Problem 1

a)

Since we use $\|\cdot\|_\infty$ ~~the~~ norm, $\|A\| = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|$ $n=2$

Since $1+2=3 > \varepsilon$ (by assumption):

$$\|A\| = \max_{i=1,2} \sum_{j=1}^{2} |a_{ij}| = \sum_{j=1}^{2} |a_{1j}| = |1| + |2| = \underline{3}$$

Now, find A^{-1}:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & \varepsilon & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{\varepsilon} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{2}{\varepsilon} \\ 0 & 1 & 0 & \frac{1}{\varepsilon} \end{array}\right) \quad \text{So } A^{-1} = \begin{pmatrix} 1 & -\frac{2}{\varepsilon} \\ 0 & \frac{1}{\varepsilon} \end{pmatrix}$$

Since $1 + \frac{2}{\varepsilon} > \frac{1}{\varepsilon}$ $(\varepsilon > 0)$, we get:

$$\|A^{-1}\| = \max_{i=1,2} \sum_{j=1}^{2} |b_{ij}| = \sum_{j=1}^{2} |b_{1j}| = |1| + \left|\frac{2}{\varepsilon}\right| = 1 + \frac{2}{\varepsilon}. \quad \left(A^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right)$$

Finally, $K_\infty(A) = \|A\| \cdot \|A^{-1}\| = 3\left(1 + \frac{2}{\varepsilon}\right) = 3 + \frac{6}{\varepsilon}$ ■

b) We apply the following theorem:

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \le K_\infty(A) \cancel{\frac{\|\delta b\|}{\|b\|}} \frac{\|\delta c\|_\infty}{\|c\|_\infty}$$

Then, $K_\infty(A) = 3 + \frac{6}{\varepsilon} = 3 + \frac{6}{10^{-4}} = 3 + 6\cdot 10^4$,

$\frac{\|\delta c\|_\infty}{\|c\|_\infty} = 0.001$.

Hence,

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \le (3+6\cdot 10^4)\cdot 10^{-3} = 3\cdot 10^{-3} + 60 \approx 60$$

We can only guarantee that the relative ~~error~~ error propagated into the solution is less than around 60. ∎

Problem 2

a)

$$x^{(1)} = f(x^{(0)}) = f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}(1-0^2\cdot 1)/4\\(0^2+1^2+1)/8\end{pmatrix} = \begin{pmatrix}1/4\\2/8\end{pmatrix} = \underline{\begin{pmatrix}1/4\\1/4\end{pmatrix}}$$

b) First, we need to show $f: D \to D$, $D = [0,1]^2$

Since, $(a,b) \in [0,1]^2$:

$(1-a^2b)/4 \leq (1-0)/4 = \frac{1}{4} \in [0,1]$

$(1-a^2b)/4 \geq (1-1)/4 = 0 \in [0,1]$

$(a^2+b^2+1)/8 \leq (1+1+1)/8 = \frac{3}{8} \in [0,1]$

$(a^2+b^2+1)/8 \geq (0+0+1)/8 = \frac{1}{8} \in [0,1]$

Hence, $f: D \to D$.

Now, find $L = \max\limits_{(a,b)\in D} \cancel{\|J_f(x)\|}\ \|J_f(a,b)\|$ in $\|\cdot\|_\infty$ norm

$$J_f = \begin{pmatrix}\frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b}\\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b}\end{pmatrix} = \begin{pmatrix}-\frac{ab}{2} & -\frac{a^2}{4}\\ \frac{a}{4} & \frac{b}{4}\end{pmatrix} =: \begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}$$

We need

$$\max_{(a,b)\in D} \max_{i=1,2} \sum_{j=1}^{2} |c_{ij}|$$

Test:
$$\max_{(a,b)\in D} \sum_{j=1}^{2} |c_{1j}| = \max_{(a,b)\in D} \frac{a}{4} + \frac{b}{4} = \frac{1}{2}$$

$$\max_{(a,b)\in D} \sum_{j=1}^{2} |c_{2j}| = \max_{(a,b)\in D} \frac{ab}{2} + \frac{a^2}{4} = \frac{3}{4}$$

Hence,

$$L = \max_{(a,b)\in D} \|J_f(a,b)\| = \frac{3}{4}$$

which is the contraction constant.

