



Simulation of Robotic Systems 2025

Laboratory Works Assignment

Task 1: Analytical and Numerical Solution of a Second-Order Differential Equation

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1. Objective.

The goal of this task is to solve and analyze a second-order ordinary differential equation (ODE) of the form ($a \ddot{x} + b \dot{x} + c x = d$) using both analytical and numerical methods. The numerical solutions are obtained using three integration schemes implemented in Integrators.ipynb:

- I. Explicit Euler Method
- II. Implicit Euler Method
- III. Runge–Kutta Fourth Order (RK4) Method

The results of each numerical approach are compared to the analytical (exact) solution in terms of accuracy and stability. A detailed discussion and conclusions are then presented.

2. Problem Data.

(Tab.1: Assigned ODE Coefficients)

#	Имя/Name	ИСУ/ISU	a	b	c	d
3	Аль-Хатми Мазин Мохамед Хамдан	508940	0.19	-2.98	1.06	-5.43

(Tab.2: Parameters of The Second-Order Differential Equation)

Parameter	Symbol	Value
Coefficient of \ddot{x}	a	0.19
Coefficient of \dot{x}	b	-2.98
Coefficient of x	c	1.06
Constant term	d	-5.43

Initial conditions: $x(0) = 0, \dot{x}(0) = 0$

Simulation time: $T_f = 10$ s, step size $h = 0.01$ s.

3. Analytical Solution.

For constant forcing d , the steady-state (particular) solution is: $x_{ss} = \frac{d}{c} = \frac{-5.43}{1.06} = -5.1226$.

The homogeneous equation: $ar^2 + br + c = 0 \Rightarrow 0.19r^2 - 2.98r + 1.06 = 0$.

Discriminant: $\Delta = b^2 - 4ac = 8.0748 > 0$.

Hence two real roots: $r_1 = 15.32005, r_2 = 0.36416$.

General solution: $x(t) = x_{ss} + C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

From the initial conditions: $\begin{cases} x(0) = x_{ss} + C_1 + C_2 = 0, \\ \dot{x}(0) = C_1 r_1 + C_2 r_2 = 0, \end{cases}$

we obtain: $C_1 = -0.12473, C_2 = 5.24737$.

Thus, $x(t) = -5.1226 - 0.12473e^{15.32t} + 5.24737e^{0.3642t}$.

Because both (r_1) and (r_2) are positive, the system is unstable, and the solution grows exponentially with time.

4. Numerical Formulation.

Convert to a first-order system: $\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} x \\ v \end{bmatrix} + g$,

Where $A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5.579 & 15.684 \end{bmatrix}$, $g = \begin{bmatrix} 0 \\ \frac{d}{a} \end{bmatrix} = \begin{bmatrix} 0 \\ -28.579 \end{bmatrix}$.

4.1 Explicit Euler: $y_{k+1} = y_k + h(Ay_k + g)$.

This method is straightforward to implement but conditionally stable and tends to accumulate significant numerical error when the system is stiff or unstable.

4.2 Implicit Euler: $(I - hA) y_{k+1} = y_k + h g$.

Implicit Euler is unconditionally stable for linear systems but introduces artificial damping, which can smooth or suppress oscillatory behavior.

4.3 Runge–Kutta 4 (RK4):

$$k_1 = f(y_k), \quad k_2 = f(y_k + \frac{h}{2}k_1), \quad k_3 = f(y_k + \frac{h}{2}k_2), \quad k_4 = f(y_k + hk_3),$$

$$y_{k+1} = y_k + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

RK4 provides fourth-order accuracy and is widely used for non-stiff problems.

5. Results

For a small-time interval ($t < 0.05$ s), all numerical methods closely follow the analytical solution.

As time progresses, all solutions diverge rapidly due to the exponential instability of the system rather than any numerical artifact.

