# Expected Values

$$E[\cancel{x}\, g(x,y)] = \sum_x \sum_y g(x,y)\, P_{x,y}(x,y).$$

$$E[g(x,y)] = \int\int g(x,y)\, f_{x,y}(x,y)\, dy\, dx$$

* $\gamma_{x,y} = E[XY] \rightarrow$ correlation of x & y

* $Cov[x,y] = E[XY] - \mu_x \mu_y$

* $\rho_{xy} = \dfrac{Cov[X,Y]}{\sigma_x \sigma_y}$

$\rightarrow -1 \leq \rho_{x,y} \leq +1$

# Ex 5.17 (Ex 4.12)



$P_{x,y}(x,y)$

| | $y=0$ | $y=1$ | $y=2$ | $P_x(x)$ |
|---|---|---|---|---|
| $x=0$ | 0.01 | 0 | 0 | $\rightarrow$ 0.01 |
| $x=1$ | 0.09 | 0.09 | 0 | $\rightarrow$ 0.18 |
| $x=2$ | ~~0.1~~ 0 | ~~0.09~~ 0 | 0.81 | $\rightarrow$ 0.81 |
| $P_y(y) \rightarrow$ | 0.1 | 0.09 | 0.81 | |

$r_{xy}$, $Cov[x,y]$, $\rho_{xy}$

$$\rho_{xy} = \frac{Cov[x,y]}{\sigma_x \sigma_y}$$

$$Cov[x,y] = r_{xy} - \mu_x \mu_y$$
$$= E[xy] - \mu_x \mu_y$$

$$E[xy] = \sum_x \sum_y xy\, P_{x,y}(x,y)$$

$$= (0)(0)\,0.01 + (1)(0)\,0.09$$
$$+ (1)(1)\,0.09 + (2)(0)\,\cancel{0.1}\ 0$$
$$\cancel{+ (1)(1)\,0.09} + (2)(1)\,\cancel{0.09}\ 0$$
$$+ (2)(2)\,0.81 = 9$$

$$\mu_x = \sum x P_x(x)$$

$$= 0(0.01) + (1)\,0.18 + (2)(0.81)$$

$$= \sum\sum x P_{x,y}(x,y) = (1)\,0.09 + (1)\,0.09 + 2(0.81)$$

$$= \cancel{(0)(\ ) + 0 + (1)} \qquad = \checkmark$$

$$\mu_y = \cancel{\sum\sum} \sum y P_y(y)$$

$$= (0)\,0.01 + (1)(0.09) + (2)(0.81)$$

$$Cov[x,y] = E[xy] - \mu_x \mu_y \neq \cancel{0}\ 0.$$

you do ~~not~~ need $\sigma_x$ & $\sigma_y$

$$\sigma_x^2 = E[x^2] - \mu_x^2$$

$$\sigma_y^2 = E[y^2] - \mu_y^2$$

...

HW Q 5-8 (A) (Q 4.7)

part A

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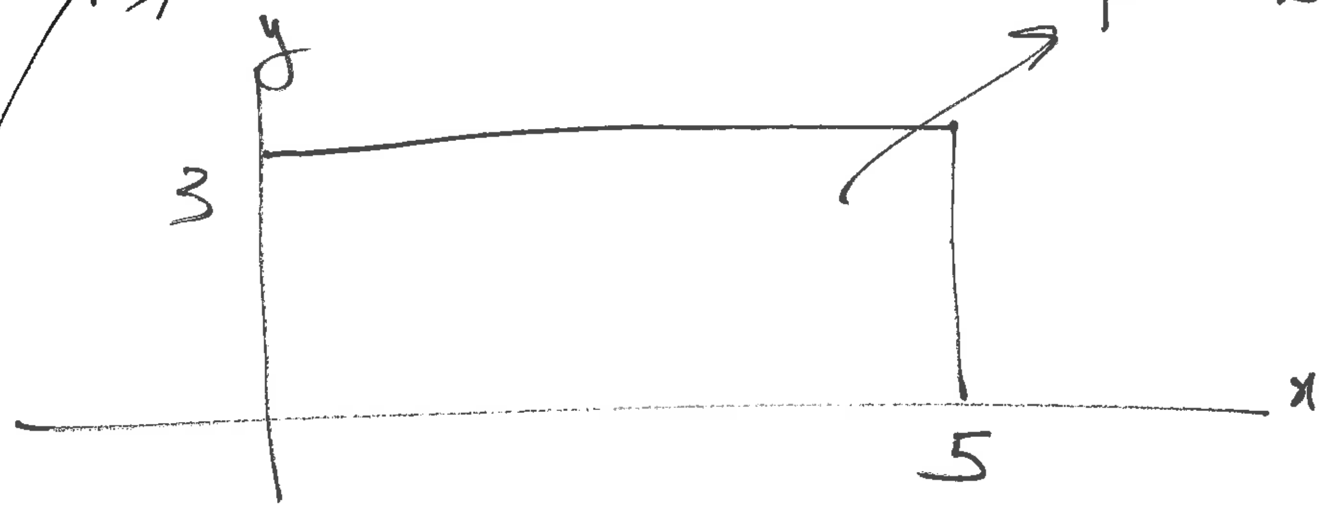
Ex:- $f_{x,y}(x,y) = \begin{cases} \frac{1}{15}, & 0 \le x \le 5,\ 0 \le y \le 3 \\ 0, & \text{Otherwise} \end{cases}$

