Lecture 16

(cont. from last time)

$$\varphi = 13 - 3x_2 - x_4 - x_6$$

$$x_3 = 1 + x_2 + 3x_4 - 2x_6$$
$$x_1 = 2 - 2x_2 - 2x_4 + x_6$$
$$x_5 = 1 + 5x_2 + 2x_4$$

$$\underbrace{B = \{3, 1, 5\},\ N = \{2, 4, 6\}}$$

Remember, B and N represents a vertex

Consider perturbation $\Delta c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

How much can we change the first comp. of c s.t. the given sets B and N still represents an optimal solution?

$$\Delta c_B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Delta c_N = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{z}_N = (B^{-1}N)^T (c_B + t\Delta c_B) - (c_N + t\Delta c_N) \geq 0$$

Wanted!

$$(B^{-1}N)^T = \begin{pmatrix} -1 & 2 & -5 \\ -3 & 2 & -2 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\Delta z_N = (B^{-1}N)^T \underbrace{\Delta c_B}_{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\tilde{z}_N = z_N^* + t\Delta z_N = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \geq 0$$

Wanted!

$$\left.\begin{array}{l} 3 + 2t \geq 0 \longrightarrow t \geq -\frac{3}{2} \\ 1 + 2t \geq 0 \longrightarrow t \geq -\frac{1}{2} \\ 1 - t \geq 0 \longrightarrow t \leq 1 \end{array}\right\} -\frac{1}{2} \leq t \leq 1$$

$t\Delta c_B = t\begin{pmatrix}0\\1\\0\end{pmatrix}$, coeff. $c_1 = 5$ can vary between $\left[5-\frac{1}{2},\ 5+1\right] = \left[\frac{9}{2},\ 6\right]$

So if c_1 varies between $9/2$ and 6, we are still in the same corner of the problem.

In general for a perturbation $t\Delta c$:

$$\left(\min_{j\in N} \frac{-\Delta z_j}{z_j^*}\right)^{-1} \leq t \leq \left(\max_{j\in N} \frac{-\Delta z_j}{z_j^*}\right)^{-1}$$

then the current sets $\mathcal{B}$ and $\mathcal{N}$ represents opt. solutions.

Analogously, changing (P) and (D) for general perturbations $t\Delta b$.

$$\text{If } \left(\min_{i \in \mathcal{B}} \frac{-\Delta x_i}{x_i^*} \right)^{-1} \leq t \leq \left(\max_{i \in \mathcal{B}} \frac{-\Delta x_i}{x_i^*} \right)^{-1}$$

then the current sets $\mathcal{B}$ and $\mathcal{N}$ represent optimal solutions.

The homotopy method

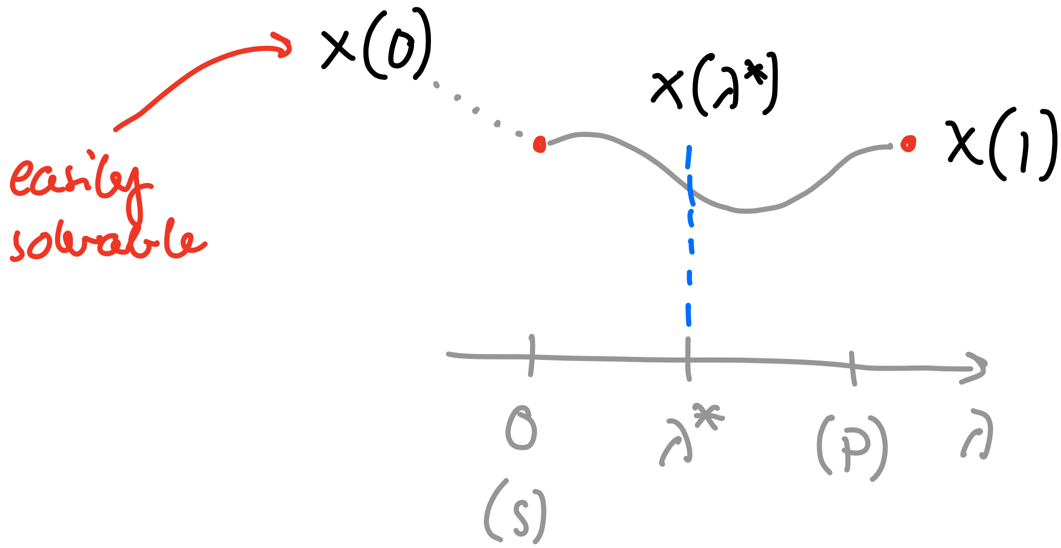
It is used in several areas of applied mathematics where a "difficult" problem (P) has to be solved.

