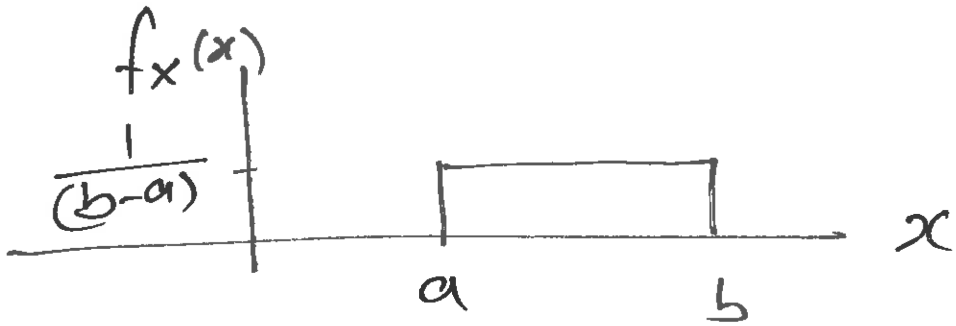
Families of Continuous RVs

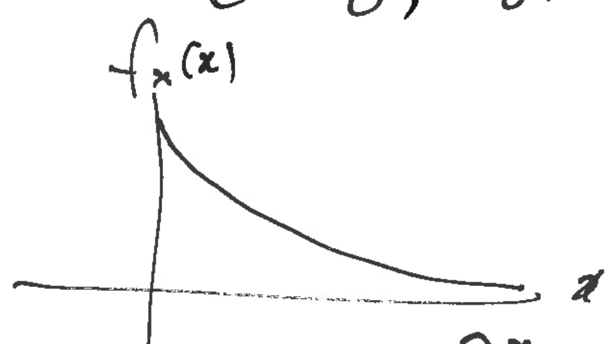
1. ~~Uni~~ Continuous Uniform (a, b)



$$\mu_X = \frac{a+b}{2}, \quad \text{var}[X] = \frac{(b-a)^2}{12}$$

2. Exponential (λ)

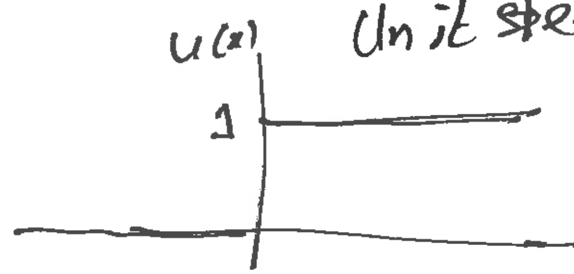
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherw.} \end{cases}$$



$$f_X(x) = \lambda e^{-\lambda x} u(x)$$

$\downarrow$

Unit step



$$\mu_X = \frac{1}{\lambda}$$

$$\text{Var}[X] = \frac{1}{\lambda^2}$$

3. Gaussian RV → Normal RV

4.6 (3.5)

Not^n:

