_0(\theta)\,\frac{\sin(\frac{2}{5}\pi)}{\sin(\frac{1}{5}\pi)}\,\sinh(\tfrac{3}{5}\theta + \mathrm{i}\tfrac{1}{5}\pi)\,, \tag{6.53c}$$



$$= S_{11}^{01}(\theta) = -\tfrac{1}{2}S_0(\theta)\left(\frac{\sin(\frac{2}{5}\pi)}{\sin(\frac{1}{5}\pi)}\right)^{1/2}\sinh(\tfrac{3}{5}\theta)\,, \tag{6.53d}$$



$$= \mathrm{S}_{11}^{00}(\theta) = \mathrm{i}\tfrac{1}{2}S_0(\theta)\,\frac{\sin(\frac{2}{5}\pi)}{\sin(\frac{1}{5}\pi)}\,\sinh(\tfrac{3}{5}\theta - \mathrm{i}\tfrac{1}{5}\pi)\,. \tag{6.53e}$$

$S_0(\theta)$ is the following function:

$$S_0(\theta) = -[\sinh(\tfrac{3}{10}(\theta - \mathrm{i}\pi))\sinh(\tfrac{3}{10}(\theta - \tfrac{2}{3}\pi\mathrm{i}))]^{-1}w(\theta,\tfrac{3}{5})w(\theta,\tfrac{1}{10})w(\theta,\tfrac{7}{10})\,, \tag{6.54}$$

where

$$w(\theta, x) = \sinh(\tfrac{3}{10}\theta + \mathrm{i}\pi x)/\sinh(\tfrac{9}{10}\theta - \mathrm{i}\pi x) .$$

The above amplitudes satisfy the unitarity equations:

$$\begin{gathered} S_{11}^{00}(\theta)S_{11}^{00}(-\theta) + S_{11}^{01}(\theta)S_{11}^{10}(-\theta) = 1 , \quad S_{11}^{10}(\theta)S_{11}^{01}(-\theta) + S_{11}^{11}(\theta)S_{11}^{11}(-\theta) = 1 , \\ S_{11}^{10}(\theta)S_{11}^{00}(-\theta) + S_{11}^{11}(\theta)S_{11}^{10}(-\theta) = 0 , \quad S_{10}^{11}(\theta)S_{10}^{11}(-\theta) = 1 , \quad S_{00}^{11}(\theta)S_{00}^{11}(-\theta) = 1 . \end{gathered} \tag{6.55}$$

They are periodic functions along the imaginary axis of θ with period $10\pi\mathrm{i}$. Restricting our attention to the physical sheet, $0 \leq \mathrm{Im}\, \theta \leq \mathrm{i}\pi$, the poles of the S-matrix are located at $\theta = \frac{2}{3}\pi\mathrm{i}$ and $\theta = \frac{1}{3}\pi\mathrm{i}$. The first pole corresponds to a bound state in the direct channel while the second one is the singularity due to the particle exchanged in the crossed process (see fig. 19). The residues at $\theta = \frac{2}{3}\pi\mathrm{i}$ are given by

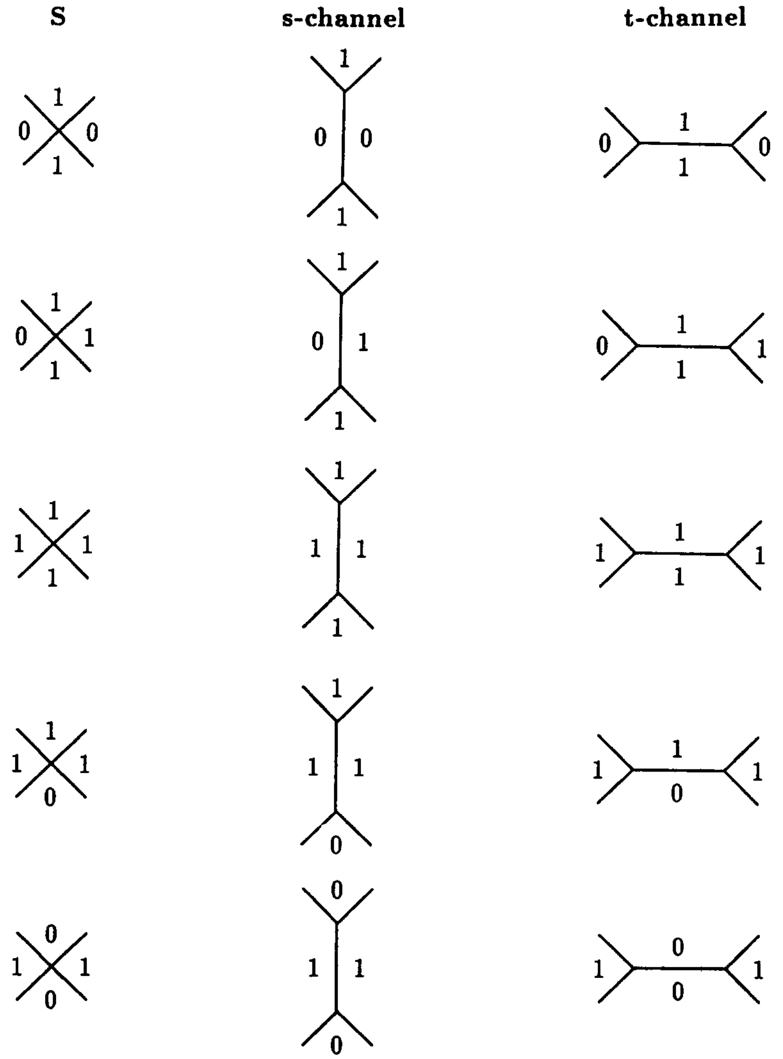


Fig. 19. Intermediate states in the s-channel and T-channel of the RSOS S-matrix for the ψ-deformation of the model $\mathcal{M}_{3,5}$.

$$
\begin{aligned}
r_1 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{00}^{11}(\theta) = 0\,,\\
r_2 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{01}^{11}(\theta) = -\mathrm{i}[s(\tfrac{1}{5})/s(\tfrac{2}{5})]^2\omega\,,\\
r_3 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{11}^{11}(\theta) = -\mathrm{i}\omega\,,\\
r_4 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{11}^{01}(\theta) = -[s(\tfrac{1}{5})/s(\tfrac{2}{5})]^{1/2}\omega\,,\\
r_5 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{11}^{00}(\theta) = \mathrm{i}[s(\tfrac{1}{5})/s(\tfrac{2}{5})]\omega\,,
\end{aligned}
\tag{6.56}
$$

where $s(x) \equiv \sin(\pi x)$ and

$$
\omega = \tfrac{5}{3}\, s(\tfrac{2}{5}) s(\tfrac{3}{10}) / s(\tfrac{1}{10})\,. \tag{6.57}
$$

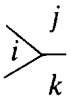
Notice the absence of a bound state in the direct channel of the amplitude S_{00}^{11}. This is due to the decoupling of the unphysical state $|K_{00}\rangle$. The residues are not all independent since a relation among them is fixed by crossing symmetry

$$
S_{11}^{11}(\mathrm{i}\pi-\theta) = S_{11}^{11}(\theta)\,,\quad S_{11}^{00}(\mathrm{i}\pi-\theta) = a^2 S_{00}^{11}(\theta)\,,\quad S_{11}^{01}(\mathrm{i}\pi-\theta) = a S_{01}^{11}(\theta)\,, \tag{6.58}
$$

where

$$
a = -\mathrm{i}[s(\tfrac{2}{5})/s(\tfrac{2}{5})]^{1/2}\,. \tag{6.59}
$$

The values of the residues are in correspondence with the coupling constants g_{ijk} of an underlying field theory. The important point here is that the coupling constants g_{ijk} are not necessarily symmetric nor real, since they define an interaction between the domain walls of asymmetric vacua of a non-unitary theory [164]. Using the following convention for the ordering of the indexes in g_{ijk}, i.e. starting from the left value and turning clockwise,

$g_{ijk} =$ 

,

there are the following self-consistent assignments

$$
\begin{aligned}
&g_{111} = \mathrm{i}\sqrt{\omega}\,,\quad g_{101} = \sqrt{[s(\tfrac{1}{5})/s(\tfrac{2}{5})]\omega}\,,\\
&g_{011} = \mathrm{i}[s(\tfrac{1}{5})/s(\tfrac{2}{5})]\sqrt{\omega}\,,\quad g_{110} = \mathrm{i}[s(\tfrac{1}{5})/s(\tfrac{2}{5})]\sqrt{\omega}\,.
\end{aligned}
\tag{6.60}
$$

Some of the coupling constants are therefore imaginary.

The physical picture coming from the above scattering theory has been verified by the CSTA in ref. [164]. An interesting link between the conformal data and the scattering theory is obtained by analysing the asymptotic behaviour of the phase shifts $\delta_i(\theta)$ defined by

$$
S_{00}^{11}(\theta) \equiv \exp[2\mathrm{i}\delta_0(\theta)]\,,\quad S_{01}^{11}(\theta) \equiv \exp[2\mathrm{i}\delta_1(\theta)]\,. \tag{6.61}
$$

The same functions $\delta_i(\theta)$ also appear diagonalizing the 2×2 symmetric S-matrix which rules the dynamics of the kinks with boundary conditions equal to the vacuum $a=1$,

$$\begin{pmatrix} S_{11}^{11}(\theta) & S_{11}^{01}(\theta) \\ S_{11}^{01}(\theta) & S_{11}^{00}(\theta) \end{pmatrix} . \tag{6.62}$$

Asymptotically, they have the following limits

$$\lim_{\theta\to\infty} \exp[2\mathrm{i}\delta_0(\theta)] = \exp(\tfrac{2}{5}\pi\mathrm{i}) , \quad \lim_{\theta\to\infty} \exp[2\mathrm{i}\delta_1(\theta)] = \exp(\tfrac{1}{5}\pi\mathrm{i}) . \tag{6.63}$$

These asymptotic values (6.63) can be used for the definition of a generalized bilinear commutation relation for a set of two fields $\vartheta_i(t,x)$ $(i=1,2)$ [100–102],

$$\vartheta_i(t,x)\vartheta_j(t,y) = \vartheta_j(t,y)\vartheta_i(t,x)\exp[2\pi\mathrm{i}s_{ij}\epsilon(x-y)] . \tag{6.64}$$

The generalized "spin" s_{ij} is thus extracted from the asymptotic behaviour of the S-matrix. A consistent assignment is given by

$$s_{11} = \tfrac{1}{5} = \delta_0(\infty)/\pi , \quad s_{12} = 0 , \quad s_{22} = \tfrac{1}{10} = \delta_1(\infty)/\pi . \tag{6.65}$$

The monodromy properties of the fields ϑ_i are those of the chiral field $\Gamma = \Phi_{1/5,0}$ of the original CFT $\mathcal{M}_{3,5}$. In fact, the operator product expansion of Γ with itself reads

$$\Gamma(z)\Gamma(0) = \frac{1}{z^{2/5}}\,1 + \frac{C_{\Gamma\Gamma\Gamma}}{z^{1/5}}\,\Gamma(0) + \cdots , \tag{6.66}$$

where $C_{\Gamma\Gamma\Gamma}$ is the structure constant of the OPE algebra. Moving z around the origin, $z\to \mathrm{e}^{2\pi\mathrm{i}}z$, the phase acquired from the first term on the right-hand side of (6.66) comes from the conformal dimension of the operator itself. In contrast, the phase obtained from the second term is due to the presence of the operator Γ. A similar structure appears in the monodromy relation of the fields ϑ_i: the first phase is relative to the amplitude $\mathrm{e}^{2\mathrm{i}\delta_0}$, where there is no bound state in the s-channel (the "identity term" in (6.66)), whereas the second phase comes from the amplitude $\mathrm{e}^{2\mathrm{i}\delta_1}$ where a bound state appears for $\theta = \frac{2}{3}\pi\mathrm{i}$ (the "Γ term" in (6.66)). In the scaling limit, the fields ϑ_i should thus reduce to the operator $\Gamma(z)$.

As last remark, let us notice that the amplitudes $S_{ab}^{cd}(\theta)$ of this model are particular solutions of the Yang–Baxter equations for the "hard square lattice model" [1]. The non-unitarity properties of the theory implies that not all Boltzmann weights are real functions when computed for imaginary values of the rapidity θ. A classification of the consistent sets of S-matrices originating from the hard square lattice model has been pursued in ref. [165].

6.4. Integrable deformations of the Ising model

The two-dimensional Ising model is defined by the microscopic Hamiltonian

$$H = -\beta \sum_{\langle i,j\rangle} \sigma_i\sigma_j + h\sum_i \sigma_i , \tag{6.67}$$

with the spin variables $\sigma_i = \pm 1$. In the thermodynamical limit the system undergoes a second-order phase transition at $\beta = \beta_c$ and $h = 0$ with a logarithmic singularity in the specific heat [1, 73–81]. As shown in appendix A, the scaling field theory of the two-dimensional Ising model is the theory of free massive Majorana fermions

$$\mathscr{L} = \tfrac{1}{2}\psi\, \partial\psi/\partial\bar{z} + \tfrac{1}{2}\bar{\psi}\, \partial\bar{\psi}/\partial z + \mathrm{i}m\bar{\psi}\psi\,. \tag{6.68}$$

The mass parameter measures the deviation of the temperature from its critical value

$$m = T - T_c\,. \tag{6.69}$$

At the critical point, we have free massless fermions with equations of motion

$$\partial\psi/\partial\bar{z} = \partial\bar{\psi}/\partial z = 0\,. \tag{6.70}$$

Hence, ψ is a holomorphic field and $\bar{\psi}$ an anti-holomorphic one. Their two-point functions are

$$\langle \psi(z_1)\psi(z_2)\rangle = 1/(z_1 - z_2)\,, \quad \langle \bar{\psi}(\bar{z}_1)\bar{\psi}(\bar{z}_2)\rangle = 1/(\bar{z}_1 - \bar{z}_2)\,. \tag{6.71}$$

From the expression of the stress-energy tensor

$$T(z) = -\tfrac{1}{2}{:}\psi(z)\,\frac{\partial}{\partial z}\,\psi(z){:} \tag{6.72}$$

and its two-point function

$$\langle T(z_1)T(z_2)\rangle = 1/4(z_1 - z_2)^4\,, \tag{6.73}$$

we can extract the value of the central charge of the model, $c = \frac{1}{2}$. Therefore, the scaling limit of the Ising model is described by the first minimal unitary model $\mathscr{M}_{3,4}$ of CFT [12]. The Kac table of this model is shown in table 5.

The field with dimension $\Delta_{2,1} = \Delta_{1,3} = \frac{1}{2}$ and $\bar{\Delta} = 0$ is correctly identified with the Onsager spinor $\psi(z)$ of the Ising model. The scalar combination

$$\varepsilon(z, \bar{z}) = \mathrm{i}{:}\,\bar{\psi}(\bar{z})\psi(z){:} \tag{6.74}$$

corresponds to the scaling limit of the energy operator of the lattice model. The critical exponent of the specific heat, $\alpha = 0$, can be extracted from its two-point function.

The spin sector of the model is given by the magnetization operator $\sigma(z, \bar{z})$ and by the disorder field

Table 5
Kac table of the Ising model.

$\frac{1}{2}$	0
$\frac{1}{16}$	$\frac{1}{16}$
0	$\frac{1}{2}$

$\mu(z, \bar{z})$. These fields have spin zero, i.e.

$$\Delta_\sigma = \bar{\Delta}_\sigma , \quad \Delta_\mu = \bar{\Delta}_\mu , \tag{6.75}$$

and the Kramers–Wannier symmetry fixes their anomalous dimensions to be equal. Explicitly,

$$\Delta_\sigma = \Delta_\mu = \tfrac{1}{16} , \tag{6.76}$$

as can be seen from the exact solution of the model for the two-point function

$$\langle \sigma(z, \bar{z})\sigma(0,0)\rangle = |z|^{-1/4} . \tag{6.77}$$

The fields ψ, σ and μ are not mutually local: in fact, the correlation function

$$\langle \psi(z)\sigma(z_1)\cdots \sigma(z_n)\mu(z_{n+1}\cdots \mu(z_m)\rangle \tag{6.78}$$

is a double-valued analytic function of z that acquires a phase factor (-1) after the analytic continuation around any singular point z_k [77]. This translates into the following operator product expansion

$$\begin{aligned} &\psi(z)\sigma(0,0) = \mu(0,0)/\sqrt{z} + \cdots , \quad \psi(z)\mu(0,0) = \sigma(0,0)/\sqrt{z} + \cdots , \\ &\sigma(z, \bar{z})\mu(0,0) = z^{3/8}\bar{z}^{-1/8}[\psi(z) + \cdots] + z^{-1/8}\bar{z}^{-3/8}[\bar{\psi}(\bar{z}) + \cdots] . \end{aligned} \tag{6.79}$$

There are three different sets of local fields which can describe the critical regime of the Ising model. They can be easily selected out by analyzing the effect of boundary conditions chosen for the transfer matrix of the model [24]. We denote by Z_{XY} the partition function for an $l \times l'$ rectangle with boundary conditions of types X, Y on the two pairs of opposite sides. P stays for periodic boundary conditions whereas A for antiperiodic ones. Let τ be the ratio of two dimensions

$$\tau = l'/l . \tag{6.80}$$

With periodic boundary condition in both directions, the only combination invariant under the full modular group generated by

$$\tau \to 1/\tau , \quad \tau \to \tau + \mathrm{i} , \tag{6.81}$$

is given [24] as

$$Z_{PP} = |\chi_0|^2 + |\chi_{1/16}|^2 + |\chi_{1/2}|^2 . \tag{6.82}$$

Therefore the local set of fields with these boundary conditions are

$$\{A_1\} = \{1, \sigma, \varepsilon\} , \tag{6.83}$$

or, equivalently,

$$\{A_1\} = \{1, \mu, \varepsilon\} . \tag{6.84}$$

The OPE algebra is given by

$$[\varepsilon] * [\varepsilon] = [1] , \quad [\varepsilon] * [\sigma] = [\sigma] , \quad [\sigma] * [\sigma] = [1] + [\varepsilon] . \tag{6.85}$$

These fusion rules are compatible with the spin reversal symmetry ($\sigma \to -\sigma$) and duality ($\varepsilon \to -\varepsilon$).

If we choose antiperiodic boundary conditions on the horizontal sides of the strip and periodic boundary conditions on the other direction the partition function is given by [24]

$$Z_{AP} = |\chi_{1/16}|^2 + \chi^*_{1/2}\chi_0 + \chi^*_0\chi_{1/2} . \tag{6.86}$$

Inverting the role of the two directions, we get

$$Z_{PA} = |\chi_0|^2 - |\chi_{1/16}|^2 + |\chi_{1/2}|^2 . \tag{6.87}$$

The combination

$$Z_{PA} + Z_{AP} = |\chi_0 + \chi_{1/2}|^2 \tag{6.88}$$

is invariant under a subgroup of the modular group generated by

$$\tau \to 1/\tau , \quad \tau \to \tau + 2\mathrm{i} , \tag{6.89}$$

and gives rise to the free fermion description of the Ising model. In this description the set of local fields coincides with

$$\{A_3\} = \{1, \psi, \bar{\psi}, \varepsilon\} , \tag{6.90}$$

and the fusion rules are

$$\begin{gathered} [\psi] * [\psi] = [1] , \quad [\bar{\psi}] * [\bar{\psi}] = [1] , \quad [\psi] * [\bar{\psi}] = [\varepsilon] , \\ [\varepsilon] * [\psi] = [\bar{\psi}] , \quad [\varepsilon] * [\bar{\psi}] = [\psi] . \end{gathered} \tag{6.91}$$

The computation of the correlation functions of the critical Ising model has been carried out in refs. [12, 82]. In the next sections, we will discuss the scaling region around the critical point spanned by the thermal and magnetic deformations of the Ising model.

6.4.1. Thermal deformation

The thermal deformation of the critical point action of the Ising model corresponds to the quantum field theory defined by the massive fermionic system, eq. (6.68). The interchange between high-temperature phase ($T > T_c$) and low-temperature phase ($T < T_c$) is equivalent to reversing the sign of the mass term. Since the two phases are related by duality, it is sufficient to discuss only one of them, say the high-temperature phase.

The fermionic sector of the model is of course a free field theory with an S-matrix equal to the identity. However, the magnetization sector corresponds to a Z_2-invariant field theory of a single (self-conjugate) boson field σ. It gives rise to the simplest nontrivial scattering theory with no bound states and an S-matrix [81, 143]

$$S = -1 . \tag{6.92}$$

This result can be obtained as a limit case of the general Z_n invariant models solved by Köberle and Swieca [143]. These models have $n-1$ particles with mass spectrum

$$m_a = \sin(\pi a/n) , \quad a = 1, 2, \ldots, n-1 . \tag{6.93}$$

The S-matrix for the fundamental particle is given by

$$S_{11} = \tanh[\tfrac{1}{2}(\theta + i2\pi/n)]/\tanh[\tfrac{1}{2}(\theta - i2\pi/3)] = f_{2/n} . \tag{6.94}$$

Choosing $n = 2$, we recover the scattering amplitude (6.92)

The check of the correct central charge recovered in the ultraviolet limit is easy: the constant value of the pseudo-energy ε_1 is just $\varepsilon_1 = 0$ and substituting this in eq. (4.39), we get

$$c = (6/\pi^2)L(\tfrac{1}{2}) = \tfrac{1}{2} , \tag{6.95}$$

in agreement with the prediction of CFT.

6.4.2. Magnetic deformation

Let S^* be the critical action of the Ising model. Its perturbation by the relevant operator $\Phi_{1,2} = \sigma(x)$,

$$S = S^* + h \int \sigma(x)\, d^2x , \tag{6.96}$$

couples the model to an external magnetic field h. The value of the temperature is that of the critical point $T = T_c$. The analysis of the quantum field theory originating from the action (6.96) has been worked out in a remarkable paper by Zamolodchikov [30], which initiated the whole area of study.

The magnetic deformation of the Ising model preserves a number of nontrivial local conserved charges present at the conformal point. The lowest degrees of these integrals of motion can be obtained by applying the counting argument [30]. The first representatives are

$$s = 1, 7, 11, 13, 17, 19 . \tag{6.97}$$

Notice the lacking of spins s having 3 and 5 as divisors. The absence of degrees s which are multiples of 3 is easily explained by postulating the existence of a fundamental particle A_1 (with mass m_1) that possesses the Φ^3 property, i.e. A_1 itself appears as bound state in the $A_1 A_1$ scattering with resonance angle $u_{11}^1 = \frac{2}{3}\pi i$. This is compatible with the explicit breaking of the Z_2 symmetry of the conformal action. The absence of degrees s divisible by 5 can be explained by conjecturing the existence of a second particle state A_2 (with mass m_2) that, together with A_1, gives rise to the following subset of bootstrap fusions:

$$A_1 \times A_1 \to A_1 + A_2 , \quad A_2 \times A_2 \to A_1 . \tag{6.98}$$

Let u_{11}^2 be the resonance angle corresponding to the bound state A_2 that appears in the scattering amplitude $S_{11}(\theta)$ of the fundamental particle and u_{22}^1 the resonance angle associated to A_1 in the scattering amplitude $S_{22}(\theta)$. Using the variables

$$y_1 = \exp(\tfrac{1}{2}\mathrm{i}u_{11}^2) , \quad y_2 = \exp(\tfrac{1}{2}\mathrm{i}u_{22}^1) , \tag{6.99}$$

the bootstrap consistency equations associated to the bootstrap fusions (6.98) are given by (see section 3.5)

$$y_1^s + y_1^{-s} = (m_2/m_1)^s \chi_s^2/\chi_s^1 , \quad y_2^s + y_2^{-s} = (m_1/m_2)\chi_s^1/\chi_s^2 . \tag{6.100}$$

For s running on the set (6.97), a nontrivial solution is given by

$$y_1 = \exp(\tfrac{1}{5}\mathrm{i}\pi) , \quad y_2 = \exp(\tfrac{2}{5}\mathrm{i}\pi) . \tag{6.101}$$

Therefore the mass ratio is fixed to be

$$m_2/m_1 = 2\cos\tfrac{1}{5}\pi . \tag{6.102}$$

Collecting these results, we conclude that in the scattering amplitude $S_{11}(\theta)$ of the fundamental particle there are poles with positive residues at the resonance angles

$$\theta = \mathrm{i}u_{11}^1 = \tfrac{2}{3}\pi\mathrm{i} , \quad \theta = \mathrm{i}u_{11}^2 = \tfrac{2}{5}\pi\mathrm{i} , \tag{6.103}$$

together with the cross-channel poles (with negative residues) at

$$\theta = \mathrm{i}\bar{u}_{11}^1 = \tfrac{1}{3}\mathrm{i}\pi , \quad \theta = \mathrm{i}\bar{u}_{11}^2 = \tfrac{3}{5}\pi\mathrm{i} . \tag{6.104}$$

However, it is possible to satisfy the bootstrap equation

$$S_{11}(\theta) = S_{11}(\theta - \tfrac{1}{3}\mathrm{i}\pi)S_{11}(\theta + \tfrac{1}{3}\mathrm{i}\pi) \tag{6.105}$$

with only the poles (6.103) and (6.104). It is thus necessary to include further additional poles. The minimal way to satisfy (6.105), without spoiling the conserved charges with degrees (6.97), is to introduce a pole at $\theta = \frac{1}{15}\mathrm{i}\pi$ (with positive residue) and its crossed symmetrical one at $\theta = \frac{14}{15}\mathrm{i}\pi$ (with negative residue). The scattering S-matrix for the fundamental particle conjectured by Zamolodchikov [30] is thus

$$S_{11}(\theta) = f_{1/3}(\theta) f_{2/5}(\theta) f_{1/15}(\theta) . \tag{6.106}$$

The bootstrap tree generated by this fundamental amplitude closes within eight particle states

$A_1, A_2, \ldots, A_8$ and mass spectrum collected in table 6. The full set of S-matrices is gathered in appendix C, together with the bootstrap fusions and the resonance angles. The mass ratios and the S-matrices are related to the "root system" of E_8. This algebraic structure can be traced back to the equivalent realization of the Ising model in terms of the coset construction $(E_8)_1 \otimes (E_8)_1/(E_8)_2$ (see appendix C).

The thermodynamical Bethe ansatz analysis of the scattering model corresponding to the magnetic deformation of the Ising model confirms that the ultraviolet central charge is equal to 1/2 [51]. Numerical checks of the mass spectrum have been obtained in refs. [137, 138, 184].

6.5. Integrable deformations of the tricritical Ising model

The Tricritical Ising Model (TIM) is the second model in the minimal unitary conformal series [12, 13]. Its central charge is $c = \frac{7}{10}$, and there are four relevant scaling fields. It represents the universality class of the Landau–Ginzburg Φ^6 theory

$$\int \mathcal{D}\Phi \, \exp\Big(\int [(\nabla\Phi)^2 + \lambda_6\Phi^6 + \lambda_4\Phi^4 + \lambda_3\Phi^3 + \lambda_2\Phi^2 + \lambda_1\Phi] \, \mathrm{d}^2 r\Big) \tag{6.107}$$

at its critical point $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ [34]. This Lagrangian describes the continuum limit of microscopic models with a tricritical point, among them the Ising model with annealed vacancies. Its Hamiltonian is given by [60, 61]

$$\mathcal{H} = -\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j t_i t_j - \mu \sum_i t_i \,. \tag{6.108}$$

β is the inverse temperature, μ the chemical potential, σ_i the Ising spins and t_i is the vacancy variable*). The model has a tricritical point (β_0, μ_0) related to the spontaneous symmetry breaking of Z_2 symmetry. The phase diagram is shown in fig. 20. The dashed line corresponds to a critical line of second-order phase transition: the large distance behaviour on this line is controlled by the Ising fixed point. The solid line is, on the contrary, a line of first-order phase transition.

At the critical point (β_0, μ_0), TIM can be described by the following scaling fields: the energy density $\varepsilon(z, \bar{z})$ with anomalous dimensions $(\Delta, \bar{\Delta}) = (\frac{1}{10}, \frac{1}{10})$, the vacancy operator or subleading energy operator $t(z, \bar{z})$ with $(\Delta, \bar{\Delta}) = (\frac{3}{5}, \frac{3}{5})$, the irrelevant field ε'' with $(\Delta, \bar{\Delta}) = (\frac{3}{2}, \frac{3}{2})$, the magnetization field (or order parameter) $\sigma(z, \bar{z})$ with $(\Delta, \bar{\Delta}) = (\frac{3}{80}, \frac{3}{80})$, and the so-called subleading magnetization operator $\alpha(z, \bar{z})$ with anomalous dimensions $(\frac{7}{16}, \frac{7}{16})$ (see table 7). All these fields enter the modular

*) As usual, $\sigma = \pm 1$ and $t = 0, 1$.

Table 6
Mass spectrum of the Ising model in a magnetic field.

$m_1 = M$	1
$m_2 = 2M\cos(\frac{1}{5}\pi)$	1.61803
$m_3 = 2M\cos(\frac{1}{30}\pi)$	1.98904
$m_4 = 4M\cos(\frac{1}{5}\pi)\cos(\frac{7}{30}\pi)$	2.40487
$m_5 = 4M\cos(\frac{1}{5}\pi)\cos(\frac{2}{15}\pi)$	2.95629
$m_6 = 4M\cos(\frac{1}{5}\pi)\cos(\frac{1}{30}\pi)$	3.21834
$m_7 = 4M\cos^2(\frac{1}{5}\pi)\cos(\frac{7}{30}\pi)$	3.89116
$m_8 = 8M\cos^2(\frac{1}{5}\pi)\cos(\frac{2}{15}\pi)$	4.78338

Table 7
Kac table of the tricritical Ising model.

$\frac{3}{2}$	$\frac{7}{16}$	0
$\frac{6}{10}$	$\frac{3}{80}$	$\frac{1}{10}$
$\frac{1}{10}$	$\frac{3}{80}$	$\frac{6}{10}$
0	$\frac{7}{16}$	$\frac{3}{2}$

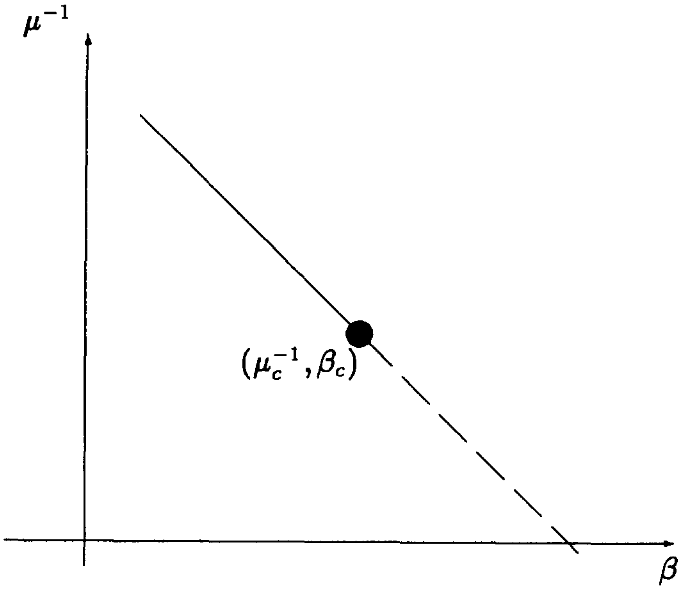


Fig. 20. Phase diagram of the tricritical Ising model.

invariant partition function [24, 25]. The Z_2 original symmetry of the Hamiltonian splits the set of operators into two classes, odd and even. The spin operators are odd while the energy operator, the vacancy operator and the irrelevant field ε'' are even. In the subalgebra of these even fields, there is a second Z_2 symmetry given by the Kramers–Wannier duality, under which ε and ε'' are odd whereas t is even:

$$D^{-1}\varepsilon D = -\varepsilon\,, \quad D^{-1}tD = t \tag{6.109}$$

In the spin sector, the application of Kramers–Wannier duality generates the dual disorder fields.

A peculiar feature of TIM is the presence of two other infinite-dimensional symmetries in addition to that of the Virasoro algebra: a local fermionic supersymmetry (generated by $T(z)$ and $G(z) = \Phi_{3/2,0}$) and a hidden E_7 structure related to the equivalent construction of TIM in terms of the coset model $(\hat{E}_7)_1 \otimes (\hat{E}_7)_1/(\hat{E}_7)_2$. We refer to appendix B for the discussion of the tricritical Ising model in terms of this coset construction. Here we briefly analyze the superconformal properties of the model.

The fermionic symmetry is that of $N = 1$ supersymmetry [64]. This consists of the Neveu–Schwarz and Ramond algebras. The Virasoro representations organize therefore into irreducible representations of $N = 1$ superconformal theory.

The Neveu–Schwarz (NS) sector is given by the even subsector of the Z_2 symmetry: the energy operator and the vacancy operator build up a superfield, where the latter one plays the role of the highest component

$$\Phi(z, \bar{z}) = \varepsilon(z, \bar{z}) + \theta\Psi(z, \bar{z}) + \overline{\theta\Psi}(z, \bar{z}) + \theta\bar{\theta}t(z, \bar{z})\,. \tag{6.110}$$

The NS sector of the theory is solved with the generalized null-vector method [65].

The Hilbert space of the superconformal theories also contains irreducible representations of the Ramond algebra, corresponding to the *spin fields* [64]. They are non local with respect to the fermionic part of the superfields. To solve the theory in the Ramond sector, it is more convenient to use a Coulomb gas formalism, in which the Ramond fields are represented by the Ising order–disorder fields times the usual bosonic vertex [66]. In the present case, the Ramond fields are the original Z_2 odd operators.

