4-9 → Conditional Distributions 7/26

↳ Conditioned on a RV (1)

$X, Y \rightarrow$ 2 RVs.

Marginals / Joint

$P_X(x)$, $P_Y(y)$, $P_{X,Y}(x,y)$

$P_{X,Y|B}(x,y) \rightarrow$ 4.8

$P_{X|Y}(x|y) \rightarrow$ PMF of X when $Y = y$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

If X & Y are continuous

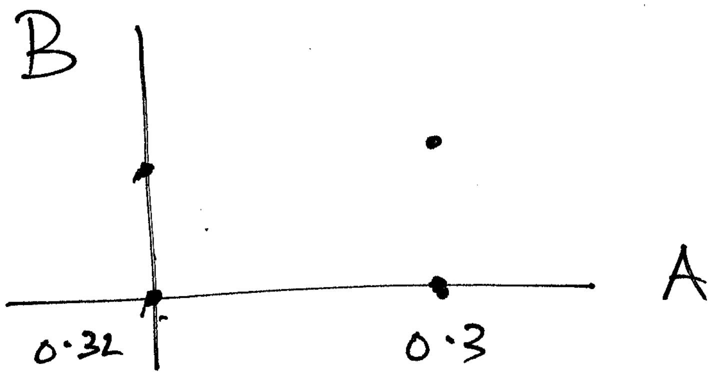
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Q 4.9

(A)
$$P_A(a) = \begin{cases} 0.4, & a=0 \\ 0.6, & a=2 \\ 0, & \text{otherwise} \end{cases}$$

$$P_{B/A}(b|0) = \begin{cases} 0.8, & b=0 \\ 0.2, & b=1 \\ 0, & \text{oth.} \end{cases}$$

$$P_{B/A}(b|2) = \begin{cases} 0.5, & b=0 \\ 0.5, & b=1 \\ 0, & \text{oth.} \end{cases}$$



$$P_{A,B}(a,b) = P[A=a \ \& \ B=b]$$

$$P_{A,B}(0,0) = P[A=0 \ \& \ B=0]$$

$$= \underbrace{P[B=0/A=0]}_{0.8} \cdot \underbrace{P[A=0]}_{0.4}$$

$$= 0.32$$

$\cancel{P}$ $P_{A,B}(2,0) = P[A=2 \ \& \ B=0]$

$= P[B=0/A=2] \cdot P[A=2]$

$= 0.5 \times 0.6 = 0.3$

$\vdots$

$P_{A,B}(0,1) = \checkmark$

$P_{A,B}(2,1) = \checkmark$

(2) $E[B/A=2] \longrightarrow$ use $P_{B/A}(b/2)$

$\hookrightarrow = 0.5$

(3) $P_{A|B}(a|0) = ?$

$$= \frac{P_{A,B}(a,0)}{P_B(0)}$$

$= \checkmark$

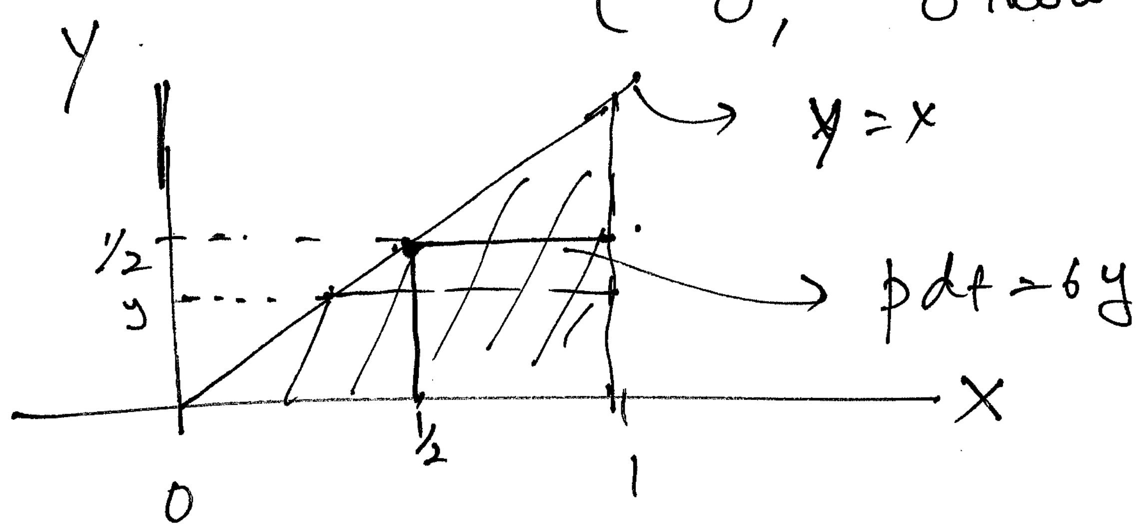
(4) use (3) to find $\mathrm{var}[A \mid B=0]$

