3.5 (2.5) ✓

$X \rightarrow$ Discrete RV
given $P_X(x)$

$Y = g(X)$

$P_Y(y)$?

X	Y	Prob.

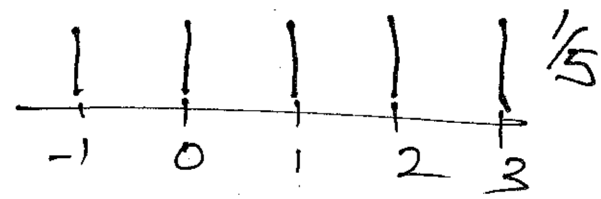
Space of Y

✓ Table gives us $P_Y(y)$

Ex:- X is discrete Uniform from -1 to 3

$Y = X^2 + 4$, $P_Y(y)$?

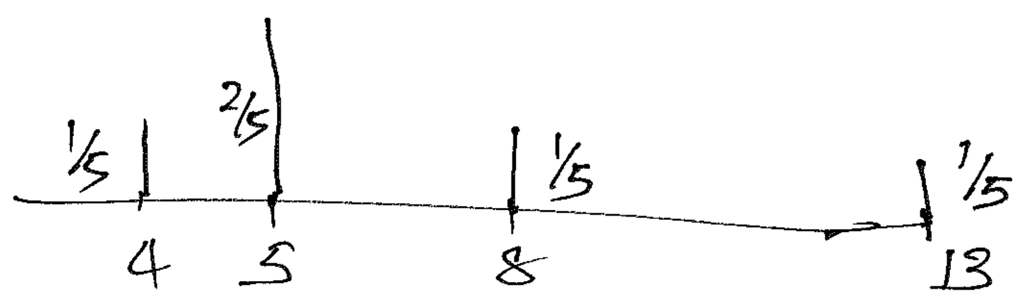
$P_X(x)$



X		Y		Prob
-1	$\rightarrow$	5	$\rightarrow$	$1/5$
0	$\rightarrow$	4	$\rightarrow$	$1/5$
1	$\rightarrow$	5	$\rightarrow$	$1/5$
2	$\rightarrow$	8	$\rightarrow$	$1/5$
3	$\rightarrow$	13	$\rightarrow$	$1/5$

Mutually Exclusive

$P_Y(y)$



3.6 (2.6) Expected value of $Y = g(X)$ ③

of a Derived RV

is called a <u>derived RV</u>

$$E[Y] = \mu_Y$$

Expected value of Y → Mean of Y

<u>Expt. No</u>	X		Y
1	x_1	$\longrightarrow$	$g(x_1)$
2	x_2	$\longrightarrow$	$g(x_2)$
⋮			
N	x_N	$\longrightarrow$	$g(x_N)$

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{g(x_i)}{N} \rightarrow \mu_Y$$

To find μ_Y

we can first find $P_Y(y)$

& then $\mu_Y = \sum_y y \cdot P_Y(y)$

Method 1

Method 2 without finding the PMF of Y

$$\mu_Y = E[Y] = \boxed{E[g(X)] = \sum_x g(x) P_X(x)}$$

Ex:- Go back to the previous Example & find μ_Y

Method 1

$$\mu_Y = \sum_y y \cdot P_Y(y) = 4(1/5) + 5(2/5) + 8(1/5) + 13(1/5)$$

Method 2

$$\mu_Y = E[Y] = E[X^2+4]$$
$$= \sum_{x \to -1,0,1,2,3} (x^2+4)\cdot P_X(x)$$

$$= (5)(1/5) + 4(1/5) + (5)(1/5) + (8)(1/5) + (13)(1/5)$$

3.8 (2.8) Variance, Standard Deviation (Std)

