Project 3, Fys4460

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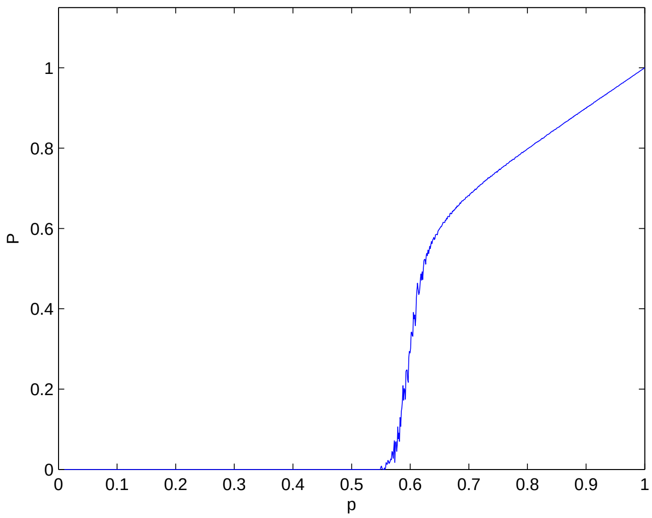

Figure 1: Probability for a site to belong to the spanning cluster plotted against probability for a site to be occupied.

(a) In this project an $L_x \times L_y$ matrix was generated. Each site in the matrix was either occupied or unoccupied. The probability for a site to be occupied was given by the probability p. The occupied sites were connected to the nearest neighbors, creating clusters. If a cluster could connect from one side of the matrix to the other, it was called a percolating cluster or a spanning cluster. The probability $P(p, L)$ for a site to be a part of the percolating cluster was found using simulations for different p. $P(p, L)$ was found by dividing the area of the spanning cluster by the total area. For each p, the results were averaged over 50 experiments, in order to get good results. The system was a 100×100 matrix. The result can be seen in figure 1. The probability for a site to be a part of the spanning cluster is zero before the spanning cluster exists. The critical value of p that gives the spanning cluster, is written as p_c. When $p = 1$ all the sites are occupied, and the probability for a site to be part of the spanning cluster was also 1.

A simulation was also run for $L_x \gg L_y$. The spanning cluster was defined when either of the edges of the matrix were connected by the spanning cluster. It was possible that more than one spanning cluster could exist. Then the areas of the spanning clusters were added together. The result can be seen in figure 2. It is similar to the result for the $L_x = L_y$ system.

(b) When $p > p_c$, the probability that a site belongs to the spanning cluster $P(p, L)$ can be written like this:

$$P(p, L) \sim (p - p_c)^{\beta}$$

It is given that $p_c = 0.59275$. Taking the logarithm of both sides of the above

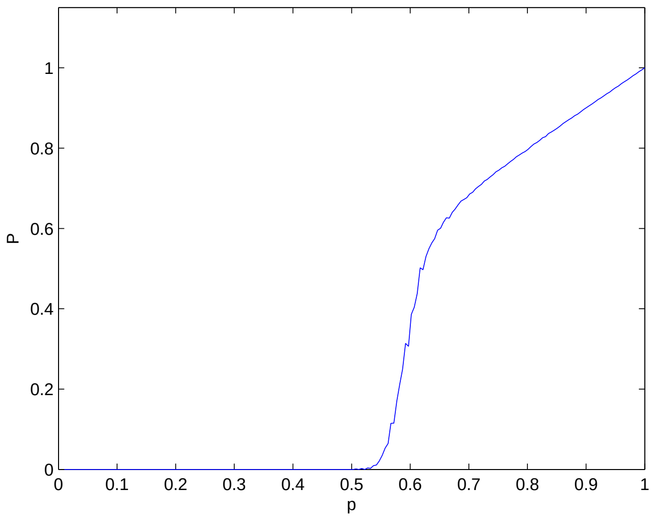

Figure 2: Probability for a site to belong to the spanning cluster plotted against probability for a site to be unoccupied. The system has a size 50×500.

equation gives

$$\log P(p, L) \sim \log(p - p_c)^\beta = \beta \log(p - p_c)$$

This linear relation between the logarithms of $P(p, L)$ and p was used to find a value for β. The results from the simulation in (a) and the given value of p_c was used. Then a line was fitted to the dataset. The plot can be seen in figure 3. The result was $\beta = 0.265 \pm 0.002$. The uncertainties are from the Matlab calculation of the fitted line.

(c) A distribution on the form $f(u) \propto u^\alpha$ was given. α is unknown. The relation between the actual distribution and the cumulative distribution was given by this equation

$$f_z(z) = \frac{dP(Z > z)}{dz}$$

This can also be written as

$$\int_1^z f_z(z)dz = P(Z > z)$$

The integral runs from the smallest to the largest value in the distribution.

The distribution $f \propto u^\alpha$ has to be normalized:

$$\int_1^\infty f = \int_1^\infty Au^\alpha du = 1$$

$$\int_1^\infty Au^\alpha du = \left[\frac{A}{\alpha + 1} u^{\alpha+1}\right]_1^\infty = \frac{A}{\alpha + 1} \cdot 0 - \frac{A}{\alpha + 1} = 1$$

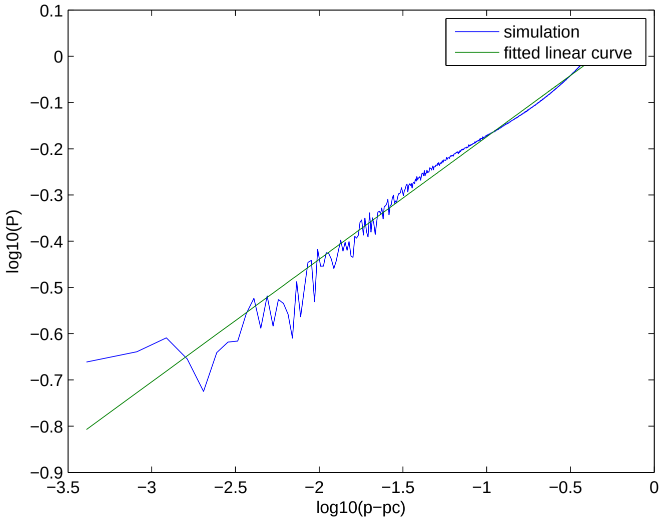

Figure 3: The logarithm of $P(p, L)$ plotted against the logarithm of $(p - p_c)$.

