Mixed RVs

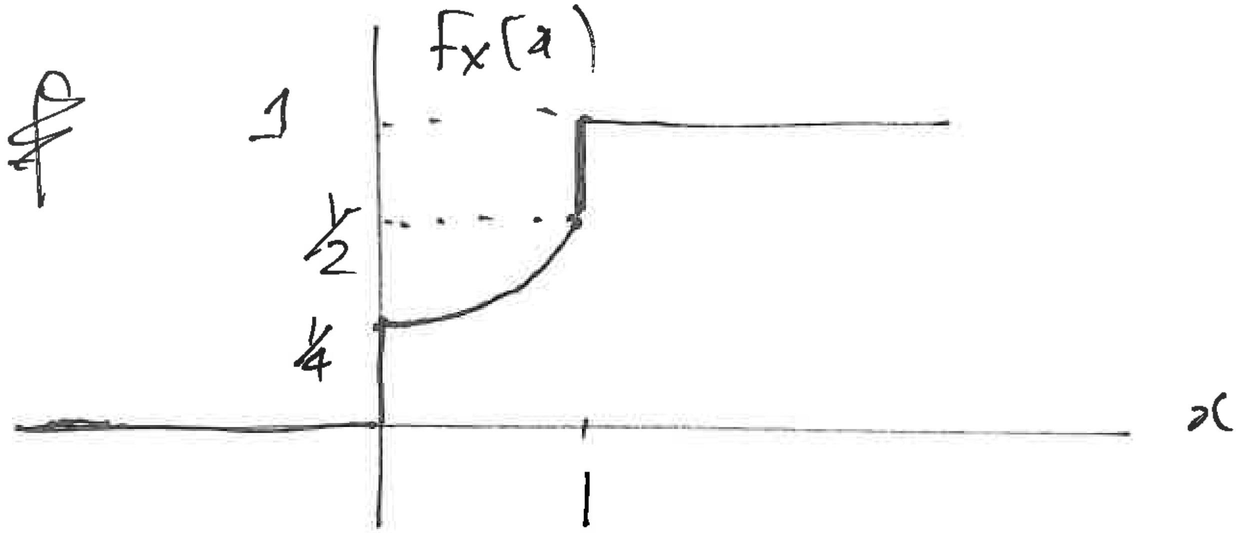
$F_X(x) \longrightarrow$ continuous parts + jumps

$f_X(x)$ Delta func$^{\underline{ns}}$

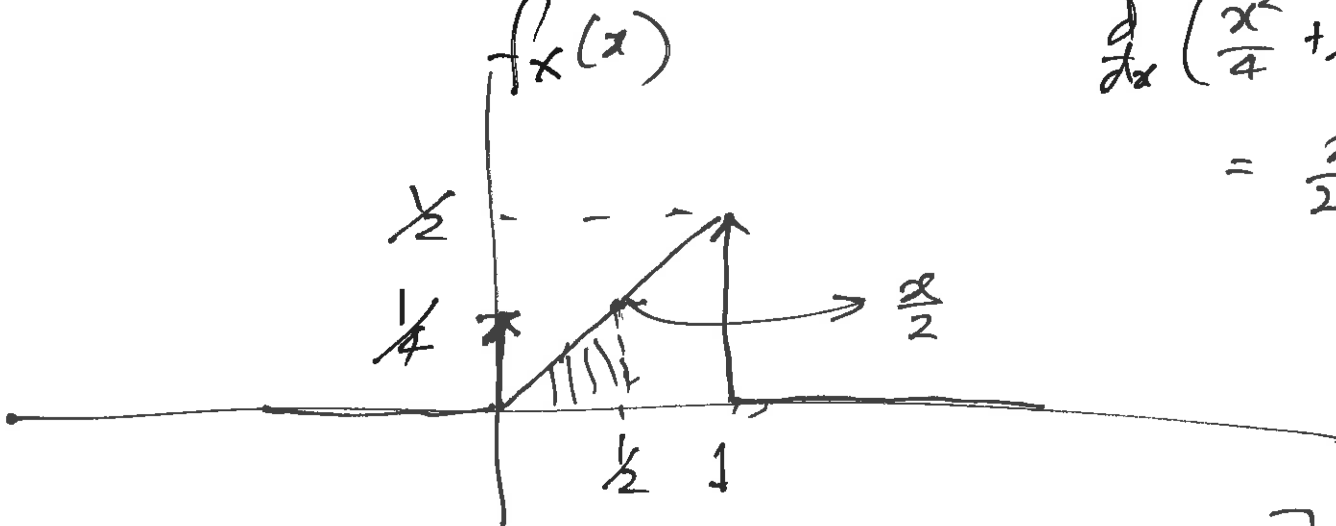
eg:-

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4} + \frac{1}{4}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