If X is a Discrete RV

$\mu_X \to$ Mean $= E[X]$

Variance of $X \longrightarrow Var[X]$

Std of $X \longrightarrow \sigma_X$

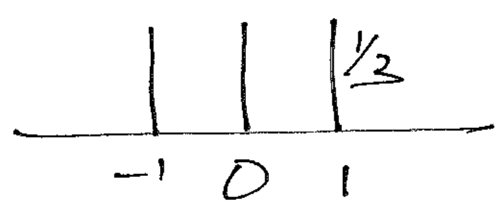
Variance of $X \to Var[X]$ is 9

Measure of the spread how much individual dis varies around μ_X

Ex:- X is discrete uniform from -1 to $+1$

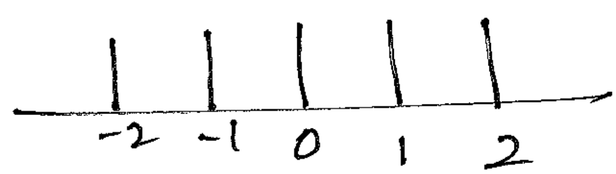
Y . . . . from -2 to $+2$

$P_X(x)$



$\mu_X = 0$

$P_Y(y)$



$\mu_Y = 0$

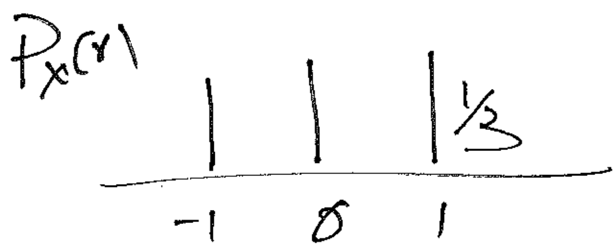
Y has a wider spread

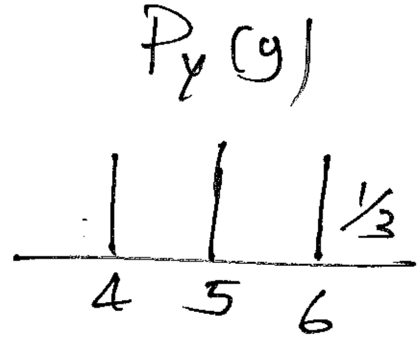
$\therefore$ $Var[Y]$ should be larger than $Var[X]$

Def^n : $Var[X] = E[(X - \mu_X)^2]$

Ex:- ~~P~~ X is discrete uniform from -1 to $+1$

Y is discrete uniform from 4 to 6





$\mu_X = 0$ $\qquad$ $\mu_Y = 5$

We will see $Var[X] = Var[Y]$

$$Var[X] = E[(X-0)^2]$$

$$= E[X^2]$$

$$= \sum_x x^2 \cdot P_X(x)$$

$$= (-1)^2 \tfrac{1}{3} + (0)^2 \tfrac{1}{3} + (1)^2 (\tfrac{1}{3})$$

(8)

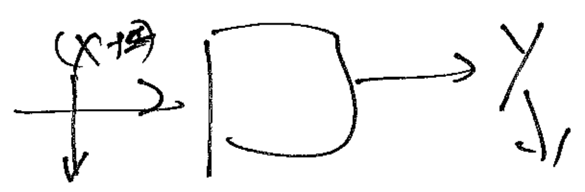
$$\mathrm{Var}[Y] = E\left[(Y-\mu_Y)^2\right]$$