It was assumed that $\alpha < -1$. The normalization constant A is

$$A = -(\alpha + 1)$$

In order to find α the equation

$$\int_1^z f_z(z)dz = P(Z > z)$$

has to be solved. The integral can be solved like this

$$\int_1^z -(\alpha+1)u^\alpha du = \left[-\frac{\alpha+1}{\alpha+1}u^{\alpha+1}\right]_1^z = -z^{\alpha+1} + 1 = P$$

The equation can be written on the form

$$z^{\alpha+1} = 1 - P$$

Taking the logarithm of this gives

$$\log z^{\alpha+1} = \log(1 - P)$$

$$(\alpha + 1)\log z = \log(1 - P)$$

Plotting this makes it possible to find α.

(d) In order to plot the actual distribution, the z-values were placed in bins of exponentially increasing size, and the content of each bin was divided by the size of each bin. Each value was also divided by the total number of z-values, in order to normalize the distribution. Then the distribution was plotted with a

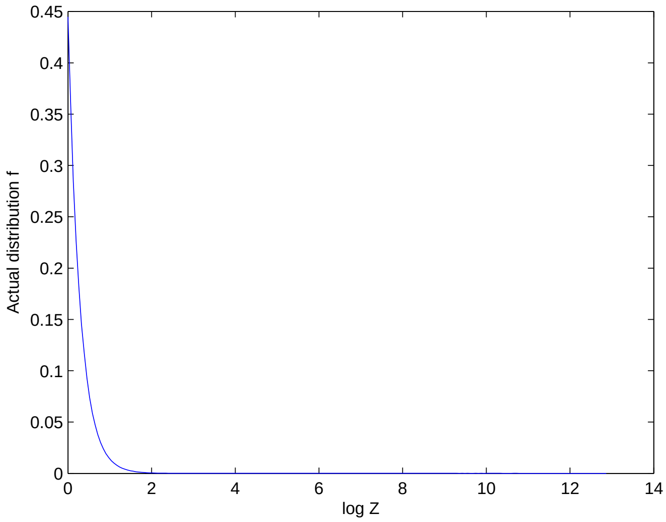

Figure 4: The actual distribution f_z, plotted with logarithmic binning.

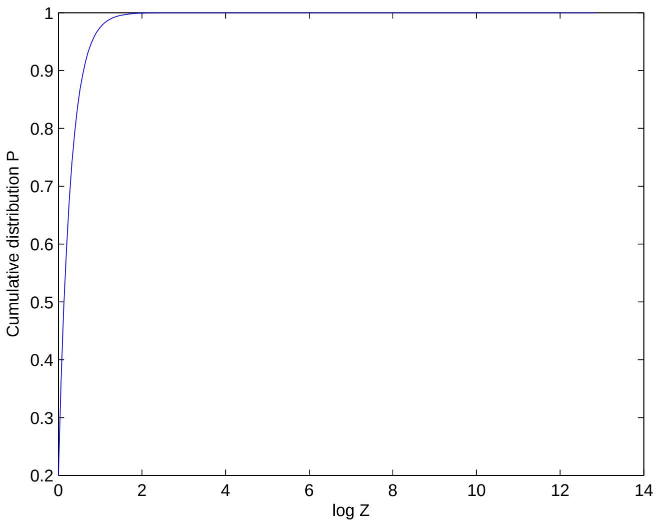

Figure 5: The cumulative distribution P, plotted with logarithmic binning.

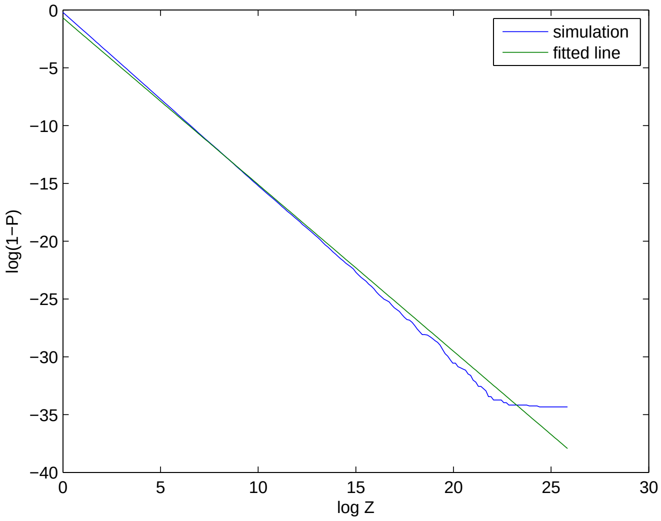

Figure 6: The linear relation between the z-values and the cumulative distribution.

logarithmic x-axis. The plot can be seen in figure 4. The cumulative distribution was found by summing all the values of the smaller bins. The distribution was normalized again, by dividing the values of each bin by the sum of the actual distribution. The plot of the cumulative distribution can be seen in figure 5.

The plot used to find α can be seen in figure 6. The value for α was found to be $\alpha = -2.44 \pm 0.02$.

(e) $n(s,p)$ is the probability for a site to be part of a cluster of size s for a given probability p. It is also called the cluster number density. $n(s,p)$ can be approximated by

$$n(s,p) \approx \frac{N_s}{L^d}$$

N_s is the number of clusters of size s, and d is the dimension of the system, in this case $d = 2$. The percolating cluster was removed, because the size of it is unknown. In figure 7, $n(s,p)$ is plotted against s for different p-values. Logarithmic binning was used. The smaller clusters are more probable for smaller p, while the larger clusters are more probable for larger p. The largest clusters are never found for the smaller p.

