If x & y are independent

7/28
①

$$f_{x,y}(x,y) = f_x(x) \cdot f_y(y)$$

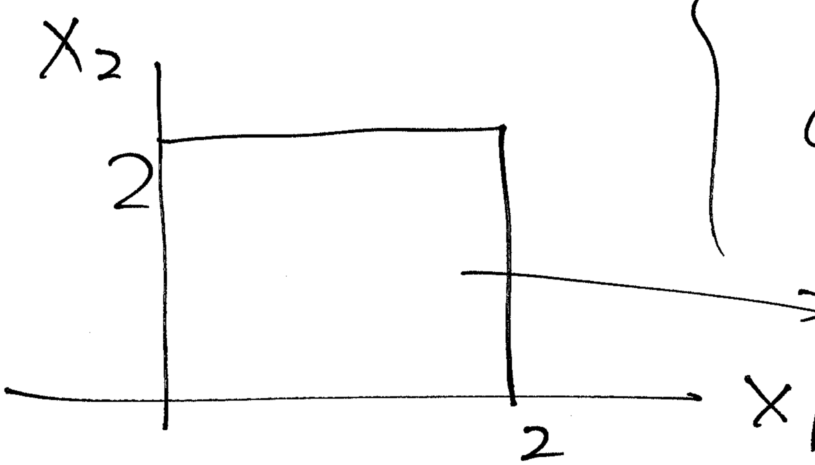
$$P[x \le a \;\; \& \;\; y \ge b] = P[x \le a] \cdot P[y \ge b]$$

Q 4.10
(B) $X_1, X_2 \rightarrow$ Independent & Identically distributed (iid)

$$f_x(x) = \begin{cases} 1 - x/2, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{x_1,x_2}(x_1,x_2) = f_{x_1}(x_1) \cdot f_{x_2}(x_2)$$

$$= \begin{cases} \left(1 - \frac{x_1}{2}\right)\left(1 - \frac{x_2}{2}\right), & 0 \le x_1 \le 2, \; 0 \le x_2 \le 2 \\ 0, & \text{oth} \end{cases}$$



$\rightarrow$ pdf $= \left(1 - \frac{x_1}{2}\right)\left(1 - \frac{x_2}{2}\right)$

$Z = \text{Max}(X_1, X_2) \rightarrow$ Z varies 0 to 2 ②

CDF of Z, $F_Z(z) = P[Z \le z]$

$$= P[X_1 \le z \ \& \ X_2 \le z]$$

$$= P[X_1 \le z] \cdot P[X_2 \le z]$$

$$= \left[\int_{-\infty}^{z} f_{X_1}(x_1)\, dx_1 \, dx_1 \right]^2$$

$$= \left[\int_{0}^{z} \left(1 - \frac{x_1}{2}\right) dx_1 \right]^2, \quad 0 \le z \le 2$$

$\vdots$

$=$ ✓ (in z)

4.11 Bivariate Gaussian RVs (Jointly Gaussian RVs)

If x is Gaussian $\rightarrow$ X is $N(\mu, \sigma)$.

If X & Y are jointly Gaussian

$f_{X,Y}(x,y) \rightarrow$ follows the pdf Eq. (4.145)

5 parameters.

$\underbrace{\mu_1, \sigma_1}_{\text{of } X}$; $\underbrace{\mu_2, \sigma_2}_{\text{of } Y}$; ρ $\rightarrow$ Correlation coeff

If X & Y are jointly Gaussian

$N(\underbrace{\mu_1, \mu_2}_{\text{Mean values}} ; \underbrace{\sigma_1, \sigma_2}_{\text{std. Devs.}} ; \rho)$ $\rightarrow$ Correlation

If X & Y are jointly Gaussian.

1. X is $N(\mu_1, \sigma_1)$

Y is $N(\mu_2, \sigma_2)$

Correlation coeff. $\rho_{X,Y} = \rho$

2. $f_{X|Y}(x|y)$ is also Normal.