According to Banach's fixed point theorem, since $D = [0,1]^2$ is closed, and $f : D \to D$ is a contraction (continuosly differentiable and Lipschitz contstant $L = \frac{3}{4}$), f has a unique fixed point, and $x^{(k+1)} = f(x^{(k)})$ converges to that point. ■ (Also since $x^{(0)} \in D$).

c) $\|x^{(k)} - x\|_\infty \overset{!}{\leq} 10^{-3}$

$$\|x^{(k)} - x\|_\infty = \lim_{l\to\infty} \|x^{(l)} - x^{(k)}\|_\infty \leq \|x^{(1)} - x^{(0)}\|_\infty \frac{L^{k+1}}{1-L}$$

$$= \left\|\frac{1}{4}\binom{1}{1} - \binom{0}{1}\right\|_\infty \frac{(3/4)^{k+1}}{1/4} = \frac{3}{4}\cdot 4\left(\frac{3}{4}\right)^{k+1} = 4\cdot\left(\frac{3}{4}\right)^{k+2}$$

When $k = 30$, this expression is less than 10^{-3}, so we need approximately 30 iterations to guarantee $\|x^{(k)} - x\|_\infty \leq 10^{-3}$ ■

Problem 3a) Since p_n interpolates f in $n+1$ points:

$$\|f-p_n\|_\infty \le \frac{M_{n+1}}{(n+1)!}\,|\pi_{n+1}(x)|$$

Where $M_{n+1} = \sup_{x\in[0,1]} \left|\frac{(-1)^{n+1}\cdot(n+1)!}{(1+x)^{n+2}}\right| \quad \left(= \sup_{x\in[0,1]} |f^{(n+1)}(x)|\right)$

$$\sup_{x\in[0,1]} \left|\frac{(-1)^{n+1}(n+1)!}{(1+x)^{n+2}}\right| = \left|\frac{(n+1)!}{1^{n+2}}\right| = (n+1)!$$

Then,

$$\|f-p_n\|_\infty \le \frac{(n+1)!}{(n+1)!}\cdot|\pi_{n+1}(x)| \le \frac{(n+1)!}{n^{n+1}}$$ Since x_n are equispaced.

If we choose $n=\cancel{10}\,15$, $\frac{(n+1)!}{n^{n+1}} = \frac{(16)!}{15^{16}} \approx 3.2\cdot 10^{-6} < 10^{-4}$,
hence we have found the desired n.

b) If we choose $\pi_{n+1}(x) = 2^{-n}\,T_{n+1}(x)$, $|\pi_{n+1}(x)|$ will be as small as possible. (T_n is the n-th Chebyshoff polynomial). Then:

~~$\|f-p_n\|_\infty \le \frac{M_{n+1}}{(n+1)!}\,|\pi_{n+1}(x)| = |2^{-n}T_n$~~

$\hat{x}_0,\dots,\hat{x}_n$ are the zeros of T_{n+1}: $\hat{x}_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)$, $j=0,\dots,n$

But $\hat{x}_0,\dots,\hat{x}_n \in [-1,1]$, so we have to let $x_0,\dots,x_n$ be defined: $x_j = \frac{\hat{x}_j+1}{2}$ as our points of interpolation. Then,

$$\|f-p_n\|_\infty \le |\pi_{n+1}(x)| = \left(\frac{1}{2}\right)^{n+1}\cdot 2^{-n}\,|T_{n+1}(2x_j-1)| \le \underline{\underline{\frac{1}{2^{2n+1}}}}$$

