Problem 7.2.8

From defn 3.8, the pdf of W is

$$f_W(w) = \frac{1}{\sqrt{32\pi}} e^{-w^2/32}$$

(a) Since W has expected value $\mu = 0$, $f_W(w)$ is symmetric about $w = 0$. Hence $P[C] = P[W > 0] = 1/2$. From defn 3.15, the conditional pdf of W given C is

$$f_{W/C}(\omega) = \begin{cases} f_W(\omega)/P[C]. & \omega \in C. \\ 0 & \text{Otherwise} \end{cases} = \begin{cases} \frac{2e^{-\omega^2/32}}{\sqrt{32\pi}} & ;\omega > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) The conditional expected value of W given C is

$$E[W/C] = \int_{-\infty}^{\infty} \omega f_{W/C}(\omega)\, d\omega = \frac{2}{4\sqrt{2\pi}} \int_0^{\infty} \omega e^{-\omega^2/32}\, d\omega$$

Making the substitution $v = \omega^2/32$ we obtain

$$E[W^2/C] = \int_{-\infty}^{\infty} \omega^2 f_{W/C}(\omega)\, d\omega = 2 \int_0^{\infty} \omega^2 f_W(\omega)\, d\omega$$

We observe that $\omega^2 f_W(\omega)$ is an even function. Hence.

$$E[W^2/C] = 2 \int_0^{\infty} \omega^2 f_W(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} \omega^2 f_W(\omega)\, d\omega = E[W^2] = \sigma^2 = 16.$$

Lastly, the conditional variance of W given C is

$$\text{Var}[W/C] = E[W^2/C] - (E[W/C])^2 = 16 - 32/\pi$$

$$= 5.81.$$

Problem 7.2.9

(a) We first find the conditional pdf of T. The pdf of T is

$$f_T(t) = \begin{cases} 100e^{-100t} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

The conditioning event has probability

$$P[T > 0.02] = \int_{0.02}^{\infty} f_T(t)\, dt = -e^{-100t}\Big|_{0.02}^{\infty} = e^{-2}.$$

From defn 3.15, the conditional pdf of T is

$$f_{T/T>0.02}(t) = \begin{cases} \dfrac{f_T(t)}{P[T>0.02]} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 100\, e^{-100(t-0.02)} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases}$$

The conditional expected value of T is

$$E[T/T>0.02] = \int_{0.02}^{\infty} t\,(100)\, e^{-100(t-0.02)}\, dt.$$

The substitution $\tau = t - 0.02$ yeilds,

$$E[T/T>0.02] = \int_{0}^{\infty} (\tau + 0.02)(100)\, e^{-100\tau}\, d\tau$$

$$= \int_{0}^{\infty} (\tau + 0.02)\, f_T(\tau)\, d\tau = E[T+0.02]$$

$$= \underline{\underline{0.03}}.$$

(b) The conditional second moment of T is

$$E[T^2/T>0.02] = \int_{0.02}^{\infty} t^2 (100)\, e^{-100(t-0.02)}\, dt$$

The substitution $\tau = t - 0.02$ yeilds,

$$E[T^2/T>0.02] = \int_{0}^{\infty} (\tau + 0.02)^2 (100)\, e^{-100\tau}\, d\tau.$$

$$= \int_0^\infty (\tau + 0.02)^2 f_T(\tau)\, d\tau$$

$$= E\left[(T+0.02)^2\right].$$

Now we can evaluate the conditional variance.

$$\mathrm{Var}\left[T / T > 0.02\right] = E\left[T^2 / T > 0.02\right] - \left(E\left[T / T > 0.02\right]\right)^2$$

$$= E\left[(T+0.02)^2\right] - \left(E\left[T+0.02\right]\right)^2$$

$$= \mathrm{Var}\left[T + 0.02\right].$$

$$= \mathrm{Var}[T] = 0.0001$$

Problem 7.3.3

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)}, & x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Given the event $A = \{X+Y \leq 1\}$, we wish to find $f_{X,Y|A}(x,y)$.

First,

$$P[A] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\, dy\, dx$$
$$= 1 - 3e^{-2} + 2e^{-3}$$

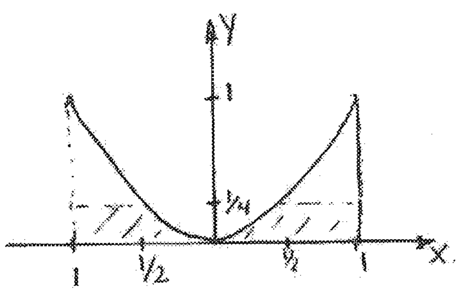
so then,

$$f_{X,Y|A}(x,y) = \begin{cases} \dfrac{6e^{-(2x+3y)}}{1-3e^{-2}+2e^{-3}} & x+y \leq 1,\ x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 7.3.8

$$f_{XY}(x,y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1;\ 0 \leq y \leq x^2 \\ 0 & \text{, otherwise.} \end{cases}$$

(a) Let $A = \{Y \leq 1/4\}$ has $P[A]$.



