Moment Generating Func^n (MGF)
of X

$$\Phi_X(s) = E[e^{sX}]$$

$$= \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$$

eg:- X is exponential (λ)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & o/w \end{cases}$$

$$\Phi_X(s) = \int_0^{\infty} \lambda e^{-\lambda x} e^{sx}\, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - s)x}\, dx$$

$$= \lambda \left. \frac{e^{-(\lambda - s)x}}{-(\lambda - s)} \right|_0^\infty$$

$$\Phi_x(s) = \frac{\lambda}{\lambda - s}$$

X_1 & X_2 are independent

$$X = X_1 + X_2$$

pdf of X?

MGF of X

$$\Phi_X(s) = E[e^{sX}]$$

$$= E[e^{s(X_1 + X_2)}]$$

$$= E[e^{sX_1} e^{sX_2}]$$

X_1 & X_2 are independent

$$= E[e^{sX_1}] E[e^{sX_2}]$$

$$\Phi_X(s) = \Phi_{X_1}(s)\,\Phi_{X_2}(s)$$

MGF of the sum of independent RVs is the product of the individual MGFs

eg:- X_1 is exponential (λ_1)

X_2 " " (λ_2)

X_1 & X_2 are independent

Find the pdf of $X = X_1 + X_2$

$$\Phi_X(s) = \Phi_{X_1}(s)\,\Phi_{X_2}(s)$$

$$= \frac{\lambda_1}{\lambda_1 - s} \cdot \frac{\lambda_2}{\lambda_2 - s}$$

partial fractions

$$= \frac{A\lambda_1}{\lambda_1 - s} + \frac{B}{\lambda_2 - s}$$

A & B are constants $\rightarrow (\lambda_1 \,\&\, \lambda_2)$

$$f_X(x) = \begin{cases} \frac{A}{\lambda_1} \cdot \lambda_1 e^{-\lambda_1 x} + \frac{B}{\lambda_2} \cdot \lambda_2 e^{-\lambda_2 x} & , x \geq 0 \\ 0 & , \text{oth.} \end{cases}$$ (4)

EXAM

1. TWO RVs

   ↓

   Marginals, conditionals, Expected values, correlation, Covariance, correlation coeff. Jointly Gaussian.