$N(\tilde{\mu}_1(y), \tilde{\sigma}_1)$

$$\tilde{\mu}_1(y) = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y - \mu_2).$$

$$\tilde{\sigma}_1^2 = \sigma_1^2(1-\rho^2)$$

3. $f_{Y|X}(y|x)$ is also Normal

$N(\tilde{\mu}_2(x), \tilde{\sigma}_2)$

$$\tilde{\mu}_2(x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1)$$

$$\tilde{\sigma}_2^2 = \sigma_2^2(1-\rho^2)$$

ex:- X and Y are jointly Gaussian with joint pdf

$$N(1, 2;\ 1, 2;\ 0.5)$$

Find (a) $P[X > 3]$

(b) $P[X > 3 \mid Y = 3]$

X is $N(1, 1)$ Y is $N(2, 2)$

$\rho = 0.5$

(a) $Z = \dfrac{X - \mu}{\sigma}$

$Z = \dfrac{X - 1}{1}$

$P[X > 3] = P[Z > 2] = Q(2)$

$$P[X > 3 \mid Y = 3]$$

$$f_{X|Y}(x \mid 3) \text{ is } N(\tilde{\mu}_1, \tilde{\sigma}_1)$$

$$\tilde{\mu}_1 = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y - \mu_2)$$

$$= 1 + 0.5 \left(\frac{1}{2}\right)(3 - 2)$$

$$\tilde{\sigma}_1^2 = \sigma_1^2 (1 - \rho^2)$$

$$= (1)(1 - 1/4)$$

$$N(\tilde{\mu}_1, \tilde{\sigma}_1)$$

$$P[X > 3 \mid Y = 3]$$

Q 4.11

(a) X is $N(0,1)$ $\qquad$ X & Y are jointly Gaussi.

Y is $N(0,1)$

$\rho = 0.5$

Joint pdf of X & Y is

$N(0, 0 ; 1, 1 ; \frac{1}{2})$

(b) $f_{X|Y}(x|2)$ is $N(\tilde{\mu}_1, \tilde{\sigma}_1)$

$y=2.$

use expressions.

$$\tilde{\mu}_1 = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y - \mu_2)$$

$$= 0 + \frac{1}{2}(1)(2-0)$$

$$\tilde{\sigma}_1^2 = \sigma_1^2(1-\rho^2)$$

$$= 1(1 - \frac{1}{4})$$

If X & Y are jointly Gaussian & uncorrelated $\rightarrow \rho = 0$.

$N(\mu_1, \mu_2 ; \sigma_1, \sigma_2 ; 0)$

From the joint pdf, it follows

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi \sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}$$

$$= f_X(x) \cdot f_Y(y)$$

X & Y are independent.

* If X & Y are jointly Gaussian & uncorrelated then they are independent too.

— Not necessarily true for any 2 RVs.

6.2, 6.6, 6.7

Sum of 2 RVs

X, Y are 2 independent RVs

$W = X + Y$

pdf of W ?

CDF of W, $F_W(w) = P[W \le w]$

$$= P[X + Y \le w]$$

$$= P[Y \le w - X]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_{X,Y}(x, y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_X(x) \cdot f_Y(y)\, dy\, dx.$$

pdf $f_W(w) = \frac{d}{dw} F_W(w)$

$\vdots$

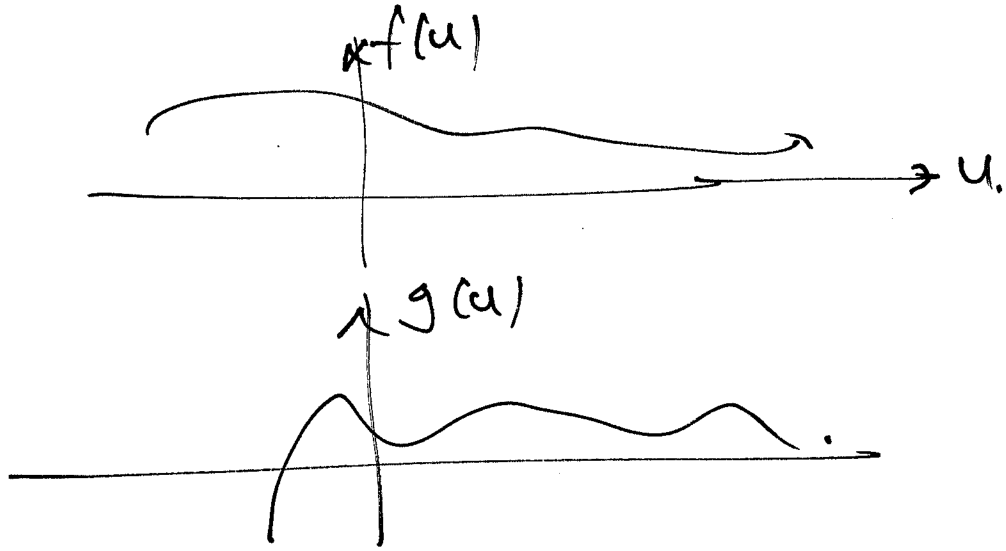
$$f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_{\cancel{W}}(w-x)\, dx$$

