Libxmotion Implementation Notes

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Contents

1 MEKF

This section is about the MEKF implementation in libxmotion. Most of the equations and derivation steps are directly from [1].

1.1 Quaternion

The definition of a quaternion is given by:

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ \boldsymbol{q}_{2:4} \end{bmatrix} \tag{1}$$

The relevant operations on quaternions that are used for MEKF are given as follows:

- Multiplication:

$$\boldsymbol{q} \otimes \boldsymbol{p} = \begin{bmatrix} q_1 p_1 - \boldsymbol{q}_{2:4} \boldsymbol{p}_{2:4} \\ q_1 \boldsymbol{p}_{2:4} + p_1 \boldsymbol{q}_{2:4} + \boldsymbol{q}_{2:4} \times \boldsymbol{p}_{2:4} \end{bmatrix} \tag{2}$$

- Inverse:

$$\boldsymbol{q}^{-1} = \begin{bmatrix} q_1 \\ -\boldsymbol{q}_{2:4} \end{bmatrix} \tag{3}$$

A quaternion represents a rotation of one frame with respect to another. The inverse of a quaternion represents the rotation in the opposite direction. The multiplication of two quaternions represents the composition of two rotations.

We have the following properties of quaternions:

$$\boldsymbol{q} \otimes \boldsymbol{q}^{-1} = \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix} \tag{4}$$

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{1}{2} \boldsymbol{q} \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix} \\ &= \frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}) \boldsymbol{q} \end{aligned} \tag{5}$$

where ω is the angular velocity vector of the body frame with respect to the inertial frame and Ω is defined as

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\boldsymbol{\omega} \\ \boldsymbol{\omega} & -[\boldsymbol{\omega}_\times] \end{bmatrix} \tag{6}$$

$$\boldsymbol{\omega}_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{7}$$

1.2 Error State Model

The MEKF is used to estimate the attitude of the body frame with respect to the inertial frame. The main difference between MEKF and other Kalman filters for attitude estimation is that MEKF does not use the quaternion in the state vector directly.

Normally, we would have the state dynamics as

$$\dot{\hat{\boldsymbol{q}}} = \frac{1}{2}\hat{\boldsymbol{q}} \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix} \tag{8}$$

$$\dot{\hat{\boldsymbol{v}}}^i = \boldsymbol{R}_b^i(\hat{\boldsymbol{q}})\hat{\boldsymbol{f}}^b + \hat{\boldsymbol{g}}^i \tag{9}$$

$$\dot{\hat{\boldsymbol{r}}}^i = \hat{\boldsymbol{v}}^i \tag{10}$$

where $\boldsymbol{R}_b^i$ is the transformation matrix that transforms a vector from the body frame to the inertial frame, $\hat{\boldsymbol{q}}$ is the estimated quaternion, $\hat{\boldsymbol{v}}$ is the estimated velocity, and $\hat{\boldsymbol{r}}$ is the estimated position.

With MEKF, we use the error quaternion, which is the difference between the estimated quaternion and the true quaternion, as well as error velocity and position.

The error quaternion is given as

$$\boldsymbol{q} = \hat{\boldsymbol{q}} \otimes \delta\boldsymbol{q} \tag{11}$$

$$\Rightarrow \delta\boldsymbol{q} = \hat{\boldsymbol{q}}^{-1} \otimes \boldsymbol{q} \tag{12}$$

where $\hat{\boldsymbol{q}}$ is the estimated quaternion and $\delta\boldsymbol{q}$ is the error quaternion.

The error state vector is then defined as

$$\delta\boldsymbol{x} = \begin{bmatrix} \delta\boldsymbol{q} \\ \delta\boldsymbol{v} \\ \delta\boldsymbol{r} \\ \boldsymbol{\beta}_{\boldsymbol{\omega}} \\ \boldsymbol{\beta}_{\boldsymbol{f}} \\ \boldsymbol{\beta}_{\boldsymbol{m}} \end{bmatrix} \tag{13}$$

where $\delta\boldsymbol{q}$ is the error quaternion, $\delta\boldsymbol{v}$ is the error in velocity, $\delta\boldsymbol{r}$ is the error in position, $\boldsymbol{\beta}_{\boldsymbol{\omega}}$ is the bias in the angular velocity, $\boldsymbol{\beta}_{\boldsymbol{f}}$ is the bias in the specific force, and $\boldsymbol{\beta}_{\boldsymbol{m}}$ is the bias in the magnetometer.

Then we need to derive the dynamics of the error state $\dot{\boldsymbol{x}}$. We can do it by examining each component of the state vector separately.