$\rho_{x,y} = ?$ $\quad \dfrac{Cov[x,y]}{\sigma_x \sigma_x}$

pdf $= \frac{1}{15}$




* $Cov[x,y] = E[xy] - \mu_x \mu_y$

$$E[xy] = \int_0^5 \int_0^3 xy \cdot \frac{1}{15}\, dy\, dx$$

$$= \frac{1}{15} \int_0^5 x\, dx \int_0^3 y\, dy = \frac{1}{15} \times \frac{25}{2} \times \frac{9}{2}$$

(5)

$$\mu_x = \int x\, f_x(x)\, dx \qquad f_x(x) = \int_0^3 \frac{1}{15}\, dy$$

$0 \le x \le 5$

$$= \int_0^5 x \cdot \frac{1}{5}\, dx \qquad = \frac{1}{15} \cdot 3$$

$$= \frac{1}{5} \cdot \frac{25}{2} = \frac{5}{2} \qquad = \begin{cases} \frac{1}{5}, & 0 \le x \le 5 \\ 0, & o.w. \end{cases}$$

$$\mu_y = \int y \cdot f_y(y)\, dy \qquad f_y(y) = \int_0^5 \frac{1}{15}\, dx$$

$0 \le y \le 3$

$$= \frac{1}{3} \cdot \frac{9}{2} = \frac{3}{2} \qquad = \begin{cases} \frac{1}{3}, & 0 \le y \le 3 \\ 0, & o.w. \end{cases}$$

$$Cov[x,y] = E[xy] - \mu_x \mu_y$$

$$= \frac{1}{15} \times \frac{25}{2} \times \frac{9}{2} - \frac{5}{2} \times \frac{3}{2}$$

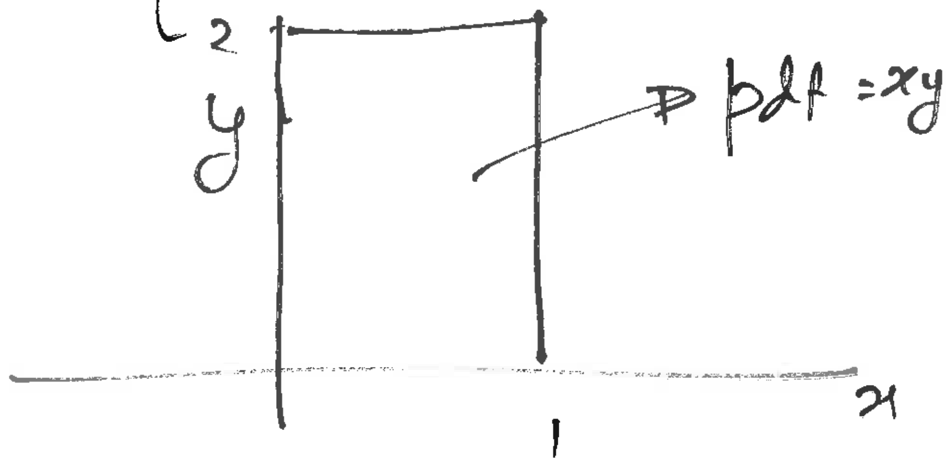
$$= 0$$

No need to calculate $\sigma_x$ & $\sigma_y$

$\rho_{xy} = 0 \longrightarrow$ x & y are uncorrelated

Q 5.8 (Q 4.7)
part B

$$f_{X,Y}(x,y) = \begin{cases} xy, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$