General idea: define a "simple" problem (S).

(simple $\rightarrow$ can be easily solved)

Now connect both problems, e.g.

"$\lambda(P) + (1-\lambda)(S)$".



start with $x(0)$ (easily obtained) and solve the problems $\lambda(P) + (1-\lambda)(S)$ in between (P) and (S) for $0 < \lambda < 1$.

In practice we will only solve finitely many problems. We use a warm start strategy. use solution $x(\lambda^k)$ of $\lambda^k(P) + (1-\lambda^k)(S)$ for solving $\lambda^{k+1}(P) + (1-\lambda^{k+1})(S)$.

Note that in general, the homotopy method works only under certain conditions. For linear programs it works well!

Example

$$\begin{aligned} \max \quad & -2x_1 + 3x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \le -1 \\ & -x_1 - 2x_2 \le -2 \\ & x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

We can use the hom. m. to calculate a feasible starting point.

Starting dict. is neither feasible for (P) or (D).

our "difficult" problem

How to construct a simple related problem (S)?

We add $\mathbb{R}^+$ to each coeff. of b and subtract it from each coeff. of c.

$$\zeta = -\overbrace{(2 + \mu)}^{\geq 0 \text{ wanted}} x_1 - \overbrace{(-3 + \mu)}^{\geq 0 \text{ wanted}} x_2$$

$$x_3 = -1 + \mu + x_1 - x_2 \qquad \geq 0 \text{ wanted}$$

$$x_4 = -2 + \mu + x_1 + 2x_2$$

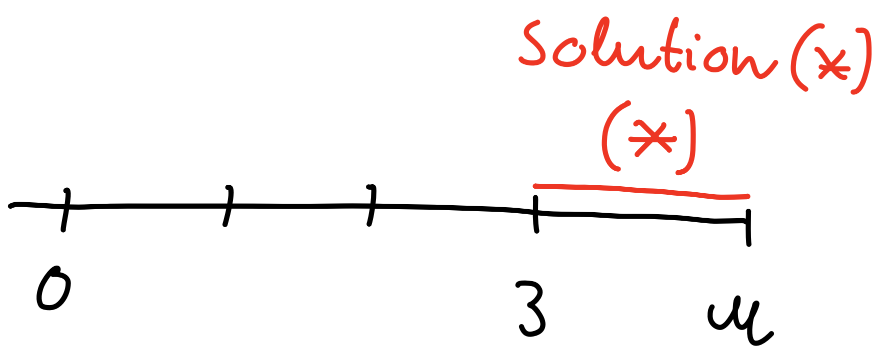
$$x_5 = 1 + \mu \qquad - x_2$$

Note! if $\mu = 0$ we get our original problem!

Also, it would be sufficient to perturb/add μ only to neg. coeff. of b.

For $\mu \geq 3$ the dict. is optimal with solutions:

$$x^*(\mu) = \begin{pmatrix} 0 \\ 0 \\ -1+\mu \\ -2+\mu \\ 1+\mu \end{pmatrix}, \quad z^*(\mu) = \begin{pmatrix} 2+\mu \\ 3+\mu \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



- Choose $\mu \geq 3$ in order to define the simple problem (S) for applying the homotopy method.
- At $\mu = 0$ we have the original "difficult" problem.
- What happens in $[0, 3]$?
- For $\mu < 3$ $(*)$ is not a solution since dual variable $y_2 = -3 + \mu < 0$ becomes infeasible for (D).

