Conditional pdfs

X is a continuous RV

$f_X(x)$ is given

B is an event of X

$f_{X|B}(x)$

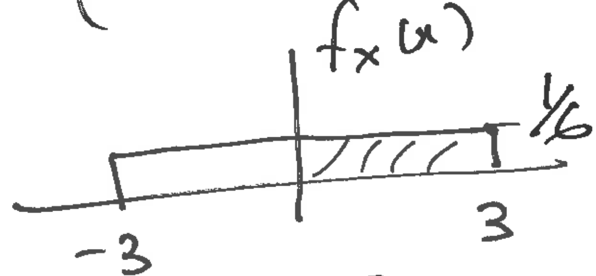
[Recall: $P_{X|B}(x)$

eg:- X is continuous uniform from -3 to $+3$

B: $X \geq 0$

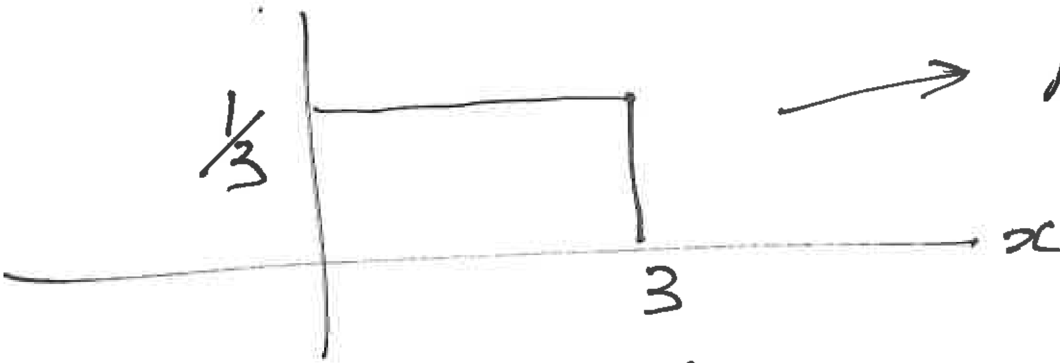
$$f_{X|X\geq 0}(x) = \begin{cases} \dfrac{f_X(x)}{P[B]}, & x \in B \\ 0, & \text{otherw.} \end{cases}$$

$P[B]$?



$P[X \geq 0] = 1/2$

$$f_{X|X\geq 0}(x) = \begin{cases} \dfrac{1/6}{1/2}, & 0 \leq x \leq 3 \quad \cancel{x \geq 0} \\ 0, & \text{o/w.} \end{cases}$$



$$\mu_{X|X\geq 0} = ?$$

$$E[X|X\geq 0] = \int_{-\infty}^{\infty} x \cdot f_{X|X\geq 0}(x)\, dx$$

$$= \int_{0}^{3} x \cdot \tfrac{1}{3}\, dx = \frac{0+3}{2} = \frac{3}{2}$$

$$Var[X|X\geq 0] = \frac{(b-a)^2}{12} = \frac{(3-0)^2}{12} = \checkmark$$

X is a Continuous RV

$B_1, B_2 \ldots B_N$ are mutually Exclusive & collectively Exhaustive events of X