(3) $f_X(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$ ④

$$f_{Y/X}(y/x) = \begin{cases} \frac{2y}{x^2}, & 0 \le y \le x, \; 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X,Y}(x,y) = f_{Y/X}(y/x) \cdot f_X(x)$$

$$= \begin{cases} 6y, & 0 \le y \le x, \; 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

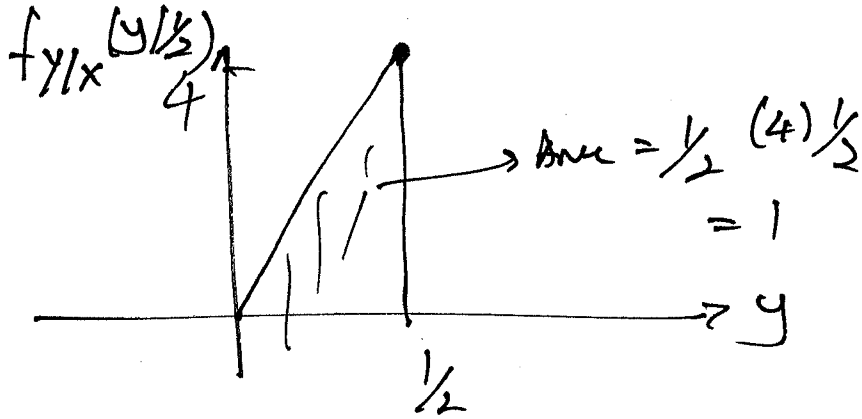


$f_{Y/X=1/2} \longrightarrow f_{Y/X}(y/\cancel{x}) = ?$

$= \checkmark$

$$f_{Y/X}(y/\tfrac{1}{2}) = \begin{cases} 8y, & 0 < y < \tfrac{1}{2} \\ 0, & \text{o/w} \end{cases}$$

$$\frac{2y}{1/4} = 8y$$



(3) $f_{X/Y}(x/\tfrac{1}{2}) \longrightarrow$ X varies from $\tfrac{1}{2}$ to 1

$$= \frac{f_{X,Y}(x, \overset{\frac{1}{2}}{y})}{f_Y(\underset{\frac{1}{2}}{y})}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$

$$= \int_y^1 6y\, dx$$

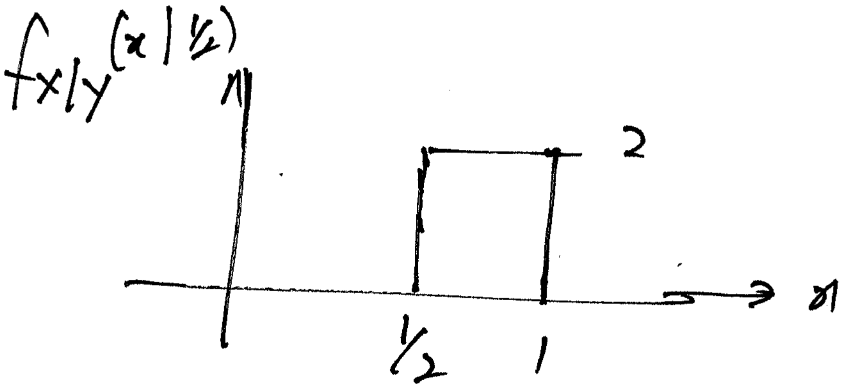
$$= 6y \cdot x \Big|_y^1$$

$$= 6y(1-y),$$

$$f_Y(y) = \begin{cases} 6y(1-y), & \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X/Y}(x \mid 1/2) = \begin{cases} \left.\dfrac{6y}{6y(1-y)}\right|_{y=1/2}, & 1/2 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2, & 1/2 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$



$$\operatorname{Var}[X \mid Y = 1/2] = \frac{(1 - 1/2)^2}{12}$$

4.10 Independence

Recall : If A & B are independent events
ch. 1

$$P[A/B] = P[A] \quad , \quad P[B/A] = P[B]$$

$$P[AB] = P[A] \cdot P[B]$$

X & Y are discrete.