$\rho_{X,Y} = ?$

$\hookrightarrow \; = E[XY] - \mu_X \mu_Y$

$$E[XY] = \int_0^1 \int_0^2 xy \cdot xy \; dy \, dx$$

$$= \int_0^1 x^2 dx \int_0^2 y^2 dy = \frac{x^3}{3}\Big|_0^1 \; \frac{y^3}{3}\Big|_0^2$$

$$= \frac{1}{3}\left(\frac{8}{3}\right) = \frac{8}{9}$$

$$\mu_x = \int_0^1\int_0^2 x \cdot xy \, dy \, dx.$$

$E[X]$

$$= \int_0^1 x^2 \, dx \int_0^2 y \, dy = \frac{1}{3} \frac{(2^2)}{2}$$

$$= \frac{2}{3}$$

$$\mu_y = \int_0^1\int_0^2 y \cdot xy \, dy \, dx$$

$$= \int_0^1 x \, dx \int_0^2 y^2 \, dy = \frac{1}{2} \times \frac{8}{3} = \frac{4}{3}$$

$$\mathrm{Cov}[X,Y] = \frac{8}{9} - \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) = 0.$$

$$\sigma_x^2 = E[X^2] - \mu_x^2$$

$$= \int_0^1\int_0^2 x^2 \cdot xy \, dy \, dx$$

$X$ & $Y$ are 2 RVs

$W = X + Y$

$\mu_W = ? = E[W] = E[X+Y]$

$\mu_W = \mu_X + \mu_Y$

$Var[W] = ?$

$= E[W^2] - \mu_W^2$

$= E[(X+Y)^2] - (\mu_X + \mu_Y)^2$

$= E[X^2 + 2XY + Y^2] - (\mu_X^2 + 2\mu_X\mu_Y + \mu_Y^2)$

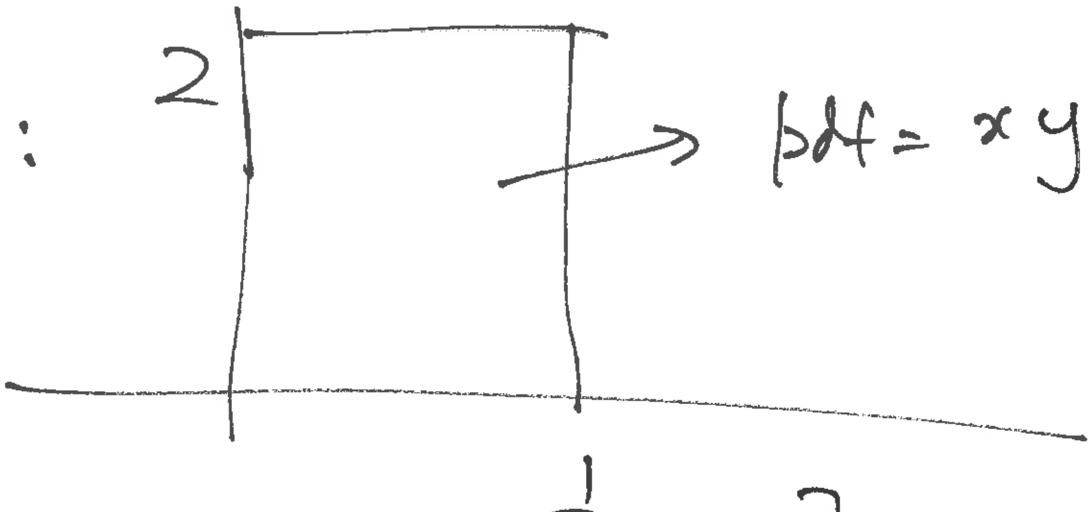
$= E[X^2] - \mu_X^2 + E[Y^2] - \mu_Y^2$
$+ 2[E[XY] - \mu_X\mu_Y]$

$Var[X+Y] = Var[X] + Var[Y] + 2\,Cov[X,Y]$

When $X$ & $Y$ are uncorrelated

$$Var[X+Y] = Var[X] + Var[Y]$$

**HW:**




Find the $V[X+2Y]$

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7.3 Conditional PMF & pdf
(4.8) conditional

– condition on an event B

Recall: if X is a RV
B → Event of X

$$P_{X|B}(x) = \begin{cases} \dfrac{P_X(x)}{P[B]} & x \in B \\ 0, & \text{oth.} \end{cases}$$