$f_X(x)$

If x is Normal

$N(\mu, \sigma)$

μ → Mean of x

σ → Std. Dev.^e of X

*

pdf:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

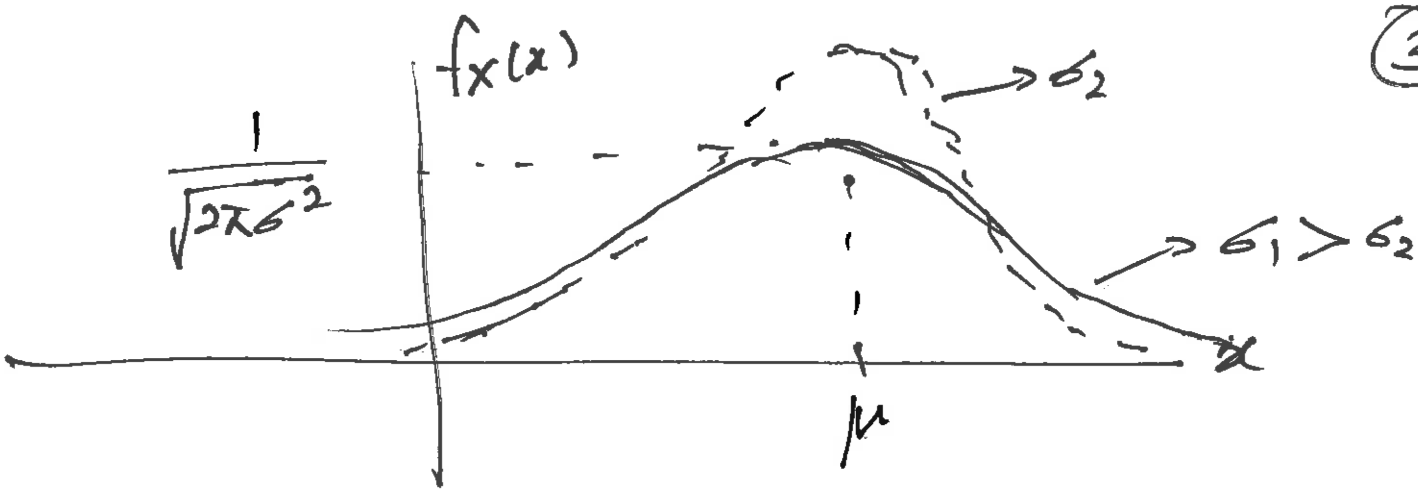
→ for all x

X can take any real value from $-\infty$ to $+\infty$

Ex → Thermal, Noise in Comm. Systems

↓

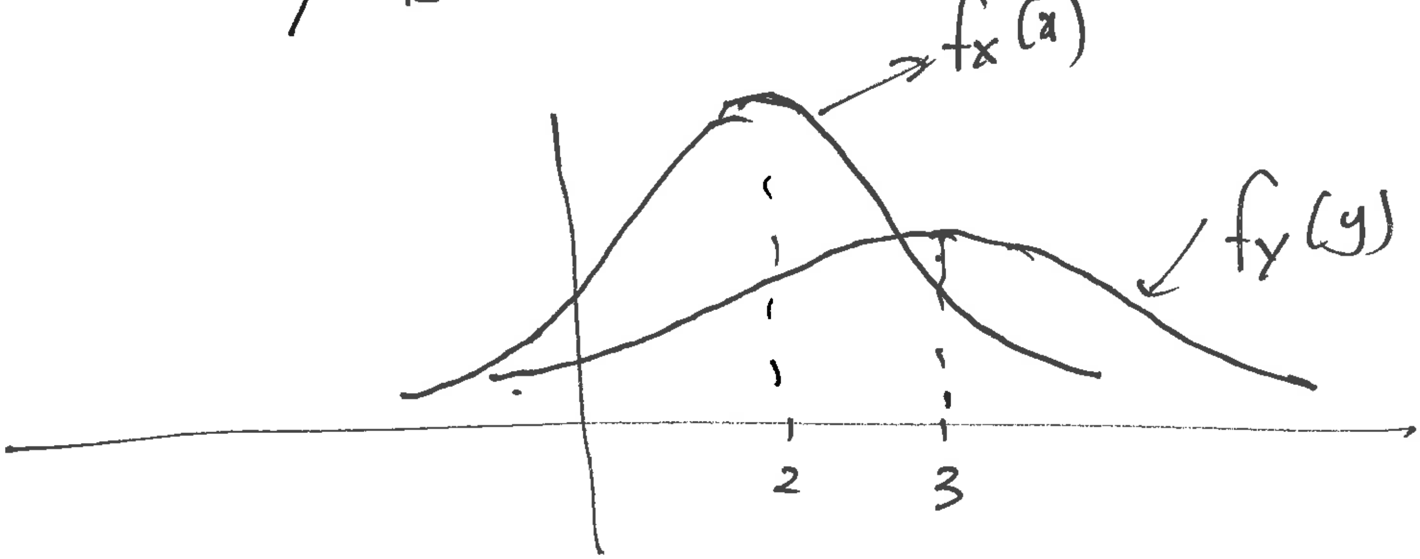
Channel Noise.



$f_X(x)$ is Max^m at $x = \mu$
" " even symmetric around $x = \mu$

$f_X(x) \to 0$ as $x \to -\infty$ & $x \to +\infty$

eg:- X is $N(2,1)$
Y is $N(3,2)$



Area under $f_X(x) = 1 \rightarrow$ for any ④
(μ, σ) Combinat

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$$

$\mu = 0, \sigma = 1$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu$$

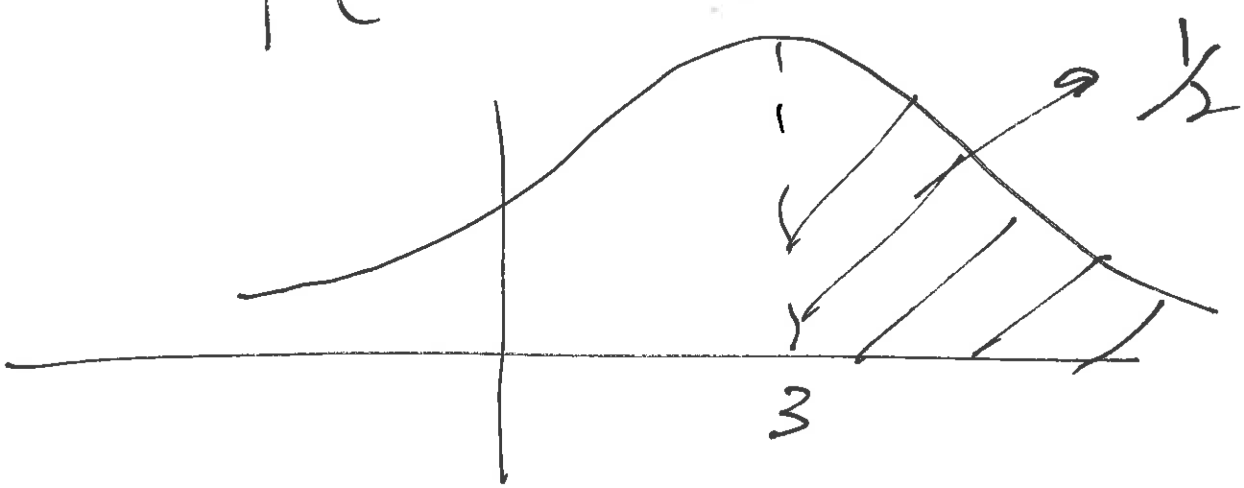
$$\int x f_X(x) dx = E[X] = \mu_X$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = E[X^2] = (\mu^2 + \sigma^2)$$

$$\underbrace{Var[X]}_{\sigma_x^2} = E[X^2] - \mu_x^2 \longrightarrow E[x^2] = \mu_x^2 + \sigma_x^2$$

eg:- X is $N(3,2)$

$$P[X \geq 3] = \tfrac{1}{2}$$



$$P[X < 3] = \tfrac{1}{2}$$

$$P[X > 4] = ?$$

$$= \int_4^\infty \frac{1}{\sqrt{2\pi(2)^2}} e^{-\frac{(x-3)^2}{2(4)}} dx$$

$\downarrow$

Need numerical integration

In general X is $N(\mu, \sigma)$

$$P[X > a], \quad a \neq \mu$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

If we are to tabulate the probs.

~~we need~~ Direct Method:

use 3 parameters μ, σ & a

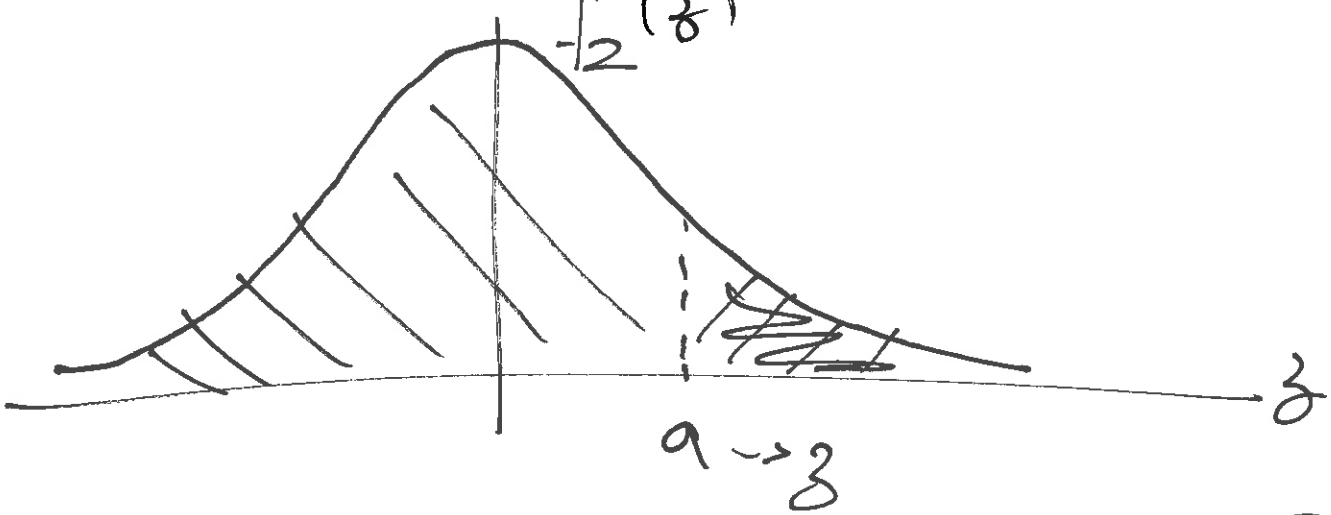
Can be simplified

Choose one simple combination of μ & σ

Choose. $\mu = 0$, $\sigma = 1$

$N(0, 1) \longrightarrow$ is called the Standard Normal RV $\rightarrow$ Denoted by Z

