Prob. 6.6.2 (2nd Edition) 7/29

①

Ex :- Calls $\rightarrow$ V or D

$P(V) = 0.8, \quad P[D] = 0.2$

$K_n \rightarrow$ No. of ~~Data~~ voice calls in n calls $\rightarrow$ Binomial (n, p)

$p = 0.8$

(a) $E[K_{100}] = np$
$= (100)(0.8)$
$= 80$

$\mu_x = np$
$Var[x] = np(1-p)$

(b) $\sigma_{K_{100}} = ?$

$Var[K_{100}] = 100(0.8)(0.2)$
$= 16$

$\sigma_{K_{100}} = 4$

(c) $P[K_{100} \geq 18]$ using CLT

$K_{100} = X_1 + X_2 \cdots + X_{100}$

$$X_i = \begin{cases} 1, & i^{th} \text{ call is Voice} \\ 0, & \cdots \text{ Data} \end{cases}$$

Bernoulli

K_{100} is approximately $N(\mu, \sigma)$ using CLT

$$\mu = E[x_1 + x_2 \cdots + x_{100}]$$

$$= 100\,\mu_x$$ → Mean of $x_i = p = 0.8$

$$= 80$$

$$\text{Var}[x] = 100\,\text{Var}[x]$$ → x_i's are iids

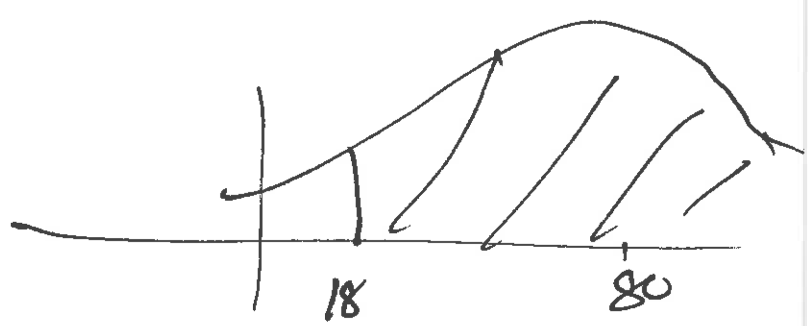
If x is Bernlli (p)
$\mu_x = p$
$\text{Var}[x] = p(1-p)$

$$= 100\,(0.8)\,(0.2)$$

$$= \frac{80}{5} = 16 \longrightarrow \sigma_x = 4$$

K_{100} is $N(80, 4)$

$P[K_{100} \geq 18] =$ ✓



$P[16 \leq K_{100} \leq 24] =$ ✓

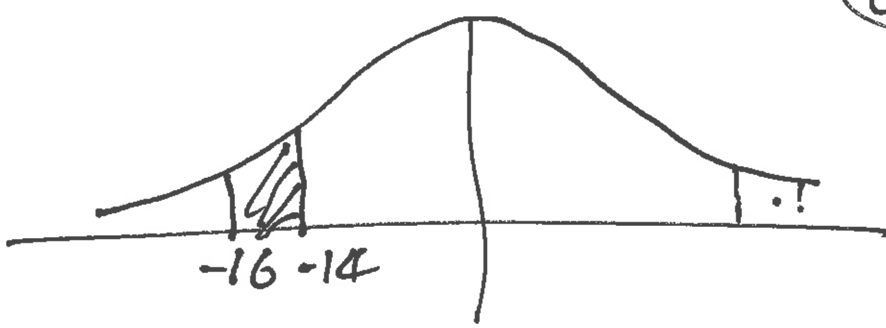
$$Z = \frac{K_{100} - 80}{4}$$

$$P[\ 16 \le K_{100} \le 24\ ]$$

$$= P\left[\frac{-64}{4} \le Z \le \frac{-56}{4}\right]$$

$$= P[\ -16 \le Z \le -14\ ] =$$

$$\Phi(14) - \Phi(16)$$



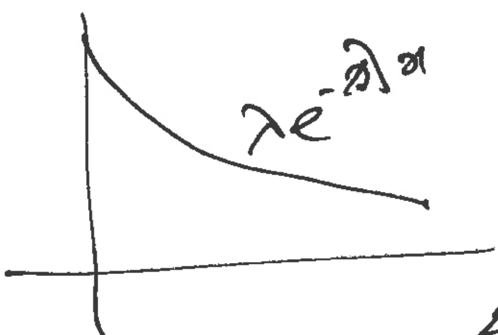
6-6-3 (2nd)