$f_X(x)$

$f_{X|B}(x)$

2 RVs $X$ & $Y$

Given: $P_{X,Y}(x,y)$

B : Event of $X$ & $Y$ → B is an event of X and/or Y

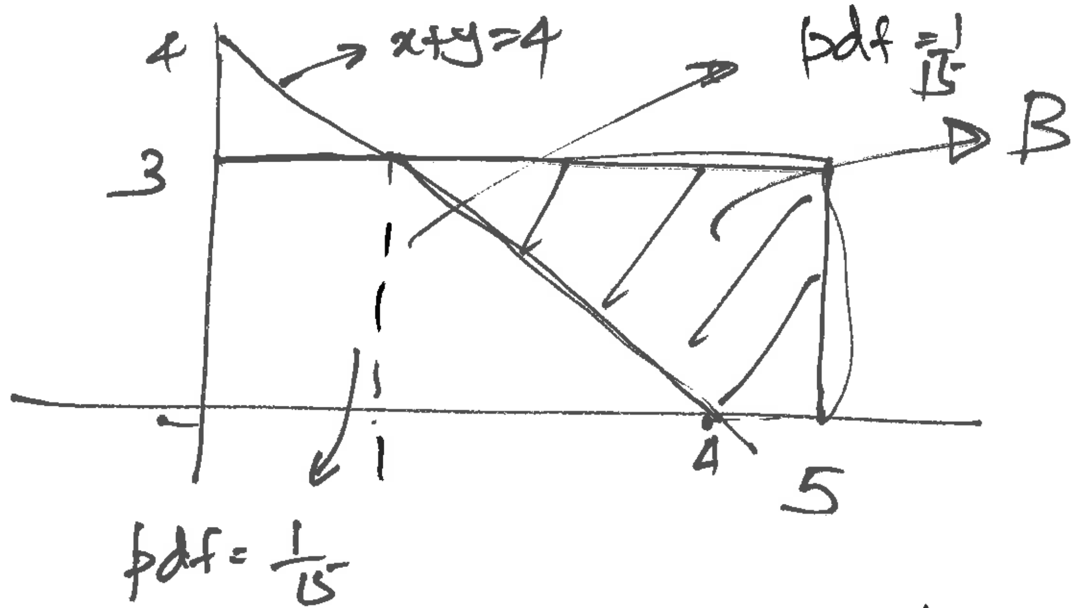
Discrete → $P_{X,Y|B}(x,y) = \begin{cases} \dfrac{P_{X,Y}(x,y)}{P[B]}, & x,y \in B \\ 0, & \text{oth.} \end{cases}$

$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]}, & x,y \in B \\ 0, & \text{oth.} \end{cases}$

B is an event of X & Y

eg:- $f_{X,Y}(x,y) = \begin{cases} \frac{1}{15}, & 0 \leq x \leq 5,\ 0 \leq y \leq 3 \\ 0, & \text{oth.} \end{cases}$

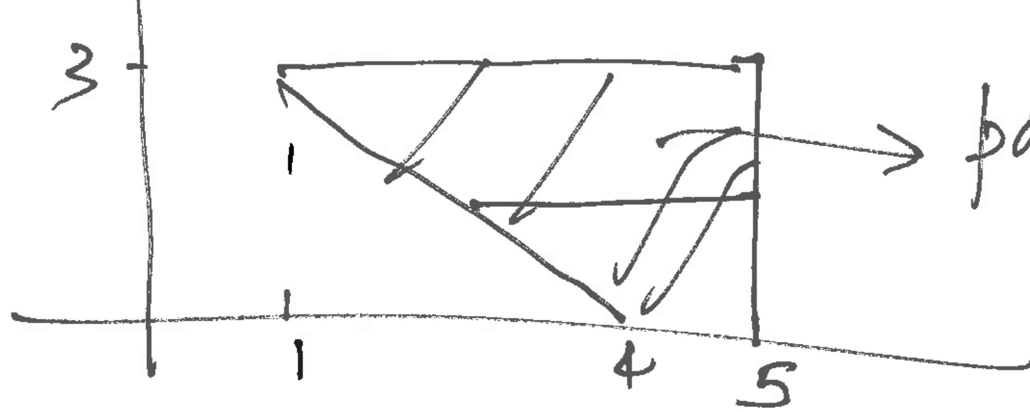
$\hat{f}_{X,Y|\underbrace{X+Y \geq 4}_{B}}(x,y)$ ?