$$P[A] = 2 \cdot \int_0^{1/2} \int_0^{x^2} \frac{5x^2}{2}\, dy\, dx + 2 \cdot \int_{1/2}^{1} \int_0^{1/4} \frac{5x^2}{2}\, dy\, dx$$
$$= \int_0^{1/2} 5x^4\, dx + \int_{1/2}^{1} \frac{5x^2}{4}\, dx$$
$$= x^5 \Big|_0^{1/2} + \frac{5}{12} x^3 \Big|_{1/2}^{1}$$
$$= \underline{19/48}$$

This implies

$$f_{XY|A}(x,y) = \begin{cases} f_{X,Y}(x,y)/P[A] & , (x,y) \in A \\ 0 & , \text{otherwise} \end{cases}$$
$$= \begin{cases} 120x^2/19 & , -1 \leq x \leq 1,\ 0 \leq y \leq x^2,\ y \leq 1/4 \\ 0 & , \text{otherwise} \end{cases}$$

(b)

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\, dx$$
$$= 2\int_{\sqrt{y}}^{1} \frac{120x^2}{19}\, dx = \begin{cases} \frac{80}{19}(1 - y^{3/2}) & ;\ 0 \leq y \leq 1/4 \\ 0 & ,\ \text{otherwise} \end{cases}$$

(c) The conditional expectation $E[Y/A]$:

$$E[Y/A] = \int_0^{1/4} y \, \frac{80}{19} \cdot (1 - y^{3/2}) \, dy = \frac{80}{19}\left(\frac{y^2}{2} - \frac{2y^{7/2}}{7}\right)\Big|_0^{1/4}$$

$$= \underline{\underline{\frac{65}{532}}}$$

(d). To find $f_{X/A}(x)$, we can write $f_{X/A}(x) = \int_{-\infty}^{\infty} f_{X,Y/A}(x,y)\, dy$.

However, when we substitute $f_{X,Y/A}(x,y)$, the limits will depend on the value of x.

When $|x| \le 1/2$,

$$f_{X/A}(x) = \int_0^{x^2} \frac{120x^2}{19}\, dy = \frac{120x^4}{19}$$

Whe $-1 \le x \le -1/2$ or $1/2 \le x \le 1$,

$$f_{X/A}(x) = \int_0^{1/4} \frac{120x^2}{19}\, dy = \frac{30x^2}{19}$$

$\therefore$ The complete expression for Conditional PDF of X give A

$$f_{X/A}(x) = \begin{cases} 30x^2/19, & -1 \le x \le -1/2 \\ 120x^4/19, & -1/2 \le x \le 1/2 \\ 30x^2/19, & 1/2 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(e) Conditional mean $E[X/A]$.

$$E[X/A] = \int_{-1}^{-1/2} \frac{30x^3}{19}\, dx + \int_{-1/2}^{1/2} \frac{120x^5}{19}\, dx + \int_{1/2}^{1} \frac{30x^3}{19}\, dx.$$

$$= \underline{\underline{0}}.$$

Problem 7.5.1

Joint PDF of X and Y,

$$f_{XY}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

For $0 \le y \le 1$

For $0 \le y \le 1$;

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_y^1 2\,dx = 2(1-y)$$

for $y<0$ or $y>1$, $f_Y(y) = 0$

$\therefore$ The complete expression for the marginal PDF of Y

$$f_Y(y) = \begin{cases} 2(1-y) & , 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 4.24, the conditional PDF of X given Y is

$$f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \left(\frac{1}{1-y}\right) ; & y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

i.e Since $Y \le X \le 1$, X is uniform over $[y,1]$ when $Y=y$.

The conditional expectation of X given $Y=y$;

$$E[X/Y=y] = \int_{-\infty}^{\infty} x\, f_{X/Y}(x/y)\,dx.$$

$$= \int_y^1 \frac{x}{1-y}\,dx = \left.\frac{x^2}{2(1-y)}\right|_y^1 = \frac{1+y}{2}.$$

In fact, since we know that the conditional PDF of X is uniform over $[y,1]$ when $Y=y$, it is not really necessary to perform the above calculation.

Problem 7.5.3

(A) First we observe that A takes on values $S_A = \{-1, 1\}$ while B takes on values from $S_B = \{0, 1\}$. To construct a table describing $P_{A,B}(a,b)$ we built a table for all possible values of pairs $(A,$
The general form of the entries is

$P_{A,B}(a,b)$	$b=0$	$b=1$
$a=-1$	$P_{B/A}(0/-1) \cdot P_A(-1)$	$P_{B/A}(1/-1) \cdot P_A(-1)$
$a=1$	$P_{B/A}(0/1) \cdot P_A(1)$	$P_{B/A}(1/1) \cdot P_A(1)$

Problem 7.6.2

It is given that

$$\mu_X = \mu_Y = 0 \quad , \quad \sigma_X^2 = \sigma_Y^2 = 1$$

From theorem 4.30, the conditional expectation of Y given x is

$$E[Y/x] = \tilde{\mu}_Y(x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) = \rho x$$

since it is given that $E[Y/x] = x/2$; we can find $\rho = 1/2$.

∴ From Definition 4.17, the joint PDF is

$$f_{XY}(x,y) = \frac{1}{\sqrt{3\pi^2}} e^{-2(x^2 - xy + y^2)/3}$$