(f) In this exercise, $n(s,p_c;L)$ was estimated for different values of L. There is an ansatz that states that

$$n(s,p) = s^{-\tau} F(s/s_\xi)$$

If F is constant, then this relation can be used to find τ.

$$\log n(s,p_c) = \log s^{-\tau} + \log F$$

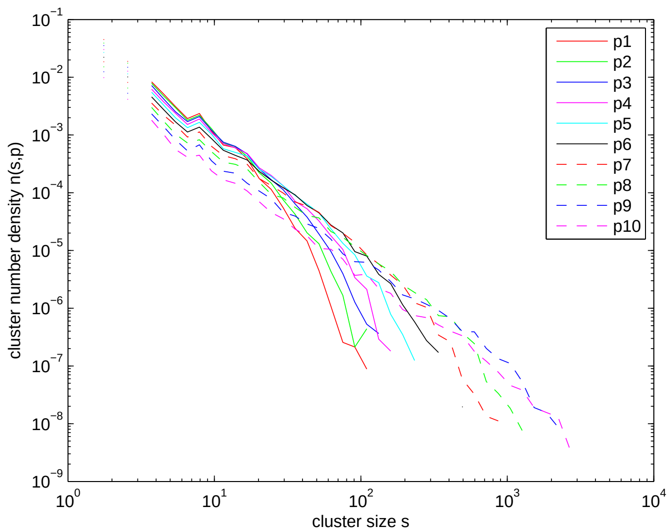

Figure 7: $n(s,p)$ plotted against s for different values of p around p_c. The values for p are increasing from $p \approx 0.39$ to $p \approx 0.59$. The axes are logarithmic.

$$\log n(s, p_c) = -\tau \log s + \log F$$

This linear relation can be used to find τ. In figure 8 a plot of this relation can be seen for different L. They fall almost on top of each other for the smaller cluster sizes, but the larger systems also have larger cluster sizes that the smaller systems cannot have. Using the results for the largest system, τ was found to be 1.87±0.02.

(g) The characteristic length s_ξ is defined as $s_\xi \sim |p-p_c|^{-1/\sigma}$. In order to find σ, another definition of s_ξ can be used: s_ξ is the value for which $n(s,p)/n(s,p_c) = F(s/s_\xi) = 0.5$.

In figure 9 $F(s/s_\xi)$ is plotted against s using the equation $F(s/s_\xi) = n(s,p)/n(s,p_c)$. In figure 10 the equation $F(s/s_\xi) = s^\tau n(s,p)$ is used. Only $p-$values below p_c were used. The two plots have the same form, but they seem to have slightly different values on the y-axis. The two methods give a little different results.

Figure 11 shows s_ξ plotted as a function of p. s_ξ increases as p approaches p_c. These values were found using the equality $n(s,p)/n(s,p_c) = F(s/s_\xi) = 0.5$, and were used to estimate σ in

$$s_\xi \sim |p - p_c|^{-1/\sigma}$$

Taking the logarithm on both sides of the equation gives

$$\log s_\xi \sim -\frac{1}{\sigma} \log |p - p_c|$$

A plot of $\log s_\xi$ as a function of $\log |p - p_c|$ in figure 12 gave the value $\sigma \approx 1.84$.

In figure 13 $s^\tau n(s,p)$ was plotted as a function of $s|p-p_c|^{1/\sigma}$, giving a data collapse plot. Different values for τ and σ were tried out, to try to get the different data sets to lie on top of each other. The values that gave the best results,

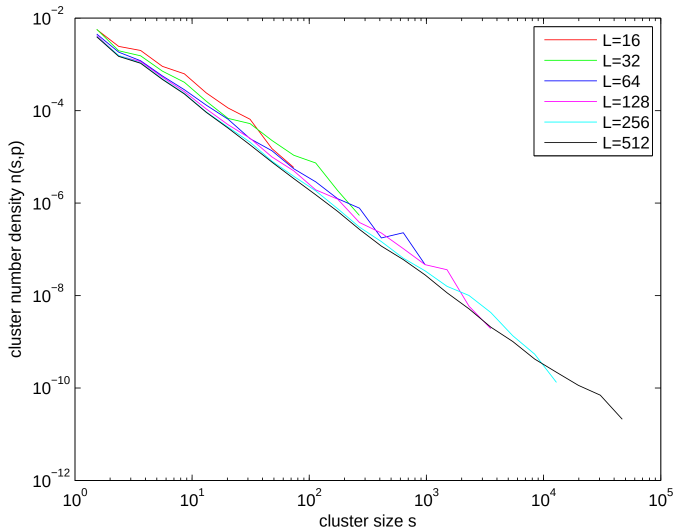

Figure 8: The logarithm of $n(s,p)$ plotted against the logarithm of s.

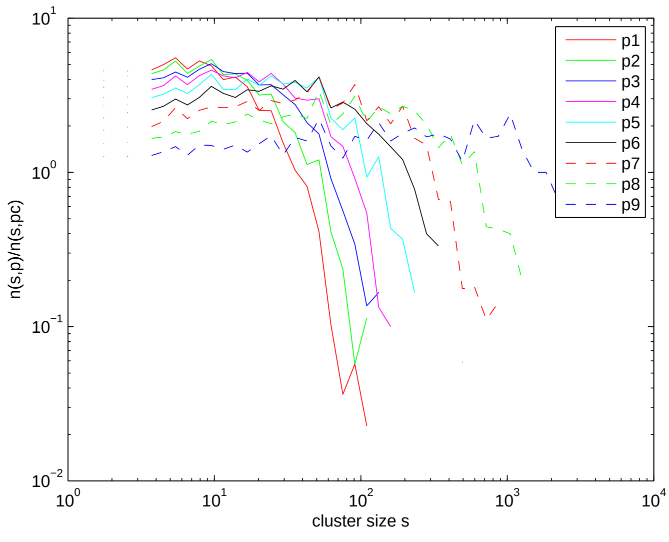

Figure 9: $F(s/s_\xi) = n(s,p)/n(s,p_c)$ plotted against s. The values for p are increasing from $p \approx 0.39$ to $p \approx 0.59$.

