Finding the Radiation from Schwarzschild Binaries through EFTs

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The following is an attempted condensed summary of the relevant parts of Michele Levi's "Effective Field Theories of Post-Newtonian Gravity: A Comprehensive Review" followed by questions. We will try to use the full formalism developed there to recreate the lowest order chirp signal for the gravitational wave. Next step will be including spin or electric charge for the black hole components (NB, we are using $(+---)$ and $c = 1$).

1 Short summary

1.1 Integrating out the Schwarzschild radius

The physics of black hole binaries in the Post-Newtonian regime operates at 3 clearly distinguished scales. As such, effective field theories are appropriate tools to deal with the unwieldy equations, for which we integrate out heavy degrees of freedom.

First, the individual black hole's (or neutron stars') physics, we would like to model as that of points interacting gravitationally. They are not points, and so in addition to the point action, there would be Wilson coefficients parametrising the non-point effects of them. Starting from the black hole's full action

$$S[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R[g_{\mu\nu}], \tag{1}$$

we separate the metric into a weak field part $\bar{g}$ and a strong field part g^s. Rather than starting from the full action and integrating out this heavy degree of freedom (top-down), we parametrise its effect by expanding the action in terms of the light field action and all possible other operators allowed by the symmetries of our theory, giving

$$\begin{aligned} S_{\text{eff}} &= -\frac{1}{16\pi G}\int d^4x\sqrt{g}R[\bar{g}_{\mu\nu}] + \sum_{i=1}^{\infty}\mathcal{C}_i(R_s)\int d\sigma\mathcal{O}_i(\sigma) && (2)\\ &= S[\bar{g}_{\mu\nu}] + S_{pp}[\bar{g}_{\mu\nu},(y_a)^\mu,(e_a)^\mu_A] && (3)\end{aligned}$$

$\mathcal{C}_i$ would be the Wilson coefficients, dependent on effects on the scale of the black hole (star), σ the world line parameter and $\mathcal{O}_i$ the operators that is not necessarily renormalisable, it being an effective theory valid at some low energy. $(y_a)^\mu$ is the worldline while $(e_a)^\mu_A$ is the tetrad basis. From exploring the lowest order terms and dimensional arguments one can however tell that the first terms dependent on the internal physics of the objects (the higher Wilson coefficients) does not become of importance until post-Newtonian order PN(5) $\sim v^{10}$ from leading order, so we are content with the point action.

1.2 Integrating out the orbital scale

Our next step is to model the two points orbiting around each other as themselves a single point, parametrising again the non-point effects using Wilson coefficients. Our scheme this time will be to use both a bottom-up and a top-down approach and then to finally match the two using Feynman diagrams, perturbatively in the velocity parameter v, to determine the Wilson coefficients of the bottom-up approach, to a given order.

Towards this aim, we separate the (light-field) metric into its different parts

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + \tilde{h}_{\mu\nu}, \tag{4}$$

where η is the Minkowski metric that prevails asymptotically as the other terms go to zero, H is the orbital modes and $\tilde{h}$ the radiative modes. The scale dependence of these latter two differ; For the orbital modes we have $\partial_t H \sim v/r$, $\partial_i H \sim 1/r$, where r is the separation, while $\partial_\mu \tilde{h} \sim v/r$, making the latter soft with respect to the first.

First, top-down, combining the two black holes' point actions which will constitute the higher energy physics

$$S_{\text{eff}} = S[\bar{g}_{\mu\nu}] + \sum_{a=1}^{2} S_{pp}[\bar{g}_{\mu\nu}, (y_a)^\mu, (e_a)^\mu_A], \tag{5}$$

we explicitly integrate out the orbital (hard) modes (using $\tilde{g} = \eta + \tilde{h}$)

$$\exp\left(iS_{\text{eff}}[\tilde{g}_{\mu\nu}, (y_c)^\mu, (e_c)^\mu]\right) = \int dH_{\mu\nu} \exp\left(iS_{\text{eff}}[\bar{g}_{\mu\nu}, \dots]\right). \tag{6}$$

On the other hand, through the bottom-up approach, we find that the same action should be equal to

$$S_{\text{eff}}[\tilde{g}_{\mu\nu}, (y_c)^\mu, (e_c)^\mu] = S[\tilde{g}_{\mu\nu}] + S_{pp(comp)}[\tilde{g}_{\mu\nu}, (y_c)^\mu, (e_c)^\mu_A]. \tag{7}$$

We will return to the details of the matching through Feynman diagrams later on.

1.3 Integrating out the radiation modes, and finding the observables

In the final step, we integrate out the radiation modes from the action above to leave us with a dissipative action where we can find the energy output through the optical theorem (the imaginary part of the effective action). In order to do this we will make use of the closed time path method to be discussed later. We can use the Hamiltonian (obtained through a Legendre transform from the effective Lagrangian) in the EOB method for the complete waveform. The emission plus the radiation field gives us finally the gravitational wave observable.