$$= \sum_{y \to 4,5,6} (y-5)^2 \cdot P_Y(y)$$

$$= (-1)^2 \cdot \tfrac{1}{3} + (0)^2 \cdot \tfrac{1}{3} + (1)^2 \tfrac{1}{3}$$

Same as $\mathrm{Var}[X]$

* Calculating $E[g(x)]$ is a linear operation



$\times C$ cY constant

* $E\left[c_1 g_1(x) + c_2 g_2(x)\right]$

$$= c_1 E[g_1(x)] + c_2 E[g_2(x)]$$

$$X \longrightarrow \boxed{(\ )^2} \longrightarrow Y$$

$$2 \longrightarrow 4$$

Nonlinear

$$(2\times 2) \longrightarrow 16 \neq 2\times 4$$

eg:- $E[2X^2 + 3X + 4]$

$$= 2E[X^2] + 3\underbrace{E[X]}_{\mu_X} + \underbrace{4E[1]}_{4}$$

$E[\text{constant}] = \text{constant}$

$$\text{Var}[X] = E[(X-\mu_X)^2] \neq (\underbrace{E[X-\mu_X]}_{\Downarrow})^2$$

$$E[X-\mu_X] = \underbrace{E[X]}_{\mu_X} - \mu_X$$

$$= 0$$

$$\text{Var}[x] = E\left[(x-\mu_x)^2\right]$$

$$= E\left[x^2 - 2x\underbrace{\mu_x}_{\text{constant}} + \mu_x^2\right]$$

$$= E[x^2] - 2\mu_x \underbrace{E[x]}_{\mu_x} + \mu_x^2$$

$$\text{Var}[x] = E[x^2] - \mu_x^2$$

Usually used to calculate Var[x]

$$\text{Var}[x] \geq 0$$

x is a constant

(11)

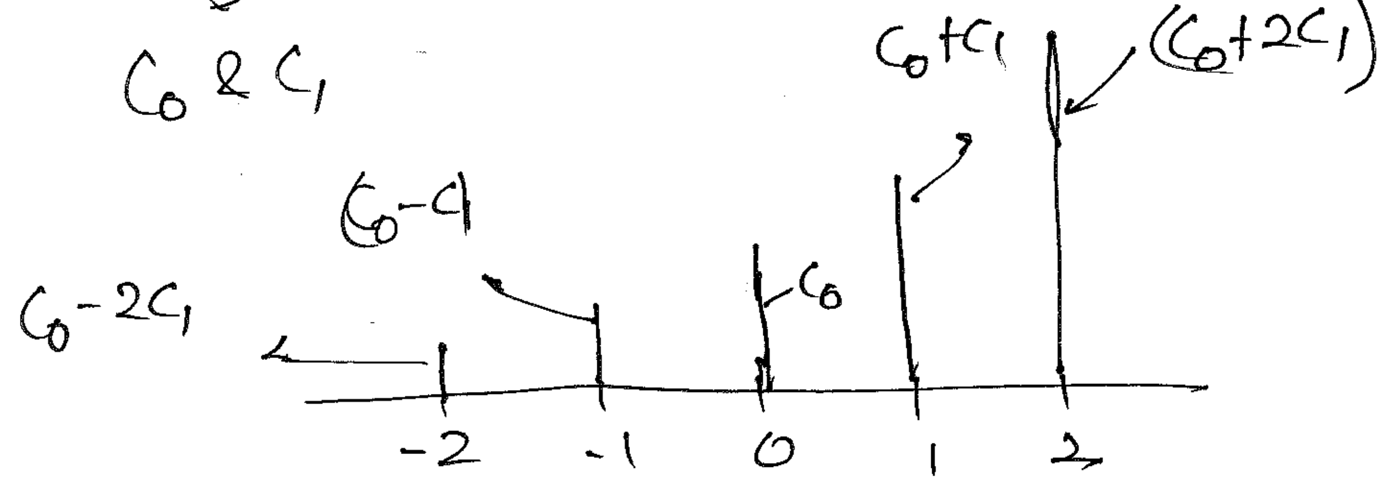
eg:-

$$P_X(x) = \begin{cases} (c_0 + c_1 x), & x = -2, -1, 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

