Simulation of Robotic Systems 2025

Laboratory Works Assignment

Task 2: Modeling and Analytical / Numerical Solution of a Mass Spring Damper System

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1. Introduction.

The objective of this assignment is to formulate, solve, and analyze the differential equations of a mass spring damper system using parameters provided for the individual variant. Both rotational and translational forms of the system are investigated. Analytical and numerical methods are applied, and the results of several numerical integration techniques are compared with the analytical solution. All derivations follow the procedure described in lecture 1.pdf using the Lagrange formalism.

2. System Description and Parameters.

(Tab.1: Given parameters of the mass–spring–damper system (Variant 1))

Parameter	Symbol	Value	Unit	Description
Description	m	0.3	kg	Translating body
Spring constant	k	7.4	N/m or N·m/rad	Linear or torsional stiffness
Damping coefficient	b	0.04	N·s/m or N·m·s/rad	Linear or torsional damping
Length (for pendulum)	l	0.69	m	Distance from pivot to mass center
Initial angle	θ_0	−0.81684	rad	Pendulum deflection
Initial displacement	x_0	0.93	m	Spring displacement

3. Derivation of the Governing Equations.

3.1 Variant 1 — Rotational (Pendulum with Torsional Spring–Damper)

Generalized coordinate: θ $\quad I\ddot{\theta} + b\dot{\theta} + k\theta + mgl\sin\,\theta = 0, I = ml^2$

This nonlinear equation results from

$$K = \frac{1}{2}ml^2\dot{\theta}^2, P = mgl(1 - \cos\,\theta) + \frac{1}{2}k\theta^2, D = \frac{1}{2}b\dot{\theta}^2.$$

Linearized form (small angles): $I\ddot{\theta} + b\dot{\theta} + (k + mgl)\theta = 0.$

3.2 Variant 2 — Translational (Mass–Spring–Damper)

Generalized coordinate: x $\quad m\ddot{x} + b\dot{x} + kx = 0.$

This is a classical second-order linear ODE describing a damped harmonic oscillator.

4. Analytical Solution.

For the linear system, define $\omega_n = \sqrt{\frac{k}{m}}, \zeta = \frac{b}{2\sqrt{km}}, \omega_d = \omega_n\sqrt{1-\zeta^2}$.

With $x(0) = x_0$and $\dot{x}(0) = 0$, $x(t) = e^{-\zeta\omega_n t}[x_0\cos(\omega_d t) + \frac{\zeta\omega_n x_0}{\omega_d}\sin(\omega_d t)]$.

Numerically, $\omega_n = 4.966$ rad/s, $\zeta = 0.0134$, $\omega_d = 4.965$ rad/s.

For Variant 1 (small-angle approximation):

$$I = ml^2 = 0.14283, \omega_{n,\theta} = \sqrt{\frac{k + mgl}{I}} = 8.13 \text{ rad/s}, \zeta_\theta = 0.0172.$$

The nonlinear equation containing $\sin\theta$has no closed-form analytical solution when damping and torsion are both present. It must be solved numerically (e.g., using RK4). However, the linearized solution is accurate for small oscillations ($|\theta_0| < 15°$); for $|\theta_0| \approx 47°$, some discrepancy is expected.

5. Numerical Integration Methods

The ODE $m\ddot{x} + b\dot{x} + kx = 0$was solved using three explicit methods with step $h = 0.01s$over $t \in [0,10]s$.

5.1 Explicit Euler

$$\begin{cases} x_{n+1} = x_n + h\,v_n, \\ v_{n+1} = v_n + h\left(-\frac{b}{m}v_n - \frac{k}{m}x_n\right) \end{cases}$$

Simple but conditionally stable; large $h$leads to rapid numerical damping and phase lag.

5.2 Implicit Euler

$$\begin{cases} x_{n+1} = x_n + h\,v_{n+1}, \\ v_{n+1} = \dfrac{v_n - (hk/m)x_n}{1 + (hb/m) + (h^2k/m)}. \end{cases}$$

Unconditionally stable but excessive artificial damping.

5.3 Runge–Kutta 4 (RK4)

Standard fourth-order algorithm:

$$
\begin{aligned}
k_1 &= f(x_n, v_n),\\
k_2 &= f\left(x_n + \frac{h}{2}v_n, v_n + \frac{h}{2}k_1\right),\\
k_3 &= f\left(x_n + \frac{h}{2}v_n, v_n + \frac{h}{2}k_2\right),\\
k_4 &= f(x_n + hv_n, v_n + hk_3),\\
v_{n+1} &= v_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),\\
x_{n+1} &= x_n + hv_n + \frac{h^2}{6}(k_1 + 2k_2 + 2k_3 + k_4),
\end{aligned}
$$

with $f(x, v) = -(b/m)v - (k/m)x$.

This scheme achieves excellent accuracy and stability for oscillatory systems.

6. Results and Comparison.

(Tab.2: Comparison of numerical integration methods and their performance)

Method	Order	Stability	Accuracy (vs Analytical)	Comments
Explicit Euler	1	Conditional	Low	Fast but numerically damped
Implicit Euler	1	Unconditional	Moderate	Stable yet overdampe
RK4	4	Stable	Very High	Matches analytical curve almost exactly

Observations

- The Explicit Euler solution lags behind the analytical curve and shows amplitude decay.
- The Implicit Euler result is more stable but over-damped, smoothing out oscillations.
- The RK4 trajectory is practically identical to the analytical solution for $h = 0.01$s.
- Reducing himproves accuracy for Euler methods but increases computation time.
- The linearized analytical and nonlinear numerical pendulum models agree closely for small times but diverge slightly due to the $\sin\theta$ nonlinearity

7. Discussion.

The experiments confirm that Lagrange-based modeling yields correct second-order ODEs describing the coupled mass, spring, and damping effects. The nonlinear rotational system lacks a closed-form solution because of the sine term; analytical progress requires small-angle linearization. In contrast, the translational system is linear, admitting an exact analytical expression that serves as a reliable benchmark.

Among numerical solvers:

- Explicit Euler is simple yet inaccurate for oscillations.
- Implicit Euler is robust but excessively damped.
- RK4 provides a superior balance of precision and efficiency.

These findings correspond to the expectations outlined in the lecture material and demonstrate the necessity of choosing numerical schemes based on both system dynamics and desired accuracy.

8. Conclusions.

The modeling and analysis performed in this work confirm that the equations of motion for both the rotational and translational mass–spring–damper systems can be accurately derived using the Lagrange formulation. The analytical solutions exist only for the linearized cases, whereas the full nonlinear pendulum equation, due to its sine term and damping, requires numerical integration. Comparisons between numerical and analytical results demonstrate that the Runge–Kutta 4 (RK4) method provides excellent accuracy and stability, closely reproducing the analytical trajectory even for relatively large time steps. In contrast, both Explicit and Implicit Euler methods exhibit noticeable numerical errors: the explicit version introduces phase lag and amplitude decay, while the implicit version adds excessive artificial damping. Overall, the study highlights that higher-order integration schemes such as RK4 or adaptive solvers like MATLAB’s ode45 are far better suited for simulating oscillatory mechanical systems, ensuring reliable agreement with theoretical predictions and validating the correctness of the derived mathematical models.