$P[X > \frac{1}{2}]$, μ_X & $Var[X]$



$$\frac{d}{dx}\left(\frac{x^2}{4}+\frac{1}{4}\right) = \frac{x}{2}$$



$$P\left[X > \frac{1}{2}\right] = 1 - P\left[X \le \frac{1}{2}\right]$$

$$= 1 - \left[\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \frac{1}{4}\right]$$

Delta

=

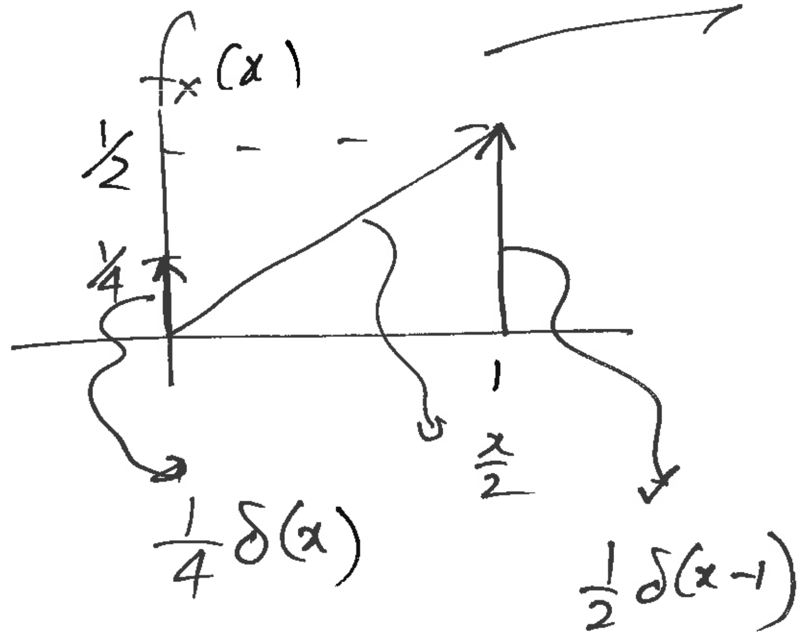
$$* \quad \int_{-\infty}^{\infty} f(a)\,\delta(x-a)\,dx = f(a)$$

$$\int_{a_-}^{a_+} f(x)\,\delta(x-a)\,dx$$

$$a_+ \to a+\varepsilon \qquad a_- \to a-\varepsilon$$

$$\int_{-\infty}^{\infty} (x^2+4)\,\delta(x+2)\,dx = \left[(-2)^2+4\right] \qquad x=-2$$

μ_x & $Var[x]$



$$\mu_x = E[X]$$

$$= \int_{-\infty}^{\infty} x\cdot f_x(x)\,dx$$

$$= \underbrace{\int_0^1 x\cdot\frac{x}{2}\,dx}_{\text{continuous part}} + \overbrace{\int_{-\infty}^{\infty} x\cdot\left[\frac{1}{4}\delta(x)+\frac{1}{2}\delta(x-1)\right]dx}^{\text{Delta func}}$$

$$= \frac{1}{2}\,\frac{x^3}{3}\Big|_0^1 + \frac{1}{4}\int_{-\infty}^{\infty} x\,\delta(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} x\,\delta(x-1)\,dx$$ (4)

$$\checkmark \quad + \frac{1}{4}(0) + \frac{1}{2}(1) = \checkmark$$

$$Var[X] = E[X^2] - \mu_x^2$$

$$E[X^2] = \int_0^1 x^2 \cdot \frac{x}{2}\,dx + \int_{-\infty}^{\infty} x^2 \cdot \left[\frac{1}{4}\delta(x) + \frac{1}{2}\delta(x-1)\right]dx$$

$$= \frac{1}{2}\,\frac{x^4}{4}\Big|_0^1 + \frac{1}{4}(0)^2 + \frac{1}{2}(1)^2$$

6.2 (3.7) Derived RV

$X \rightarrow$ RV $\rightarrow$ continuous

$Y = g(X) \longrightarrow$ Derived RV

~~Recall:~~ Given $f_X(x)$ & $g(x)$

Find the pdf of Y

Recall: In the discrete case

Given $P_X(x)$ & $g(x)$] used a Table.

