$X, Y \rightarrow$ 2 RVs

Conditional PMF

$P_{X,Y|B}(x,y) \rightarrow$

$$= \begin{cases} \dfrac{P_{X,Y}(x,y)}{P[B]}, & x \in B \\ 0, & \text{o/w.} \end{cases}$$

$P_{X|Y}(x|y) \rightarrow$ PMF of X when $Y = y$

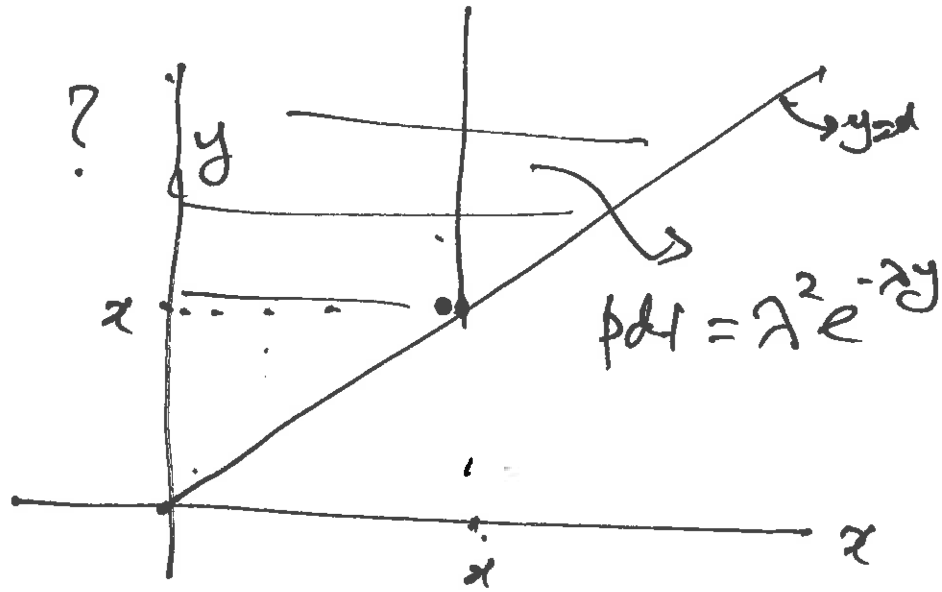
$$= \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Ex 7.17 (Ex 4.21) ②

Ex:- $f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & \boxed{0 \le x \le y} \\ 0, & \text{otherwise} \end{cases}$

$f_{Y/X}(y/x)$?



$= \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{x}^{\infty} \lambda^2 e^{-\lambda y}\,dy$$

$$= \lambda^2 \left.\frac{e^{-\lambda y}}{-\lambda}\right|_x^{\infty} = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & \text{oth.} \end{cases}$$

$$f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

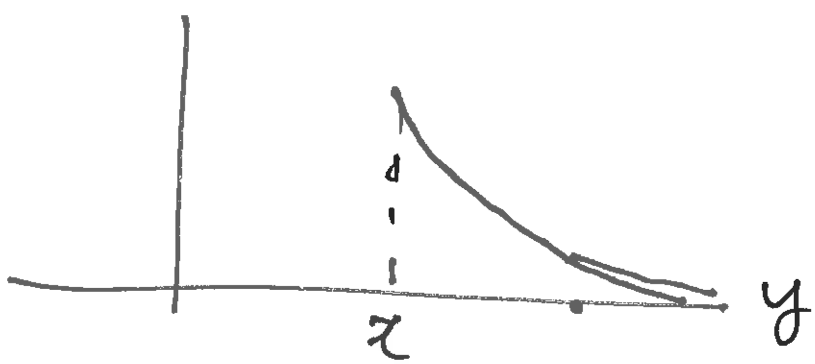
$$= \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)}$$

for $x > 0$

$$f_{Y/X}(y/x) = \begin{cases} \lambda e^{-\lambda(y-x)}, & y \geq x \\ 0, & \text{otherwise} \end{cases}$$

