Problem 4.2.4

In this problem, the CDF of W is,

$$F_W(w) = \begin{cases} 0 & w < -5 \\ (w+5)/8 & -5 \le w < -3 \\ 1/4 & -3 \le w < 3 \\ 1/4 + 3(w-3)/8 & 3 \le w < 5 \\ 1 & w \ge 5 \end{cases}$$

Each question can be answered directly from this CDF.

(a) $P[W \le 4] = F_W(4) = 1/4 + 3/8 = 5/8$

(b) $P[-2 < W \le 2] = F_W(2) - F_W(-2) = 1/4 - 1/4 = 0$

(c) $P[W > 0] = 1 - P[W \le 0] = 1 - F_W(0) = 3/4$

(d) By inspection of $F_W(w)$, we observe that $P[W \le a] = F_W(a) = 1/2$

for a in the range $3 \le a \le 5$.

In this range, $F_W(a) = 1/4 + 3(a-3)/8 = 1/2$.

$\Rightarrow \quad a = 11/3$ //.

Problem 4.3.1

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) From the above pdf we can determine the value of c by integration the pdf and setting it equal to 1

$$\Rightarrow \int_0^2 cx\,dx = 2c = 1. \quad \Rightarrow c = 1/2$$

(b) $P[0 \le x \le 1] = \int_0^1 \frac{x}{2} dx = 1/4$

(c) $P[-1/2 \le x \le 1/2] = \int_0^{1/2} \frac{x}{2} dx = 1/16$

(d) The CDF of X is found by integrating the pdf from 0 to x

$$F_X(x) = \int_0^x f_X(x')\,dx' = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \le x \le 2 \\ 1 & x > 2. \end{cases}$$

Problem 4.4.1

$$f_X(x) = \begin{cases} 1/4 & -1 \le x \le 3 \\ 0 & \text{Otherwise} \end{cases}$$

We recognize that X is a uniform random variable from $[-1, 3]$.

(a) $E[X] = 1$ and $Var[X] = \frac{(3+1)^2}{12} = 4/3$.

(b) The new random variable Y is defined as $Y = h(X) = X^2$.

Therefore $h(E[X]) = h(1) = 1$.

(a) and $E[h(X)] = E[X^2] = Var[X] + E[X]^2 = 4/3 + 1 = 7/3$

(c) $E[Y] = E[h(X)] = E[X^2] = 7/3$

$Var[Y] = E[X^4] - E[X^2]^2 = \int_{-1}^{3} \frac{x^4}{4} dx - \frac{49}{9} = \frac{61}{5} - \frac{49}{9}$ //

Problem 4.4.3

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 2 \\ 1 & x \ge 2. \end{cases}$$

(a) To find $E[X]$, we first find the PDF by differentiating the above CDF.

$$\Rightarrow \quad f_X(x) = \begin{cases} 1/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad E[X] = \int_0^2 \frac{x}{2}\,dx = 1 .$$

(b) $$E[X^2] = \int_0^2 \frac{x^2}{2}\,dx = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2} \times \frac{x^3}{3}\Big|_0^2 = \frac{1}{2} \times \frac{8}{3} = \frac{4}{3}$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{4}{3} - 1 = 1/3 .$$

Problem 4.4.7

To find the moments, we first find the pdf of U by taking the derivative of $F_U(u)$. The CDF and corresponding pdf are.

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 , & 3 \le u < 5 \\ 1 & u \geq 5 \end{cases}$$

$$\Rightarrow \quad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \geq 5 . \end{cases}$$

(a) The expected value of U is

$$E[U] = \int_{-\infty}^{\infty} u f_U(u)\,du = \int_{-5}^{-3} u/8\,du + \int_3^5 \frac{3u}{8}\,du$$

$$= \frac{u^2}{16}\Big|_{-5}^{-3} + \frac{3u^2}{16}\Big|_3^5 = 2$$

(b) The second moment of U is

$$E[U^2] = \int_{-\infty}^{\infty} u^2 f_U(u)\,du = \int_{-5}^{-3} \frac{u^2}{8}\,du + \int_{3}^{5} \frac{3u^2}{8}\,du$$

$$= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} = 49/3.$$

The variance of U is $\mathrm{Var}[U] = E[U^2] - (E[U])^2 = 37/3$

(c) Note that $2^U = e^{(\ln 2)U}$.