Given: $f_{X|B_i}(x), \quad i = 1, \cdots N$

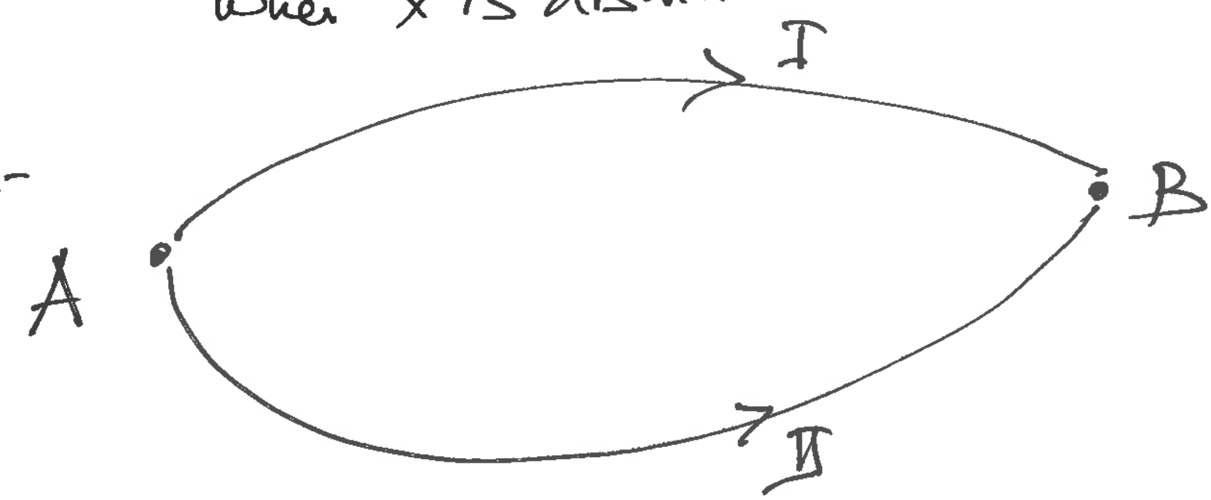
$P[B_i], \quad i = 1, \cdots N$

$$\boxed{f_X(x) = \sum_{i=1}^{N} f_{X|B_i}(x)\, P[B_i]}$$

[Recall: $P_X(x) = \sum_{i=1}^{N} P_{X|B_i}(x)\, P[B_i]$]

when x is discrete.

eg:-



Given Along path I : Time taken (T) is continuous uniform from 2 to 5 hrs.

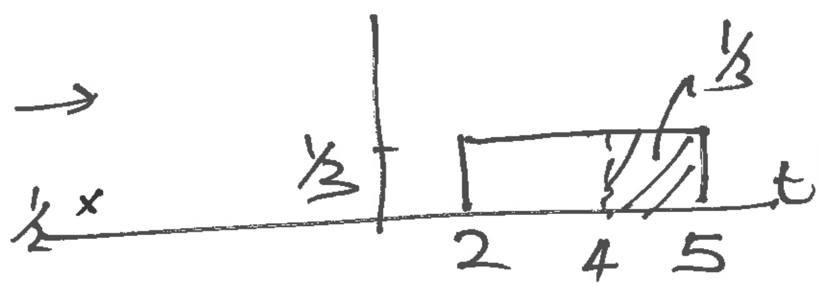
Along " II : . . . (T) is continuous uniform from 1 to 7 hrs

Driver Randomly chooses a path

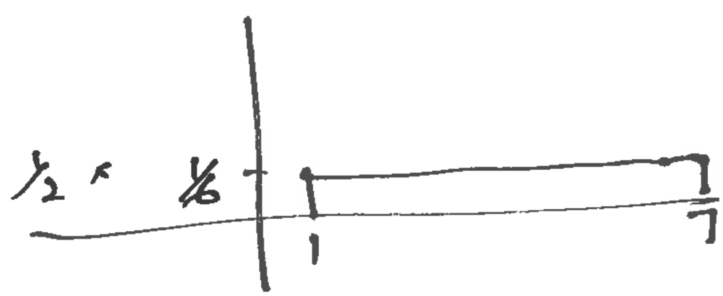
(1) Find the ~~average~~ mean & variance of time taken to reach point B

(2) If ~~he~~ the Driver arrived at B after 4 hrs, find the prob. that he chose path I