Shifted Exponential

$E[Y \mid X=x]$



without the shift $\longrightarrow$ Mean $= \frac{1}{\lambda}$

$$E[Y \mid X=x] = \left(\frac{1}{\lambda} + x\right)$$

$$\mathrm{Var}[Y \mid X = x] = \frac{1}{x^2}$$

[Note: $E[X + c] = \mu_X + c$
$\mathrm{Var}[X + c] = \mathrm{Var}[X]$]

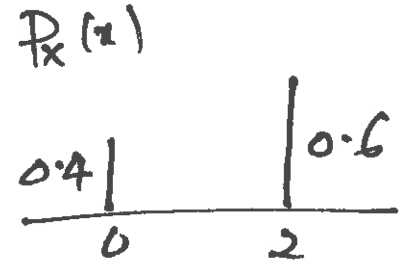
Recall: $Y = aX + b$

$$\mu_Y = a\mu_X + b$$

$$\mathrm{Var}[Y] = a^2 \, \mathrm{Var}(X)$$

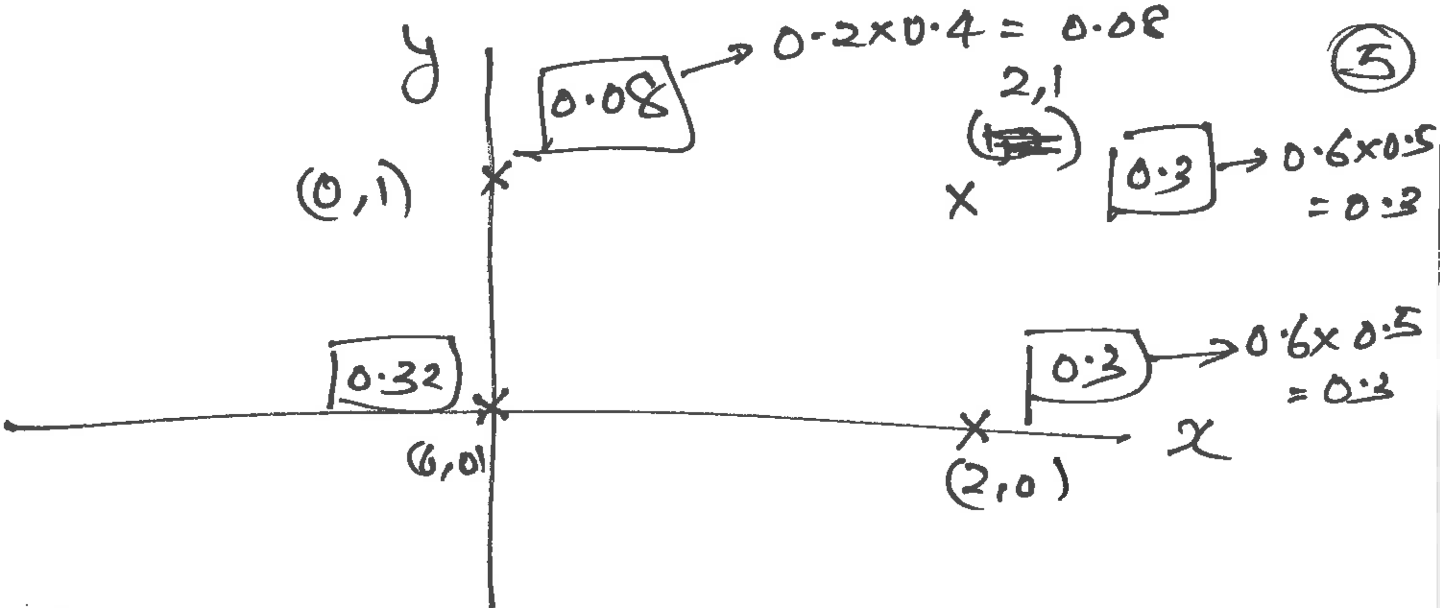
Q7.4 (Q4.9)

$$P_X(0) = 1 - P_X(2) = 0.4$$



$$P_{Y|X}(y|0) = \begin{cases} 0.8, & y = 0 \\ 0.2, & y = 1 \\ 0, & \text{oth.} \end{cases}$$

$$P_{Y|X}(y|2) = \begin{cases} 0.5, & y = 0 \\ 0.5, & y = 1 \\ 0, & \text{oth.} \end{cases}$$



