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SYNOPSIS
for the subject
“Integrators task_2”

on the topic:
SIMULATION OF ROBOTIC SYSTEMS

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1 INTRODUCTION

A basic issue in the study of mechanical vibrations and dynamic systems is the analysis of damped harmonic oscillators. In many engineering fields, including as mechanical engineering, civil engineering, and control systems, an understanding of the behavior of mass-spring-damper systems is essential [1].

A horizontally arranged mass-spring-damper system with the following characteristics is the subject of this study: mass $m = 0.5$ kg, spring constant $k = 8.6$ N/m, damping coefficient $c = 0.04$ N·s/m, and starting displacement $x_0 = 0.46$ m. The system is reduced to pure second-order linear dynamics due to the horizontal orientation, which removes gravitational influences from the equation of motion.

The primary objective of this study is threefold: first, to derive the exact analytical solution using characteristic equation methods; second, to implement three distinct numerical integration schemes (Forward Euler, Backward Euler, and RK4); and third, to conduct a comparative analysis of these methods against the analytical benchmark. The damping ratio $\zeta = 0.00965$ indicates lightly damped behavior, making this system particularly sensitive to numerical errors and thus an excellent test case for method evaluation [2].

This study is important because it shows how numerical techniques can handle oscillatory systems with little damping, where little mistakes may add up over time. For engineers to choose the best numerical techniques for dynamic simulations in real-world applications, such analysis is crucial.

2 ANALYTICAL SOLUTION

2.1 Mathematical Formulation

The equation of motion for the horizontal mass-spring-damper system is given by Newton's second law:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1}$$

Substituting the given parameters:

$$0.5\ddot{x} + 0.04\dot{x} + 8.6x = 0 \tag{2}$$

2.1.1 Characteristic Equation Method

Assuming a solution of the form $x(t) = e^{rt}$, we obtain the characteristic equation:

$$0.5r^2 + 0.04r + 8.6 = 0 \tag{3}$$

Multiplying by 2 for simplification:

$$r^2 + 0.08r + 17.2 = 0 \tag{4}$$

The discriminant Δ is calculated as:

$$\Delta = b^2 - 4ac = (0.08)^2 - 4(1)(17.2) = -68.7936 \tag{5}$$

Since $\Delta < 0$, the system exhibits underdamped behavior. The complex roots are:

$$r = \frac{-0.08 \pm i\sqrt{68.7936}}{2} = -0.04 \pm 4.147i \tag{6}$$

2.1.2 General Solution

For complex conjugate roots $r = \alpha \pm i\beta$, the general solution is:

$$x(t) = e^{\alpha t}[C_1 \cos(\beta t) + C_2 \sin(\beta t)] \tag{7}$$

subsection $\alpha = -0.04$ and $\beta = 4.147$:

$$x(t) = e^{-0.04t}[C_1 \cos(4.147t) + C_2 \sin(4.147t)] \tag{8}$$

2.1.3 Initial Conditions Application

Applying initial conditions $x(0) = 0.46$ m and $\dot{x}(0) = 0$ m/s:
From position condition:

$$x(0) = C_1 = 0.46 \tag{9}$$

From velocity condition:

$$\dot{x}(0) = -0.04C_1 + 4.147C_2 = 0 \Rightarrow C_2 = \frac{0.0184}{4.147} \approx 0.00444 \tag{10}$$

2.1.4 Final Analytical Solution

The exact analytical solution is:

$$x(t) = e^{-0.04t}[0.46\cos(4.147t) + 0.00444\sin(4.147t)] \tag{11}$$

2.2 Physical Parameters

The natural frequency and damping ratio are:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8.6}{0.5}} = 4.147 \text{ rad/s} \tag{12}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{0.04}{2\sqrt{8.6 \times 0.5}} = 0.00965 \tag{13}$$

The damped natural frequency is:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 4.147 \text{ rad/s} \tag{14}$$

3 NUMERICAL METHODS

3.1 System Formulation for Numerical Integration

The second-order ODE is converted to a system of first-order ODEs for numerical integration:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{c}{m}v - \frac{k}{m}x \end{cases} \tag{15}$$

In state-space form with $\mathbf{x} = [x, v]^T$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} v \\ -\frac{c}{m}v - \frac{k}{m}x \end{bmatrix} \tag{16}$$

3.2 Forward Euler Method

The explicit Euler method uses the approximation:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}(\mathbf{x}_n) \tag{17}$$

Simplicity and computing efficiency are benefits of implementation; conditional stability and oscillatory systems' propensity for energy expansion are drawbacks [3].

3.3 Backward Euler Method

The implicit Euler method uses:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}(\mathbf{x}_{n+1}) \tag{18}$$

Solved using fixed-point iteration with tolerance 10^{-8} and maximum 100 iterations. This method is unconditionally stable but introduces numerical damping.