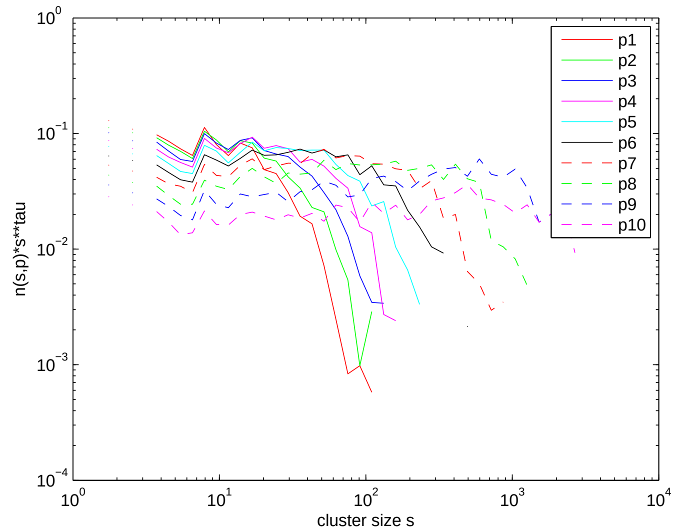

Figure 10: $F(s/s_\xi) = s^\tau n(s,p)$ plotted against s. The values for p are increasing from $p \approx 0.39$ to $p \approx 0.59$.

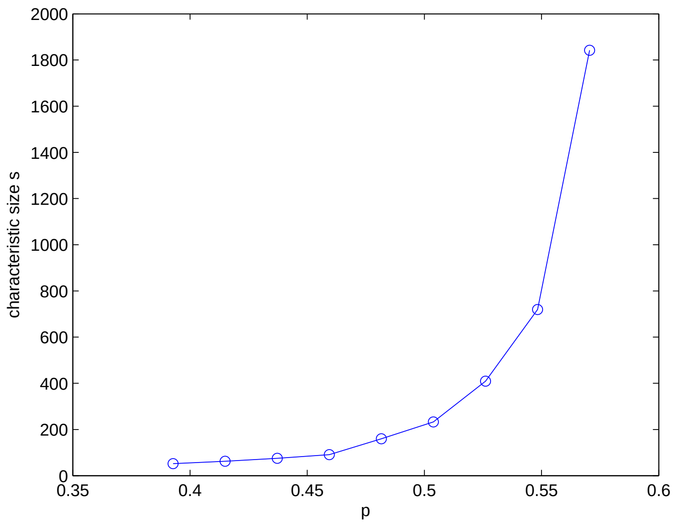

Figure 11: The characteristic size s_ξ as a function of p.

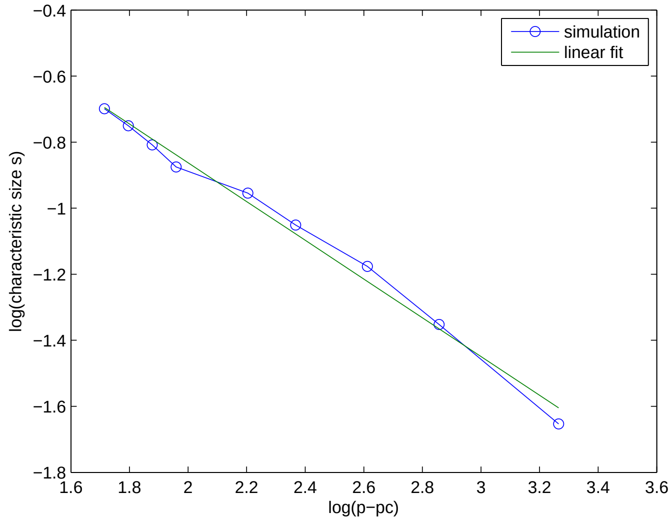

Figure 12: $\log s_\xi$ as a function of $\log |p - p_c|$ with a linear fit in order to find σ.

and were used in the plot were $\tau = 1.87$ and $\sigma = 0.4$. These two methods gave very different results for σ. The second method is probably a better method, because the values that were found for s_ξ were probably not completely accurate.

(h) The mass of the percolating cluster can be found with

$$M(p, L) = P(p, L)L^d$$

where $P(p, L)$ is the probability for a site to belong to the percolating cluster and L^d is the size of the system. There is an assumption that $M(p, L)$ can be written on the form

$$M(p, L) \propto L^D$$

The constant D can be found by taking the logarithm of both sides:

$$\log M(p, L) \propto D \log L$$

This has been plotted in figure 14, giving the result $D = 1.89 \pm 0.05$.

(i) In figure 15 $p_{\Pi=x}$ for $x = 0.8$ and $x = 0.3$ is plotted against L. Π is the probablity of having a spanning cluster, and p is the probability that a site is occupied. The values were found by averaging over 50 experiments. The plots are the most different when the system is small. When the system size inceases, the plots seem to approach the same value.

(j) The finite size scaling theory gives the equation

$$p_{x_1} - p_{x_2} = (C_{x_1} - C_{x_2})L^{-1/\nu}$$

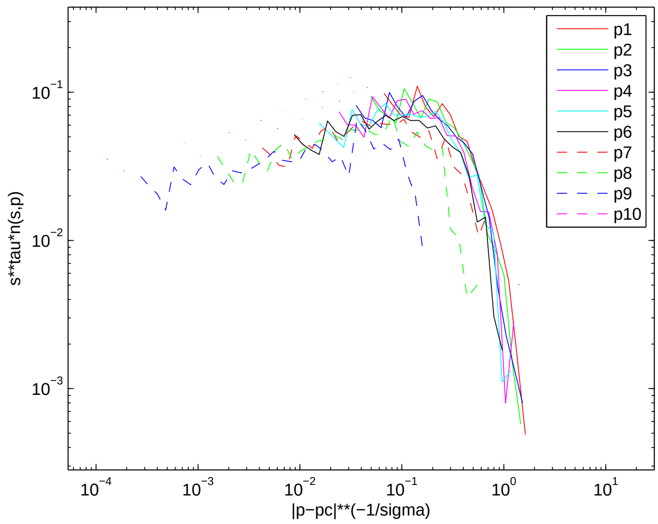

Figure 13: $s^{\tau}n(s,p)$ as a function of $s|p-p_c|^{1/\sigma}$ for different values of p. In this plot $\tau = 1.87$ and $\sigma = 0.4$. The values for p are increasing from $p \approx 0.39$ to $p \approx 0.59$.