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$



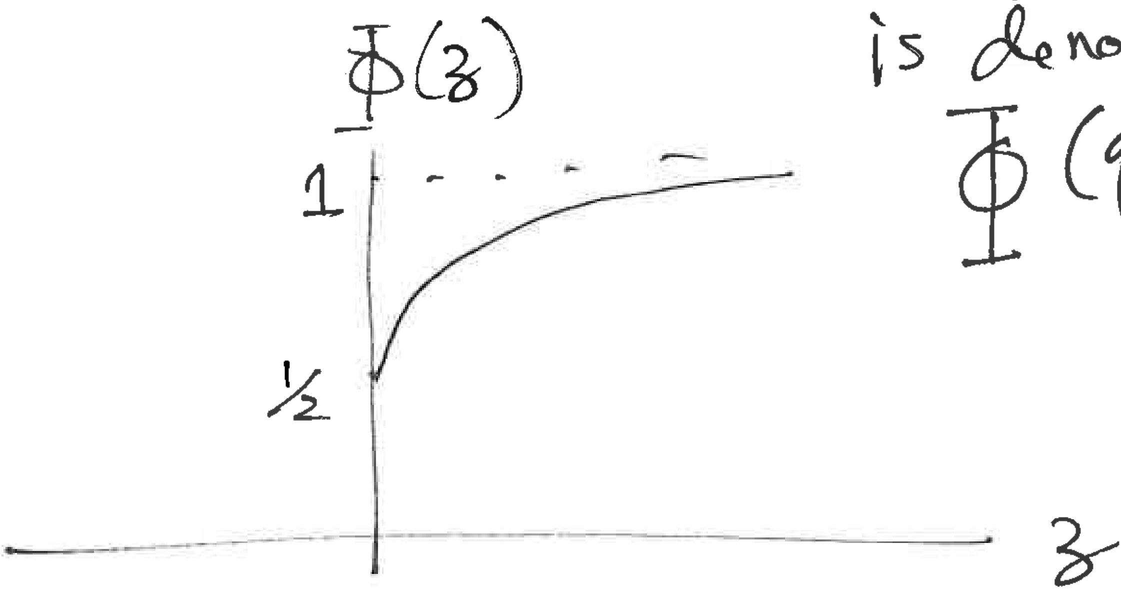
~~$P[Z > a]$~~ Tabulate $P[Z \leq a]$

$F_Z(a) \longrightarrow$ CDF

is denoted by

$\Phi(a)$

$a \to z$



Note $\Phi(0) = \frac{1}{2}$ $\quad \Phi(\infty) = 1$

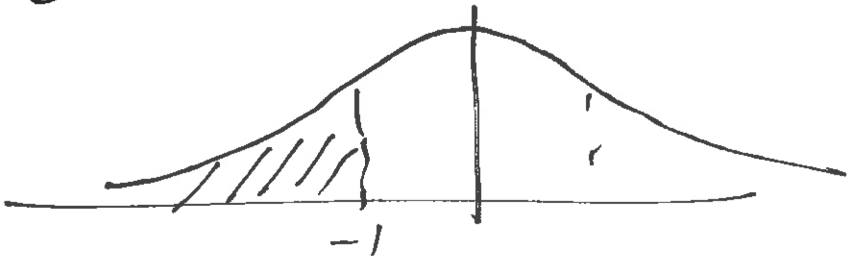
$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{\cancel{z}u^2}{2}}\, d\cancel{z}u$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du$$

$$= \frac{1}{2} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-\frac{u^2}{2}}\, du}_{\text{ready for Numerical integration}}$$