$\Downarrow$

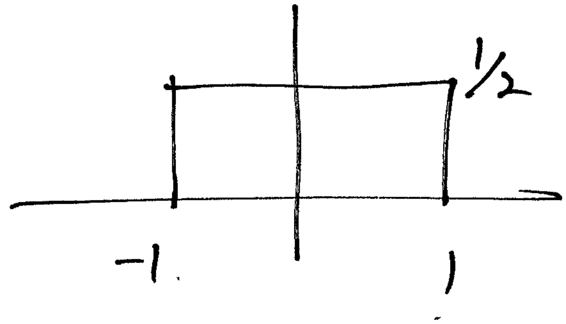
Convolution integral

$(*)$

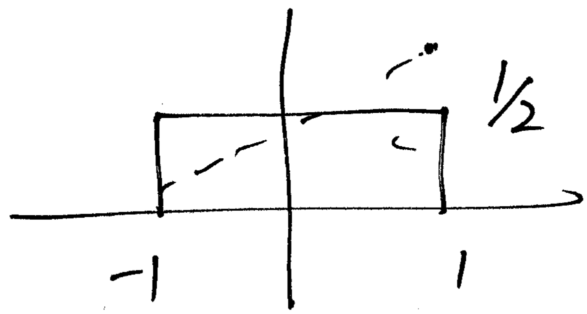
$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$



