

# Abstract Kleisli structures on 2-categories

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## Plan

1. Review & reformulate the definition of AKS on categories
2. Review & extend theory of AKS on categories
3. Define AKS on 2-categories
4. State analogous results for AKS on 2-categories.

Def (Führmann 1999)

An AKS consists of

$$A \xrightarrow{\quad Q \quad} A \quad \text{"force"}$$
$$A \xrightarrow{\quad \sqrt{\varepsilon} \quad} A \quad \text{1}_A$$

$$X \xrightarrow{\phi_X} QX \quad \text{"thunk"}$$

such that  $(A, Q, \varepsilon, \phi_Q)$

is a comonad  $\mathcal{S}$

$\phi_X : X \rightarrow QX$  is a  
coalgebra.

# Part 1 Reformulating the definition of AKS on categories

A commutative diagram illustrating the relationships between objects  $A_0$ ,  $A$ ,  $A^Q$ , and  $A_0^Q$ .  
 The top row shows a sequence of morphisms:  $A_0 \rightarrow A \rightarrow A^Q$ .  
 The bottom row shows a sequence of morphisms:  $A_0 \rightarrow A$ .  
 Vertical arrows  $Q_0$  and  $u^Q$  map  $A_0$  to  $A^Q$  and  $A$  respectively.  
 A dashed arrow labeled  $\phi_-$  maps  $A$  to  $A^Q$ .  
 Two blue circles, labeled 1 and 2, are placed near the dashed arrow.

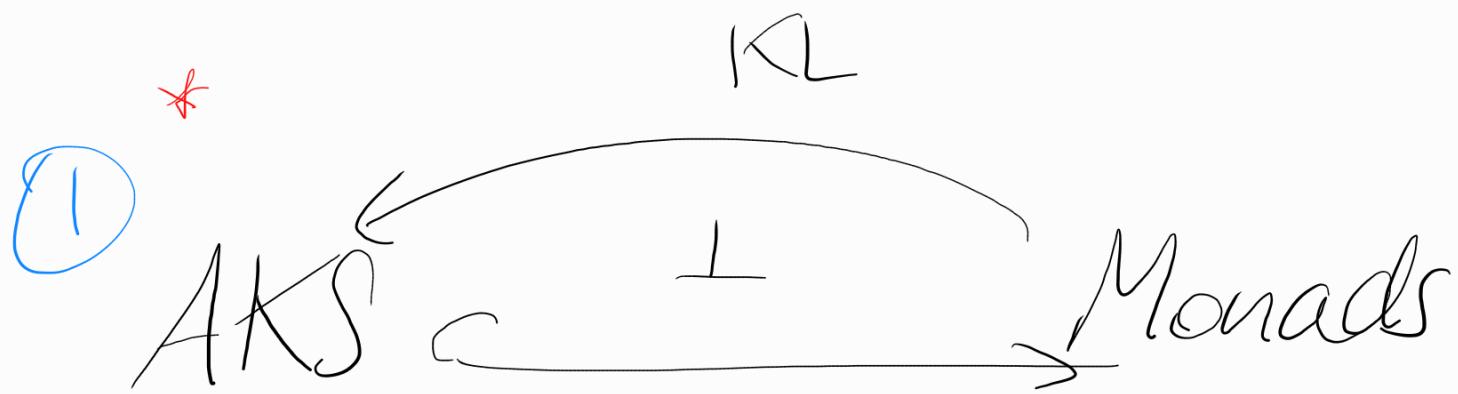
①  $\phi_{QX}$  multiplication for  
a comonad.

2  $\phi_x$  is a coalgebra.

- Note that  $\phi_X : X \rightarrow QX$   
not natural in  $X$
- Morphisms  $f: X \rightarrow Y$  satisfying  

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi_X \downarrow & & \downarrow \phi_Y \\ QX & \xrightarrow{\quad} & QY \\ & Qf & \end{array}$$
 are called "thunkable."
- $A \xrightleftharpoons[T]{Q} A$   
 $\phi$  bijective on objects
- all examples are Kleisli categories of monads.