$$P[Z \cancel{<} \le -1] = \Phi(-1) \quad ?$$

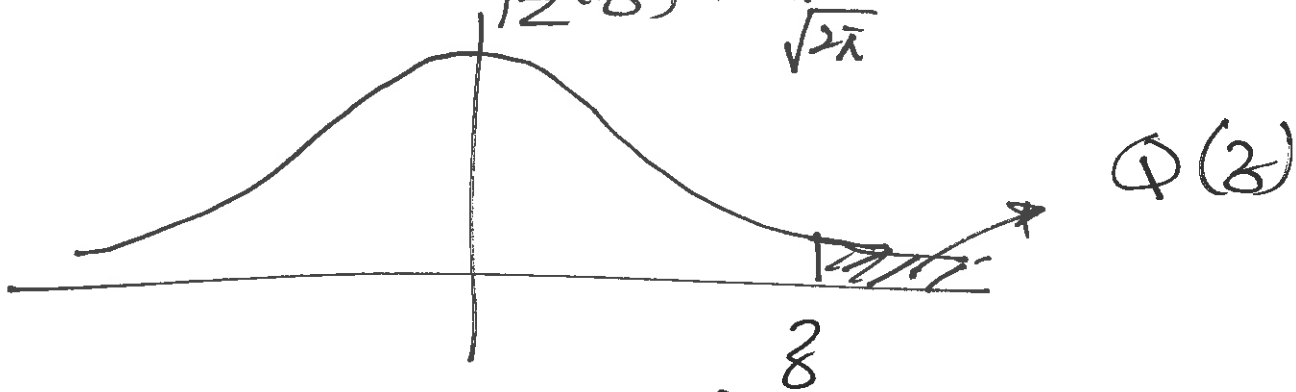
$\hookrightarrow$ is not on the Table



$\Phi(z)$ is used for moderate values of z

For larger values of z

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

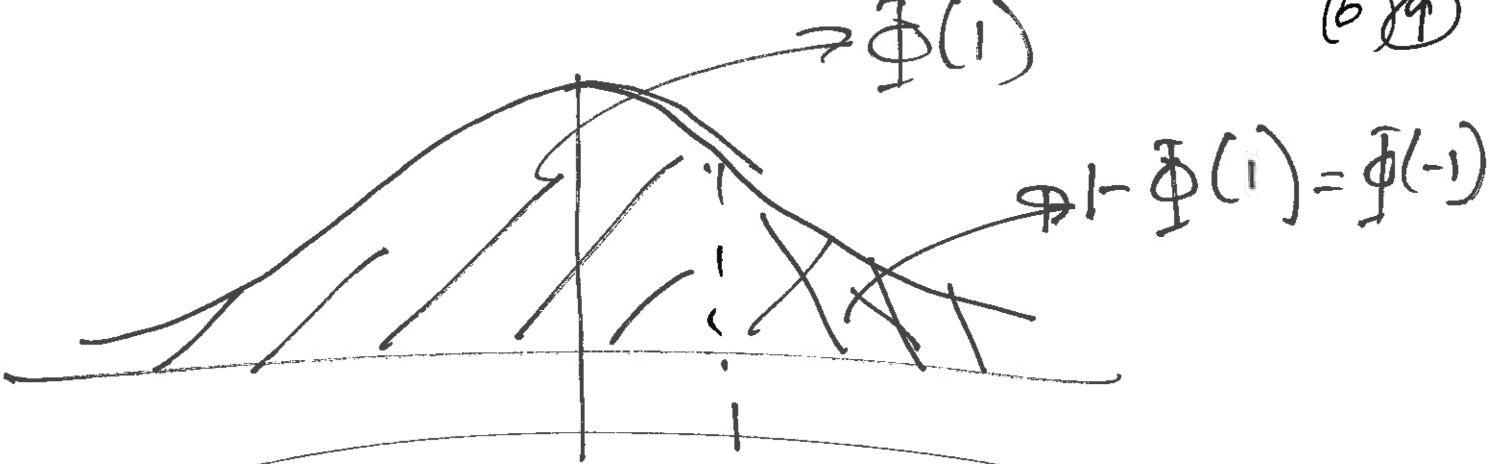


$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{u^2}{2}} du$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-\frac{u^2}{2}} du$$

Ready for Numerical Integration

6 9



$$\Phi(-z) = 1 - \Phi(z)$$

eg:- X is $N(\mu, \sigma)$ 10

$P[X > a]$

Find the relationship betⁿ X & Z

$$* \quad \boxed{Z = \frac{X - \mu}{\sigma}}$$

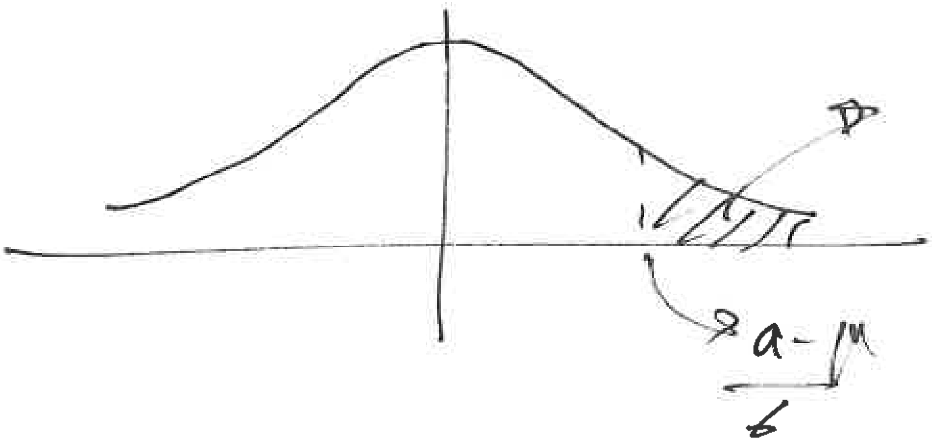
Exp No	X	Z
1		
2		
.		
.		
.	$N(\mu, \sigma)$	$N(0,1)$

$$\mu_Z = E[Z] = E\left[\frac{X - \mu}{\sigma}\right]$$

$$= \frac{1}{\sigma}\left[\underbrace{E[X]}_{\mu} - \mu\right] = 0$$

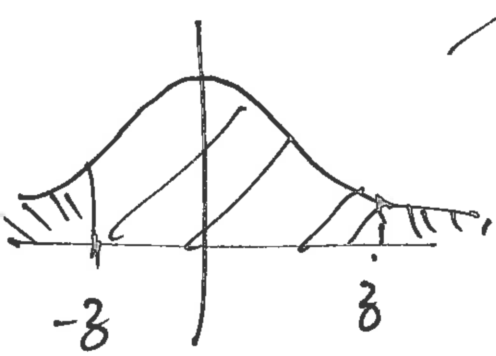
$$P[X > a] = P\left[Z > \frac{a-\mu}{\sigma}\right]$$

$$=$$

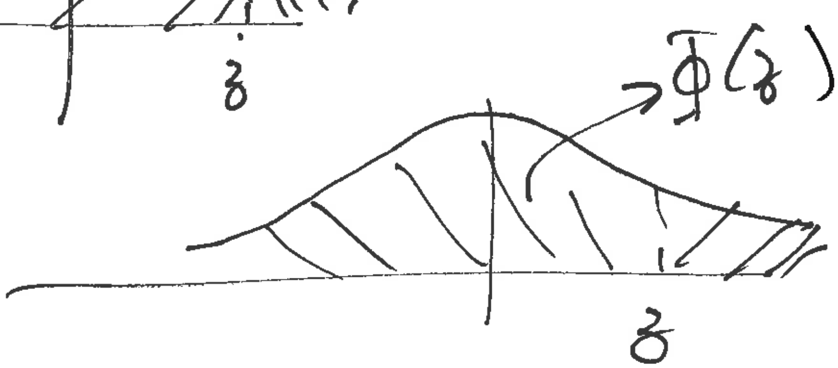


$$\Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$\Phi(-z) = 1 - \Phi(z)$$



$$\Phi(-z) = 1 - \Phi(z)$$



eg:- X is $N(\underset{\mu}{3}, \underset{\sigma}{2})$

$$P[X^2 > 4]$$ → bring down to x

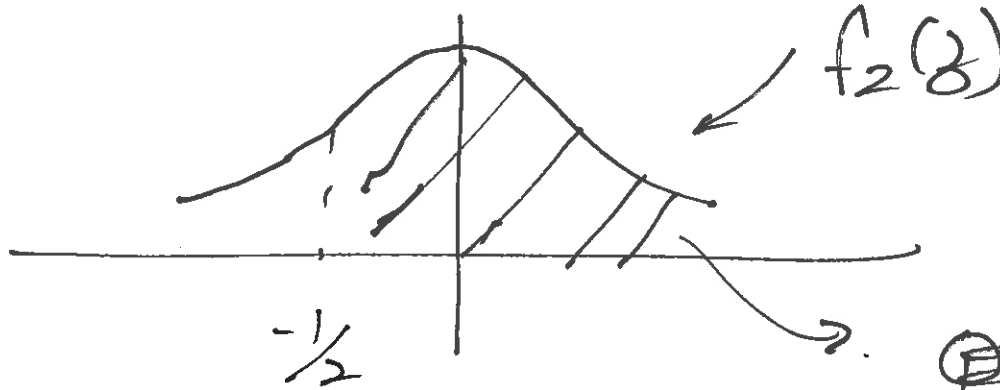
$$= P[\underbrace{X > 2 \text{ or } X < -2}_{\text{Union mutually Exclusive}}]$$

