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Dugga 1: Logic, sets and probability

Pass: at least 60% correct (i.e. 24p). Pass with distinction: at least 80% correct on the two tests added. When in doubt about the interpretation of a question, make reasonable assumptions and motivate those. If you get stuck on a task, try to solve other tasks first, then go back. Please read the whole exam before beginning.

General rules: Mobile phones must be switched off.

Tools: Pen and paper. Calculators are not required.

Predicate calculus (10p)

1. Show, using a truth table, the truth values of the basic logic operations. (6p)

   a) $A \land B$
   b) $A \lor B$
   c) A xor B
   d) $\neg A$
   e) $A \rightarrow B$
   f) $A \leftrightarrow B$

A	B	$(A \land B)$	$(A \lor B)$	(A xor B)	$\neg A$	$A \rightarrow B$	$A \leftrightarrow B$
1	1	1	1	0	0	1	1
1	0	0	1	1	0	0	0
0	1	0	1	1	1	1	0
0	0	0	0	0	1	1	1

2. Break down and check, with a truth table, whether the following statement is always valid. (4p)
   (A xor B) $\leftrightarrow ((A \lor B) \land \neg(A \land B))$
   (i.e. the logic equivalence between the statements "*A xor B*" and "*(A or B) and not (A and B)*")

A	B	(A xor B)	$(A \lor B)$ (1)	$(A \land B)$ (2)	$(1) \land \neg(2)$
1	1	0	1	1	0
1	0	1	1	0	1
0	1	1	1	0	1
0	0	0	0	0	0

Set theory (10p)

3. For *finite sets* A and B, show for each of the following statements if it is true for all A and B, false for all A and B, or true for some A and B. (4p)

   a) $A \subseteq (A \cup B)$
   b) $(A - B) \subseteq A$
   c) $|A \cup B| = |A| + |B| - |A \cap B|$
   d) $|A - B| = |B - A|$

> a) A is always a subset of a union between itself and any B.
> b) (A-B) must include A or less
> c) True for all A and B, Venn diagram
> d) Can be true for disjunct A and B where |A|=|B|, or empty sets.

4. For sets $A=\{x \in \mathbb{N} | \; x \leq 3\}$ and B={a,b,c}, list the elements given by the following statements. (6p)

   a) A
   b) $A \cup B$
   c) $A \times B$
   d) P(A), *i.e. the power set of A*

> a) A = {1, 2, 3}
> b) A ∪ B= {1, 2, 3, a, b, c}
> c) A × B = {(1, a), (1, b) … }
> d) P(A) = {∅, {1}, [{2}, {3}, {1,2} … }

Probability (20p)

4. In one turn of a dice game, two six sided dice are cast, the two values are then added up. If the combined value of the dice exceed 7, player one gets a point. If the combined value is lower than 7, player 2 gets a point. (6p)
   a. Define a suitable sample space, S, for one turn in this game.
   b. Show (using a table or event tree) the number of outcomes in S that favour player one and player two, respectively.
   c. Calculate by relative frequencies the probabilities P(player one wins) and P(player two wins). Answers should be given as a rational number with relevant calculations/reasoning.

a. T = {1, 2, 3, 4, 5, 6}, S = TxT = {(1, 1), (1, 2), (1,3), (1, 4) … (6,6)}

b.

a + b	**1**	**2**	**3**	**4**	**5**	**6**
1	*2*	*3*	*4*	*5*	*6*	7
2	*3*	*4*	*5*	*6*	7	8
3	*4*	*5*	*6*	7	8	9
4	*5*	*6*	7	8	9	10
5	*6*	7	8	9	10	11
6	7	8	9	10	11	12

c. P(A) = P(B) = 15/36

5. At a company with 100 employees, the staff is (for tax reasons) divided into two overlapping work groups. 80 people work in production roles and 40 people with support roles (i.e. sales, financial management, maintenance etc). (6p)
   a. How many people take on the roles of both work groups?
   b. Are the probabilities of an employee belonging to one or the other work group statistically independent?
   c. What is the probability of an employee working in a support role given that we know they already work in a production role, and vice versa (*i.e. $P(A|B)$ and $P(B|A)$*)? Answer in percentages with relevant calculations.

> a. $|S| = |A \cup B| = |A| + |B| - |AB| \Rightarrow 100 = 80 + 40 - |AB| \Rightarrow$ **$|AB| = 20$**
> b. If stat. indep. $P(AB) = P(A)P(B) \Rightarrow |AB|/|S| = (|A|/|S|) * (|B|/|S|)$
>   **No**, LHS: $|AB|/|S|=20\%$, RHS: $(|A|/|S|) * (|B|/|S|) = 32\%$
> c. $P(A|B) = P(AB)/P(B) = (|AB|/|S|) * (|S|/|B|) = 20/80 =$ **25%**
> $P(B|A) = P(AB)/P(A) = (|AB|/|S|) * (|S|/|A|) = 20/40 =$ **50%**

6. In some population group, 10% have disease Z. All are given a screening test. Of those who have the disease, 90% will get a positive screening result (i.e. the test say they have the disease). Of those who don't have the disease, 10% will get a positive screening result. When a person gets a positive screening result, what is the probability that they actually have the disease? Use Bayes' rule and the theorem of total probability to find the answer. (8p)
   a. Derive Bayes' rule from the equation for conditional probability. *i.e. $P(A|B) = P(AB)/P(B)$*
   b. Define the events A and B.
   c. Find numbers for the probabilities needed to use Bayes' rule (i.e. having the disease, getting a positive result, getting a positive result given that one has the disease etc).
   d. What is the probability of having the disease, given a positive screening result?

> a. $P(A|B) = P(AB)/P(B) \Leftrightarrow P(AB) = P(A|B)P(B) = P(B|A)P(A) \Rightarrow$ **$P(A|B) = P(B|A)P(A)/P(B)$**
> b. A (or any arbitrary character) is defined as having the disease
>   B (or any arbitrary character) is defined as getting a positive result
> c. $P(A)=0.1$
> $P(B|A) = 0.9$
> $P(B|A^c) = 0.1$
> $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 0.9*0.1 + 0.1(1-0.1) = 0.09 + 0.09 = 0.18$ (total probability theorem)
> d. $P(A|B) = P(B|A)P(A)/P(B) = 0.9*0.1/0.18 = 50\%$ (Bayes' rule)