TTK 4250 Assignment 3 Øystein Kolbø

1 Getting the a priori likelihood. Conditional

$$P(R_{k+1}) = P(S \mid R_k)P(R_k) + P(E \mid \bar{R}_k)P(\bar{R}_k)$$

$$= P_S \cdot r_k + P_E(1 - r_k)$$

$$\Rightarrow r_{k+1|k} = P_S r_k + P_E(1 - r_k)$$

Getting the a posteriori probability

$M_{k+1} = D_{k+1} \cup F_{k+1}$ where M measurement

Must take into account that we can ~~measure a~~ get a measurement ~~andthen~~ or not get a measurement. From bayes rule, we have that

$$P(R \mid M) = \frac{P(M \mid R)P(R)}{P(M)}$$

$$P(R \mid \bar{M}) = \frac{P(\bar{M} \mid R)P(R)}{P(\bar{M})}$$

For finding the probabilities, the total probability and bayes law is used

$$P(M \mid R) = P(D \cup F \mid R)$$

$$P(\bar{M} \mid R) = P(\bar{D} \cap \bar{F} \mid R)$$

$$P(M) = P(M \mid R)P(R) + P(M \mid \bar{R})P(\bar{R})$$

$$P(\bar{M}) = P(\bar{M} \mid R)P(R) + P(\bar{M} \mid \bar{R})P(\bar{R})$$

$$P(M|R) = P(D|R) + P(F|R) - \overset{0}{P(F \cap D\ |R)}$$

$$= P_D + P(F|R)$$

For it to trigger a "false alarm" while the ship is in the area, the ship cannot be detected

$$\Rightarrow P(F|R) = P(F|\bar{D})\, P(\bar{D}|R)$$

$$= P_{FA}\,(1 - P_D)$$

$$\Rightarrow P(M|R) = P_D + P_{FA}\,(1 - P_D)$$

$$P(M|\bar{R}) = P(D|\bar{R}) + P(F|\bar{R}) = P_{FA}$$

$$P(\bar{M}|R) = 1 - P(M|R) = 1 - P_D - P_{FA}\,(1 - P_D)$$

$$P(\bar{M}|\bar{R}) = 1 - P(M|\bar{R}) = 1 - P_{FA}$$

$$P(M) = P(M|R)\,P(R) + P(M|\bar{R})\,P(\bar{R})$$

$$= (P_D + P_{FA}\,(1 - P_D))\,(P_S\, r + P_E\,(1-r))$$

$$+ P_{FA}\,(1 - P_S\, r - P_E\,(1-r))$$

$$= (P_S\, r + P_E\,(1-r))\,(P_D + P_{FA}\,(1 - P_D) - P_{FA}) + P_{FA}$$

$$= (P_S\, r + P_E\,(1-r))\, P_D\,(1 - P_{FA}) + P_{FA}$$

$$P(\bar{M}) = P(\bar{M}|R)\,P(R) + P(\bar{M}|\bar{R})\,P(\bar{R})$$

$$= (1 - P_D - P_{FA}\,(1 - P_D))\,(P_S\, r + P_E\,(1-r))$$

$$+ (1 - P_{FA})\,(1 - P_S\, r - P_E\,(1-r))$$

$$= (P_S\, r + P_E\,(1-r))\,(1 - P_D - P_{FA}\,(1 - P_D) + P_{FA} - 1)$$

$$+ 1 - P_{FA}$$

$$= (P_S\, r + P_E\,(1-r))\,(P_{FA} - 1)(\cancel{1-}P_D) + 1 - P_{FA}$$

All of these expressions give that

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$$P(R|M) = \frac{P(M|R)P(R)}{P(M)}$$

$$= \frac{(P_D + P_{FA}(1-P_D))(r P_S + P_E(1-r))}{P_S r + P_E(1-r) \; P_D(1-P_{FA}) + P_{FA}}$$

$$P(R|\bar{M}) = \frac{P(\bar{M}|R)P(R)}{P(\bar{M})}$$

$$= \frac{(1-(P_D + P_{FA}(1-P_D))(P_S r + P_E(1-r))}{(P_S r + P_E(1-r))(P_{FA}-1) \; P_D + 1 - P_{FA}}$$

I don't see a way to simplify these expressions.

I have also dropped the subscript, such that

$P(R|M) \iff P(R_{k+1}|M_{k+1})$

and etc.