Problem 4

Know that given φ_0, φ_1

$$p(x) = \frac{\langle f, \varphi_0\rangle}{\langle \varphi_0, \varphi_0\rangle}\varphi_0 + \frac{\langle f, \varphi_1\rangle}{\langle \varphi_1, \varphi_1\rangle}\varphi_1$$

$$\langle f, \varphi_0\rangle = \langle e^x, 1\rangle = \int_0^1 x \cdot e^x dx = \left[e^x(x-1)\right]_0^1 = 1$$

$$\langle f, \varphi_1\rangle = \langle e^x, x - \tfrac{2}{3}\rangle = \langle e^x, x\rangle - \tfrac{2}{3}\langle e^x, \varphi_0\rangle = \int_0^1 x^2 \cdot e^x dx - \tfrac{2}{3}$$

$$= \left[e^x(x^2 - 2x + 2)\right]_0^1 - \tfrac{2}{3} = e - 2 - \tfrac{2}{3} = e - \tfrac{8}{3}$$

$$\langle \varphi_0, \varphi_0\rangle = \int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}$$

$$\langle \varphi_1, \varphi_1\rangle = \int_0^1 x(x-\tfrac{2}{3})^2 dx = \int_0^1 x^3 - \tfrac{4}{3}x^2 + \tfrac{4}{9}x\,dx = \left[\tfrac{x^4}{4} - \tfrac{4x^3}{9} + \tfrac{2x^2}{9}\right]_0^1 = \tfrac{1}{36}$$

Then, $p(x) = \frac{1}{\frac{1}{2}}\varphi_0 + \frac{e - \frac{8}{3}}{\frac{1}{36}}\varphi_1 = 2 + 36(e - \tfrac{8}{3})(x - \tfrac{2}{3})$

$$= 2 + (36e - 96)x - 24e + 64 = \underline{(36e - 96)x - 24e + 66}$$

Problem 5

Since the order is always at least $n+1=2$, we need $I(f)=I_1(f)$ $\forall f \in P_2$. It is enough to check for $f(x)=x^2$.

We want $I_1(f) = I(f) = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

Define $w_0 = \int_{-1}^{1} L_0(x)dx = \int_{-1}^{1} \frac{x-x_1}{-\frac{2}{3}-x_1}dx = \left[\frac{x^2}{2(-\frac{2}{3}-x_1)} - \frac{x \cdot x_1}{(-\frac{2}{3}-x_1)}\right]_{-1}^{1}$

$= -\frac{2x_1}{(-\frac{2}{3}-x_1)} = \underline{\frac{2x_1}{\frac{2}{3}+x_1}}$

$w_1 = \int_{-1}^{1} L_1(x)dx = \int_{-1}^{1} \frac{x+\frac{2}{3}}{x_1+\frac{2}{3}}dx = \left[\frac{x^2}{2(x_1+\frac{2}{3})} + \frac{\frac{2}{3}x}{x_1+\frac{2}{3}}\right]_{-1}^{1} = \frac{4}{3(x_1+\frac{2}{3})}$

Then, $I_1(f) = w_0 f(-\frac{2}{3}) + w_1 f(x_1) \overset{!}{=} \frac{2}{3}$

$\Leftrightarrow \frac{2x_1}{\frac{2}{3}+x_1}\left(\frac{2}{3}\right)^2 + \frac{2\cdot 2}{3} \frac{x_1^2}{x_1+\frac{2}{3}} = \frac{2}{3}$

$\Leftrightarrow \frac{2x_1(\frac{2}{3})}{\frac{2}{3}+x_1} + \frac{2x_1^2}{x_1+\frac{2}{3}} = 1 \Leftrightarrow \frac{2x_1(\frac{2}{3}+x_1)}{\frac{2}{3}+x_1} = 1$

This is solved when $\underline{\underline{x_1 = \frac{1}{2}}}$

This choise of $x_1 = \frac{1}{2}$ will guarantee $I_1(f) = I(f)$ $\forall f \in P_2$, which means the order of $I_1(f)$ is at least 3.