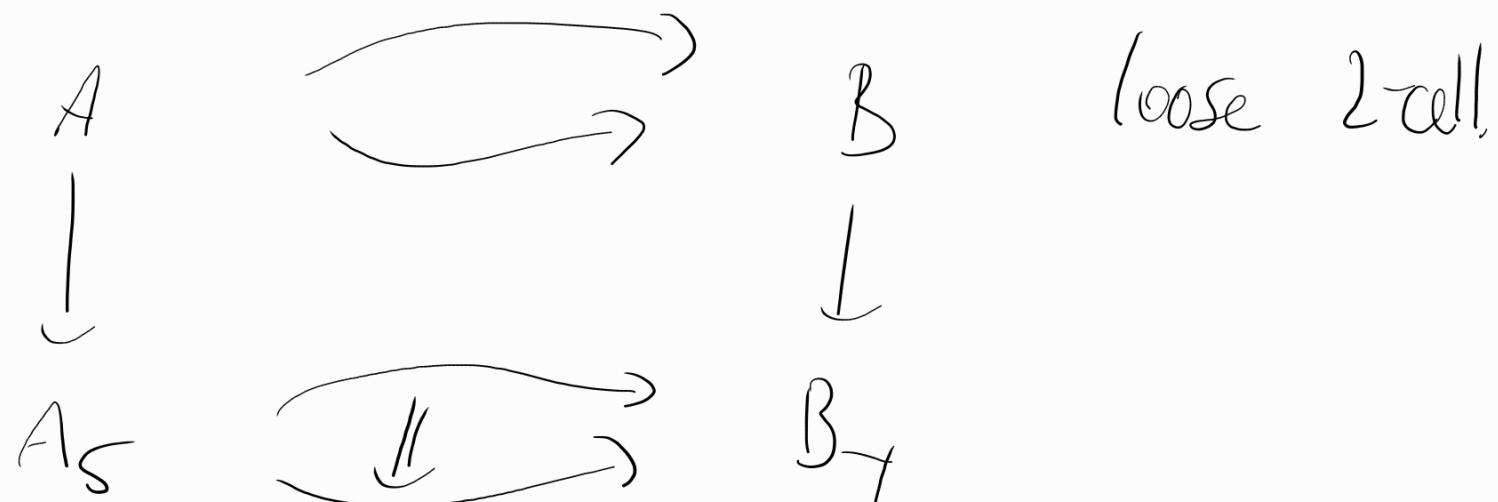
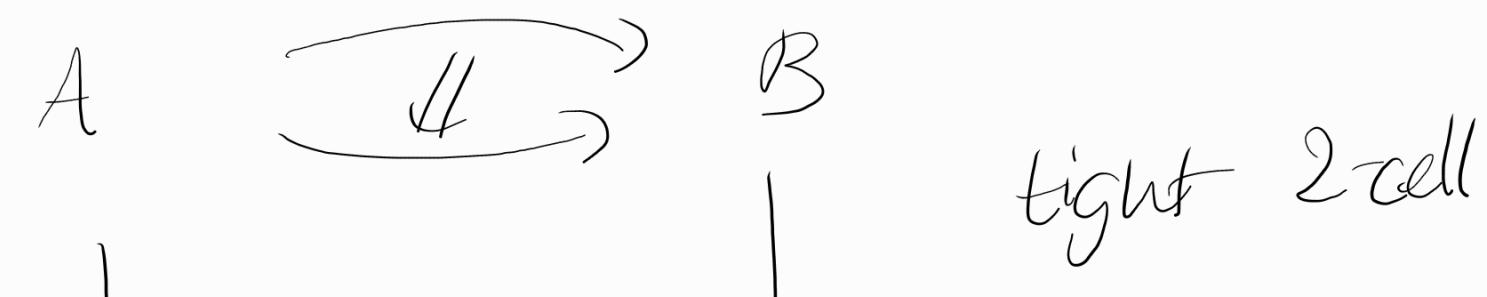