$$P[B] = \frac{1}{15}\left(\text{Shaded Area}\right) = 1/2$$

Divides the rectangle into half of the rectangle

$f_{x,y|x+y\geq 4}(x,y)$




$$\text{pdf} = \frac{1/15}{1/2} = \frac{2}{15}$$

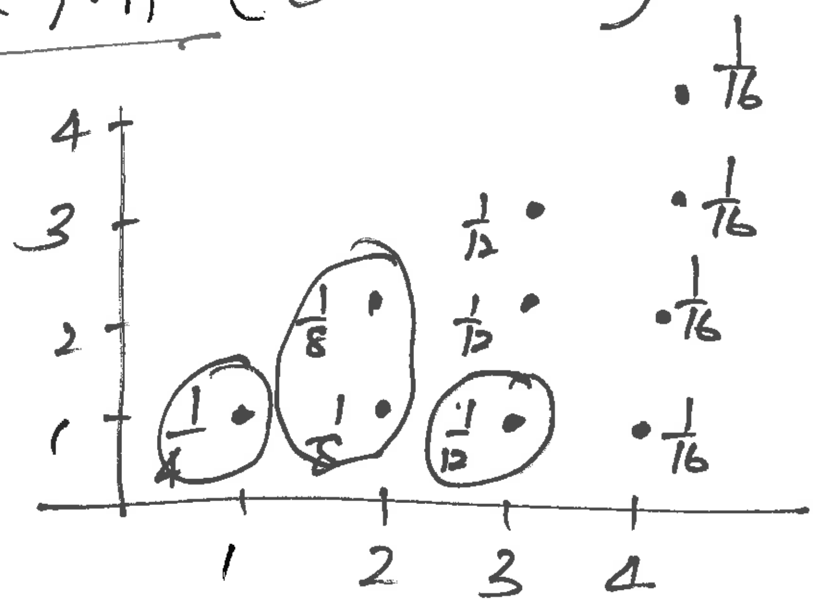
$$\mu_{X|X+Y\geq 4} = E[X \mid X+Y\geq 4]$$

$$= \int_0^3 \int_{4-y}^5 x \cdot \underbrace{f_{x,y|X+Y\geq 4}(x,y)}_{=\frac{2}{15}}\, dx\, dy$$

$$= \int_0^3 \int_{4-y}^5 x \cdot \frac{2}{15}\, dx\, dy$$

$$=$$

Ex 7.11 (Ex 4.13)

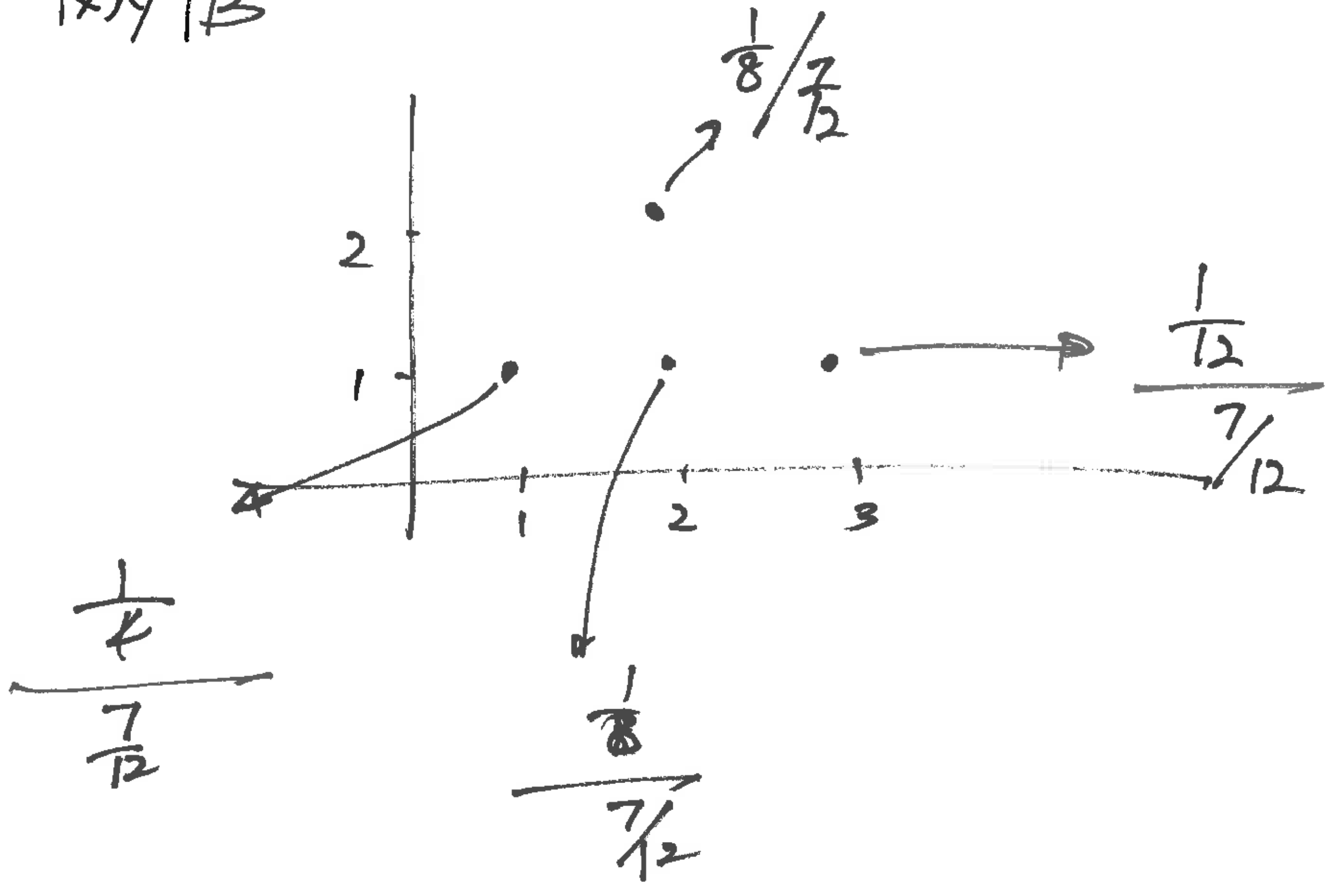



B: $X+Y \le 4$

$$P[B] = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{12} = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$$