we found $P_Y(y)$

2 cases:

1 Case 1

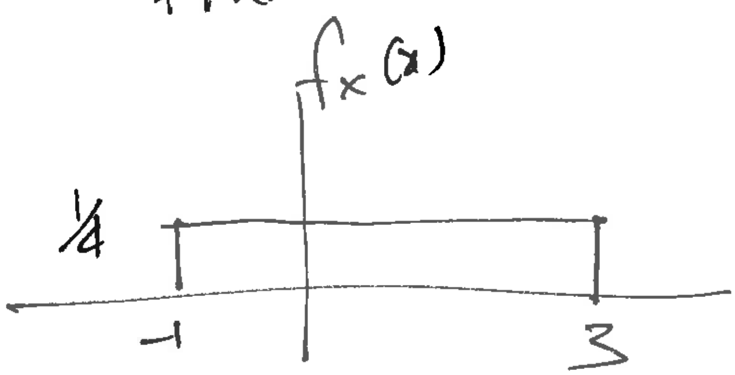
Y is a linear func^n of X

$$Y = aX + b$$ ($a, b \rightarrow$ constants)

* ~~P~~ Shape of $f_Y(y)$ is the same as the shape of $f_X(x)$

eg:- If X is continuous uniform $\rightarrow$ $f_Y(y)$ is also continuous uniform eg:- X is continuous uniform from -1 to +3 (6)

find the pdf of $y = 3x - 7$

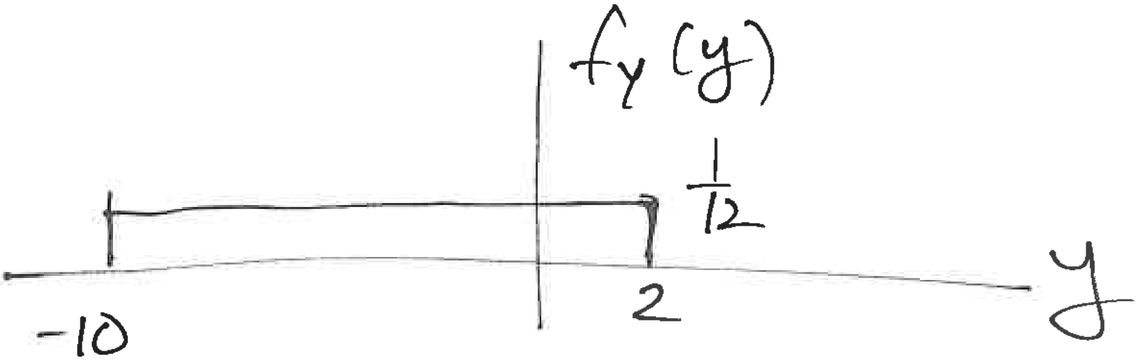


linear

$\therefore$ y is continuous uniform

Smallest $y = 3(-1) - 7 = -10$

largest $y = 3(3) - 7 = 2$



In general, $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

$Y = aX + b \rightarrow$ linear relationship

$$\mu_Y = E[Y] = E[ax + b]$$

$$= a\,E[x] + b$$

$$\mu_Y = a\mu_x + b$$

$$Var[y] = ?$$

$$= E[Y^2] - \mu_Y^2$$

$$E[y^2] = E[(ax+b)^2]$$

$$= E[a^2x^2 + 2abx + b^2]$$

$$= a^2 E[x^2] + 2ab\mu_x + b^2$$

~~$E[y^2]$~~

$$Var[y] = a^2 E[x^2] + 2ab\mu_x + b^2 - (a\mu_x + b)^2$$

$$= a^2 E[x^2] + 2ab\mu_x + b^2 - (a^2\mu_x^2 + 2ab\mu_x + b^2)$$

$$= a^2 \underbrace{\left\{ E[x^2] - \mu_x^2 \right\}}_{Var[x]}$$

If $y = ax + b$

$\therefore$ $Var[y] = a^2 Var[x]$

$\mu_y = a\mu_x + b$

$\sigma_y = |a| \sigma_x$

Note

independent of b

~~b in~~

b is just a shift & does not influence the spread.