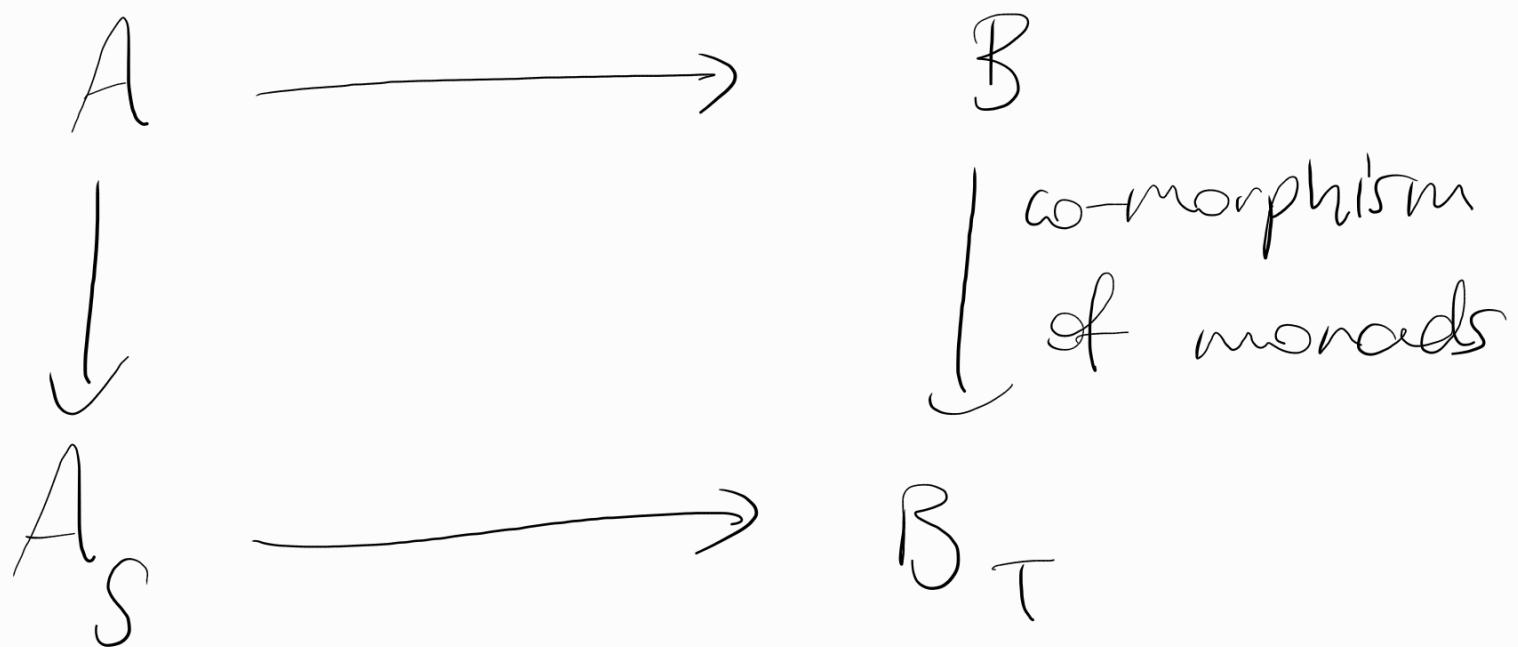
# Part 2 Extending results on AKS on categories.