$$P_{X,Y}(x,y) = P[X = x \ \& \ Y = y]$$

~~If~~ X & Y are independent iff

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y) \quad \text{for all } x \ \& \ y.$$

If X & Y continuous

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

* If X & Y are independent

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{f_X(x) \cdot f_Y(y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = f_X(x)$$

$$f_{Y|X}(y|x) = f_Y(y)$$

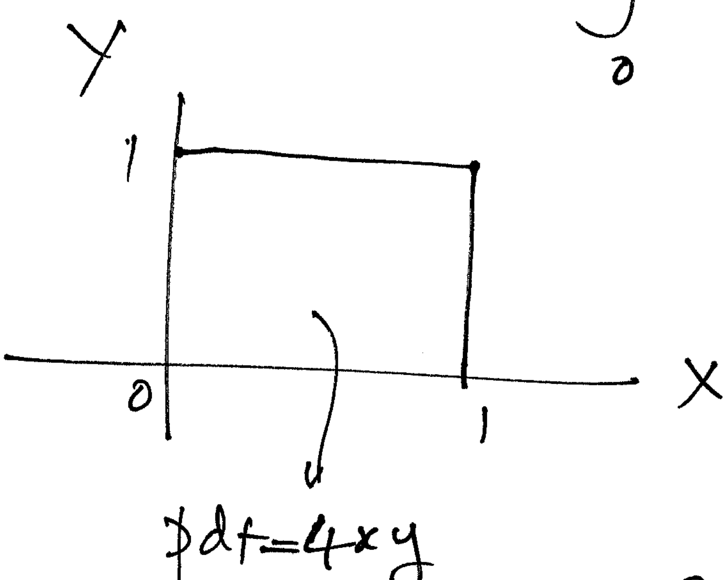
Ex :- $f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \le x \le 1, \ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

Are X & Y independent ?

Find $f_X(x)$ & $f_Y(y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$= \int_0^1 4xy\, dy$$



$$= 4x \cdot \frac{y^2}{2}\Big|_0^1$$

$$= \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{o/w.} \end{cases}$$

Similarly $f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{o/w.} \end{cases}$

$\therefore$ ~~X & Y are indept.~~

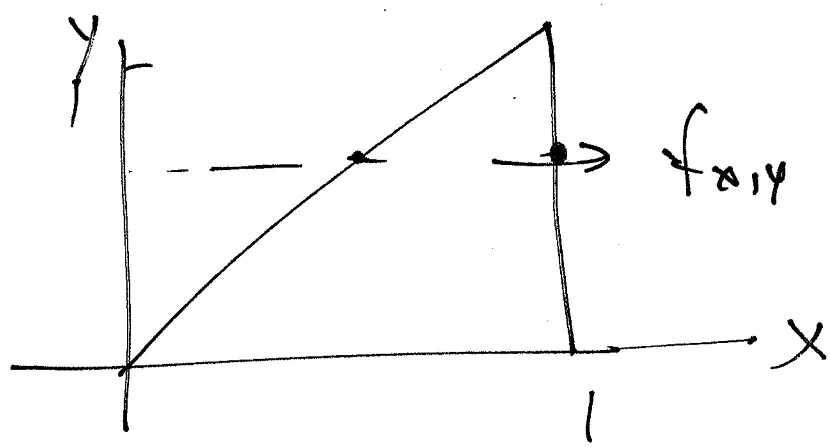
$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$\therefore$ X & Y are independ.

$$4xy = \underbrace{(2x)}_{f_X(x)} \underbrace{(2y)}_{f_Y(y)}$$

Are X & Y independent ?

eg:-



$f_{X,Y}(x,y) = 2$

$1(2)$

$\sqrt{2}\sqrt{2}$

Even though $f_{X,Y}(x,y) = g(x) \cdot h(y)$

X & Y are not independent.

$f_{X|Y}(x|y)$ depends on y.

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

Are X & Y independent ?

$$f_{X,Y}(x,y) = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{N(0,1)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}}_{N(0,1)}$$

X & Y are independent and they are Gaussian.

