Derivation of the Lagrangian from Newton's laws

Oxide

> When any real progress is made, we unlearn and learn anew what we thought we knew before.
>
> Henry David Thoreau

Prerequisite Mathematics

I will use simple concepts from multivariable calculus freely throughout this article, so you should know what they mean.

- Partial Derivatives:

  If I have a function f of two variables x, y, then I can treat y as a constant and differentiate it with respect to x, or the other way around. For example:

  $$f(x, y) = x^2 \sin y$$

  The partial derivatives with respect to x, y, denoted $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ respectively are the following:

  $$\frac{\partial f}{\partial x} = 2x \sin y$$
  $$\frac{\partial f}{\partial y} = x^2 \cos y$$

- Gradient:

  I can make a list of all the partial derivatives of f, and call that list its "gradient", denoted by ∇f.

  $$\nabla f = \big(2x \sin y, x^2 \cos y\big).$$

  In general, for an arbitrary function $f(x_1, x_2, \ldots, x_n)$:

  $$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Note: Saying $F = \nabla V$ is like a 3-dimensional generalization of $f = v'$, where v is the antiderivative of f. This will be important later.

Introduction

We shall start by first listing out Newton's laws for our convenience:

- An object will remain at rest or in a state of uniform motion unless acted on by an external force.
- An external force acting on an object will induce an acceleration inversely proportional to its mass (ie. $F = ma$)
- For every action there is an equal and opposite reaction.

Interpretation

There is an insight hidden in Newton's laws of motion. Let's consider one overarching concept: **change**.

- Newton's first law describes a quantity that doesn't change unless acted on by an external force. Let's call this quantity **momentum**.
- Newton's second law then describes the change in momentum with respect to time as an indication of some external **force**. (ie $F = \frac{dp}{dt}$)
- Newton's third law tells us that changes must be balanced. That means a change in one momentum must be balanced by a change in another. This is just the **conservation of momentum**.

We have only considered change with respect to time and momentum as some general property that changes under application of a force. Let's call this the *time perspective*.

Let's consider change from a *space perspective*:

- Newton's first law would identify an invariant spatial property. Let's just call this a **potential**. A constant potential is unchanging throughout space.
- Newton's second law would state that any change in the potential is associated with a **force**. (ie $F = -\nabla\Phi$)
- Newton's third law states that the change in potential will be balanced by a similar change in potential. This is just **conservation of energy**. It tells us that there must be some property associated with a particle that balances the change in potential. We call this kinetic energy.

So far we have managed to express Newton's laws of motion in terms of the generalized properties of momentum and potential and the concept of change.

Let's look at the two versions of Newton's second law, in one dimension for simplicity:

Time perspective:

$$F = \frac{dp}{dt}.$$

Space perspective:

$$F = -\frac{d\Phi}{dx}.$$

Kinetic energy and momentum

A natural question is: How is the momentum related to the quantity that we call kinetic energy? The answer is as follows:

Consider a change in the potential moving along some path:

$$\begin{aligned}\frac{d\Phi}{ds} &= -\nabla\Phi \\ d\Phi &= -\nabla\Phi \cdot ds\end{aligned}$$

As we saw in Newton's third law (space perspective), the change in the potential must be balanced out by a change in the particle's kinetic energy, say T.

$$\begin{aligned}dT &= F \cdot ds \\ \frac{dT}{ds} &= F\end{aligned}$$

Now this step was quite tricky, and somewhat interesting. If you remember the work-energy chapter taught in 11th, you will remember that to derive the work-energy theorem we first proved that $dK = F \cdot ds$. We did the same thing here, but with a simpler derivation.

The trick here was to turn the three dimensional problem into a one dimensional one, described by a particular path in space (the path that the particle takes). This means we are only considering motion along the direction of the force. Remember this as it will become important later.

With respect to a path through space, position is one property, but it alone is insufficient to determine the next point. We need some pointer to the next point in space as we move in time. That pointer is simply velocity, v.