$P_{X,Y|B}(x,y)$

$P_{X,Y|B}(x,y)$




$$E[XY|B] = \sum_x \sum_y xy \cdot P_{X,Y|B}(x,y)$$

$$= (1)(1)\frac{\frac{1}{4}}{\frac{7}{12}} + (2)(1)\frac{1/8}{\frac{7}{12}} + (2)(2)\frac{\frac{1}{8}}{\frac{7}{12}}$$

$$+ (3)(1)\frac{1/12}{7/12}$$

- Condition on an event ✓ 

7.4 (4.9) Condition on a RV

$X, Y \rightarrow$ 2 RVs

Discrete case:

Fix $Y = y$ $\rightarrow$ Fix $Y$ at $y$

↳ $X$ is the only RV left.

PMF of ~~X~~ when $Y$ is Fixed at $y$

Not$^n$: $P_{X|Y}(x|y)$ $\rightarrow$ PMF of $X$ when $Y = y$

↳ Conditional

eg: $P_{X|Y}(x|2)$

PMF of $X$

Conditioned on $Y$

$P_{X|Y}(x|y)$ = PMF of $X$ when $Y=y$

$\downarrow$

one RV $\hookrightarrow X$

$= P[X=x \mid Y=y]$

$$= \frac{P[X=x \ \& \ Y=y]}{P[Y=y]}$$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$$

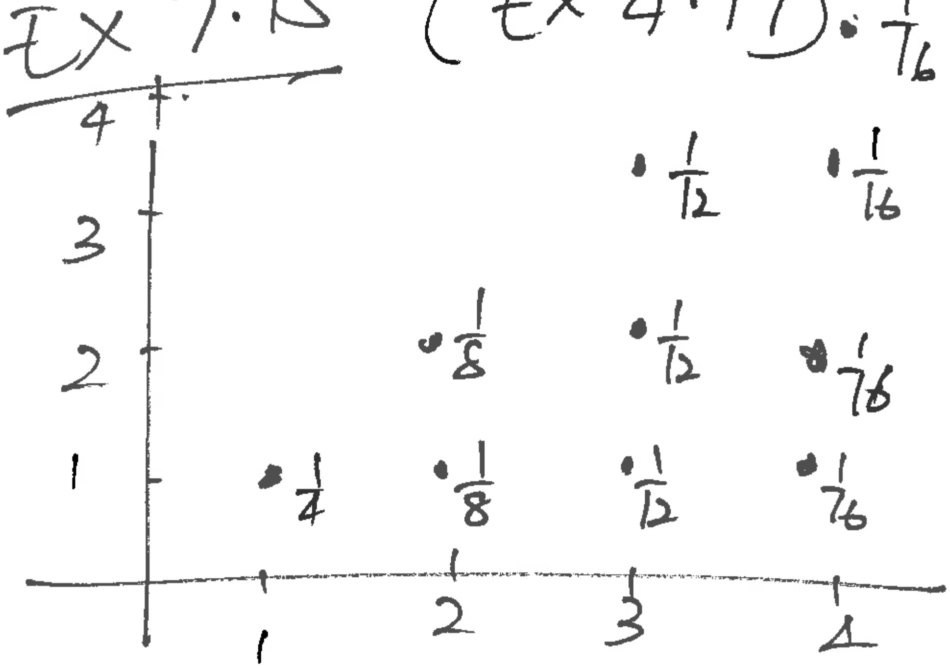
$\downarrow$

$Y$ is the RV

$$P_{X|Y}(x|y)\,P_Y(y) = P_{Y|X}(y|x)\,P_X(x)$$

$\hookrightarrow$ PMF version of Bayes' Thm.

EX 7.15 (EX 4.17)




$P_{Y|X}(y|3)$ ?

