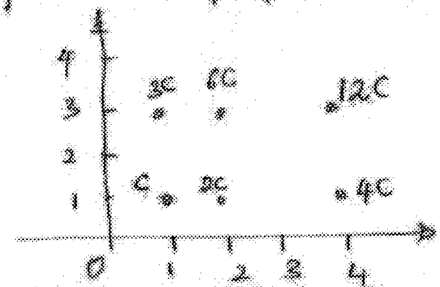
Problem 5.2.1

In this problem, it is helpful to label points with non zero probability on the X, Y plane:



(a) We must choose c so that the PMF sums to one;

$$\sum_{x=1,2,4}\sum_{y=1,3} P_{XY}(x,y) = c\sum_{x=1,2,4} x \sum_{y=1,3} y$$

$$= c\left[1(1+3) + 2(1+3) + 4(1+3)\right] = 28c.$$

$$= 1$$

$$\Rightarrow \quad c = 1/28.$$

(b) The event $\{Y < X\}$ has probability;

$$P[Y<X] = \sum_{x=1,2,4}\sum_{y<x} P_{XY}(x,y) = \frac{1(0)+2(1)+4(1+3)}{28} = \frac{18}{28}$$

(c) The event $\{Y > X\}$ has probability;

$$P[Y>X] = \sum_{x=1,2,4}\sum_{y>x} P_{XY}(x,y) = \frac{1(3)+2(3)+4(0)}{28}$$

$$= 9/28.$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y=X] = \sum_{x=1,2,4}\sum_{y=x} P_{XY}(x,y) = \frac{1(1)+2(0)}{28}$$

$$= 1/28$$

The indirect way is to use the previous results and the observation that

$$P[Y=X] = 1 - P[Y<X] - P[Y>X]$$
$$= \left(1 - 18/28 - 9/28\right) = 1/28 .$$

(e)
$$P[Y=3] = \sum_{x=1,2,3} P_{XY}(x,3) = \frac{(1)(3)+(2)(3)+(4)(3)}{28}$$
$$= \frac{21}{28} = 3/4 .$$

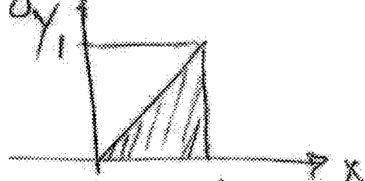
Problem 5.4.2

$$f_{XY}(x,y) = \begin{cases} cxy^2 & 0 \le x,y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

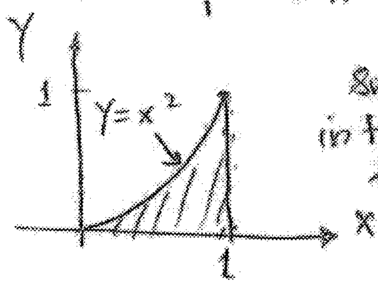
(a) To find the constant c integrate $f_{XY}(x,y)$ over the all possible values of X and Y to get

$$1 = \int_0^1 \int_0^1 c\,xy^2\,dx\,dy = c/6$$
$$\Rightarrow c = 6 .$$

(b) The probability $P[X \geq Y]$ is the integral of the joint PDF $f_{XY}(x,y)$ over the indicated shaded region.

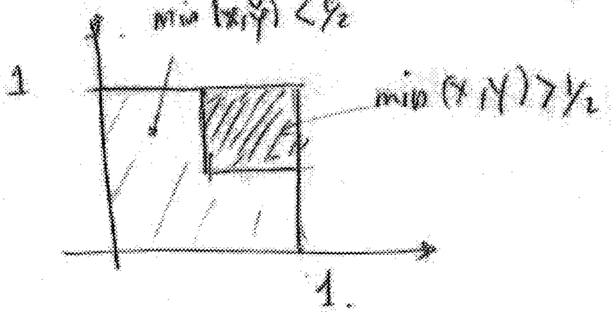


$$P[X \geq Y] = \int_0^1 \int_0^x 6xy^2\,dy\,dx$$
$$= \int_0^1 2x^4\,dx$$
$$= 2/5$$



Similarly, to find $P[Y \le X^2]$ we can integrate over the shaded region in the figure. $P[Y \le X^2] = \int_0^1 \int_0^{x^2} 6xy^2\,dy\,dx$
$$= 1/4 .$$

(c) Here we can choose to either integrate $f_{XY}(x,y)$ over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing.