$$(\alpha, \beta, \gamma, \delta, \phi) \mapsto \alpha_\phi$$

② The essential image is characterized by any of the following equivalent conditions

- $X \xrightarrow{\eta_X} TX \xrightarrow{T\eta_X} T^2X$  equalizer  
 $\Downarrow_{TX}$
- $B \rightarrow B_T$  faithful &  
full on thunkable morphisms.
- $B \rightarrow (B_T)^T$  fully faithful
- $B \rightarrow ((B_T)^T)^T$  fully faithful
- \* with strict morphisms of monads Führmann (1999)
- \* with co-morphisms of monads either possible 2-cell (M. 2024)  
Prop 2.7.



## Part 3

# Defining AKS on 2 categories.

• A pseudocomonad

$$(A, Q, \varepsilon, \delta, \lambda, \vartheta, \beta)$$

• A lifting

$$\begin{array}{ccccc} A_0 & \xrightarrow{\text{inc}} & A & \xrightarrow{F^Q} & A^Q \\ \downarrow \rho_0 & & \nearrow \phi & & \downarrow u^Q \\ A_0 & \xrightarrow{\quad} & A & \xrightarrow{\quad} & \end{array}$$

inc

In more detail?

$$\begin{array}{ccc} X & \xrightarrow{\phi} & QX \\ \downarrow \text{id}_X & \swarrow \cong u_X & \downarrow e_X \quad \phi_X \\ & X & \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\phi_X} & QX \\ & \downarrow \cong m_X & \downarrow s_X \\ QX & \xrightarrow{\phi_{QX}} & Q^2X \end{array}$$