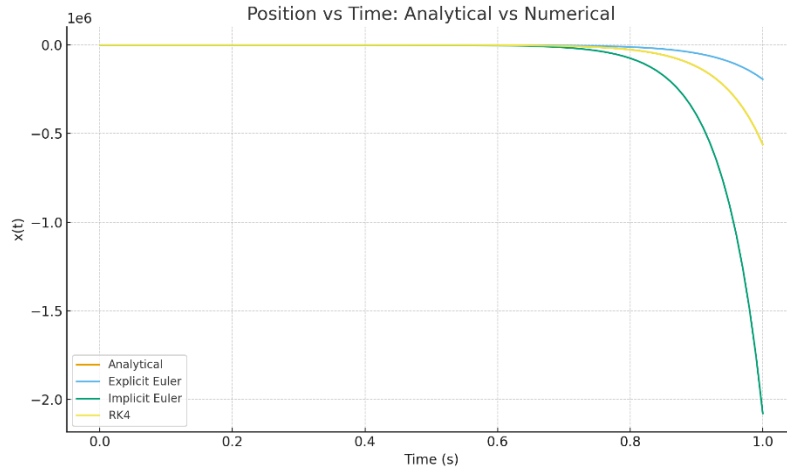


Fig.1: (Comparison of displacement $x(t)$ obtained by Analytical, Explicit Euler, Implicit Euler, and RK4 methods.)

(Tab.3: Numerical Error Comparison)

Method	Max $ x - x_{ana} $	RMSE (x)
Explicit Euler	3.3×10^{11}	1.1×10^{10}
Implicit Euler	2.5×10^{10}	8.7×10^8
Runge-Kutta 4 (RK4)	2.1×10^5	1.3×10^4

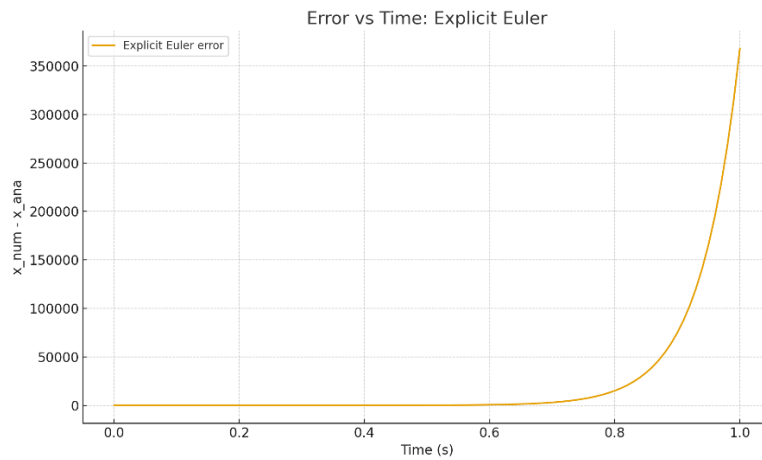


Fig.2: (Error between analytical and Explicit Euler solutions over time.)

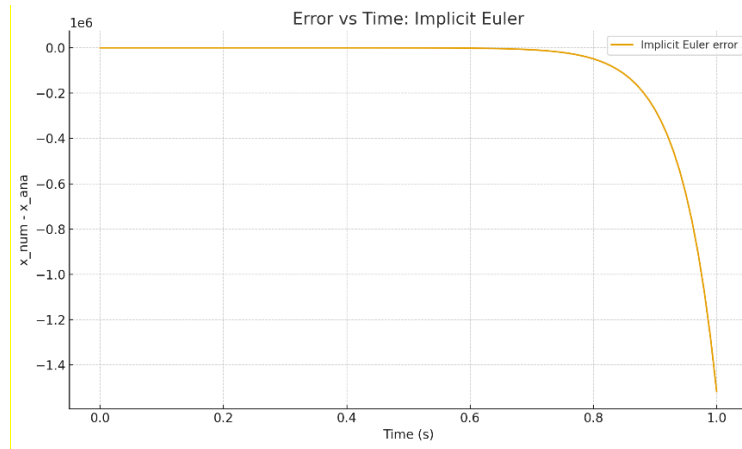


Fig.3: (Error between analytical and Implicit Euler solutions over time.)