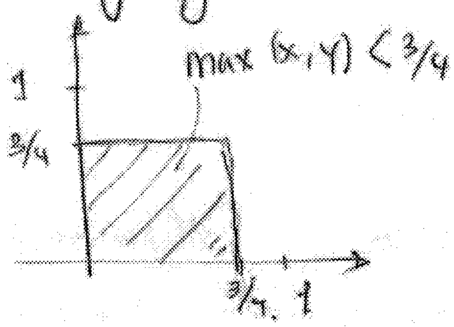
$$P[\min(X,Y) \le 1/2]$$

$$= 1 - P[\min(X,Y) > 1/2]$$

$$= 1 - \int_{1/2}^{1}\int_{1/2}^{1} 6xy^2 \, dx\, dy$$

$$= 1 - \int_{1/2}^{1} \frac{9y^2}{4} \, dy = 11/32 .$$

(d) The probability $P[\max(X,Y) \le 3/4]$ can be found by integrating over the shaded region shown below.



$$P[\max(X,Y) \le 3/4] = P[X \le 3/4 , Y \le 3/4]$$

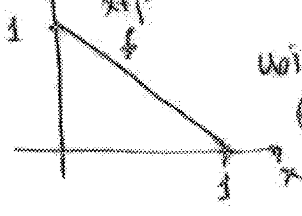
$$= \int_0^{3/4}\int_0^{3/4} 6xy^2 \, dx\, dy$$

$$= \left(x^2 \Big|_0^{3/4}\right)\left(y^3 \Big|_0^{3/4}\right).$$

$$= (3/4)^5 = 0.237$$

Problem 5.5.3

$$f_{XY}(x,y) = \begin{cases} 2 & x+y \le 1,\ x,y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



Using the figure to the left we can find the marginal PDFs by integrating over the appropriate regions

$$f_X(x) = \int_0^{1-x} 2\, dy = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

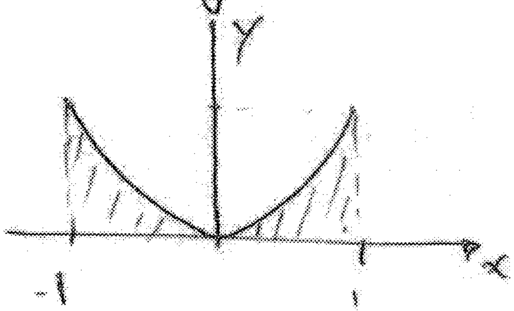
Likewise for $f_Y(y)$;

$$f_Y(y) = \int_0^{1-y} 2\,dx = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{Otherwise.} \end{cases}$$

Problem 5.5.5

The joint pdf of X and Y and the region of nonzero probability are

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & , -1 \le x \le 1,\ 0 \le y \le x^2 \\ 0 & \text{Otherwise} \end{cases}$$

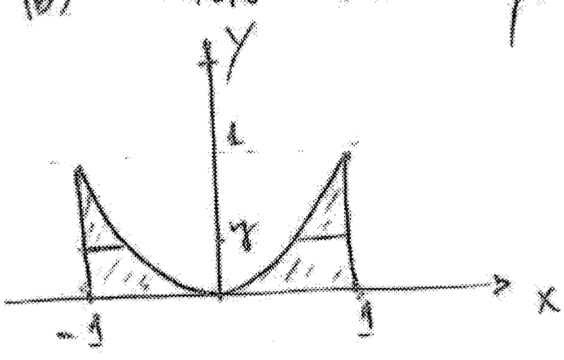


We can find the appropriate marginal pdf's by integrating the joint pdf.

(a) The marginal pdf of x is

$$f_X(x) = \int_0^{x^2} \frac{5x^2}{2}\,dy = \begin{cases} 5x^4/2 & -1 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Note that $f_Y(y) = 0$ for $y > 1$ or $y < 0$. For $0 \le y \le 1$



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$

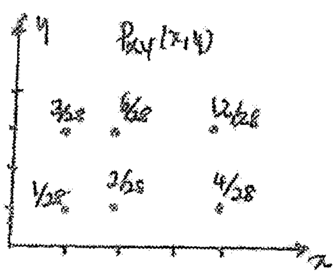
$$= \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2}\,dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2}\,dx$$

$$= 5(1-y^{3/2})/3.$$

∴ The complete expression for marginal pdf of Y;

$$\therefore f_Y(y) = \begin{cases} 5(1-y^{3/2})/3 & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 5.8.2



In Problem 4.2.1 (HW#4), we found the joint PMF $P_{XY}(x,y)$ as shown. Also the expected values and variances were

$$E[X] = 3 \quad \& \quad Var[X] = 10/7$$
$$E[Y] = 5/2 \quad \quad Var[Y] = 3/4$$

We use these result to solve this problem.