$$P[X > 0] - P[X < 0] = 1/8$$

Find $f_X(x)$, μ_X, $Var[X]$

$\to c_0$ & c_1



$$(c_0 - 2c_1) + (c_0 - c_1) + c_0 + (c_0 + c_1) + (c_0 + 2c_1) = 1$$

$$5c_0 = 1 \longrightarrow c_0 = \frac{1}{5}$$ ✓

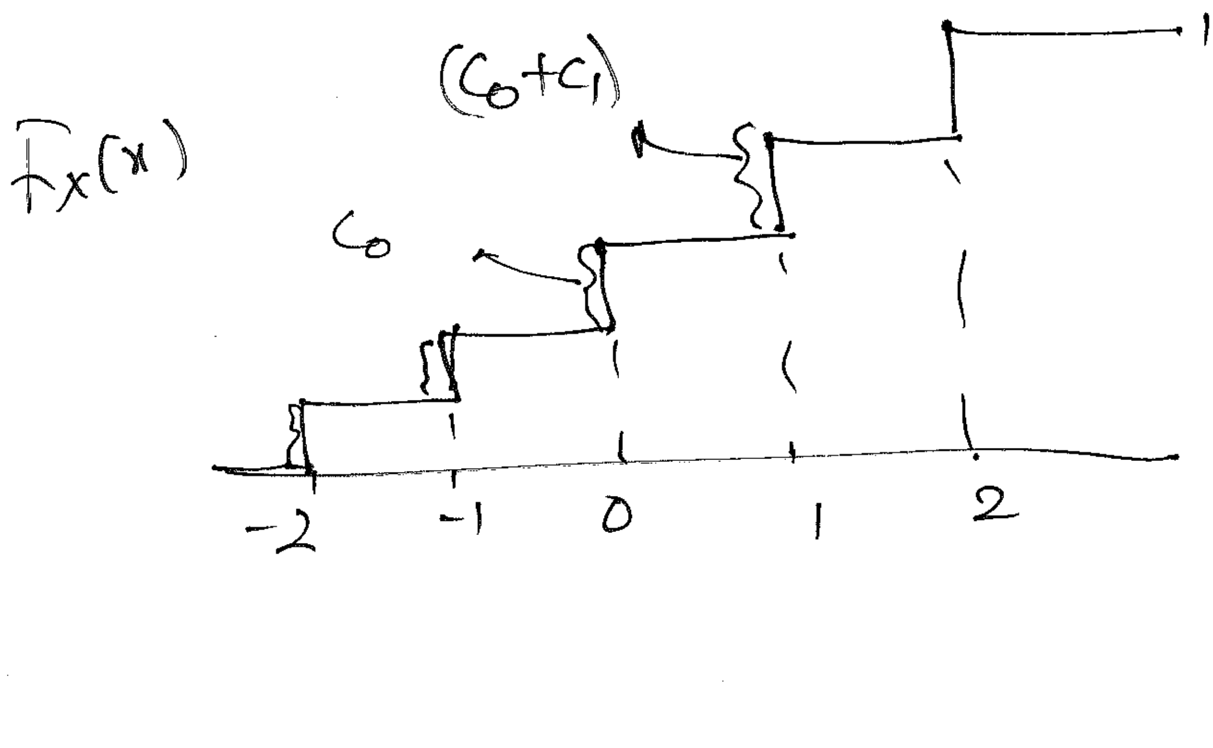
$$P[X>0] - P[X<0] = 1/8$$

$$P[X>0] = (c_0 + c_1) + (c_0 + 2c_1)$$
$$= (2c_0 + 3c_1)$$

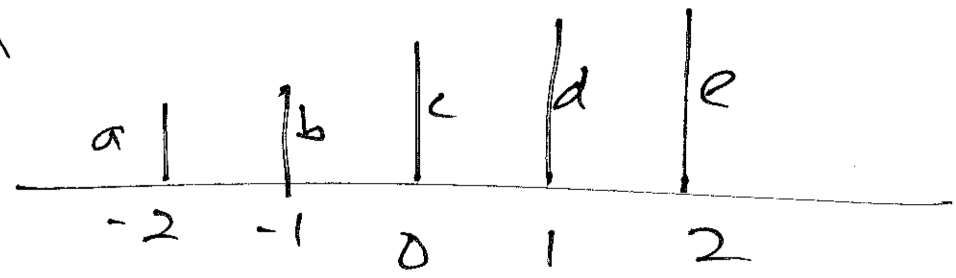
$$P[X<0] = (c_0 - 2c_1) + (c_0 - c_1)$$
$$= (2c_0 - 3c_1)$$

$$\therefore\ P[X>0] - P[X<0] = 6c_1 = 1/8$$

$$c_1 = \frac{1}{48}$$



$P_x(x)$



$a = c_0 - 2c_1$
$b = c_0 - c_1$
$c = c_0$
$d = c_0 + c_1$
$e = c_0 + 2c_1$

$$\mu_x = \sum x \cdot P_x(x)$$
$$= (-2)a + (-1)b + 0 + (1)d + 2(e)$$

$$Var[x] = E[x^2] - \mu_x^2$$

$$E[x^2] = \sum x^2 P_x(x)$$
$$= (-2)^2 a + (-1)^2 b + 0 + (1)^2 d + (2)^2 e$$
$$= \checkmark = f$$