Given: $f_{T|I}(t) \rightarrow$



$f_{T|II}(t) \rightarrow$

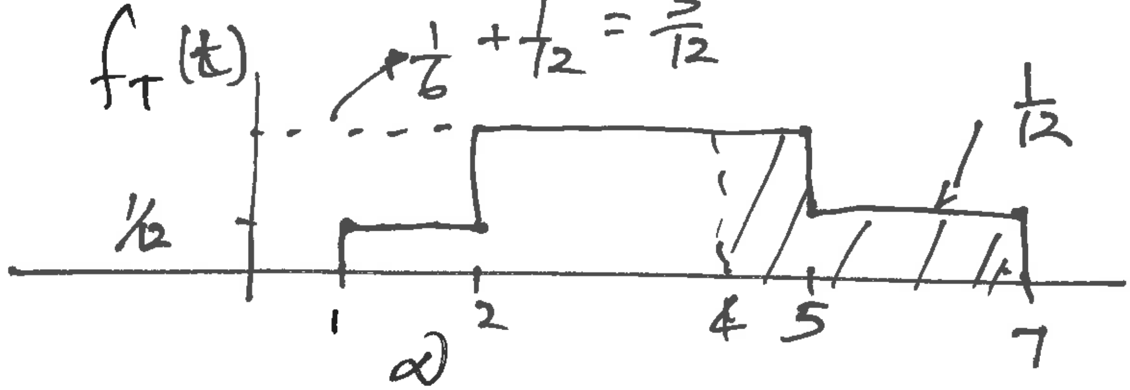


$P[I] = P[II] = \frac{1}{2}$

(1) $E[T] = \mu_T = ?$

$Va[T] = ?$

$$f_T(t) = f_{T|I}(t)\, P[I] + f_{T|II}(t)\, P[II]$$



$$\mu_T = E[T] = \int_{-\infty}^{\infty} t \cdot f_T(t)\,dt$$

$$= \int_1^2 t \cdot \frac{1}{12}\,dt + \int_2^5 t \cdot \frac{3}{12}\,dt + \int_5^7 t \cdot \frac{1}{12}\,dt$$

$=$

$$Va[T] = E[T^2] - \mu_T^2$$

$\downarrow$

HW

(2) Given $T>4$

$$P[I \mid T>4] = ?$$

$$= \frac{P[I \;\&\; T>4]}{P[T>4]} = \frac{P[T>4 \text{ along path } I]}{P[T>4]}$$

$$= \frac{1/3}{\left[\frac{3}{12}(1) + \frac{1}{12}(2)\right]}$$

$\frac{1}{3} \longrightarrow$ Read from $f_{T/I}(t)$

$\left[\frac{3}{12}(1) + \frac{1}{12}(2)\right] \longrightarrow$ Read from $f_T(t)$

Hypothesis testing $\longrightarrow$ Detection

H_0: Null Hypothesis $\longrightarrow$ path I

H_1: Alternate ... $\longrightarrow$ path II

Based on an observation, we need to select the Null or Reject it

$T>4$

Detⁿ:

Digital comm:

Receiver

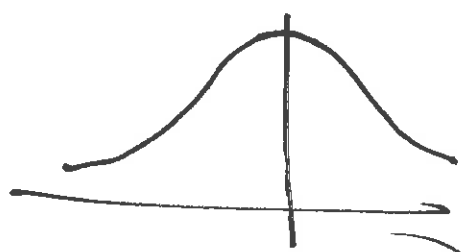
$$\left.\begin{array}{l} 1 \longrightarrow +A \\ 0 \longrightarrow -A \end{array}\right\} + N = R(t)$$

Gaussian

H_0: '0' was transmitted

H_1: '1' " "