$$= P[x > 2] + P[x < -2]$$

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 3}{2}$$

$$P[x > 2] = P\left[Z > \frac{-1}{2}\right]$$

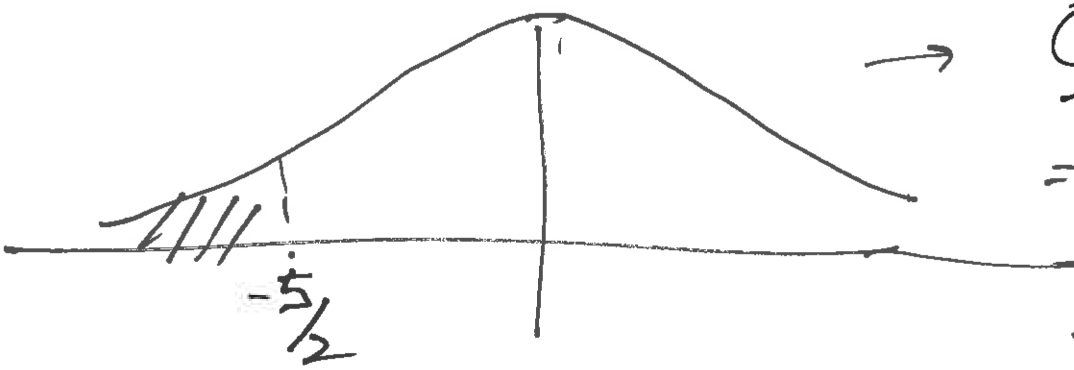
13



~~$\Phi(z$~~

$$\phi(-1/2) = 1 - \phi(1/2) = \Phi(1/2)$$

$$P[X < -2] = P\left[Z < -\frac{5}{2}\right]$$



$$\Phi(-5/2) = 1 - \Phi(5/2) = \phi(5/2)$$

$$\therefore P[X^2 > 4] = \Phi(1/2) + \phi(5/2)$$

eg:- X is $N(3, 2)$

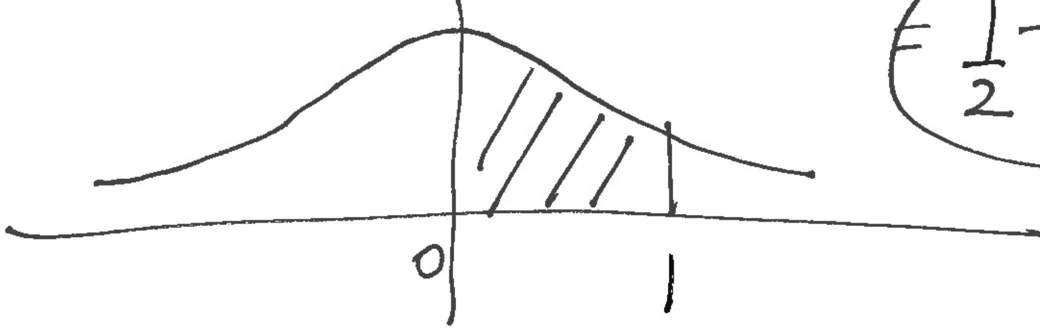
$$P\left[\,|X-4| < 1\,\right]$$

$$= P\left[-1 < X-4 < 1\right]$$

$$= P\left[3 < X < 5\right]$$

$$Z = \frac{X-3}{2}$$

$$= P\left[0 < Z < 1\right]$$

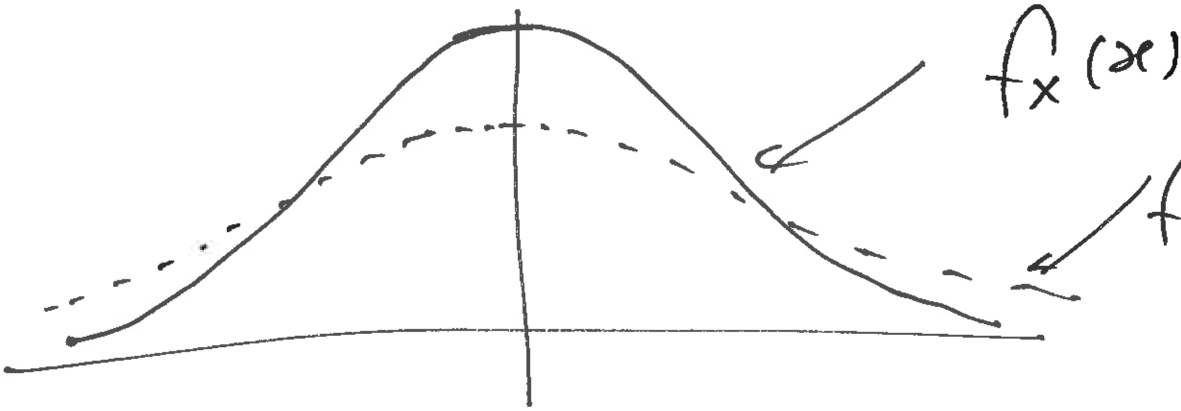


$$= \frac{1}{2} - \Phi(1)$$

Q 4.6 (3.5) 14

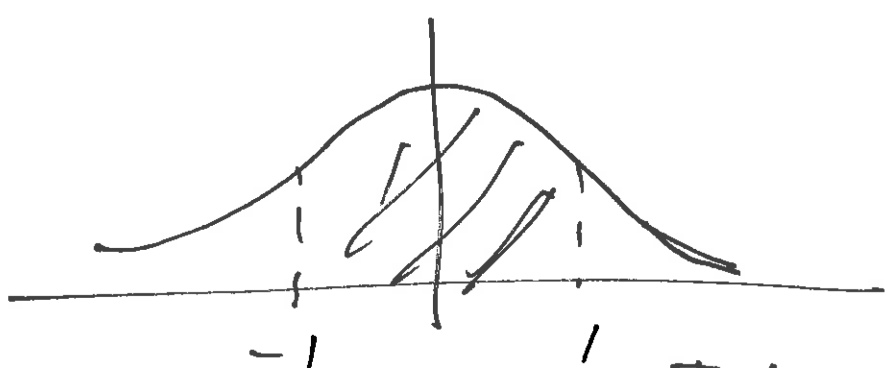
(b)

X is $N(0,1)$ $\quad Y$ is $N(0,2)$



$$P[-1 < X < 1] = P[|X| < 1]$$

$$Z = X \qquad = P[-1 < Z < 1]$$



$$\Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)]$$

Or $\quad 1 - 2\Phi(1)$

$P[-1 < Y < 1]$

$Z = \frac{Y - 0}{2} = \frac{Y}{2}$

$P\left[-\frac{1}{2} < Z < \frac{1}{2}\right] = 1 - 2\Phi(\frac{1}{2})$

4.7 (3.6) Mixed RVs

In ch. 3 (ch. 2) $\longrightarrow$ Discrete RVs

Sofar in ch. 4 (ch. 3) $\rightarrow$ Continuous RVs

Lifetime (T) of a machine manufactured by a Compan

If the machine is Defective $T = 0$.