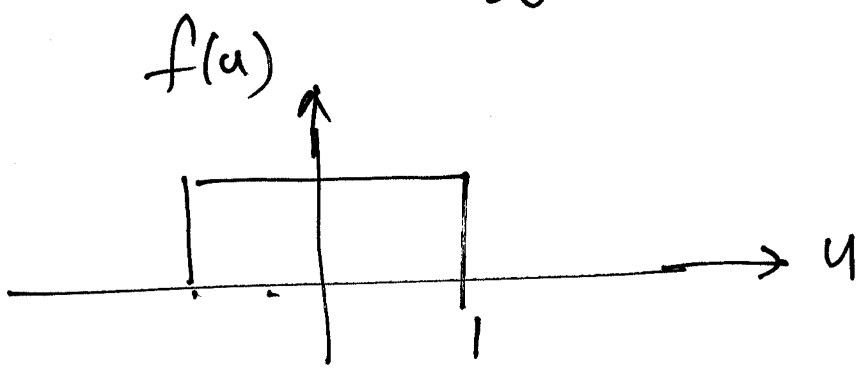
eg:- $f(x)$



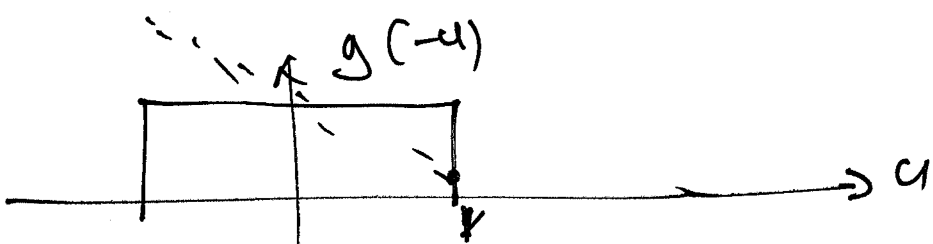
$g(x)$

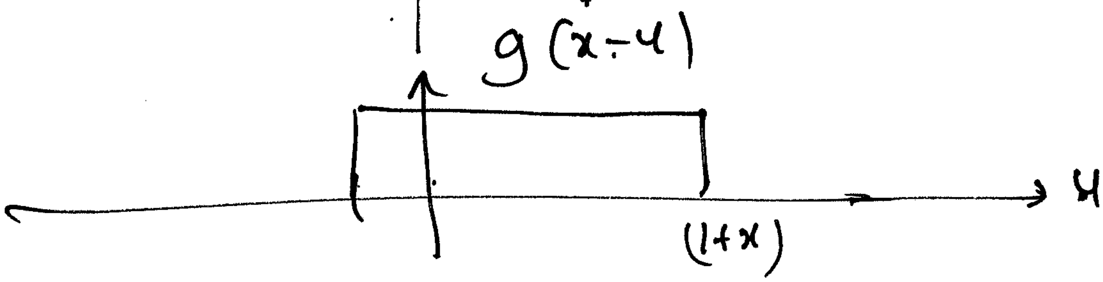


$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$



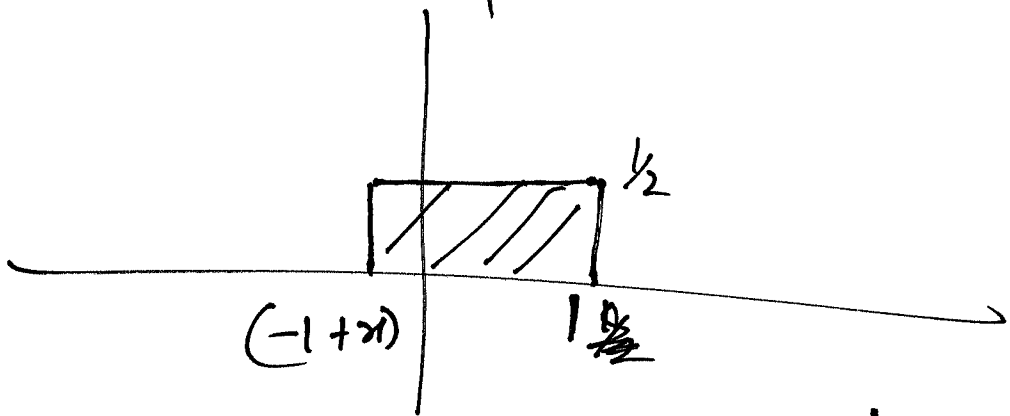
1





2

$f(u)\, g(x-u)$

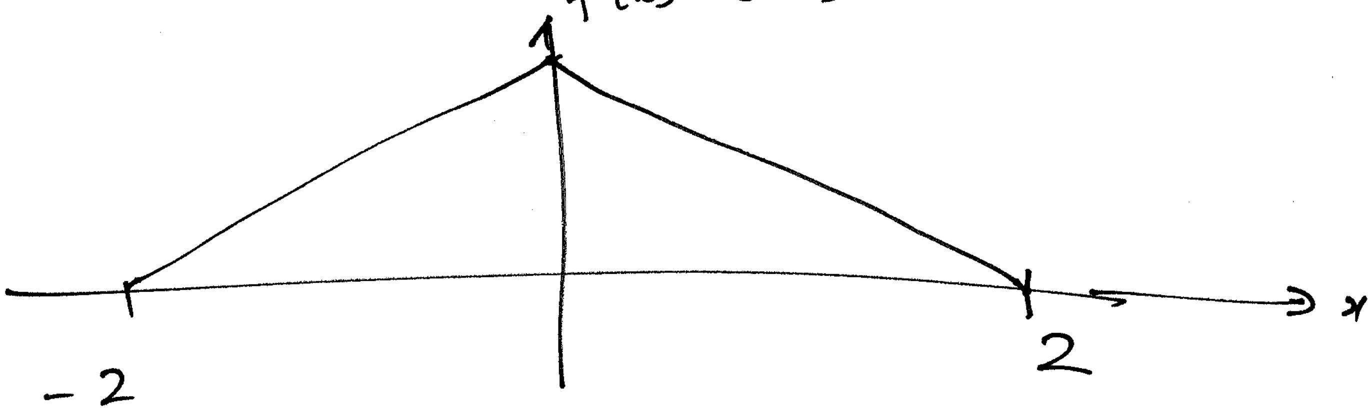


① × ②

$$\text{Area} = \tfrac{1}{2}\left[1 - (-1+x)\right]$$

$$= \tfrac{1}{2}(2 - x).$$

$f(x) * g(x)$



① $f(x) * g(x) = g(x) * f(x)$

② width of $f(x) * g(x)$ = width of $f(x)$ + width of $g(x)$

③ $f(x) * g(x) * f(x_0) \longrightarrow$ width $= 6$

Convolve

If X_1 & X_2 are iid

pdf of $X_1 + X_2$ is the convolution of the pdfs of X_1 & X_2

n iids $X_1, X_2, X_3 \cdots X_n$.