$$Var[x] = f - \mu_x^2$$

Std ~~d~~ of X

σ_X = positive square root of $Var[X]$

$\sigma_X = \sqrt{Var[X]}$ → tells us about the spread again

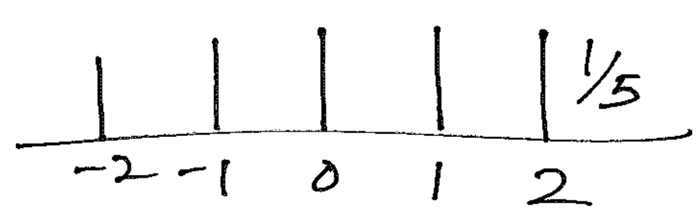
μ_X, $Var[X]$, σ_X → are all statistics of X

3.~~8~~9 (2.~~8~~9) Conditional PMF

X is a RV

$P_X(x)$ → is the PMF of X

eg :-



→ Discrete uniform

Given that ~~If~~ $X \geq 0 \longrightarrow$ condition B

Given condition $B, \longrightarrow X$ is still a Discrete RV

$P_X(x)$ & $P_X(x)$ given B are different

$P_X(x) \downarrow$ Unconditional PMF

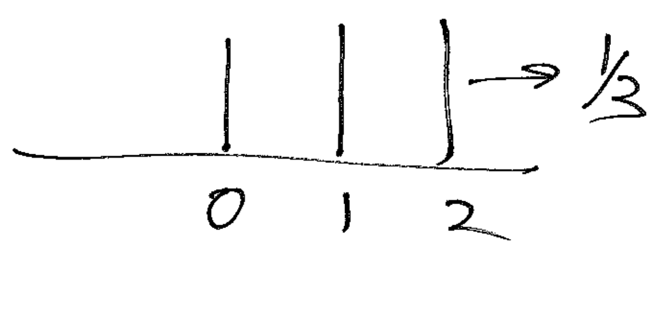
$P_X(x)$ given B $\downarrow$ is called a conditional PMF

Not^n:

$$P_{X|B}(x)$$

$$P_{X|X\geq 0}(x)$$

(first 2 lines ~~cannot~~ are gone)



$P_{X|B}(x) = ?$

Recall $P_X(x) = P[X = x]$

$$P_{X|B}(x) = P[X = x \mid B]$$

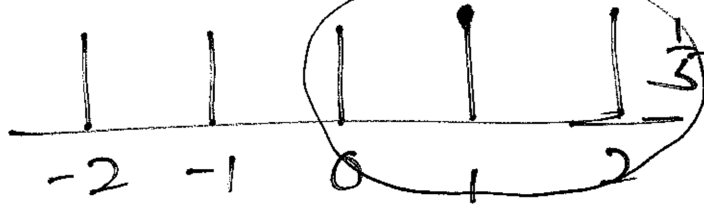
$$= \frac{P[X = x \,\&\, B]}{P[B]}$$

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{P[B]}, & x \in B \\ 0, & \text{otherwise} \end{cases}$$