This algebraic approach can be rephrased in terms of a supersymmetric Lagrangian formalism. In fact, TIM describes the nontrivial fixed point of the following Landau–Ginzburg supersymmetric action of one real scalar superfield [34]

$$\mathcal{S} = \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, [\tfrac{1}{2} \mathrm{D}\Phi\, \bar{\mathrm{D}}\Phi + gW(\Phi)]\,. \tag{6.111}$$

$\mathrm{D} = \partial/\partial\theta + \theta\, \partial/\partial z$ is the usual covariant derivative and $W(\Phi) = \Phi^3$. The Lagrangian (6.111) has the advantage of showing directly which are the deformations of the theory that preserve supersymmetry. They are the F-components of the superfields. In the case of TIM this is just the field $t(z, \bar{z})$. Since this field is even under duality, its insertion into the action

$$\mathcal{S} \to \mathcal{S} + \lambda \int \mathrm{d}^2 z\, t(z, \bar{z}) \tag{6.112}$$

shifts the theory along the phase transition lines. The sign of the deformation determines whether the off-critical system reaches the line of second-order phase transition or that of first order. If $\lambda > 0$, we have a situation of spontaneously supersymmetry breaking [67]: the scalar field becomes massive but the fermion field remains massless (goldstino). The renormalization group trajectory ends at the fixed point of the Ising model. In the infrared limit the scalar particle becomes infinitely massive and decouples from the theory and we obtain the usual massless description of Ising model. If $\lambda < 0$, the model describes the scaling region of the first-order phase transition line. The corresponding massive quantum field theory will be described in section 6.5.2.

After this discussion about the properties at the critical point, we now turn to the analysis of the massive theories arising from different deformations of TIM. The counting argument selects three of them as integrable directions in the phase diagram, those of the leading thermal operator, the sub-leading thermal operator and the sub-leading magnetization (tables 8–11). The leading magnetic perturbation seems to be non-integrable: this is supported by the counting argument (table 8a) and by an explicit check of the absence of conserved currents of spin 5 and spin 7 [134, 63]. Further support of this picture comes from the numerical analysis of this deformation [134].

Table 8

Dimensions of the spaces $\hat{\Lambda}_{s+1}$ and $\hat{\varphi}_{(3/80)s}$. The counting argument gives no evidence of conserved currents for this perturbation.

s	**1**	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\hat{\Lambda}_{s+1}$	1	0	1	0	2	0	3	1	4	2	6	3	9	6	12	9	18	14
$\tilde{\varphi}(3/80)s$	0	1	1	1	2	2	3	4	5	6	8	10	12	15	19	22	28	34

Table 9

Dimensions of the spaces $\hat{\Lambda}_{s+1}$ and $\hat{\varphi}_{(1/10)s}$. In boldface are the spins of the conserved currents found by the counting argument.

s	**1**	2	3	4	**5**	6	**7**	8	**9**	10	**11**	12	**13**	14	15	16	17	18
$\hat{\Lambda}_{s+1}$	1	0	1	0	2	0	3	1	4	2	6	3	9	6	12	9	18	14
$\hat{\varphi}(1/10)s$	0	0	1	1	1	2	2	3	3	5	5	8	8	11	13	17	19	25

Table 10
Dimensions of the spaces $\hat{\Lambda}_{s+1}$ and $\hat{\varphi}_{(7/16)s}$. In boldface are the spins of the conserved currents found by the counting argument.

s	**1**	2	3	4	**5**	6	**7**	8	9	10	**11**	12	**13**	14	15	16	17	18
$\hat{\Lambda}_{s+1}$	1	0	1	0	2	0	3	1	4	2	6	3	9	6	12	9	18	14
$\hat{\varphi}(7/16)s$	0	0	1	1	1	2	2	2	4	4	5	7	8	10	13	15	18	23

Table 11
Dimensions of the spaces $\hat{\Lambda}_{s+1}$ and $\hat{\varphi}_{(6/10)s}$. In boldface are the spins of the conserved currents found by the counting argument.

s	**1**	2	**3**	4	**5**	6	**7**	8	**9**	10	**11**	12	13	14	15	16	17	18
$\hat{\Lambda}_{s+1}$	1	0	1	0	2	0	3	1	4	2	6	3	9	6	12	9	18	14
$\hat{\varphi}_{(6/10)s}$	0	1	0	2	1	2	2	4	3	6	5	9	9	13	13	20	20	28

6.5.1. *Leading thermal perturbation*

Let us consider the theory defined by the action

$$\mathcal{S} = \mathcal{S}^* + \lambda \int d^2z\, \varepsilon(z, \bar{z}) , \tag{6.113}$$

For $\lambda > 0$ the system is in the Z_2-symmetric high-temperature phase. The behaviour of the system in the low-temperature phase ($\lambda < 0$) is related by duality to the previous one. In the following we consider the massive quantum field theory defined by (6.113) with λ positive.

As shown in appendix C, the field ε is associated to the adjoint of E_7. Following the original suggestion of Eguchi–Yang [189] and Hollowood–Mansfield [190], the off-critical massive system shares the same grading of conserved currents as the affine Toda field theory for the algebra E_7, i.e. the spins of the higher conserved currents are equal to the exponents of the E_7 algebra modulo the Coxeter numbers $h = 18$, i.e.

$$s = 1, 5, 7, 9, 11, 13, 17 \quad (\mathrm{mod}\, 18) . \tag{6.114}$$

The existence of these non-trivial conserved currents can be explicitly checked for the lowest values of s using the counting argument. The presence of these higher conserved currents implies the elasticity of the scattering processes of the massive excitations. To compute the mass spectrum and the scattering amplitudes, it is important to observe that the "fundamental particle" of this massive theory cannot be a bound state of itself because in the set of conserved spins there is the spin $s = 9$. As explained by Fateev and Zamolodchikov [159], one should expect the occurrence of this situation because the Z_2 symmetry is exact even away from criticality and can be used as good quantum number for labelling the states. The "fundamental particle" is expected to be Z_2 odd and therefore cannot satisfy the Φ^3 property. However, the existence of Z_2-even particles with the Φ^3 property is not in contradiction with the conserved charges provided the operator $\mathcal{Q}_9$ annihilates these states. Let us assume that the lightest of such particles, denoted by A_2, appears as bound state in the scattering amplitude of the Z_2-odd "fundamental particle" A_1. Since $\gamma_9^1 \neq 0$ but $\gamma_9^2 = 0$, using eqs. (3.38) we obtain for the resonance angle u_{11}^2 the condition

$$\cos(\tfrac{9}{2} u_{11}^2) = 0 . \tag{6.115}$$

As shown in ref. [159], in the range $0 < u_{11}^2 < \pi$ the solution of (6.115) which gives rise to a consistent system is

$$u_{11}^2 = \tfrac{5}{9}\pi \,. \tag{6.116}$$

This fixes the mass ratio of the particles to be

$$m_2 = 2\cos(\tfrac{5}{18}\pi)m_1 \,. \tag{6.117}$$

Using eq. (3.23) we see that the existence of a pole at $\theta = \mathrm{i}\frac{5}{9}\pi$ in S_{11} with positive residue implies a pole in S_{12} at $\theta = \mathrm{i}\frac{5}{9}\pi$ with negative residue, corresponding to the singularity due to the particle A_1 in the cross-channel. With these conditions, the bootstrap equations for the amplitudes S_{11} and S_{12} become

$$S_{12}(\theta) = S_{11}(\theta + \mathrm{i}\tfrac{5}{18}\pi)S_{11}(\theta - \mathrm{i}\tfrac{5}{18}\pi) \,, \tag{6.118}$$

$$S_{11}(\theta) = S_{11}(\theta + \mathrm{i}\tfrac{4}{9}\pi)S_{12}(\theta - \mathrm{i}\tfrac{5}{18}\pi) \,. \tag{6.119}$$

We cannot satisfy these equation with only one pole in S_{11} and S_{12}. The minimal way to fulfill them is to introduce an additional pole at $\theta = \mathrm{i}\frac{1}{9}\pi$ with positive residue in S_{11} (and correspondingly a pole at $\theta = \mathrm{i}\frac{8}{9}\pi$ with negative residue) and a pole at $\theta = \mathrm{i}\frac{7}{18}\pi$ with positive residue (and at $\theta = \mathrm{i}\frac{13}{18}\pi$ with negative residue) in S_{12}. The pole at $\theta = \mathrm{i}\frac{1}{9}\pi$ in S_{11} corresponds to a new Z_2-even bound state A_4 with mass

$$m_4 = 2\cos(\tfrac{1}{18}\pi)m_1 \,. \tag{6.120}$$

The pole at $\theta = \mathrm{i}\frac{7}{18}\pi$ in S_{12} represents another Z_2-odd particle A_3 with mass

$$m_3 = 2\cos(\tfrac{1}{9}\pi)m_1 \,. \tag{6.121}$$

We can now use the bootstrap equations in order to compute the other scattering amplitudes. The bootstrap closes with seven particles. The complete set of S-matrices was computed by Christe and Mussardo [157] and can be found in appendix B. The list of the resonance angles is given in table 12 whereas the masses of the particles together with their Z_2 parity are collected in table 13. The seven

Table 12
List of the poles u_{ij}^k of odd order of the S-matrices S_{ij}. In parentheses are those of order >1.

Pole	Set of u_{ij}^k	Pole	Set of u_{ij}^k
$\frac{1}{9}\pi$	$u_{11}^4, u_{22}^5, u_{33}^7$	$\frac{11}{18}\pi$	$(u_{46}^6), (u_{57}^7), u_{14}^3, u_{23}^3$
$\frac{1}{6}\pi$	u_{14}^6, u_{25}^7	$\frac{2}{3}\pi$	$u_{44}^4, (u_{77}^7), u_{16}^5, (u_{36}^5), (u_{55}^5), u_{22}^2$
$\frac{2}{9}\pi$	u_{16}^7, u_{44}^7	$\frac{13}{18}\pi$	$u_{12}^1, (u_{45}^4), u_{25}^4, (u_{67}^6), (u_{37}^6)$
$\frac{5}{18}\pi$	u_{23}^6	$\frac{7}{9}\pi$	$(u_{66}^4), (u_{47}^5), (u_{77}^5), u_{13}^2, u_{24}^2, u_{33}^2, u_{15}^3, (u_{56}^3)$
$\frac{1}{3}\pi$	u_{13}^5	$\frac{5}{6}\pi$	$u_{34}^1, u_{23}^1, (u_{57}^4), u_{17}^6, u_{45}^2, (u_{67}^3), u_{26}^3$
$\frac{7}{18}\pi$	$(u_{45}^7), u_{12}^3$	$\frac{8}{9}\pi$	$u_{56}^1, u_{35}^1, u_{16}^4, u_{47}^4, u_{27}^5, u_{36}^2, (u_{77}^2)$
$\frac{4}{9}\pi$	$u_{22}^4, u_{15}^6, (u_{36}^7), u_{24}^5$	$\frac{17}{18}\pi$	$u_{14}^1, u_{46}^1, u_{67}^1, u_{57}^2, u_{25}^2, u_{37}^3$
$\frac{5}{9}\pi$	$u_{13}^4, (u_{35}^6), (u_{66}^7), (u_{27}^7), (u_{44}^5), u_{11}^2$		

Table 13
Mass spectrum of the E_7 Toda system.

$m_1 = M$	1	odd
$m_2 = 2M\cos(\frac{5}{18}\pi)$	1.28557	even
$m_3 = 2M\cos(\frac{1}{9}\pi)$	1.87938	odd
$m_4 = 2M\cos(\frac{1}{18}\pi)$	1.96961	even
$m_5 = 4M\cos(\frac{1}{18}\pi)\cos(\frac{5}{18}\pi)$	2.53208	even
$m_6 = 4M\cos(\frac{2}{9}\pi)\cos(\frac{1}{9}\pi)$	2.87938	odd
$m_7 = 4M\cos(\frac{1}{18}\pi)\cos(\frac{1}{9}\pi)$	3.70166	even

particles can be written in terms of two triplets and one singlet [157],

$$(Q_1, Q_2, Q_3) \equiv (m_6, m_3, m_1)\,, \quad (K_1, K_2, K_3) \equiv (m_2, m_4, m_7)\,, \quad (N) \equiv (m_5)\,. \tag{6.122}$$

The first triplet consists of particles that are odd under the Z_2 symmetry. The other triplet and the singlet are Z_2 even. The "bootstrap fusions' involving $[N]$ and $[N, K_i]$ form closed subsets

$$\begin{aligned} &N \cdot N = N\,, \quad N \cdot K_A = K_1 + K_2 + K_3\,, \\ &K_A \cdot K_{A+1} = K_A + N\,, \quad K_A \cdot K_A = K_A + K_{A+1} + N\,. \end{aligned} \tag{6.123}$$

The remaining particles only couple to the previous one. We obtain

$$\begin{aligned} &K_A \cdot Q_A = Q_{A+1}\,, \quad K_A \cdot Q_{A+1} = Q_1 + Q_2 + Q_3\,, \\ &K_A \cdot Q_{A-1} = Q_{A-1} + Q_{A+1}\,, \quad Q_A \cdot Q_A = K_{A-1} + K_{A+1}\,, \\ &Q_A \cdot Q_{A+1} = K_A + K_{A-1} + N\,, \quad N \cdot Q_A = Q_{A-1} + Q_{A+1}\,. \end{aligned} \tag{6.124}$$

We will comment further on these relations in the next chapter devoted to the affine Toda field theories. It is worth noticing here that these bootstrap fusions are a subset of the tensor product decomposition of the associate representations of E_7 [226, 227] (see appendix A).

The mass spectrum has been numerically confirmed by Lässig, Mussardo and Cardy in ref. [134]. An independent check has also been done by Von Gehlen by using finite-size analysis on the Blume–Capel model [62]. The TBA computation confirms that the ultraviolet behaviour of the above scattering theory is controlled by the CFT of the tricritical Ising model [51].

6.5.2. Sub-leading thermal perturbation

The scattering theory relative to this integrable deformation of TIM has been discussed originally by Zamolodchikov [179]. The off-critical theory is defined by the action

$$\mathcal{S} = \mathcal{S}^* + \lambda \int d^2z\, t(z, \bar{z})\,. \tag{6.125}$$

Besides a number of local bosonic conserved currents with spins $s = 1, 3, 5, 7, \ldots$, the field theory defined by (6.125) also possesses two integrals of motion Q and $\bar{Q}$ with spin $s = \pm\frac{1}{2}$ (see chapter 2).

This is because the field $t(z, \bar{z})$ is the upper component of the superfield $\Phi(z, \bar{z})$ in eq. (6.110) and therefore the off-critical action (6.125) is invariant under a global supersymmetry generated by

$$Q = \int (G \, dz + \bar{\Psi} \, d\bar{z}) , \quad \bar{Q} = \int (\bar{G} \, d\bar{z} + \Psi \, dz) . \tag{6.126}$$

For $\lambda < 0$, the model develops a finite correlation length. The field $t(z, \bar{z})$ is invariant with respect to the duality transformation and its insertion into the action shifts the theory along the self-dual line of the first-order phase transition line, where three phases simultaneously coexist. The corresponding Landau–Ginzburg potential is shown in fig. 21. There are three degenerate ground states denoted by $|-1\rangle$, $|0\rangle$ and $|1\rangle$. The massive elementary excitations consist in four different kinks connecting two neighbouring vacua. We denote them by

$$K_{0,+1} , \quad K_{0,-1} , \quad K_{+1,0} , \quad K_{-1,0} . \tag{6.127}$$

They have the same mass M. An asymptotic multi-particle state is given by a sequence of kink configurations

$$|K_{\sigma_0,\sigma_1}(\theta_1) K_{\sigma_1,\sigma_2}(\theta_2) \cdots K_{\sigma_{n-1},\sigma_n}(\theta_n)\rangle . \tag{6.128}$$

In order to have a finite energy, the neighbouring "vacua" σ_i must satisfy the condition

$$|\sigma_i - \sigma_{i+1}| = 1 . \tag{6.129}$$

The action of Q and $\bar{Q}$ on the asymptotic states is given by [179]

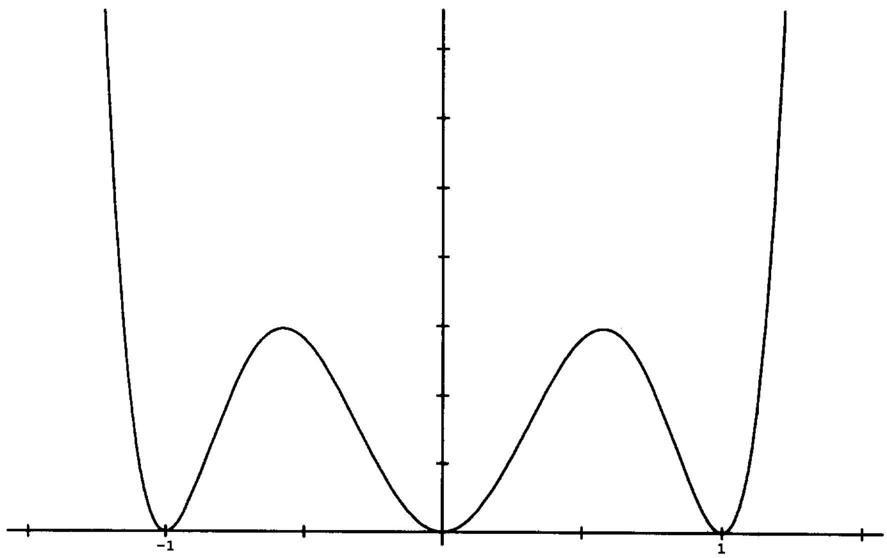


Fig. 21. Landau–Ginzburg potential for the $\Phi_{1,3}$ deformation of the tricritical Ising model in the massive regime.

$$
\begin{aligned}
&Q|K_{\sigma_0,\sigma_1}(\theta_1)K_{\sigma_1,\sigma_2}(\theta_2)\cdots K_{\sigma_{n-1},\sigma_n}(\theta_n)\rangle = \sum_{i=1}^{n} M^{1/2}\beta(\sigma_i,\sigma_{i+1})\,\mathrm{e}^{\theta_i/2} \\
&\quad\times|K_{-\sigma_0,-\sigma_1}(\theta_1)\cdots K_{-\sigma_{i-1},-\sigma_i}(\theta_i)K_{-\sigma_i,\sigma_{i+1}}\cdots K_{\sigma_{n-1},\sigma_n}(\theta_n)\rangle\,, \\
&\bar{Q}|K_{\sigma_0,\sigma_1}(\theta_1)K_{\sigma_1,\sigma_2}(\theta_2)\cdots K_{\sigma_{n-1},\sigma_n}(\theta_n)\rangle = \sum_{i=1}^{n} M^{1/2}\bar{\beta}(\sigma_i,\sigma_{i+1})\,\mathrm{e}^{-\theta_i/2} \\
&\quad\times|K_{-\sigma_0,-\sigma_1}(\theta_1)\cdots K_{-\sigma_{i-1},-\sigma_i}(\theta_i)K_{-\sigma_i\sigma_{i+1}}\cdots K_{\sigma_{n-1},\sigma_n}(\theta_n)\rangle\,,
\end{aligned}
\tag{6.130}
$$

where

$$
\beta(\sigma,\sigma') = \sigma + \mathrm{i}\sigma'\,,\quad \bar{\beta}(\sigma,\sigma') = \sigma - \mathrm{i}\sigma'\,. \tag{6.131}
$$

Since the operators Q and $\bar{Q}$ are integrals of motion, they must commute with the S-matrix of the model, defined by

$$
\begin{aligned}
&|K_{0,+s}(\theta_1)K_{+s,0}(\theta_2)\rangle = A_0(\theta_1-\theta_2)|K_{0,+s}(\theta_2)K_{+s,0}(\theta_1)\rangle + A_1(\theta_1-\theta_2)|K_{0,-s}(\theta_2)K_{-s,0}(\theta_1)\rangle\,, \\
&|K_{+s,0}(\theta_1)K_{0,+s}(\theta_2)\rangle = B_0(\theta_1-\theta_2)|K_{+s,0}(\theta_2)K_{0,+s}(\theta_1)\rangle\,, \\
&|K_{+s,0}(\theta_1)K_{0,-s}(\theta_2)\rangle = B_1(\theta_1-\theta_2)|K_{+s,0}(\theta_2)K_{0,-s}(\theta_1)\rangle\,.
\end{aligned}
\tag{6.132}
$$

The above amplitudes are further restricted by the unitarity and the crossing symmetry conditions

$$
\begin{aligned}
&A_0(\theta)A_0(-\theta) + A_1(\theta)A_1(-\theta) = 1\,,\quad B_0(\theta)B_0(-\theta) + B_1(\theta)B_1(-\theta) = 1\,, \\
&B_0(\theta) = A_0(\mathrm{i}\pi-\theta)\,,\quad B_1(\theta) = A_1(\mathrm{i}\pi-\theta)\,.
\end{aligned}
\tag{6.133}
$$

The minimal solution is given by [179]

$$
\begin{aligned}
&A_0(\theta) = \mathrm{e}^{-\mathrm{i}\gamma\theta}\cosh\tfrac{1}{4}\theta\,\Xi(\theta)\,,\quad A_1(\theta) = \mathrm{e}^{-\mathrm{i}\gamma\theta}\sinh\tfrac{1}{4}\theta\,\Xi(\theta) \\
&B_0(\theta) = \mathrm{e}^{\mathrm{i}\gamma\theta}(\cosh\tfrac{1}{4}\theta - \mathrm{i}\sinh\tfrac{1}{4}\theta)\Xi(\theta)\,,\quad B_1(\theta) = \mathrm{e}^{\mathrm{i}\gamma\theta}(\cosh\tfrac{1}{4}\theta + \mathrm{i}\sinh\tfrac{1}{4}\theta)\Xi(\theta)\,,
\end{aligned}
\tag{6.134}
$$

where $\mathrm{e}^{2\pi\mathrm{i}\gamma} = 2$ and $\Xi(\theta)$ is the following meromorphic function

$$
\Xi(\theta) = (\cosh\tfrac{1}{2}\theta)^{-1/2}\exp\left(\frac{\mathrm{i}}{4}\int_0^\infty \frac{\mathrm{d}t}{t}\sin(\theta t/\pi)\cosh^2\tfrac{1}{2}t\right). \tag{6.135}
$$

It satisfies the functional equations

$$
\Xi(\theta)\Xi(-\theta) = 1/\cosh^2\tfrac{1}{2}\theta\,,\quad \Xi(\theta) = \Xi(\mathrm{i}\pi-\theta)\,. \tag{6.136}
$$

The scattering amplitudes are periodic functions of θ with period equal to $8\pi\mathrm{i}$. They do not have poles on the physical sheet and therefore no additional bound states appear in the spectrum. It is worth noticing that the same scattering amplitudes are recovered by applying the general reduction procedure for the $\Phi_{1,3}$ deformation discussed in section 3.8.1, the only difference being in the notation used for denoting the kink states. The above picture of this off-critical theory has been confirmed both by a numerical analysis [134] and by the TBA approach [53].

6.5.3. Sub-leading magnetic perturbation

The perturbing field has anomalous dimensions $(\Delta, \bar{\Delta}) = (\frac{7}{16}, \frac{7}{16})$ and is odd with respect to the Z_2 spin-reversal transformation. Hence, this deformation explicitly breaks the Z_2 symmetry of the tricritical point and the corresponding massive theory can exhibit the Φ^3 property. The counting argument supports this picture, giving for the spin of the conserved currents the values $s = (1, 5, 7, 11, 13)$ [30, 134]. The interesting features of this massive field theory have been first outlined in ref. [134] where the model was studied by the conformal space truncation approach. The lowest energy levels with periodic boundary condition are given in fig. 22. The theory presents two degenerate ground states (which correspond to the minima of the asymmetric double-well Landau–Ginzburg potential in fig. 23) and a single excitation B of mass m below the threshold at $2m$. The twofold degeneracy of the vacuum permits two fundamental kink configurations $|K_+\rangle$ and $|K_-\rangle$ and, possibly, bound states thereof. If the two vacua were related by a symmetry transformation, i.e. if we were in the situation of a spontaneously broken symmetry, a double degeneracy of the breather-like bound state $|B\rangle$ would be expected in the infrared regime $R \to \infty$. However, the absence of a Z_2 symmetry makes it possible that in this case only one of the two asymptotic states $|K_+K_-\rangle$ or $|K_-K_+\rangle$ is coupled to the bound state $|B\rangle$. This is confirmed by the explicit solution of the model, proposed in [166] along the line of Smirnov's RSOS reduction of the Zhiber–Mikhailov–Shabat model. In this case, the only possible values of a_i which label the vacuum states in the RSOS S-matrix (see section 3.8.2) are 0 and 1. The one-particle states are thus the vectors: $|K_{01}\rangle$, $|K_{10}\rangle$ and $|K_{11}\rangle$. They correspond to the states that we previously denoted as $|K_+\rangle$, $|K_-\rangle$ and $|B\rangle$ respectively. All of them have the same mass m. Notice that the state

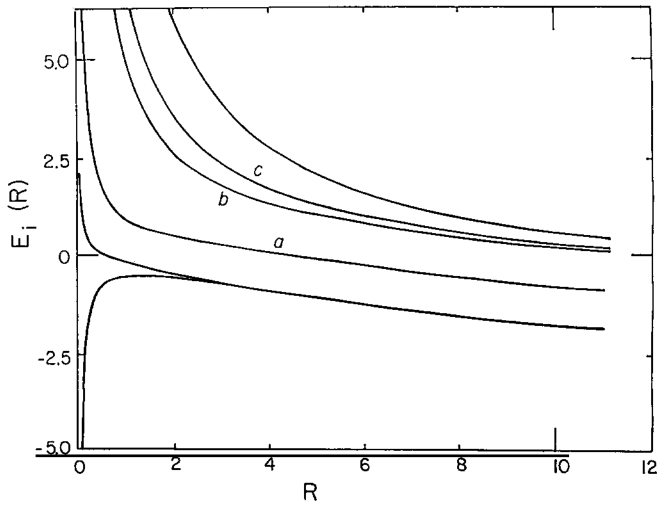


Fig. 22. First energy levels of the Hamiltonian associated to the sub-leading magnetic deformation of the tricritical Ising model.

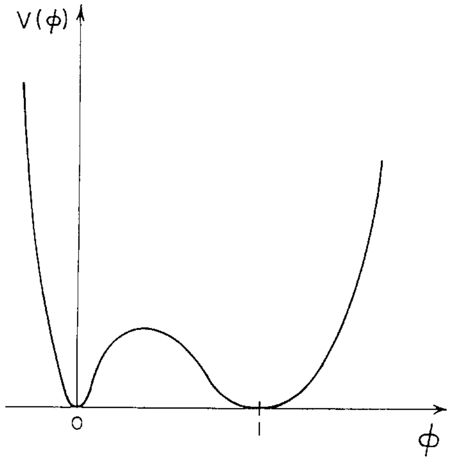


Fig. 23. Landau–Ginzburg potential for the sub-leading magnetic deformation of the tricritical Ising model.

$|K_{00}\rangle$ is projected out because of the reduction. The scattering processes are given by

$$
\begin{aligned}
&K_{01}(\theta_1)K_{10}(\theta_2)\rangle = S_{00}^{11}(\theta_1-\theta_2)|K_{01}(\theta_2)K_{10}(\theta_1)\rangle\,,\\
&|K_{01}(\theta_1)K_{11}(\theta_2)\rangle = S_{01}^{11}(\theta_1-\theta_2)|K_{01}(\theta_2)K_{11}(\theta_1)\rangle\,,\\
&|K_{11}(\theta_1)K_{10}(\theta_2)\rangle = S_{10}^{11}(\theta_1-\theta_2)|K_{11}(\theta_2)K_{10}(\theta_1)\rangle\,,\\
&|K_{11}(\theta_1)K_{11}(\theta_2)\rangle = S_{11}^{11}(\theta_1-\theta_2)|K_{11}(\theta_2)K_{11}(\theta_1)\rangle + S_{11}^{10}(\theta_1-\theta_2)|K_{10}(\theta_2)K_{01}(\theta_1)\rangle\,,\\
&|K_{10}(\theta_1)K_{01}(\theta_2)\rangle = S_{11}^{00}(\theta_1-\theta_2)|K_{10}(\theta_2)K_{01}(\theta_1)\rangle + S_{11}^{10}(\theta_1-\theta_2)|K_{11}(\theta_2)K_{11}(\theta_1)\rangle\,.
\end{aligned}
\tag{6.137}
$$

Explicitly, the above amplitudes are given by

$$
0 \overset{1}{\underset{0}{\times}} 0 = S_{00}^{11}(\theta) = \mathrm{i}\,\tfrac{1}{2}S_0(\theta)\sinh(\tfrac{9}{5}\theta - \mathrm{i}\,\tfrac{1}{5}\pi)\,, \tag{6.138a}
$$

$$
0 \overset{1}{\underset{1}{\times}} 1 = S_{01}^{11}(\theta) = -\mathrm{i}\,\tfrac{1}{2}S_0(\theta)\sinh(\tfrac{9}{5}\theta + \mathrm{i}\,\tfrac{1}{5}\pi)\,, \tag{6.138b}
$$

$$
1 \overset{1}{\underset{1}{\times}} 1 = S_{11}^{11}(\theta) = \mathrm{i}\,\tfrac{1}{2}S_0(\theta)\,\frac{\sin(\frac{1}{5}\pi)}{\sin(\frac{2}{5}\pi)}\,\sinh(\tfrac{9}{5}\theta - \mathrm{i}\,\tfrac{2}{5}\pi)\,, \tag{6.138c}
$$

$$
1 \overset{1}{\underset{0}{\times}} 1 = S_{11}^{01}(\theta) = -\mathrm{i}\,\tfrac{1}{2}S_0(\theta)\left(\frac{\sin(\frac{1}{5}\pi)}{\sin(\frac{2}{5}\pi)}\right)^{1/2}\sinh(\tfrac{9}{5}\theta)\,, \tag{6.138d}
$$

$$
1 \overset{0}{\underset{0}{\times}} 1 = S_{11}^{00}(\theta) = -\mathrm{i}\,\tfrac{1}{2}S_0(\theta)\,\frac{\sin(\frac{1}{5}\pi)}{\sin(\frac{2}{5}\pi)}\,\sinh(\tfrac{9}{5}\theta + \mathrm{i}\,\tfrac{2}{5}\pi)\,. \tag{6.138e}
$$

The function $S_0(\theta)$ which implements the unitarity condition reads

$$S_0(\theta) = -[\sinh(\tfrac{9}{10}(\theta - \mathrm{i}\pi))\sinh(\tfrac{9}{10}(\theta - \tfrac{2}{3}\pi\mathrm{i}))]^{-1} w(\theta, -\tfrac{1}{5}) w(\theta, \tfrac{1}{10}) w(\theta, \tfrac{3}{10})$$
$$\times t(\theta, \tfrac{2}{9}) t(\theta, -\tfrac{8}{9}) t(\theta, \tfrac{7}{9}) t(\theta, -\tfrac{1}{9})\,, \tag{6.139}$$

where

$$w(\theta, x) = \frac{\sinh(\frac{9}{10}\theta + \mathrm{i}\pi x)}{\sinh(\frac{9}{10}\theta - \mathrm{i}\pi x)}\,, \quad t(\theta, x) = \frac{\sinh(\frac{1}{2}(\theta + \mathrm{i}\pi x))}{\sinh(\frac{1}{2}(\theta - \mathrm{i}\pi x))}\,.$$

Note that a non-trivial charge conjugation operator appears in the crossing symmetry transformations. However, it is always possible to implement the crossing symmetry in a standard fashion by using "gauge" transformed amplitudes, as clarified in refs. [95, 166]. This consists in a change of basis in the space of asymptotic states and, correspondingly

$$\tilde{S}^{a_k a'_k}_{a_{k-1} a_{k+1}}(\theta_k - \theta_{k+1}) = (-1)^{(a_k - a'_k)/2}$$
$$= \left(\frac{[2a_k + 1]_q [2'_k + 1]_q}{[2a_{k-1} + 1]_q [2a_{k+1} + 1]_q}\right)^{-\theta/2\pi\mathrm{i}} S^{a_k a'_k}_{a_{k-1} a_{k+1}}(\theta_k - \theta_{k+1})\,, \tag{6.140}$$

where

$$[y]_q = (q^{y/2} - q^{-y/2})/(q^{1/2} - q^{-1/2})\,.$$

In the new basis, the crossing relations become trivial

$$\tilde{S}^{a_k a'_k}_{a_{k-1} a_{k+1}}(\mathrm{i}\pi - \theta) = \tilde{S}^{a_{k-1} a_{k+1}}_{a'_k a_k}(\theta)\,. \tag{6.141}$$

But, the price we pay by performing such procedure is that the new amplitudes have an oscillatory behaviour for $\theta \to \infty$, which might be inconvenient for a comparison of the ultraviolet limit of the S-matrix with the underlying CFT (see below).