2 More details

2.1 Smallest scale

For the point action (the Einstein-Hilbert action for a point particle), we would simply have

$$S_{\mathrm{pp}} = -m \int d\tau = -m \int d\sigma \sqrt{u^2}, \tag{8}$$

where $u = \frac{dy}{d\sigma}$. This coincides with what we would naively expect, knowing that the path taken in GR is the one that maximises the proper time. The first non-minimal terms of unpointlyness one can argue, as does Levi, are the terms

$$S_{\mathrm{pp}} \supset C_E \int d\sigma \frac{E_{\mu\nu}^2(y^2(\sigma))}{[\sqrt{u^2}]^3} + (E \to B)\,, \tag{9}$$

where E and B are the electric and magnetic projections of the Riemann tensor respectively. These, one can check, for black holes, have an additional velocity dependence relative to the leading order term of v^{10} and so are only made important at PN5 (for the Schwarzschild case). We are therefore happy simply using the action (8) for the individual black holes[1].

2.2 Orbital scale

Now that we have our effective theory at the black hole scale, we zoom out to the orbital scale and use the previous in constructing our more UV-complete theory that we want to find the effective theory for (top-down) by integrating out the orbital (hard) modes. We then find, using (5) and (6),

$$\begin{aligned} &\exp\left(iS_{\mathrm{eff}}[\tilde{g}_{\mu\nu}, (y_c)^\mu, (e_c)^\mu]\right) \\ &= \int dH_{\mu\nu} \exp\left(-\frac{i}{16\pi G}\int d^4x\sqrt{g}R[\bar{g}_{\mu\nu}] - i\sum_{i=1}^{2} m_i \int d\sigma_i \sqrt{u_i^2}\right). \end{aligned} \tag{10}$$

[1] There are also some dissipative, potentially unknown, microphysics happening on the black hole boundaries that requires careful attention. Levi finds however that these do not become of significance until PN6.5

On the other hand, we construct the bottom-up effective theory at the same scale as in (7), where

$$S_{\mathrm{pp(comp)}} = -\int dt \sqrt{\tilde{g}_{00}}$$
$$\times \left[M(t) - \sum_{l=2}^{\infty} \left(\frac{1}{l!} I^L(t) \nabla_{L-2} E_{i_{l-1} i_l} - \frac{2l}{(l+1)!} J^L(t) \nabla_{L-2} B_{i_{l-1} i_l} \right) \right], \quad (11)$$

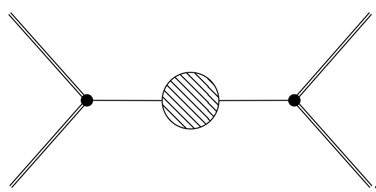
where we note that the time-dependence of the mass and current multipole tensors I and J are because of the dissipation we are anticipating for the next section when we integrate out the radiation field, while E and B are the electric and magnetic projections of the Riemann tensor as before.

2.2.1 Feynman diagrams

Now the point is to find the lowest order operators of (11) expressed through the variables in (10) using Feynman diagrams. We remember from the path integral formalism that we may write n-point correlation functions like

$$\langle \phi_1 \dots \phi_n \rangle = \frac{\int \mathcal{D}\phi \exp(iS) \phi_1 \dots \phi_n}{\int \mathcal{D} \exp(iS)}, \quad (12)$$

and we are therefore lucky to already have the top-down action in the exponential (10). We now want to graphically represent the worldline of each of the components and account for their interaction, like (time vertically)



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where the centre blob hides all their possible interactions while the double lines are their world lines. These are the conservative interactions, but there would also be ones were radiation escapes accounting for the dissipative sector.

Gauge Choices

Before we start, we need to pick appropriate coordinates. In [Levi,2020] it is mentioned that the non-relativistic character of the lowest order terms, and their instantaneous propagators motivate a Kaluza-Klein 3+1 parametrisation of the metric, which we write like

$$ds^2 = e^{2\phi} \left(dt - A_i dx^i \right)^2 - e^{-2\phi} \gamma_{ij} dx^i dx^j, \quad (13)$$

where $\gamma_{ij} = \delta_{ij} + \sigma_{ij}$. We pick the (fully) harmonic gauge, which amounts to adding to the action a gauge fixing term

$$S_{GF} = \frac{1}{32\pi G} \int d^4x \sqrt{g} g_{\mu\nu} \Gamma^\mu \Gamma^\nu, \tag{14}$$

$$\Gamma^\mu \equiv \Gamma^\mu_{\rho\sigma} g^{\rho\sigma}. \tag{15}$$

Finally, we need to choose an internal Lorentz gauge for the tetrad field. Because of the non-relativistic limit, the time direction being singled out, a good choice, Levi argues, is