§6 X & Y are independent.

$$\mathrm{Cov}[X,Y] = E[XY] - \mu_X \mu_Y$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot \underbrace{f_{X,Y}(x,y)}_{f_X(x)\, f_Y(y)} \, dy\, dx$$

$$= \underbrace{\int_{-\infty}^{\infty} x\, f_X(x)\, dx}_{\mu_X} \cdot \underbrace{\int_{-\infty}^{\infty} y\, f_Y(y)\, dy}_{\mu_Y}$$

$$E[XY] = E[X] \cdot E[Y]$$

Note: $E[g(X)\, h(Y)] = E[g(X)] \cdot E[h(Y)]$

eg:- $E[X^5 \sqrt{Y}] = E[X^5] \cdot E[\sqrt{Y}]$

$\therefore \ \mathrm{Cov}[X,Y] = 0 \Rightarrow X$ & Y are uncorrelated

$\therefore$ If X & Y are independent, they are uncorrelated too.

But the converse is not necessarily true.

Independence is a stronger condⁿ.

If X & Y are independent

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

Ex:

$P_{X,Y}(x,y)$	$y=-1$	$y=0$	$y=1$
$x=-1$	0	0.25	0
$x=1$	0.25	0.25	0.25

Are X & Y independent?

Are X & Y uncorrelated?

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y) \rightarrow \text{is this true? for all } x \,\&\, y$$

$$P_{X,Y}(-1,-1) = 0.$$

$$P_X(-1) = 0.25, \quad P_Y(-1) = 0.25$$

$$\therefore \quad P_{X,Y}(-1,-1) \neq P_X(-1) \cdot P_Y(-1)$$

$\therefore$ X & Y are not independent.

$$\mathrm{Cov}[X,Y] = E[XY] - \mu_X \mu_Y$$

$$E[XY] = \sum_x \sum_y xy \cdot P_{X,Y}(x,y)$$

$$= (-1)(0)(0.25) + (1)(-1)(0.25)$$
$$+ (-1)(0)(0.25) + (1)(0)(0.25)$$
$$+ (-1)(0)(0) + (1)(1)(0.25)$$

$$= -0.25 + 0.25 = 0.$$

$$\mu_Y = \sum_y y \cdot P_Y(y) \qquad P_Y(y) = \begin{cases} 0.25, & y=-1 \\ 0.5, & y=0 \\ 0.25, & y=1 \\ 0, & \text{o/w} \end{cases}$$

$$= (-1)(0.25) + (0)(0.5) + 1(0.25)$$

$$= 0.$$

$$\mathrm{Cov}[X,Y] = 0.$$

$\therefore$ X & Y are uncorrelated

X & Y are independent

A is an event of X only

eg: $A: \; a < X \le b$

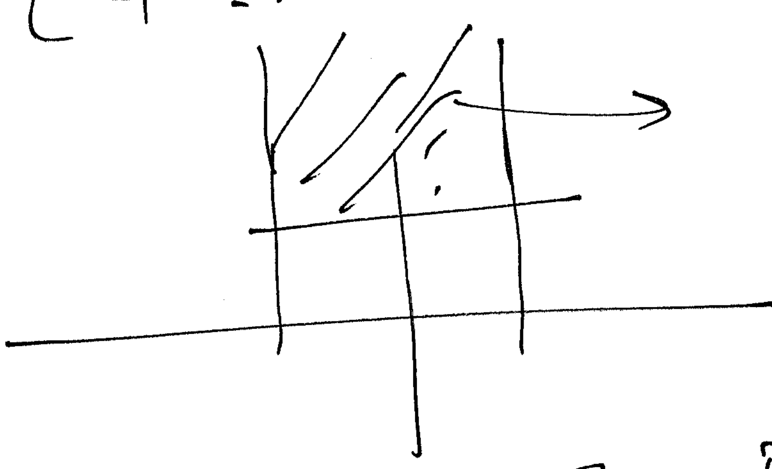
B is an event of Y only

eg:- $B: \; c < Y \le d$

Events A & B are also independent.

eg:- If X & Y are independent

$P[-1 \le X \le 1 \;\&\; Y > 2]$



integrate over R

$= P[-1 \le X \le 1] \cdot P[Y > 2]$

use $f_X(x)$ & $f_Y(y)$ to calculate each probability