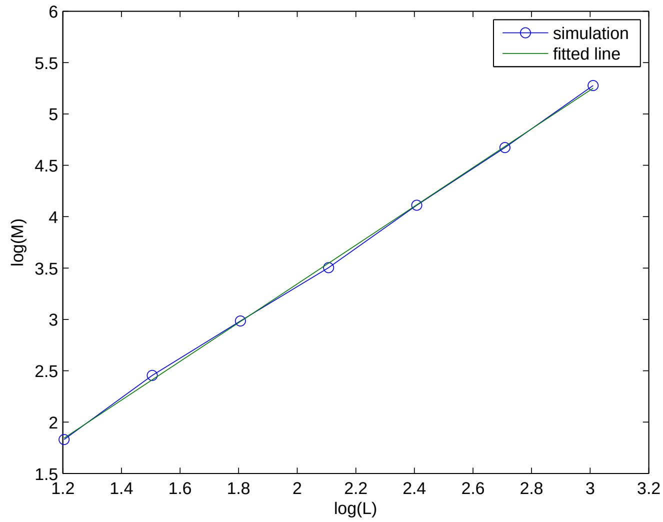

Figure 14: The logarithm of the mass of the spanning cluster as a function of the logarithm of the length of the system.

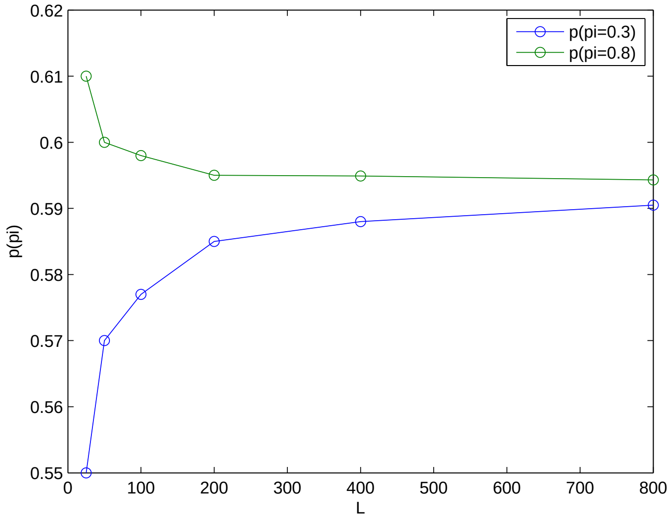

Figure 15: The probability of a site being occupied when the probability of having a spanning cluster is 0.3 and 0.8, plotted against L.

Taking the logarithm on both sides gives

$$\log(p_{x_1} - p_{x_2}) = -\frac{1}{\nu}\log L + \log(C_{x_1} - C_{x_2})$$

In figure 16 $\log(p_{\Pi=0.8} - p_{\Pi=0.3})$ is plotted as a function of $\log L$ to find ν. ν was found to be approximately 1.28. This is a little lower than the expected value of $\nu = 1.33$.

(k) The scaling theory also give the equation

$$p_{\Pi=x} = p_c + C_x L^{-1/\nu}$$

In figure 17 $p_{\Pi=0.3}$ and $p_{\Pi=0.8}$ is plotted as functions of $L^{-1/\nu}$ using $\nu = 1.33$. The intersection between the two plots gave $p_c \approx 0.593$. The two plots did not intersect exactly on the $y-$axis, so the value is not exact. This is approximately the same as the given value.

In figure 18 a data collapse plot for $\Pi(p, L) = \Phi(u) = \Phi[(p - p_c)L^{1/\nu}]$ can be seen. $\Pi(p, L)$ was plotted against $(p - pc)L^{1/\nu}$ for the different values of L in order to find the form of the function Φ.

(l) In figure 19 the singly connected bonds of a spanning cluster can be seen. The singly connected bonds are found by starting two walkers from one side of the spanning cluster. The first walker prefers to go to the right, the other prefers to go to the left. They both walk to the other side of the spanning cluster. The singly connected bonds are the sites that both of them have to pass through.

(m) The mass of the singly connected bonds M_{sc} was found for $p = p_c$ and different system sizes L. The results from 100 experiments were averaged over

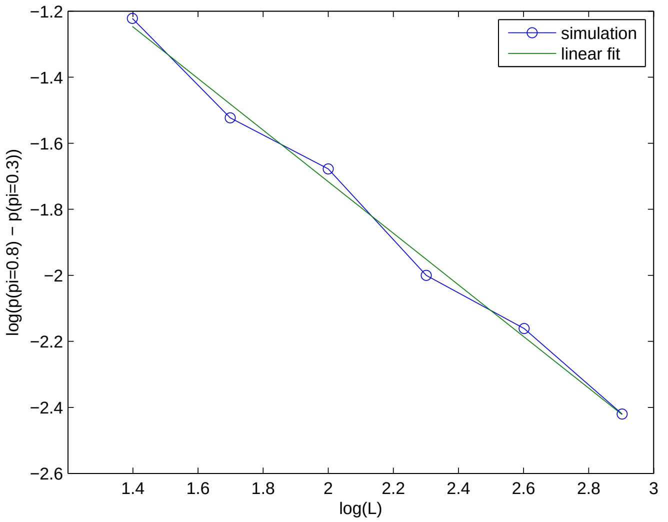

Figure 16: $\log(p_{\Pi=0.8} - p_{\Pi=0.3})$ plotted against $\log L$ with a linear fit.