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a) $x_{k+1} = F x_k + v_k \quad , \quad v_k \sim N(0, Q)$

$z_k = [I_2 \;\; O_2] \, z_k + w_k , \quad w_k \sim N(0, R)$

Using that $H = [I_2 \;\; O_2]$ and that F and Q given in the book

~~$z_1 = H(Fx_1 + v_1)$~~

~~$z_0 = F$~~

$z_1 = H x_1 + w_1 = p_1 + w_1$

$z_0 = H x_0 + w_0 \quad , F x_0 + v_0 = x_1 \iff x_0 = F^{-1}(x_1 - v_0)$

$= H F^{-1}(x_1 - v_0) + w_0$

According to matlab, $HF^{-1} = [I_2 \;\; -TI_2]$

$\Rightarrow HF^{-1} x_1 = p_1 - T u_1$

$\Rightarrow z_0 = p_1 - T u_1 - HF^{-1} v_0 + w_0$

b) Using a) we have that

$$\hat{x}_1 = \begin{bmatrix} K_{p1} & K_{p0} \\ K_{u1} & K_{u0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$$

To have an unbiased estimator, we know that

$$E[\hat{x}_1] = x_1$$

$$\hat{x}_1 = \begin{bmatrix} K_{p1} z_1 + K_{p0} z_0 \\ K_{u1} z_1 + K_{u0} z_0 \end{bmatrix}$$

$$= \begin{bmatrix} K_{p1} (H x_1 + w_1) + K_{p0} (H F^{-1}(x_1 - v_0) + w_0) \\ K_{u1} (H x_1 + w_1) + K_{u0} (H F^{-1}(x_1 - v_0) + w_0) \end{bmatrix}$$

$$E[\hat{x}_1] = \begin{bmatrix} K_{p1} H E[x_1] + K_{p0} H F^{-1} E[x_1] \\ K_{u1} H E[x_1] + K_{u0} H F^{-1} E[x_1] \end{bmatrix}$$

$$= \begin{bmatrix} K_{p1} p_1 + K_{p0} (p_1 - T u_1) \\ K_{u1} p_1 + K_{u0} (p_1 - T u_1) \end{bmatrix} = \begin{bmatrix} p_1 \\ u_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (K_{p1} + K_{p0} - I) p_1 - K_{p0} T u_1 \\ (-I - K_{u0} T) u_1 + (K_{u1} + K_{u0}) p_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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b) This gives that the positional estimate is dependent on the velocity measurement/estimate. To eliminate this dependence, we set

$$K_{p0} = 0_2$$

$$\Rightarrow \quad K_{p1} = I_2$$

From the velocity estimation we have that

$$\left.\begin{array}{l} K_{u1} + K_{u0} = 0_2 \\ I_2 = -K_{u0}\, T \end{array}\right\} \quad \begin{array}{l} K_{u0} = -\frac{1}{T} I_2 \\ K_{u1} = \frac{1}{T} I_2 \end{array}$$

c) From the lectures, we know that

$$\operatorname{cov}[\hat{x}] = E\left[(\hat{x} - x)(\hat{x} - x)^T\right]$$

$$\hat{x} - x = \begin{bmatrix} K_{p1} z_1 + K_{p0} z_0 - p_1 \\ K_{u1} z_1 + K_{u0} z_0 - u_1 \end{bmatrix}$$

$$= \begin{bmatrix} K_{p1}(Hx_1 + w_1) + K_{p0}(Hx_0 + w_0) - p_1 \\ K_{u1}(Hx_1 + w_1) + K_{u0}(Hx_0 + w_0) - u_1 \end{bmatrix}$$

$$= \begin{bmatrix} K_{p1} p_1 + K_{p1} w_1 + K_{p0} H x_0 + K_{p0} w_0 - p_1 \\ K_{u1} p_1 + K_{u1} w_1 + K_{u0} H x_0 + K_{u0} w_0 - u_1 \end{bmatrix}$$