The amplitudes (3.3) are periodic along the imaginary axis of θ with period $10\pi\mathrm{i}$. The whole structure of poles and zeros is quite rich. On the physical sheet, $0 \leq \mathrm{Im}\,\theta \leq \mathrm{i}\pi$, the poles of the S-matrix are located at $\theta = \frac{2}{3}\pi\mathrm{i}$ and $\theta = \frac{1}{3}\pi\mathrm{i}$ (fig. 24). The first pole corresponds to a bound state in the direct channel whereas the second one is the singularity due to the particle exchanged in the crossed process. The residues at $\theta = \frac{2}{3}\pi\mathrm{i}$ are given by

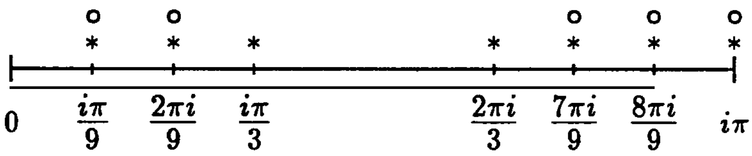


Fig. 24. Pole structure of $S_0(\theta)$ in the RSOS S-matrix of the sub-leading magnetic deformation of the tricritical Ising model.

$$
\begin{aligned}
r_1 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{00}^{11}(\theta) = 0\,, \\
r_3 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{01}^{11}(\theta) = \mathrm{i}[s(\tfrac{2}{5})/s(\tfrac{1}{5})]^2\omega\,, \\
r_3 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{11}^{11}(\theta) = \mathrm{i}\omega\,, \\
r_4 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{11}^{01}(\theta) = \mathrm{i}[s(\tfrac{2}{5})/s(\tfrac{1}{5})]^{1/2}\omega\,, \\
r_5 &= \mathrm{Res}_{\theta=2\pi\mathrm{i}/3}\, S_{11}^{00}(\theta) = \mathrm{i}[s(\tfrac{2}{5})/s(\tfrac{1}{5})]\omega\,,
\end{aligned}
\tag{6.142}
$$

where

$$
\omega = \frac{5}{9}\,\frac{s(\frac{1}{5})s(\frac{1}{10})s(\frac{4}{9})s(\frac{1}{9})s^2(\frac{5}{18})}{s(\frac{3}{10})s(\frac{1}{18})s(\frac{7}{18})s^2(\frac{2}{9})}\,, \tag{6.143}
$$

and $s(x) \equiv \sin(\pi x)$. In the amplitude S_{00}^{11} there is no bound state in the direct channel but only the singularity coming from the state $|K_{11}\rangle$ exchanged in the t-channel. This is easily seen from fig. 19 where we stretch the original amplitudes along the vertical direction (s-channel) and along the horizontal one (t-channel). Since the state $|K_{00}\rangle$ is not physical, the residue in the direct channel is zero.

The above scattering theory easily explains the spectrum obtained by the CSTA. In fact, the one-particle line a of fig. 22 corresponds to the state $|K_{11}\rangle$. This energy level is not doubly degenerate because the state $|K_{00}\rangle$ is forbidden by the RSOS selection rules. With periodic boundary conditions, the kink states $|K_{01}\rangle$ and $|K_{10}\rangle$ are projected out and $|K_{11}\rangle$ is only one-particle state that can appear in the spectrum.

The same analysis of the phase-shifts that we pursued for the $\Phi_{2,1}$ deformation of the minimal model $\mathcal{M}_{3,5}$ can be repeated here. For real values of θ, the amplitudes $S_{00}^{11}(\theta)$ and $S_{01}^{11}(\theta)$ are numbers of modulus 1 and can be parameterized as

$$
S_{00}^{11}(\theta) \equiv \exp[2\mathrm{i}\delta_0(\theta)]\,, \quad S_{01}^{11}(\theta) \equiv \exp[2\mathrm{i}\delta_1(\theta)]\,. \tag{6.144}
$$

The non-diagonal sector of the scattering processes is characterized by the 2×2 symmetric S-matrix

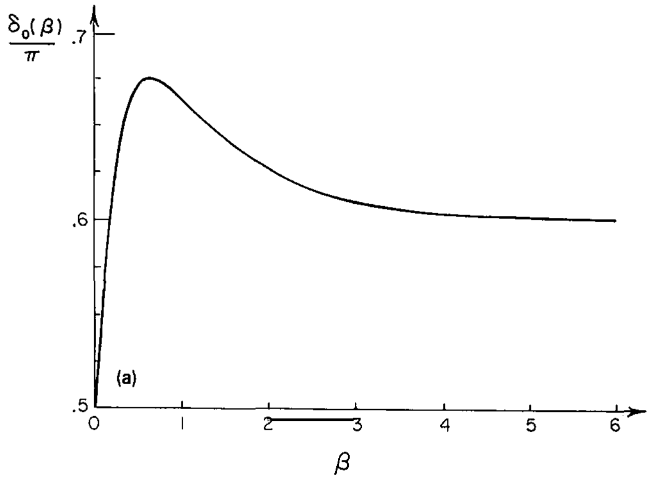
$$
\begin{pmatrix} S_{11}^{11}(\theta) & S_{11}^{01}(\theta) \\ S_{11}^{01}(\theta) & S_{11}^{00}(\theta) \end{pmatrix}. \tag{6.145}
$$

The eigenvalues of this matrix coincide with the same functions in (6.144),

$$
\begin{pmatrix} \mathrm{e}^{2\mathrm{i}\delta_0(\theta)} & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\delta_1(\theta)} \end{pmatrix}. \tag{6.146}
$$

The phase shifts, for positive values of θ, are shown in fig. 25. Asymptotically, they have the limits

$$
\lim_{\theta\to\pm\infty} \exp[2\mathrm{i}\delta_0(\theta)] = \exp(\pm\tfrac{6}{5}\pi\mathrm{i})\,, \quad \lim_{\theta\to\pm\infty} \exp[2\mathrm{i}\delta_1(\theta)] = \exp(\pm\tfrac{3}{5}\pi\mathrm{i})\,. \tag{6.147}
$$



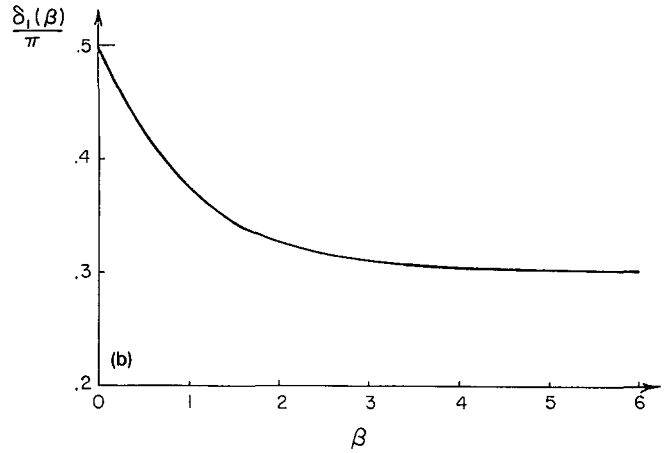


Fig. 25. Phase-shifts of the RSOS S-matrix of the sub-leading magnetic deformation of the tricritical Ising model.

There is a striking difference between the two phase shifts: while $\delta_1(\theta)$ is a monotonic decreasing function, starting from its value at zero energy $\delta_1(0) = \frac{1}{2}\pi$, $\delta_0(\theta)$ shows a maximum for $\theta \sim \frac{1}{3}\pi$ and then decreases to its asymptotic value $\frac{3}{5}\pi$. Its values are always larger that $\delta_0(0) = \frac{1}{2}\pi$. Such different behaviour of the phase shifts is related to the presence of a zero very close to the real axis in the amplitude $e^{2i\delta_0(\theta)}$, i.e. at $\theta = i\frac{1}{9}\pi$. This zero competes with the pole at $\theta = i\frac{1}{3}\pi$ in creating a maximum in the phase shift. Similar behaviour also occurs in non-relativistic cases [169] and in the case of breather-like S-matrices which contain zeros [170]. The presence of such a zero is deeply related to the absence of the pole in the s-channel of the amplitude $e^{2i\delta_0(\theta)}$. For the amplitude $e^{2i\delta_1(\theta)}$, the zero is located at $\theta = \frac{4}{9}\pi i$ (between the two poles) and therefore its contribution to the phase shift is damped with respect to the one resulting from the poles. The overall effect is a monotonic decreasing phase shift.

Coming back to the 2×2 S-matrix of eq. (6.145), the unitary transformation we have to perform to diagonalize it turns out to be independent of the rapidity θ. A basis of eigenvectors is given by

$$\begin{aligned} |\phi_1(\theta_1)\phi_1(\theta_2)\rangle &= \frac{1}{\sqrt{1+a^2}}\left(|K_{11}(\theta_1)K_{11}(\theta_2)\rangle + a|K_{10}(\theta_1)K_{01}(\theta_2)\rangle\right), \\ |\phi_1(\theta)\phi_1(\theta_2)\rangle &= \frac{1}{\sqrt{1+a^2}}\left(a|K_{11}(\theta_1)K_{11}(\theta_2)\rangle - |K_{10}(\theta_1)K_{01}(\theta_2)\rangle\right), \end{aligned} \tag{6.148}$$

where

$$a = -(2\cos\tfrac{1}{5}\pi)^{-1/2}. \tag{6.149}$$

The "kinks" ϕ_1 and ϕ_2 have the generalized bilinear commutation relation [100, 101, 93]

$$\phi_i(t,x)\phi_j(t,y) = \phi_j(t,y)\phi_i(t,x)\exp[2\pi \mathrm{i} s_{ij}\,\varepsilon(x-y)], \tag{6.150}$$

with s_{ij} given by

$$s_{11} = \tfrac{3}{5} = \delta_0(\infty)/\pi, \quad s_{12} = 0, \quad s_{22} = \tfrac{3}{10} = \delta_1(\infty)/\pi. \tag{6.151}$$

It is easy to prove that these monodromy properties coincide with those of the chiral field $\Psi = \Phi_{6/10,0}$ of the original CFT of the TIM, i.e.

$$\Psi(z)\Psi(0) = \frac{1}{z^{2/5}}\,\mathbf{1} + \frac{C_{\Psi\Psi\Psi}}{z^{1/5}}\,\Psi(0) + \cdots, \tag{6.152}$$

where $C_{\psi\Psi\Psi}$ is the structure constant of the OPE algebra.

A different S-matrix for this deformation of the tricritical Ising model has been proposed by Zamoldchikov [167]. A comparison between the two scattering models has been performed in ref. [166] by using the CSTA approach and the predictions of finite-size theory [168]. We refer the reader to the original literature for the discussion of this problem.

6.6. Thermal perturbation of the three-state Potts model

The partition function of the lattice formulation of the 3-state Potts model is given by

$$Z(\beta) = \sum_{\{\sigma\}} \exp\Big(\beta \sum_{\langle i,j\rangle} \tfrac{1}{2}(\sigma_i\bar{\sigma}_j + \text{c.c.})\Big), \tag{6.153}$$

where $\sigma = \exp(\mathrm{i}\varphi)$ and $\varphi = 0, \pm\frac{2}{3}\pi$. The model is invariant under the group of permutations S_3. The group S_3 is isomorphic to the dihedral group D_3 hence is a semidirect product of two abelian groups, Z_3 and Z_2 with generators ϑ and C

$$\vartheta^3 = C^2 = 1, \quad C\vartheta = \bar{\vartheta}C. \tag{6.154}$$

The operator C corresponds to the charge conjugation symmetry. The irreducible representations of S_3 are both one-dimensional and bidimensional.

The three-state Potts model also possesses order–disorder symmetry (self-duality) with the self-dual point equal to

$$\beta_c = \tfrac{2}{3}\ln(\sqrt{3}+1)\,. \tag{6.155}$$

At $\beta = \beta_c$, the model undergoes a second order phase transition [145]. The corresponding critical theory is described by a subset of operators of the minimal models $\mathcal{M}_{5,6}$ with central charge $c = \frac{4}{5}$ [35, 24]. These operators enter the modular invariant partition function of the type (D, A) in the notation of Cappelli, Itzykson and Zuber [25] and they transform according to the irreducible representations of the group S_3. There are two spin density doublets corresponding to the primary fields $\sigma = \Phi_{1/15,1/15}$ and $\sigma' = \Phi_{2/3,2/3}$ which transform according to the bidimensional representation of S_3. The chiral fields $V = \Phi_{7/5,2/5}$ and $\bar{V} = \Phi_{2/5,7/5}$ with spin 1 together with the spin-3 fields $W = \Phi_{3,0}$ and $\bar{W} = \Phi_{0,3}$ are neutral under Z_3 but they change sign under the charge conjugation C. The list of operators also includes a set of three scalar energy operators $\varepsilon = \Phi_{2/5,2/5}$, $\varepsilon' = \Phi_{7/5,7/5}$, $\varepsilon'' = \Phi_{3,3}$ which are invariant under the whole group S_3.

Equivalently, we can consider the three-state Potts model as the class of universality of the critical Landau–Ginzburg Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi)(\partial_\mu \Phi^*) + \lambda[(\Phi)^3 + (\Phi^*)^3]\,, \tag{6.156}$$

where Φ is a complex scalar field [16]. The leading magnetic operators σ are identified with the fields Φ and Φ^* whereas the second magnetic doublet σ' corresponds to $(\Phi^*)^2\Phi$ and $\Phi^*\Phi^2$. The chiral operators V and W correspond to $(\Phi^*\,\partial\Phi - \Phi\,\partial\Phi^*)$ and $(\Phi^*\,\partial^3\Phi - \Phi\,\partial^3\Phi^*)$, respectively. The relevant operator ε of the energy sector is easily identified with $\Phi^*\Phi$.

The Potts model can be shifted away from its critical temperature by adding to the Hamiltonian the relevant operator ε

$$H = H_c + \lambda \int \varepsilon(x)\, \mathrm{d}^2x\,. \tag{6.157}$$

The field theory defined by (6.157) is integrable and characterized by a purely elastic S-matrix [143, 146, 147]. In terms of the Landau–Ginzburg description, the thermal perturbation of the three-state Potts model is equivalent to an insertion of a mass term $m^2\Phi^*\Phi$ into the Lagrangian (6.156). The perturbed conformal field theory is still invariant under S_3 and the massive particles are characterized by their properties under S_3. The fundamental asymptotic states of the scattering theory form a doublet of particle–antiparticle $(A, \bar{A})$ of mass m. They form a representation of S_3 with the properties

$$\vartheta A = \omega A\,, \quad \vartheta\bar{A} = \bar{\omega}\bar{A}\,, \quad CA = \bar{A}\,, \tag{6.158}$$

where $\omega = \exp(\frac{2}{3}\pi\mathrm{i})$. The most general two-particle S-matrix is given by

$$\begin{aligned} |A(\theta_1)A(\theta_2)\rangle_{\text{in}} &= u(\theta_{12})|A(\theta_1)A(\theta_2)\rangle_{\text{out}}\,, \\ |A(\theta_1)\bar{A}(\theta_2)\rangle_{\text{in}} &= t(\theta_{12})|A(\theta_1)\bar{A}(\theta_2)\rangle_{\text{out}} + r(\theta_{12})|\bar{A}(\theta_1)A(\theta)\rangle_{\text{out}}\,. \end{aligned} \tag{6.159}$$

As consequence of the higher integrals of motion, the coefficient of reflection vanishes and the S-matrix is completely diagonal [146]. The crossing symmetry implies

$$t(\theta) = u(\mathrm{i}\pi - \theta)\,, \tag{6.160}$$

and unitarity leads to

$$t(\theta)t(-\theta) = 1\,, \quad u(\theta)u(-\theta) = 1\,. \tag{6.161}$$

The minimal solution of these equations is given by [146]

$$u(\theta) = \frac{\sinh(\frac{1}{2}\theta + \frac{1}{3}\mathrm{i}\pi)}{\sinh(\frac{1}{2}\theta - \frac{1}{3}\mathrm{i}\pi)}\,, \quad t(\theta) = \frac{\sinh(\frac{1}{2}\theta + \frac{1}{6}\mathrm{i}\pi)}{\sinh(\frac{1}{2}\theta - \frac{1}{6}\mathrm{i}\pi)}\,. \tag{6.162}$$

The antiparticle $\bar{A}$ appears as a bound state of two particles A and vice versa.

The finite-size effects of the relativistic theory defined by (6.162) can be analyzed by means of the TBA [48]. The integral equations for the two species of particles are given by

$$mR\cosh\theta + \varepsilon_A + \varphi_{AA} * L_A + \varphi_{A\bar{A}} * L_{\bar{A}} = 0\,, \quad mR\cosh\theta + \varepsilon_{\bar{A}} + \varphi_{\bar{A}A} * L_A + \varphi_{\bar{A}\bar{A}} * L_{\bar{A}} = 0\,, \tag{6.163}$$

where

$$L_A = \ln[1 + \exp(-\varepsilon_A)],\ L_{\bar{A}} = \ln[1 + \exp(-\varepsilon_{\bar{A}})]\,, \tag{6.164}$$

$$\begin{aligned} \varphi_{AA}(\theta) = \varphi_{\bar{A}\bar{A}} &= -\frac{\sqrt{3}}{2\cosh\theta + 1}\,, \\ \varphi_{A\bar{A}}(\theta) = \varphi_{\bar{A}A} &= -\frac{\sqrt{3}}{2\cosh\theta - 1}\,. \end{aligned} \tag{6.165}$$

The ground state energy of the theory on a cylinder with width R is thus given by

$$E(R) = m\int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2\pi}\cosh\theta\,[L_A(\theta) + L_{\bar{A}}(\theta)]\,. \tag{6.166}$$

In order to study the thermodynamics of the charge symmetric sector we impose the condition

$$\varepsilon_A(\theta) = \varepsilon_{\bar{A}}(\theta)\,. \tag{6.167}$$

The analysis in the other sectors has been pursued in refs. [56, 57]. Taking into account eq. (6.167), the TBA equations (6.163) reduce to a single integral relation

$$-Rm\cosh\theta + \varepsilon + \varphi * \ln(1 + \mathrm{e}^{-\varepsilon}) = 0\,, \tag{6.168}$$

where

$$\varphi(\theta) = \varphi_{AA}(\theta) + \varphi_{A\bar{A}} = -\frac{2\sqrt{3}\sinh(2\theta)}{\sinh(3\theta)} . \tag{6.169}$$

In the ultraviolet limit $R \to 0$, the pseudo-energy ε goes to the limiting value

$$\varepsilon_0 = \ln[(\sqrt{5}+1)/2] , \tag{6.170}$$

and the ground energy takes the scaling form

$$E(R) = -2\pi/15R , \tag{6.171}$$

i.e. the massive field theory reduces to the critical three-state Potts model with central charge $c = \frac{4}{5}$.

6.7. Thermal perturbation of the three-state tricritical Potts model

The tricritical version of the three-state Potts model is identified with a subset of the minimal conformal model $\mathcal{M}_{6,7}$ [13]. Similarly to the Potts model, its tricritical version is invariant under the permutation group S_3. Its thermal perturbation is realized by adding to the critical action the energy operator $\Phi_{1,2}$ with anomalous dimensions $(\Delta, \bar{\Delta}) = (\frac{1}{7}, \frac{1}{7})$. This field is the most relevant S_3 invariant operator present in the Kac table of the model.

The off-critical model is an integrable quantum field theory and its exact S-matrix has been determined in refs. [156, 159]. Following ref. [156], we assume the existence of two doublets $(A_a, A_{\bar{a}})$ and $(A_b, A_{\bar{b}})$ with bootstrap fusions

$$A_a \times A_a \to A_{\bar{a}} + A_{\bar{b}} , \quad A_b \times A_b \to A_{\bar{a}} + A_{\bar{b}} , \tag{6.172}$$

and masses m_a and m_b ($m_a < m_b$). The analysis of the corresponding consistency equations (see section 3.5.2) leads to the following resonance angles:

$$\bar{U}^{a}_{a\bar{b}} = \tfrac{1}{12}\pi , \quad \bar{U}^{\bar{b}}_{a\bar{b}} = \tfrac{5}{12}\pi , \quad \bar{U}^{a}_{a\bar{a}} = \tfrac{1}{3}\pi . \tag{6.173}$$

According to the analysis of ref. [144], when the anti-particle $A_{\bar{a}}$ appears as a bound state in the scattering of two particles $A_a A_a$, the corresponding reflection amplitude vanishes, i.e. $S^R_{aa} = 0$. Therefore we are left with the following amplitudes for the lowest mass doublet $(A_a, A_{\bar{a}})$:

$$|A_a(\theta_1)A_a(\theta_2)\rangle_{\text{in}} = S_{aa}(\theta_{12})|A_a(\theta_1)A_a(\theta_2)\rangle_{\text{out}} , \quad |A_a(\theta_1)A_{\bar{a}}(\theta_2)\rangle_{\text{in}} = S^{\mathrm{T}}_{a,\bar{a}}(\theta_{12})|A_a(\theta_1)A_{\bar{a}}(\theta_2)\rangle_{\text{out}} . \tag{174}$$

The fusion $a \times a \to \bar{a}$ implies

$$S^{\mathrm{T}}_{a\bar{a}}(\theta) = S_{aa}(\theta - \mathrm{i}\tfrac{1}{3}\pi)S_{aa}(\theta + \mathrm{i}\tfrac{1}{3}\pi) , \quad S_{aa}(\theta) = S^{\mathrm{T}}_{a\bar{a}}(\theta - \mathrm{i}\tfrac{1}{3}\pi)S^{\mathrm{T}}_{a\bar{a}}(\theta + \mathrm{i}\tfrac{1}{3}\pi) . \tag{6.175}$$

Equivalently,

$$S_{aa}(\theta)S_{aa}(\theta - \mathrm{i}\tfrac{2}{3}\pi)S_{aa}(\theta + \mathrm{i}\tfrac{2}{3}\pi) = 1 . \tag{6.176}$$

The minimal solution of these equations that satisfies the unitary condition is

$$S_{aa}(\theta)=\frac{\sinh(\frac{1}{2}\theta+\mathrm{i}\frac{1}{3}\pi)\sinh(\frac{1}{2}\theta+\mathrm{i}\frac{1}{12}\pi)\sinh(\frac{1}{2}\theta+\mathrm{i}\frac{1}{4}\pi)}{\sinh(\frac{1}{2}\theta-\mathrm{i}\frac{1}{3}\pi)\sinh(\frac{1}{2}\theta-\mathrm{i}\frac{1}{12}\pi)\sinh(\frac{1}{2}\theta-\mathrm{i}\frac{1}{4}\pi)}\equiv s_{2/3}(\theta)s_{1/6}(\theta)s_{1/2}(\theta)\,. \tag{6.177}$$

S_{aa} has two simple poles with positive residue: the one at $\theta=\mathrm{i}\frac{2}{3}\pi$ corresponds to the particle $A_{\bar{a}}$ whereas the other one at $\theta=\mathrm{i}\frac{1}{6}\pi$ corresponds to the particle $A_{\bar{b}}$. Their mass ratio is

$$m_{\bar{b}}=m_b=2m_a\cos(\tfrac{1}{12}\pi)\,. \tag{6.178}$$

The additional pole at $\theta=\mathrm{i}\frac{1}{12}\pi$ has a negative residue and represents a bound state in the crossing channel. In fact,

$$S^{\mathrm{T}}_{a\bar{a}}(\theta)=S_{aa}(\mathrm{i}\pi-\theta)=-s_{1/3}(\theta)s_{1/2}(\theta)s_{5/6}(\theta)\,, \tag{6.179}$$

which has a simple pole with positive residue at $\theta=\mathrm{i}\frac{1}{2}\pi$. This pole is associated with a new neutral particle A_c, with mass

$$m_c=2m_a\cos(\tfrac{1}{4}\pi)\,. \tag{6.180}$$

The scattering amplitude $S_{\bar{a}b}$ is obtained from

$$S_{\bar{a}b}(\theta)=S_{\bar{a}\bar{a}}(\theta-\mathrm{i}\tfrac{1}{12}\pi)S_{\bar{a}b}(\theta+\mathrm{i}\tfrac{1}{12}\pi)\,.$$

The result is

$$S_{\bar{a}b}(\theta)=s_{3/4}(\theta)s_{1/4}(\theta)s_{1/12}(\theta)s_{5/12}(\theta)s^2_{7/12}(\theta)\,. \tag{6.181}$$

The pole structure of $S_{\bar{a}b}$ shows the presence of a new neutral particle A_d, entering the fusions

$$A_{\bar{a}}\times A_b\rightarrow A_c+A_d$$

with mass

$$m_d=4m_a\cos(\tfrac{1}{12}\pi)\cos(\tfrac{1}{4}\pi)\,. \tag{6.182}$$

It is possible to show that this set of six particles $\{A_a, A_{\bar{a}}, A_b, A_{\bar{b}}, A_c, A_d\}$ closes the bootstrap process and the full set of S-matrices is

$$S_{aa}=(\tfrac{1}{6})(\tfrac{2}{3})(\tfrac{1}{2})\,,\quad S_{\bar{a}\bar{a}}=S_{aa}\,,\quad S^{\mathrm{T}}_{a\bar{a}}=-(\tfrac{1}{3})(\tfrac{5}{6})(\tfrac{1}{2})\,,$$

$$S_{ab}=S_{\bar{a}\bar{b}}=(\tfrac{1}{4})(\tfrac{3}{4}(\tfrac{7}{12})(\tfrac{11}{12})(\tfrac{5}{12})^2\,,\quad S_{a\bar{b}}=S_{\bar{a}b}=(\tfrac{1}{12})(\tfrac{1}{4})(\tfrac{3}{4})(\tfrac{5}{12})(\tfrac{7}{12})^2\,,$$

$$S_{ac} = S_{\bar{a}c} = (\tfrac{1}{4})(\tfrac{3}{4})(\tfrac{5}{12})(\tfrac{7}{12}) , \quad S_{ad} = S_{\bar{a}d} = (\tfrac{1}{6})(\tfrac{5}{6})(\tfrac{1}{3})^2(\tfrac{2}{3})^2(\tfrac{1}{2})^2 ,$$

$$S_{bb} = (\tfrac{5}{6})(\tfrac{1}{6})^2(\tfrac{1}{3})^2(\tfrac{1}{6})^2(\tfrac{2}{3})^3(\tfrac{1}{2})^3 , \quad S^{\mathrm{T}}_{b\bar{b}} = -(\tfrac{1}{6})(\tfrac{5}{6})^2(\tfrac{2}{3})^2(\tfrac{1}{3})^3(\tfrac{1}{2})^3 ,$$

$$S_{bc} = S_{\bar{b}c} = (\tfrac{1}{6})(\tfrac{5}{6}(\tfrac{1}{2})^2(\tfrac{2}{3})^2(\tfrac{1}{3})^2 , \quad S_{bd} = S_{\bar{b}d} = (\tfrac{1}{12})(\tfrac{11}{12})(\tfrac{1}{4})^3(\tfrac{1}{4})^3(\tfrac{5}{12})^4(\tfrac{7}{12})^4 ,$$

$$S_{cc} = -(\tfrac{1}{6})(\tfrac{5}{6})(\tfrac{1}{3})(\tfrac{2}{3})(\tfrac{1}{2})^2 , \quad S_{cd} = (\tfrac{1}{12})(\tfrac{11}{12})(\tfrac{1}{4})^2(\tfrac{3}{4})^2(\tfrac{5}{12})^3(\tfrac{7}{12})^3 ,$$

$$S_{dd} = -(\tfrac{1}{6})^3(\tfrac{5}{6})^3(\tfrac{1}{3})^5(\tfrac{2}{3})^5(\tfrac{1}{2})^6 ,$$

where we use the short notation $(x) \equiv s_x(\theta)$. The resonance angles are collected in table 14. The TBA analysis of the ultraviolet limit of the above massive QFT supports the interpretation of the above scattering theory as off-critical model of the thermal perturbation of the Potts model [51]. Notice that the above scattering amplitudes are the minimal S-matrices of the E_6 affine Toda field theory. This is not surprising since the tricritical Potts model may be obtained as the coset construction $(E_6)_1 \otimes (E_6)_1 / (E_6)_2$.

6.8. Thermal perturbation of the O(n) critical models

The O(n) model is characterized by the isotropic ferromagnetic interaction of n-component spin variables S_i. The partition function of such a model on the "honeycomb" lattice is defined by [87]

$$Z(k, n) = \int \prod_i \mathrm{d}S_i \prod_{\langle k,j \rangle} (1 + kS_k \cdot S_j) . \tag{6.184}$$

Table 14
Resonance angles of the $E_6^{(1)}$ S-matrices.

$u^{\bar{a}}_{aa} = \frac{2}{3}\pi$	$u^{c}_{a\bar{a}} = \frac{1}{2}\pi$	$u^{\bar{a}}_{ab} = \frac{11}{12}\pi$	$u^{c}_{a\bar{b}} = \frac{3}{4}\pi$	$u^{a}_{ac} = \frac{3}{4}\pi$	$u^{b}_{ad} = \frac{5}{6}\pi$
$u^{b}_{aa} = \frac{1}{6}\pi$		$u^{\bar{b}}_{ab} = \frac{7}{12}\pi$	$u^{d}_{a\bar{b}} = \frac{1}{4}\pi$	$u^{b}_{ac} = \frac{5}{12}\pi$	
	$u^{a}_{\bar{a}\bar{a}} = \frac{2}{3}\pi$	$u^{c}_{ab} = \frac{3}{4}\pi$	$u^{a}_{\bar{a}\bar{b}} = \frac{11}{12}\pi$	$u^{\bar{a}}_{ac} = \frac{3}{4}\pi$	$u^{\bar{b}}_{ad} = \frac{5}{6}\pi$
	$u^{b}_{\bar{a}\bar{a}} = \frac{1}{6}\pi$	$u^{d}_{\bar{a}b} = \frac{1}{4}\pi$	$u^{b}_{\bar{a}\bar{b}} = \frac{7}{12}\pi$	$u^{\bar{b}}_{\bar{a}c} = \frac{5}{12}\pi$	
		$u^{\bar{a}}_{bb} = \frac{5}{12}\pi$	$u^{d}_{b\bar{b}} = \frac{1}{2}\pi$	$u^{a}_{bc} = \frac{5}{6}\pi$	$u^{a}_{bd} = \frac{11}{12}\pi$
		$u^{\bar{b}}_{bb} = \frac{2}{3}\pi$			$u^{b}_{bd} = \frac{3}{4}\pi$
			$u^{a}_{\bar{b}\bar{b}} = \frac{5}{12}\pi$	$u^{\bar{a}}_{\bar{b}c} = \frac{5}{6}\pi$	$u^{\bar{a}}_{\bar{b}d} = \frac{11}{12}\pi$
			$u^{b}_{\bar{b}\bar{b}} = \frac{2}{3}\pi$		$u^{\bar{b}}_{\bar{b}d} = \frac{3}{4}\pi$
				$u^{c}_{cc} = \frac{2}{3}\pi$	$u^{c}_{cd} = \frac{11}{12}\pi$
				$u^{d}_{cc} = \frac{1}{6}\pi$	$u^{d}_{cd} = \frac{7}{12}\pi$
					$u^{c}_{dd} = \frac{5}{6}\pi$
					$u^{d}_{dd} = \frac{2}{3}\pi$

Making use of

$$\int \mathrm{d}S = 1\,, \quad \int \mathrm{d}S\; S^a S^b = \delta^{ab}\,, \tag{6.185}$$

the partition function can be expressed as a sum over all closed non-intersecting loops on the lattice

$$Z(k, n) = \sum_G k^L n^r\,. \tag{6.186}$$

L is the total number of bonds and r is the number of loops. Equation (6.186) defines the O(n) model for arbitrary values of n. For $n \to 0$, the correlation functions of (6.186) describe the statistics of the self-avoiding polymer chains [88]. The series (6.186) is convergent for $k < k_{\mathrm{c}}$ where

$$k_{\mathrm{c}} = [2 + (2 - n)^{1/2}]^{-1/2}\,. \tag{6.187}$$

Hence, in the range $-2 < n < 2$, the model presents a second-order phase transition [87]. The corresponding conformal field theory has been identified by Dotsenko and Fateev [20]. Its central charge is given by

$$c = 1 - 6/p(p+1)\,, \tag{6.188}$$

where p is a function of n,

$$n = 2\cos(\pi/p)\,. \tag{6.189}$$

The scaling limit of the energy operator on the lattice corresponds to the primary operator $\Phi_{1,3}$ in the conformal model. Hence, the thermal perturbation of the critical point action is described by

$$A = A_{\mathrm{c}} + \tau \int \Phi_{1,3}(x)\, \mathrm{d}^2x\,. \tag{6.190}$$

The S-matrix of the massive excitations of this model has been proposed by Zamolodchikov [89]. On the basis of the form of the partition function (6.186), he argued that it is possible to interpret the loops as the trajectories of a set of n particles that belong to the vector representation of O(n). Hence, the scattering matrix for the process $A_{i_1}(\theta_1)A_{i_2}(\theta_2) \to A_{j_1}(\theta_1)A_{j_2}(\theta_2)$ can be decomposed as

$$S_{i_1 i_2}^{j_1 j_2}(\theta) = S_0(\theta)\delta_{i_1}^{j_1}\delta_{i_2}^{j_2} + S_1(\theta)\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} + S_2(\theta)\delta_{i_1 i_2}\delta^{j_1 j_2}\,. \tag{6.191}$$

The property that the loops entering (6.186) are non-intersecting paths implies

$$S_0(\theta) = 0\,. \tag{6.192}$$

The remaining amplitudes satisfy the crossing symmetry relation

$$S_1(\theta) = S_2(\mathrm{i}\pi - \theta)\,. \tag{6.193}$$

The general solution of the Yang–Baxter equations is given by

$$S_1(\theta) = \mathrm{i}\sinh[(\mathrm{i}\pi - \theta)/p]R(\theta)\,, \quad S_2(\theta) = \mathrm{i}\sinh(\theta/p)R(\theta)\,, \tag{6.194}$$

where $R(\theta)$ is an arbitrary crossing symmetric function,

$$R(\theta) = R(\mathrm{i}\pi - \theta)\,. \tag{6.195}$$

It can be fixed by imposing the unitary equations

$$S_1(\theta)S_1(-\theta) = 1\,, \quad S_1(\theta)S_2(-\theta) + S_2(\theta)S_1(-\theta) + nS_2(\theta)S_2(\theta) = 0\,. \tag{6.196}$$

The second equation of (6.196) is automatically satisfied by (6.194), using eq. (6.189). The first equation of (6.196) implies

$$R(\theta)R(-\theta) = -\{\sinh[(\mathrm{i}\pi - \theta)/p]\sinh[(\mathrm{i}\pi + \theta)/p]\}^{-1}\,. \tag{6.197}$$

The minimal solution of (6.195) and (6.197) is given by

$$\begin{aligned} R(\theta) = {} & \frac{1}{\sin[\pi(1/p - \theta/\mathrm{i}\pi p)]}\,\frac{\Gamma(1-\theta/\mathrm{i}\pi p)}{\Gamma(1+\theta/\mathrm{i}\pi p)} \\ & \times \prod_{k=1}^{\infty} \frac{\Gamma(2k/p - \theta/\mathrm{i}\pi p)\Gamma(1 + 2k/p - \theta/\mathrm{i}\pi p)}{\Gamma(2k/p + \theta/\mathrm{i}\pi p)\Gamma(1 + 2k/p + \theta/\mathrm{i}\pi p)} \\ & \times \frac{\Gamma((2k-1)/p + \theta/\mathrm{i}\pi p)\Gamma(1 + (2k-1)/p + \theta/\mathrm{i}\pi p)}{\Gamma((2k-1)/p - \theta/\mathrm{i}\pi p)\Gamma(1 + (2k-1)/p - \theta/\mathrm{i}\pi p)}\,. \end{aligned} \tag{6.198}$$

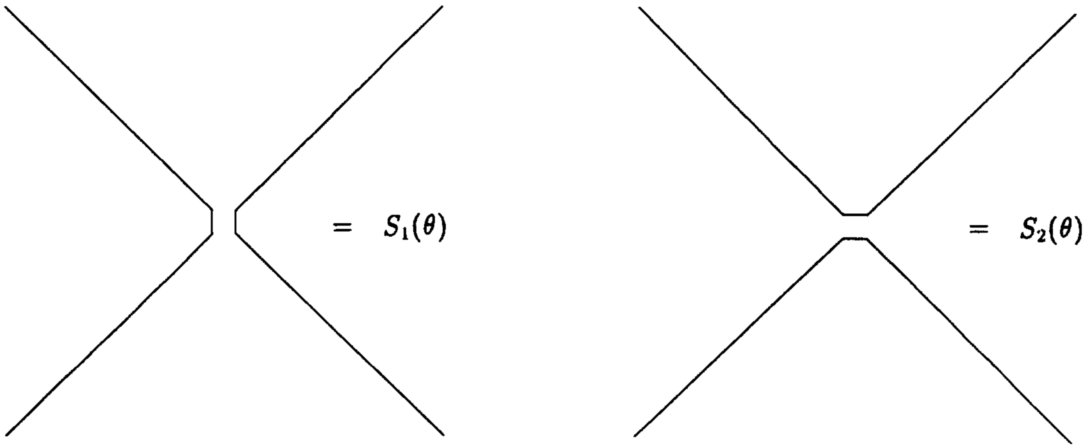


Fig. 26. Polymer interactions.