eg:- X is $N(3,2)$

$$Y = \underset{a}{4}X \underset{b}{-7} \longrightarrow f_Y(y)?$$

linear

Y is also Normal

Y is $N(\mu_Y, \sigma_Y)$

μ_Y & σ_Y = ?

$$\mu_Y = 4(\underset{3}{\mu_X}) - 7 = 5$$

$$\sigma_Y = |4|(2) = 8$$

$\therefore$ Y is $N(5,8)$

Case 2: When $g(x)$ is Non-linear

$f_X(x)$ & $f_Y(y)$ can have different shapes

$Y = g(x)$ is given
$f_X(x)$ is given
Find $f_X(y)$

Find the range ~~of~~ of Y

1. Start with the CDF of Y

$$F_Y(y) = P[Y \le y]$$

in the range of Y

$$= P[g(x) \le y]$$

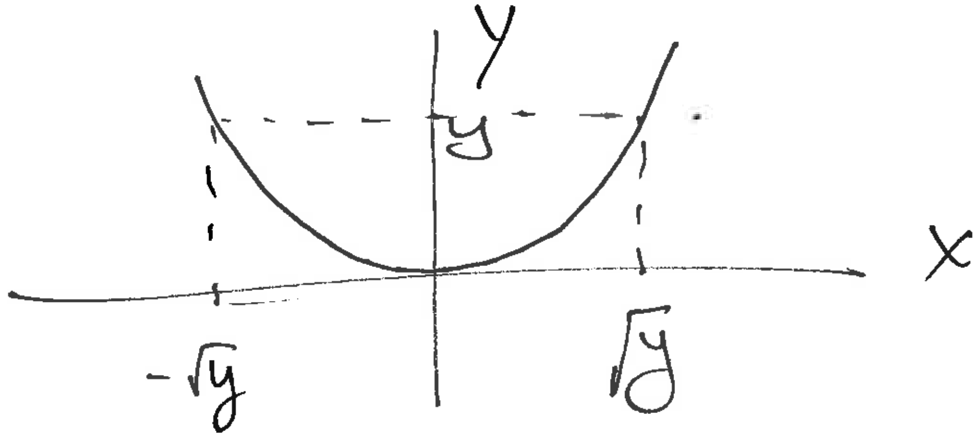
$\vdots$

Express as a prob. of X

eg:- $Y = X^2$

$$F_Y(y) = P[X^2 \le y]$$

$$= P[-\sqrt{y} \le X \le \sqrt{y}]$$



2. Calculate that Prob. using $f_X(x)$

in the Example: $\displaystyle\int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx$