$$P_{X,Y}(x,y)$$

$$P_{X,Y}(0,0) = P[X=0 \,\&\, Y=0]$$

$$= \underbrace{P[Y=0|X=0]}_{0.8}\;\underbrace{P[X=0]}_{0.4}$$

$$= 0.32$$

$$P_{X|Y}(x|0) = ?$$

$$= \frac{P_{X,Y}(x,0)}{P_Y(0)}$$

$$P_{X|Y}(0|0) = \frac{0.32}{0.32+0.3};$$

$$P_{X|Y}(2|0) = \frac{0.3}{0.32+0.3}$$

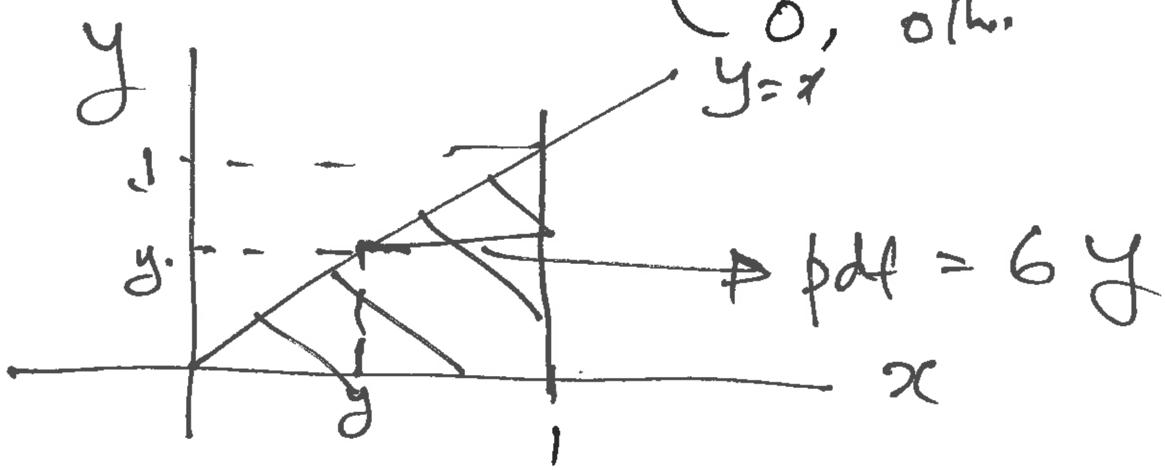
(B) $$f_X(x) = \begin{cases} 3x^2, & 0 \le x < 1 \\ 0, & o/w \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{x^2}, & 0 \le y \le x \\ 0, & o/w \end{cases}$$

$$= \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$\therefore f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x)$$

$$= \begin{cases} 6y, & 0 \le x \le 1 \ \& \ 0 \le y \le x \\ 0, & o/w \end{cases}$$



7

$$f_{X|Y}\left(x \mid \tfrac{1}{2}\right) = ?$$

$$= \frac{f_{X,Y}\left(x, \tfrac{1}{2}\right)}{f_Y\left(\tfrac{1}{2}\right)} \qquad f_{X,Y}\left(x, \tfrac{1}{2}\right) = 6\left(\tfrac{1}{2}\right) = 3$$

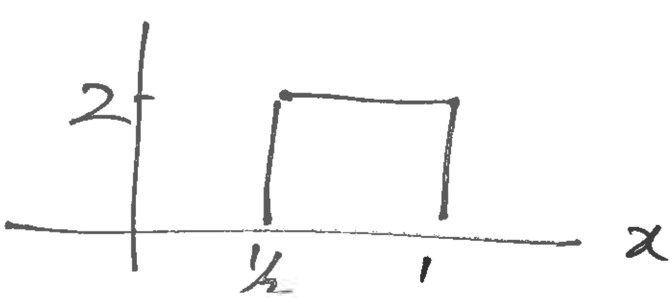
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$f_Y(y) = \int_{y}^{1} 6y\,dx = \begin{cases} 6y(1-y), & 0 \le y \le 1 \\ 0, & \text{oh} \end{cases}$$

$\cancel{4}$

$$f_Y\left(\tfrac{1}{2}\right) = 6\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \frac{3}{2}$$

$$f_{X|Y}\left(x \mid \tfrac{1}{2}\right) = \frac{3}{\frac{3}{2}} = \begin{cases} 2, & \tfrac{1}{2} \le x \le 1 \\ 0, & \text{oh} \end{cases}$$



$$\mu_{X|Y=\frac{1}{2}} = \frac{3}{4}$$

HW: X is uniform from 0 to 1 (8)

When $X=x$, Y is uniform from x to 1

$f_{X|Y}(x|y)$?