1.2.1 Error Quaternion Dynamics

The error quaternion dynamics is derived as follows:

$$\begin{aligned} \delta\boldsymbol{q} &= \hat{\boldsymbol{q}}^{-1} \otimes \boldsymbol{q} \\ \Rightarrow \delta\dot{\boldsymbol{q}} &= \hat{\boldsymbol{q}}^{-1} \otimes \dot{\boldsymbol{q}} + \dot{\hat{\boldsymbol{q}}}^{-1} \otimes \boldsymbol{q} \end{aligned} \tag{14}$$

After the following steps described in [1], we will eventually get

$$\delta\dot{\boldsymbol{q}}_{2:4} \cong -\hat{\boldsymbol{\omega}}_{\times}\delta\boldsymbol{q}_{2:4} + \frac{1}{2}\delta\boldsymbol{\omega} \tag{15}$$

with the fact that $\delta q_1 = 1$ and the assumption that $\delta\boldsymbol{q}$ is small.

Here we replace the error states $\delta\boldsymbol{q}_{\{2:4\}}$ with a vector of small angles to further simplify the equations.

$$\begin{aligned} \boldsymbol{\alpha} &= 2\delta\boldsymbol{q}_{2:4} \\ \Rightarrow \dot{\boldsymbol{\alpha}} &= -\hat{\boldsymbol{\omega}}_{\times}\boldsymbol{\alpha} + \delta\boldsymbol{\omega} \end{aligned} \tag{16}$$

1.2.2 Error Velocity and Position Dynamics

For the velocity and position error, we have

$$\delta \boldsymbol{v} = \boldsymbol{v} - \hat{\boldsymbol{v}} \tag{17}$$

$$\delta \boldsymbol{r} = \boldsymbol{r} - \hat{\boldsymbol{r}} \tag{18}$$

Based on Equation 9 and Equation 10, we can derive the dynamics of the velocity and position error as

$$\delta \dot{\boldsymbol{v}} = -\boldsymbol{R}_b^i(\hat{\boldsymbol{q}})\hat{\boldsymbol{f}}_\times^b \boldsymbol{\alpha} + \boldsymbol{R}_b^i \delta \boldsymbol{f} \tag{19}$$

$$\delta \dot{\boldsymbol{r}} = \delta \boldsymbol{v} \tag{20}$$

1.2.3 Sensor Bias Dynamics

The angular rate error model is given by

$$\boldsymbol{\omega} = \hat{\boldsymbol{\omega}} - \boldsymbol{\beta}_\omega - \boldsymbol{\eta}_\omega \tag{21}$$

$$\dot{\boldsymbol{\beta}}_\omega = \boldsymbol{\nu}_\omega \tag{22}$$

Together with the definition

$$\boldsymbol{\omega} = \hat{\boldsymbol{\omega}} + \delta \boldsymbol{\omega} \tag{23}$$

We can get

$$\delta \boldsymbol{\omega} = -\boldsymbol{\beta}_\omega - \boldsymbol{\eta}_\omega \tag{24}$$

$$\delta \dot{\omega} = -\boldsymbol{\nu}_\omega \tag{25}$$

The linear acceleration error model is given by

$$\boldsymbol{f} = \hat{\boldsymbol{f}} - \boldsymbol{\beta}_f - \boldsymbol{\eta}_f \tag{26}$$

$$\dot{\boldsymbol{\beta}}_f = \boldsymbol{\nu}_f \tag{27}$$

Together with the definition

$$\boldsymbol{f} = \hat{\boldsymbol{f}} + \delta \boldsymbol{f} \tag{28}$$

We can get

$$\delta \boldsymbol{f} = -\boldsymbol{\beta}_f - \boldsymbol{\eta}_f \tag{29}$$

The magnetometer bias is treated as a slowly diverging random walk process driven by the noise process $\boldsymbol{\nu}_m$

$$\dot{\boldsymbol{\beta}}_m = \boldsymbol{\nu}_m \tag{30}$$

1.2.4 Full Error State Dynamics

According to the results above, we redefine the error state as

$$\delta \boldsymbol{x} = \begin{bmatrix} \boldsymbol{\alpha} \\ \delta \boldsymbol{v} \\ \delta \boldsymbol{r} \\ \boldsymbol{\beta_\omega} \\ \boldsymbol{\beta_f} \\ \boldsymbol{\beta_m} \end{bmatrix} \tag{31}$$