Consider $W = X_1 + X_2 + \cdots + X_n$

$$= \sum_{i=1}^{n} x_i$$

pdf of W can be found by convolving the pdfs of $\underbrace{X_1, X_2 \cdots X_n}_{(n-1) \text{ convolutions}}$

width of the pdf of W increases approaches infinity as $n \to \infty$

Central Limit Theorem.

If $W = \sum_{i=1}^{n} x_i$ $\quad E[x_i] = \mu_x$

$Var[x_i] = Var[x]$
for all i

If x_is are iids pdf of W

approaches a Gaussian distribution
as $n \to \infty$

Note: the above ~~ap~~ pproximation. is true regardless of the individual pdf of ~~#~~ any X_i

W is $N(\mu_W, \sigma_W)$.

$$\mu_W = E[W] = E\left[\sum_{i=1}^{n} x_i\right]$$

$$= n\,\mu_x$$

$$Var[W] = Var[x_1 + x_2 \cdots + x_n]$$

$$= Var[x_1] + Var[x_2] \cdots + Var[x_n]$$

(because x_is are independent)

$$Var[W] = n\ Var[x]$$
$$= n\,\sigma_x^2.$$

Consider a Modified W

$$Z = \frac{1}{n} \sum_{i=1}^{n} x_i$$
$$= \frac{1}{n} W$$

If W is Gaussian Z is also Gauss.

$$\mu_Z = E[Z] = \frac{1}{n} \cdot E[W]$$
$$= \frac{1}{n} \cdot n\mu_x$$
$$= \mu_x$$

$$Var[Z] = \left(\frac{1}{n}\right)^2 Var[W]$$
$$= \frac{1}{n^2} \cdot n\ Var[x]$$
$$= \frac{1}{n} Var[x].$$

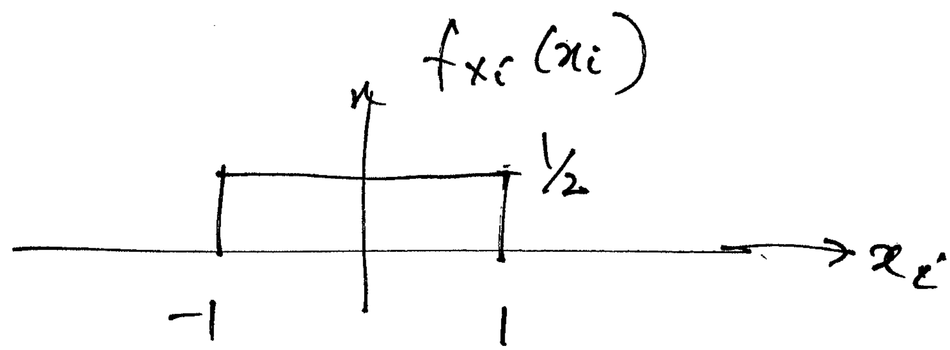
Ex :- X_is ($i=1, \dots 100$)

X_is are iid.

$$\boxed{W = \frac{1}{100}\sum_{i=1}^{100} X_i}$$

Find $P\left[\underbrace{\frac{1}{100}\sum_{i=1}^{100} x_i}_{W} > \frac{1}{2}\right]$

$f_{x_i}(x_i)$ is uniform from -1 to $+1$



W is approximated to a Gaussian Distribution

(follows from the Central Limit Th)

Need to find $P[W > \frac{1}{2}]$

W is $N(\mu_W, \sigma_W)$

$$\mu_W = \mu_{x_i} = 0.$$

$$\mathrm{Var}[W] = \frac{1}{n} \mathrm{Var}[X]$$

$$= \frac{1}{100} \frac{(2)^2}{12} = \checkmark$$

$$\sigma_W = \checkmark$$

$$P[W > 1/2] =$$

$$Z = \frac{W - \mu_W}{\sigma_W} = \frac{W}{\sigma_W}$$

$$P[W > 1/2] = P\left[Z > \frac{1/2}{\sigma_W}\right]$$

$$= \Phi\left(\frac{1}{2\sigma_W}\right)$$

- Can use the Central Limit Thm when X_i's are discrete too eg:- A company manufactures 1000 chips every day.

Prob. of ~~any~~ a ~~chip~~ defective chip $= 1\%$

Chips are independent.