Notice that at $n = 1$

$$S_1(\theta) + S_2(\theta) = -1 \,, \tag{6.199}$$

i.e. the S-matrix reduces to that of thermal perturbation of Ising model. In the limit $n \to 0$, the amplitudes S_1 and S_2 can be interpreted as the two possible interactions of the polymer chains shown in fig. 26. A different interpretation of the limit $n \to 0$ has been proposed in ref. [90].

7. Affine Toda field theories

7.1. Basic properties

Much attention has been paid recently to the Affine Toda Field Theories (ATFT) as possible integrable Lagrangian theories for the off-critical models coming from deformations of CFT [157–159, 171, 172, 174, 176, 177, 184, 185, 189, 190]. Following Eguchi and Yang, the argument is the following: consider a CFT described by a coset $\mathcal{G}_k \otimes \mathcal{G}_1 / \mathcal{G}_{k+1}$ where $\mathcal{G}$ is a Lie algebra with rank r. In the usual decomposition of the first two algebras with respect to the third one, one can identify a field Φ paired with the adjoint representation of $\mathcal{G}_{k+1}$. Its anomalous dimension is given by $\Delta = (k+1)/(k + \tilde{\psi} + 1)$, where $\tilde{\psi}$ is the dual Coxeter number of the algebra $\mathcal{G}$. Using a Fegin–Fuchs construction in terms of free boson fields ϕ_i $(i = 1, \ldots, r)$, Φ can be represented by the vertex operator

$$\Phi = \exp(-\mathrm{i}\beta \boldsymbol{\alpha}_{r+1} \cdot \boldsymbol{\phi}) \,. \tag{7.1}$$

In (7.1) β is fixed to be

$$\beta = -\sqrt{(k + \tilde{\psi})/(k + \tilde{\psi} + 1)} \,, \tag{7.2}$$

and $\boldsymbol{\alpha}_{r+1}$ is the highest root of $\mathcal{G}$. Its expression in terms of the simple roots α_i is

$$\alpha_{r+1} = -\sum_{i=1}^{r} q_i \alpha_i \,. \tag{7.3}$$

The set of integers $\{q_i\}$ is specific to each algebra. The simple roots α_i enter the expression of the screening operators*)

$$\Phi_{\alpha_i} = \exp(\mathrm{i}\beta \boldsymbol{\alpha}_i \cdot \boldsymbol{\phi}) \,. \tag{7.4}$$

The effective Hamiltonian of the off-critical coset model $\mathcal{G}_k \otimes \mathcal{G}_1 / \mathcal{G}_{k+1}$ perturbed by the operator Φ is thus given by

*) We use the normalization $|\alpha_i|^2 = 2$.

$$H = \oint dz \left(\sum_{i=1}^{r} \exp(i\beta \boldsymbol{\alpha}_i \cdot \boldsymbol{\phi}) + \exp(-i\beta \boldsymbol{\alpha}_{r+1} \cdot \boldsymbol{\phi}) \right). \tag{7.5}$$

This is the affine Toda Hamiltonian (with imaginary coupling) based on the algebra $\mathcal{G}$. There is a large literature on these integrable models [213–215, 218–224]. Classically, the conserved charges have spins equal to the Coxeter exponents of $\mathcal{G}$ modulo the Coxeter number (see table 15).

There exists a strong belief among workers in the field that the off-critical description of the models of CFT is obtained by a quantum group reduction on the states of the Hamiltonian (7.5). An explicit realization of this reduction for SU(3) has been investigated in ref. [97]. However, a general proof of this mechanism for an arbitrary root system is still lacking. Rather than presenting a discussion on this interesting problem, we have decided to illustrate the properties of ATFT with real coupling, i.e. those defined by an analytic continuation $\beta \rightarrow -i\beta$. The reason is that they provide interesting integrable QFT whose S-matrices can be computed exactly. This allows us to formulate questions concerning the meaning of bound states, conserved charges, and perturbative calculations in a manageable setting.

For ATFT with real coupling, the Hamiltonian presents a ground state shifted from the origin: the minimum is at the points

$$\phi_{i0} \equiv \sum_j \alpha_i^j \phi_0^j = \tfrac{1}{2} \ln(q_i N), \quad N = \prod_{i=1}^{r} (n_i)^{-n_i/h}. \tag{7.6}$$

With the shift $\phi_j \rightarrow \phi^j - \phi_0^j$ the Lagrangian can be rewritten as

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{r} (\partial_\mu \phi_i)^2 - \frac{m_0^2}{\beta^2} \sum_{i=1}^{r+1} q_i [\exp(\beta \alpha_i^j \phi^j) - 1], \tag{7.7}$$

where m_0^2 sets the mass scale. In the case of non-simply laced algebras, for α_{r+1} one can also take the

Table 15
Coxeter numbers and exponents for affine Dynkin diagrams.

Algebra	h	Exponents
$A_r^{(1)}$	$r+1$	$1, 2, \ldots, r$
$A_{2r}^{(2)} \equiv A_{2r}/Z_2$	$4r+2$	$1, 3, 5, \ldots, 2r-1, 2r+3, \ldots, 4r+1$
$B_r^{(1)}$	$2r$	$1, 3, 5, \ldots, 2r-1$
$\tilde{B}_r \equiv D_{r+1}^{(2)}$	$2r+2$	$1, 3, 5, \ldots, 2r+1$
$C_r^{(1)}$	$2r$	$1, 3, 5, \ldots, 2r-1$
$\tilde{C}_r \equiv A_{2r-1}^{(2)}$	$4r-2$	$1, 3, 5, \ldots, 4r-3$
$D_r^{(1)}$	$2r-2$	$1, 3, 5, \ldots, 2r-3, r-1$
$E_6^{(1)}$	12	$1, 4, 5, 7, 8, 11$
$E_7^{(1)}$	18	$1, 5, 7, 9, 11, 13, 17$
$E_8^{(1)}$	30	$1, 7, 11, 13, 17, 19, 23, 29$
$G_2^{(1)}$	6	$1, 5$
$\tilde{G}_2 \equiv D_4^{(3)}$	12	$1, 5, 7, 11$
$F_4^{(1)}$	12	$1, 5, 7, 11$
$\tilde{F}_4 \equiv E_6^{(2)}$	18	$1, 5, 7, 11, 13, 17$

maximal short root. These second systems will be called twisted Toda systems and denoted with a tilde, $\tilde{\mathcal{G}}$. In both cases, the extended root system forms the Dynkin diagram of an affine (twisted) Lie algebra (see refs. [103, 187]). The basic properties of the integer numbers $\{q_i\}$ are (with $q_{r+1}=1$)

$$\sum_{i=1}^{r+1} q_i \alpha_i = 0\,, \quad \sum_{i=1}^{r+1} q_i = h\,. \tag{7.8}$$

For all untwisted models, h is equal to ψ, the Coxeter number of $\mathcal{G}$. In the twisted situation, it will coincide with the dual Coxeter number $\tilde{\psi}$, either of $\mathcal{G}$ itself or of another non-simply laced Lie algebra [158].

A geometrical interpretation of the interaction couplings obtained by expanding the exponential terms in (7.7) can be given as follows. Let us consider an electrostatic charge distribution given by $r+1$ electric positive charges $\{q_i\}$, placed at α_i. The total charge of this system will be h. The condition (7.8) is nothing but the definition of the charge-center frame and diagonalizing the quadratic term is equivalent to choosing the axes along the principal axes of the ellipsoid of the quadrupole moment. The higher couplings are thus the higher moments of this charge distribution.

It is important to distinguish between the affine Toda field theories constructed in terms of simply laced algebras (ADE) from those constructed in terms of the non-simply laced algebras B, C, F and G. The distinguished role of the ADE series arises from the fact that they give rise to consistent systems both at classical and quantum level. About the non-simply laced algebras, it turns out that although classically every-non-simply-laced theory can be obtained by the so-called "folding procedure" of the ADE series, this folding does not preserve the consistency of the quantum theory. Other degrees of freedom are necessary in order to construct a consistent quantum field theory for these models [178]. We will discuss the folding of non-simply laced ATFT in the next section and we will comment on the consistency of the quantum theories in section 8.7.2.

7.2. Non-simply laced ATFT: folding procedure

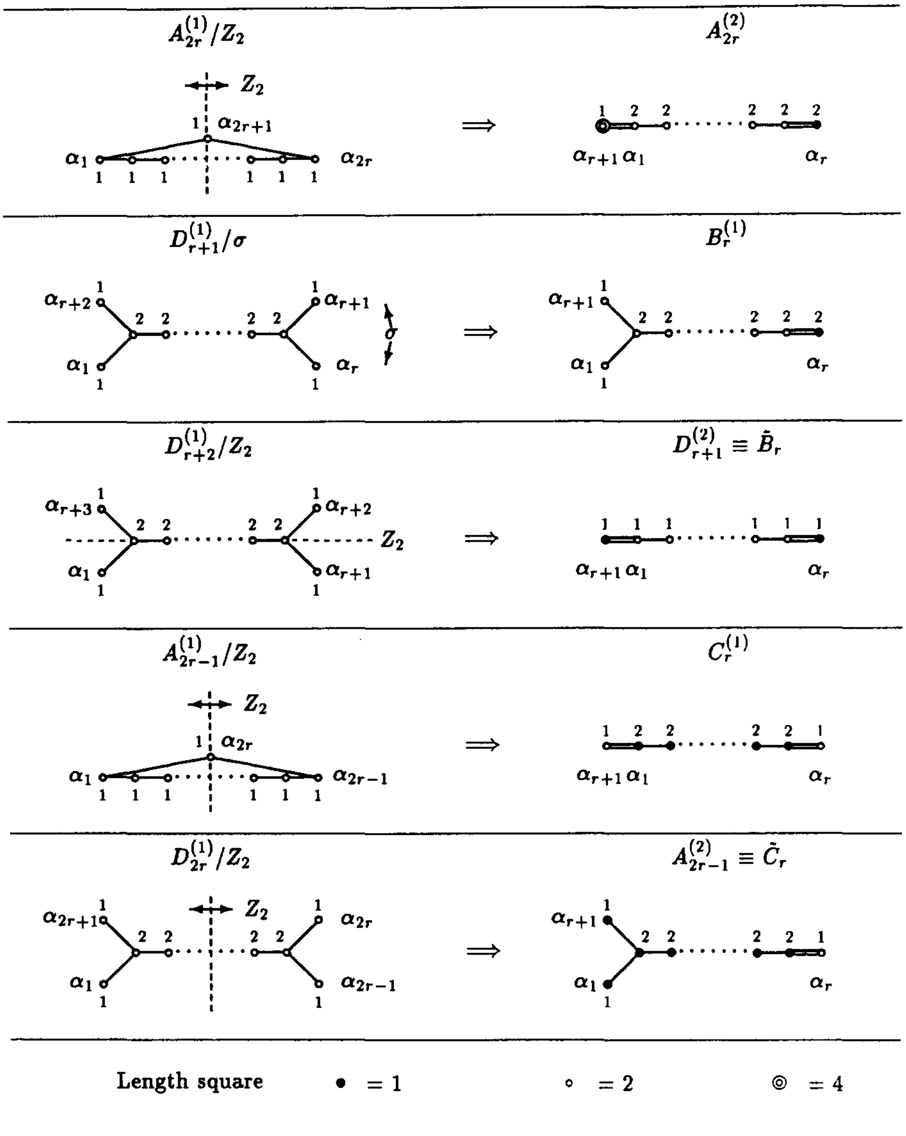
The Lagrangian of ATFT of non-simply laced algebras can be obtained by folding a Lagrangian based on simply laced root systems [158, 171, 223]. Namely, the discrete automorphisms of the Dynkin diagrams of the simply laced ATFT allow us to organize the fields in terms of conjugacy classes. The equations of motion respect this structure and therefore, if a field initially belongs to a subspace which is invariant under a discrete automorphism, its classical evolution remains in this subspace as well. This means that, for what the classical equations of motion concerns, we can consistently restrict the initial Lagrangian to the invariant subspaces. This reduction gives rise to the Lagrangians of non-simply laced ATFT. The complete list of these foldings is in tables 16 and 17. Rather than extensively analyzing these reductions, we concentrate on some significant examples.

We start with the most simple example, the $A_{2r}^{(2)}$ models versus the $A_{2r}^{(1)}$ ones. Their affine Dynkin diagrams are given in table 16. The simply laced $A_{2r}^{(1)}$ models have $2r$ particles, organized in degenerate pairs $(i, \bar{i} \equiv 2r+1-i)$, with masses

$$m_i = 2M \sin[\pi i/(2r+1)]\,, \quad 1 \leq i \leq r\,, \tag{7.9}$$

and $m_{\bar{i}} = m_i$. The mass degeneracy can be removed by imposing the following constraint on the fields ϕ_a,

Table 16
Foldings of simply laced Dynkin diagrams: the principal series. Near to the roots are the numbers q_i.

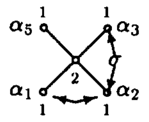
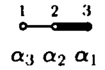
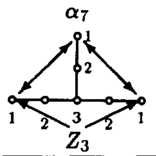
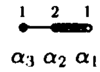
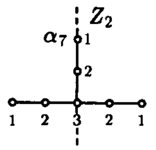
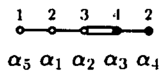
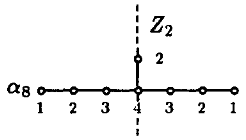
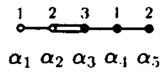


$$\sum_a \alpha_i^a \phi_a \equiv \sum_a \alpha_{2r+1-i}^a \phi_a\,, \quad 1 \leq i \leq r\,. \tag{7.10}$$

This amounts to folding the affine algebra with respect to the Z_2 symmetry of the Dynkin diagram, as depicted in tables 16 and 17. This operation projects out the odd sector of the original $A_{2r}^{(1)}$ theory. The remaining even fields must be rescaled by a factor 1/2 to ensure a correct normalization of the kinetic term in the new Lagrangian, that is the one for the $A_{2r}^{(2)}$ model. This non-simply laced model has r non degenerate masses given in eq. (7.9). Classically the conserved spins are

$$s = 1, 3, \ldots, 2r-1, 2r+3, \ldots, 4r+1 \quad (\mathrm{mod}\, 4r+2)\,. \tag{7.11}$$

Table 17
Foldings of simply laced Dynkin diagrams: the exceptional series. Near to the roots are the numbers q_i.

$D_4^{(1)}/\sigma$		$G_2^{(1)}$
	$\Longrightarrow$	
$E_6^{(1)}/Z_3$		$D_4^{(3)} \equiv \tilde{G}_2$
	$\Longrightarrow$	
$E_6^{(1)}/Z_2$		$F_4^{(1)}$
	$\Longrightarrow$	
$E_7^{(1)}/Z_2$		$E_6^{(2)} \equiv \tilde{F}_4$
	$\Longrightarrow$	
Length square	• = 1 or 2/3	∘ = 2

Let us consider another non-simply laced model, i.e. the twisted version G_2 (also denoted as $D_4^{(3)}$ in refs. [103, 187]). The parent theory is $E_6^{(1)}$ and this model is obtained by making a folding of the $E_6^{(1)}$ ATFT with respect to the Z_3 symmetry of the Dykin diagram. By folding of $E_6^{(1)}$ with respect to the Z_2 symmetry of its axis, we obtain as resulting model the ATFT on the root system $F_4^{(1)}$.

Our last example is F_4. This twisted model can be obtained from the $E_7^{(1)}$ model using the Z_2 symmetry of the root system. As shown in ref. [157], the particles of $E_7^{(1)}$ are classified according to their Z_2 parity: three particles are odd and four are even. The odd ones are projected out by the folding operation and only the four even particles appear in the twisted $\tilde{F}_4$ model.

7.3. Mass spectrum

Expanding the potential term in the Lagrangian (7.7) in a power series, we have

$$V(\phi) = m_0^2 \frac{1}{2} \sum_{i=1}^{r+1} q_i \alpha_i^a \alpha_i^b \phi^a \phi^b + m_0^2 \beta \frac{1}{6} \sum_{i=1}^{r+1} q_i \alpha_i^a \alpha_i^b \alpha_i^c \phi^a \phi^b \phi^c + \cdots \tag{7.12}$$

The quadratic term gives rise to the mass-matrix

$$M_{ab}^2 = m_0^2 \sum_{i=1}^{r+1} q_i \alpha_i^a \alpha_i^b . \tag{7.13}$$

The classical mass spectrum $\{m_i\}$ is determined by the roots of its characteristic equation

$$\|\boldsymbol{M}^2 - x \cdot \mathbf{1}\| = 0 . \tag{7.14}$$

Equation (7.14) is a polynomial of order r whose general form is

$$\mathcal{P}(x) = x^r - p_1 x^{r-1} - p_2 x^{r-2} - \cdots - p_r . \tag{7.15}$$

The first coefficient p_1 is simply the trace of $\boldsymbol{M}^2$ and for the simply laced algebras is twice the Coxeter number. The other coefficients p_i can be expressed in terms of the traces of higher powers of $\boldsymbol{M}^2$ as well. In order to simplify the notation, we put $\mathcal{M} = \boldsymbol{M}^2$. Their expression is thus given by

$$kp_k = a_k - p_1 a_{k-1} - \cdots p_{k-1} a_1 , \tag{7.16}$$

where

$$a_1 = \mathrm{Tr}\, \mathcal{M} = \sum_i m_i^2 , \quad a_2 = \mathrm{Tr}\, \mathcal{M}^2 = \sum_i m_i^4, \ldots, \quad a_n = \mathrm{Tr}\, \mathcal{M}^n = \sum_i m_i^{2n} . \tag{7.17}$$

As shown in ref. [228], it is convenient to introduce a matrix $\mathcal{N}$ directly related to the Dynkin diagrams. Its matrix elements are given by

$$\mathcal{N}_{ij} = (q_i \alpha_i, \alpha_j) = \sum_{k=1}^{n} q_i \alpha_i^k \alpha_j^k . \tag{7.18}$$

It is easy to prove that

$$\mathrm{Tr}\, \mathcal{M}^s = \mathrm{Tr}\, \mathcal{N}^s , \quad s = 1, 2, \ldots, n . \tag{7.19}$$

Hence, the characteristic equation of $\mathcal{M}$ coincides with that of $\mathcal{N}$. However, $\mathcal{N}$ is a $(n+1) \times (n+1)$ matrix whereas $\mathcal{M}$ is only a $n \times n$ matrix but α_0 is a linear combination of the other simple roots α_i. Consequently, $\mathcal{N}$ is a singular matrix. One of its eigenvalues is zero while the remaining ones coincide with the eigenvalues of $\mathcal{M}$.

In the basis of its eigenvectors, $M_{ij}^2 = m_i^2 \delta^{ij}$. The mass spectrum is degenerate whether the group of automorphisms of the non-affine Dynkin diagram of the Lie algebra is non-trivial. In these cases, it may be simpler to organize particles in complex conjugate pairs. In the case of simply laced theories an interesting result is that the masses can be organized in a vector

$$\boldsymbol{m} = (m_1, m_2, \ldots, m_r) ,$$

which is an eigenvector of the incidence matrix of the algebra $\mathcal{G}$ [44, 171, 174, 177]. Indeed, $\boldsymbol{m}$ is the Perron–Frobenius eigenvector of $\mathcal{G}$ and its components can be associated with the spots of the Dynkin diagram [225]. On the other hand, the spots of the Dynkin diagram define the fundamental representations in the case of simply laced theories of the algebra $\mathcal{G}$. We have in this way a correspondence between the particle with mass m_i and the relative representation of $\mathcal{G}$. This observation will be useful in the discussion of the bootstrap fusions and the decay processes in presence of multiple deformations of CFT.

In the next subsections, following ref. [228], we systematically analyse the mass spectrum for all Lie algebras. Similar computation can also be found in refs. [171, 220]. The results are given in table 18 and table 19. Let us start with the ADE series.

7.3.1. $A_n^{(1)}$ series

For the $A_n^{(1)}$ series, the matrix $\mathcal{N}$ reduces to the Cartan matrix of the affine Lie algebra. Hence, the characteristic equation associated to $\mathcal{N}$ is

$$\mathcal{Q}_{n+1}(x) = \begin{vmatrix} 2-x & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2-x & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2-x & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 2-x & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2-x \end{vmatrix} . \tag{7.20}$$

Table 18
Masses of models related by folding.

$A_{2r}^{(1)}$	$2M\sin(\pi i/(2r+1)),\quad 1\le i\le 2r$	$A_{2r}^{(2)}$	$4M\sin(\pi i/(2r+1)),\quad 1\le i\le r$
$D_{r+1}^{(1)}$	$M, M, 2M\sin(\pi i/2r),\quad 1\le i\le r-1$	$B_r^{(1)}$	$M, 2M\sin(\pi i/2r),\quad 1\le i\le r-1$
$D_{r+2}^{(1)}$	$M, M, 2M\sin(\pi i/(2r+2)),\quad 1\le i\le r$	$D_{r+1}^{(2)}\equiv\tilde{B}_r$	$\sqrt{2}M\sin(\pi i/(2r+2)),\quad 1\le i\le r$
$A_{2r-1}^{(1)}$	$2M\sin(\pi i/2r),\quad 1\le i\le 2r-1$	$C_r^{(1)}$	$2M\sin(\pi i/2r),\quad 1\le i\le r$
$D_{2r}^{(1)}$	$M, M, 2M\sin(\pi i/2(2r-1)),\quad 1\le i\le 2r-2$	$A_{2r-1}^{(2)}\equiv\tilde{C}_r$	$M/\sqrt{2}, \sqrt{2}M\sin(\pi i/(2r-1)),\quad 1\le i\le r-1$
$D_4^{(1)}$	$M, M, M, \sqrt{3}M$	$G_2^{(1)}$	$M, \sqrt{3}M$
$E_6^{(1)}$	$m_a = m_{\bar{a}} = M$ $m_b = m_{\bar{b}} = 2M\cos(\frac{1}{12}\pi)$ $m_c = 2M\cos(\frac{1}{4}\pi)$ $m_d = 4M\cos(\frac{1}{12}\pi)\cos(\frac{1}{4}\pi)$	$D_4^{(3)}\equiv\tilde{G}_2$ $F_4^{(1)}$	m_c, m_d m_a, m_b, m_c, m_d
$E_7^{(1)}$	$m_1 = M$ $m_2 = 2M\sin(\frac{2}{9}\pi)$ $m_3 = 2M\cos(\frac{1}{9}\pi)$ $m_4 = 2M\cos(\frac{1}{18}\pi)$ $m_5 = 4M\cos(\frac{1}{18}\pi)\sin(\frac{2}{9}\pi)$ $m_6 = \frac{1}{2}M\sin(\frac{1}{18}\pi)$ $m_7 = 4M\cos(\frac{1}{18}\pi)\cos(\frac{1}{9}\pi)$	$E_6^{(2)}\equiv\tilde{F}_4$	m_2, m_4, m_5, m_7

Table 19
$E_8^{(1)}$ mass spectrum.

$m_1 = M$	$m_5 = 4M\cos(\frac{1}{5}\pi)\cos(\frac{2}{15}\pi)$
$m_2 = 2M\cos(\frac{1}{5}\pi)$	$m_6 = 4M\cos(\frac{1}{5}\pi)\cos(\frac{1}{30}\pi)$
$m_3 = 2M\cos(\frac{1}{30}\pi)$	$m_7 = 2M\cos^2(\frac{1}{5}\pi)\cos(\frac{7}{30}\pi)$
$m_4 = 4M\cos(\frac{1}{5}\pi)\cos(\frac{7}{30}\pi)$	$m_8 = 8M\cos^2(\frac{1}{5}\pi)\cos(\frac{2}{15}\pi)$

Putting $2y = 2 - x$, it is possible to show (see e.g. ref. [229]) that

$$\mathcal{Q}_{n+1} = 2[\mathcal{T}_{n+1}(y) - 1] , \tag{7.21}$$

where $\mathcal{T}_{n+1}(y)$ is the Chebyshev polynomial of the first kind,

$$\mathcal{T}_{n+1}(\cos\theta) = \cos(n+1)\theta . \tag{7.22}$$

The mass spectrum of the $A_n^{(1)}$ series is given by the n non-zero roots of the equation $\mathcal{T}_{n+1}(y) = 1$, i.e.

$$m_k^2 = 4\sin^2[k\pi/(n+1)] , \quad k = 1, 2, \ldots, n . \tag{7.23}$$

7.3.2. $D_n^{(1)}$ *series*

For this series, the expression of $\mathcal{N}$ is the following:

$$\mathcal{N} = \begin{vmatrix} 4 & -2 & 0 & \cdots & 0 & 0 & -2 & -2 \\ -2 & 4 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & \cdots & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \end{vmatrix} . \tag{7.24}$$

The characteristic equation has the form

$$\|\mathcal{N} - x \cdot \mathbf{1}\| = 2^{n+2}(y-1)(2y-1)^2 \mathcal{U}_{n-2}(y) = 0 , \tag{7.25}$$

where we put $x = 4(1-y)$. $\mathcal{U}_m$ is the Chebyshev polynomial of the second kind. The roots of (7.25) are given by

$$y_{n+1} = 1 \;\rightarrow\; x_{n+1} = 0 , \quad y_n = \tfrac{1}{2} \;\rightarrow\; x_n = 2 , \quad y_{n-1} = \tfrac{1}{2} \;\rightarrow\; x_{n-1} = 2 \tag{7.26}$$

and by $\mathcal{U}_{n-2}(y) = 0$, i.e.

$$y_k = \cos\left(\frac{k\pi}{n-1}\right) \;\rightarrow\; x_k = 8\sin^2\left(\frac{k\pi}{2(n-1)}\right) , \quad k = 1, 2, \ldots, n-2 . \tag{7.27}$$

The first root in (7.26) is irrelevant for the computation of the mass spectrum. The resulting mass spectrum is given in table 18.

7.3.3. E_n *series*

The analysis of this exceptional series is carried out by considering each of the characteristic equations separately.

(i) The characteristic equation for the mass spectrum of the ATFT built on the E_6 algebra is

$$\begin{aligned} \|\mathcal{M} - x \cdot \mathbf{1}\| &= x^6 - 24x^5 + 216x^4 - 936x^3 + 2052x^2 - 2160x + 864 \\ &= (x^2 - 12x + 24)(x^2 - 6x + 6)^2 . \end{aligned} \tag{7.28}$$

There are two doublets of degenerate particles plus two other particles with different masses. The mass spectrum can be found in table 18.

(ii) The characteristic equation for the ATFT built on the E_7 algebra is

$$\begin{aligned} \|\mathcal{M} - x\cdot\mathbf{1}\| &= x^7 - 36x^6 + 504x^5 - 3552x^4 + 13536x^3 - 27648x^2 + 27648x - 10368 \\ &= (x-6)(x^3 - 18x^2 + 72x - 72)(x^3 - 12x^2 + 36x - 24)\,. \end{aligned} \tag{7.29}$$

The mass spectrum is given in table 18. The Z_2 symmetry of the affine E_7 diagram classify the particles into odd and even ones [157].

(iii) For the ATFT on E_8 we have

$$\begin{aligned} \|\mathcal{M} - x\cdot\mathbf{1}\| &= x^8 - 60x^7 + 1440x^6 - 18000x^5 + 1257440x^4 - 518400x^3 + 1166400x^2 - 1296000x \\ &\quad + 518400 \\ &= (x^4 - 30x^3 + 240x^2 - 720x + 720)(x^4 - 30x^3 + 300x^2 - 1080x + 720)\,. \end{aligned} \tag{7.30}$$

The masses can be found in table 19.

These cases exhaust the mass spectrum of the simply-laced Toda field theory. Now we turn to the non-simply laced ATFT.

7.3.4. B_n series

The matrix $\mathcal{N}$ for the $B_n^{(1)}$ series is given by

$$\mathcal{N} = \begin{vmatrix} 4 & -2 & 0 & \cdots & 0 & 0 & -2 & -2 \\ -2 & 4 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \end{vmatrix}\,. \tag{7.31}$$

Then we have

$$x\mathcal{B}(x) = x\det(\mathcal{M} - x\cdot\mathbf{1}) = \det(\mathcal{N} - x\cdot\mathbf{1}')\,, \tag{7.32}$$

where $\mathbf{1}$ is the $n\times n$ unit matrix and $\mathbf{1}'$ is the $(n+1)\times(n+1)$ unit matrix. Defining $x = 4(1-y)$, we can write (7.32) in the following form

$$\mathcal{B}_n(x) = 2^n(1-2y)\mathcal{U}_n(y)\,. \tag{7.33}$$

Therefore the classical mass spectrum is given by

$$m_n^2 = 2\,,\quad m_k^2 = 8\sin^2(k\pi/2n)\,,\quad k = 1, 2, \ldots, n-1\,. \tag{7.34}$$

7.3.5. C_n series

For the $C_n^{(1)}$ series the matrix $\mathcal{N}$ has the form

$$\mathcal{N} = \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & -2 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \end{vmatrix} . \tag{7.35}$$

Let us introduce the determinant

$$x\mathscr{C}_n(x) = x \det(\mathcal{M} - x \cdot \mathbf{1}) = \det(\mathcal{N} - x \cdot \mathbf{1}') . \tag{7.36}$$

Using the variable $y = 1 - x/2$, we can write $\mathscr{C}_n(x)$ in terms of the Chebyschev polynomial of the second kind,

$$\mathscr{C}_n(x) = (-1)^n 2(y+1)\mathscr{U}_{n-1}(-y) . \tag{7.37}$$

The classical mass ratios are given by

$$m_k^2 = 4\sin^2(k\pi/2n) , \quad k = 1, 2, \ldots, n . \tag{7.38}$$

The $\tilde{C}_n$ twisted series has the following matrix $\mathcal{N}$:

$$\mathcal{N} = \begin{vmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & 1 \end{vmatrix} . \tag{7.39}$$

With the position $2 - x = 2y$, the characteristic equation takes the form

$$\det(\mathcal{N} - x \cdot \mathbf{1}') = -x \det(\mathcal{M} - x \cdot \mathbf{1}) = 2(y-1)[\mathscr{U}_n(y) + \mathcal{M}_{n-3}(u)] . \tag{7.40}$$

The mass ratios are easily computed

$$m_1^2 = 1 , \quad m_k^2 = 4\sin^2[k\pi/(2n-1)] , \quad k = 1, 2, \ldots, n-1 . \tag{7.41}$$

7.3.6. G_2

We start with the untwisted G_2 ATFT. This theory can be obtained by folding the ATFT constructed on $D_4^{(1)}$ [158, 171, 223]. The mass matrix in the diagonal basis is

$$\mathcal{M} = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix} . \tag{7.42}$$

Therefore the mass ratio is

$$m_2 = \sqrt{3}\, m_1 \,. \tag{7.43}$$

The mass matrix of the twisted G_2 is given, on the contrary, by

$$\mathcal{M} = \begin{pmatrix} 1 & -\sqrt{1/3} \\ -\sqrt{1/3} & 3 \end{pmatrix}. \tag{7.44}$$

Then the mass ratio is

$$m_2/m_1 = 2\cos(\pi/12) = (1+\sqrt{3})/\sqrt{2}\,. \tag{7.45}$$

This theory is obtained by folding the $E_6^{(1)}$ ATFT [158, 171, 223].