$Y = X^2$

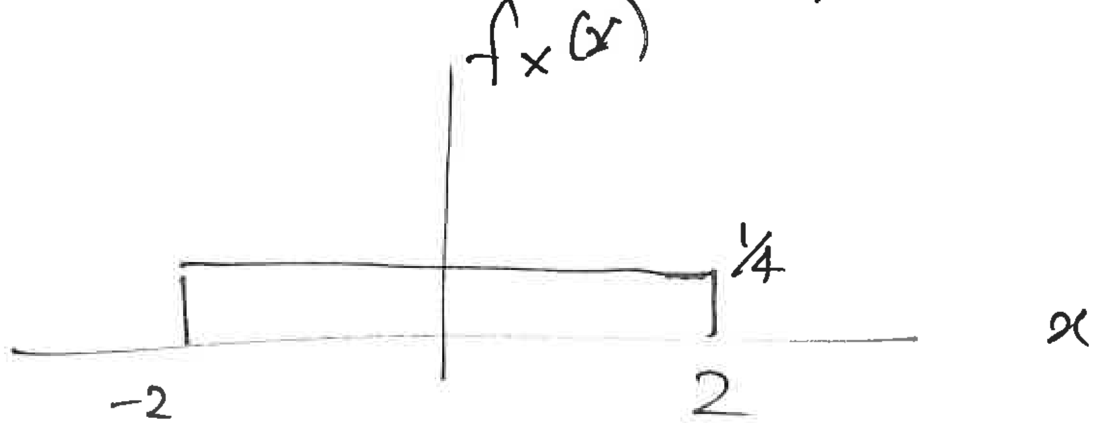
as a func^n of y

3. We found $F_Y(y)$ as a func^n of y

$$\therefore\ f_Y(y) = \frac{d}{dy} F_Y(y)$$

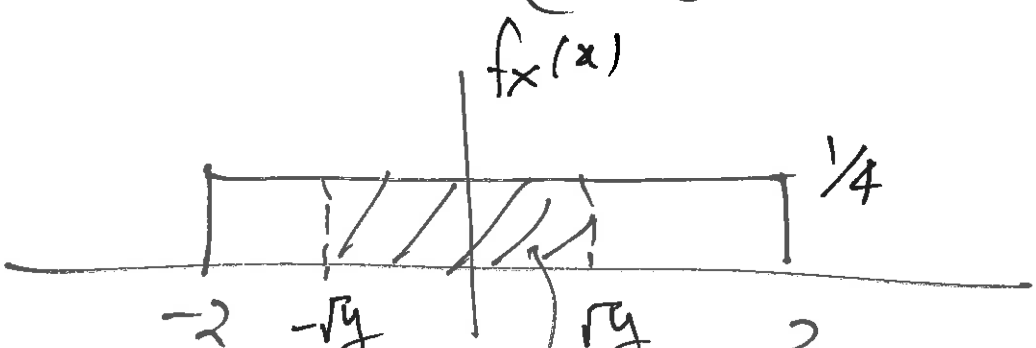
ex: X is continuous uniform from -2 to $+2$

Find the pdf of $Y = X^2$



Y varies from 0 to 4

$$F_Y(y) = P[Y \le y]$$
$$= P[X^2 \le y]$$
$$= P[-\sqrt{y} \le X \le \sqrt{y}]$$



$$= \frac{1}{4}(2\sqrt{y})$$

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{\sqrt{y}}{2}, & 0 \leq y < 4 \\ 1, & y \geq 4 \end{cases} \longrightarrow \text{No jumps}$$

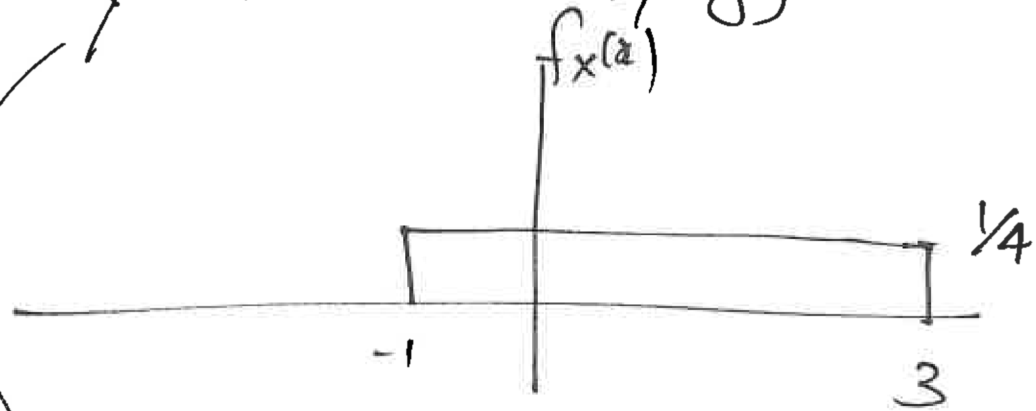
$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \begin{cases} \frac{1}{2} \cdot \frac{1}{2\sqrt{y}}, & 0 \leq y < 4 \\ 0, & \text{o/w}. \end{cases}$$