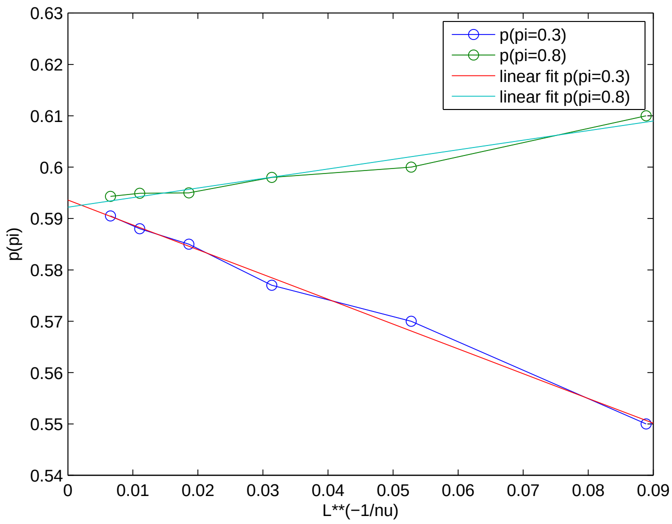

Figure 17: $p_{\Pi=0.3}$ and $p_{\Pi=0.8}$ plotted as functions of $L^{-1/\nu}$ with linear fits.

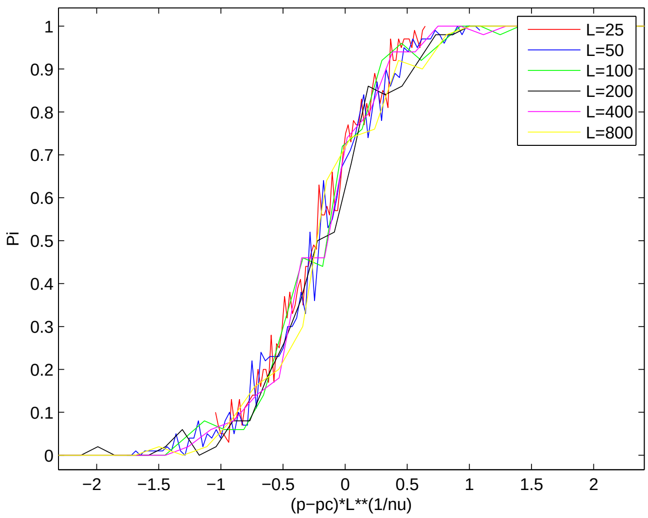

Figure 18: Data collapse plot for $\Pi(p, L) = \Phi(u)$.

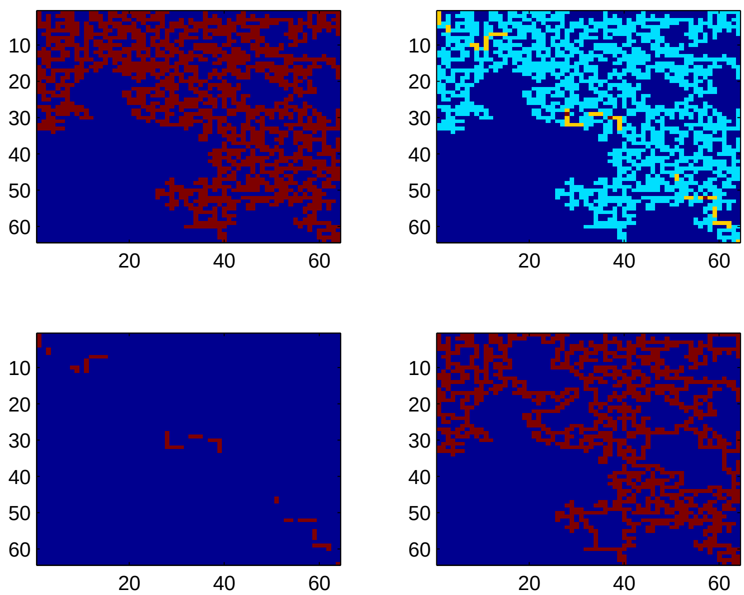

Figure 19: Plots of the singly connected bonds of a spanning cluster. Upper left: The spanning cluster. Upper right: The spanning cluster with the singly connected bonds in a different color. Lower left: Only the singly connected bonds. Lower right: The paths of the two walkers.

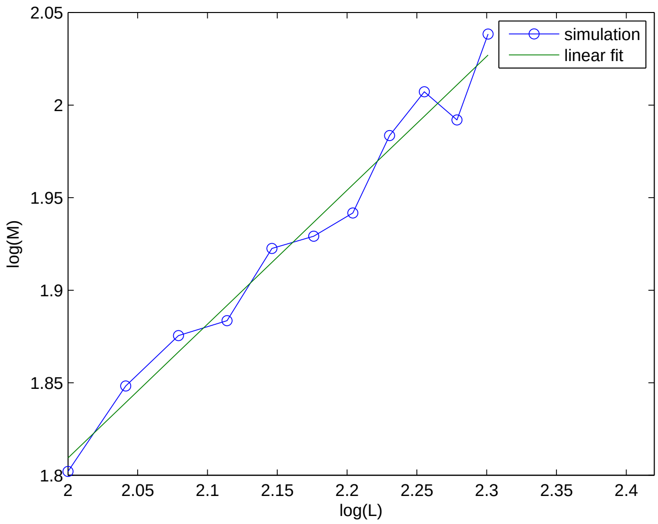

Figure 20: The logarithm of M_{sc} plotted as a function of the logarithm of the system size L with a fitted line.

in order to find M_{sc} for each L. The relation between the mass and the system size L can be written like this: $M_{sc} \propto L^{D_{sc}}$. Using the linear fit in figure 20, the exponent D_{sc} was found to be $D_{sc} = 0.72 \pm 0.09$.