Combining equation Equation 16 Equation 19 Equation 20 Equation 25 Equation 27 Equation 30, together with Equation 24 and Equation 29, we get the full dynamics of the error state:

$$\delta \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{\alpha}} \\ \delta \dot{\boldsymbol{v}} \\ \delta \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{\beta}}_{\boldsymbol{\omega}} \\ \dot{\boldsymbol{\beta}}_{\boldsymbol{f}} \\ \dot{\boldsymbol{\beta}}_{\boldsymbol{m}} \end{bmatrix} = \begin{bmatrix} -\hat{\boldsymbol{\omega}}_\times & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ -\boldsymbol{R}_b^i(\hat{\boldsymbol{q}})\hat{\boldsymbol{f}}^b_\times & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\boldsymbol{R}_b^i(\hat{\boldsymbol{q}}) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \delta \boldsymbol{v} \\ \delta \boldsymbol{r} \\ \boldsymbol{\beta_\omega} \\ \boldsymbol{\beta_f} \\ \boldsymbol{\beta_m} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{\eta_\omega} \\ -\boldsymbol{R}_b^i(\hat{\boldsymbol{q}})\boldsymbol{\eta}_f \\ \mathbf{0}_{3\times3} \\ \boldsymbol{\nu}_\omega \\ \boldsymbol{\nu}_f \\ \boldsymbol{\nu}_m \end{bmatrix} \tag{32}$$

The 18-error-state model is linear and time-varying with respect to the error states. Following the standard representation of a linear time-varying system, we can write the error state dynamics as

$$\delta \dot{\boldsymbol{x}} = \boldsymbol{F} \delta \boldsymbol{x} + \mathtt{W} \tag{33}$$

where matrices $\boldsymbol{F}$ is given by

$$\boldsymbol{F}\left(\hat{\boldsymbol{q}}, \hat{\boldsymbol{\omega}}, \hat{\boldsymbol{f}}\right) = \begin{bmatrix} -\hat{\boldsymbol{\omega}}_\times & \mathbf{0} & \mathbf{0} & -\boldsymbol{I}_{3\times3} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{R}_b^i(\hat{\boldsymbol{q}})\hat{\boldsymbol{f}}^b_\times & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{R}_b^i(\hat{\boldsymbol{q}}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{3\times3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{34}$$

and $\mathtt{W}$ is the process noise. According to Equation 32, we have

$$\mathtt{W} = \begin{bmatrix} -\boldsymbol{\eta_\omega} \\ -\boldsymbol{R}_b^i(\hat{\boldsymbol{q}})\boldsymbol{\eta}_f \\ \mathbf{0}_{3\times3} \\ \boldsymbol{\nu}_\omega \\ \boldsymbol{\nu}_f \\ \boldsymbol{\nu}_m \end{bmatrix} \tag{35}$$

From the definitions of $\boldsymbol{\eta}_\omega$, $\boldsymbol{\eta}_f$, $\boldsymbol{\nu}_\omega$, $\boldsymbol{\nu}_f$, $\boldsymbol{\nu}_m$, we can get the covariance matrix of the process noise as

$$
\boldsymbol{Q}_c = \begin{bmatrix} \mathrm{diag}(\boldsymbol{\sigma}_\omega^2) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}\left(\boldsymbol{\sigma}_f^2\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{diag}\left(\boldsymbol{\sigma}_{\beta\omega}^2\right) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{diag}\left(\boldsymbol{\sigma}_{\beta f}^2\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{diag}\left(\boldsymbol{\sigma}_{\beta m}^2\right) \end{bmatrix} \tag{36}
$$

where $\boldsymbol{\sigma}_\omega$, $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_{\beta\omega}$, $\boldsymbol{\sigma}_{\beta f}$, $\boldsymbol{\sigma}_{\beta m}$ are the standard deviations of the white noise processes.

1.3 Measurement Model

The measurement model for the MEKF can be represented in the standard form

$$
\boldsymbol{z} = \boldsymbol{H}\delta\boldsymbol{x} + \mathtt{V} \tag{37}
$$

where $\boldsymbol{z}$ is the measurement vector, $\boldsymbol{H}$ is the measurement function, $\delta\boldsymbol{x}$ is the error state vector, and $\mathtt{V}$ is the measurement noise.