eg: X is continuous Unifm from -1 to $+3$

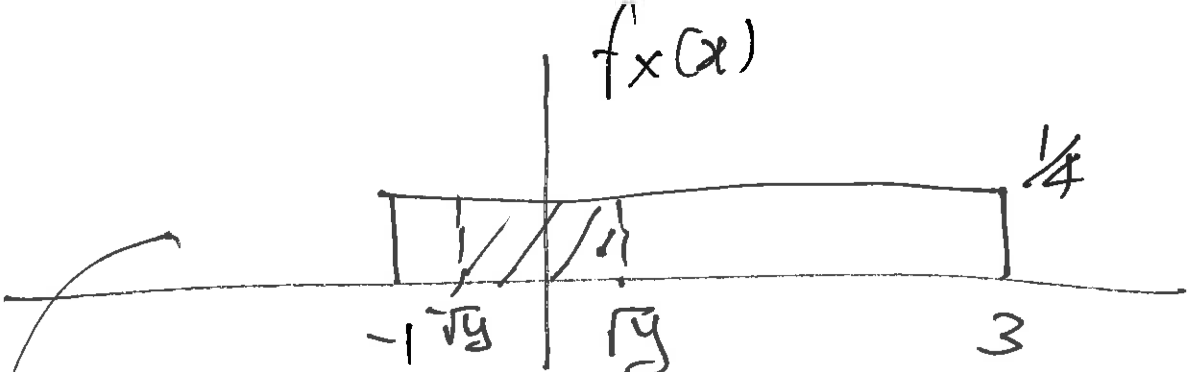
Ex 6.4 (Ex 3.26)

$$Y = X^2 \to f_Y(y)$$

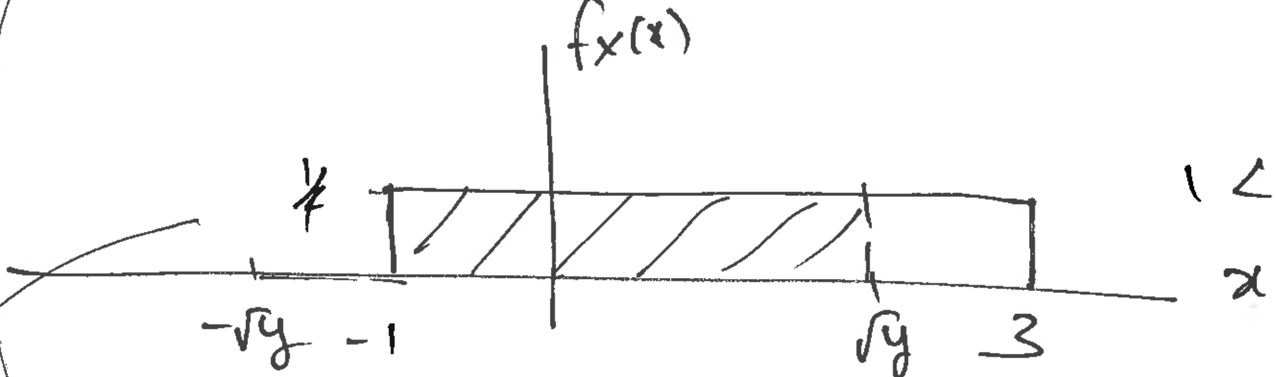


Y varies from 0 to 9

$$F_Y(y) = P[Y \le y]$$
$$= P[X^2 \le y]$$
$$= P[-\sqrt{y} \le X \le \sqrt{y}]$$



$\sqrt{y} < 1$



$1 < \sqrt{y} \le 3$

$0 \le y < 1$

$$F_Y(y) = P[-\sqrt{y} \le X \le \sqrt{y}] = \frac{1}{4}(2\sqrt{y})$$

$1 \le y \le 9$

$$F_Y(y) = P[-\sqrt{y} \le X \le \sqrt{y}]$$

$$= \frac{1}{4}(\sqrt{y}+1)$$

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{\sqrt{y}}{2}, & 0 \le y < 1 \\ \frac{1}{4}(\sqrt{y}+1), & 1 \le y < 9 \\ 1, & y \ge 9 \end{cases}$$

No jumps

$\therefore Y$ is continuous

$\Big\downarrow \frac{d}{dy}$

$$f_Y(y) = \Big\{$$

6.3 (3.7)