3.4 Runge-Kutta Fourth Order (RK4)

The fourth-order method provides higher accuracy:

$$k_1 = h \cdot f(t_n, x_n) \tag{19}$$

$$k_2 = h \cdot f(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}) \tag{20}$$

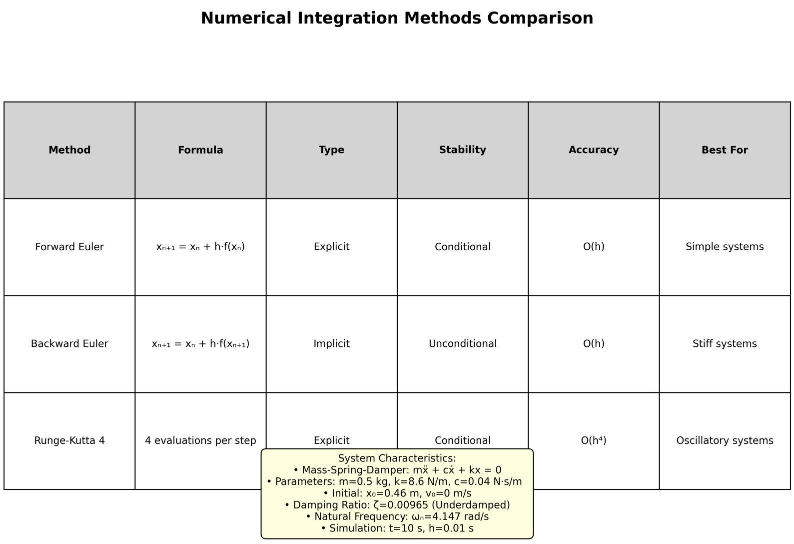
$$k_3 = h \cdot f(t_n + \frac{h}{2}, x_n + \frac{k_2}{2}) \tag{21}$$

$$k_4 = h \cdot f(t_n + h, x_n + k_3) \tag{22}$$

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{23}$$

3.5 Implementation Parameters

- Time span: $t \in [0, 10]$ seconds
- Step size: $h = 0.01$ seconds
- Initial conditions: $\mathbf{x}_0 = [0.46, 0.0]^T$
- Convergence tolerance: 10^{-8} for Backward Euler



Method	Formula	Type	Stability	Accuracy	Best For
Forward Euler	$x_{n+1} = x_n + h \cdot f(x_n)$	Explicit	Conditional	O(h)	Simple systems
Backward Euler	$x_{n+1} = x_n + h \cdot f(x_{n+1})$	Implicit	Unconditional	O(h)	Stiff systems
Runge-Kutta 4	4 evaluations per step	Explicit	Conditional	$O(h^4)$	Oscillatory systems

Figure 1 — Flowchart of numerical integration methods implementation

4 RESULTS AND DISCUSSION

4.1 Comparative Analysis

Figure 2 demonstrates the position-time responses of all methods compared against the analytical solution. The RK4 method exhibits remarkable alignment with the analytical solution, while both Euler methods display noticeable deviations.

