Non-Inertia Frames

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1 Rotating Frames

From an inertial frame, the speed is

$$\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r},$$

and the different measure of velocities are related by

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{r}.$$

The acceleration:

$$\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_S = \left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}). \tag{1}$$

2 Newton's Equation of Motion in a Rotating Frame

$$m\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{S'} = \boldsymbol{F} - 2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}).,\tag{2}$$

3 Euler Force

$$\boldsymbol{F}_{\text{euler}} = -m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}. \tag{3}$$

4 Centrifugal Force

$$\begin{aligned}\boldsymbol{F}_{\text{cent}} &= -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \\ &= -m(\boldsymbol{\omega} \cdot \boldsymbol{r})\boldsymbol{\omega} + m(\boldsymbol{\omega} \cdot \boldsymbol{\omega})\boldsymbol{r}\end{aligned} \tag{4}$$

The first term of (4) is related to the projection of $\boldsymbol{r}$ on to $\boldsymbol{\omega}$, denoted as $\boldsymbol{r}_{\shortparallel}$ by

$$-m(\boldsymbol{\omega} \cdot \boldsymbol{r})\boldsymbol{\omega} = -m\omega^2\boldsymbol{r}_{\shortparallel}$$

Since the second term is effectively $m\omega^2\boldsymbol{r}$, we can then deduce

$$\boldsymbol{F}_{\text{cent}} = m\omega^2\boldsymbol{r}_{\perp},$$

where $\boldsymbol{r}_{\perp}$ is the projection of $\boldsymbol{r}$ on to the plane perpendicular to $\boldsymbol{\omega}$.

5 Coriolis Force

$$\boldsymbol{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \boldsymbol{v} \tag{5}$$

where $\boldsymbol{v} = (\,\mathrm{d}\boldsymbol{r}/\,\mathrm{d}t)_{S'}$ is the velocity of the particle measured in the rotating frame S'.