ITMO University

Analytical and Numerical Analysis of a Mass–Spring–Damper System Using Lagrange Formalism

Course: Simulation of Robotic Systems

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Introduction

The aim of this work is to model a mass–spring–damper system, derive the corresponding ordinary differential equation (ODE) using the Lagrange method (as in `SRS/lecture_1`), and obtain its analytical solution. This analytical result is then used as a reference to evaluate several numerical integration methods from the first task: Explicit (Forward) Euler, Implicit (Backward) Euler, and the fourth–order Runge–Kutta (RK4) method. The physical system under consideration is a cart of mass m connected to a rigid wall by a linear spring and a viscous damper, moving in one horizontal direction. Gravity is taken into account in the potential energy term, in accordance with the lecture notes.

1 Model and ODE

We consider a cart of mass m connected to a wall by a linear spring and a viscous damper. The generalized coordinate is the horizontal displacement $x(t)$ of the cart from the wall.

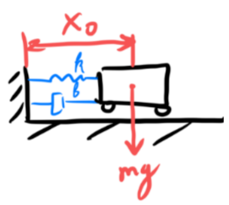

Figure 1: Mass–spring–damper mechanism.

The parameters from the variant table are

$$m = 0.6\,\mathrm{kg}, \qquad k = 16.6\,\mathrm{N/m}, \qquad b = 0.02\,\mathrm{N\,s/m}, \qquad g = 9.81\,\mathrm{m/s^2}.$$

Initial conditions:

$$x(0) = x_0 = 0.51\,\mathrm{m}, \qquad \dot{x}(0) = v_0 = 0\,\mathrm{m/s}.$$

Kinetic energy:

$$K = \frac{1}{2}m\dot{x}^2.$$

Potential energy (spring + gravity, following the lecture notes):

$$P = mgx + \frac{1}{2}kx^2.$$

Lagrangian:

$$L(x, \dot{x}) = K - P = \frac{1}{2}m\dot{x}^2 - \left(mgx + \frac{1}{2}kx^2\right).$$

The damper is modeled as a non-conservative generalized force

$$Q = -b\dot{x}.$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q$$

gives

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}, \quad \frac{\partial L}{\partial x} = -mg - kx.$$

Substituting:

$$m\ddot{x} - (-mg - kx) = -b\dot{x} \quad \Rightarrow \quad m\ddot{x} + mg + kx = -b\dot{x}.$$

The final ODE is

$$m\ddot{x} + b\dot{x} + kx + mg = 0. \tag{1}$$

With the numerical values:

$$0.6\,\ddot{x} + 0.02\,\dot{x} + 16.6\,x + 5.886 = 0.$$

2 Analytical Solution

2.1 Shift to equilibrium

The static equilibrium satisfies

$$kx_{\text{eq}} + mg = 0 \quad \Rightarrow \quad x_{\text{eq}} = -\frac{mg}{k} = -\frac{0.6 \cdot 9.81}{16.6} \approx -0.3546 \text{ m}.$$

Introduce the shifted coordinate

$$z(t) = x(t) - x_{\text{eq}}.$$

Substituting into (1) cancels the constant term and yields

$$\ddot{z} + \frac{b}{m}\dot{z} + \frac{k}{m}z = 0.$$

Define

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16.6}{0.6}} \approx 5.2599\,\text{rad/s}, \qquad \zeta = \frac{b}{2\sqrt{mk}} = \frac{0.02}{2\sqrt{0.6 \cdot 16.6}} \approx 0.00317.$$

Since $\zeta < 1$, the system is underdamped. The damped natural frequency is

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2} \approx 5.2599\,\mathrm{rad/s},$$

and

$$a = \zeta\omega_0 = \frac{b}{2m} = \frac{0.02}{1.2} = \frac{1}{60}.$$

2.2 Closed-form solution

The general underdamped solution for $z(t)$ is

$$z(t) = e^{-at}\big(C_1 \cos\omega_d t + C_2 \sin\omega_d t\big).$$

Initial conditions in terms of z:

$$z(0) = x(0) - x_{\mathrm{eq}} = 0.51 - (-0.3546) = 0.8646, \qquad \dot{z}(0) = \dot{x}(0) = 0.$$

Thus

$$C_1 = z(0) = 0.8646,$$

and from

$$\dot{z}(0) = -aC_1 + \omega_d C_2 = 0 \quad \Rightarrow \quad C_2 = \frac{aC_1}{\omega_d} \approx 0.00274.$$

Therefore,

$$z(t) = e^{-t/60}\big(0.8646\cos(5.2599t) + 0.00274\sin(5.2599t)\big),$$

and

$$x(t) = z(t) + x_{\mathrm{eq}}.$$

Differentiating $z(t)$ gives the analytical velocity

$$\dot{x}(t) = \dot{z}(t) = e^{-t/60}\Big[(-aC_1)\cos\omega_d t - (aC_2)\sin\omega_d t - C_1\omega_d \sin\omega_d t + C_2\omega_d \cos\omega_d t\Big].$$

3 Numerical Methods from the First Task

We rewrite (1) as a first-order system. Let

$$x_1 = x, \qquad x_2 = \dot{x}.$$

Then

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 - g.$$

With the numerical parameters,

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{1}{30}x_2 - 27.6667\,x_1 - 9.81.$$

We simulate on $t \in [0, 10]$ s with step $h = 0.01\,\mathrm{s}$ using three methods from the first task:

- **Forward Euler (FE):**
$$x^{n+1} = x^n + hf(x^n).$$
- **Backward Euler (BE):**
$$x^{n+1} = x^n + hf(x^{n+1}),$$
solved by fixed-point iteration at each step.
- **Runge–Kutta 4 (RK4):**
$$\begin{aligned} k_1 &= f(x^n), \\ k_2 &= f(x^n + \tfrac{h}{2}k_1), \\ k_3 &= f(x^n + \tfrac{h}{2}k_2), \\ k_4 &= f(x^n + hk_3), \\ x^{n+1} &= x^n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

The same initial state $x_1(0) = 0.51$, $x_2(0) = 0$ is used for all methods.