This implies that;

$$\int 2^u\,du = \int e^{(\ln 2)u}\,du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2}$$

The expected value of 2^U is then

$$E[2^U] = \int_{-\infty}^{\infty} 2^u f_U(u)\,du = \int_{-5}^{-3} \frac{2^u}{8}\,du + \int_{3}^{5} \frac{3\cdot 2^u}{8}\,du.$$

$$= \left.\frac{2^u}{8\ln 2}\right|_{-5}^{-3} + \left.\frac{3\cdot 2^u}{8\ln 2}\right|_{3}^{5} = \frac{2307}{256\ln 2} = 13.001.$$

Problem 4.5.10

(a) The pdf of a continuous uniform $(-5,5)$ random variable is

$$f_X(x) = \begin{cases} 1/10 & -5 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(b) For $x < -5$, $F_X(x) = 0$, For $x \ge 5$, $F_X(x) = 1$. For $-5 \le x \le 5$, the CDF is

$$F_X(x) = \int_{-5}^{x} f_X(\tau)\,d\tau = \frac{x+5}{10}$$

The complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -5 \\ (x+5)/10, & -5 \le x \le 5 \\ 1 & x > 5 \end{cases}$$

(c) The expected value of X is

$$\int_{-5}^{5} x/10 \, dx = \frac{x^2}{20}\Big|_{-5}^{5} = 0$$

Another way to obtain this answer is to use Theorem 3.6 which says the expected value of X is $E[X] = (5 + -5)/2 = 0$

(d) The fifth moment of X is

$$\int_{-5}^{5} \frac{x^5}{10} dx = \frac{x^6}{60}\Big|_{-5}^{5} = 0$$

(e) The expected value of e^X is

$$\int_{-5}^{5} \frac{e^x}{10} dx = \frac{e^x}{10}\Big|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84$$

Problem 4.6.10

In this problem, we use Theorem 3.14 and the tables for the Φ and Q functions to answer the question. Since $E[Y_{20}] = 40(20) = 800$ and $Var[Y_{20}] = 100(20) = 2000$, we can write

$$P[Y_{20} > 1000] = P\left[\frac{Y_{20} - 800}{\sqrt{2000}} > \frac{1000 - 800}{\sqrt{2000}}\right]$$
$$= P\left[Z > \frac{200}{20\sqrt{5}}\right] = Q(4.47) = 3.91 \times 10^{-6}$$

The second part is a little trickier. Since $E[Y_{25}] = 1000$, we know what the prof will spend around \$1000 will ~~roughly~~ require more than 25 years. In particular, we know that

$$P[Y_n > 1000] = P\left[\frac{Y_n - 40n}{\sqrt{100n}} > \frac{1000 - 40n}{\sqrt{100n}}\right] = 1 - \Phi\left(\frac{100-4n}{\sqrt{n}}\right)$$

$$= 0.99$$

Hence, we must find n such that

$$\Phi\left(\frac{100-4n}{\sqrt{n}}\right) = 0.01$$

Recall that $\Phi(x) = 0.01$ for a negative value of x. This is consistent with our earlier observation that we would need $n > 25$ corresponding to $100-4n < 0$. Thus, we use the identity $\Phi(x) = 1 - \Phi(-x)$ to write

$$\Phi\left(\frac{100-4n}{\sqrt{n}}\right) = 1 - \Phi\left(\frac{4n-100}{\sqrt{n}}\right) = 0.01$$

Equivalently, we have

$$\Phi\left(\frac{4n-100}{\sqrt{n}}\right) = 0.99$$

From the table of the Φ function, we have that $(4n-100)/\sqrt{n} = 2.33$, or

$$(n-25)^2 = (0.58)^2 n = 0.3393\, n$$

Solving this quadratic yields $n = 28.09$. Hence, only after 28 years are we 99 percent sure that the prof will have spent \$1000. Note that a second root of the quadratic yields $n = 22.25$.