$f_N(n) \to (0, \sigma)$



$f_{R|1}(r) \to ?$

$A + N = R \xrightarrow{\text{liner}} R$ is Gaussian

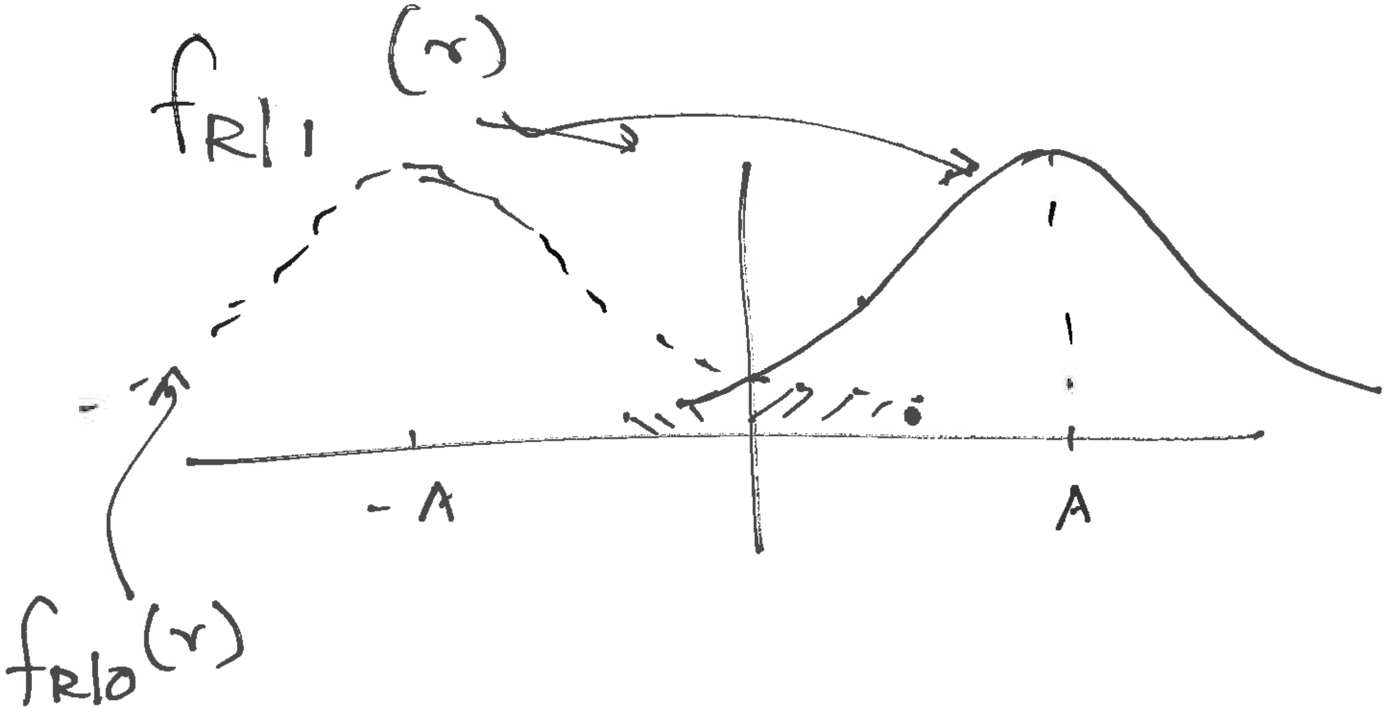
$\mu_R = A + \mu_N = A$

$Var[R] = Var[N] = \sigma^2$

$Y = aX + b$

$\mu_Y = a\mu_X + b$

$Var[Y] = a^2 Var(X)$



If $R(t) > 0 \longrightarrow$ more likely 1 was transmitted

Best Decision Rule

$$R \underset{0}{\overset{1}{\gtrless}} 0$$

$$P[R<0 \mid 1] = P[A+N<0]$$
$$= P[N<-A]$$

$$\downarrow$$

<u>HW</u>

Ch. 5: {2 RVs}

(Ch. 4)

Expt. $\longrightarrow$ 2 values (one for X
" " Y)