From the chain rule, we can manipulate $\frac{dT}{ds}$ to obtain:

$$\frac{dT}{ds} = \frac{dv}{ds}\frac{dT}{dv}$$

But, $v = \frac{ds}{dt}$, so doing a physicist trick yields us:

$$\frac{dT}{ds}\frac{dv}{ds}\frac{dT}{dv} = \frac{d(\frac{ds}{dt})}{ds}\frac{dT}{dv} = \frac{d}{dt}\frac{dT}{dv}$$

Since we had

$$\frac{dT}{ds} = F = \frac{dp}{dt}$$

We can immediately obtain the important fact that:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{dT}{dv}$$

Functionals and Lagrangians

It is now pertinent to define a functional. A functional is a type of function that takes functions as inputs and real numbers as outputs. If $C^{\infty}[0,1]$ represents the set of infinitely differentiable functions on the interval $[0,1]$, then a functional is a function $L : C^{\infty}[0,1] \longrightarrow \mathbb{R}$.

The physical meaning of this, is that in physics, we have "possible paths of motion" and want to assign a real number to each of them. The functional does exactly this.

Next, let's consider the path that a particle follows. This can be generally written as some functional of two variables: the position and the velocity (remember that we only need the initial position and initial velocity to completely determine the object's path as remarked before), i.e $L(x, v)$.

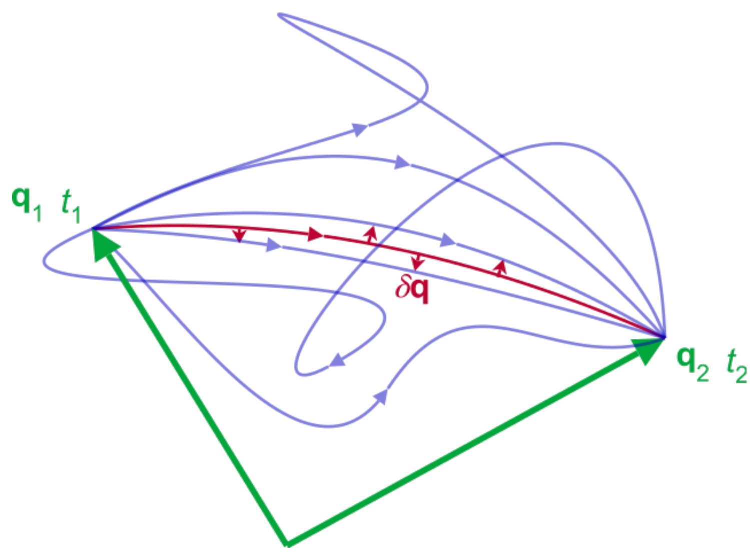

Figure 1: Possible paths for a particle

If a particle is to follow the most "natural" path, i.e the red path in the figure above, it should have no external forces on it, in other words $2F = F + F = 0$.

This means that $F + F = \frac{dp}{dt} - \nabla\Phi = 0$.

$$\begin{aligned} \frac{dp}{dt} - \nabla\Phi &= 0 \\ \frac{d}{dt}\frac{dT}{dv} - \frac{d\Phi}{ds} &= 0 \end{aligned}$$

A remark is in order, since the second equation is arguably the most relevant equation in physics. It is known as the Euler-Lagrange equation. I recommend watching this video after reading this article.

$$\begin{aligned} \frac{dT}{ds} - \frac{d\Phi}{ds} &= 0 \\ \frac{d}{ds}(T - \Phi) &= 0 \end{aligned}$$

Conclusion

This quantity $T - \Phi$, frequently written $L = T - V$ is therefore minimized over the motion and is called the *Lagrangian* of the system, and it is one of the most important quantities in classical mechanics. The motivation for choosing $L = T - V$, as opposed to something that is conserved (i.e $T + V$) is almost never given in classical mechanics courses, and the reasoning given for choosing it is that “it gives the correct equations of motion”.

I hope you enjoyed reading this derivation as much as I enjoyed writing it. Thank you for your time.