Inserting values

$$= \begin{bmatrix} p_1 + w_1 - p_1 \\ \frac{1}{T} p_1 + \frac{1}{T} w_1 - \frac{1}{T} H F^{-1}(x_1 - v_0) - \frac{1}{T} w_0 - u_1 \end{bmatrix}$$

$$= \begin{bmatrix} w_1 \\ \frac{1}{T} w_1 - \frac{1}{T} w_0 + \frac{1}{T} H F^{-1} v_0 \end{bmatrix}$$

$$(\hat{x} - x)(\hat{x} - x)^T = \begin{bmatrix} w_1 \\ \frac{1}{T}(w_1 - w_0 + HF^{-1} v_0) \end{bmatrix} \Big[w_1^T \quad \frac{1}{T}(w_1^T - w_0 + HF$$

$$= \begin{bmatrix} w_1 w_1^T & \frac{1}{T} w_1 (w_1 - w_0 + MF^{-1} v_0)^T \\ \frac{1}{T} (w_1 - w_0 + MF^{-1} v_0) w_1^T & \frac{1}{T^2} (w_1 - w_0 + MF^{-1} v_0) \cdot (w_1 - w_0 + MF^{-1} v_0)^T \end{bmatrix}$$

$$E[(\tilde{x} - x)(\tilde{x} - x)^T] =$$

$$= E \begin{bmatrix} w_1 w_1^T & \frac{1}{T} w_1 (w_1 - w_0 + MF^{-1} v_0)^T \\ \frac{1}{T} (w_1 - w_0 + MF^{-1} v_0) w_1^T & \frac{1}{T^2} (w_1 - w_0 + MF^{-1} v_0)(w_1 - w_0 + MF^{-1} v_0)^T \end{bmatrix}$$

$$= \begin{bmatrix} R & \frac{1}{T} R \\ \frac{1}{T} R & \frac{1}{T^2} (R + R + MF^{-1} E[v_0 v_0^T] (MF^{-1})^T) \end{bmatrix}$$

$$= \begin{bmatrix} R & \frac{1}{T} R \\ \frac{1}{T} R & \frac{1}{T^2} (2R + MF^{-1} Q (MF^{-1})^T) \end{bmatrix} = \mathrm{cov}[\tilde{x}]$$

d) Trying to find the ~~error~~ distribution for the actual value for x

Using from earlier that

$$\tilde{x}_1 = \begin{bmatrix} p_1 + w_1 \\ \frac{1}{T} p_1 + \frac{1}{T} w_1 - \frac{1}{T} MF^{-1} (x_1 - v_0) - \frac{1}{T} w_0 \end{bmatrix}$$

$$= \begin{bmatrix} p_1 + w_1 \\ \frac{1}{T} p_1 - \frac{1}{T} p_1 + \frac{1}{T} w_1 + u_1 - \frac{1}{T} w_0 + \frac{1}{T} MF^{-1} v_0 \end{bmatrix}$$

$$= \begin{bmatrix} p_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} w_1 \\ \frac{1}{T} (w_1 - w_0 + MF^{-1} v_0) \end{bmatrix}$$

$$\Rightarrow \hat{x}_1 = x_1 + \begin{bmatrix} w_1 \\ \frac{1}{T}(w_1 - w_0 + HF^{-1}v_0) \end{bmatrix}$$

$$\Rightarrow x_1 = \hat{x}_1 - \begin{bmatrix} w_1 \\ \frac{1}{T}(w_1 - w_0 + HF^{-1}v_0) \end{bmatrix}$$

$$E[x_1] = E[\hat{x}_1] - E\begin{bmatrix} w_1 \\ \frac{1}{T}(w_1 - w_0 + HF^{-1}v_0) \end{bmatrix} = \hat{x}_1$$

$\mathrm{cov}[x_1] = E[(x_1 - \hat{x}_1)(x_1 - \hat{x}_1)^T]$ // equivalent to expression in c) just with a different sign

$$= \begin{bmatrix} R & \frac{1}{T}R \\ \frac{1}{T}R & \frac{1}{T^2}(2R + HF^{-1}Q(HF^{-1})^T) \end{bmatrix}$$

normally distributed since $\hat{x}_1$ is a gaussian.

$$\Rightarrow x_1 \sim N\left(\hat{x}_1, \begin{bmatrix} R & \frac{1}{T}R \\ \frac{1}{T}R & \frac{1}{T^2}(2R + HF^{-1}Q(HF^{-1})^T) \end{bmatrix}\right)$$

e) Assuming that the model is totally correct and that the covariances are totally known (and wgn), the model / estimate should be optimal. This is however only theoretically possible since in reality:

- the noise will not be gaussian and white
- the system model will be simplified from an unknown nonlinear model. I write unknown, since it is impossible to know ~~exactly~~ the exact modelling parameters etc that make up the system.
- in reality the system will also be time varying, such that the variance of the measurements will increase f.ex.