Make a primal pivot step at $\mu = 3$ choosing x_2 as new BV and x_3 is new NBV.

Note! The pivot step is done even though the dict. is optimal.

In general: choose the smallest μ^* (here $\mu^* = 3$) such that the current solution is optimal $\forall \mu \geq \mu^*$.

Choose an NBV ($\to$ optimal pivot step) or BV ($\to$ dual SM step) which becomes zero.

Dictionary after pivot step:

$$\zeta = -3 + 4\mu - \mu^2 - (-1 + 2\mu) x_1 - (3 - \mu) x_3$$

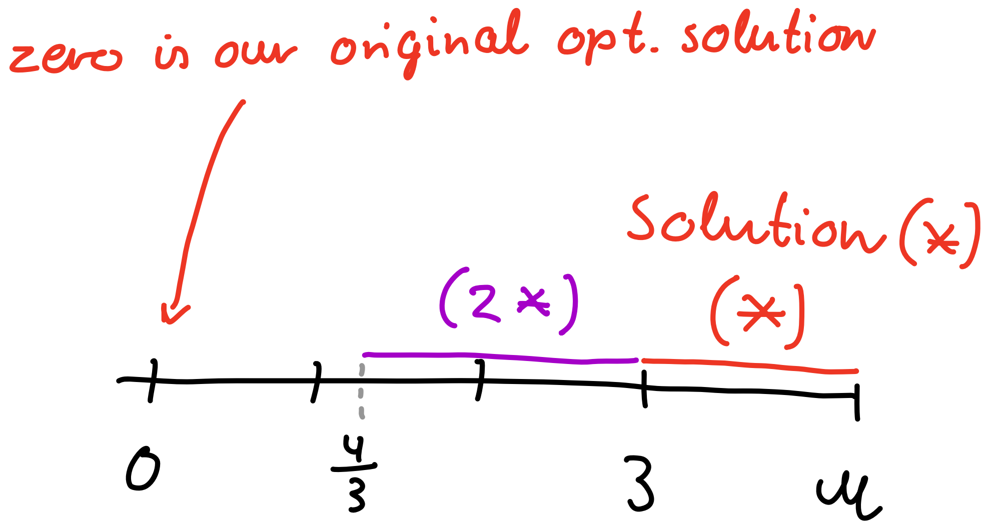
$$\begin{aligned}
x_2 &= -1 + \mu + x_1 - x_3 \\
x_4 &= -4 + 3\mu + 3x_1 - 2x_3 \\
x_5 &= 2 - x_1 + x_3
\end{aligned}$$

Dict. is optimal as long as:

$$\left.\begin{array}{lll} -1 + 2\mu \geq 0 & \longrightarrow & \mu \geq \frac{1}{2} \\ 3 - \mu \geq 0 & \longrightarrow & \mu \geq 3 \\ -1 + \mu \geq 0 & \longrightarrow & \mu \geq 1 \\ -4 + 3\mu \geq 0 & \longrightarrow & \mu \geq \frac{4}{3} \end{array}\right\} \frac{4}{3} \leq \mu \leq 3$$

With solutions $\left(\forall \mu \in \left[\frac{4}{3}, 3\right]\right)$:

$$\underbrace{x^*(\mu) = \begin{pmatrix} 0 \\ -1+\mu \\ 0 \\ -4+3\mu \\ 2 \end{pmatrix}, \quad z^*(\mu) = \begin{pmatrix} -1+2\mu \\ 0 \\ 3-\mu \\ 0 \\ 0 \end{pmatrix}}_{(2*)}$$



zero is our original opt. solution
Solution (*)
(2*)
(*)
0
4/3
3
u