5.6 (4.10) Independence

Recall: If A & B are independent events

1. $P[A|B] = P[A]$
2. $P[B|A] = P[B]$
3. $P[AB] = P[A]P[B]$

- Extend independence to RVs.

X & Y are 2 RVs.

If X & Y are independent

1. $f_{X|Y}(x|y) = f_X(x)$
2. $f_{Y|X}(y|x) = f_Y(y)$
3. $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

Same for the Discrete

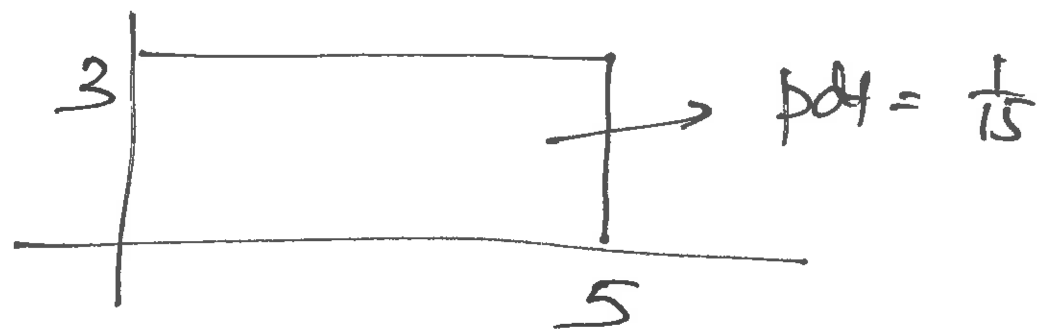
$f_{X,Y}(x,y) \longrightarrow P_{X,Y}(x,y)$

$f_{X|Y}(x|y) \rightarrow P_{X|Y}(x|y)$

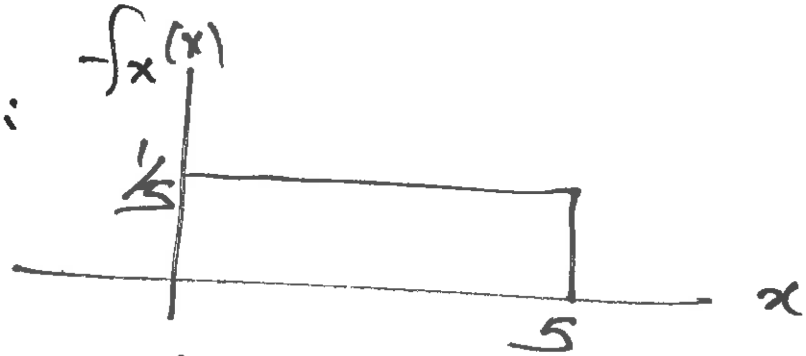
$f_{Y|X}(y|x) \longrightarrow P_{Y|X}(y|x)$

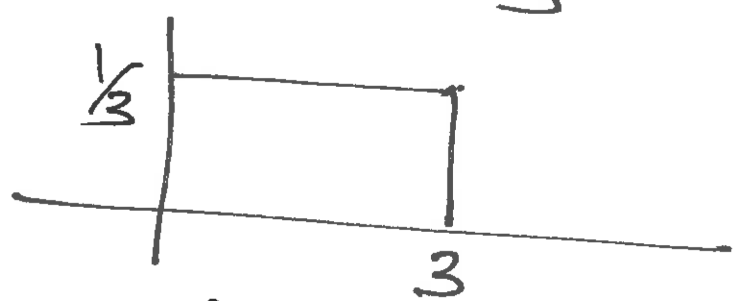
eg:- Recall:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{15}, & 0 \le x \le 5, 0 \le y \le 3 \\ 0, & \text{o/w.} \end{cases}$$



We found:

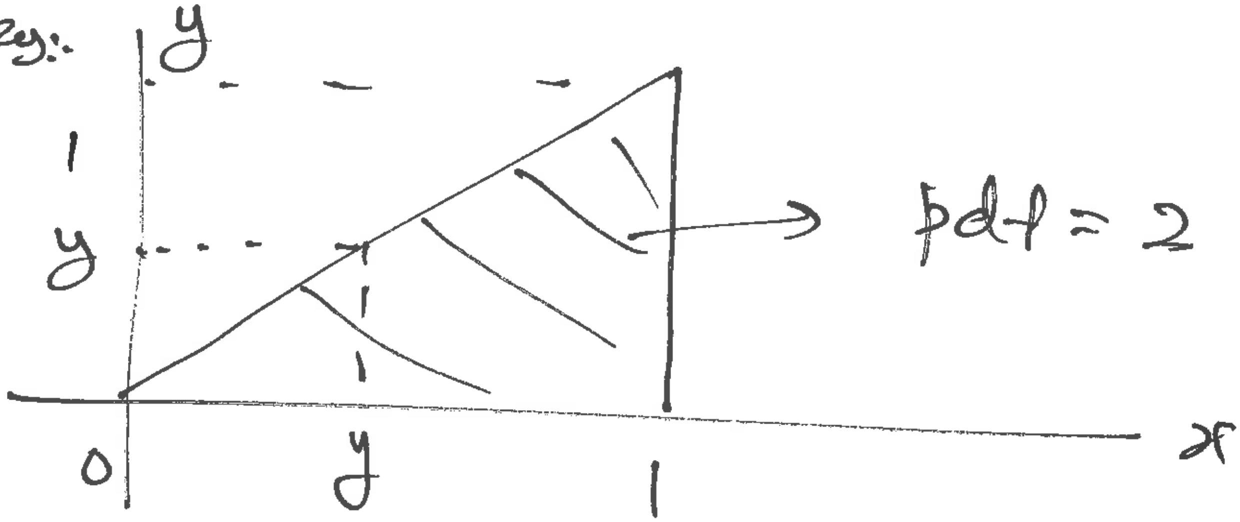




$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$\therefore$ X & Y are independent

eg:



Are X & Y independent ?

$f_X(x) \rightarrow 0 \leq x \leq 1$

$f_Y(y) \rightarrow 0 \leq y \leq 1$

$f_{X|Y}(x|y) \rightarrow$ x varies from y to 1

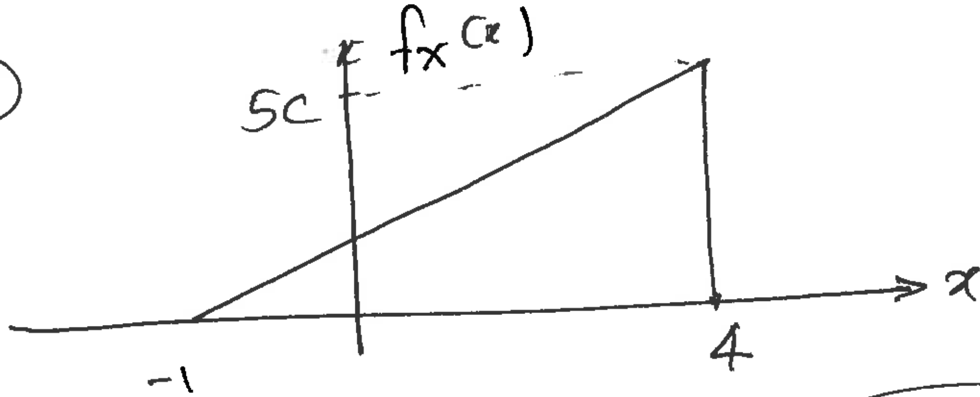
$\therefore f_X(x) \neq f_{X|Y}(x|y)$

$\therefore$ X & Y are not independent

↓

dependent.