2. Central Limit Thm.

3. Estimation → Linear Regression

4. Confidence Interval calculations

eg:- Joint pdf of X & Y is

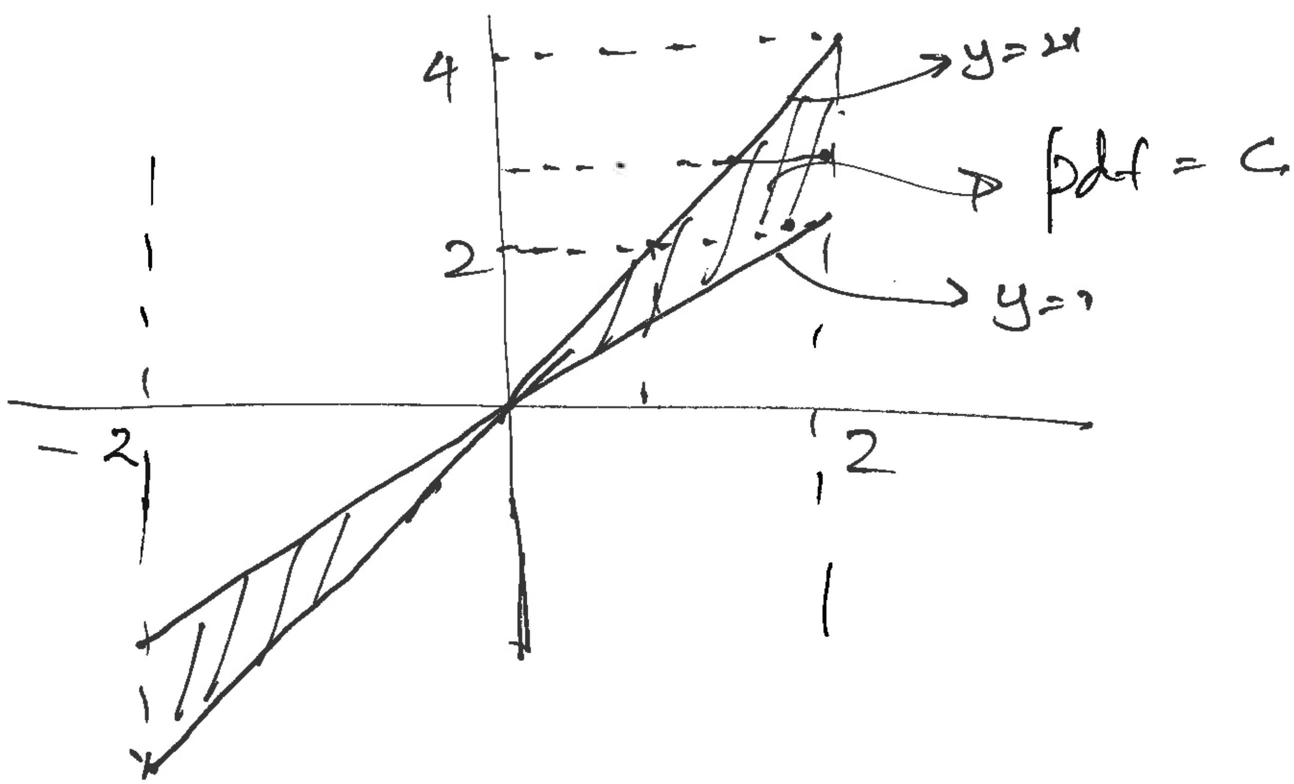
$$f_{X,Y}(x,y) = \begin{cases} c, & -2 \le x \le 2, \ \cancel{x \le y \le 2x} \ y \text{ lies bet}^n\ x \ \& \ 2x \\ 0, & \text{o/w.} \end{cases}$$

Find c

Are X & Y uncorrelated

" X & Y independent

Estimate x linearly in terms of y → Regression line?



$c[\text{Area}] = 1$

$\rightarrow 2\left[\frac{1}{2}(2)(4) - \frac{1}{2}(2)(2)\right]$

$c[8-4] = 1 \longrightarrow c = \frac{1}{4}$

Check the independence

1. $f_{X,Y}(x,y) = f_X(x) f_Y(y)$
2. $f_{X|Y}(x|y) = f_X(x)$ → varies from -2 to 2
3. $f_{Y|X}(y|x) = f_Y(y)$

$f_{X|Y}(x|2) \longrightarrow$ varies from 1 to 2

$f_{X|Y}(x|y)$ depends on y

$\therefore$ X & Y are not independent

Uncorrelated ?

Need $Cov[X,Y]$

$$Cov[X,Y] = E[XY] - \mu_X \mu_Y$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f_{X,Y}(x,y)\, dy\, dx$$

$$= 2\int_0^2 \int_x^{2x} xy\, c \, dy\, dx$$

$$= 2c\int_0^2 x \left. \frac{y^2}{2} \right|_x^{2x} dx$$

$$= \frac{2c}{2} \int_0^2 x(4x^2 - x^2)\, dx$$

$$= c \left. \frac{3x^4}{4} \right|_0^2 = \frac{3c}{\cancel{4}}(4) = 12c$$

$$\mu_X = \int_0^2 \int_x^{2x} x \cdot c \; dy \, dx + \int_{-2}^{0} \int_{2x}^{x} x \, c \; dy \, dx$$

$$= 0.$$

$f_X(x)$

$$\mathrm{Cov}[X,Y] = E[XY] - \mu_X \mu_Y$$

$$= 12c \neq 0 \rightarrow$$

X & Y are correlated.