This root is not a valid solution to our problem. Mathematically, it is a solution of our quadratic in which we chose the negative root of $\sqrt{\alpha}$. This would correspond to assuming the standard deviation of Y_n is negative.

Problem 4.6.11

We are given that there are 100,000,000 men in the United States and 23,000 of them are at least 7 feet tall, and the heights of U.S men are independent Gaussian random variables with mean 5'10".

(a) Let H denote the height in inches of a U.S male. To find σ_x, we look at the fact that the probability that $P[H \geq 84]$ is the number of men who are at least 7 feet tall divided by the total number of men (the frequency interpretation of probability). Since we measure H in inches, we have

$$P[H \geq 84] = \frac{23,000}{100,000,000} = \Phi\left(\frac{70-84}{\sigma_x}\right) = 0.00023$$

Since, $\Phi(-x) = 1 - \Phi(x) = Q(x)$

$$Q(14/\sigma_x) = 2.3 \times 10^{-4}$$

From Table 3.2, this implies $14/\sigma_x = 3.5$ or $\sigma_x = 4$.

(b) The probability that a random chosen man is at least 8 feet tall is

$$P[H \geq 96] = Q\left(\frac{96-70}{4}\right) = Q(6.5)$$

Unfortunately Table 3.2 doesn't include $Q(6.5)$, although it should be apparent that the probability is very small. In fact, $Q(6.5) = 4.0 \times 10^{-11}$.

(c) First we need to find the probability that a man is at least 7'6".

$$P[H \geq 90] = Q\left(\frac{90-70}{4}\right) = Q(5) \approx 3 \times 10^{-7} = \beta.$$

Although Table 3.2 stops at $Q(4.99)$, if you're curious, the exact value is $Q(5) = 2.87 \times 10^{-7}$.

Now we can begin to find the probability that no man is at least 7'6". This can be modeled as 100,000,000 repetitions of a Bernoulli trial with parameter $1-\beta$. The probability that no man is at least 7'6" is

$$(1-\beta)^{100,000,000} = 9.4 \times 10^{-14}$$

(d) The expected value of N is just the number of trials multiplied by the probability that a man is at least 7'6".

$$E[N] = 100,000,000 \cdot \beta = 30$$

Problem 4.7.1

(a) Using the given CDF

$$P[X < -1] = F_X(-1^-) = 0$$

$$P[X \leq -1] = F_X(-1) = -1/3 + 1/3 = 0$$

Where $F_X(-1^-)$ denotes the limiting value of the CDF found by approaching -1 from the left. Likewise, $F_X(-1^+)$ is interpreted to be the value of the CDF found by approaching -1 from the right. We notice that these two probabilities are the same and therefore the probability that X is exactly -1 is zero.

(b)

$$P[X<0] = F_X(0^-) = 1/3$$
$$P[X \leq 0] = F_X(0) = 2/3$$

Here we see that there is a discrete jump at $X=0$. Approached from the left the CDF yields a value of $1/3$ but approached from the right the value is $2/3$. This means that there is a non-zero probability that $X=0$, in fact that probability is the difference of the two values.

$$P[X=0] = P[X \leq 0] - P[X<0] = 2/3 - 1/3 = 1/3.$$

(c)

$$P[0<X\leq 1] = F_X(1) - F_X(0^+) = 1 - 2/3 = 1/3.$$
$$P[0\leq X\leq 1] = F_X(1) - F_X(0^-) = 1 - 1/3 = 2/3$$

The difference in the last two probabilities above is that the first was concerned with the probability that x was strictly greater than 0, and the second with the probability that x was greater than or equal to zero. Since the second probability is a larger set (it includes the probability that $X=0$) it should always be greater than or equal to the first probability. The two differ by the probability that $X=0$, and this difference is non-zero only when the random variable exhibits a discrete jump in the CDF.