eg:- 120 calls made

Duration of a single call ~~is~~ has a pdf of an exponential with mean 150 seconds

$$\frac{1}{\lambda} = \frac{150}{60} = \frac{5}{2}$$

$$\lambda = \frac{2}{5}$$



$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

300 minutes → \$30

Any extra minute → Cost 0.40

$$P[\text{Cost} > 36]$$

$$P\Big[T > \underbrace{\left(\frac{36-30}{0.4} + 300\right)}_{a}\Big]$$

T → Total Duration

$$\frac{36-30}{0.4} = \frac{6}{0.4}$$

$$T = X_1 + X_2 + \cdots + X_{120}$$

Total Du

$X_i \rightarrow$ Duration of the i^{th} call

iids

$\therefore$ T is approximately $N(\mu, \sigma)$

$$\mu = n\mu_X = (120)\left(\frac{1}{\lambda}\right) = 120\left(\frac{5}{2}\right)$$
$$= 300$$

$$\sigma^2 = Var[T] = n\,Var[X]$$
$$= n\left(\frac{1}{\lambda^2}\right) = \frac{120}{\left(\frac{2}{5}\right)^2}$$

$\sigma = \checkmark$

$$P[T > a] = \Phi\left(\quad b \quad\right) \checkmark$$

If X is exponential (λ) 6

$K = \lceil X \rceil$ is Geometric (p)

$p = 1 - e^{-\lambda}$

(b) $T = K_1 + K_2 \cdots + K_{120}$

K_is are iids $\rightarrow$ each Geometric (p)

$p = (1 - e^{-\frac{2}{3}})$

T is approximately $N(\mu, \sigma)$

$\mu = n \mu_K$

$\mu_K = (120) \frac{1}{p}$

$Var[K] = \sigma^2 = \frac{(120)(1-p)}{p^2}$

X is Geometric (p)

$\mu_X = \frac{1}{p}$

$Var[X] = \frac{1-p}{p^2}$

If X is a RV

$x_1, x_2 \ldots x_n \rightarrow$ Samples of X

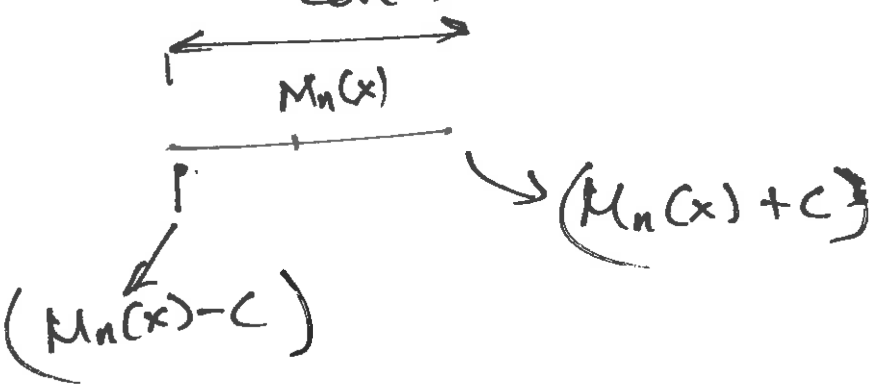
$\downarrow$

independent & identically distributed samples of

$$M_n(x) = \frac{x_1 + x_2 \ldots + x_n}{n}$$

$$P\left[\,\left|M_n(x) - \mu_x\right| \geq c\,\right] \leq \frac{Var[x]}{nc^2} = \alpha$$

confidence Interval



$$P\left[\,\left|M_n(x) - \mu_x\right| < c\,\right] \leq \underbrace{1 - \frac{Var[x]}{nc^2}}_{(1-\alpha)}$$

$(1-\alpha) \rightarrow$ Called Confidence Coefficeint

$\alpha \longrightarrow$ Confidnce level

eg:- Estimation of the Prob. of a Defective Product

$$\text{Product} \begin{cases} \rightarrow \text{Good} \\ \rightarrow \text{Defective} \end{cases}$$