If Good $T \rightarrow$ Continuous

If Defective $T = 0$

$P[T=0] \neq 0$.

T is not a continuous RV

part of it is continuous
" " " Discrete

Mixed

Continuous $\longrightarrow$ $F_X(x)$, $f_X(x)$

Discrete $\longrightarrow$ $F_X(x)$, $P_X(x)$

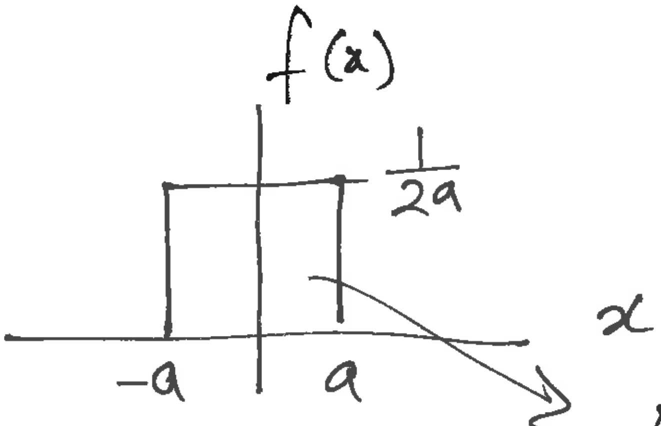
Mixed $\longrightarrow$ $F_X(x)$, $f_X(x)$

find a way to express $f_X(x)$ of the Discrete part.

Delta Func^{ns} (Impulse func^{ns}) ⑰

→ Dirac Delta

Not^{n} : $\delta(x)$



Area = 1

As $a \to 0$

$$\lim_{a \to 0} f(x) = \delta(x)$$

1. $\delta(x) = 0$, for all $x \neq 0$

2. $\delta(0) \longrightarrow \infty$

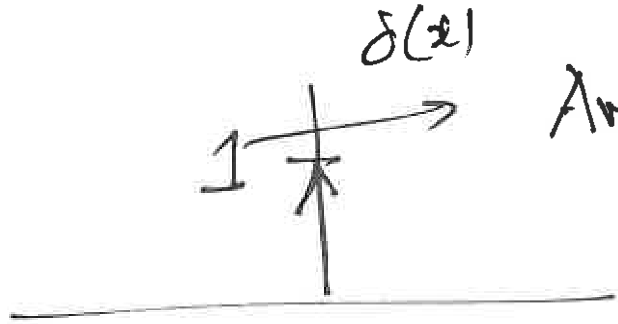
3. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