Prob 3.6.1.

Problem 4.7.6

(a) Since the conversion time cannot be negative, we know that $F_W(w) = 0$ for $w < 0$. The conversion time W is zero iff either the phone is busy, no one answers, or if the conversation time X of a completed call is zero. Let A be the event that the call is answered. Note that the event A^c implies $W = 0$. For $w \geq 0$

$$F_W(w) = P[A^c] + P[A]\, F_{W|A}(w) = (1/2) + (1/2)\, F_X(w).$$

Thus the complete CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 1/2 + (1/2) F_X(w) & w \geq 0 \end{cases}$$

(b). By taking the derivative of $F_W(w)$, the pdf of W is

$$f_W(w) = \begin{cases} (1/2)\delta(w) + (1/2) f_X(w) & \\ 0 & \text{otherwise} \end{cases}$$

Next, we keep in mind that since X must be non-negative, $f_X(x) = 0$ for $x < 0$. Hence, $f_W(w) = (1/2)\delta(w) + (1/2) f_X(w)$

(c) From the pdf $f_W(w)$, calculating the moments is straight forward.

$$E[W] = \int_{-\infty}^{\infty} w\, f_W(w)\, dw = (1/2)\int_{-\infty}^{\infty} w f_X(w)\, dw = E[X]/2.$$

The second moment is,

$$E[W^2] = \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = \frac{1}{2} \int_{-\infty}^{\infty} w f_X(w)\, dw = E[X]/2.$$

The variance of W is

$$Var[W] = E[W^2] - (E[W])^2 = E[X^2]/2 - (E[X]/2)^2$$
$$= (1/2) \cdot Var[X] + (E[X])^2/4.$$

Problem 4.7.8

Let G denote the event that the throw is good, that is, no foul occurs. The CDF of D obeys

$$F_D(y) = P[D \le y / G]\, P[G] + P[D \le y / G^c] \cdot P[G^c].$$

Given the event G,

$$P[D \le y | G] = P[X \le y - 60] = 1 - e^{-(y-60)/10} \; ; (y \geq 60)$$

Of course, for $y < 60$, $P[D \le y/G] = 0$. From the problem statement, if the throw is a foul, then $D = 0$.

This implies,
$$P[D \le y / G^c] = u(y).$$

where $u(\cdot)$ denotes the unit step function.

Since $P[G] = 0.7$, we can write

$$F_D(y) = P[G]\, P[D \le y/G] + P[G^c] \cdot P[D \le y/G^c].$$

$$= \begin{cases} 0.3\, u(y) & ; \; y < 60 \\ 0.3 + 0.7\left(1 - e^{-(y-60)/10}\right) & ; \; y \geq 60. \end{cases}$$

another way to write this CDF is;

$$F_D(y) = 0.3 u(y) + 0.7 u(y-60)\left(1 - e^{-(y-60)/10}\right).$$

However, when we take the derivative, either expression for the CDF will yeild the pdf. However, taking the derivative of the first expression perhaps may be simpler:

$$f_D(y) = \begin{cases} 0.3\,\delta(y) & y < 60 \\ 0.07\,e^{-(y-60)/10} & y \geq 60. \end{cases}$$

Taking the derivative of the second expression for the CDF is a little tricky because of the product of the exponential and the step function. However, applying the usual rule ~~of~~ for the differentiation of a product does give the correct answer:

$$f_D(y) = 0.3\,\delta(y) + 0.7\,\delta(y-60)\left(1 - e^{-(y-60)/10}\right) + 0.07\,u(y-60)\,e^{-(y-60)/10}.$$

$$= 0.3\,\delta(y) + 0.07\,u(y-60)\,e^{-(y-60)/10}.$$

The middle term $\delta(y-60)\left(1-e^{-(y-60)/10}\right)$ dropped out because at $y=60$, $e^{-(y-60)/10} = 1$