$$\tilde{e}^0_i = 0, \tag{16}$$

where $\tilde{e}^\mu_i$ is the tetrad basis of the composite particle worldline. In terms of the KK fields, the tetrad fields would then be

$$\tilde{e}^a{}_\mu = \begin{pmatrix} e^\phi & -e^\phi A_i \\ 0 & e^{-\phi}\sqrt{\gamma_{ij}} \end{pmatrix}, \tag{17}$$

where $\sqrt{\gamma_{ij}}$ is the symmetric square root of γ_{ij}. One can tell that this is a solution by requiring that the tetrad field be such that that the metric is flat

$$g_{\mu\nu} = \eta_{ab} \tilde{e}^a{}_\mu \tilde{e}^b{}_\nu, \tag{18}$$

and inserting this with the above definition of $\tilde{e}$ into the spacetime interval, which will yield back the KK metric. The motivation behind doing this is that the new vectors $x'^a = e^a{}_\mu x^\mu$ lives in the Lorentz representation where we have a good description of SO(3,1) so that we may include into our description spin. For now, however, we consider non-spinning black holes.

Action

We turn back to the point action (8), choosing a basis where the worldline parameter $\sigma = t$ and the velocity $u^\mu = \frac{dy^\mu}{dt} = (1, v^i)$, so that

$$S_{pp} = -m \int dt\, e^\phi \sqrt{(1 - 2A_i v^i)^2 - e^{-4\phi}\gamma_{ij} v^i v^j}, \tag{19}$$

which may be expanded in both the velocity and the KK fields. { NB: we should really also find the interaction terms stemming from the gauge fixing term...}

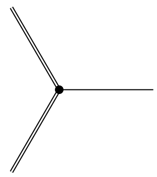
These terms would be for each of the black holes and constitute interaction terms for the fields. Establishing Feynman diagram conventions

$$\langle \phi\phi \rangle = \rule{2cm}{0.4pt}, \tag{20}$$

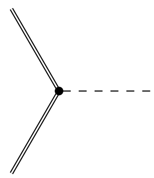
$$\langle A_i A_j \rangle = - - - - - - - - - - - -, \tag{21}$$

$$\langle \sigma_{ij}\sigma_{kl} \rangle = \sim\sim\sim\sim\sim\sim\sim\sim, \tag{22}$$

so that we have the following lowest order vertices (seen from the point action)



$$= -m \int dt\, \phi \tag{23}$$



$$= m \int dt\, A_i v^i + \mathcal{O}(v^2), \tag{24}$$

where the KK vector field interaction can be neglected at leading order with respect to the scalar field, and the tensor field comes in at an even higher order.

Propagator

Now we are only lacking the propagator(s) to be ready to start calculating diagrams. At lowest order, because of the lowest order vertices we found, we need only to care about the propagator for the KK scalar field.

We know that the propagator is the Green's function of the equation of motion for the field, and so we are really looking for Green's functions for the wave equation (because we have the fully harmonic gauge, chosen normalisation gives the $4\pi G$)

$$(\Box_x + m^2)G(x, y) = 4\pi G\, \delta(x - y), \tag{25}$$

which we can take to Fourier space to find

$$(-p^2 + m^2)G(p) = 4\pi G, \tag{26}$$

which means that

$$G(x, y) = -4\pi G \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{p^2 - m^2 \pm i\epsilon}, \tag{27}$$

where we have taken the inverse Fourier and had to include $\pm i\epsilon$ for some small ϵ because (26) is not an algebraic equation but a distributional one, and therefore (27) comes as a consequence of the Sokhotski-Plemelj theorem where the $\pm i\epsilon$ is to do countours around the complex poles. The choice of $\pm i\epsilon$ around the two poles actually amounts to the choice of propagator, resulting in either the retarded, advanced or Feynman propagators.

We remember that the orbital field modes have $k_0 \sim v/r$ while $k^i \sim 1/r$, and so we may do a Taylor in the velocity (we pick $m = 0$ and name $p = k$)

$$\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik_0 t + ik^i x_i}}{k_0^2 - k_i k^i} = \delta(t) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^2} \left(1 + \mathcal{O}(v^2)\right) \tag{28}$$

and see that the propagator at lowest order is instantaneous, motivating our KK decomposition and Schwinger gauge that both treated the time-dimension as sort of compactified.

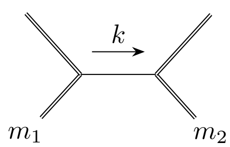
For the KK scalar field, we then have

$$= 4\pi G\, \delta(t_1 - t_2) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}}{\boldsymbol{k}^2}. \tag{29}$$

Feynman graphs

We are now in a position to start looking at Feynman graphs showing the orbital interactions perturbatively, which we may use to determine the lowest order Wilson coefficients for the composite point particle EFT. We are in practice evaluating right-hand side of (10).