(1)



$$\text{Area} = \frac{1}{2}(5)(5c) = 1 \longrightarrow \boxed{c = \frac{2}{25}}$$

$$\mu_x = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = \int_{-1}^{4} x\, c\,(x+1)\, dx$$

$$= c\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{4}$$

$$= \boxed{c\left[\left(\frac{64}{3} + 8\right) - \left(-\frac{1}{3} + \frac{1}{2}\right)\right] = a}$$

$$Var[x] = E[x^2] - \mu_x^2$$

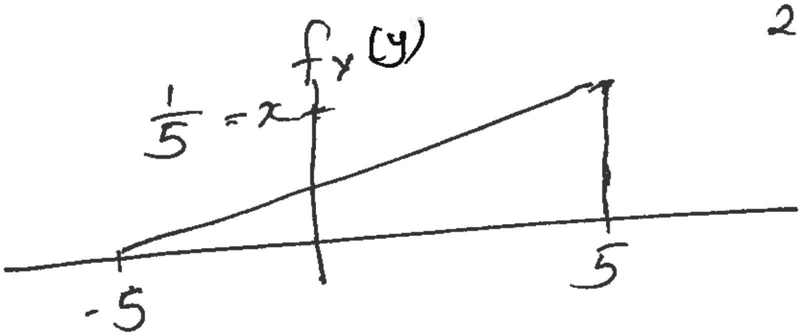
$$E[x^2] = \int_{-1}^{4} x^2\, c\,(x+1)\, dx$$

$$= c\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{4} = \boxed{c\left[\left(\frac{256}{4} + \frac{64}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right] = b}$$

$$Var[x] = b - a^2$$

② (a) (i) $Y = 2X - 3 \longrightarrow$ linear

$\hookrightarrow$ varies from $2(-1) - 3 = -5$ to $2(4) - 3 = 5$



$$\frac{1}{2}(10)\,x = 1$$

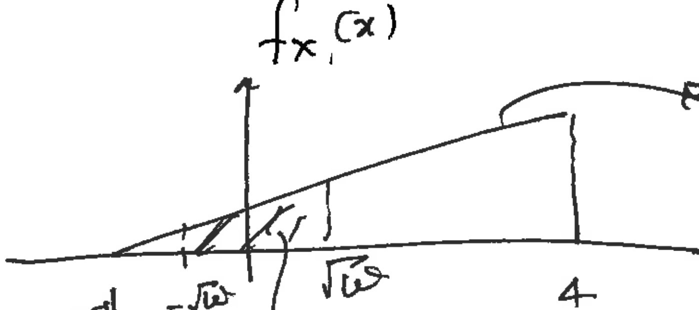
$$x = \frac{1}{5}$$

(ii) $W = X^2 \longrightarrow$ Nonlinear

$\hookrightarrow$ varies from 0 to 16

$$F_W(\omega) = P[W \le \omega] = P[X^2 \le \omega]$$
$$= P[-\sqrt{\omega} \le X \le \sqrt{\omega}]$$

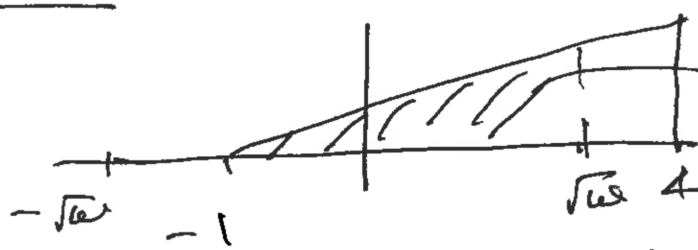
$0 \le \omega \le 1$



slope = c

$$\text{Area} = \frac{1}{2}c(\sqrt{\omega}+1)^2 - \frac{1}{2}c(1-\sqrt{\omega})^2$$

$1 \le \omega \le 16$

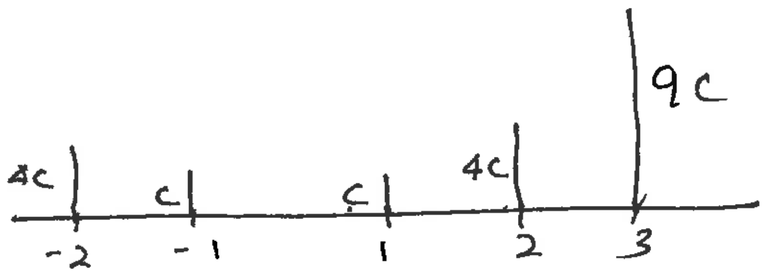


$$\text{Area} = \frac{1}{2}c(\sqrt{\omega}+1)^2$$

$$F_W(\omega) = \begin{cases} 0, & \omega < -1 \\ \frac{c}{2}\left[(\sqrt{\omega}+1)^2 - (1-\sqrt{\omega})^2\right], & 0 \le \omega \le 1 \\ \frac{1}{2}c(1+\sqrt{\omega})^2, & 1 \le \omega \le 16 \\ 1, & \omega > 16 \end{cases}$$