$$\int_{0_-}^{0_+} \delta(x)\, dx = 1$$

$0_+ \to 0 + \varepsilon$, $\varepsilon \to 0$

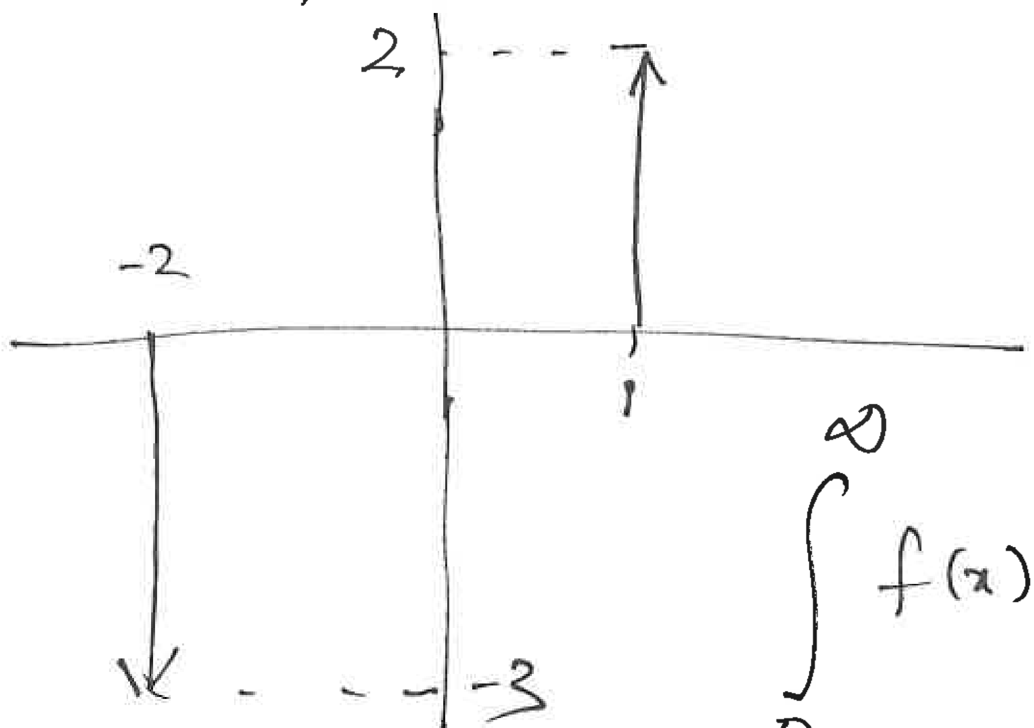
$0_- \to 0 - \varepsilon$

Graphical Representation of $\delta(x)$



Area $\rightarrow$ also called the strength of the Delta

eg:- $f(x) = 2\delta(x-1) - 3\delta(x+2)$

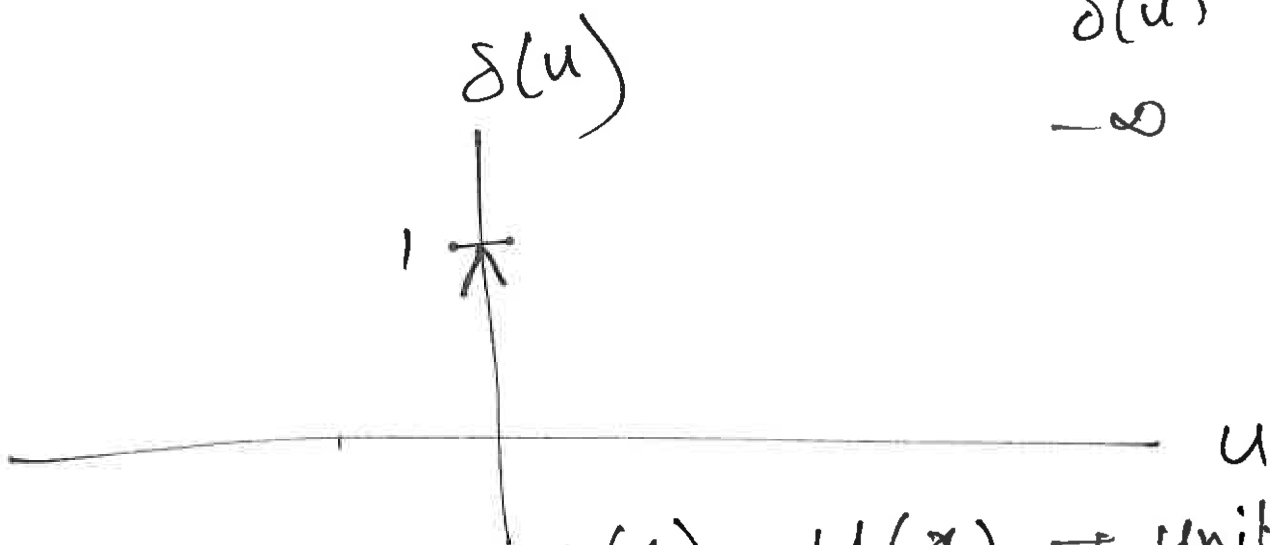


$$\int_0^\infty f(x)\,dx = 2$$

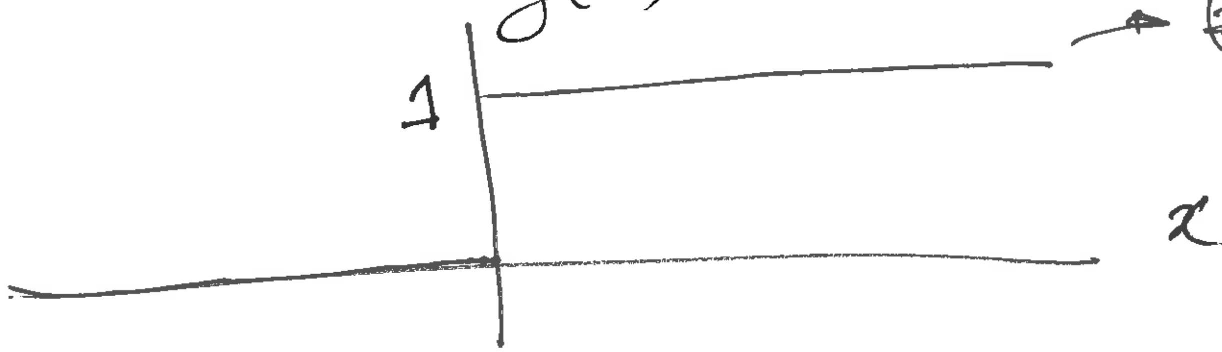
$$\int_{-\infty}^0 f(x)\,dx = -3$$

$$g(x) = \int_{-\infty}^{x} \delta(u)\, du$$

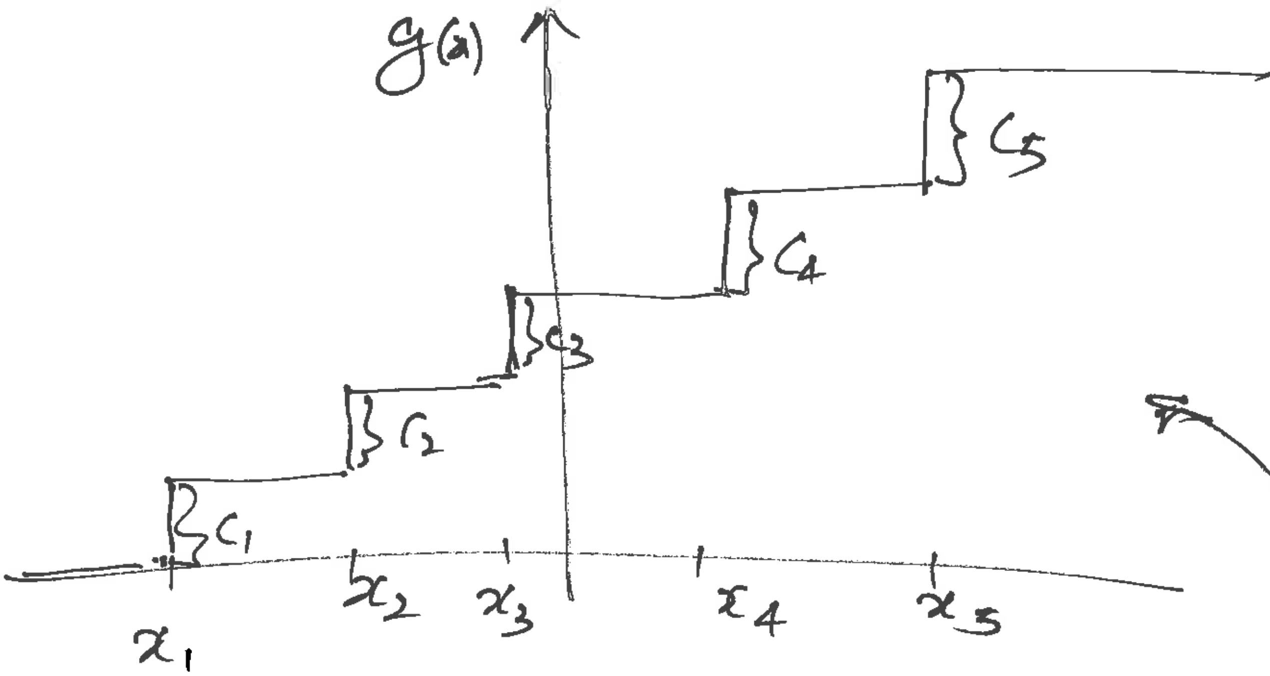
→ Area under $\delta(u)$ from $-\infty$ to x