Expt. No	X	Y
1	x_1	y_1
2	x_2	y_2
⋮		
N	x_N	y_N

X & Y are discrete

$\underbrace{P_X(x), \; P_Y(y)}_{\text{PMF}}$

Joint PMF

Not^n: $P_{X,Y}(x,y) = P[X=x \; \& \; Y=y]$

X can take values $\{x_1, x_2 \ldots x_N\}$

Y .. .. $\{y_1, y_2 \ldots y_M\}$

Every Expt. outcome is a particular (x,y) combo

NM possible Expt. outcomes,

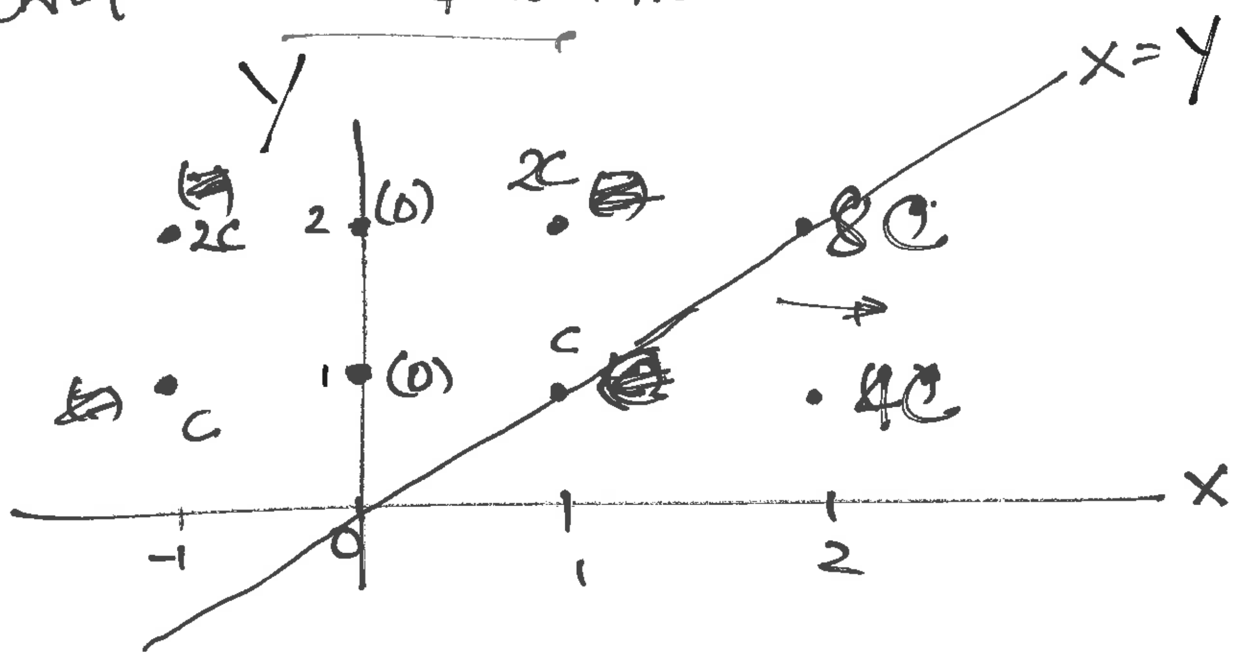
gives the prob. of each outcome

$P_{X,Y}(x,y) \longrightarrow$ for $x = x_1 \ldots x_N$, $y = y_1 \ldots y_M$

eg:- Joint PMF of X & Y is

$$P_{X,Y}(x,y) = \begin{cases} cx^2y, & x=-1,0,1,2 \\ & y=1,2 \\ 0, & \text{o/w.} \end{cases}$$

Graphical Representation



Find c & $P[X > Y]$

$$c + 2c + 0 + 0 + c + 2c + 4c + 8c = 1$$

[In ch. 3 (ch. 2), $\sum_x P_X(x) = 1$]

$\sum\sum$

$$\boxed{\sum_x \sum_y P_{x,y}(x,y) = 1}$$

$$c = \frac{1}{18}$$

$$P[X > Y] = P[X = 2 \,\&\, Y = 1] = 4c$$
$$= \checkmark$$

$P_X(x) = ?$

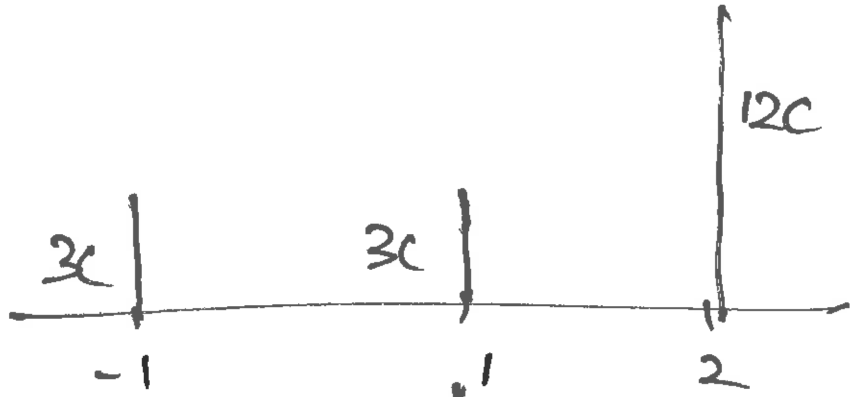
In the example
$X \to -1, 0, 1, 2$

$$P[X=-1],\ P[X=0],\ P[X=1],\ P[X=2]$$
$$\;\;\;\; = 3c \qquad\quad = 0 \qquad\quad = 3c \qquad\quad = 12c$$