(a) Random variable $W = Y/X$ has expected value,

$$E[Y/X] = \sum_{x=1,2,4} \sum_{y=1,3} \frac{y}{x} \cdot P_{XY}(x,y)$$

$$= \frac{1}{1}\cdot\frac{1}{28} + \frac{3}{1}\cdot\frac{3}{28} + \frac{1}{2}\cdot\frac{2}{28} + \frac{3}{2}\cdot\frac{6}{28} + \frac{1}{4}\cdot\frac{4}{28} + \frac{3}{4}\cdot\frac{12}{28}$$

$$= 15/14$$

(b) The correlation of X and Y is (method I)

$$r_{X,Y} = \sum_{x=1,2,4} \sum_{y=1,3} xy \, P_{XY}(x,y)$$

$$= 1\cdot1\cdot\frac{1}{28} + 1\cdot3\cdot\frac{3}{28} + 2\cdot1\cdot\frac{2}{28} + 2\cdot3\cdot\frac{6}{28} + 4\cdot1\cdot\frac{4}{28} + 4\cdot3\cdot\frac{12}{28}$$

$$= 210/28 = 15/2.$$

method II (faster calculation):

Recognizing that $P_{XY}(x,y) = xy/28$ yeilds the faster calculation

$$r_{X,Y} = E[XY] = \sum_{x=1,2,4} \sum_{y=1,3} \frac{(xy)^2}{28}$$

$$= \frac{1}{28} \sum_{x=1,2,4} x^2 \sum_{y=1,3} y^2$$

$$= \frac{1}{28}(1+2^2+4^2)(1^2+3^2) = \frac{210}{28} = \frac{15}{2}.$$

(c) Covariance of X and Y:

$$Cov[X,Y] = E[XY] - E[X]\cdot E[Y]$$

$$= \frac{15}{2} - 3\cdot\frac{5}{2} = 0$$

(d) Correlation coefficient:

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]\cdot Var[Y]}} = 0 \qquad (\because Cov[X,Y] = 0)$$

(e) Variance of X and Y could be added, because X and Y are uncorrelated.

$$Var[X+Y] = Var[X] + Var[Y]$$

$$= \frac{61}{28}$$

Problem 5.6.3

Flip a fair coin 100 times and let X be the number of heads in the first 75 flips and Y be the ~~#~~ number of heads in the last 25 flips.

We know that X and Y are independent.

$$P_X(x) = \binom{75}{x}(1/2)^{75}, \quad P_Y(y) = \binom{25}{y}(1/2)^{25}$$

$\therefore$ The joint PMF of X and Y can be expressed as the product of marginal PMFs since X & Y are independent.

$$\therefore \quad P_{XY}(x,y) = \binom{75}{x}\binom{25}{y}(1/2)^{100}.$$

Problem 5.9.1

$$f_{XY}(x,y) = c e^{-(x^2/8)-(y^2/18)}$$

The omission of any limits for the PDF indicates that it defined over all x and y. We know that $f_{XY}(x,y)$ is in the form of a bivariate Gaussian distribution so we look to Definition 4.17 and attempt to find values for σ_Y, σ_X, $E[X]$, $E[Y]$ and ρ.

We know that the constant c,

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \qquad (\because \text{ bivariate Gaussian})$$

Since the exponent of $f_{XY}(x,y)$ doesn't contain any cross terms we know that ρ must be zero, and now we have to find $E[X]$, $E[Y]$, σ_X and σ_Y.

$$\frac{1}{2}\left(\frac{x-E[X]}{\sigma_X}\right)^2 = \frac{x^2}{8}, \quad \frac{1}{2}\left(\frac{y-E[Y]}{\sigma_Y}\right)^2 = \frac{y^2}{18}$$

$$\Rightarrow \quad E[X] = E[Y] = 0$$

$$\sigma_X = \sqrt{4} = 2.$$

$$\sigma_Y = \sqrt{9} = 3$$

Putting all the pieces together, we find that $c = \frac{1}{12\pi}$, since $\rho = 0$ X & Y are independent