We already know that the lowest order interaction vertex is that of the KK scalar degree of freedom, and so the lowest order interaction between two point mass worldlines would be one exchanging a single scalar



$$= \text{Figure 1} \tag{30}$$

$$\text{Figure 1} = 4\pi G\, m_1 m_2 \int dt_1 dt_2 \delta(t_1 - t_2) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}_1(t_1) - \boldsymbol{y}_2(t_2))}}{\boldsymbol{k}^2}. \tag{31}$$

To solve this latest integral, we use at first spherical coordinates (using now $k = |\boldsymbol{k}|$)

$$\int d^3k \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2} = 2\pi \int dk d(\cos\theta) k^2 e^{ikr\cos\theta} \frac{1}{k^2} = 2\pi \int dk dx e^{ikrx}, \tag{32}$$

where x goes from -1 to 1. We perform the x integration

$$= -\int dk \frac{1}{ikr} \left(e^{ikr} - e^{ikr} \right) = -2 \int dk \frac{\sin(kr)}{kr}. \tag{33}$$

We make use of the integral that $\int_0^\infty ds\, e^{-xs} = 1/x$ to change it to

$$= -2 \int dk ds\, e^{-krs} \sin kr, \tag{34}$$

from where we can either recognise the Laplacian tranform of the sine or solve ourselves with partial integration to find

$$= -\frac{2}{r} \int ds \frac{1}{1 + s^2}, \tag{35}$$

where we recognise the integrand to be the arctan derivative and finally find

$$= \frac{\pi}{r}, \tag{36}$$

which we swap into our original expression to find

$$\text{Figure 1} = \int dt \frac{Gm_1 m_2}{r(t)}, \tag{37}$$

where we use $t_1 = t_2 = t$ and $r = |\boldsymbol{x}_1 - \boldsymbol{x}_2|$. This expression seems familiar, having the Newtonian potential for integrand.

2.2.2 Matching Coefficients

Inserting this back into the action (10), we see that we have effectively integrated out the orbital modes (which at lowest order in fields and velocity simply was exchange of a KK scalar), and recognise the effective action

$$\begin{aligned} S_{\text{eff}}[\tilde{g}_{\mu\nu}, (y_c)^\mu, (e_c)^\mu] = &-\frac{1}{16\pi G}\int d^4x\sqrt{\tilde{g}}R[\tilde{g}_{\mu\nu}] \\ &- M\int dt\sqrt{\tilde{g}_{00}} + \int dt\left(\frac{1}{2}Mv^2 + \frac{Gm_1m_2}{r}\right), \end{aligned} \tag{38}$$

where the first new term you can see in (19) is because $e^\phi = \sqrt{g_{00}}$, and we use tilde because we have integrated out the orbital modes. We then simply had $\sum_{1,2} m_i \int dt\sqrt{\tilde{g}_{00}} = M\int dt\sqrt{g_{00}}$. The kinetic term comes from the δ_{ij} part of γ_{ij}. In the bottom-up action (7), our second term therefore corresponds to the mass quadrupole term. We have thus found the first few Wilson coefficients for the orbital scale composite object EFT in terms of the "UV-physics".

2.3 Radiation scale

We now will, similarly to what we did for our previous EFT, construct both an op-down and a bottom-up EFT where the radiative sector has been removed, and then finally do the matching. We separate the orbital scale background metric like

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{39}$$

Now we are faced with the following problem. Our action, and everything we have done thus far, are assumed to be time-symmetric, because energy and angular momentum should be conserved. However, now that we wish to extract gravitational waves, we want to impose boundary conditions that removes from the system the waves, and thus energy and angular momentum. The resulting effective action will have equations of motion that are not generally real nor causal as they should be. A formalism in QFT to deal with non-conservative evolution as we are faced with here is the closed time path (CTP) formalism, typically used in non-equilibrium QFT.

3 Questions

1. Why does the terms in (9) look the way they do (especially the velocity in the denominator)?
2. Question around the viability of splitting up the metric in different components and then splitting the action likewise when the action depends upon the metric non-linearly...
3. Question on power-counting. Especially (3.5) of Levi where I find $\sim v^8$ not v^{10}...
4. Question about the different velocity dependence of orbital- and radiative field modes.
5. Use of Kaluza-Klein decomposition, and relation to KK theory. Also, how do we get (4.8)/(13)?
6. Finding the different propagators (4.15-4.17).
7. How do we typically calculate $\sqrt{g}$? I've only come across it explicitly when we vary with respect to it (and we can use the trace of determinant identity)? Best to use Mathematica?