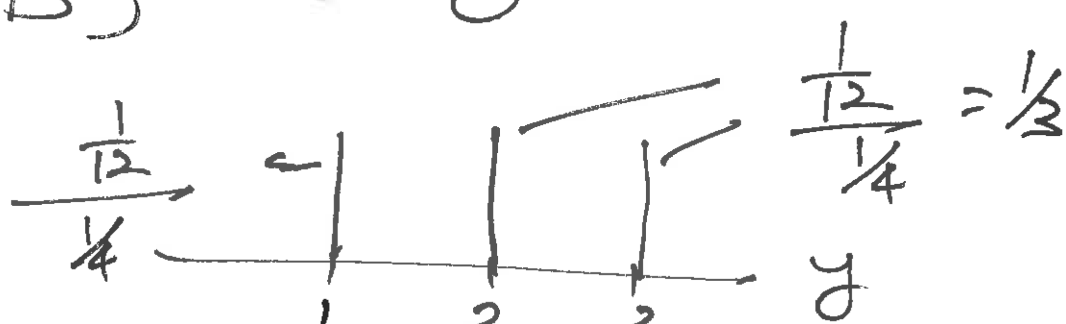
$P_{X|Y}(x|2)$ ?

#

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$$

$P_X(3) = P[X=3] = 1/4$

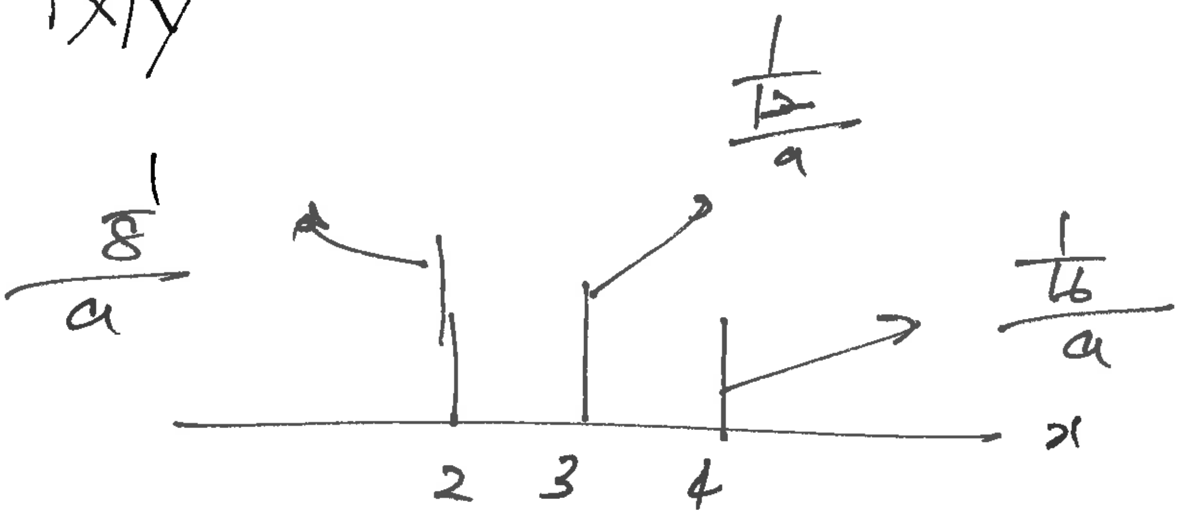
$P_{Y|X}(y|3) \to y = 1, \underline{2, 3}$



$$P_{X|Y}(x|2) = \frac{P_{X,Y}(x,2)}{P_Y(2)}$$

$$P_Y(2) = P[Y=2] = \frac{1}{8} + \frac{1}{12} + \frac{1}{16} = a$$

$P_{X|Y}(x|2) \rightarrow \quad x = 2, 3, 4$




Continuous Case

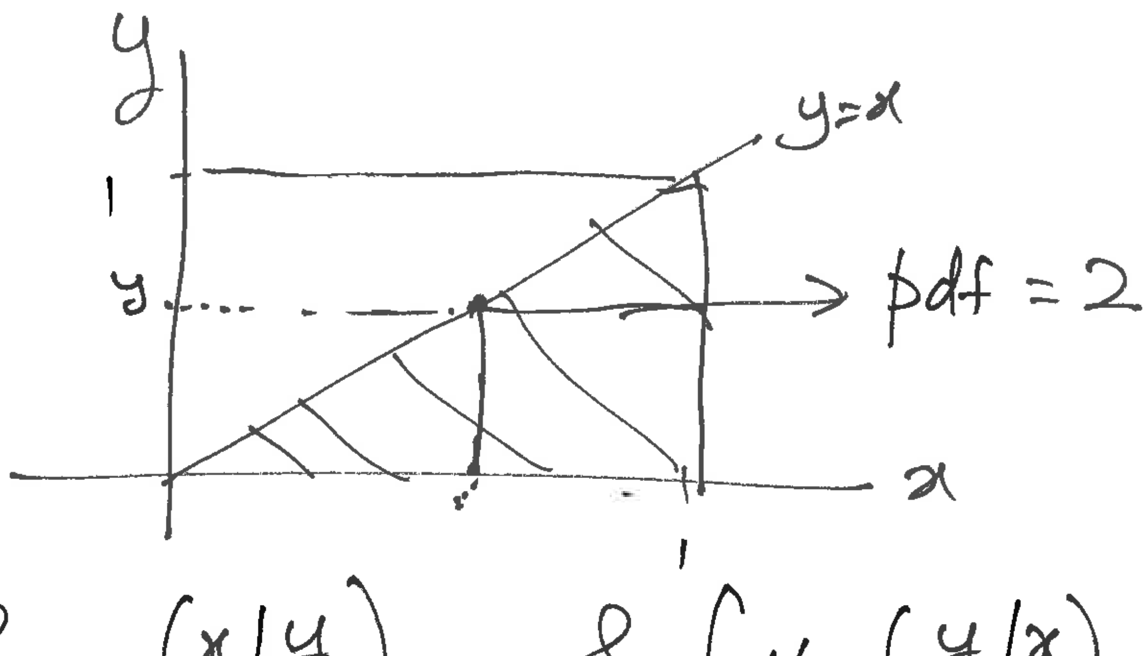
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