Find the prob. that more than 100 defective chips are manufactured on a single day.

chip $\rightarrow$ G
chip $\rightarrow$ D

$P[D] = 0.01$
$P[G] = 0.99$

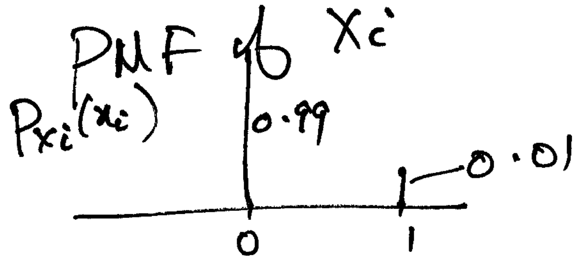
P[More than 100 defective chips]

$$= \sum_{i=101}^{1000} \binom{1000}{i} (0.01)^i (0.99)^{1000-i}$$

or $$1 - \sum_{i=0}^{100} \binom{1000}{i} (0.01)^i (0.99)^{1000-i}$$

$X_i = 0 \rightarrow G$

$X_i = 1 \rightarrow D.$



$W = X_1 + X_2 \cdots + X_{1000}.$

W represents the total No. of Defective chips.

W can be can be approximated to a Gaussian Distribution. (using the central Limit Thm)

W is $N(\mu_W, \sigma_W)$

$\mu_W = 1000\ \mu_x$

$= 1000\ (0.01)$

$= 10$

$\mu_x = \sum_i P_i x_i$

$= 0(.99) + 1(0.01)$

$\text{var}[W] = \sigma_W^2 = 1000\ (\text{var}[x_i])$

$\sigma_W^2 = 1000 \left(\text{Var}[x_i] \right)$

$\text{Var}[x_i] = E[x^2] - \mu_x^2$

$= 0^2 (0.99) + 1^2 (0.01) - \mu_x^2$

$= 0.01 - (0.01)^2 = \checkmark$

$\therefore \sigma_W = \checkmark$

$P[W > 100] = \checkmark$

$N(\mu_W, \sigma_W)$

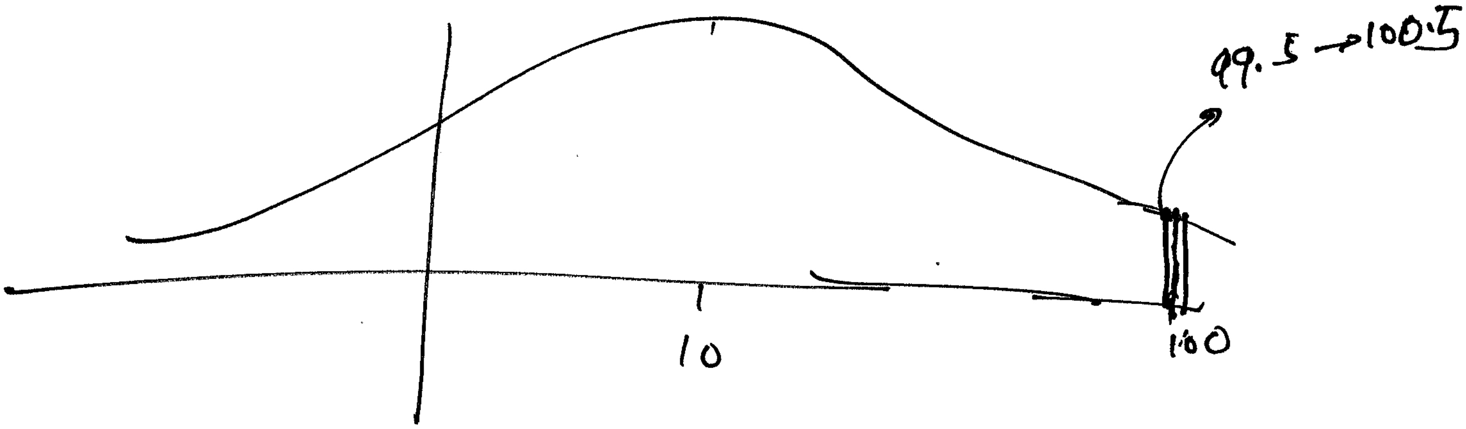
Find the prob. that exactly 100 chips are defective. (out of 1000)

should be $\binom{1000}{100} (0.01)^{100} (0.99)^{900}$

But using the Gaussian approximation.

$P[W = 100] = 0.$

Gaussian $\rightarrow$ Continuous



Using the Gaussian approximation.

$P[W = 100]$ is calculated as

$$P[99.5 < W < 100.5]$$