4 Results and Error Analysis

Figure 2 shows the displacement, velocity and phase portrait (top row) and the corresponding errors (bottom row) for the three numerical methods compared to the analytical solution.

The maximum errors obtained in the experiment are summarised in Table 1. Here e_x is the displacement error, e_v the velocity error, and $\|e\|$ the Euclidean distance to the analytical state in the phase portrait.

Discussion

From Fig. 2 and Table 1:

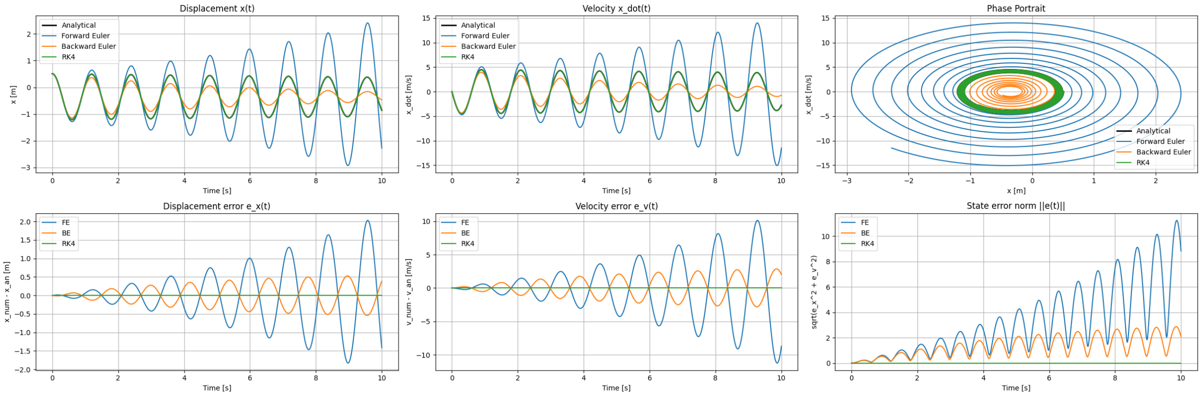

Figure 2: Top: analytical solution vs. Forward Euler (FE), Backward Euler (BE) and RK4 for displacement, velocity and phase portrait. Bottom: displacement error $e_x(t) = x_{\mathrm{num}}(t) - x_{\mathrm{an}}(t)$, velocity error $e_v(t) = \dot{x}_{\mathrm{num}}(t) - \dot{x}_{\mathrm{an}}(t)$, and state error norm $\|e(t)\| = \sqrt{e_x^2 + e_v^2}$.

Table 1: Maximum errors over $t \in [0, 10]$ s for step $h = 0.01$ s.

Method	$\max \lvert e_x \rvert$ [m]	$\max \lvert e_v \rvert$ [m/s]	$\max \lVert e \rVert$
Forward Euler	2.0293941891	11.2317623605	11.2318698263
Backward Euler	0.5404738478	2.8708950286	2.8709003925
RK4	2.42699×10^{-6}	1.24390×10^{-5}	1.24391×10^{-5}

- The **RK4** solution is visually indistinguishable from the analytical one. Both displacement and velocity errors stay below 1.3×10^{-5}, and the state error norm is of the same order. This reflects the high accuracy of a 4th-order method.
- **Forward Euler** exhibits large amplitude and phase errors. The displacement error reaches about 2.03 m and the velocity error about 11.23 m/s. The state error norm grows monotonically, indicating that the chosen step size $h = 0.01$ s is too large for this explicit first-order scheme in a lightly damped oscillatory system.
- **Backward Euler** remains stable, but introduces noticeable numerical damping. Its displacement and velocity amplitudes decay faster than in the analytical solution, leading to moderate but non-negligible errors: about 0.54 m in position and 2.87 m/s in velocity. This behaviour is consistent with the known dissipative nature of the implicit Euler method.

Overall, the error plots clearly show the trade-offs among the three integrators: simplicity and low stability region for Forward Euler, unconditional stability but overdamping for Backward Euler, and high accuracy with reasonable cost for RK4.

5 Conclusions

Using the Lagrange formulation, the mass–spring–damper system was modeled and the ODE $m\ddot{x} + b\dot{x} + kx + mg = 0$ was derived. An analytical solution was obtained by shifting to the static equilibrium and solving the resulting homogeneous second-order ODE.

The numerical investigation applied three integration methods from the first task (Forward Euler, Backward Euler, RK4) to the same ODE and initial conditions. By comparing displacement, velocity and phase portrait with the analytical solution and computing the corresponding errors, we concluded that:

- RK4 provides an excellent approximation of the true dynamics with very small errors.
- Forward Euler is straightforward to implement but unsuitable for this lightly damped oscillator at the chosen step size, producing large errors and energy growth.
- Backward Euler is robust and unconditionally stable, but its strong numerical damping causes significant deviation from the true oscillatory behaviour.

For accurate simulation of such mechanical systems with reasonable step sizes, higher-order methods like RK4 are preferable, while Euler methods may require much smaller time steps or be used only for qualitative analysis.