7.3.7. F_4

The untwisted F_4 ATFT in the diagonal basis has the mass matrix

$$\mathcal{M} = \begin{pmatrix} 3-\sqrt{3} & & & \\ & 2(3-\sqrt{3}) & & \\ & & 3+\sqrt{3} & \\ & & & 2(3+\sqrt{3}) \end{pmatrix} \tag{7.46}$$

and the mass ratios of the theory can be read directly.

For what concerns the twisted F_4 ATFT, this model is recovered as a folding of the $E_7^{(1)}$ theory [157, 158, 171, 223]. The folding is done with respect to the Z_2 symmetry of the Dynkin diagram of the affine E_7 theory. This projects out particles, which are odd with respect to the Z_2 symmetry, leaving the four even particles of $E_7^{(1)}$ as the basic fields of the twisted F_4 ATFT. Therefore, the characteristic equation for the F_4 ATFT can be read off from the characteristic equation of $E_7^{(1)}$, namely

$$\mathcal{P}(x) = x^4 - 24x^3 + 180x^2 - 504x + 432 = 0\,. \tag{7.47}$$

The corresponding mass ratios are in table 18.

7.4. *Interaction couplings*

The series expansion (7.12) gives the whole set of the interaction couplings. The Feynman rules associated to them are easily obtained. Next to the mass matrix, the first important terms are the three particle couplings

$$f^{abc} = m_0^2\beta \sum_i q_i \alpha_i^a \alpha_i^b \alpha_i^c \,. \tag{7.48}$$

These couplings can be written in terms of geometrical quantities [157, 158, 171, 172, 177]. First of all, they vanish if it is impossible to construct with the values of the masses m_a, m_b, and m_c, three sides of a triangle whose angles are rational fractions*) of π. They are also zero if forbidden by a discrete

*) This is a natural consequence of the fact that the masses are algebraic numbers.

symmetry of the affine Dynkin diagram (like in the $E_7^{(1)}$) or by another discrete symmetry of the root system (this is the case of D_n series [171, 172]). The vanishing of f^{abc} when one of three particles has a mass larger than the sum of the remaining two, ensures the stability of the spectrum at lowest order in β and is a consequence of integrability. When not zero, f^{abc} is simply proportional to the area $\mathscr{A}^{abc}$ of the above mentioned triangle. If g is simply laced, we have

$$|f^{abc}| = (4\beta/\sqrt{h})\mathscr{A}^{abc} . \tag{7.49}$$

This is the so called *area formula*. Equation (7.49) is slightly modified if $\mathscr{G}$ is non-simply laced [158, 171]. As discussed in ref. [158], the rhs of eq. (7.49) is sometimes corrected multiplicatively by a factor which takes into account the different ratio of the short root with respect to the long one. When we consider the twisted versions of the algebras, the rhs of eq. (7.49) is multiplied by $1/\sqrt{2}$ for B_n, C_n, F_4 and $1/\sqrt{3}$ for G_2. In the case of untwisted algebras, we have to distinguish two classes of particles: the first class consists of l *strong* particles (l being the number of long roots) and the second one is that of $s = r - l$ *weak* particles. The reason for this denomination is that, if any *strong* particle enters the vertex f^{abc}, then eq. (7.49) holds as it stands. But if we consider a three-coupling constant involving all weak particles we have to modify the *area formula* by a factor $1/\sqrt{2}$ for B_n, C_n, F_4 and $2/\sqrt{3}$ for G_2.

In the case of simply laced algebras, the importance of these couplings is related to their "topological nature", i.e. the exact S-matrices of these theories respect the table of the non-zero three-couplings: they give the bootstrap fusions of the models. Interesting examples are given by the exceptional series E_n which will be analyzed in the next chapter.

Concerning higher-order couplings in (7.12), they do not share any simple geometrical interpretation. However, they can be expressed in terms of some recursive equations [158]. Let us define a set of $(r+1)$-dimensional vectors $\{\xi^a\}_{a=1}^{r+1} \in \mathbb{R}^{r+1}$ by

$$(\xi^a)_i = m_a^{-1}\alpha_i^a , \quad 1 \leq a \leq r ; \quad (\xi^{r+1})_i = 1 . \tag{7.50}$$

We introduce the *reduced* n-vertex couplings $\tilde{f}$, dividing the original n-vertex couplings f by the masses of the particles involved

$$\tilde{f}^{a_1, \ldots, a_n} = \frac{f^{a_1, \ldots, a_n}}{m_{a_1} \cdots m_{a_n}} = \sum_{i=1}^{r+1} q_i \xi_i^{a_1} \cdots \xi_i^{a_n} . \tag{7.51}$$

In particular $\tilde{f}^{a_1,a_2} = \delta^{a_1,a_2}$ and $\tilde{f}^{a_1} = 0$. The ξ vectors form a basis of linearly independent vectors in $\mathbb{R}^{r+1}$ and therefore the vector γ^{ab} $(a, b \leq r)$ with components $\gamma_i^{ab} = \xi_i^a \xi_i^b$ has the decomposition

$$\gamma_i^{ab} = \xi_i^a \xi_i^b = \frac{\delta^{a,b}}{h} \xi_i^{r+1} = \sum_{x=1}^{r} \tilde{f}^{a,b,x} \xi_i^x , \quad a, b \leq r . \tag{7.52}$$

This equation allows us to compute recursively all couplings $\tilde{f}$ by using the identity

$$\tilde{f}^{a_1, \ldots, a_n} = \frac{\delta^{a_{n-1},a_n}}{h} \tilde{f}^{a_1, \ldots, a_{n-2}} + \sum_{x=1}^{r} \tilde{f}^{a_1, \ldots, a_{n-2},x} \tilde{f}^{x,a_{n-1},a_n} . \tag{7.53}$$

It is also convenient for the discussion of the renormalization properties of the quantum field theory to define a matrix Γ built up out of all the ξ vectors but ξ^{r+1},

$$\Gamma_{ij} = \sum_{p=1}^{r} \xi_i^p \xi_j^p q_j . \tag{7.54}$$

It satisfies the properties

$$\Gamma \xi^a = \xi^a , \quad 1 \leq a \leq r ; \quad \Gamma \xi^{r+1} = 0 . \tag{7.55}$$

This matrix will appear in the calculation of tadpole diagrams (see section 8.7).

8. Scattering theory of the affine Toda field theories based on simply laced root systems

The scattering theory of the affine Toda field theories based on simply laced Lie algebras $\mathcal{G}$ present an interesting feature of universality which allows us to discuss them in a unified scheme. The same does not hold for the scattering theory of the ATFT constructed on non-simply laced root systems. For this reason, in this chapter we will discuss only the ATFT related to the ADE series referring to the original literature for the analysis of the scattering theory of non-simply laced ATFT [158, 171, 178].

For simply laced ATFT, the classical masses, the three-particle couplings and the fusion angles provide a solution to the bootstrap equations for the S-matrix of the quantized theory. The exact S-matrices can be written as a product of two terms [144, 157, 158, 171]:

$$S_{ij}(\theta, \beta) = S_{ij}^{(\min)}(\theta) Z_{ij}(\theta, b(\beta)) . \tag{8.1}$$

$S_{ij}^{\min}(\theta)$, which form the so-called "minimal solutions", only depend on the resonance angles u_{ij}^k and encode the informations on the mass spectrum. These factors are independent from the coupling constant β present in the Lagrangian (7.7) and have already all poles we need in order to identify the physical bound states. The coupling constant β enters the second term Z_{ij} in (8.1) through the function $b(\beta)$. The full S-matrices (8.1), as functions of β, must have the same pole structure of $S^{(\min)}$ in the physical strip. Hence, the Z_{ij} terms may have only zeros in the physical strip (and poles outside). The Z_{ij} terms are further restricted by the requirements of unitarity, crossing and closure under the bootstrap iteration. In the limit $\beta \to 0$, the full S_{ij} must reduce to the identity operator since all couplings but the mass ratios vanish. Correspondingly, $b(\beta) \to 0$ and the Z_{ij} terms reduce to the inverse of the minimal solution

$$Z_{ij}(\theta, b(0)) = [S_{ij}^{(\min)}(\theta)]^{-1} . \tag{8.2}$$

The function $b(\beta)$ seems to have a universal expression for all simply laced ATFT. The conjectured formula is given by [144, 158, 171]

$$\pi b = (\beta^2/2h)(1 + \beta^2/4\pi)^{-1} , \tag{8.3}$$

where h is the Coxeter number of the algebra $\mathcal{G}$. This relation was initially guessed in ref. [144] for the

$A_n^{(1)}$ models using known result for the sine–Gordon model*). The function $b(\beta)$ given in eq. (8.3), being the matter of a conjecture, should at least be checked perturbatively, by calculating order by order the coefficients of the power expansion

$$b(\beta) = \beta^2 \sum_{k=0}^{\infty} b_n \beta^{2n} . \tag{8.4}$$

We will present the computation of the lowest orders in section 8.7.3.

In refs. [158, 171] the following *duality* relations for the S functions were pointed out:

$$S_{ij}(b) = S_{ij}(2/h - b) . \tag{8.5}$$

Using (8.3), this means

$$S_{ij}(\beta) = S_{ij}(4\pi/\beta) . \tag{8.6}$$

At the *self-dual* point $b = 1/h$, $\beta = \sqrt{4\pi}$, the Z_{ij} factors become the square of a meromorphic function, i.e. with double poles and double zeroes. Equation (8.6) implies that the S-matrix reduces to the identity also in the strong coupling limit, i.e. $\beta \to \infty$. The origin of this duality property is not yet clear but a possible hint comes from the finite renormalization effect, discussed in section 8.7.2.

An important step for the determination of the exact S-matrices of the ADE series is the assumption that the conserved charges of the quantum theory coincide with those of the classical theory and that the particles of these theories are uniquely identified by the conserved charges (even the mass degenerate states). This implies the absence of any reflection amplitude and the S-matrices are purely elastic. The above assumption about the quantum version of the ATFT has to be verified in the discussion of the renormalization properties of the theory. Since only the ratios of the conserved charges are determined in the scattering theory, the renormalization of the ATFT for the ADE series should reduce, on-shell, only to an overall normalization of the charges. In particular it becomes important to check that the mass ratios are preserved by the loop corrections. Equally important is the explanation of the higher-order poles present in the S-matrices in terms of Landau singularities of the Feynman graphs constructed from the Lagrangian (7.7). We will discuss the renormalization properties of ATFT and their multiple structure in section 8.7.

In the last section of this chapter we will use the correspondence between the particles and the fundamental representations of the algebra $\mathcal{G}$ in order to compute the decay processes of the particles with higher masses in CFT which have been deformed by two relevant fields.

8.1. S-matrices of the $A_n^{(1)}$ series

The S-matrices of the $A_n^{(1)}$ series have been discussed by Arinshtein et al. [144]. The Dynkin diagram of the affine algebra is completely symmetric with respect to any root. The symmetry of these models is given by the group $Z_2 \times Z_{n+1}$. The Z_{n+1} symmetry is related to the invariance of the Lagrangian (7.7) under the cyclic permutation $\phi_i \to \phi_{i+1}$. The Z_2 symmetry is given by the automorphism of the Dynkin diagram under a reflection. The models contain particles and anti-particles. We label them by

*) A similar expression also appears in the anomaly of the stress-energy tensor [190, 222].

$a = 1, 2, \ldots, n$, where $\bar{a} = n + 1 - a$, as a result of the Z_2 symmetry. In particular for odd n the particle $\phi(n+1)/2$ is self-conjugate. The masses are

$$m_a = 2M \sin[\pi a/(n+1)] , \quad a = 1, 2, \ldots, n . \tag{8.7}$$

The "minimal solution" for the scattering amplitude of the lightest particle is

$$S_{11}^{(\min)}(\theta) = s_{2/(n+1)}(\theta) . \tag{8.8}$$

It possesses a pole at $u_{11}^2 = 2\pi/(n+1)$ which corresponds to the particle A_2 appearing as bound state in this channel. The other scattering amplitudes can be inferred from the bootstrap equations

$$S_{ab}^{(\min)}(\theta) = s_{|a-b|/(n+1)}(\theta) s_{(a+b)/(n+1)}(\theta) \Bigg(\prod_{k=1}^{\min(a,b)-1} s_{(|a-b|+2k)/(n+1)}(\theta) \Bigg)^2 . \tag{8.9}$$

Notice the presence of double poles in these amplitudes which arise from multi-scattering processes. The Z term for the fundamental amplitude reads

$$Z_{11}(\theta, b(\beta)) = s_{-b}(\theta) s_{-2/(n+1)+b}(\theta) . \tag{8.10}$$

The other Z_{ab} terms are found by applying the bootstrap equations. In order to compare the exact expression of the S-matrix with the perturbative computation coming from the Lagrangian, we have to solve b for β. The conjectured close formula reads

$$\pi b(\beta) = [\beta^2/2(n+1)](1 + \beta^2/4\pi)^{-1} . \tag{8.11}$$

Notice that the minimal solution for $n = 1$, that is $S_{11} = -1$, coincides with the S-matrix for the thermal perturbation of the Ising model (see section 6.4.1) and for $n = 2$ we get the S-matrices of the critical 3-state Potts model perturbed by the leading energy density operator (see section 6.6).

8.2. *S-matrices of the* $D_n^{(1)}$ *series*

The S-matrices of these models have been computed in ref. [171]. One has to distinguish two cases, depending on whether n is even or odd.

8.2.1. $D_n^{(1)}$, *n even*

In this case the n particles are all self-conjugate and will be denoted by $1, 2, \ldots, n-2$, f_1 and f_2. Their masses are

$$m_{f_1} = m_{f_2} = 1 , \quad m_a = 2 \sin[\pi a/2(n-1)] , \quad a = 1, 2, \ldots, n-2 . \tag{8.12}$$

The minimal solutions for the scattering amplitudes of the particles A_f are given by

$$S_{f_1 f_1}^{(\min)} = S_{f_2 f_2}^{(\min)} = -S_{f_1 f_2}^{(\min)} = (-1)^{n/2} \prod_{k=1}^{n/2-1} f_{2k/(n-1)}(\theta) . \tag{8.13}$$

The corresponding Z-terms are

$$Z_{f_1,f_1}(\theta, b(\theta)) = Z_{f_2,f_2}(\theta, b(\theta)) = (-1)^{n/2} \prod_{k=0}^{n/2-1} f_{-k/(n-1)-(-1)^k b}(\theta) ,$$
$$Z_{f_1,f_2}(\theta, b(\theta)) = -(-1)^{n/2} \prod_{k=1}^{n/2-1} f_{-k/(n-1)+(-1)^k b}(\theta) . \tag{8.14}$$

The remaining minimal S-matrices are computed by the bootstrap equation. Their expression is given by

$$S_{af_1}^{(\min)}(\theta) = S_{af_2}^{(\min)}(\theta) = (-1)^a \prod_{k=0}^{a-1} f_{1/2-(a-2k)/2(n-1)}(\theta) , \quad a = 1, 2, \ldots , n-2 , \tag{8.15}$$

$$S_{ab}^{(\min)}(\theta) = f_{|a-b|/2(n-1)}(\theta) f_{(a+b)/2(n-1)}(\theta) \Bigg(\prod_{k=1}^{\min(a,b)-1} f_{(|a-b|+2k)/2(n-1)}(\theta) \Bigg)^2 . \tag{8.16}$$

8.2.2. $D_n^{(1)}$, *n odd*

In this case, there are $n-2$ particles which are self-conjugate and a doublet of particles $(f, \bar{f})$ which are conjugate to each other. The mass spectrum is again given by eq. (8.12). The basic amplitudes are

$$S_{ff}^{(\min)}(\theta) = -S_{f\bar{f}}^{(\min)}(\theta) = \sum_{k=1}^{n-2} s_{k/(n-1)}(\theta) . \tag{8.17}$$

For the Z-terms we have

$$Z_{ff}(\theta, b(\theta)) = \prod_{k=0}^{n-2} s_{-k/(n-1)-(-1)^k b}(\theta) , \quad Z_{f\bar{f}}(\theta, b(\theta)) = \prod_{k=0}^{n-2} s_{-k/(n-1)+(-1)^k b}(\theta) . \tag{8.18}$$

The scattering amplitudes of the remaining particles are equal to those computed in the case of n even.

8.3. *S-matrices of* $E_6^{(1)}$ *ATFT*

The full set of minimal S-matrices is given by the scattering amplitudes of the thermal perturbation of the tricritical 3-state Potts model, discussed in section 6.7. Here we only recall the fundamental amplitude and the corresponding Z-term. The minimal S-matrix of the lowest particle is given by

$$S_{11}^{(\min)}(\theta) = s_{1/6}(\theta) s_{1/2}(\theta) s_{2/3}(\theta) . \tag{8.19}$$

The Z-term is given by

$$Z_{11}(\theta, b(\beta)) = s_{-b}(\theta) s_{-1/6+b}(\theta) s_{-1/2+b}(\theta) s_{-2/3+b}(\theta) . \tag{8.20}$$

The bootstrap fusions of this model (which correspond to the non-zero three-point couplings) can be written in a compact way. To this aim, let us consider the characteristic equation (7.28) of this model and let us introduce the following notation: the particles whose masses are solutions of the equation

$$(x^2 - 6x + 6)^2 = 0\,, \tag{8.21}$$

as denoted as

$$m_a = m \;\to\; A_1\,, \quad m_{\bar{a}} = m \;\to\; \bar{A}_1\,, \quad m_b = 2m\cos(\tfrac{1}{12}\pi) \;\to\; A_2\,, \quad m_{\bar{b}} = m_b \;\to\; \bar{A}_2$$

(with $m^2 = 3 - 2\sqrt{3}$). Let us consider the other factor in (7.28),

$$x^2 - 12x + 24 = 0\,. \tag{8.22}$$

The corresponding particles are labelled as

$$m_c = 2m\cos(\tfrac{1}{4}\pi) \;\to\; B_1\,, \quad m_d = 4m\cos(\tfrac{1}{4}\pi)\cos(\tfrac{1}{12}\pi) \;\to\; B_2\,.$$

With these notations, the bootstrap fusions of $E_6^{(1)}$ can be written as

$$\begin{aligned}
&A_i \times A_i = \bar{A}_1 + \bar{A}_2\,, \quad A_i \times A_{i+1} = \bar{A}_1 + \bar{A}_2\,, \quad A_i \times \bar{A}_i = B_i\,,\\
&A_i \times \bar{A}_{i+1} = B_1 + B_2\,, \quad A_i \times B_i = A_1 + A_2\,, \quad \bar{A}_i \times B_i = A_1 + A_2\,,\\
&A_i \times B_{i+1} = A_{i+1}\,, \quad B_i \times B_i = B_1 + B_2\,, \quad B_i \times B_{i+1} = B_1 + B_2\,.
\end{aligned} \tag{8.23}$$

Using the above mentioned correspondence between the particles and the representations of the Lie algebra [150, 171], from table 20 we have

$$m_a, m_{\bar{a}} \;\ldots\; 27, \overline{27}\,; \quad m_c \;\ldots\; 78\,; \quad m_b, m_{\bar{b}} \;\ldots\; 351, \overline{351}\,; \quad m_d \;\ldots\; 2925\,, \tag{8.24}$$

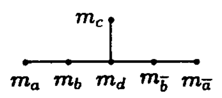
where on the right-hand sides are reported the dimensions of the irreducible representations of E_6. It is easy to check that the above fusion rules are a subset of the tensor product decomposition of the representations (8.24) of E_6 [172].

8.4. S-matrices of $E_7^{(1)}$ ATFT

The full set of minimal S-matrices is given by the scattering amplitudes of the massive excitations of

Table 20
Dynkin diagram of E_6 and assignment of the masses to the corresponding dots.

Exponents:	1, 4, 5, 7, 8, 11



the tricritical Ising model perturbed by the energy operator and has been discussed in section 6.5.1. The minimal S-matrix of the lowest particle is

$$S_{11}^{(\min)}(\theta) = -f_{1/9}(\theta) f_{5/9}(\theta) . \tag{8.25}$$

The corresponding Z-term is

$$Z_{11}(\theta, b(\beta)) = -f_{-b}(\theta) f_{-1/9+b}(\theta) f_{-5/9+b}(\theta) . \tag{8.26}$$

Using the factorization of the characteristic equation (7.29), we organize the seven particles into two triplets and one singlet [157],

$$(Q_1, Q_2, Q_3) \equiv (m_6, m_3, m_1) , \quad (K_1, K_2, K_3) \equiv (m_2, m_4, m_7) , \quad (N) \equiv (m_5) . \tag{8.27}$$

The first triplet consists of particles which are odd under the Z_2 symmetry of the Dynkin diagram of the affine algebra. The other triplet and the singlet are Z_2 even. In section 6.5.1 we have used this symmetry in order to write the bootstrap fusions of this model in a compact way. These bootstrap fusions are a subset of the tensor product decomposition of the associate representations of E_7 (see, e.g., refs. [226, 227] and appendix B).

8.5. *S-matrices of* $E_8^{(1)}$ *ATFT*

The full set of minimal S-matrices is given by the scattering amplitudes of the Ising model in a magnetic field. They can be found in appendix C. The minimal S-matrix of the lightest particle is

$$S_{11}^{(\min)}(\theta) = f_{1/3}(\theta) f_{2/5}(\theta) f_{1/15}(\theta) . \tag{8.28}$$

The corresponding Z-term reads

$$Z_{11}(\theta, b(\beta)) = f_{-b}(\theta) f_{-1/3+b}(\theta) f_{-2/5+b}(\theta) f_{-1/15+b}(\theta) . \tag{8.29}$$

The bootstrap fusions can be found in appendix C.

8.6. *Conserved charges*

The spins of the conserved charges in ATFT coincide with the exponents of the corresponding algebra, modulo the Coxeter number. But a close analysis of the conserved quantities reveals a more remarkable structure. First of all, the set of masses for the ADE series can be organized in a vector which is the Perron–Frobenius eigenvector of the Cartan matrix of the corresponding algebra $\mathcal{G}$. This result was obtained in refs. [44, 171, 174] and proved in ref. [177]. The Cartan matrix is defined by

$$C_{ij} = \alpha_i \cdot \alpha_j . \tag{8.30}$$

If we introduce the vector $\boldsymbol{m} = (m_1, m_2, \ldots, m_r)$ made out of the r-masses of the theory, we have

$$C\boldsymbol{m} = \mu_{\min}\boldsymbol{m}\,, \quad \mu_{\min} = 4\sin^2(\pi/2h)\,. \tag{8.31}$$

This observation permits to associate each mass m_i with a dot of the Dynkin diagram for $\mathcal{G}$ and hence with the corresponding fundamental representation of that algebra. Hence, one is tempted to interpret the bootstrap fusions of the bound states

$$A_i \times A_j = \sum A_k \tag{8.32}$$

as Clebsch–Gordan decomposition of the tensor product representations associated to the mass eigenstates. Actually, this is only true for the $A_n^{(1)}$ series and $D_4^{(1)}$. For the other simply laced algebras the bootstrap fusions coincide only with a subset of the Clebsch–Gordan series [172]. A general analysis of the bootstrap fusions for the simply laced ATFT has been carried out in refs. [174, 177], where their relation with the root system has been clarified.

The above observation on the masses can be generalized to the higher conserved quantities q_s^i entering the consistency equations

$$q_s^a \exp(-\mathrm{i}s\bar{u}_{ac}^b) + q_s^b \exp(\mathrm{i}s\bar{u}_{bc}^a) = q_s^{\bar{c}}\,. \tag{8.33}$$

Indeed, the vectors $\boldsymbol{q}_s = (q_s^1, q_s^2, \ldots, q_s^r)$ of the conserved charges are eigenvectors of the Cartan matrix [171, 174, 177]

$$C\boldsymbol{q}_s = \mu_s\boldsymbol{q}_s\,, \quad \mu_s = 4\sin^2(s\pi/2h)\,, \tag{8.34}$$

where the spin s takes values in the set of the r possible exponents of the algebra $\mathcal{G}$. In all ADE models there are r linear independent vectors $\boldsymbol{q}_s$ which label uniquely the particles of the theory and make the scattering processes purely elastic.

8.7. Perturbation theory

In this section we discuss some aspects of perturbation theory for ATFT and compare the results with the proposed exact solutions of the on-shell theory provided by the S-matrices computed in the previous sections.

8.7.1. Infinite renormalization of ATFT

The quantization and the renormalization of the ATFT is not completely straightforward in many cases. In particular this is true for the non simply laced models. In this section we only discuss the universal ultraviolet divergences that occur in the ATFT constructed on root systems either of simply laced algebras or non-simply laced ones. These divergences originate from the self-contractions (tadpoles) of the fields themselves (fig. 27). In those theories in which only one field is present, the tadpoles can be neglected as a consequence of normal ordering. However, if many different fields with different masses are present, it is not obvious that the finite part of the tadpoles do not result in renormalization effects, that depend on the particular vertex or propagator they are attached to. In ref. [158] it was shown that this situation cannot occur. The reasoning is the following. Consider the two possible kinds of general tadpoles which can affect a general vertex $f^{a_1,\ldots,a_n}$ (a propagator if $n=2$) (fig. 28). The bulb represents all possible loop corrections to the tadpoles. It can be connected to the

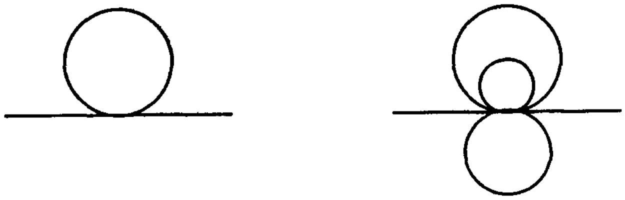


Fig. 27. Tadpole.

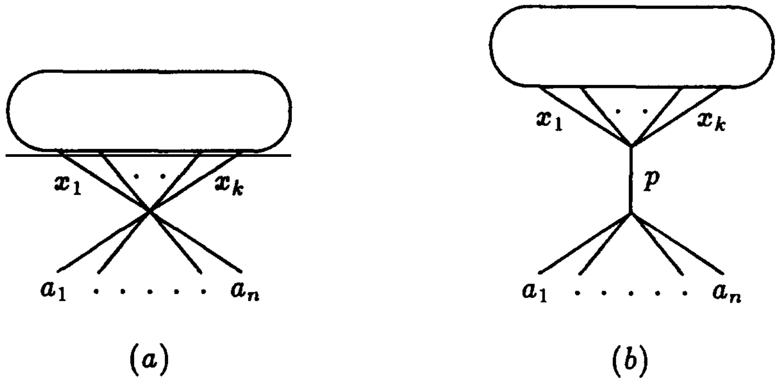


Fig. 28. General tadpole graphs.

vertex $f^{a_1,\ldots,a_n}$ either directly, via propagators with masses $m_{x_1},\ldots,m_{x_k}$ (fig. 28a), or indirectly, as shown in fig. 28b. Whatever the contribution $G(x_1,\ldots,x_k)$ of the bulb is (finite or infinite), the sum of graphs in fig. 28 amounts to computing

$$\sum_{x_1,\ldots,x_k}\left(f^{a_1,\ldots,a_n,x_1,\ldots,x_k}-\sum_p f^{a_1,\ldots,a_n,p}\,\frac{1}{m_p^2}\,f^{p,x_1,\ldots,x_k}\right)G(x_1,\ldots,x_k)\,. \tag{8.35}$$

Using the properties of $f^{a_1,\ldots,a_n}$ discussed in section 7.4, we can write this expression as

$$\sum_{x_1,\ldots,x_k}\left(\sum_{ij} q_i\xi_i^{a_1}\cdots\xi_i^{a_n}(\delta_{ij}-\Gamma_{ij})\beta_j^{x_1}\cdots\beta_j^{x_k}\right)G(x_1,\ldots,x_k)\,, \tag{8.36}$$

where the matrix Γ is defined in eq. (7.54). With eq. (7.52), we can decompose

$$\xi_j^{x_1}\cdots\xi_j^{x_k}=K^{r+1}_{x_1\cdots x_k}\xi_j^{r+1}+\sum_{y\le r}K^{y}_{x_1\cdots x_k}\xi_j^{y}\,. \tag{8.37}$$

The matrix $\delta-\Gamma$ acting on this last vector will be non-zero only if K^{r+1} is non zero. Due to the fact that all components of ξ^{r+1} are 1, we conclude that eq. (8.36) reduces to

$$\left(\sum_{x_1,\ldots,x_k}K^{r+1}_{x_1\cdots x_k}G(x_1,\ldots,x_k)\right)\tilde{f}^{a_1,\ldots,a_n}=\tilde{\mathscr{D}}\tilde{f}^{a_1,\ldots,a_n}\,, \tag{8.38}$$

i.e. it factorizes into a product of $f^{a_1,\ldots,a_n}$ times a term $\mathscr{D}$ that does not depend on the particular choice of indices $a_1,\ldots,a_n$. This shows, that the way propagators or vertices are affected by finite or infinite tadpole contributions does not depend on the particular choice of the propagator or the vertex. Divergences are now easy to evaluate. We can get rid of the divergent part of $\mathcal{D}$ by taking all propagators to be regularized with the same cut-off μ^2. This is equivalent to considering the exponents in the Lagrangian as normal ordered with respect to this arbitrary scale. The bare mass M in front of the potential in (7.7), computed with an ultraviolet cut-off Λ, is related to the dressed one as

$$M^2 \to M^2(\Lambda^2/\mu^2)^{(\beta^2/4\pi)\tilde{h}/h}\,, \tag{8.39}$$

where

$$\tilde{h} = \frac{1}{2}\sum_{a=1}^{r}\sum_{i=1}^{r+1} q_i \alpha_i^a \alpha_i^a\,. \tag{8.40}$$

In the simply laced cases, $\tilde{h} = h$ and they cancel in (8.39). The finite part of the tadpoles gives a further finite renormalization of the parameter M. Other finite renormalization effects will be discussed in the next section. It is interesting to note that β does not renormalize.

8.7.2. Finite renormalization

Except for the tadpole diagrams, all other Feynman diagrams of ATFT are finite. Hence we can compute perturbatively the finite renormalization of the theories. One important point is to check that the ratios of masses are preserved by quantum corrections because the bootstrap can close after a finite number of steps only if these ratios take very particular values. In the cases of ADE Toda field theories the mass ratios, as determined from the exact solution of the S-matrix problem, coincide with those obtained at the classical level. Therefore the mass ratios must remain stable with respect to the quantum corrections.

The masses of the quantum ATFT are given by the poles of the full propagators. At the lowest order in β, the self-energy diagrams are given by the diagram shown in fig. 29 with the external momentum put on-shell. Summing up all intermediate contributions, a nice universal result appears for the ATFT of simply laced type [158, 171],

$$\Delta m_i^2/m_i^2 = (\beta^2/4h)\cot(\pi/h)\,, \tag{8.41}$$

i.e. the mass shift is given by the area of the regular planar polyhedron with h equal sides and perimeter β. This result seems to indicate the existence of a universal renormalization function $\Xi(\beta, h)$, that only depends on β and h, and connects any bare mass to its renormalized one, as

$$(m_a)_{\text{ren}} = (m_a)_0\,\Xi(\beta, h)\,. \tag{8.42}$$

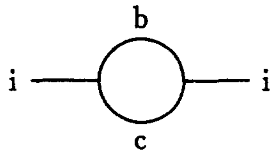


Fig. 29. One-loop mass correction.

An interesting open problem is to find its closed form.

The analysis made so far on the mass ratios of ADE Toda field theories does not hold for the non-simply laced ones. In fact, order one loop, the quantum corrections spoil the classical mass ratios [158, 171]. This renormalization of the mass ratios conflicts with the quantum integrability of the models. The only case where this problem seems absent is the $A_{2n}^{(2)}$ series where the renormalization properties are induced, in a consistent way, by those of the parent $A_n^{(1)}$ theories. In all other cases, the folding of the simply laced ATFT not only removes the non-invariant particles from the set of asymptotic states, but also forbids them to appear as virtual intermediate particles. These virtual states are very important in the parent theories and give rise to the universal mass renormalization (8.41). Their absence essentially explains the lack of an analogous formula for the non-simply laced ATFT. These difficulties for a consistent quantum field theory of the non-simply laced ATFT are unpleasant but not surprising. The unpleasant fact is that there exist some minimal models of CFT that can be constructed in terms of a coset construction on non-simply laced algebras and, moreover, their Kac table presents a relevant field that can be used to write the Lagrangian of the corresponding ATFT. For instance, these are the cases of the minimal model $\mathcal{M}_{9,10}$ (obtained as coset construction on the exceptional algebra G_2) and of the minimal model $\mathcal{M}_{10,11}$ (obtained as coset construction on the exceptional algebra F_4). On the other hand, the difficulties pointed out for the non-simply laced ATFT are not surprising because analogous complications exist at the level of conformal field theory models (see, e.g. refs. [231, 232] for the coset construction on the non-simply laced algebras). The quantum consistency of the scattering theory for the non-simply laced ATFT requires the introduction of additional degrees of freedom, as shown ref. [178].