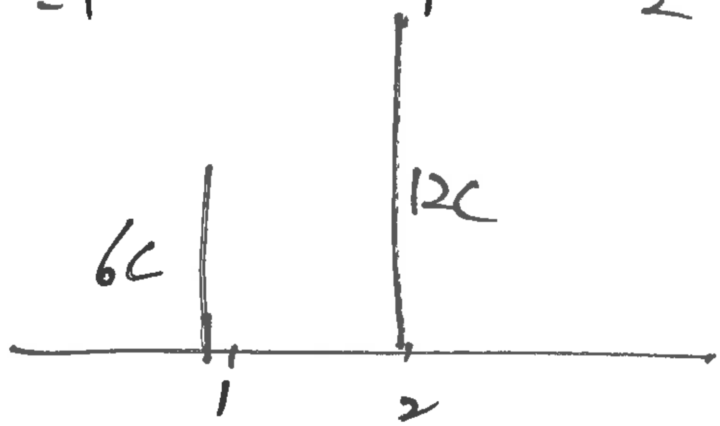
Marginal PMF of X
$$P_X(x) = \sum_y P_{x,y}(x,y)$$

Marginal PMF of Y
$$P_Y(y) = \sum_x P_{x,y}(x,y)$$

$P_X(x)$



$P_Y(y) =$



$P_{X,Y}(x,y)$ can be expressed on Table.

	$y=y_1$	$y=y_2$		$y=y_N$
$x=x_1$	$P_{X,Y}(1,1)$			
$x=x_2$				
.			$P_{X,Y}(x_i,y_j)$	
.				
$x=x_N$				

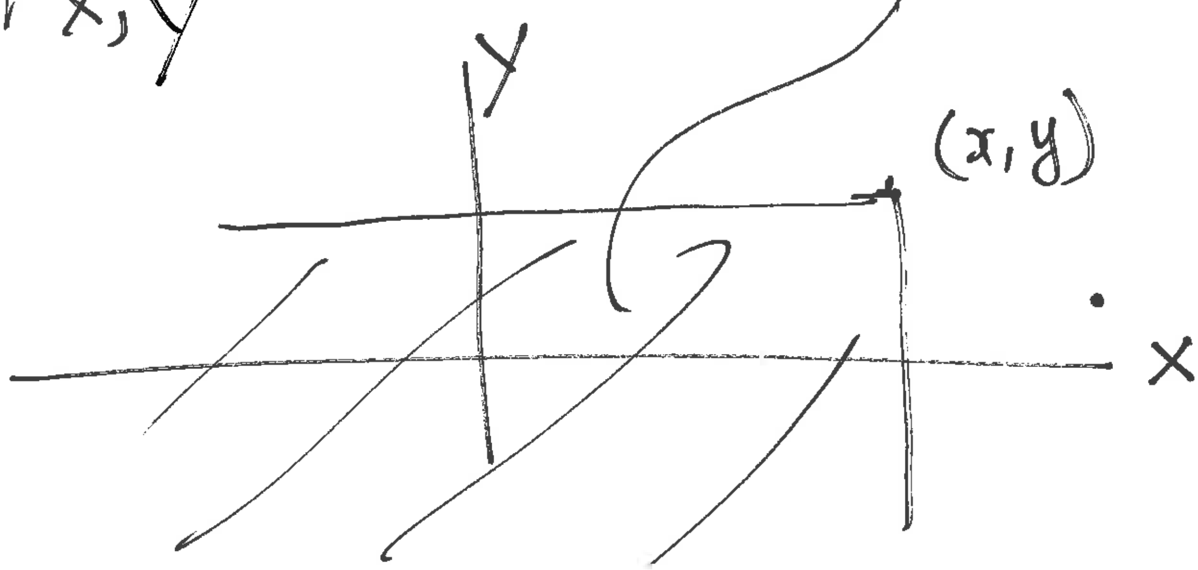
	$y=1$	$y=2$	
$x=-1$	c	$2c$	$\to 3c$
$x=0$	0	0	$\to 0$
$x=1$	c	$2c$	$\to 3c$
$x=2$	$4c$	$8c$	$\to 12$
	$\downarrow$ $6c$	$\downarrow$ $12c$	