$$f_W(\omega)=\begin{cases}\frac{c}{2}\left[2(\sqrt{\omega}+1)\left(\frac{1}{2\sqrt{\omega}}\right)-2(1-\sqrt{\omega})\left(-\frac{1}{2\sqrt{\omega}}\right)\right], & 0\le\omega\le 1\\[2ex] \frac{c}{2}\left[2(1+\sqrt{\omega})\left(\frac{1}{2\sqrt{\omega}}\right)\right], & 1\le\omega\le 16\\[2ex] 0, & \text{otherw.}\end{cases}$$

(b) $P_X(x)$

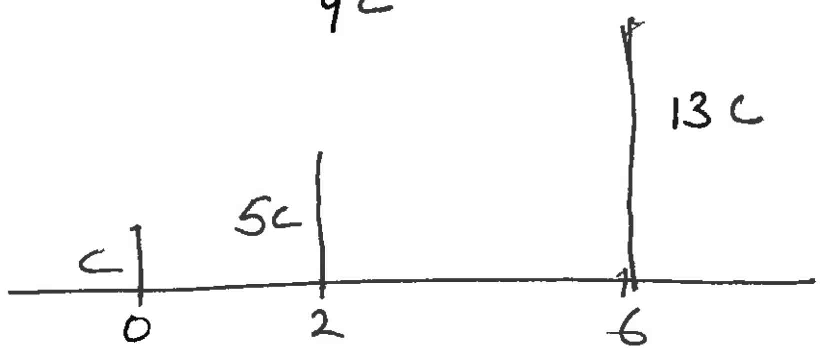


$$4c+c+c+4c+9c=1 \longrightarrow c=\frac{1}{19}$$

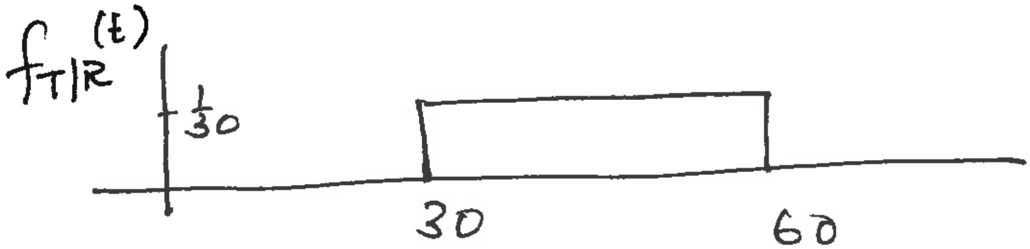
$$Y=(X^2-X)$$

X	Y	Prob.
-2	6	4c
-1	2	c
1	0	c
2	2	4c
3	6	9c

$P_Y(y)$

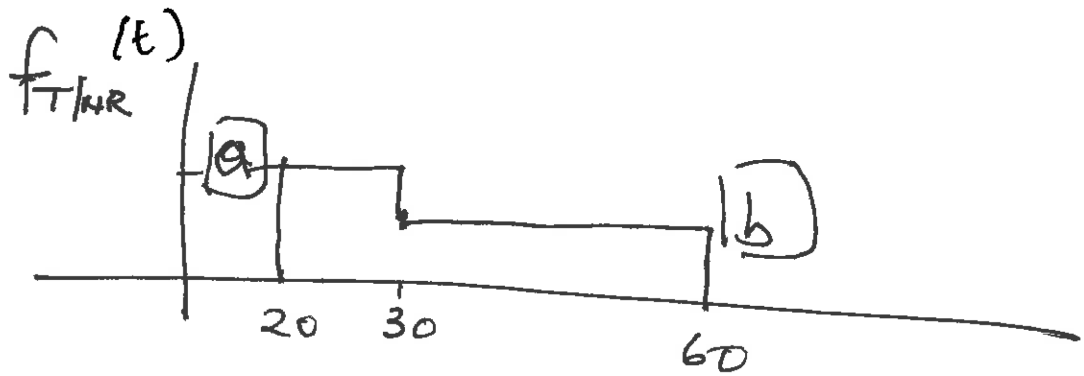


③



$$f_T(t) = f_{T|R}(t)\,\underbrace{P[R]}_{0.2} + f_{T|NR}(t)\,\underbrace{P[NR]}_{0.8}$$