X is Bernoulli (p) RV

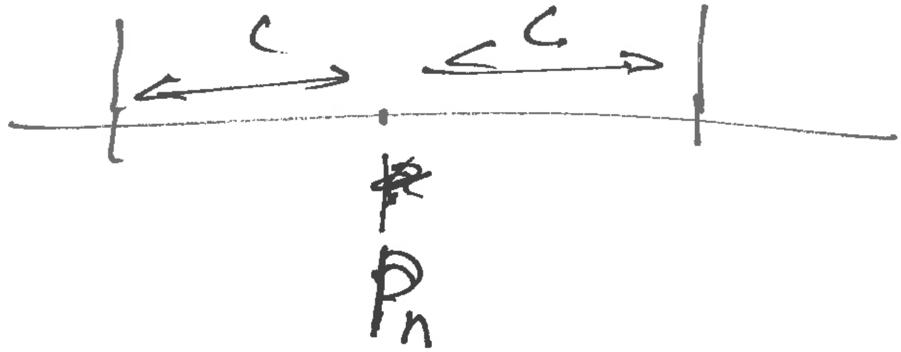
Trying to estimate p

$X_1, X_2 \; . \; . \; . \; , X_n$

Confidnce interval $\rightarrow$ $C = 0.01$

Confidnce Coeff. $= 0.999$

Find n



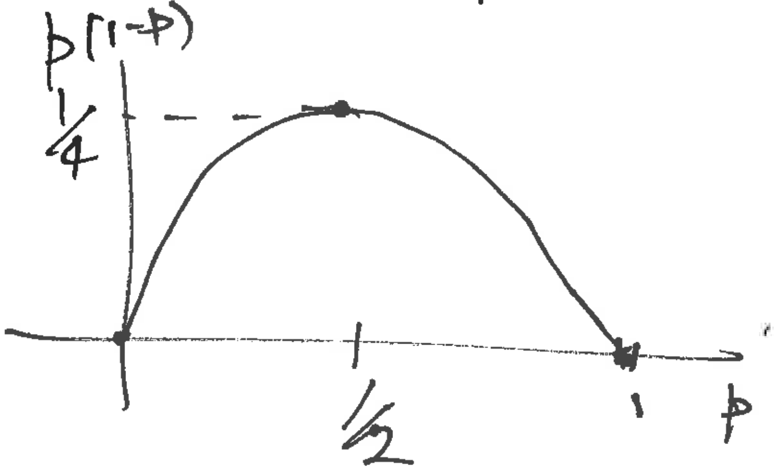
$$P\left[\,|\hat{p} - p| \geq c\,\right] \leq \frac{Var[X]}{nc^2}$$

$X \rightarrow$ Bernoulli

$Var[X] = p(1-p)$

$$\alpha = \frac{Var[X]}{nc^2} = \frac{p(1-p)}{n(0.01)^2}$$

$$\max_{p} \left\{ \frac{p(1-p)}{n(0.01)^2} \right\} = \frac{1/4}{n(0.01)^2}$$



$1 - \alpha = 0.999$

$\alpha = 0.001$

$$\frac{1}{4n(0.01)^2} \leq 0.001$$

$$n \geq \frac{1}{4\times(0.01)^2\,(0.001)}$$

$$n \geq \frac{10^7}{4} = 2.5\times10^6$$

7.4.1 (2nd)

Ex: X is Bernoulli

$$P_X(x) = \begin{cases} 0.1, & x=0 \\ 0.9, & x=1 \\ 0, & \text{o/w} \end{cases}$$

(a) $E[X] = p = 0.9$, $\quad P_X(1) = 0.9$

$$E[X] = P_X(1)$$

(b) Use Chebeshev inequality to find α

$$P\left[\,\left| M_{90}(x) - P_X(1) \right| \geq 0.05\right] \leq \alpha$$

$$\alpha = \frac{Var[X]}{nc^2} = \frac{p(1-p)}{nc^2} = \frac{0.9(0.1)}{90\,(0.05)^2}$$

(C) Find the minimum n to ensure

$$P\left[\,|M_n(x) - P_x(1)| \geq 0.03\,\right] \leq \overset{0.1}{\cancel{0.01}}$$

$$\alpha = 0.1 = \frac{p(1-p)}{n\,(0.03)^2}$$

$$\frac{0.9(0.1)}{n\,(0.03)^2} \leq 0.01$$

$$n \geq (\quad)$$

Moment Generating Func^n (MGF) of a RV X

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$X \longrightarrow$ RV

$M_n = E[X^n] =$ n^{th} Moment of X

$M_1 = \mu_x$

2nd Mom $\leftarrow M_2 - \mu_x^2 = Var[X]$

Def^n:

$$\Phi_X(s) = E[e^{sX}]$$

$$= \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

$M_n = E[X^n]$

$C_n = E[(X - \mu_x)^n]$

Central Moments

Recall: $\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt$ 13

$\Phi_X(s) \longrightarrow$ can calculate if $f_X(x)$ is kno

$$M_n = \frac{d^n}{ds^n} \Phi_X(s) \Big|_{s=0}$$