$P_X(x)$ — Add Alay row

$P_Y(y)$ → " " colums

Joint CDF

$$F_{X,Y}(x,y) = P[X \le x \;\&\; Y \le y]$$



$$F_{X,Y}(\infty,\infty) = 1$$

$$F_{X,Y}(-\infty,-\infty) = 0$$

$$F_{X,Y}(x,-\infty) = 0, \quad F_{X,Y}(-\infty,y) = 0$$

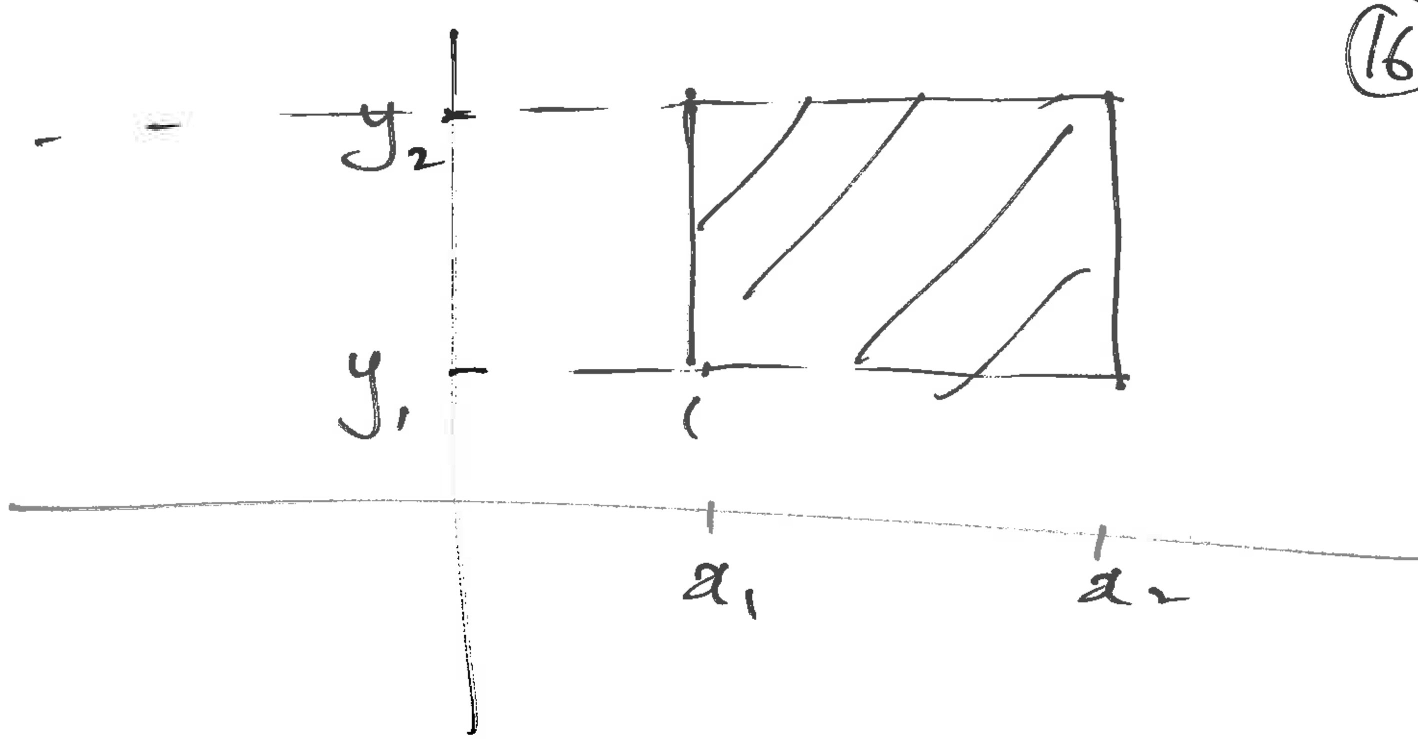
$$F_{X,Y}(x,\infty) = P[X \le x \text{ \& } Y \le \infty]$$

$$= P[X \le x]$$

$$= F_X(x)$$

Marginal CDF of X

$$F_{X,Y}(\infty,y) = F_Y(y)$$



Given $f_{X,Y}(x,y)$

$$P[x_1 \le X \le x_2 \;\&\; y_1 < Y \le y_2]$$

$$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2)$$

$$- F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$$

When X & Y are continuous

Joint pdf.