Since this is the unique solution of x_1 such that the order is at least 3, ~~~~ we have to hope that the order is 4 for the same x_0, x_1.
But since the order can be maximum $2(n+1) = 4$, this is the largest possible order for $p=1$, which means it has to be the Gauss quadrature rule. But since the weight function here is $w(x) = 1$, which is symmetric, the quad. points must ~~~~ also be symmetric (i.e. $x_1 = -x_0$) which is not the case. Hence, since x_0, x_1 are ~~not~~ not Gauss quad points, the order cannot be 4. $\blacksquare$
(order $= 2(n+1) \Leftrightarrow$ Gauss quad rule)

Problem 6 a)

Say we want to approximate $f(x)$ using a Taylor expansion, using the point $a = (\frac{1}{2}, \dots, \frac{1}{2})$ and order 1. We get

$$T_n f(x) = f(\tfrac{1}{2}, \dots, \tfrac{1}{2}) + \nabla f(a)(x_1 - \tfrac{1}{2}, \dots, x_n - \tfrac{1}{2})$$

Then, to approximate $\int_{[0,1]^d} f(x)\,dx$, we integrate:

$$\int_{[0,1]^d} T_1 f(x)\,dx = \int_{[0,1]^d} f(\tfrac{1}{2}, \dots, \tfrac{1}{2})\,dx + \nabla f(\tfrac{1}{2}, \dots, \tfrac{1}{2}) \int_{[0,1]^d} (x - \tfrac{1}{2}, \dots, x - \tfrac{1}{2})\,dx$$

But $\int_{[0,1]^d} (x - \frac{1}{2}, \dots, x - \frac{1}{2})\,dx = 0$, so we get

$$\int_{[0,1]^d} T_1 f(x)\,dx = \int_{[0,1]^d} f(\tfrac{1}{2}, \dots, \tfrac{1}{2})\,dx = f(\tfrac{1}{2}, \dots, \tfrac{1}{2}) = I_0(f).$$

To analyze the error, we look at ~~the~~ the remainder of the multi-dimensional Taylor expansion:

$$|R(x)| \leq \tfrac{1}{2} \|\nabla^2 f\|_\infty \|x - a\|_\infty^2$$

Since $x - a = (x_1 - \frac{1}{2}, \dots, x_n - \frac{1}{2})$, and $x \in [0,1]^d$, $\|x - a\|_\infty \leq \frac{1}{2}$. Hence,

$$\cancel{|R(x)|}\; |I(f) - I_0(f)| = |R(x)| \leq \tfrac{1}{2} \|\nabla^2 f\|_\infty \cdot (\tfrac{1}{2})^2 = \tfrac{1}{8} \|\nabla^2 f\|_\infty \quad \blacksquare$$

b) Composite midpoint method:

$$I_{0,m}(f) = \frac{1}{m^d}\sum_{i_1=1}^{m} \cdots \sum_{i_d=1}^{m} f\left(\frac{i_1-\frac{1}{2}}{m}, \dots, \frac{i_d-\frac{1}{2}}{m}\right)$$

Error:

$$|I(f) - I_{0,m}(f)| = \left|\int_{[0,1]} f(x)\,dx - \frac{1}{m^d}\sum_{i_1=1}^{m} \cdots \sum_{i_d=1}^{m} f\left(\frac{i_1-\frac{1}{2}}{m}, \dots, \frac{i_d-\frac{1}{2}}{m}\right)\right|$$

$$\cancel{= \left|\int_0^1 \cdots \int_0^1 f(x_1,\dots,x_n)\,dx_1 \cdots dx_n - \frac{1}{m}\sum\sum \int_0^1 f\left(\frac{i-\frac{1}{2}}{m}, x_2, \dots, x_n\right) dx_2 \cdots dx_n \right.}$$