Another important test for the quantum integrability of the ATFT is the stability of the particles. In many systems there are particles above the threshold of the lightest particle. They do not decay classically because the three-couplings of the possible decays are zero. Any decay at the quantum level would rule out the existence of an infinite number of conserved quantities. The calculations made at one-loop order show that they are still stable [158]. This amounts to compute the Feynman diagrams shown in fig. 30. We will comment further on the decay processes in section 8.8, when we consider the off-critical theories with two different perturbing fields.

8.7.3. Pole structure

The exact S-matrices computed in the previous sections contain a rich structure of singularities. In the standard interpretation of these singularities, the simple poles are associated with the bound states in the direct or in the cross channel, depending on the sign of the residue. But, as we observed in chapter 3, an interesting feature is given by their multiple pole structure. In the case of $A_n^{(1)}$ series there are simple and double poles, in the $D_n^{(1)}$ series there are singularities up to 4th order poles, whereas in the E_6 theory there are poles up 6th order, in E_7 up to 8th and in E_8 up to 12th order. A consistent identification of these exact S-matrices with those obtained by the LSZ reduction of the Lagrangian

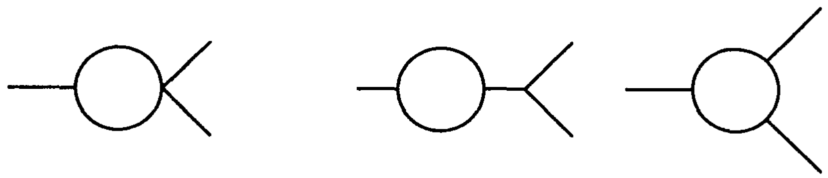


Fig. 30. Vertex corrections.

(7.7) requires at least a check with the lowest orders in perturbation theory and an explanation of all singularities in terms of Landau singularities of the Feynman diagrams [211]. The general analysis was reviewed in chapter 3 and therefore here we present, as a simple example of the general structure of the simply laced ATFT, the case of ATFT on the root system of $E_7^{(1)}$.

The full S-matrix for the lightest particle of $E_7^{(1)}$ is given by

$$S_{11} = f_{1/9}(\theta) f_{b-1/9}(\theta) f_{5/9}(\theta) f_{b-5/9}(\theta) f_{-b}(\theta) . \tag{8.43}$$

In the physical sheet we have two simple poles with positive residue at $\theta = \mathrm{i}\pi/9$ and $5\mathrm{i}\pi/9$ which correspond to physical particles. How the S-matrix depends on β through the function $b(\beta)$ can be determined using perturbation theory. At the tree level we have the graphs of fig. 31. First of all, one can easily verify the elasticity of all the scattering processes. For the allowed amplitudes the poles come from the internal propagators. On the other hand, we can see where the poles are located just expanding S. In our example, we obtain

$$\hat{S}_{11} = \mathrm{i}\pi b \left(\frac{1}{\sinh(\theta - \frac{1}{9}\mathrm{i}\pi)} - \frac{1}{\sinh(\theta - \frac{8}{9}\mathrm{i}\pi)} + \frac{1}{\sinh(\theta - \frac{5}{9}\mathrm{i}\pi)} - \frac{1}{\sinh(\theta - \frac{4}{9}\mathrm{i}\pi)} - \frac{2}{\sinh(\theta)} \right) + \cdots \tag{8.44}$$

The pole with negative residue $-2\mathrm{i}\pi b$ at $\theta = 0$, that comes from $f_{-b}(\theta)$, is a consequence of the Jacobian that appears expressing $S(\theta)$ in terms of the Mandelstam variable s,

$$\mathcal{S}(s) = 4 m_i m_j \sinh(\theta_{ij}) S(\theta_{ij}) . \tag{8.45}$$

Perturbation expansion (i.e. the graphs in fig. 31) shows that the residue at this pole is only non zero in S_{aa} for any a. The two other poles with positive residues $\mathrm{i}\pi b$ are identified with the poles in the direct channel (fig. 31b) where the internal propagators correspond to the particles with masses m_2 and m_4. It is interesting to note that the bootstrap condition which acts multiplicatively on the S-matrices, is instead additive at the perturbative level, mapping poles with different physical meanings into each other. The residues are always simple multiples of b, although the graphs of perturbation theory have very different propagators and vertices f^{abc}. This fact leads to the *area formula* (7.49) for the couplings f^{abc}, because the same area appears in eq. (8.45) and in the residue of the propagators. With this direct comparison, we can consistently obtain the lowest term of the function $b(\beta)$. For any simply laced root system we obtain [158, 171]

$$\pi b = \beta^2/2h + \cdots \tag{8.46}$$

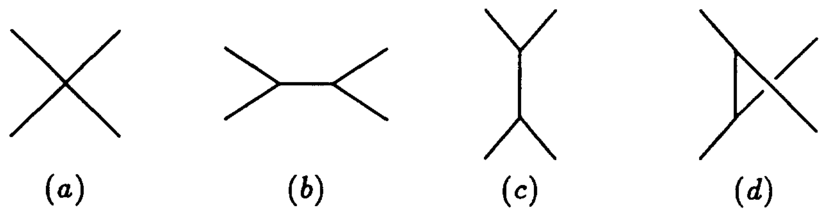


Fig. 31. Tree level scattering processes.

Braden and Sazaki [175] have further pursued the computation checking the function $b(\beta)$ at the second order in perturbation theory,

$$\pi b = (\beta^2/2h)(1 - \beta^2/4\pi + \cdots)\,. \tag{8.47}$$

This result is in agreement with the corresponding order in the expansion of the conjectured exact form of the function $b(\beta)$,

$$\pi b = (\beta^2/2h)(1 + \beta^2/4\pi)^{-1}\,. \tag{8.48}$$

The structure of higher-order poles has been analyzed in detail in ref. [172].

8.8. Decay processes in theories with two perturbing fields

In this section we go back to the case of CFT theories perturbed away from the critical point by two relevant fields. We assume that these perturbations are in correspondence with some fields in the Toda field theories. Explicitly we consider the case of the Ising model and the tricritical Ising model.

Suppose we perturb the Ising model at the critical point by inserting a magnetic field and also increasing its temperature above T_c. The magnetic field is the most relevant perturbation associated with E_8. One might wonder if the structure of E_8 remains a reliable description of the model, at least to the first order in $(T - T_c) \equiv \delta T$. We have to check whether or not the particles of E_8 above threshold remain stable.

The probability of the decay processes is given by

$$\mathrm{d}P = \frac{1}{2M}\, \|{}_{\mathrm{in}}\langle h|l_1 l_2\rangle_{\mathrm{out}}\|^2\, \mathrm{d}\mathcal{R}^{(2)}\, hs \simeq \frac{(\delta T)^2}{2M}\, \|\langle h|\Phi_{1/2,1/2}|l_1 l_2\rangle\|^2\, \mathrm{d}\mathcal{R}^{(2)}\,, \tag{8.49}$$

where h is a particle with mass M and l_1, l_2 are two particles with masses m_1, m_2 in which the particle h can decay. Assuming the independence of the matrix element from the momentum, we can compute the total two-body phase space,

$$\mathcal{R}^{(2)} = \int (2\pi)^2 \delta^2(p_1 + p_2 - p)\, \frac{\mathrm{d}p_1}{(2\pi)2E_1}\, \frac{\mathrm{d}p_2}{(2\pi)2E_2} = \frac{1}{2}\, \frac{1}{M|p|}\, \theta(M^2 - (m_1 + m_2)^2)\,, \tag{8.50}$$

where

$$|p| = \sqrt{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}/2M\,.$$

For dimensional reasons, the matrix element $\langle h|l_1 l_2\rangle$ is proportional to M^2 and can be estimated by the Wigner–Eckart theorem. What we need to know is the quantum number of the quantities involved in (8.49). The transformation properties of the operator $\Phi_{1/2,1/2}$ with respect to the group E_8 are worked out in appendix C. They coincide with those of the particle with mass m_2 in the spectrum of the Ising model in a magnetic field. Hence, we can use the bootstrap fusions given in appendix C for obtaining the allowed decay processes. The full list is presented in table 21.

It is worth noting that although each particle with mass above threshold decays, there exist however

Table 21
Possible decay processes for the E_8 masses of the Ising model in the presence of the energy operator $\Phi_{1/2,1/2}$.

$m_8 \to m_1 m_1$	no	$m_8 \to m_2 m_4$	yes	$m_7 \to m_2 m_2$	yes
$m_8 \to m_1 m_2$	no	$m_8 \to m_2 m_5$	yes	$m_7 \to m_2 m_3$	yes
$m_8 \to m_1 m_3$	no	$m_8 \to m_3 m_3$	yes	$m_6 \to m_1 m_1$	yes
$m_8 \to m_1 m_4$	yes	$m_8 \to m_3 m_4$	no	$m_6 \to m_1 m_2$	yes
$m_8 \to m_1 m_5$	yes	$m_7 \to m_1 m_1$	no	$m_6 \to m_1 m_3$	yes
$m_8 \to m_1 m_6$	yes	$m_7 \to m_1 m_2$	yes	$m_5 \to m_1 m_1$	yes
$m_8 \to m_2 m_2$	yes	$m_7 \to m_1 m_3$	yes	$m_5 \to m_1 m_2$	yes
$m_8 \to m_2 m_3$	yes	$m_7 \to m_1 m_4$	yes	$m_4 \to m_1 m_1$	yes

Table 22
Possible decay processes for the E_7 masses of the TIM in the presence of the density operator $\Phi_{6/10,6/10}$.

$m_7 \to m_1 m_1$	yes	$m_7 \to m_2 m_2$	yes	$m_6 \to m_2 m_2$	no
$m_7 \to m_1 m_2$	no	$m_7 \to m_2 m_3$	no	$m_5 \to m_1 m_1$	yes
$m_7 \to m_1 m_3$	yes	$m_7 \to m_2 m_4$	yes	$m_5 \to m_1 m_2$	no
$m_7 \to m_1 m_4$	no	$m_6 \to m_1 m_1$	no		
$m_7 \to m_1 m_5$	no	$m_6 \to m_1 m_2$	yes		

some selection rules in these processes. For instance, the particle with the highest mass m_8 cannot decay into the lightest ones $m_1 m_1$.

Let us consider now the tricritical Ising model perturbed by the energy operator $\Phi_{1/10,1/10}$ and by the vacancy density operator $\Phi_{6/10,6/10}$. The density energy operator is most relevant, and fixes the E_7 structure of the model. The transformation properties of the field $\Phi_{6/10,6/10}$ with respect to the group E_7 coincide with those of the particle with mass m_4. This particle is even under the Z_2 symmetry of the model. Using the bootstrap fusion of the model, we obtain the allowed decays given in table 22. In this case the selection rule acting in the decay of these particles is just the Z_2 parity of the model.

9. Correlation functions and form factors

In the previous chapters, we have characterized massive integrable field theories in terms of their scattering data, i.e. their properties on mass-shell. An important problem is to complete the analysis of such models by computing the correlation functions of local fields. Despite the existence of an exact S-matrix of these integrable models, the investigation of the off-shell behaviour reveals to be a difficult task and closed expressions for the correlators have not been found up to now. An exception consists in the tour de force calculation of the correlation functions of the Ising model [78–81].

One possible approach to get informations about the off-critical correlators is to use the spectral density representation. This means that we decompose the correlation functions into an infinite sum over multiparticle intermediate states, each of those contributions being given by the corresponding form factors. For instance, in the case of two-point functions we have

$$\langle \mathcal{O}_a(p)\mathcal{O}_a(0)\rangle = G_a(p) = \int \frac{\rho_a(\kappa^2)\,\mathrm{d}\kappa^2}{p^2 - \kappa^2 + \mathrm{i}\varepsilon}\,, \tag{9.1}$$

with the spectral functions given by

$$\rho_a(\kappa^2) = \sum_n \frac{1}{n!} \int \prod_i \frac{d^2 p_i}{(2\pi)^2} \, \delta(\kappa^0 - \Sigma_i \, p_i^0) \delta(\kappa^1 - \Sigma_i \, p_i^1) |F_n^a|^2 \, . \tag{9.2}$$

The form factors F_n^a are matrix elements of the quantum fields $\mathcal{O}_a$ between asymptotic scattering states,

$$F_n^a = {}_{\text{out}}\langle p_1, p_2, \dots p_m | \mathcal{O}_a(0) | p_{m+1}, \dots p_n \rangle_{\text{in}} \, . \tag{9.3}$$

As a result of unitarity and CPT invariance, the form factors obey the Watson equations [68], which, for systems with factorized elastic S-matrices, become rather simple functional equations [83, 84]. In addition, the form factors corresponding to multiparticle asymptotic states are related by LSZ reduction and bootstrap equations to those with fewer particles.

These two types of conditions, together with analyticity requirements, have been used to determine some of the form factors in several theories [69–71, 83–86, 147]. In certain cases the results may be tested against exact or perturbative solutions.

The form factor approach for computing the off-critical correlators presents, however, some shortcomings. The first one is related to the arbitrariness inherent in solving the Watson equations – analogously to the CDD arbitrariness present in any S-matrix. In fact, it is often necessary to make certain "minimality" assumptions in order to find the correct form factors of a specific theory. The second source of difficulties is that, although the form factors of a theory may be computed exactly, the difficult step remains in finding a closed expression for the infinite sum over the intermediate particle number.

The goal of this chapter is to briefly review the basic properties of the form factors in a factorized scattering theory and to show – taking the Ising model as example – how they can be used for studying the ultraviolet limit of a massive field theory.

9.1. *Equations for the form factors*

The form factors are matrix elements of local operators $\mathcal{O}(x)$ between out-states and in-states. We define the functions

$$F_n = \langle 0 | \mathcal{O}(0) | p_1, \dots, p_n \rangle_{\text{in}} \, . \tag{9.4}$$

If $\mathcal{O}$ has spin s, Lorentz invariance implies that F_n is of the form $e^{s\theta_1}$ times a function depending only on the differences $\theta_{ij} = \theta_i - \theta_j$. This function is the boundary value on the real axis of an analytic function of the θ_{ij}. The most general n-particle form factors are

$${}_{\text{out}}\langle p_1, \dots, p_m | \mathcal{O}(0) | p_{m+1}, \dots, p_n \rangle_{\text{in}} \, . \tag{9.5}$$

Crossing invariance allows us to express them as analytic continuation of (9.4), and they are equal to

$$F_n(\theta_{ij}, \mathrm{i}\pi - \theta_{rs}, \theta_{kl}) \, , \tag{9.6}$$

where $1 \le i < j \le m$, $1 \le r \le m < s \le n$, and $m < k < l \le n$. We have assumed, for simplicity, a theory with self-conjugate particles.

The Watson equations are derived by inserting into (9.5) a complete set of in-states before the operator $\mathcal{O}$, and of out-states after this operator. We use then the definition of the S-matrix

$$S_n(p_1, \ldots, p_n) = {}_{\text{out}}\langle p_1, \ldots, p_n | p_1, \ldots, p_n \rangle_{\text{in}} , \tag{9.7}$$

together with the factorization property

$$S_n(p_1, \ldots, p_n) = \prod_{i<j} S_2(p_i, p_j) . \tag{9.8}$$

The matrix element with in- and out-states interchanged is obtained by CPT invariance from (9.5) by changing the signs of all the θ_{ij}. Thus, the final form of the Watson equations is

$$F_n(\theta_{ij}, \mathrm{i}\pi - \theta_{rs}, \theta_{kl}) = \Big(\prod_{i<j} S(\theta_{ij})\Big) F_n(-\theta_{ij}, \mathrm{i}\pi + \theta_{rs}, -\theta_{kl}) \Big(\prod_{k<l} S(\theta_{kl})\Big) . \tag{9.9}$$

In the case $n = 2$, these equations simplify to

$$F_2(\theta) = F_2(-\theta) S_2(\theta) , \quad F_2(\mathrm{i}\pi - \theta) = F_2(\mathrm{i}\pi + \theta) . \tag{9.10}$$

It was shown in ref. [69] that the general solutions of this system of equations have the form

$$F_n = K_n \prod_{i<j} F_{\min}(\theta_{ij}) , \tag{9.11}$$

where $F_{\min}(\theta)$ has the properties that by itself satisfies (9.10), is analytic in $0 \leq \mathrm{Im}\, \theta \leq 2\pi$, and has no zeros in $0 < \mathrm{Im}\, \mu < 2\pi$. These requirements uniquely fix this function. The remaining factors K_n then satisfy the Watson equations with $S_2 = 1$, which implies that they are completely symmetric, periodic functions of the θ_i.

The other constraint on the K_n is that they must contain all the physical poles expected in the form factor under consideration. Kinematic poles are expected at $\theta_{ij} = \mathrm{i}\pi$, with the corresponding residues which link $(n+2)$-particle form factors with n-particle form factors [71]

$$\mathrm{Res}_{\theta'=0}\, F^{n+2}_{a,a,a_1,\ldots,a_n}(\theta' + \mathrm{i}\pi, \theta, \theta_1, \ldots, \theta_n) = \mathrm{i}\Big(1 - \prod_{i=1}^{n} S_{a_i a}(\theta - \theta_i)\Big) F^{n}_{a_1,a_2,\ldots,a_n}(\theta_1, \ldots, \theta_n) . \tag{9.12}$$

This corresponds to a zero-angle scattering of the particle A_a – a process which comes from the two different kinematical situations depicted in fig. 32.

Eventual additional poles depend on the operator $\mathcal{O}$, on its transformation properties under any global symmetries the theory may possess and on the structure of the bootstrap fusions. If there exists a set of operators which share the same symmetry properties, they will present the same pole structure in the K_n and will only differ in the numerator which multiplies these poles. Moreover, their specific expression is further constrained by the requirement that the residues of the poles are proportional to form factors with fewer particles. If particles A_i and A_j give rise to a bound state A_c, the corresponding amplitude presents a pole at $\theta = \mathrm{i}u^c_{ij}$, with residue equal to the product of on-mass-shell three-point vertices (fig. 33)

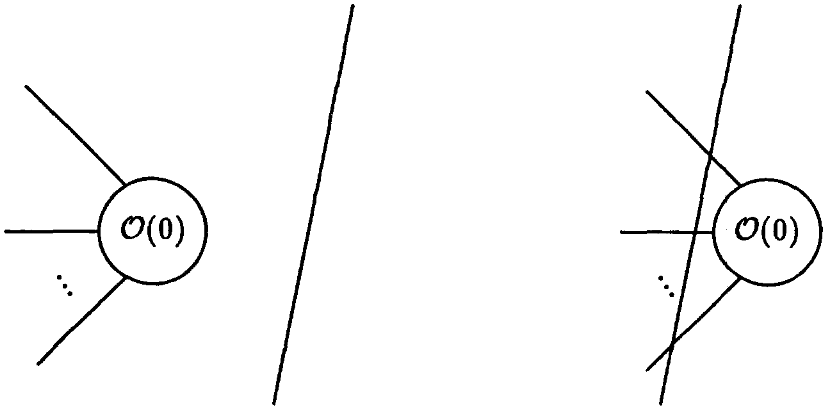


Fig. 32. Two different kinematical situations for the scattering process at zero rapidity.

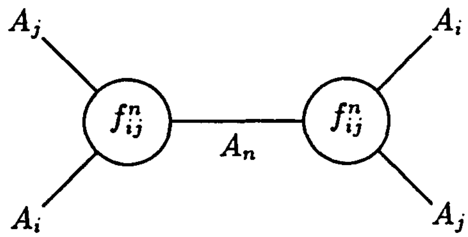


Fig. 33. Bound-state pole in scattering amplitude.

$$\mathrm{Res}_{\theta=\mathrm{i}u^c_{ij}}\, S_{ij}(\theta) = \mathrm{i} f^c_{ij} f^c_{ij} \,. \tag{9.13}$$

Hence, we obtain the following relationship between a n-particle and $(n-1)$-particle form factors (fig. 34)

$$\mathrm{i}\,\mathrm{Res}_{\varepsilon=0}\, F^n_{a_1,a_2,\ldots,a_i a_j}(\theta_1, \theta_2, \ldots, \theta_c + \mathrm{i}\bar{u}^j_{ic} - \varepsilon/2, \theta_c - \mathrm{i}\bar{u}^i_{jc} + \varepsilon/2) = f^c_{ij} F^{n-1}_{a_1,a_2,\ldots,a_c}(\theta_1, \theta_2, \ldots, \theta_c)\,. \tag{9.14}$$

After this general discussion on the matrix elements of the local operators $\mathcal{O}$, we return to the problem of the arbitrariness present in solving the Watson equations and to the analysis of the asymptotic

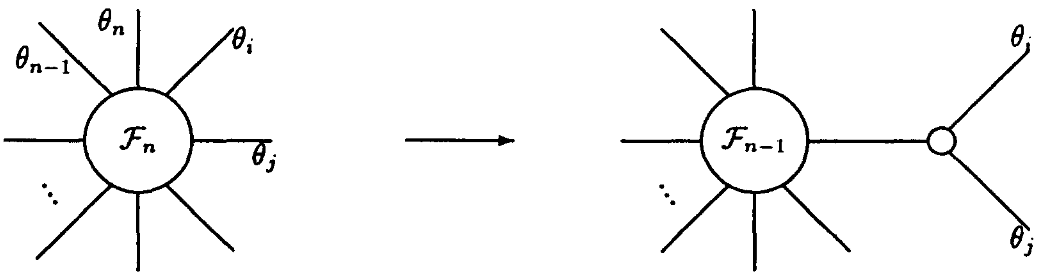


Fig. 34. Bootstrap equation for the form factor $\mathcal{F}_n$.

behaviour of the two-point function (9.1). As a nontrivial example we will consider the spin–spin correlation function of the Ising model at a noncritical value of the temperature [70, 71].

9.2. Magnetization correlation functions in the Ising model

In general, the Watson equations and the LSZ reduction formulae form a linear system of equations, whose solutions therefore span a linear space. In deriving the equations, the only important fact that matters is the knowledge of the symmetry properties of the operator $\mathcal{O}$ which may constrain the form of the couplings to the asymptotic states. A family of operators which share the same symmetry therefore satisfies the same set of equations. At the conformal point, such families are given by the Verma modules of primary fields. It is thus natural to think that this type of organization of the fields at the conformal point has some implication also for the fields out of the critical point and that form factors of the off-critical fields originating from the same conformal family satisfy the same Watson equations, i.e. the space of solutions to the Watson + LSZ system is isomorphic to the space of descendent operators. This problem has been investigated by Cardy and Mussardo [70] for the Ising conformal field theory perturbed by the energy operator. These authors have shown that there exists a natural grading of the space of solutions to the equations for the form factors and that the dimension at the level $(n, \bar{n})$ agrees with that of the corresponding Virasoro representation. Moreover, the analysis of the ultraviolet behavior of the form factors corresponding to deformed descendant operators leads to scaling dimensions which are precisely shifted by n and $\bar{n}$ with respect to those of the primary operator. These findings were interpreted as the result of a pair of Virasoro algebras acting on the space of form factors in the non-critical theory. We briefly summarize the analysis developed in [70].

The thermal perturbation of the Ising model preserves the original Z_2 spin symmetry of the critical point, therefore the operators may be labelled by their parity quantum number. The S-matrix in the spin sector of the thermal perturbed Ising model is given by $S = -1$ and does not present any additional bound state. The minimal solution of (9.10) is simply

$$F_{\min}(\theta) = \sinh \tfrac{1}{2}\theta \,. \tag{9.15}$$

For $\mathcal{O}$ we take a Z_2-odd operator originating from some field in the conformal tower of the primary magnetization operator. The Z_2 symmetry implies that F_n vanishes when n is even. The pole structure of K_n may be deduced as follows. There should be poles in every three-body channel. One may argue that no explicit poles should occur in n-body channels with $n > 3$, because crossing would then imply the existence of inelastic processes. Using the fact that

$$(p_i + p_j + p_k)^2 - 1 = 8 \cosh \tfrac{1}{2}\theta_{ij} \cosh \tfrac{1}{2}\, \theta_{jk} \cosh \tfrac{1}{2}\theta_{ki} \,, \tag{9.16}$$

we see that all possible three-body poles may be taken into account by letting

$$K_n = R_n \Big(\prod_{i<j} \cosh \tfrac{1}{2}\theta_{ij} \Big)^{-1} \,, \tag{9.17}$$

where the function R_n has no singularities. Note that when n is odd, the denominator in (9.17) is periodic in each rapidity variable θ_i, and therefore so must be R_n. We may therefore consider it as having a Taylor expansion in the variables e^{θ_i} and $e^{-\theta_i}$. Actually, it is possible to prove that in order for the ultraviolet behavior of the two-point function to be power-law bounded, this expansion should in fact terminate, so that R_n may be written in the form

$$R_n = P_n(p_1, \ldots, p_n) \exp\Big(-N \sum_i \theta_i\Big) \tag{9.18}$$

for some integer N. Here P_n is a totally symmetric polynomial in the variables $p_i = e^{\theta_i}$. If the spin of $\mathcal{O}$ is s, P_n must in fact be homogeneous of degree $s + N$. We will focus on the case $N = 0$. Using the bootstrap equations for the form factors F_n (this time, they relate F_n with F_{n-2} because the model presents no bound states but singularities in the three-particle channels, see fig. 35), it is possible to show that the P_n satisfy the recursion equations

$$P_n(p_1, -p_1, p_3, \ldots, p_n) = 2\mathrm{i} P_{n-2}(p_3, \ldots, p_n)\,. \tag{9.19}$$

Note that $\deg P_n = \deg P_{n-2}$, which implies that it is possible to find a solution with $\deg P_n = s$, independent of n. In other theories, the situation is different and the degree of P_n increases with n. In the case $s = 0$, which corresponds to the primary magnetization operator, we see that P_n is just a constant, proportional to $(2\mathrm{i})^{n/2}$. This result was previously obtained in ref. [69]. The general solution of (9.19) can be written as a sum of products of the form $\sigma_{k_1}\sigma_{k_2}, \ldots$, with $\Sigma_i k_i = s$, where the σ_k are the elementary symmetric polynomials in the momenta p_i,

$$\sigma_1 = p_1 + p_2 + \cdots, \quad \sigma_2 = p_1p_2 + p_1p_3 + \cdots, \quad \sigma_3 = p_1p_2p_3 + \cdots \tag{9.20}$$

Then, before the bootstrap equations (9.19) are used, the dimension of the space of solutions to the Watson equations at this level is given by the number $P(s)$ of partitions of s. But, imposing the bootstrap equations (9.19), the number of independent solutions becomes equal to the number $q(s)$ of partition of s into odd integers [70]. The generating function of this partition (see e.g. ref. [72]) coincides, up to a prefactor, with the character of the magnetic primary field of the Ising model,

$$\sum_{s=0}^{\infty} q(s) x^s = \prod_{r=1}^{\infty} (1 + x^r)\,. \tag{9.21}$$

From this result, one can infer that the dimension of the space of solutions to the equations for the off-critical form factors of the most relevant operators with spin s is equal to that of the space of such

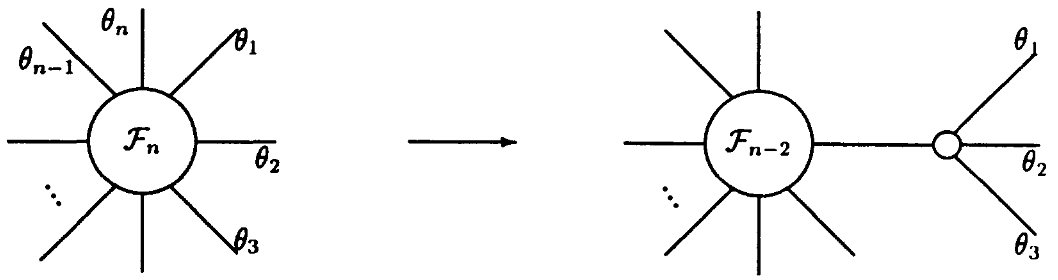


Fig. 35. Bootstrap equation for the form factor $\mathcal{F}_n$ in the thermal perturbation of the Ising model.

operators in the conformal field theory, establishing therefore an isomorphism between the conformal and the off-critical theory.

Let us analyze the ultraviolet behaviour of the two-point function of the leading primary magnetization operator. Its expression, in real Euclidean space reads

$$G(r)=\sum_n \frac{1}{n!}\int \sum_i \frac{\mathrm{d}\theta_i}{4\pi}\exp\Big(-|r|\sum_i \cosh\theta_i\Big)|F_n|^2 . \tag{9.22}$$

The F_n are given by

$$F_n = P_n(p_1, p_2, \ldots, p_n)\prod_{i<j}\tanh\tfrac{1}{2}\theta_{ij} , \tag{9.23}$$

with $p_n=(2\mathrm{i})^{n/2}$. Then, if we define

$$V(\theta)=-\ln\tanh^2\tfrac{1}{2}\theta , \quad U(y)=\mathrm{e}^{-y} , \quad l=\ln(2/r) , \tag{9.24}$$

the above expression may be rewritten in the very suggestive form [70]

$$G(r)=\tfrac{1}{2}[\Xi(1/2\pi, l)+\Xi(-1/2\pi, l)] . \tag{9.25}$$

where

$$\Xi(z,l)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\int\prod_i \mathrm{d}\theta_i \exp\Big(-\sum_i [U(\theta_i+l)+U(l-\theta_i)]\Big)\exp\Big(-\sum_{i<j}V(\theta_i-\theta_j)\Big) . \tag{9.26}$$

This is nothing but the grand partition function for a one-dimensional gas of particles, of fugacity z, interacting with each other via a two-body potential V, and with two walls, located at $\pm l$, with a potential U. Both of these potentials are repulsive and short-range. In the limit $l\to\infty$, we have

$$-\ln\Xi \sim -2pl+2f_{\mathrm{s}}+\mathrm{O}(\mathrm{e}^{-l/\xi}) , \tag{9.27}$$

where p is the pressure, f_{s} is the surface or boundary contribution to the free energy, and ξ is of the order of the correlation length. We see therefore that each term in (9.25) should have a power law dependence on r, with an exponent proportional to the pressure p. Since we expect the gas with positive fugacity (corresponding to the first term in (9.25)) to have larger pressure (a result which may be checked within the virial expansion [71]), we have the prediction that, as $r\to 0$,

$$G(r)\sim \mathrm{const}/r^{2p} . \tag{9.28}$$

Hence the overall scaling dimension of the operator is nothing but the pressure of this fictitious gas, with fugacity $z=1/2\pi$. In order to find its value, we can take advantage of the short range character of the potential and solve the one-dimensional gas in the approximation of nearest neighbour interaction. In this approximation, the pressure solves the following equation

$$2\pi = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-px} \tanh^2 \tfrac{1}{2}x \,, \tag{9.29}$$

whose numerical solution is

$$p = 0.125294 \,. \tag{9.30}$$

This gives a remarkably good approximation to the exact answer, namely $p = 1/8$. The validity of the nearest neighborhood approximation can be consistently checked by computing the density of the particles of the one-dimensional gas. The result is $\langle N \rangle / l = 0.1019$, which shows that the gas is really dilute. An exact expression of p, which coincides with the expected value $p = 1/8$, has been found in ref. [71] by summing the cluster expansion series.

Let us turn now to correlation functions of descendent operators, whose form factors differ from those of the primary operator by special combinations of polynomials in the variables e^{θ_i} and $\mathrm{e}^{-\theta_i}$, of respective degrees n and $\bar{n}$. Their correlation functions can be written as expectation values of these polynomials in the fictitious gas ensemble [70]

$$\Xi(z, l) \langle P(\{\mathrm{e}^{\theta_i}\}, \{\mathrm{e}^{-\theta_i}\}) \rangle \,, \tag{9.31}$$

where the partition function is, as before, essentially the two-point function of the primary operator, and P is homogeneous of degrees $2n, 2\bar{n}$, respectively, in the two sets of variables. They can be put in the following form

$$\exp[2(n + \bar{n})l] \langle P(U(l - \theta_i), U(l + \theta_i)) \rangle \,. \tag{9.32}$$

The latter expectation value, which is related to the probability for finding a given number of particles close to one or other of the walls, is finite in the thermodynamic limit $l \to \infty$. Thus, in that limit, the scaling dimension of an operator whose form factors contains polynomials of degree $(n, \bar{n})$ has a scaling dimension shifted by $(n + \bar{n})$ from that of the primary field. Of course, it is clear from the Lorentz transformation properties of its form factors that it corresponds to spin $(n - \bar{n})$. This shows the isomorphism between the descendent fields of the magnetization operators at the conformal point and the off-critical fields for the deformed theory.

10. Conclusions

We have analyzed massive integrable models originating from the deformations of minimal models of CFT. The case where only one relevant field is used to move the system away from criticality shows a rich pattern. The infinite set of non-critical conserved currents allows us to compute the exact S-matrix of the corresponding relativistic field theory and the mass spectrum of the excitations. These are related to the different correlation lengths of the statistical systems. The bootstrap approach, successfully applied to the computation of the scattering matrices, also provides powerful means for studying off mass-shell properties of integrable theories. The finite-size corrections can be analyzed in terms of the thermodynamical Bethe ansatz whereas the problem of off-critical correlators can be formulated in terms of the form-factor method.

An interesting result of the S-matrix approach is that the scattering data of several massive deformations are related to the root systems of simply laced algebras. We have discussed in detail the statistical models corresponding to the exceptional algebras E_n: the Ising model in a magnetic field background (E_8), the tricritical Ising model (E_7) and the tricritical Potts model (E_6), both in their high-temperature phase. These additional symmetries of the models permit to write the bootstrap fusions of the scattering processes in a very compact way.