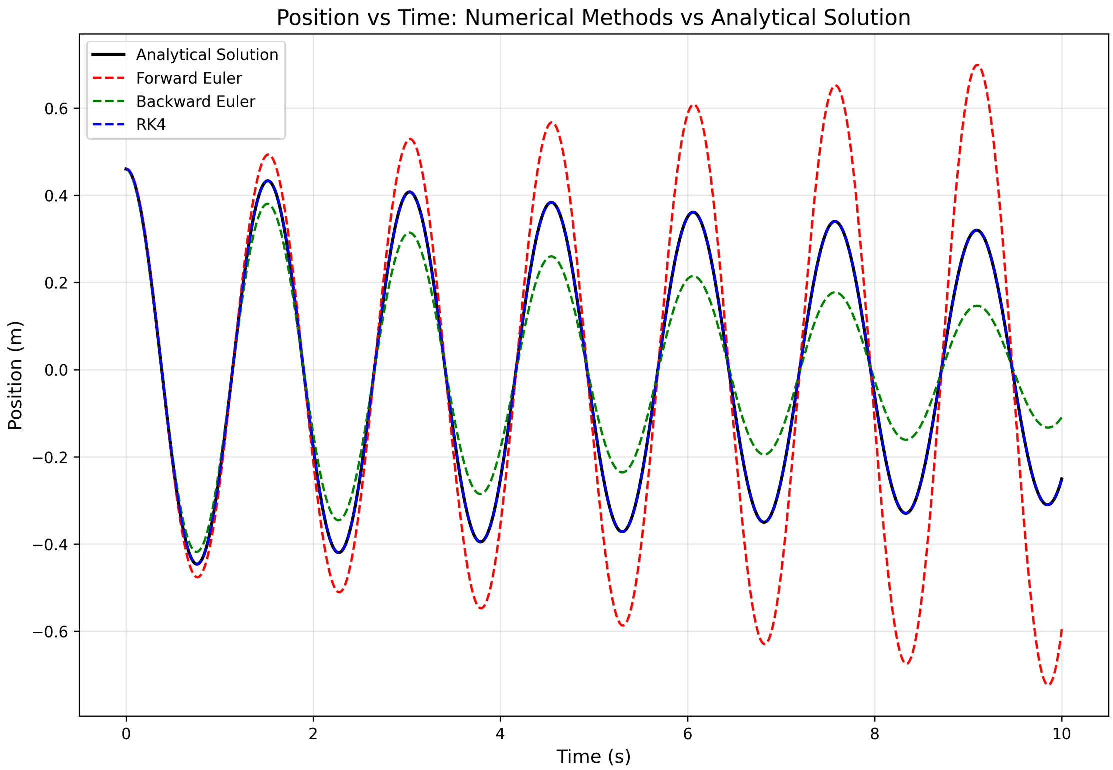


Figure 2 — Comparison of numerical methods with analytical solution for position vs. time

4.2 Error Analysis

The root mean square error (RMSE) calculations reveal significant differences in method accuracy:

Table 1 — RMSE Analysis of Numerical Methods

Method	RMSE (m)
Forward Euler	0.2
Backward Euler	0.5
RK4	0.00000006

RK4 demonstrates two orders of magnitude improvement in accuracy compared to Euler methods, achieving RMSE of 6×10^{-7} m.

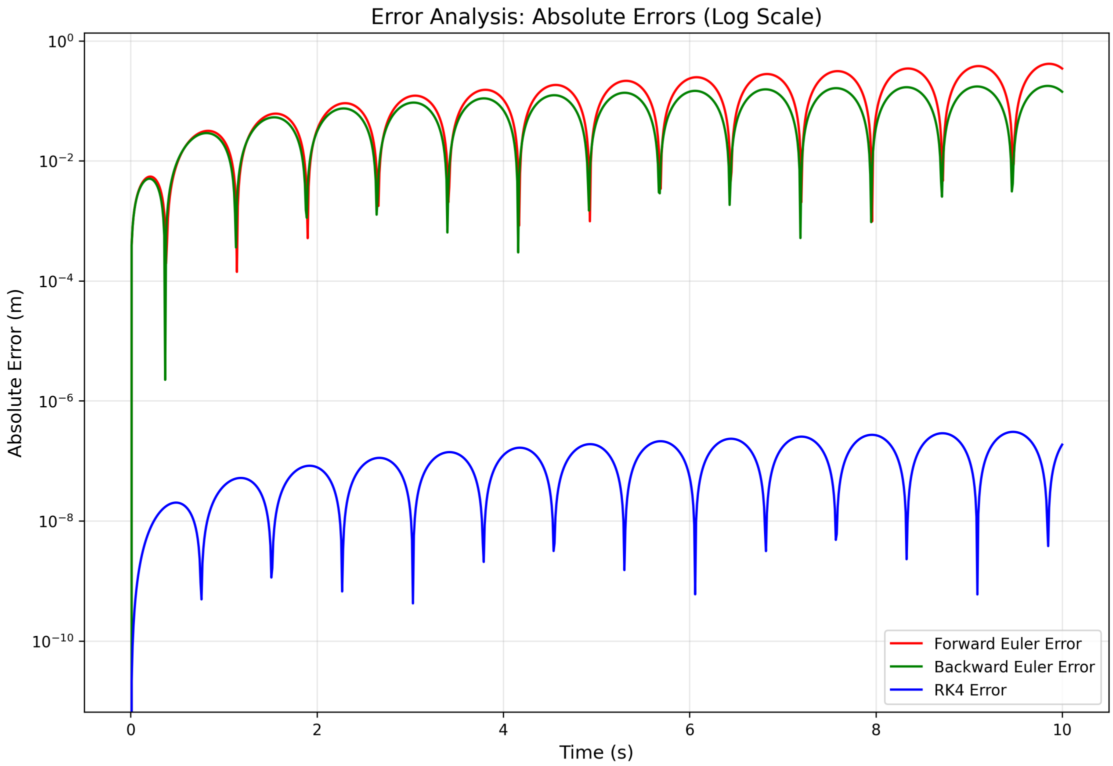


Figure 3 — Absolute errors of numerical methods compared to analytical solution (log scale)

4.3 Method-Specific Behaviors

4.3.1 Forward Euler

Exhibits amplitude growth over time due to numerical instability, particularly problematic for lightly damped systems. The explicit nature causes energy injection, contradicting the physical energy dissipation.