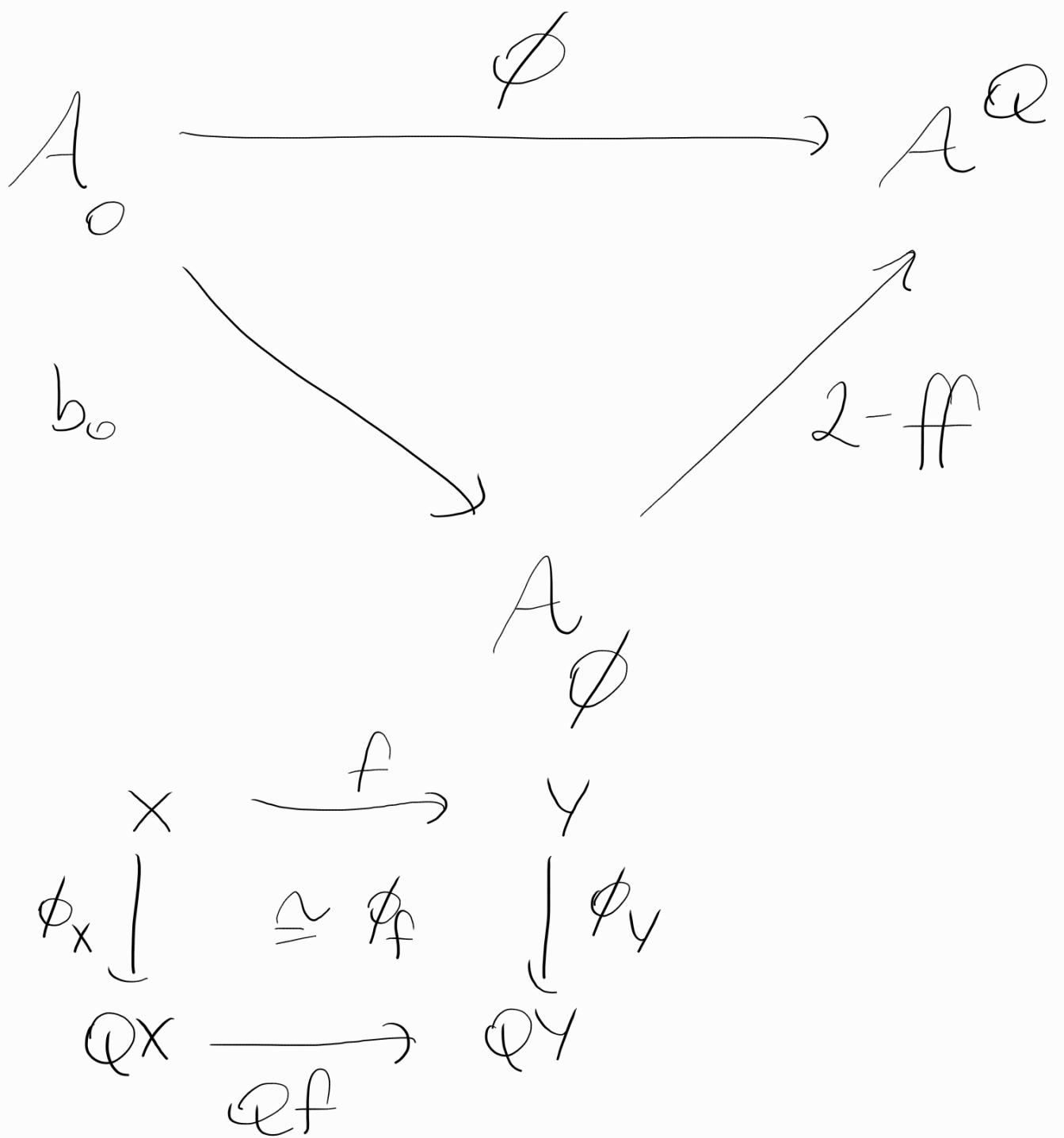
such that •  $\phi_{QX} = \mu_X$

$$• u_{QX} = \gamma_X$$

$$• m_{QX} = \delta_X$$

• the pseudo coalgebra axioms hold.

Can form the 2-category  
 $A_\phi$  of "morphisms equipped with  
thinking & thinkable 2-cells"



# Sidenotes

- Neither Kleisli  
bicats nor  
free ps-algebras  
are Kleisli objects  
in the enriched sense.  
(Gambino Lobbik  
2022)
- Comparison  
from enriched  
Kleisli is a  
biequiv iff  
left adj is  
biess surj on  
objects.  
(M. 2023)

Kleisli : free pseudalgebras

$$A_\phi \begin{array}{c} \swarrow \\ \vdash \\ \searrow \end{array} A$$

pseudoadjunction

Inducing  $(\alpha, \varepsilon, \delta, ?, \gamma, \beta)$   
on  $A_\phi^\vee$

2-AKL  $\longrightarrow$  Pseudomonads

$$\begin{array}{ccc} A_\phi & \longrightarrow & B_\psi \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Pseudomonad  
on  $A_\phi$ .

Part 4 Analogous results for AKS  
on 2-categories

(M. 2024 Thm 6.1)

Have co-morphism of pseudomonads.

$$\begin{array}{ccc} A & \longrightarrow & (A_s)_{S\eta} \\ \downarrow & & \downarrow \\ A_s & \xrightarrow{\quad} & A_S \end{array}$$

which is a biequivalence  
iff the following equivalent  
conditions hold

- The data

$$\begin{array}{ccc}
 & \eta \rightarrow S & S_2 \\
 & \downarrow & \searrow \\
 \frac{1}{A} & \cong 2\eta & S^2 \\
 & \eta \rightarrow S & 2S
 \end{array}$$

is an isobidescent cone

for

$$\begin{array}{ccccc}
 S & \xrightarrow{S_2} & S^2 & \xleftarrow{S_3} & S^3 \\
 \xleftarrow{\mu} & & \xrightarrow{S_{23}} & & \xleftarrow{\mu_3} \\
 \eta_S & & & & \eta_{S^2}
 \end{array}$$

- $A \rightarrow A_S$  faithful on 2-cells, full on thunkable 2-cells & on morphisms which admit a thunking.

$$\begin{array}{c}
 A \rightarrow (A_S)^S \\
 A \rightarrow (A^S)^S
 \end{array}$$

bi-faithfully faithful

bi-faithfully faithful.

Th<sup>m</sup> 6.8 (M. 2024)

$$A \rightarrow (As)_{Sn}$$
$$\downarrow \qquad \qquad \qquad \downarrow$$
$$As \xrightarrow{1} As$$

is the  
unit of

a reflection to

$$2\text{-AKS} \xrightarrow{\leftarrow \dashv \perp} \text{Pseudomonads}$$

NB. This is actually  
a Gray-adjunction.

# Future Work 1

- Führmann also studied monoidal versions
- I have recently shown how monoidal structures extend to Kleisli objects for pseudomonads
- To do's Study monoidal versions of AKS on 2-categories -

## Future Work 2

- Ikonicoff & Lemay (Lemay 2023) used AKS to characterise when a (co)kinds' category supports structure of differentiation.
- I am currently developing the theory of differential bicategories
- Could seek a categorification of these results.