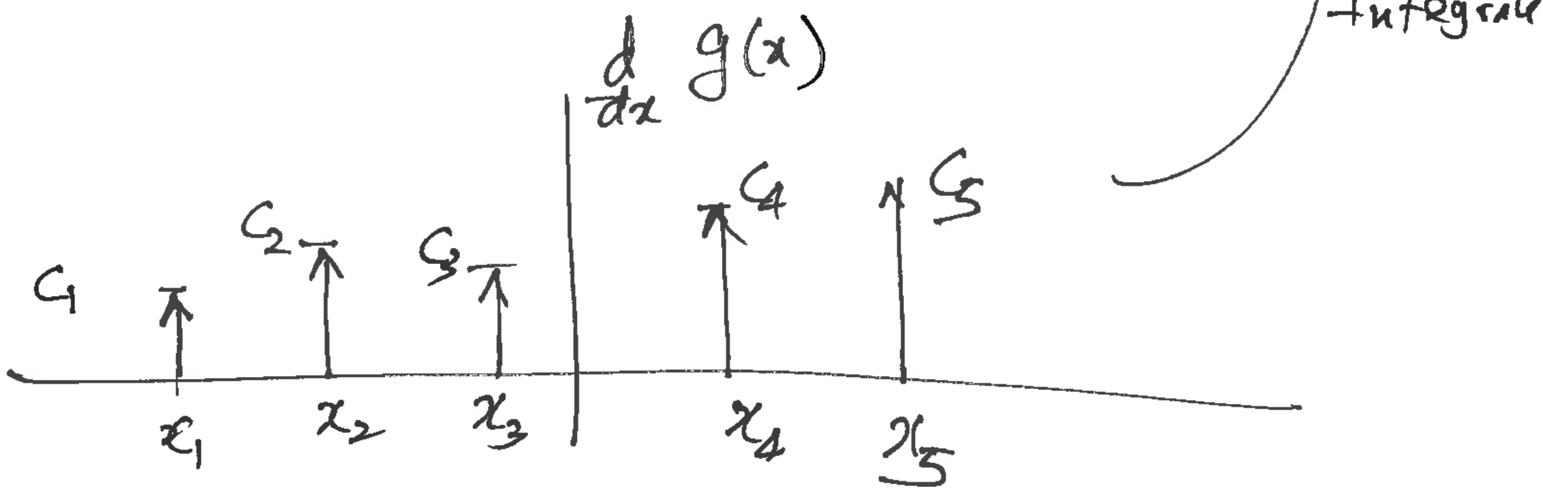


$g(x) = U(x)$ → unit step



$$\frac{d}{dx} U(x) = \delta(x)$$



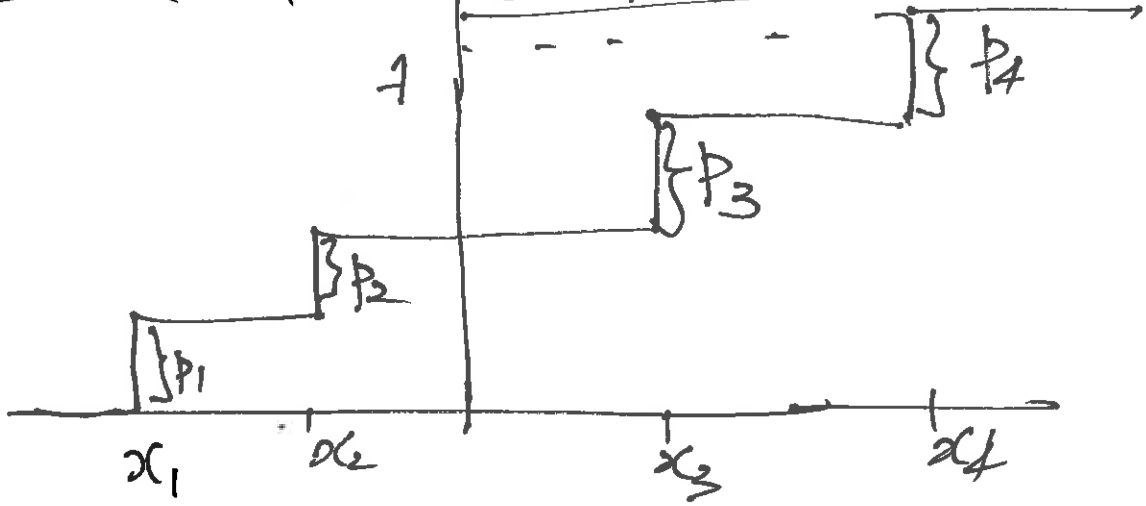


Note: $g(x)$ is similar to the CDF of a Discrete RV

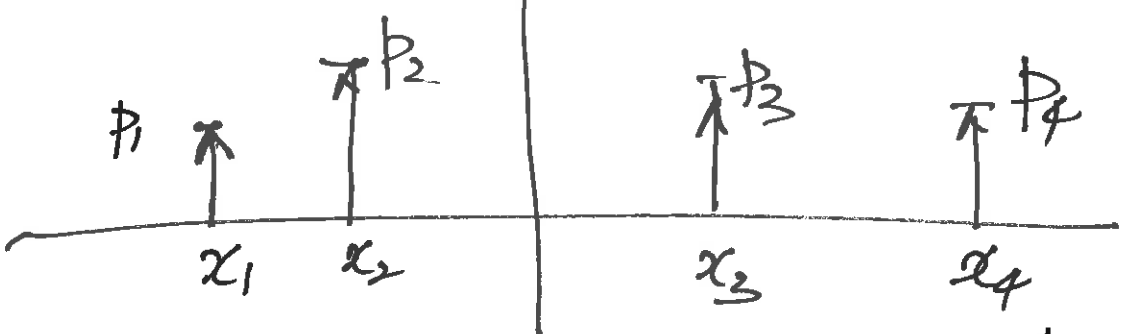
$\therefore \frac{d}{dx} g(x) \rightarrow$ in Delta fuc^{ns}

corresponds to the pdf $f_X(x)$

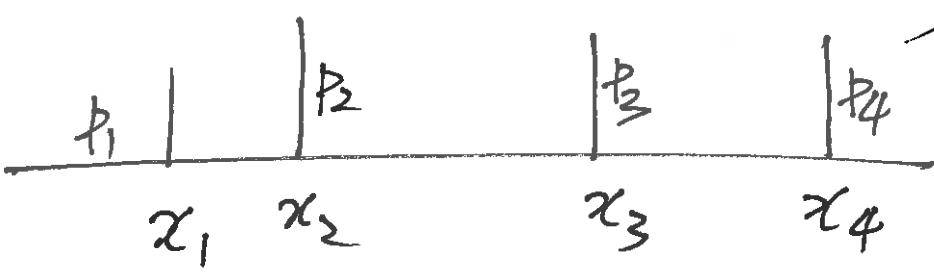
X is a discrete RV → $F_X(x)$



$f_X(x)$ → pdf



$P_X(x)$



lines become Delta fucns on $f_X(x)$

eg:-

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2} + \frac{1}{2}x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

$f_X(x) = \ ?$