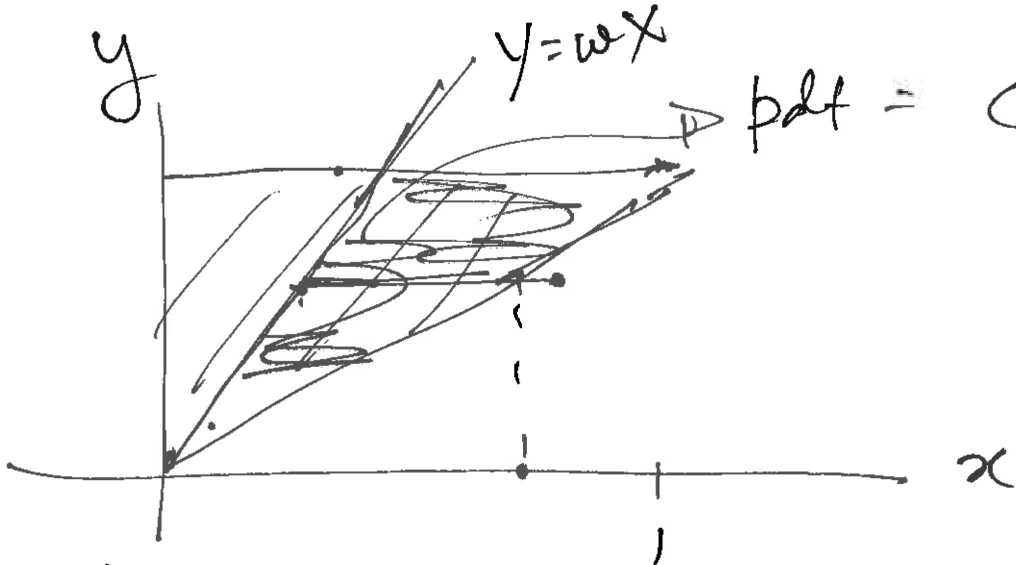
$$\left[x_L(y) - \mu_X\right] = \frac{\rho_{x,y}\,\sigma_X}{\sigma_Y} \left[y - \mu_Y\right]$$

Find μ_Y, σ_X, $\sigma_Y \rightarrow \rho_{X,Y}$

$$\frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[Y]}$$

eg :- $$f_{X,Y}(x,y) = \begin{cases} cx, & 0 \le x \le y \le 1 \\ 0, & o/w. \end{cases}$$ 9

Find c & the pdf of $\frac{Y}{X}$



$$\int_0^1 \int_x^1 cx \, dy \, dx = 1$$

$$c \int_0^1 x(1-x) \, dx = 1$$

$$c \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 1 \longrightarrow$$

$$c \left[\frac{1}{2} - \frac{1}{3} \right] = 1$$

$c =$ ✓

$W = \frac{Y}{X} \rightarrow$ w varies from 1 to infinity (10)

$W \geq 1$

CDF of W, $F_W(w) = P[W \leq w]$

$= P\left[\frac{Y}{X} \leq w\right]$

$\underline{w \geq 1}$

$= P[\underbrace{Y \leq wX}_{R}]$

$= \iint_R f_{X,Y}(x,y)\,dy\,dx$

$= \int_0^1 \int_{\frac{y}{w}}^{y} cx\,dx\,dy$

$= c\int_0^1 \frac{x^2}{2}\Big|_{\frac{y}{w}}^{y}\,dy$

⋮

$\cancel{f_W(w)} =$

$$F_W(\omega) = \frac{c}{2}\int_0^1 \left(y^2 - \frac{y^2}{\omega^2}\right) dy$$

$$= \frac{c}{2}\left[\frac{y^3}{3} - \frac{y^3}{3\omega^2}\right]_0^1$$

$$F_W(\omega) = \frac{c}{6}\left[1 - \frac{1}{\omega^2}\right]$$

$$F_W(\omega) = \begin{cases} 0, & \omega < 1 \\ \frac{c}{6}\left[1 - \frac{1}{\omega^2}\right] & \omega \geq 1 \end{cases}$$

$$f_W(\omega) = \begin{cases} \frac{c}{6}\left[\frac{2}{\omega^3}\right], & \omega \geq 1 \\ 0, & \text{o.w.} \end{cases}$$

Ex: X & Y are jointly Gaussian with joint pdf

$N(2, 3; 1, 2; 0)$

Find $P[|X| > 3 \ \& \ Y < 2]$

X & Y are uncorrelated & Jointly Gauss

$\therefore$ X & Y are indepen

$= P[|X| > 3] \cdot P[Y < 2]$