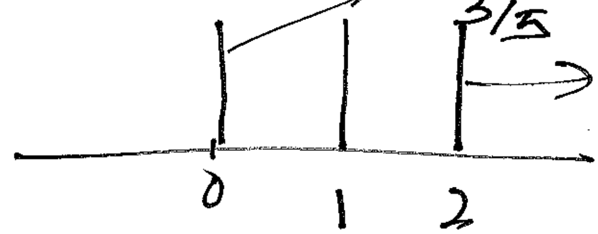
Go back to the example

$P[B] = P[X \geq 0] = \frac{3}{5}$

$P_X(x)$



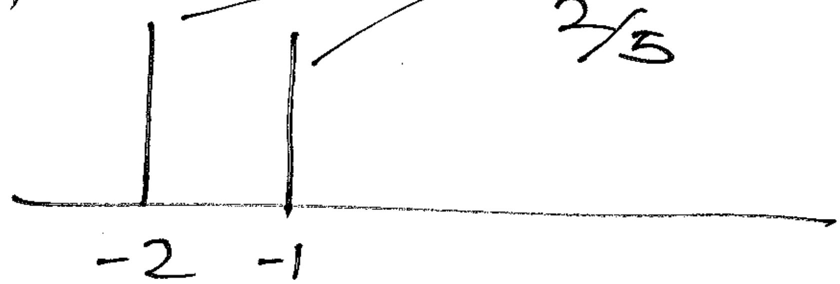
$P_{X|X\geq 0}(x)$



$\frac{1/5}{3/5} = 1/3$

$\frac{1/5}{3/5} = 1/3$

$P_{X|\underbrace{X<0}_{C}}(x)$



$\frac{1/5}{2/5} = 1/2$

$P[C] = \frac{2}{5}$

Mean of X when $X \geq 0$ $\longrightarrow$ Conditional Mean

$\mu_{X|X\geq 0}$

Std. of X when $X \geq 0$ $\longrightarrow$ $\sigma_{X|X\geq 0}$

$\mathrm{Var}[X]$ when $X \geq 0$

$\mathrm{Var}[X \mid X \geq 0] \rightarrow$ Conditional Variance

$\mu_{X \mid X \geq 0} = ?$

$$E[X] = \mu_X = \sum_x x \cdot P_X(x)$$

$$\mu_{X \mid B} = \sum_x x \cdot P_{X \mid B}(x)$$

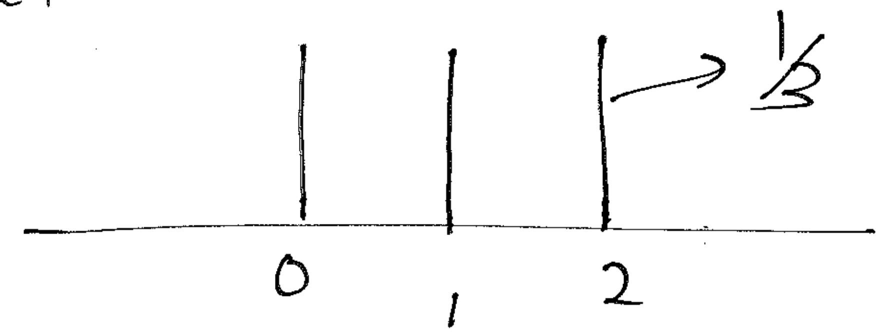
$$\mathrm{Var}[X \mid B] = E[X^2 \mid B] - \mu_{X \mid B}^2$$

$$E[X^2 \mid B] = \sum x^2 \, P_{X \mid B}(x)$$

In the previous example

find $\mu_{X|X\geq 0}$ & $\mathrm{var}[X \mid X \geq 0]$

$P_{X|X\geq 0}(x)$



$$\mu_{X|X\geq 0} = E[X \mid X \geq 0] = 1$$

$$= \sum x \cdot P_{X|X\geq 0}(x)$$

$$= 0\left(\tfrac{1}{3}\right) + (1)\left(\tfrac{1}{3}\right) + (2)\left(\tfrac{1}{3}\right)$$

$$\mathrm{var}[X \mid X \geq 0] = E[X^2 \mid X \geq 0] - \mu^2_{X|X\geq 0}$$

Complete at home.