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

$$\left[f_X(x) = \frac{d}{dx} F_X(x) \right]$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$$

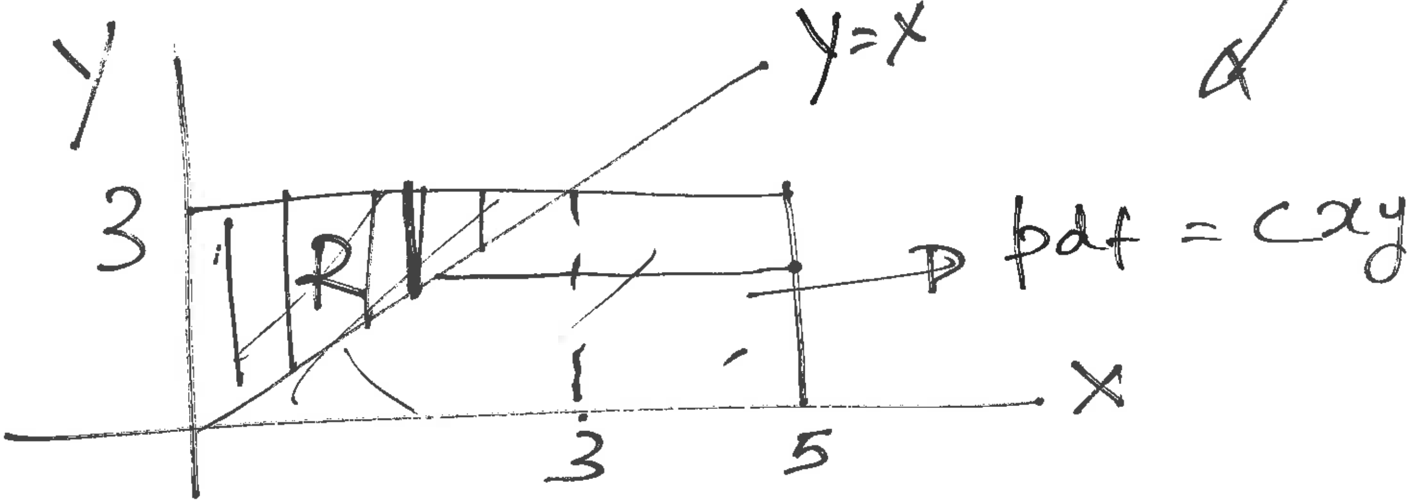
Recall: $\displaystyle\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$$F_{X,Y}(\infty,\infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx = 1$$

eg:- Joint pdf of X & Y is

$$f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \le x \le 5 \\ & 0 \le y \le 3 \\ 0, & o/w. \end{cases}$$

Find C & $P[Y > X]$



$$\int_0^5 \int_0^3 cxy \, dy \, dx = 1$$

$$c \int_0^5 \int_0^3 xy \, dy \, dx = 1$$

$$c \int_0^5 x \left. \frac{y^2}{2} \right|_0^3 dx = 1$$

$$\frac{9c}{2} \left. \frac{x^2}{2} \right|_0^5 = 1$$

$$\frac{9c}{4}(25) = 1$$

$$c = \frac{4}{9 \times 25}$$

$$P[Y > X]$$

Recall $P[a < X \leq b] = \int_a^b f_X(x)\, dx$

$$P \quad \iint_R f_{X,Y}(x,y)\, dy\, dx$$

$R \longrightarrow y > x$

$$\int_0^3 \int_x^3 c\, x y \, dy\, dx$$

Or

$P[Y>X]$

$$= \int_0^3 \int_0^y cxy \, dx \, dy$$

$P[X>Y] = 1 - P[Y>X]$

$$\int_0^3 \int_y^5 cxy \, dx \, dy$$