$$f_{T|NR}(t) = \frac{f_T(t) - 0.2\, f_{T|R}(t)}{0.8}$$



$$\boxed{a = \frac{\frac{1}{40} - \frac{1}{30}(0.2)}{0.8}} \qquad \boxed{b = \frac{\frac{1}{40}}{0.8}}$$

(a) $$\mu_T = \int_{-\infty}^{\infty} f_{T|NR}(t)\,dt = \boxed{10a + 30b = \mu_T}$$

$$Var[T] = E[T^2] - \mu_T^2$$

$$E[T^2] = a^2(10) + b^2(30)$$

$$Var[T] = (10a^2 + 30b^2) - (10a + 30b)^2$$

(b) $$P[R \mid T > 50] = \frac{P[R \,\&\, T > 50]}{P[T > 50]}$$

$$= \frac{P[R] \int_{50}^{\infty} f_{T|R}(t)\,dt}{\frac{1}{40}(10)}$$

$$= \frac{0.2 \times \cancel{\frac{1}{40}}\frac{1}{30}(10)}{\frac{1}{40}(10)} = \boxed{0.2 \times \frac{4}{3}}$$

④ (a) X is N ~~(3,2)~~ $(2,3)$

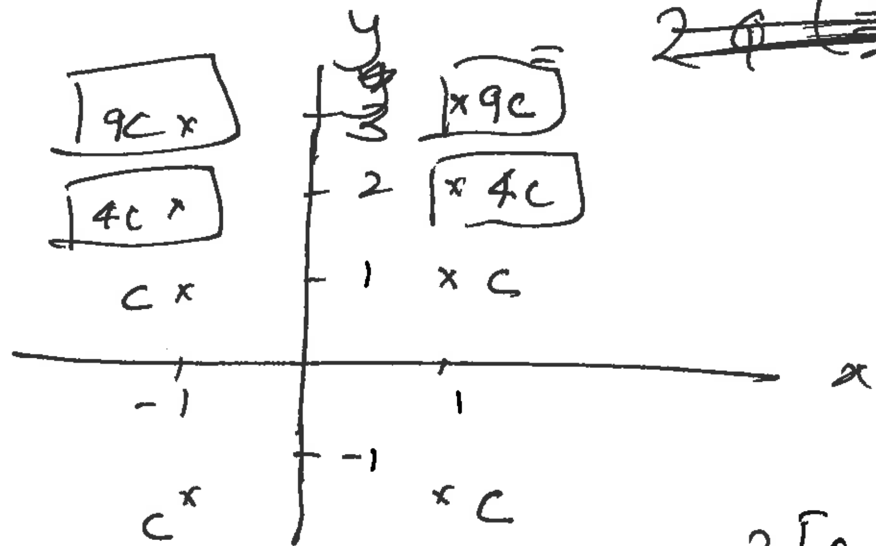
$$P[|x-3|>3] = P[x-3<-3] + P[x-3>3]$$

$$= P[x<0] + P[x>6]$$

$$z = \frac{x-2}{3} \qquad = P\left[z < -\frac{2}{3}\right] + P\left[z > \frac{4}{3}\right]$$

$$\cancel{2\varphi\left(\frac{3}{\ }\right)} = \boxed{\varphi\left(\frac{2}{3}\right) + \varphi\left(\frac{4}{3}\right)}$$

(b)



$$2[c+c+4c+9c] = 1$$

$$\boxed{c = \frac{1}{30}}$$

$$P[x^2+y^2>4] = 18c + 8c$$

$$= 26c = \frac{26}{30} = \boxed{\frac{13}{15}}$$