We have also considered the Ising model and the tricritical Ising model in the case of multi-coupling deformations of the corresponding critical CFT. The problem under consideration has been to see, whether there exist integrable directions in the plane of the phase diagram other than those along the axis of the temperature and, for Ising, the axis of magnetic field whereas for the tricitical Ising model the axis of the chemical potential of the vacancies. In these cases, dimensional analysis shows that the possible conserved spins might only occur at the values given by the Coxeter exponents of E_8 (Ising) and E_7 (TIM). The explicit computation for the lowest values of the spins shows that conserved currents do not exist when the fixed point action is perturbed by two different relevant fields. The reason lies in the different null-vector conditions satisfied by the two different operators. One might consider the less relevant field as a perturbation of the theory defined by the most relevant one. However, this perturbation really ruins the structure of the original theory. In fact, in the original theories some particle masses are above the threshold of the lightest one and their stability is related to the root system of the algebras E_8 and E_7. When the second relevant field is present all these heavy particles decay.

The integrability of these models is still an open problem, even though we believe that the argument given in the text for the lowest spin can be generalized to higher level in order to show the absence of any conserved currents common to both perturbations. Assuming the non-integrability of these models, it would be interesting to study them in some detail as prototype realistic models which show a much richer behaviour than the integrable ones. One would expect the appearance of resonances as well as bound states with an S-matrix no longer being elastic.

Concerning recent developments in the analysis of integrable deformations of CFT, the S-matrix approach has recently been generalized by Zamolodchikov and Zamolodchikov [233] to massless but non-scale invariant QFT which correspond to renormalization group trajectories flowing between two fixed points of infinite correlation length. Although the status of this proposal needs further investigation, the present understanding is supported by the analysis of the (infinite) system of integral Bethe equations which gives the correct value of the fixed-point central charges.

In conclusion, the S-matrix approach sets a reference ground for developing ideas and techniques in off-critical statistical models and offers a new viewpoint on the space of two-dimensional quantum field theories.

Acknowledgement

I thank F. Colomo, A. Fring, A. Koubek, M.J. Martins, A. Schwimmer, G. Sotkov and Al.B. Zamolodchikov for helpful comments. Discussions on the subject with J.L. Cardy and S.K. Yang were extremely valuable.

Appendix A

In this appendix we discuss the fermionic representation of the two-dimensional Ising model. Let us consider a square lattice of $N = n^2$ spins consisting of n rows and n columns, with periodic boundary conditions. Let μ_a $(a = 1, 2, \ldots, n)$ denote the collection of all spin coordinates of the a-row

$$\mu_a = \{\sigma_1, \sigma_2, \ldots \sigma_n\}_{a\text{-row}} . \tag{A.1}$$

A spin configuration is specified by a set of $\{\mu_1, \ldots, \mu_n\}$. The ath row interacts only with the nearest rows labelled by $(a-1)$ and $(a+1)$. Let $E(\mu_a, \mu_{a+1})$ be the interaction energy between adjacent rows and $E(\mu_a)$ be the interaction energy of the spins within the ath row (eventually coupled to an external magnetic field B). Explicitly,

$$E(\mu, \mu') = -\varepsilon_1 \sum_{k=1}^{n} \sigma_k \sigma'_k , \tag{A.2}$$

$$E(\mu) = -\varepsilon_2 \sum_{k=1}^{n} \sigma_k \sigma_{k+1} - B \sum_{k=1}^{n} \sigma_k , \tag{A.3}$$

where ε_1 and ε_2 are the coupling constants in the horizontal and vertical directions, respectively. The total energy can be written as

$$E(\mu_1, \ldots, \mu_n) = \sum_{a=1}^{n} [E(\mu_a, \mu_{a+1}) + E(\mu_a)] , \tag{A.4}$$

and the partition function is then

$$Z = \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \exp[-\beta E(\mu_1, \ldots, \mu_n)] . \tag{A.5}$$

Let P be the $2^n \times 2^n$ transfer matrix operator defined by the matrix elements

$$\langle \mu | P | \mu' \rangle = \exp[-\beta(E(\mu, \mu') + E(\mu))] . \tag{A.6}$$

Then

$$Z = \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \langle \mu_1 | P | \mu_2 \rangle \langle \mu_2 | P | \mu_3 \rangle \cdots \langle \mu_n | P | \mu_1 \rangle = \sum_{\mu_1} \langle \mu_1 | P^n | \mu_1 \rangle = \mathrm{Tr}\, P^n . \tag{A.7}$$

The operator P can be written as

$$P = V_3 V_2 V_1 , \tag{A.8}$$

where V_i are the $2^n \times 2^n$ matrices,

$$\langle \sigma_1 \cdots \sigma_n | V_1 | \sigma'_1 \cdots \sigma'_n \rangle = \prod_{k=1}^{n} \exp(\beta \varepsilon_1 \sigma_k \sigma'_k) , \tag{A.9}$$

$$\langle \sigma_1 \cdots \sigma_n | V_2 | \sigma'_1 \cdots \sigma'_n \rangle = \delta_{\sigma_1 \sigma'_1} \cdots \delta_{\sigma_n \sigma'_n} \prod_{k=1}^{n} \exp(\beta \varepsilon_2 \sigma_k \sigma_{k+1}) , \tag{A.10}$$

$$\langle \sigma_1 \cdots \sigma_n | V_1 | \sigma'_1 \cdots \sigma'_n \rangle = \delta_{\sigma_1 \sigma'_1 \cdots \delta_{\sigma_n \sigma'_n}} \prod_{k=1}^{n} \exp(\beta B \sigma_k) . \tag{A.11}$$

A more convenient expression for these matrices is obtained by introducing the following operators defined in terms of direct products

$$\tilde{\sigma}_1(a) = 1 \times 1 \times \cdots \times \overset{a}{\tilde{\sigma}_1} \times 1 \cdots \times 1 , \tag{A.12}$$

$$\tilde{\sigma}_2(a) = 1 \times 1 \times \cdots \times \overset{a}{\tilde{\sigma}_2} \times 1 \cdots \times 1 , \tag{A.13}$$

$$\tilde{\sigma}_3(a) = 1 \times 1 \times \cdots \times \overset{a}{\tilde{\sigma}_3} \times 1 \cdots \times 1 , \tag{A.14}$$

where σ_i are the usual Pauli matrices. For $a \neq b$ it is easy to check that

$$[\tilde{\sigma}_i(a), \tilde{\sigma}_j(b)] = 0 , \tag{A.15}$$

whereas for the same a, the $\tilde{\sigma}_i(a)$ satisfy

$$[\tilde{\sigma}_i(a), \tilde{\sigma}_j(b)] = 2\mathrm{i} \varepsilon_{ijk} \tilde{\sigma}_k(a) , \tag{A.16}$$

$$\{\tilde{\sigma}_i(a), \tilde{\sigma}_j(a)\} = 2\delta_{ij} . \tag{A.17}$$

In terms of $\tilde{\sigma}_i(a)$, P can be written as

$$P = c \prod_{a=1}^{n} \exp[\beta B \tilde{\sigma}_3(a)] \exp[\beta \varepsilon_2 \tilde{\sigma}_3(a) \tilde{\sigma}_3(a+1)] \exp[\theta \tilde{\sigma}_1(a)] , \tag{A.18}$$

where

$$c = [2 \sinh(2\beta\varepsilon_1)]^{n/2} , \quad \tanh \theta = \exp(-2\beta\varepsilon_1) .$$

The corresponding Hamiltonian can be defined by

$$P = c\, \mathrm{e}^{aH} , \tag{A.19}$$

where a is the lattice spacing. In the continuum limit ($a \to 0$) and in absence of the magnetic field ($B = 0$), we have

$$H = \sum_{a=1}^{n} [\tilde{\eta} \tilde{\sigma}_1(a) + \eta \tilde{\sigma}_3(a) \tilde{\sigma}_3(a+1)] , \tag{A.20}$$

where $\tilde{\eta}$ and η are new coupling constants (functions of the previous ones).

We can define the *dual operators* by

$$\tilde{\mu}_3(r+\tfrac{1}{2}) = \prod_{\rho=-\infty}^{r} \tilde{\sigma}_1(\rho)\,, \tag{A.21}$$

$$\tilde{\mu}_1(r+\tfrac{1}{2}) = \tilde{\sigma}_3(r)\tilde{\sigma}_3(r+1)\,. \tag{A.22}$$

It is easy to check the relations

$$\begin{gathered} \tilde{\mu}_3^2 = \tilde{\mu}_1^2 = 1\,, \quad \tilde{\mu}_3(r-\tfrac{1}{2})\tilde{\mu}_3(r+\tfrac{1}{2}) = \tilde{\sigma}_1(r)\,, \quad [\tilde{\mu}_1(r+\tfrac{1}{2}), \tilde{\mu}_3(r'+\tfrac{1}{2})] = 2\delta_{r,r'}\,, \\ [\tilde{\mu}_3(r+\tfrac{1}{2}), \tilde{\mu}_3(r'+\tfrac{1}{2})] = 0\,, \quad [\tilde{\mu}_3(r+\tfrac{1}{2}), \tilde{\sigma}_1(r')] = 0\,. \end{gathered} \tag{A.23}$$

The Hamiltonian (A.20), in terms of the dual operators, reads

$$H = \sum_r [\tilde{\eta}\tilde{\mu}_3(r-\tfrac{1}{2})\tilde{\mu}_3(r+\tfrac{1}{2}) + \eta\tilde{\mu}_1(r+\tfrac{1}{2})]\,. \tag{A.24}$$

The Kramers–Wannier symmetry is expressed by the equations

$$\tilde{\mu}_1 \leftrightarrow \tilde{\sigma}_1\,, \quad \tilde{\mu}_3 \leftrightarrow \tilde{\sigma}_3\,, \quad \eta \leftrightarrow \tilde{\eta} \tag{A.25}$$

and can be made manifest by this expression of the Hamiltonian

$$H = \frac{1}{2}\sum_r [\tilde{\eta}\tilde{\sigma}_1(r) + \eta\tilde{\mu}_1(r+\tfrac{1}{2})]\,. \tag{A.26}$$

The equations of motion are

$$(\partial/\partial t)\tilde{\sigma}_3(r) = [H, \tilde{\sigma}_3(r)] = \tilde{\eta}\tilde{\sigma}_1(r)\tilde{\sigma}_3(r)\,, \tag{A.27}$$

$$(\partial/\partial t)\tilde{\mu}_3(r+\tfrac{1}{2}) = [H, \tilde{\mu}(r+\tfrac{1}{2})] = \eta\tilde{\sigma}_3(r)\tilde{\sigma}_3(r+1)\tilde{\mu}_1(r+\tfrac{1}{2})\,. \tag{A.28}$$

Let us put

$$u(r) = \tilde{\mu}_3(r-\tfrac{1}{2})\tilde{\sigma}_3(r)\,, \tag{A.29}$$

$$v(r) = \tilde{\mu}_3(r+\tfrac{1}{2})\tilde{\sigma}_3(r)\,. \tag{A.30}$$

The equations of motion for the new variables become

$$\partial u(r)/\partial t = \tilde{\eta}v(r) - \eta v(r-1)\,, \tag{A.31}$$

$$\partial v(r)/\partial t = \tilde{\eta}u(r) - \eta u(r-1)\,, \tag{A.32}$$

and in the continuum limit we obtain

$$\partial u(r)/\partial t = (\tilde{\eta} - \eta)v(r) + \eta\, \partial v(r)/\partial r\,, \tag{A.33}$$

$$\partial v(r)/\partial t = (\tilde{\eta} - \eta)u(r) - \eta\, \partial u(r)/\partial r\,. \tag{A.34}$$

The two fields $u(r)$ and $v(r)$ can be organized as

$$\psi(r) = \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}, \tag{A.35}$$

with anticommutation relations

$$\{u(r), u(r')\} = 2\delta_{r,r'}\,, \tag{A.36}$$

$$\{v(r), v(r')\} = 2\delta_{r,r'}\,. \tag{A.37}$$

A compact form of the equations of motion is given by

$$(\gamma^0\, \partial/\partial t + \gamma^3\, \partial/\partial r - m)\psi = 0\,, \tag{A.38}$$

where

$$m = \tilde{\eta} - \eta\,, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

γ^0 and γ^3 being the Euclidean γ matrices.

Therefore, the Ising model is equivalent to a free Majorana fermionic system. At the critical point, defined by $\tilde{\eta} = \eta$, the mass parameter vanishes.

Appendix B

Here we discuss some properties of the tricritical Ising model. First of all, we consider the coset construction of the tricritical Ising models in terms of the exceptional group E_7

$$M_4 = (E_7)_1 \otimes (E_7)_1/(E_7)_2\,. \tag{B.1}$$

We need the following results from the theory of the affine Kac–Moody algebras [230]. The central charge of a conformal field theory constructed on an affine Lie algebra G at level k is given by

$$c_G = k|G|/(k + \tilde{\psi}_G)\,, \tag{B.2}$$

where $|G|$ is the dimension of the algebra and $\tilde{\psi}_G$ the dual Coxeter number. The unitarity condition for the CFT restricts the highest weight representations $|(\lambda)\rangle$ which can appear at the level k. Denoting with ω the highest root, the allowed representations $|\lambda\rangle$ at the level k must satisfy

$$2\omega \cdot \lambda/\omega^2 \leq k\,. \tag{B.3}$$

Their dimension is

$$\Delta_\lambda = \frac{C_\lambda/\omega^2}{k+\tilde{\psi}_G}\,, \tag{B.4}$$

where C_λ is the quadratic Casimir in the representation $\{\lambda\}$.

Using a subgroup $\mathrm{H} \subset \mathrm{G}$ we can construct a CFT on the coset group G/H. The coset theory G/H has central charge equal to [230]

$$c_{\mathrm{G/H}} = c_{\mathrm{G}} - c_{\mathrm{H}} = k_{\mathrm{G}}|\mathrm{G}|/(k_{\mathrm{G}} + \tilde{\psi}_{\mathrm{G}}) - k_{\mathrm{H}}|\mathrm{H}|/(k_{\mathrm{H}} + \tilde{\psi}_{\mathrm{H}}) \tag{B.5}$$

and its representations h^k are obtained by the decomposition of the Hilbert space

$$|c_{\mathrm{G}}, \lambda_{\mathrm{G}}\rangle = \oplus_k [|c_{\mathrm{G/H}}, h^k_{\mathrm{G/H}}\rangle \otimes |c_{\mathrm{H}}, \lambda^k_{\mathrm{H}}\rangle]\,. \tag{B.6}$$

In the case of the TIM, $\tilde{\psi} = 18$ and eq. (B.5) gives $c = \frac{7}{10}$. At level $k = 1$ the possible representations are the identity 1 and the representation Π_6 with scaling dimension 0 and $\frac{3}{4}$ respectively

$$(\mathrm{E}_7)_1 \to \{1, \Pi_6\} = \{0, \tfrac{3}{4}\}\,. \tag{B.7}$$

Their components $(n_1, n_2, \ldots, n_7)$(n_i integer) with respect to the simple roots of E_7 are [226, 227]

$$1 \to (0,0,0,0,0,0,0)\,, \quad \Pi_6 \to (0,0,0,0,0,1,0)\,. \tag{B.8}$$

At the level $k = 2$ one finds the representations

$$(\mathrm{E}_7)_2 \to \{1, \Pi_1, \Pi_2, \Pi_5, \Pi_6\} = \{0, \tfrac{9}{10}, \tfrac{21}{16}, \tfrac{7}{5}, \tfrac{57}{80}\}\,. \tag{B.9}$$

The corresponding fundamental weights are

$$\Pi_1 \to (1,0,0,0,0,0,0)\,, \quad \Pi_2 \to (0,1,0,0,0,0,0)\,, \quad \Pi_5 \to (0,0,0,0,1,0,0)\,. \tag{B.10}$$

Π_1 is the adjoint representation. Using eq. (B.6) we can recover the scaling dimensions of the TIM,

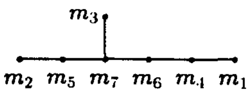
$$\begin{aligned} &(0)_1 \times (0)_1 = [(0)_{\mathrm{TIM}} \otimes (0)_2] + [(\tfrac{1}{10})_{\mathrm{TIM}} \otimes (\Pi_1)_2] + [(\tfrac{6}{10})_{\mathrm{TIM}} \otimes (\Pi_5)_2]\,, \\ &(0)_1 \times (\tfrac{3}{4})_1 = [(\tfrac{7}{16})_{\mathrm{TIM}} \otimes (\Pi_2)_2] + [(\tfrac{3}{80})_{\mathrm{TIM}} \otimes (\Pi_6)_2]\,, \\ &(\tfrac{3}{4})_1 \times (\tfrac{3}{4})_1 = (\tfrac{3}{2})_{\mathrm{TIM}} \otimes (0)_2\,. \end{aligned} \tag{B.11}$$

Note that the operator of the energy density $\Phi_{1/10,1/10}$ is associated to the adjoint of E_7. This is the necessary condition in order to write the ATFT Lagrangian on E_7 for the non-critical model.

The value of the masses given in table 18 form the components of the Perron–Frobenius vector of the algebra E_7, with the following correspondence to the representations of E_7 (see table 23) [225–227] (we use the dimension of the representations to denote them)

Table 23
Dynkin diagram of E_7 and assignment of the masses to the corresponding dots.

Exponents:	1, 5, 7, 9, 11, 13, 17



$$
\begin{aligned}
&m_1 \to 56\,, \quad m_2 \to 133\,, \quad m_3 \to 912\,, \quad m_4 \to 1539\,, \\
&m_5 \to 8645\,, \quad m_6 \to 27664\,, \quad m_7 \to 365750\,.
\end{aligned}
\tag{B.12}
$$

From eq. (B.11), the vacancy density operator $\Phi_{6/10,6/10}$ is paired with (Π_5) that has the same quantum number of the particle with mass m_4. The list of all minimal S-matrices of the quantum E_7 Toda system (which coincide with the thermal deformation of the TIM) is given in terms of the following particular subset of functions f_x:

$$
f_{1/18},\, f_{5/18},\, f_{7/18},\, f_{1/6},\, F_{2/9},\, f_{4/9},\, f_{8/9},\, f_{2/3},\, f_{1/2}\,. \tag{B.13}
$$

The positions of their poles are those given in table 12. The complete set of two-particle functions S_{AB} reads:

$$
\begin{aligned}
&S_{11} = f_{4/9} f_{8/9}\,, \quad S_{12} = f_{5/16} f_{7/18}\,, \quad S_{13} = f_{2/3} f_{2/9} f_{4/9}\,, \quad S_{14} = f_{1/6} f_{1/18} f_{7/18} f_{1/2}\,, \\
&S_{15} = f_{2/9} f_{4/9} (f_{2/3})^2\,, \quad S_{16} = f_{2/3} f_{2/9} f_{8/9} (f_{4/9})^2\,, \quad S_{17} = f_{1/6} f_{1/2} (f_{5/18})^2 (f_{7/18})^2\,, \quad S_{22} = f_{2/3} f_{4/9} f_{8/9}\,, \\
&S_{23} = f_{1/6} f_{5/18} f_{7/18} f_{1/2}\,, \quad S_{24} = f_{2/9} f_{4/9} (f_{2/3})^2\,, \quad S_{25} = f_{1/6} f_{1/18} f_{5/18} f_{1/2} (f_{7/18})^2\,, \\
&S_{26} = f_{1/6} f_{1/2} (f_{5/18})^2 (f_{7/18})^2\,, \quad S_{27} = f_{8/9} (f_{2/9})^2 (f_{2/3})^2 (f_{4/9})^3\,, \quad S_{33} = f_{2/9} f_{8/9} (f_{2/3})^2 (f_{4/9})^2\,, \\
&S_{34} = f_{1/6} f_{1/2} (f_{5/18})^2 (f_{7/18})^2\,, \quad S_{35} = f_{8/9} (f_{2/9})^2 (f_{2/3})^2 (f_{4/9})^3\,, \quad S_{36} = f_{8/9} (f_{2/9})^2 (f_{4/9})^3 (f_{2/3})^3\,, \\
&S_{37} = f_{1/18} (f_{1/2})^2 (f_{1/6})^2 (f_{5/18})^3 (f_{7/18})^4\,, \quad S_{44} = f_{2/3} f_{2/9} (f_{8/9})^2 (f_{4/9})^3\,, \quad S_{45} = f_{1/6} f_{1/2} (f_{5/18})^3 (f_{7/18})^3\,, \\
&S_{46} = f_{1/18} (f_{1/2})^2 (f_{1/6})^2 (f_{5/18})^2 (f_{7/18})^3\,, \quad S_{47} = f_{8/9} (f_{2/9})^3 (f_{4/9})^4 (f_{2/3})^4\,, \\
&S_{55} = (f_{2/9})^2 (f_{8/9})^2 (f_{2/3})^3 (f_{4/9})^4\,, \quad S_{56} = f_{8/9} (f_{2/9})^3 (f_{2/3})^4 (f_{4/9})^4\,, \\
&S_{57} = f_{1/18} (f_{1/2})^3 (f_{1/6})^3 (f_{5/18})^4 (f_{7/18})^5\,, \quad S_{66} = (f_{8/9})^2 (f_{2/9})^3 (f_{2/3})^4 (f_{4/9})^5\,, \\
&S_{67} = f_{1/18} (f_{1/2})^3 (f_{1/6})^3 (f_{5/18})^5 (f_{7/18})^6\,, \quad S_{77} = (f_{8/9})^3 (f_{2/9})^5 (f_{2/3})^7 (f_{4/9})^8\,.
\end{aligned}
\tag{B.14}
$$

Appendix C

Here we derive similar results of the previous appendix for the case of the Ising model. We consider the coset

$$M_3 = (E_8)_1 \otimes (E_8)_1 / (E_8)_2 \,. \tag{C.1}$$

The Coxeter number is now $\tilde{\psi} = 1/18$. Using eq. (B.5) we get $c = \frac{1}{2}$. At the level $k = 1$ there is only the identity representation 1 with dimension 0. At the level $K = 2$ there are three representations

$$(E_8)_2 \rightarrow \{1, \Pi_1, \Pi_7\} = \{0, \tfrac{15}{16}, \tfrac{3}{2}\} \,. \tag{C.2}$$

Π_1 is the adjoint representation of E_8. The Ising model is recovered by the decomposition

$$(0)_1 \times (0)_1 = [(0)_{Is} \otimes (0)_2] + [(\tfrac{1}{16})_{Is} \otimes (\tfrac{15}{16})_2] + [(\tfrac{1}{2})_{Is} \otimes (\tfrac{3}{2})_2] \,. \tag{C.3}$$

The spin operator $\Phi_{1/16,1/16}$ is associated to the adjoint Π_1 and following Eguchi and Yang the off-critical Ising model in the presence of a magnetic field is described by the ATFT based on E_8. The Perron–Frobenius vector of the masses is associated to the following representations of E_8 [225, 226, 227] (see table 24):

$$\begin{gathered} m_1 \rightarrow 248 \,, \quad m_2 \rightarrow 3875 \,, \quad m_3 \rightarrow 30380 \,, \quad m_4 \rightarrow 147250 \,, \quad m_5 \rightarrow 2450240 \,, \\ m_6 \rightarrow 6696000 \,, \quad m_7 \rightarrow 146325270 \,, \quad m_8 \rightarrow 6899079264 \,. \end{gathered} \tag{C.4}$$

From eq. (C.3), the energy density operator $\Phi_{1/2,1/2}$ is associated to the representation Π_7, i.e. to the particle with mass m_2.

The square of the masses $\{m_1, m_6, m_5, m_7\}$ are the roots of the quartic polynomial

$$P_1 = x^4 - 30x^3 + 300x^2 - 1080x + 720 \tag{C.5}$$

and we introduce the notation

$$(m_1, m_6, m_5, m_7) \rightarrow (A_1, A_2, A_3, A_4) \,. \tag{C.6}$$

The square of the masses $\{m_2, m_3, m_8, m_4\}$ are the roots of the other quartic polynomial

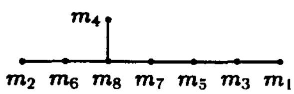
$$P_1 = x^4 - 30x^3 + 240x^2 - 720x + 720 \tag{C.7}$$

and for them we introduce the notation

$$(m_2, m_3, m_8, m_4) \rightarrow (B_1, B_2, B_3, B_4) \,. \tag{C.8}$$

Table 24
Dynkin diagram of E_8 and assignment of the masses to the corresponding dots.

Exponents:	1, 7, 11, 13, 17, 19, 23, 29



The bootstrap fusions of the E_8 systems can be written as

$$
\begin{aligned}
&A_i \times A_i = A_i + B_i + B_{i+1}\,, \quad A_i \times A_{i+1} = A_{i+2} + A_{i+3} + B_{i+3}\,,\\
&A_i \times A_{i+2} = A_{i+1} + A_{i+3} + B_{i+1} + B_{i+3}\,, \quad A_i \times A_{i+3} = A_{i+1} + A_{i+2} + B_{i+2}\,,\\
&B_i \times B_i = A_i + A_{i+1} + A_{i+2} + B_i + B_{i+3}\,, \quad B_i \times B_{i+1} = A_i + A_{i+1} + B_{i+1}\,,\\
&B_i \times B_{i+2} = A_{i+1} + A_{i+3}\,, \quad B_i \times B_{i+3} = A_i + B_i + A_{i+3}\,, \quad A_i \times B_i = A_i + B_i + B_{i+1} + B_{i+3}\,,\\
&A_i \times B_{i+1} = A_i + A_{i+2} + B_i + B_{i+3}\,, \quad A_i \times B_{i+2} = B_{i+2} + A_{i+3}\,,\\
&A_i \times B_{i+3} = A_{i+1} + A_{i+2} + B_i + B_{i+1} + B_{i+3}
\end{aligned}
\tag{C.9}
$$

and they are a subset of the tensor product decomposition of the E_8 representations given in eq. (C.4) [226, 227].

In order to simplify the expression of the S-matrices, in this appendix we denote the basic functions f_x simply by (x) (with the identification $(x) \equiv (1-x)$). With this abbreviation, the complete set of two particle S_{AB}-matrices for the Ising model in a magnetic field reads:

$$
\begin{aligned}
&S_{11} = (\tfrac{1}{3})(\tfrac{2}{5})(\tfrac{1}{15})\,, \quad S_{12} = (\tfrac{4}{15})(\tfrac{3}{5})(\tfrac{4}{5})(\tfrac{7}{17})\,,\\
&S_{13} = (\tfrac{1}{10})(\tfrac{1}{30})(\tfrac{3}{10})(\tfrac{17}{30})(\tfrac{11}{30})^2\,, \quad S_{14} = (\tfrac{1}{6})(\tfrac{3}{10})(\tfrac{7}{30})(\tfrac{17}{30})(\tfrac{19}{30})(\tfrac{1}{2})^2\,,\\
&S_{15} = (\tfrac{2}{15})(\tfrac{1}{15})(\tfrac{7}{15})(\tfrac{11}{15})(\tfrac{1}{3})^2(\tfrac{2}{5})^2\,, \quad S_{16} = (\tfrac{1}{6})(\tfrac{7}{10})(\tfrac{11}{30})(\tfrac{1}{2})^2(\tfrac{7}{30})^2(\tfrac{17}{30})^2\,,\\
&S_{17} = (\tfrac{1}{10})(\tfrac{1}{6})(\tfrac{7}{30})(\tfrac{1}{2})^2(\tfrac{7}{10})^2(\tfrac{11}{30})^2(\tfrac{17}{30})^2\,, \quad S_{18} = (\tfrac{2}{15})(\tfrac{1}{5})^2(\tfrac{2}{5})^2(\tfrac{2}{3})^2(\tfrac{11}{15})^2(\tfrac{7}{15})^3\,,\\
&S_{22} = (\tfrac{1}{15})(\tfrac{1}{5})(\tfrac{2}{3})(\tfrac{7}{15})(\tfrac{11}{15})(\tfrac{2}{5})^2\,, \quad S_{23} = (\tfrac{1}{6})(\tfrac{11}{30})(\tfrac{3}{10})(\tfrac{1}{2})^2(\tfrac{7}{30})^2(\tfrac{17}{30})^2\,,\\
&S_{24} = (\tfrac{1}{10})(\tfrac{1}{6})(\tfrac{7}{30})(\tfrac{1}{2})^2(\tfrac{11}{30})^2(\tfrac{3}{10})^2(\tfrac{17}{30})^2\,, \quad S_{25} = (\tfrac{2}{15})(\tfrac{1}{5})^2(\tfrac{2}{5})^2(\tfrac{4}{15})^2(\tfrac{2}{3})^2(\tfrac{7}{15})^2\,,\\
&S_{26} = (\tfrac{1}{30})(\tfrac{1}{10})(\tfrac{1}{6})(\tfrac{1}{2})^2(\tfrac{3}{10})^2(\tfrac{7}{30})^2(\tfrac{11}{30})^3(\tfrac{17}{30})^3\,, \quad S_{27} = (\tfrac{1}{10})(\tfrac{7}{30})^2(\tfrac{1}{6})^2(\tfrac{3}{10}{}^3(\tfrac{11}{30})^3(\tfrac{17}{30})^3(\tfrac{1}{2})^4\,,\\
&S_{28} = (\tfrac{1}{15})(\tfrac{2}{15})^2(\tfrac{1}{5})^2(\tfrac{4}{15})^3(\tfrac{2}{3})^4(\tfrac{2}{5})^4(\tfrac{8}{15})^4\,, \quad S_{33} = (\tfrac{2}{15})(\tfrac{4}{15})(\tfrac{8}{15})(\tfrac{1}{15})^2(\tfrac{2}{5})^3(\tfrac{2}{3})^3\,,\\
&S_{34} = (\tfrac{2}{15})(\tfrac{1}{5})^2(\tfrac{2}{5})^2(\tfrac{2}{3})^2(\tfrac{4}{15})^2(\tfrac{8}{15})^3\,, \quad S_{35} = (\tfrac{1}{30})(\tfrac{1}{6})(\tfrac{7}{30})(\tfrac{1}{10})^2(\tfrac{1}{2})^2(\tfrac{3}{10})^3(\tfrac{17}{30})^3(\tfrac{19}{30})^4\,,\\
&S_{36} = (\tfrac{2}{15})(\tfrac{2}{3})^2(\tfrac{1}{5})^3(\tfrac{2}{5})^3(\tfrac{4}{15})^3(\tfrac{7}{15})^4\,, \quad S_{37} = (\tfrac{1}{15})(\tfrac{1}{5})^2(\tfrac{2}{15})(\tfrac{4}{15})^3(\tfrac{2}{3})^4(\tfrac{1}{5})^4(\tfrac{7}{15})^4\,,\\
&S_{38} = (\tfrac{1}{10})(\tfrac{1}{6})^3(\tfrac{3}{10})^4(\tfrac{7}{30})^4(\tfrac{11}{30})^4(\tfrac{17}{30})^5(\tfrac{1}{2})^6\,, \quad S_{44} = (\tfrac{1}{15})(\tfrac{2}{15})(\tfrac{1}{5})^2(\tfrac{4}{15})^2(\tfrac{2}{3})^3(\tfrac{2}{5})^3(\tfrac{7}{15})^3\,,\\
&S_{45} = (\tfrac{1}{10})(\tfrac{1}{6})^2(\tfrac{7}{30})^3(\tfrac{3}{10})^3(\tfrac{11}{30})^3(\tfrac{1}{2})^4(\tfrac{17}{30})^4\,, \quad S_{46} = (\tfrac{1}{15})(\tfrac{1}{5})^2(\tfrac{2}{15})^2(\tfrac{4}{15})^3(\tfrac{2}{3})^4(\tfrac{2}{5})^4(\tfrac{7}{15})^4\,,\\
&S_{47} = (\tfrac{1}{15})(\tfrac{2}{15})^2(\tfrac{1}{5})^3(\tfrac{2}{3})^4(\tfrac{4}{15})^4(\tfrac{2}{5})^5(\tfrac{7}{15})^5\,, \quad S_{48} = (\tfrac{1}{30})(\tfrac{1}{10})^2(\tfrac{1}{6})^3(\tfrac{7}{30})^4(\tfrac{3}{10})^5(\tfrac{1}{2})^6(\tfrac{11}{30})^6(\tfrac{17}{30})^6\,,\\
&S_{55} = (\tfrac{1}{15})^2(\tfrac{2}{15})^2(\tfrac{1}{5})^2(\tfrac{4}{15})^3(\tfrac{7}{15})^4(\tfrac{2}{3})^5(\tfrac{1}{5})^5\,, \quad S_{56} = (\tfrac{1}{10})(\tfrac{1}{6})^3(\tfrac{3}{10})^4(\tfrac{11}{30})^4(\tfrac{7}{30})^4(\tfrac{17}{30})^5(\tfrac{1}{2})^6\,,\\
&S_{57} = (\tfrac{1}{30})(\tfrac{1}{10})^2(\tfrac{1}{6})^3(\tfrac{7}{30})^4(\tfrac{3}{10})^5(\tfrac{1}{2})^6(\tfrac{11}{30})^6(\tfrac{17}{30})^6\,, \quad S_{58} = (\tfrac{1}{15})(\tfrac{2}{15})^3(\tfrac{1}{5})^5(\tfrac{2}{3})^6(\tfrac{4}{15})^6(\tfrac{2}{5})^7(\tfrac{7}{15})^8\,,
\end{aligned}
$$

$$S_{66} = (\tfrac{1}{15})^2(\tfrac{2}{15})^2(\tfrac{1}{5})^3(\tfrac{4}{15})^4(\tfrac{2}{3})^5(\tfrac{7}{15})^5(\tfrac{2}{5})^6\,, \quad S_{67} = (\tfrac{1}{15})(\tfrac{2}{15})^3(\tfrac{1}{5})^4(\tfrac{4}{15})^5(\tfrac{2}{3})^6(\tfrac{2}{5})^6(\tfrac{7}{15})^7\,,$$

$$S_{68} = (\tfrac{1}{30})(\tfrac{1}{10})^3(\tfrac{1}{6})^4(\tfrac{7}{30})^5(\tfrac{3}{10})^7(\tfrac{1}{2})^8(\tfrac{11}{30})^8(\tfrac{17}{30})^8\,, \quad S_{77} = (\tfrac{1}{15})^2(\tfrac{2}{15})^3(\tfrac{1}{5})^5(\tfrac{4}{15})^6(\tfrac{2}{3})^7(\tfrac{2}{5})^8(\tfrac{7}{15})^8\,,$$

$$S_{78} = (\tfrac{1}{30})(\tfrac{1}{10})^3(\tfrac{1}{6})^5(\tfrac{7}{30})^7(\tfrac{3}{10})^8(\tfrac{11}{30})^9(\tfrac{1}{2})^{10}(\tfrac{17}{30})^{10}\,, \quad S_{88} = (\tfrac{1}{15})^3(\tfrac{2}{15})^5(\tfrac{1}{5})^7(\tfrac{4}{15})^9(\tfrac{2}{3})^{11}(\tfrac{2}{5})^{12}(\tfrac{7}{15})^{12}\,.$$

Appendix D

In this appendix we present the computations related to the conserved charge with spin $s = 7$. We use an algebraic method that leads to an expression for the current P_8 different from that given in ref. [189], eq. (2.39), but the result can be obtained from the expression (2.39) as well.