pdf of $X \rightarrow$ is the only RV
when $Y$ is set at $y$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Ex 7·16 (Ex 4·19)

$$f_{X,Y}(x,y) = \begin{cases} \cancel{2}\ 2, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$




$f_{X|Y}(x|y)$ & $f_{Y|X}(y|x)$ ?

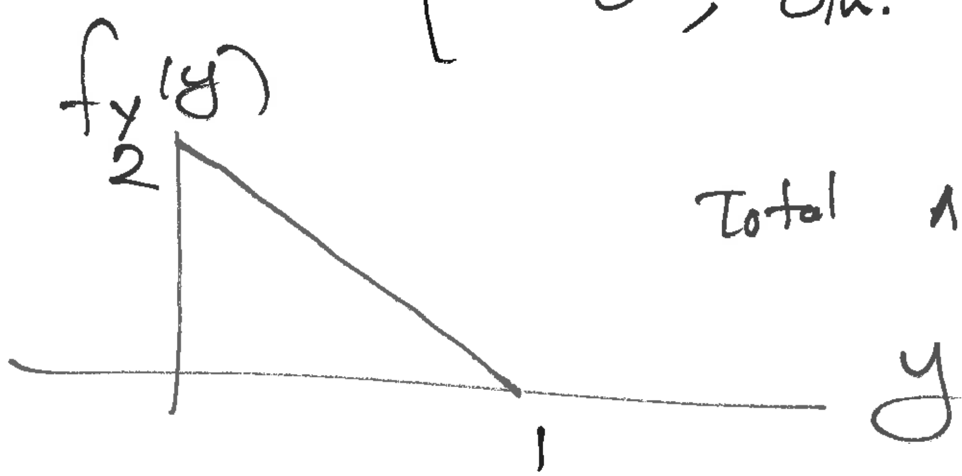
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx.$$

$0 \le y \le 1$

$$= \int_y^1 2\,dx$$

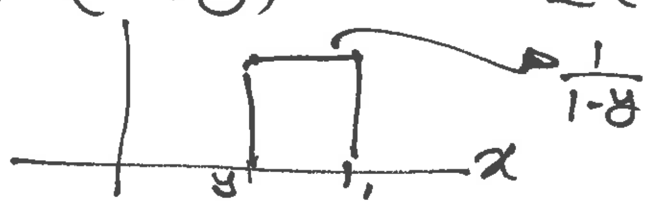
$$= \begin{cases} 2(1-y), & 0 \le y \le 1 \\ 0, & o/w. \end{cases}$$




Total Area = 1

0 → 1

$$f_{X|Y}(x|y) = \frac{2}{2(1-y)}, \quad y \le x \le 1$$



When Y is set at y

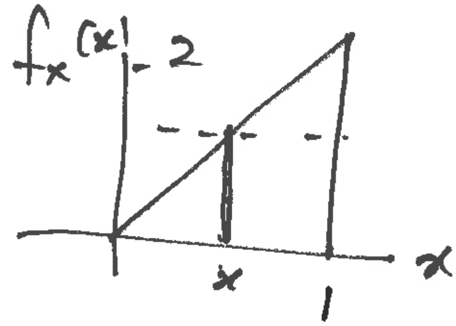
X is uniform from y to 1

$$\mu_{X|Y=1/2} = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_0^x 2\,dy = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & o/w \end{cases}$$

$0 \to 1$




$$f_{Y|X}(y|x) = \begin{cases} \frac{2}{2x}, & 0 \le y \le x \\ 0, & o/w. \end{cases}$$