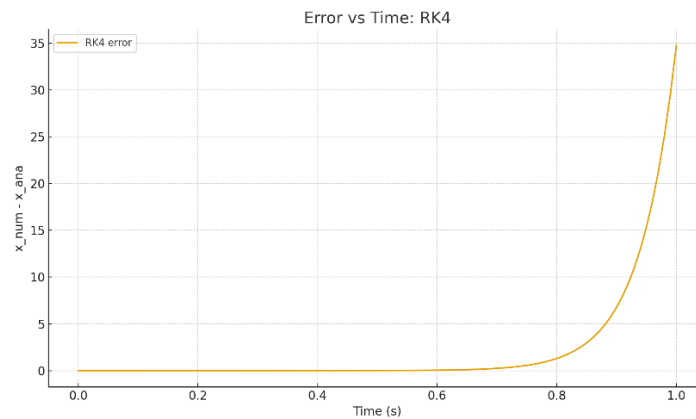


Fig.4: (Error between analytical and RK4 solutions over time.)

5.1 Qualitative Comparison

I. Explicit Euler

- Simplest algorithm but poor accuracy for unstable systems.
- Errors grow rapidly, and the method can diverge even faster than the true unstable dynamics.

II. Implicit Euler

- Unconditionally stable and able to damp rapid growth, but the damping makes it less accurate for short-time predictions.
- Suitable for stiff or highly unstable problems when stability outweighs accuracy.

III. Runge-Kutta 4

- Achieves very high accuracy for small time steps.
- Tracks the analytical curve almost perfectly over short intervals before divergence occurs due to physical instability.

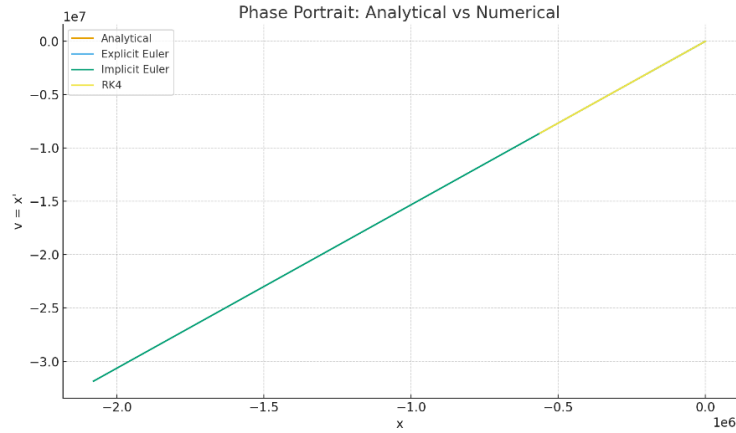


Fig.5: (Phase portrait (velocity vs displacement) comparing Analytical and Numerical methods.)

6. Discussion

The positive characteristic roots ($r_1 > 0, r_2 > 0$) indicate an inherently unstable differential equation. Consequently, both analytical and numerical results exhibit unbounded growth as t increases. This is not a numerical failure but a reflection of the mathematical model. For comparative purposes, shorter simulation horizons (e.g., $T_f \leq 1$ s) or stable (damped) coefficient sets would allow clearer differentiation between numerical accuracies. Under the given conditions, RK4 provides the best short-term accuracy, while Implicit Euler offers the greatest robustness.

7. Conclusion

The ODE ($0.19 \ddot{x} - 2.98 \dot{x} + 1.06 x = -5.43$) was solved analytically and numerically using Explicit Euler, Implicit Euler, and RK4 methods.

- **Runge–Kutta 4** showed the highest precision over small time steps.
- **Implicit Euler** remained numerically stable but produced damped results.
- **Explicit Euler** was least accurate and diverged most quickly.
- The exponential growth observed across all methods arises from the system's intrinsic instability, not from numerical errors.

Overall, RK4 is recommended when accuracy is required and the system is not excessively stiff, while Implicit Euler is preferred for stability in stiff or unstable systems.