~~Let~~ X is a discrete RV

$B_1, B_2 \ldots B_N$ are N events of X

→ Mutually Exclusive

& collectively Exhaustive

eg:- In the previous example

B_1: $X \geq 0$, B_2: $X < 0$

Say: we are given

$P_{X|B_i}(x)$ for $i = 1, \ldots N$

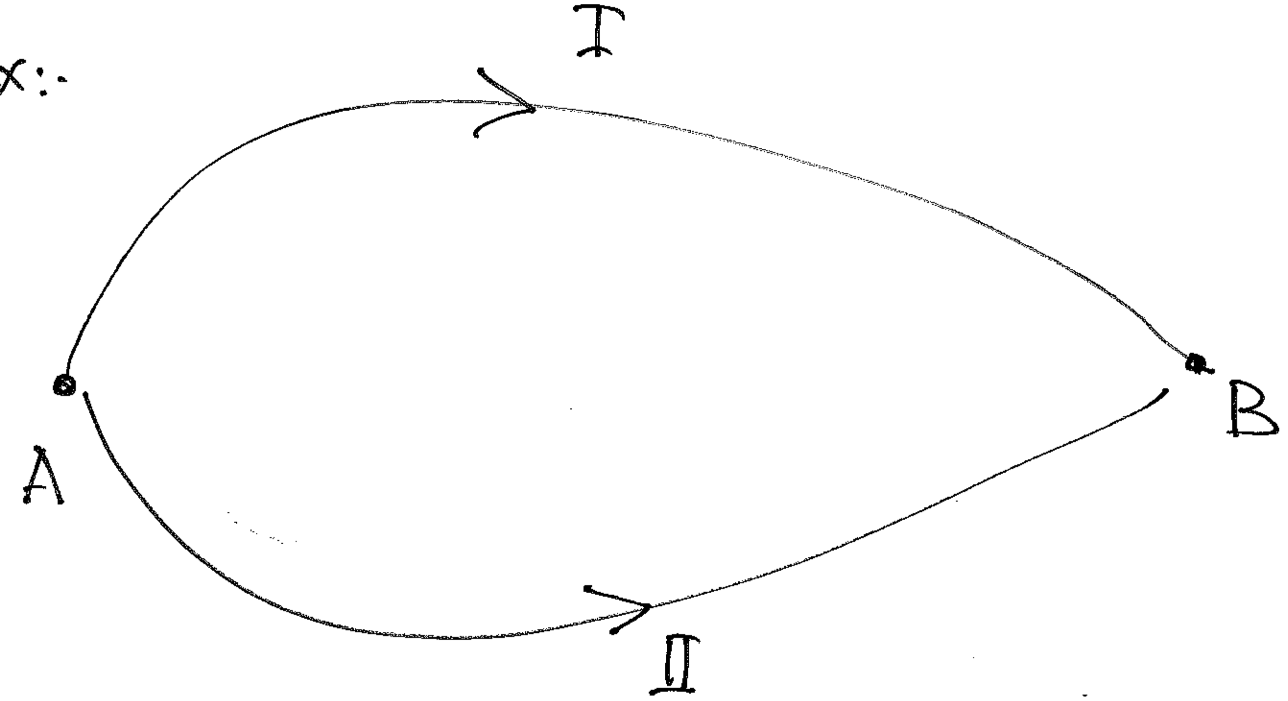
& $P[B_i]$ for $i = 1, \ldots N$

$P_X(x)$?

$$P_X(x) = \sum_{i=1}^{N} P_{X|B_i}(x) P[B_i]$$

PMF version of Total Prob.

Ex::



Along path I: No. of red lights is discrete uniform from 3 to 6

Along path II: " " " " discrete uniform 1 to 8

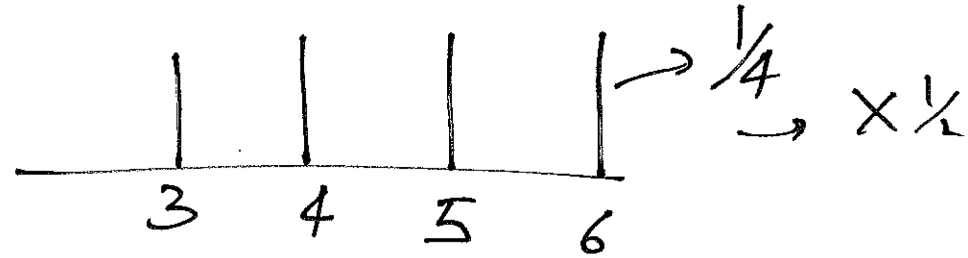
Driver chooses a path randomly

(a) Find the prob. that the driver will see more than 5 red lights.