eg:- X is continuous uniform from -1 to $+3$

$$Y = \begin{cases} +1, & X \geq 0 \\ -1, & X < 0 \end{cases}$$

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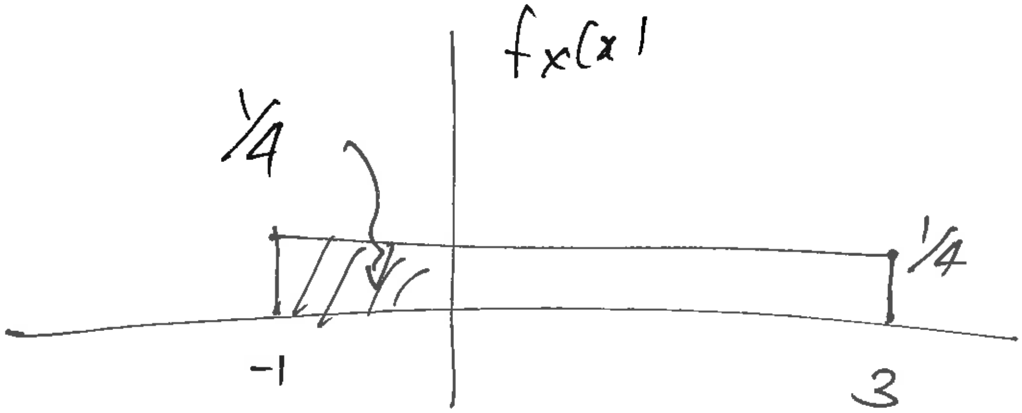
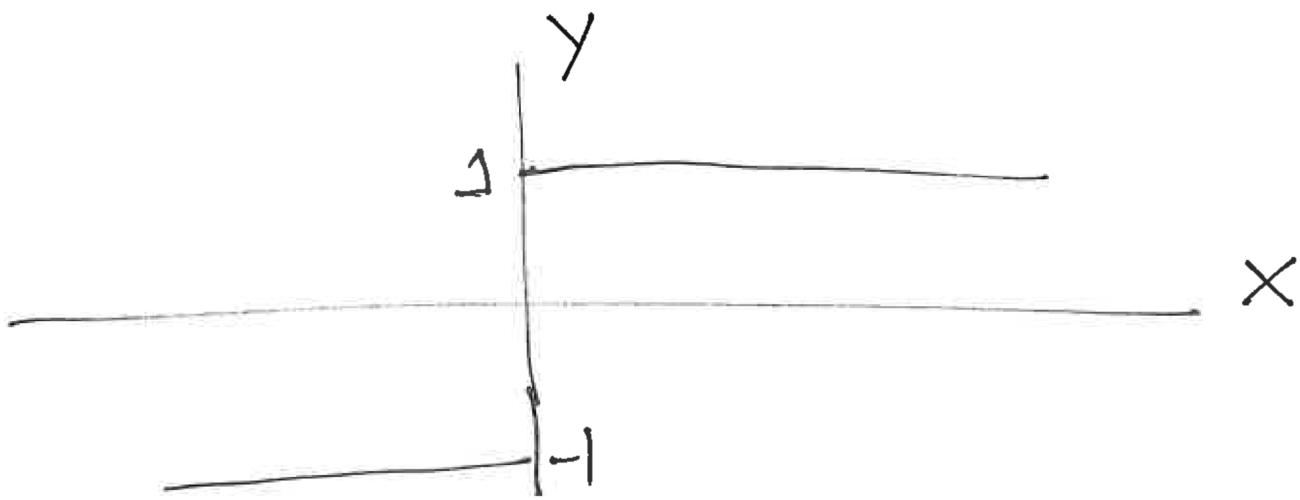
(a)





Y can take only 2 values $y = 1$ or $y = -1$

$\therefore$ Y is discrete.

$P_Y(y)$? $\longrightarrow$ 2 lines at 1 & -1

$P_Y(-1) = P[Y = -1] = P[X < 0] = 1/4$

$P_Y(1) = P[Y = 1] = P[X \geq 0] = 3/4$

$$\mu_y = \sum_y y \cdot P_Y(y) = (-1)\tfrac{1}{4} + (1)\tfrac{3}{4}$$

HW X is continuous uniform from -2 to $+2$

$$Y = \begin{cases} 4, & X \geq 1 \\ 2x, & -1 < X < 1 \\ -4, & X \leq -1 \end{cases}$$

Mixed.

$f_Y(y)$? μ_Y & $Var[Y]$

Conditional pdfs

Recall: If X is discrete.

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{P[B]}, & x \in B \\ 0, & \text{o/w.} \end{cases}$$

B is an event of X

$$P_X(x) = \sum_{i=1}^{N} P_{X|B_i}(x)\, P[B_i]$$

B_i's are mutually Exclusive & collectively Exhaustive