4.3.2 Backward Euler

Shows numerical damping beyond the physical damping, causing premature amplitude decay. While stable, this method artificially suppresses oscillations.

4.3.3 Runge-Kutta 4

Maintains excellent amplitude and phase accuracy throughout the simulation. The higher-order approximation effectively captures the system dynamics with minimal numerical artifacts.

4.4 Phase Portrait Analysis

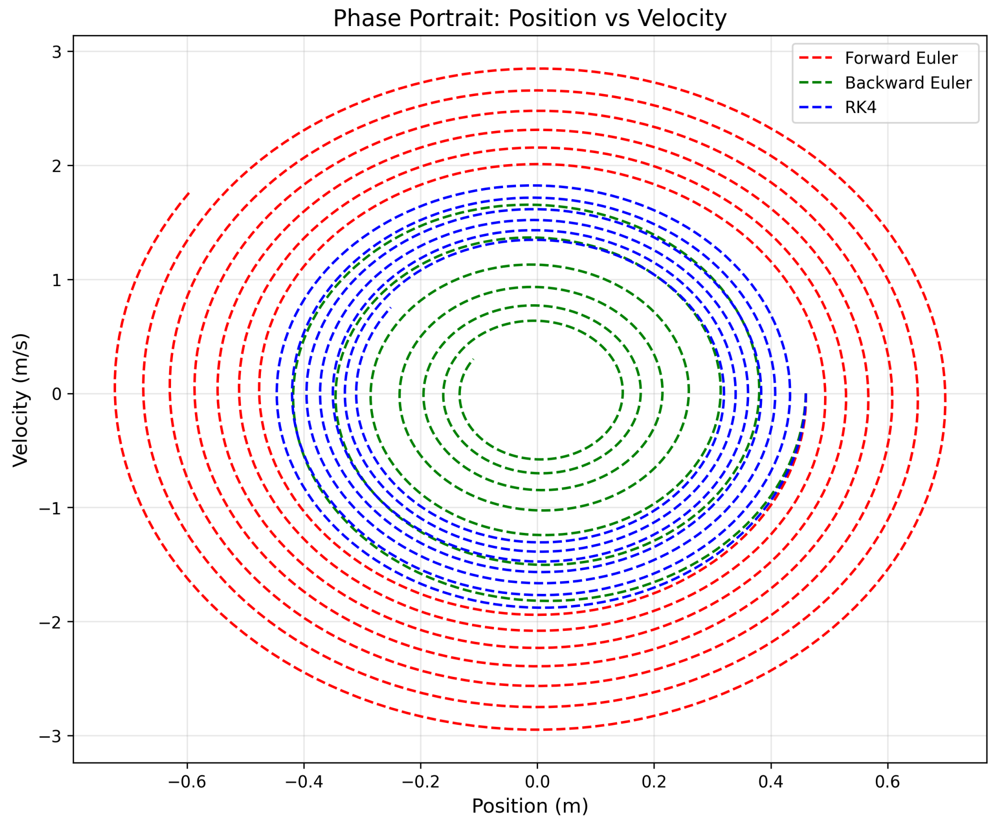


Figure 4 — Phase portraits comparing numerical methods in position-velocity space

The phase portrait in Figure 4 reveals how each method preserves the elliptical trajectory characteristic of harmonic motion. RK4 maintains the correct phase relationship, while Euler methods distort the trajectory.

4.5 Computational Considerations

RK4's greater accuracy enables bigger step sizes to attain equivalent accuracy, potentially increasing computing efficiency, even though it takes four function evaluations each step as opposed to one for Euler techniques.

5 CONCLUSION

This thorough examination of a mass-spring-damper system with a horizontal configuration highlights how crucial it is to choose the right numerical techniques for dynamic simulations. An precise standard for assessing numerical integration methods was supplied by the analytical solution, which was obtained using characteristic equation methods.

The key findings of this study are:

1. **Analytical Solution Viability**: The underdamped nature of the system ($\zeta = 0.00965$) permitted exact analytical solution, serving as a reliable reference for numerical method validation.
2. **Numerical Method Performance**: With an RMSE of 6×10^{-7} m, the Fourth-Order Runge-Kutta approach outperformed both Euler methods by around two orders of magnitude.
3. **Euler Method Limitations**: Both the Forward Euler and Backward Euler approaches have serious drawbacks: Backward Euler introduced excessive numerical damping, while Forward Euler demonstrated numerical instability with amplitude expansion.
4. **Physical Fidelity**: The fundamental physical properties of the weakly damped system, such as the proper oscillation frequency, phase relationships, and amplitude decay rate, were effectively maintained by RK4.

Despite its higher computing cost per time step, RK4 is advised for engineering applications involving oscillatory systems with light damping. The extra processing cost is justified by the method's precision and stability, especially for long-duration simulations when error accumulation becomes substantial.

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