At level 8, the general expression of a quasi-primary operator in the family of the identity operator $I(z)$ is

$$A_8^{(a)} = a_1 L_{-8} + a_2 L_{-6} L_{-2} + a_3 L_{-5} L_{-3} + a_4 L_{-4}^2 + a_5 L_{-4} L_{-2}^2 + a_6 L_{-3}^2 L_{-2} + a_7 L_{-2}^4\,. \tag{D.1}$$

The coefficients $\{a_i\}$ are fixed by the condition of the quasi-primary fields, namely

$$L_{\pm 1} A_8^{(a)} = 0\,, \quad L_0 A_8^{(a)} = 0\,. \tag{D.2}$$

From this we obtain the equations

$$9a_1 + 5a_4 + 27a_7 = 0\,, \quad 7a_2 + 4a_6 + 24a_7 = 0\,, \quad 6a_3 + 10a_4 + 3a_5 = 0\,, \quad 5a_5 + 8a_6 + 18a_7 = 0\,. \tag{D.3}$$

At the level 8 there exist three quasi-primary operators in the family of the identity operator. We choose

$$\begin{aligned} A_8^{(1)} &= L_{-4}^2 - \tfrac{5}{3} L_{-5} L_{-3} - \tfrac{4}{9} L_{-8}\,, \\ A_8^{(2)} &= L_{-3}^2 L_{-2} - \tfrac{4}{7} L_{-6} L_{-2} + \tfrac{4}{5} L_{-5} L_{-3} - \tfrac{8}{5} L_{-4} L_{-2}^2\,, \\ A_8^{(3)} &= L_{-2}^4 - \tfrac{18}{5} L_{-4} L_{-2}^2 + \tfrac{9}{5} L_{-5} L_{-3} - \tfrac{24}{7} L_{-6} L_{-2} - 3L_8\,. \end{aligned} \tag{D.4}$$

A linear combination of these operators,

$$P_8 = \alpha A_8^{(1)} + \beta A_8^{(2)} + \gamma A_8^{(3)}\,, \tag{D.5}$$

is conserved under the perturbation of the field $\Phi_{1,3}$, due to the null-vector condition satisfied by this field. To fix the coefficient α, β, and γ we have to compute

$$\partial_{\bar z} P_8 = L_{-1}[\cdots] + B \tag{D.6}$$

and to demand that the piece B (which is not a total derivative) is zero. This can be achieved due to the null-vector condition of the field $\Phi_{1,3}$. The computation is done using the operators D_n defined by the commutation relations with the L_ns [29],

$$[D_m, L_n] = ((1-\Delta)(n+1)+m)D_{n+m}\,, \quad D_0 = \partial_{\bar{z}}\,, \quad D_{-n}I(z) = [\lambda/(n-1)!]L_{-1}^{n-1}\Phi\,.$$

For the Ising model the result is given by

$$P_8^{(1)} = A_8^{(1)} + \frac{968240}{166869}A_8^{(2)} - \frac{69160}{18541}A_8^{(3)}\,, \tag{D.7}$$

whereas for the TIM

$$P_8^{(1)} = A_8^{(1)} + \frac{83537300}{12310833}A_8^{(2)} - \frac{53253200}{12310833}A_8^{(3)}\,. \tag{D.8}$$

Both currents are not conserved under the perturbation of the respective $\Phi_{1,2}$ operator, $\Phi_{1/16,1/16}$ for the Ising model and $\Phi_{1/10,1/10}$ for the TIM. In fact, the linear combination which is conserved under the perturbation of the Ising model with a magnetic field is

$$P_8^{(2)} = A_8^{(1)} - \frac{1585360}{12003709}A_8^{(2)} - \frac{4426240}{12003709}A_8^{(3)}\,. \tag{D.9}$$

This is the conserved current of the E_8 group. The conserved current of the group E_7 (TIM perturbed with energy density operator $\Phi_{1/10,1/10}$) is given by

$$P_8^{(2)} = A_8^{(1)} - \frac{764750}{8735637}A_8^{(2)} - \frac{3458000}{8735637}A_8^{(3)}\,. \tag{D.10}$$

References

[1] R.J. Baxter, Exactly Solvable Models in Statistical Mechanics (Academic Press, New York, 1982).
[2] A.Z. Patashinski and V.L. Pokrovskii, Fluctuation Theory of Phase Transitions (Pergamon Press, Oxford, 1979).
[3] L.P. Kadanoff et al., Rev. Mod. Phys. 39 (1967) 395.
[4] D.J. Amit, Field Theory, the Renormalization Group and Critical Phenomena (McGraw-Hill, New York, 1978).
[5] C. Domb and M.S. Green, Phase Transition and Critical Phenomena (Academic Press, New York, 1983).
[6] H. Stanley, Introduction to Phase Transition and Critical Phenomena (Clarendon, Oxford, 1971).
[7] G. Parisi, Statistical Field Theory (Addison-Wesley, Reading, 1988).
[8] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Oxford Univ. Press, Oxford, 1989).
[9] C. Itzykson and J.M. Drouffe, Statistical Field Theory (Cambridge Univ. Press, Cambridge, 1989).
[10] K.G. Wilson and J. Kogut, Phys. Rep. 12 (1974) 200.
[11] A.M. Polyakov, JETP Lett. 12 (1970) 381; Sov. Phys. JETP 39 (1974) 10.
[12] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.
[13] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575.
[14] C. Itzykson, H. Saleur and J.B. Zuber, Conformal Invariance and Applications to Statistical Mechanics, (World Scientific, Singapore 1988).
[15] Vl.S. Dotsenko, Advanced Studies Pure Math. 16 (1988) 123.
[16] J.L. Cardy, in: Champs, Cordes et Phénomènes Critiques/Field, String and Critical Phenomena, eds E. Brezin and J. Zinn-Justin (Elsevier, Amsterdam, 1989); Phase Trans. Critical Phenom. 11 (1987) 55.
[17] P. Ginsparg, in: Champs, Cordes et Phénomènes Critiques/Field, String and Critical Phenomena, eds E. Brezin and J. Zinn-Justin (Elsevier, Amsterdam, 1989).

[18] P. Goddard and D. Olive, Int. J. Mod. Phys. A 1 (1986) 304.
[19] V. Kac, Lecture Notes in Physics, Vol. 94 (Springer, Berlin, 1979) p. 441;
B.L. Feigin and D.B. Fuchs, Funct. Analysis and Appl. 16 (1982) 114.
[20] Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B 240 [FS12] (1984) 312; B 251 [FS13] (1985) 691; Phys. Lett. B 154 (1985) 291.
[21] M. Bauer, P. Di Francesco, C. Itzykson and J.B. Zuber, Phys. Lett. B 260 (1991), 323.
[22] E. Verlinde, Nucl. Phys. B 300 (1988), 360.
[23] A. Rocha-Caridi, in: Vertex Operators in Mathematics and Physics, eds J. Lepowski, S. Mandelstam and I.M. Singer (Springer, Berlin, 1985).
[24] J.L. Cardy, Nucl. Phys. B 270 [FS16] (1986) 186; B 275 [FS17] (1986), 200.
[25] A. Cappelli, C. Itzykson and J.B. Zuber, Nucl. Phys. B 280 [FS18] (1987) 445.
[26] A.B. Zamolodchikov, Theor. Math. Phys. 63 (1985), 1205.
[27] D. Friedan, Z. Qiu and S. Shenker, Phys. Lett. B 151 (1985) 37.
[28] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP 62 (1984) 215; JETP 63 (1986) 913.
[29] A.B. Zamolodchikov, JETP Lett. 46 (1987) 160;
Int. J. Mod. Phys. A 3 (1988) 743.
[30] A.B. Zamolodchikov, Advanced Studies Pure Math. 19 (1989) 641.
[31] G.E. Andrew, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193.
[32] D. Huse, Phys. Rev. B 30 (1984) 3908.
[33] V. Pasquier, Nucl. Phys. B 285 [FS19] (1987) 162.
[34] A.B. Zamolodchikov, Sov. J. Nucl. Phys. 44 (1986) 529.
[35] Vl.S. Dotsenko, Nucl. Phys. B 235 [FS11] (1984) 54.
[36] A.B. Zamolodchikov, JETP Lett. 43 (1986), 730.
[37] A. Cappelli, D. Friedan and J.I. Latorre, Nucl. Phys. B 352 (1991), 616.
[38] J.L. Cardy, Phys. Rev. Lett. 60 (1988), 2709.
[39] J.L. Cardy, J. Phys. A 21 (1988) L797.
[40] S.C. Lee, W.C. Su and Y.C. Kao, Phys. Rev. D 41 (1990), 1241.
[41] D. Freedman, J.I. Latorre and X. Vilasis, Mod. Phys. Lett. A 6 (1991), 531.
[42] M.T. Grisaru, A. Lerda, S. Penati and D. Zanon, Nucl. Phys. B 342 (1990), 564.
[43] Al.B. Zamolodchikov, Resonance Factorized Scattering and Roaming Trajectories, ENS-LPS-335 (1991).
[44] R. Shankar, Phys. Lett. B 92 (1980) 333.
[45] R. Dashen, S.K. Ma and H.J. Bernstein, Phys. Rev. 187 (1969) 345.
[46] C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312.
[47] C.N. Yang and C.P. Yang, J. Math. Phys. 10 (1969) 1115.
[48] Al.B. Zamolodchikov, Nucl. Phys. B 342 (1990), 695.
[49] Al.B. Zamolodchikov, Nucl. Phys. B 253 (1991), 391; Nucl. Phys. B 358 (1991) 524.
[50] M.J. Martins, Phys. Rev. Lett. 65 (1990) 2091; Phys. Lett. B 240 (1990) 404; H. Itoyama and P. Moxhay, Phys. Rev. Lett. 65 (1990) 2102.
[51] T.R. Klassen and E. Melzer, Nucl. Phys. B 338 (1990) 485.
[52] P. Christe and M.J. Martins, Mod. Phys. Lett. A 5 (1990) 2189.
[53] Al.B. Zamolodchikov, Nucl. Phys. B 358 (1991) 497; B 366 (1991) 122.
[54] V.A. Fateev and Al.B. Zamolodchikov, Integrable perturbations of Z_N parafermion models and O(3) sigma Model, ENS-LPS-341/1991;
C. Ahn and S. Nam, Thermodynamic Bethe ansatz equations for perturbed minimal conformal field theories, SNUTP-91-30.
[55] P. Forgacs, F. Niedermayer and P. Weisz, Nucl. Phys. B 367 (1991) 123, 144;
P. Hasenfratz, M. Maggiore and F. Niedermayer, Phys. Lett. B 245 (1990) 522;
P. Hasenfratz and F. Niedermayer, Phys. Lett. B 245 (1990) 529.
[56] M.J. Martins, Phys. Rev. Lett. 67 (1991) 419.
[57] P. Fendley, Excited-state thermodynamics, BUHEP-91-16.
[58] P. Fendley and K. Intriligator, Scattering and thermodynamics of fractional-charged supersymmetry solitons, BUHEP-91-17, HUTP-91/A043.
[59] L. Lewis, Dilogarithms and associate functions (MacDonald, London, 1958); Polylogarithms and associated functions (North-Holland, Amsterdam, 1981).
[60] M. Blume, Phys. Rev. 141 (1966) 517;
H.W. Capel, Physica 32 (1966) 866;
R.B. Griffiths, Physica 33 (1967) 689;
M. Blume, V.J. Emery and R.B. Griffiths, Phys. Rev. A 4 (1971) 1071;
M.T. Tejwani, O. Ferreira and O.E. Vilches, Phys. Rev. Lett. 44 (1980) 152.
[61] I.D. Lawrie and S. Sarbach, in: Phase Transitions and Critical Phenomena, Vol. 9, eds C. Domb and J. Lebowitz (Academic Press, New York, 1984).
[62] G. von Gehlen, Nucl. Phys. B 330 (1990), 741.
[63] M. Henkel, Phys. Lett. B 247 (1990) 567.
[64] D. Friedan, Z. Qiu and S.H. Shenker, Phys. Lett. B 151 (1985), 37.

[65] Z. Qiu, Nucl. Phys. B 270 [FS16] (1986) 205.
[66] G. Mussardo, G. Sotkov and M. Stanishov, Nucl. Phys. B 305 [FS23] (1988) 69; Phys. Lett. B 195 (1987) 397; G. Mussardo, Magister thesis ISAS-Trieste (1987) unpublished.
[67] D.A. Kastor, E.J. Martinec and S.H. Shenker, Nucl. Phys. B 316 (1989), 590.
[68] K.M. Watson, Phys. Rev. 95 (1954) 228.
[69] B. Berg, M. Karowski and P. Weisz, Phys. Rev. D 19 (1979) 2477; M. Karowski and P. Weisz, Nucl. Phys. B 139 (1978), 445; M. Karowski, Phys. Rep. 49 (1979) 229.
[70] J.L. Cardy and G. Mussardo, Nucl. Phys. B 340 (1990), 387.
[71] V.P. Yurov and Al.B. Zamolodchikov, Int. J. Mod. Phys. A 6 (1991), 3419.
[72] G.E. Andrew, The theory of partitions (Addison-Wesley, Reading, 1976).
[73] L. Onsager, Phys. Rev. 65 (1944) 117.
[74] C.N. Yang, Phys. Rev. 85 (1952) 808.
[75] M. Kac and J. Ward, Phys. Rev. 88 (1952) 1332.
[76] T. Schultz, D.C. Mattis and E.H. Lieb, Rev. Mod. Phys. 36 (1964) 856.
[77] L.P. Kadanoff and H. Ceva, Phys. Rev. B 3 (1971), 3918.
[78] B.M. McCoy and T.T. Wu, The Two Dimensional Ising Model (Harvard Univ. Press, Cambridge, 1982).
[79] T.T. Wu, B.M. McCoy, C.A Tracy and M. Barouch, Phys. Rev. B 13 (1976) 316; B.M. McCoy, C. Tracy and T.T. Wu, Phys. Rev. Lett. 38 (1977) 793; B.M. McCoy and T.T. Wu, Phys. Rev. Lett. 45 (1980) 675; Phys. Rev. D 18 (1978) 1259; J.H.H. Perk, Phys. Lett. A 79 (1990) 1; B.M McCoy, J.H.H. Perk and T.T. Wu, Phys. Rev. Lett. 46 (1981) 757.
[80] A. Luther and I. Peschel, Phys. Rev. B 12 (1975) 3908; M. Bander and C. Itzykson, Phys. Rev. D 15 (1977) 463; C. Itzykson and J.B. Zuber, Phys. Rev. D 15 (1977) 2875.
[81] M. Sato, T. Miwa and M. Jimbo, Proc. Japan Acad. A 53 (1977) 147, 153, 183.
[82] P. Di Francesco, H. Saleur and J.B. Zuber, Nucl. Phys. B 290 [FS20] (1987) 527.
[83] F.A. Smirnov, J. Phys. 18 (1984) L 873; 19 (1986) L 575.
[84] A.N. Kirillov and F.A. Smirnov, Phys. Lett. B 198 (1987) 506; Int. J. Mod. Phys. A 3 (1988), 731.
[85] Al.B. Zamolodchikov, Nucl. Phys. B 348 (1991) 619.
[86] P. Christe, Int. J. Mod. Phys. A 6 (1991), 5271.
[87] B. Nienhuis, J. Stat. Phys. 34 (1984) 731.
[88] P.G. de Gennes, Phys. Lett. A 38 (1972) 339.
[89] A.B. Zamolodchikov, Mod. Phys. Lett. A 6 (1991) 1807.
[90] H. Saleur, Polymers and Percolation in Two Dimensions and Twisted $N = 2$ Supersymmetry, YCTP-P38-91.
[91] N.Yu. Reshetikhin and F.A. Smirnov, Commun. Math. Phys. 131 (1990) 157.
[92] F.A. Smirnov, Int. J. Mod. Phys. A 4 (1989) 4231; Nucl. Phys. B 337 (1990) 156.
[93] F.A. Smirnov, Int. J. Mod. Phys. A 6 (1991) 1407.
[94] A. LeClair, Phys. Lett. B 230 (1989), 103.
[95] D. Bernard and A. LeClair, Nucl. Phys. B 340 (1990) 721.
[96] T. Eguchi and S.K. Yang, Phys. Lett. B 235 (1990) 282.
[97] T. Nakatsu, Nucl. Phys. B 356 (1991) 499.
[98] G. Felder, Nucl. Phys. B 317 (1989) 215.
[99] A.G. Izergin and V.E. Korepin, Comm. Math. Phys. 79 (1981) 303.
[100] A. Swieca, Fortschr. Phys. 25 (1977) 303.
[101] M. Karowski and H.J. Thun, Nucl. Phys. B 190 [FS3] (1981) 61.
[102] F.A. Smirnov, Comm. Math. Phys. 132 (1990) 415.
[103] V. Kac, Infinite Dimensional Lie Algebras (Cambridge Univ. Press, Cambridge, 1985).
[104] J.B. McGuire, J. Math. Phys. 5 (1964) 622.
[105] C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312.
[106] R.J. Baxter, Ann. Phys. 70 (1972) 193.
[107] L.A. Takhajan and L.D. Faddev, Russian Math. Surveys 34 (1980) 688.
[108] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.
[109] P.P. Kulish, Theor. Math. Phys. 26 (1976) 132.
[110] D. Iagolnitzer, Phys. Rev. D 18 (1978) 1275.
[111] R. Shankar and E. Witten, Phys. Rev. D 17 (1978) 2134.
[112] P. Di Vecchia and S. Ferrara, Nucl. Phys. B 130 (1977) 93.
[113] E. Witten, Phys. Rev. D 16 (1977) 2991.
[114] O. Alvarez, Phys. Rev. D 17 (1978) 1123.

[115] K. Schoutens, Nucl. Phys. B 344 (1990) 665.
[116] P. Fendley, W. Lerche, S.D. Mathur and N.P. Warner, Phys. Lett. B 243 (1990) 257; Nucl. Phys. B 348 (1991), 66.
[117] P. Fendley, Phys. Lett. B 250 (1990) 96.
[118] W. Lerche and N.P. Warner, Nucl. Phys. B 358 (1991) 571.
[119] C. Ahn, Nucl. Phys. B 354 (1991) 57.
[120] C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312;
R.J. Baxter, Ann. Phys. (NY) 70 (1972) 323.
[121] B. Schroer, T.T. Truong and P. Weisz, Phys. Lett. B 63 (1976), 422.
[122] P.P. Kulish, E.R. Nisimov, Theor. Math. Phys. 29 (1976); JETP Lett. 24 (1977) 220.
[123] S. Ferrara, L. Girardello, S. Sciuto, Phys. Lett. B 76 (1978) 303.
[124] R. Sasaki, I. Yamanaka, Comm. Math. Phys. 108 (1987) 691.
[125] K. Kobayashi and T. Uematsu, Phys. Lett. B 264 (1991) 107
[126] G.F. Chew, S-matrix Theory of Strong Interactions (Benjamin, New York).
[127] G.F. Chew and M. Jacob, Strong Interaction Physics (Benjamin, New York).
[128] M.R. Garey and D.S. Johnson, Computers and Intractability. A Guide to the Theory of NP-Completeness, (Freeman and Co., San Francisco, 1979).
[129] C.H. Papadimitriou and K. Steiglitz, Combinatorial Optimization. Algorithms and Complexity (Prentice-Hall, Englewood Cliffs, 1982).
[130] A. Salomaa, Computation and Automata (Cambridge Univ. Press, Cambridge, 1985).
[131] G.S. Boolos and R.C. Jeffrey, Computability and Logic (Cambridge Univ. Press, Cambridge, 1970).
[132] A. Turing, Proc. London Math. Soc. Ser. 42 (1936) 230, 43 (1936) 544.
[133] V.P. Yurov and Al.B. Zamolodchikov, Int. J. Mod. Phys. A 5 (1990) 3221.
[134] M. Lässig, G. Mussardo and J.L. Cardy, Nucl. Phys. B 348 (1991) 591.
[135] M. Lässig and G. Mussardo, Comp. Phys. Comm. 66 (1991) 71.
[136] M. Lässig and M.J. Martins, Nucl. Phys. B 354 (1991) 666.
[137] V.P. Yurov and Al.B. Zamolodchikov, Int. J. Mod. Phys. A 6 (1991) 4557.
[138] I.R. Sagdeev and A.B. Zamolodchikov, Mod. Phys. Lett. B 3 (1989) 1375.
[139] M.J. Martins, The off-critical behaviour of the multicritical Ising models, UCSBTH 91–45.
[140] L.D. Landau and E.M. Lifshitz, Quantum Mechanics (Pergamon Press, Oxford, 1977).
[141] M. Lüscher, Commun. Math. Phys. 105 (1986) 153.
[142] M. Lüscher and U. Wolff, Nucl. Phys. B 339 (1991) 222.
[143] R. Köberle and J.A. Swieca, Phys. Lett. B 86 (1979) 209.
[144] A.E. Arinshtein, V.A. Fateyev and A.B. Zamolodchikov, Phys. Lett. B 87 (1979), 389.
[145] R.J. Baxter, J. Phys. C 6 (1973), L445.
[146] A.B. Zamolodchikov, Int. J. Mod. Phys. A 3 (1988), 743.
[147] L. Chim and A. Zamolodchikov, Integrable Field Theory of q-State Potts Model with $0<q<4$, preprint RU-91-24.
[148] J.L. Cardy and G. Mussardo, Phys. Lett. B 225 (1989), 275.
[149] G. Mussardo, in: Differential Geometric Methods in Theoretical Physics. Physics and Geometry, NATO ASI Ser. B. Physics, Vol. 245, eds L.L. Chau and W. Nahm (Plenum Press, New York, 1990).
[150] P.G. Freund, T.R. Klassen and E. Melzer, Phys. Lett. B 229 (1989) 243.
[151] C.N. Yang and T.D. Lee, Phys. Rev. 87 (1952) 404.
[152] T.D. Lee and C.N. Yang, Phys. Rev. 87 (1952) 410.
[153] M.E. Fisher, Phys. Rev. Lett. 40 (1978) 1610.
[154] J.L. Cardy, Phys. Rev. Lett. 54 (1985) 1354.
[155] M. Karowski and P. Weisz, Nucl. Phys. B 139 (1978) 445; M. Karowski, Phys. Rep. 49 (1979) 229;
B. Berg, M. Karowski and P. Weisz, Phys. Rev. D 19 (1979) 2477.
[156] G. Sotkov and C.J. Zhu, Phys. Lett B 229 (1989) 391.
[157] P. Christe and G. Mussardo, Nucl. Phys. B 330 (1990) 465.
[158] P. Christe and G. Mussardo, Int. J. Mod. Phys. A 5 (1990) 4581.
[159] V.A. Fateev and A.B. Zamolodchikov, Int. J. Mod. Phys. A 5 (1990) 1025.
[160] A. Koubek, G. Mussardo and R. Tateo, Bootstrap Trees and Consistent S Matrices, Int. J. Mod. Phys. A 7 (1992) 3435.
[161] A Koubek and G. Mussardo, Phys. Lett. B 266 (1991) 363.
[162] M.J. Martins, Phys. Lett. B 262 (1991) 39.
[163] A. Koubek, M.J. Martins and G. Mussardo, Nucl. Phys. B 368 (1992) 591.
[164] G. Mussardo, Integrable deformations of the non-unitary minimal conformal model $\mathcal{M}_{3,5}$, NORDITA 91/54, Int. J. Mod. Phys. A, to appear.
[165] A. Koubek and G. Mussardo, Interpretation of IRF Models as Scattering Amplitudes of Perturbed Conformal Field Theories, ISAS 20/92/EP.
[166] F. Colomo, A. Koubek and G. Mussardo, Phys. Lett. B 274 (1992) 367; On the S-matrix of the sub-leading magnetic deformation of the tricritical Ising model in two dimensions, Int. J. Mod. Phys. A, to appear.

[167] A.B. Zamolodchikov, S-matrix of the subleading magnetic perturbation of the tricritical Ising, PUTP 1195–90.
[168] M. Lüscher, Comm. Math. Phys. 104 (1986) 177; in: Progress in Gauge Field Theory (Cargèse 1983), eds G. 't Hooft et al. (Plenum, New York, 1984); in: Champs, Cordes, et Phénomènes Critiques, Proc. 1988 Les Houches Summer School, E. Brezin and J. Zinn-Justin (North-Holland, Amsterdam, 1989).
[169] L.I. Schiff, Quantum Mechanics (McGraw-Hill, New York, 1968).
[170] F. Colomo and G. Mussardo, unpublished.
[171] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Nucl. Phys. B 338 (1990) 689.
[172] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Nucl. Phys. B 356 (1991) 469.
[173] H.W. Braden and R. Sazaki, Affine Toda Perturbation Theory, YITP/U-91-40.
[174] P. Dorey, Nucl. Phys. B 358 (1991) 654; Root Systems and Purely Elastic S-matrices. 2, Kyoto preprint RIMS-787 (1991).
[175] H.W. Braden and R. Sazaki, Phys. Lett. B 255 (1991) 343.
[176] O. Babelon and L. Bonora, Phys. Lett. B 244 (1991) 220; B 267 (1991) 71.
[177] M.D. Freeman, Phys. Lett. B 261 (1991) 57;
A. Fring, H.C. Liao and D.I. Olive, Phys. Lett. B 266 (1991) 82;
A. Fring and D.I. Olive, The fusing rule and the scattering matrix of Affine Toda Theory, Imperial/TP/91-92/08;
H.W. Braden, A Note on Affine Toda Coupling, Edinburgh preprint 91–01.
[178] G.W. Delius, M.T. Grisaru, S. Penati and D. Zanon, Phys. Lett. B 256 (1991) 164;
C. Destri, H.J. de Vega and V.A. Fateev, Phys. Lett. B 256 (1991) 173.
[179] A.B. Zamolodchikov, Moscow preprint (October 1989).
[180] E. Ogievetsky and P. Wiegmann, Phys. Lett. B 168 (1986) 360.
[181] E. Ogievetsky, N. Reshetikhin and P. Wiegmann, Nucl. Phys. B 280 [FS18] (1987) 45.
[182] N.Yu. Reshetikhin and P.B. Wiegmann, Phys. Lett. B 189 (1987) 125.
[183] E.I. Ogievetsky, J. Phys. G 12 (1986) L105.
[184] M. Henkel and H. Saleur, J. Phys. A 23 (1990) 791.
[185] C. Destri and H.J. de Vega, Phys. Lett. B 233 (1989) 336.
[186] O. Babelon, Phys. Lett. B 215 (1988) 329; Nucl. Phys. B 230 [FS10] (1984) 241.
[187] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces (Academic Press, New York, 1978).
[188] V.A. Fateev and A.B. Zamolodchikov, Nucl. Phys. B 280 [FS18] (1987) 644.
[189] T. Eguchi S.K. Yang, Phys. Lett. B 224 (1989) 373.
[190] T.J. Hollowood and P. Mansfield, Phys. Lett. 226 B (1989) 73.
[191] S.K. Yang, private communication.
[192] S. Coleman, Phys. Rev. D 11 (1975), 2088.
[193] S. Coleman and H.J. Thun, Comm. Math. Phys. 61 (1978) 31.
[194] C.J. Goebel, Prog. Theor. Phys. Suppl. 86 (1986) 261.
[195] G. Källen and J. Toll, J. Math. Phys. 6 (1965) 299.
[196] R.E. Cutkosky, J. Math. Phys. 1 (1960), 429.
[197] P.P. Kulish and E.R. Nisimov, Teor. Math. Fiz. 29 (1976) 161.
[198] R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D 11 (1975) 3424.
[199] V.E. Korepin and L.D. Faddev, Teor. Math. Fiz. 25 (1975) 147.
[200] R. Sasaki and I. Yamanaka, Comm. Math. Phys. 108 (1987) 691.
[201] E.K. Sklyanin, L.A. Takhtadzhyan and L.D. Faddev, Teor. Math. Fiz. 40 (1980) 688;
L.D. Faddev and L.A. Takhtajan, Russian Math. Survey 34 (1979).
[202] L.D. Faddev, Hamiltonian Method in the Theory of Solitons (Springer, Berlin 1987).
[203] V. Drinfeld, Proc. Int. Congr. Math. Berkeley (1986) pp. 1, 698.
[204] S. Majid, Int. J. Mod. Phys. A 5 (1990) 91.
[205] L.D. Faddev, N.Yu. Reshetikhin and L.A. Takhtajan, in: Braid Groups, Knot Theory and Statistical Mechanics, eds C.N. Yang and M.L. Ge (World Scientific, Singapore, 1989).
[206] A.N. Kirillov and N.Yu. Reshetikhin, Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links, LOMI preprint E-9-88.
[207] V.I. Vichirko and N.Yu. Reshetikhin, Teor. Mat. Fiz. 56 (1983) 260.
[208] O. Babelon, Phys. Lett. B 215 (1988), 523.
[209] J.L. Gervais, Comm. Math. Phys. 130 (1990) 257.
[210] F. Smirnov and L. Takhtajan, Towards a quantum Liouville theory with $c > 1$ Colorado preprint (1990).
[211] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, The Analytic S-matrix (Cambridge Univ. Press, Cambridge 1966).
[212] V.A. Fateev and S. Lykyanov, I. J. Mod. Phys. A 3 (1988) 507; Lectures given at 2nd Spring school Contemporary Problems in Theoretical Physics, Kiev 1988, Preprint ITP-88-74-76P, Kiev (in Russian).
[213] A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelomov, Comm. Math. Phys. 79 (1981) 473.
[214] G. Wilson, Ergod. Th. Dyn. Syst. 1 (1981) 361.

[215] D. Olive and N. Turok, Nucl. Phys. B 257 [FS14] (1985) 277.
[216] L. Castillejo, R.H. Dalitz and F.J. Dyson, Phys. Rev. 101 (1956) 453.
[217] P. Mitra, Phys. Lett. B 72 (1977) 62.
[218] V.G. Drinfeld and V.V. Sokolov, Soviet Math. Dokl. 23 (1981) 457.
[219] A.N. Leznov and M.V. Saveliev, Lett. Math. Phys. 3 (1979) 489.
[220] S.A. Bulgadaev, Phys. Lett. B 96 (1980) 151; B 87 (1979) 47.
[221] A.V. Mikhailov, JETP Lett. 30 (1979) 414.
[222] P. Mansfield, Nucl. Phys. B 208 (1982), 277; Nucl. Phys. B 222 (1983) 419.
[223] D. Olive and N. Turok, Nucl. Phys. B 215 [FS7] (1983) 470.
[224] J.L. Gervais and A. Neveu, Nucl. Phys. B 224 (1983) 329; A. Bilal and J.L. Gervais, Phys. Lett. B 206 (1988) 412; A. Bilal and J.L. Gervais Nucl. Phys. 314 (1989) 646.
[225] F.M. Goodman, P. de la Harpe and V.F.R. Jones, Coxeter Graphs and Towers of Algebras (Springer, Berlin, 1989).
[226] R. Slansky, Phys. Rep. 79 (1981) 1.
[227] W.G. McKay and J. Patera, Tables of dimensions, indices, and branching rules for representations of simple Lie algebras, Lecture Note in Pure and Applied Mathematics (1981).
[228] M. Koca and G. Mussardo, Int. J. Mod. Phys. A 6 (1991), 1543.
[229] R.J. Rivlin, The Chebyshev Polynomials (Wiley, New York, 1974).
[230] P. Goddard, A. Kent and D. Olive, Comm. Math. Phys. 103 (1986) 105.
[231] D. Bernard J. Thierry-Mieg, Comm. Math. Phys. 111 (1987) 181.
[232] P. Goddard, W. Nahm, D. Olive and A. Schwimmer, Comm. Math. Phys. 107 (1987) 179.
[233] A.B. Zamolodchikov and Al.B. Zamolodchikov, Massless factorized scattering and sigma-models with topological terms, ENS-LPS-355 (1991).