$$\cancel{+ \frac{1}{m}\sum_{i_1=1}^{m}\sum_{i=1}^{m} \int_0^1 f\left(\frac{i-\frac{1}{2}}{m}, x_2, \dots, x_n\right) dx_2 \cdots dx_n - \frac{1}{m}\sum_{i_1=1}^{m} \cdots \sum_{i_d=1}^{m} f\left(\frac{i_1-\frac{1}{2}}{m}\right.}$$

$$= \left|\int_0^1 \cdots \int_0^1 f(x_1,\dots,x_n)\,dx_1 \cdots dx_n - \frac{1}{m}\sum_{i_1=1}^{m} \sum_{i_{d-1}=1}^{m} \int f\left(\frac{i_1-\frac{1}{2}}{m}, \dots, x_d\right) dx_d\right|$$

$$\cancel{\leq \left|\int_0^1 \cdots \left(\int_0^1 f(x)\,dx_d - \frac{1}{m}\sum_{i=1}^{m} f\left(\frac{i-\frac{1}{2}}{m}, x_d\right)\right) d\right.}$$

$$\cdots \leq \left|\int_0^1 \left(\int_0^1 \cdots \int_0^1 f(x)\,dx_2\,dx_3 \cdots dx_d - \frac{1}{m^{d-1}}\sum_{i=1}^{n} \cdots \sum_{i=1}^{m} f\left(\frac{i-\frac{1}{2}}{m}, x_2, \dots, x_d\right)\right) dx_d\right|$$

$$+ \left|\frac{1}{m}\sum_{i=1}^{m}\sum\left(\int_0^1 f\left(\frac{i_1-\frac{1}{2}}{m}, \frac{i_2-\frac{1}{2}}{2}, \dots, x_d\right) dx_d - \frac{1}{m}\sum_{j=1}^{m} f\left(\frac{i-\frac{1}{2}}{m}, \dots, \frac{j-\frac{1}{2}}{m}\right)\right)\right|$$

$$\leq \int_0^1 \int \frac{1}{8}\frac{1}{m^2}\left\|\frac{\partial f}{\partial x_1}\right\| dx_2 \cdots dx_n + \cdots + \frac{1}{m}\sum_{i=1}^{m} \frac{1}{8}\frac{1}{m^2}\left\|\frac{\partial f}{\partial x_d}\right\|_\infty \quad \leftarrow \text{by last problem a)}$$

$$\leq \frac{1}{8}\frac{1}{m^2}\left\|\frac{\partial f}{\partial x_1}\right\|_\infty + \cdots + \frac{1}{8}\frac{1}{m^2}\left\|\frac{\partial f}{\partial x_d}\right\|_\infty = \frac{1}{8}\frac{1}{m^2}\|\nabla f\|_\infty \quad \blacksquare$$

c) The total ~~nu~~ number of function evaluations are $(m(n+1))^d$ for d dimensions, m sub-intervals and n points per sub-interval. So for $n=0$; total ~~nu~~ number of evaluations:

$m_{tot} = m^d$, so $m = m_{tot}^{\frac{1}{d}}$. We insert into the error term:

$$\frac{\|D^2 f\|_\infty}{8m^2} = \frac{1}{8 \cdot m_{tot}^{\frac{2}{20}}} = \frac{1}{8 \cdot m_{tot}^{\frac{1}{10}}} \overset{!}{\leq} 10^{-4}$$

Solve for m_{tot}:

$$m_{tot}^{\frac{1}{10}} \geq \frac{1}{8} \cdot 10^4 = \left(\frac{5}{4}\right) \cdot 10^3$$

$$\Leftrightarrow \quad m_{tot} \geq \left(\frac{5}{4}\right)^{10} \cdot 10^{30}$$

If a laptop can do 10^{12} operations per second, then it would take at least $10^{30-12} = 10^{18}$ seconds ~~or~~ to compute, which is definetly not feasible on a modern laptop. Most laptops will operate at around this speed. ∎