In figure 21 $P_{sc} = M_{sc}/L^d$ is plotted as a function of $p - p_c$. The probability of finding singly connected bonds decreases as p increases. As the spanning cluster becomes larger, there are fewer singly connected bonds.

(n) Figure 22 shows a visualization of the currents in the spanning cluster. The backbone is the part of the spanning cluster that has a flux going through it. The dangling ends are the parts of the spanning cluster that do not have any flux.

(o) The mass of the singly connected bonds, the backbone and the dangling ends were found for different values of L. The singly connected bonds were assumed to be the sites with the maximum amount of flux, because all the flux had to pass through these sites. The backbone were the parts of the spanning cluster that had some flux, while the dangling ends were the parts of the spanning cluster that did not have flux. The results were averaged over 100 experiments. This is plotted in figure 23. This was used to find their dimensionality $M_x \propto L^{D_x}$. In order to find the exponents, the logarithm was taken on both sides of the equation, and a line was fitted to the data points, as can be seen in figure 24 and 25.

The exponential for the singly connected bonds was $D_{sc} = 0.86 \pm 0.14$. This is a slighly higher value than was found in (m), but the standard deviation is also higher. The measurements on the backbone gave $D_{bb} = 1.53 \pm 0.04$. The dangling ends gave the value $D_{de} = 2.07 \pm 0.04$.

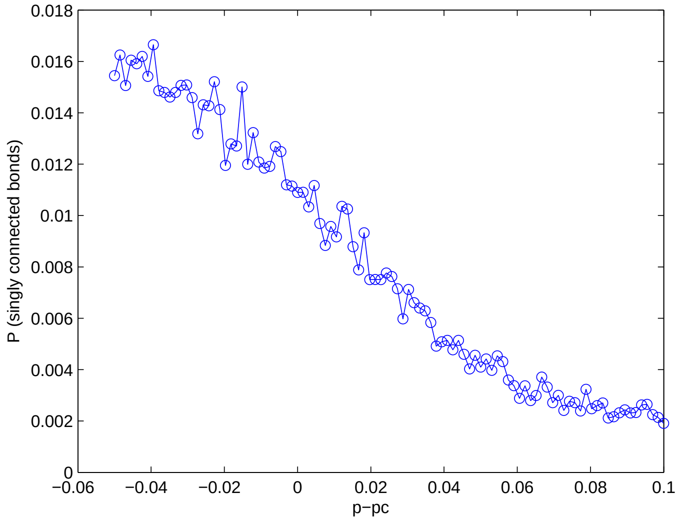

Figure 21: P_{sc} plotted as a function of $p - pc$.

(p) The conductivity of the system was also found for different $p-$values. Because the system is two-dimensional, the conductivity σ is the same as the conductance G. Since the total flux is equal to the conductance multiplied with the pressure difference $\Phi = \Delta P \cdot G$, and the pressure difference is 1, the conductivity is equal to the flux in the system. The flux going out is the same as the flux going in, so it is sufficient to find the flux in one "slize" of the system. In figure 26 the conductivity is plotted as a function of $p - p_c$. The conductivity increases as p increases, and as the spanning cluster becomes larger.

In order to determine the exponent $\tilde{\zeta_R}$ in the expression $\sigma = G \propto L^{-\tilde{\zeta_R}}$, the conductivity was measured for different values of L, for $p = p_c$. The results can be seen in figure 27. The conductivity decreases as the system size increases. The spanning cluster may have more "holes" as the system size increases. The exponent was found to be $\tilde{\zeta_R} = 1.06 \pm 0.07$.

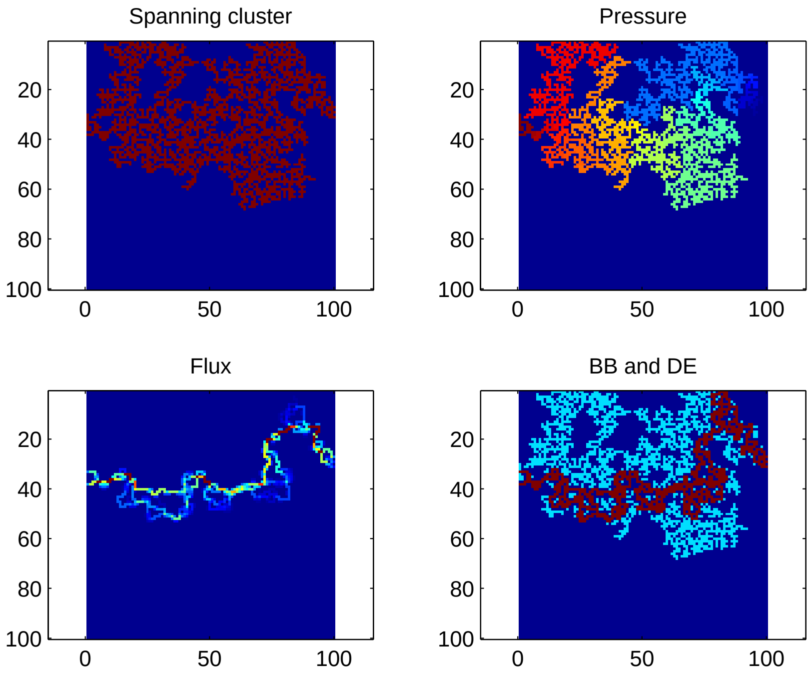

Figure 22: Flow in the spanning cluster. Upper left: spanning cluster; Upper right: pressure in the sites. The largest pressure is on the left side, and it decreases towards the right side. Lower left: flux in the spanning cluster, the different colors show the amount of flux. The red sites are the ones with the most flux. Lower right: backbone (red color) and dangling ends (light blue color) of the spanning cluster.