In this case, the gyroscope measurement is treated as control input. We have the following two types of measurements:

- Accelerometer: $\tilde{\boldsymbol{a}}^b = \boldsymbol{a}^b + \boldsymbol{R}_i^b(\boldsymbol{q}) \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \boldsymbol{\beta}_f + \boldsymbol{\eta}_f$
- Magnetometer: $\widetilde{\boldsymbol{m}}^b = \boldsymbol{R}_i^b(\boldsymbol{q})\boldsymbol{m}^i + \boldsymbol{\beta}_m + \boldsymbol{\eta}_m$

In practice, $\tilde{\boldsymbol{a}}^b$ and $\widetilde{\boldsymbol{m}}^b$ can be acquired directly from the accelerometer and magnetometer, respectively. We will need to derive the measurement model for the accelerometer and magnetometer so that we can use the current estimated state to predict the measurements in order to find the measurement residual.

Equation 37 and 38 in [1] are used to derive the measurement model for the accelerometer and magnetometer

$$
\boldsymbol{R}_i^b(\boldsymbol{q}) \cong [\boldsymbol{I} - \boldsymbol{\alpha}_\times]\boldsymbol{R}_i^b(\hat{\boldsymbol{q}}) \tag{38}
$$

$$
\boldsymbol{R}_b^i(\boldsymbol{q}) \cong \boldsymbol{R}_b^i(\hat{\boldsymbol{q}})[\boldsymbol{I} + \boldsymbol{\alpha}_\times] \tag{39}
$$

1.3.1 Magnetometer Measurement Model

The predicted magnetometer measurement is given by

$$
\widehat{\widetilde{\boldsymbol{m}}}^b = \boldsymbol{R}_i^b(\hat{\boldsymbol{q}})\boldsymbol{m}^i \tag{40}
$$

The measurement residual is then

$$
\delta\widetilde{\boldsymbol{m}}^b = \tag{41}
$$

$$
\begin{aligned} \widetilde{\boldsymbol{m}}^b &= \boldsymbol{R}_i^b(\boldsymbol{q})\boldsymbol{m}^i + \boldsymbol{\beta}_m + \boldsymbol{\eta}_m \\ &\cong [\boldsymbol{I} - \boldsymbol{\alpha}_\times]\boldsymbol{R}_i^b(\hat{\boldsymbol{q}})\boldsymbol{m}^i + \boldsymbol{\beta}_m + \boldsymbol{\eta}_m \end{aligned} \tag{42}
$$

1.3.2 Angle Measurements

2 Heading: first level

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2.1 Heading: second level

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3 Math

Inline: Let a, b, and c be the side lengths of right-angled triangle. Then, we know that: $a^2 + b^2 = c^2$

Block without numbering:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Block with numbering:

As shown in Equation 43.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{43}$$

More information:
- https://typst.app/docs/reference/math/equation/

4 Citation

You can use citations by using the `#cite` function with the key for the reference and adding a bibliography. Typst supports BibLateX and Hayagriva.

```
#bibliography("bibliography.bib")
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Single citation [2]. Multiple citations [2], [3]. In text A. Vaswani, N. M. Shazeer, N. Parmar, J. Uszkoreit, L. Jones, A. N. Gomez, L. Kaiser, and I. Polosukhin [2]

More information:
- https://typst.app/docs/reference/meta/bibliography/
- https://typst.app/docs/reference/meta/cite/

5 Figures and Tables

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Table 1: Lorem ipsum dolor sit amet.

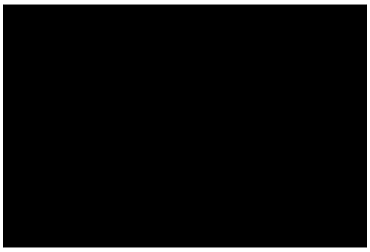

Figure 1: Lorem ipsum dolor sit amet, consectetur adipiscing.

More information

- https://typst.app/docs/reference/meta/figure/
- https://typst.app/docs/reference/layout/table/

6 Referencing

Figure 1 Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do., Table 1.

More information:

- https://typst.app/docs/reference/meta/ref/

7 Lists

Unordered list

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- Lorem ipsum dolor sit amet, consectetur adipiscing elit.

Numbered list

1. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do.
2. Lorem ipsum dolor sit amet, consectetur adipiscing elit.
3. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor.

More information:

- https://typst.app/docs/reference/layout/enum/
- https://typst.app/docs/reference/meta/cite/

Bibliography

[1] J. M. Maley, “Multiplicative quaternion extended kalman filtering for nonspinning guided projectiles,” 2013. [Online]. Available: https://apps.dtic.mil/sti/pdfs/ADA588831.pdf

[2] A. Vaswani *et al.*, “Attention is All you Need,” in *NIPS*, 2017.

[3] G. E. Hinton, O. Vinyals, and J. Dean, “Distilling the Knowledge in a Neural Network,” *ArXiv*, 2015.

APPENDIX A

A.1 Appendix section

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