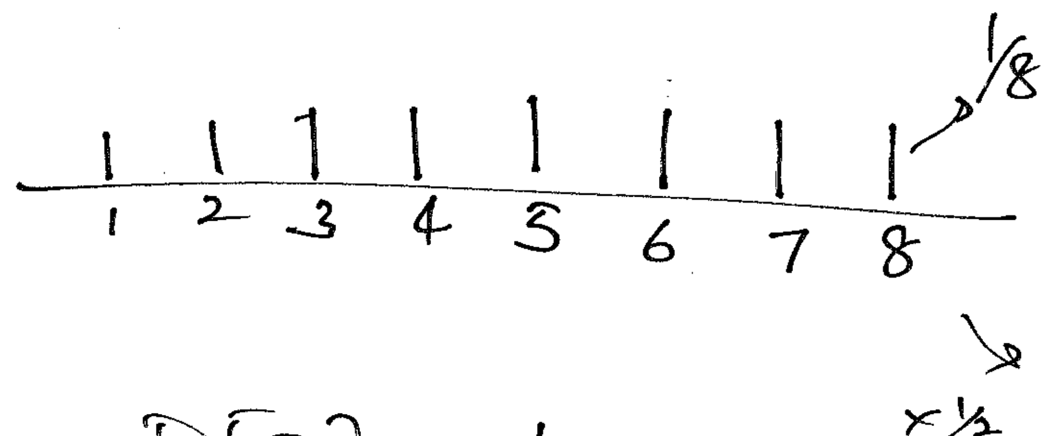
(b) Find mean no. of red lights.

$X \to$ No. of redlights

$P_{X|I}(x)$



$P_{X|II}(x)$



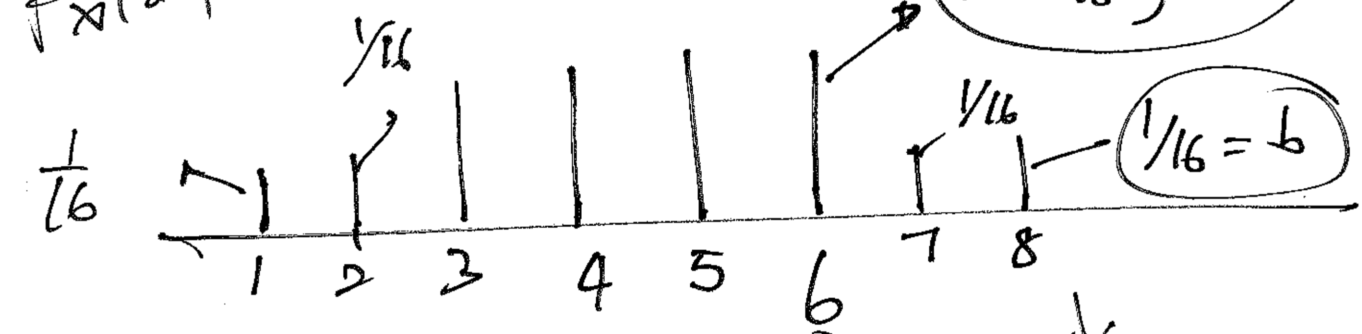
$$P[I] = P[II] = 1/2$$

$$P[X > 5] = ?$$

$$P_X(x) = ?$$

$$P_X(x) = P_{X|I}(x) \cdot \underbrace{P[I]}_{1/2} + P_{X|II}(x) \cdot \underbrace{P[II]}_{1/2}$$

$P_X(x)$



$\left(\frac{1}{8}+\frac{1}{16}\right)=a$

$1/16=b$

$$P_X(x)=\begin{cases}\frac{1}{16}, & x=1,2,7,8\\ \frac{3}{16}, & x=3,4,5,6\\ 0, & \text{o/w.}\end{cases}$$

$P[X>5]=(a+2b)$

$$\mu_X = E[X] = \sum_x x \cdot P_X(x)$$

$$=\left[(1)(b)+(2)(b)+3(a)+4(a)+5(a)+6(a)+7(b)+8(b)\right]$$

Conditional PMFs $\rightarrow$

3rd Edition: 7.1, 7.2

2nd " : 2.9

Q 7.1 (Part A)
(or Q 2.9) $\longrightarrow$ HW

$\downarrow$

Please complete at home.