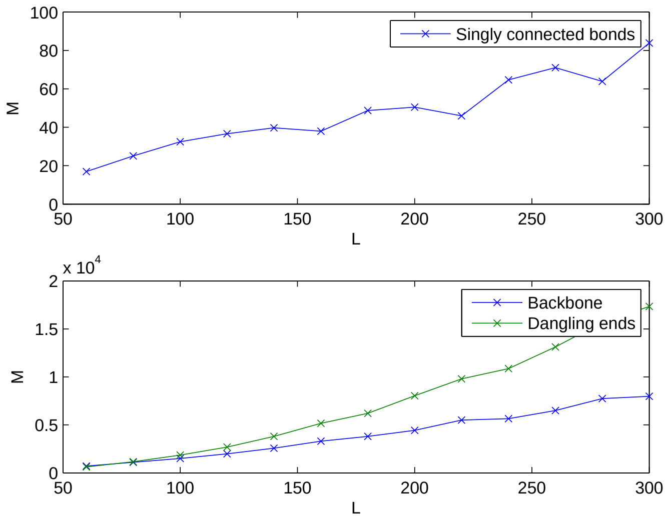

Figure 23: The mass plotted as a function of the system size L for the singly connected bonds, the backbone and the dangling ends.

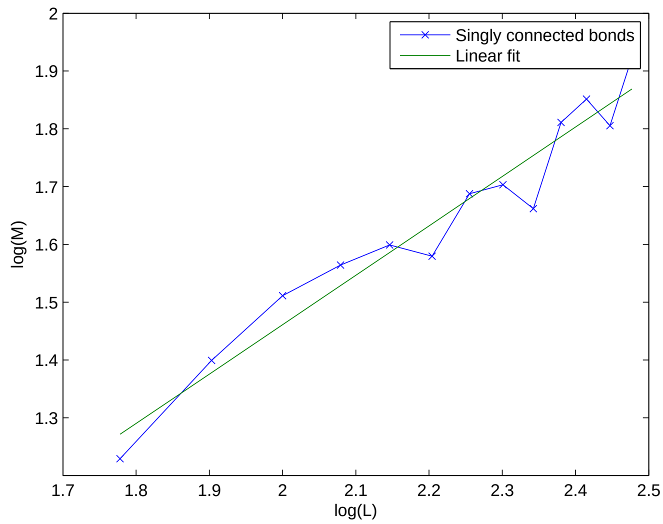

Figure 24: The logarithm of the mass plotted as a function of the logarithm of the system size L for the singly connected bonds with a fitted line.

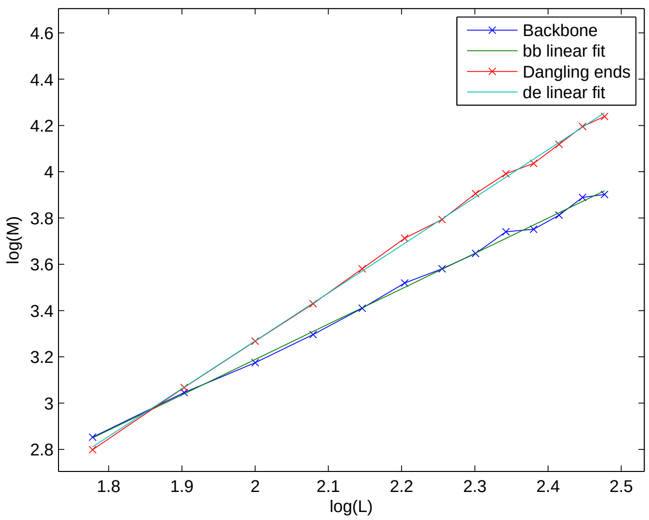

Figure 25: The logarithm of the mass plotted as a function of the logarithm of the system size L for the bacckbone and the dangling ends with fitted lines.

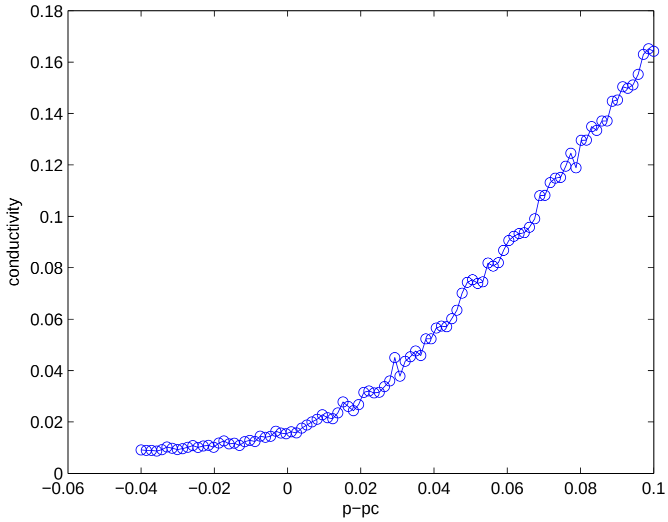

Figure 26: The conductivity σ plotted as a function of $p - p_c$.

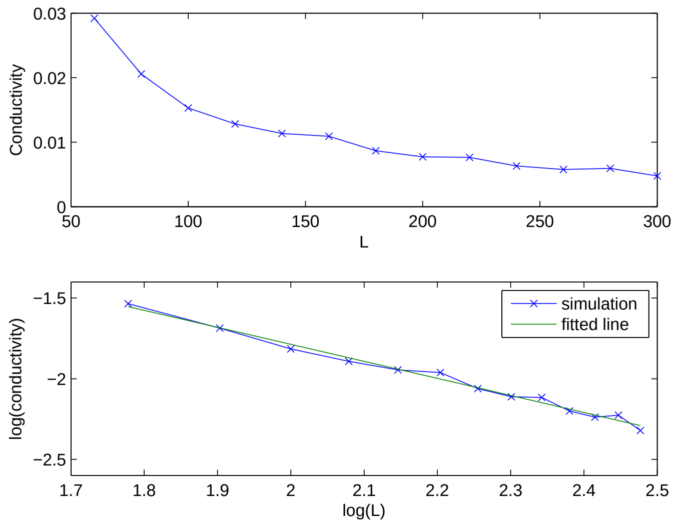

Figure 27: Top picture: the conductivity σ plotted as a function of L. Bottom picture: the logarithm of σ and L, with a fitted line.