A can be split into submatrices

$$A = \underbrace{B}_{m} \; \underbrace{N}_{n} \quad \Big] \, m$$

B is quadratic and non-singular.

Initially, first m columns is the basis variables and the first n columns is the non-basic variables.

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \quad c = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$$

These form our obj. func.

These are zero

Remember:

$$Ax = \begin{pmatrix} B & N \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = Ax_N + Ax_B = b$$

multiply from left by B^{-1}.

$$\underbrace{\underbrace{B^{-1}B}_{I^m} x_B}_{x_B} + B^{-1}N \underbrace{x_N}_{=0,\text{ because they are NBV.}} = B^{-1}b$$

$$x_B = B^{-1}b - \underbrace{B^{-1}N x_N}_{0} \qquad 2*$$

$$x_B = B^{-1}b \qquad 3*$$

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \quad \text{Feasible} \iff B^{-1} \cdot b \geq 0$$

Example:

$$A = \begin{pmatrix} -2 & 4 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

I^m (slack variables)

which two submatr. of A can be chosen as B s.t. B is non-sing.

(columns of B belong to the BV.

$$\frac{4 \cdot 3}{2 \cdot 1} = \binom{4}{2} = 6 \text{ comb. of two matr.}$$

All comb.:

1, 2
1, 3
1, 4
2, 3
2, 4
3, 4

$1\ 2 \Rightarrow x_1, x_2 \quad B_{12} \quad \begin{pmatrix} -2 & 4 \\ 0 & -1 \end{pmatrix}$

Is it non-singular (i.e. does the inverse matrix exist)?

In general, if we have $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$D^{-1} = \frac{1}{\det D} \begin{pmatrix} d & -b \\ c & a \end{pmatrix}$$

$(D^{-1} \text{ exists} \iff \det D \neq 0)$

$$\det B_{12} = 2 \rightarrow B_{12}^{-1} = \frac{1}{2} \begin{pmatrix} -1 & -4 \\ 0 & -2 \end{pmatrix}$$

Hence, we can choose x_1 and x_2 a basis variables here.

Feasibility check:

$$B_{12}^{-1} b = \frac{1}{2} \begin{pmatrix} -1 & -4 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -9/2 \\ -4/2 \end{pmatrix}$$

Infeas. because we have at lest one neg. coeff.

The vertex $\begin{pmatrix} -9/2 \\ -4/2 \\ 0 \\ 0 \end{pmatrix}$ is <u>not</u> a feasible point.

Now let's look at x_1 and x_3:

$$B_{13} = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

Determinant is zero / inverse matr. does <u>not</u> exist.

$\{x_1, x_3\}$ does not form a basis.

Now let's look at x_1 and x_4:

B_{14}^{-1} exists

not feasible.

$$x_B = B_{14}^{-1} b = \begin{pmatrix} -1/2 \\ 4/2 \end{pmatrix}$$

Now let's look at x_2 and x_3:

$$B^{-1}_{23} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

not feasible.

$$x_B = B^{-1}_{23}\, b = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$$

Now let's look at x_2 and x_4:

$$B^{-1}_{24} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 1 & 4 \end{pmatrix}$$

$$x_B = B^{-1}_{24}\, b = \begin{pmatrix} 1/4 \\ 9/4 \end{pmatrix}$$ Feasible!

$$\Longrightarrow \begin{array}{c} \begin{pmatrix} 0 \\ 1/4 \\ 0 \\ 9/4 \end{pmatrix} \end{array} \begin{array}{ll} x_1 & \text{NBV} \\ x_2 & \text{BV} \\ x_3 & \text{NBV} \\ x_4 & \text{BV} \end{array}$$

Now let's look at x_3 and x_4:

$$B_{34} = I^m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

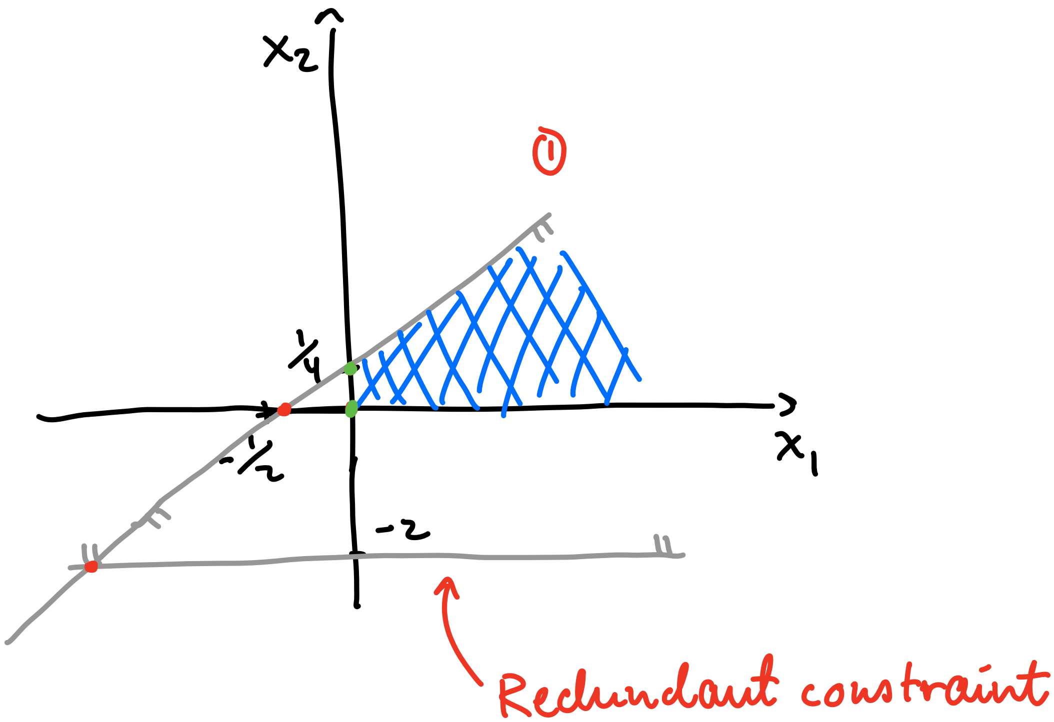
$$B_{34}^{-1} b = b \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$ feasible vertex!
(this was our starting point)

$$A = \begin{pmatrix} -2 & 4 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$-2x_1 + 4x_2 \leq 1 \quad (1)$$
$$-x_2 \leq 2 \quad (2)$$

Let's graph our program:



Only to feasible points! (green dots)

Consider the obj. func.

$$\varphi = c^T x = \left(c_B^T + c_N^T \right) \begin{pmatrix} c_B \\ c_N \end{pmatrix}$$

$$= c_B^T x_B + c_N^T x_N$$

Using formula in 2*

$$= c_B^T \left(B^{-1} b - B^{-1} N x_N \right) + c_N^T x_N$$

$$= \underbrace{c_B^T B^{-1} b}_{\text{constant}} - \left(\left(B^{-1} N \right)^T c_B - c_N \right)^T x_N$$

In the dict. we get the following:

$$\zeta = \overbrace{c_B^T B^{-1} b}^{\text{The obj. value}} - \overbrace{\left((B^{-1}N)^T c_B - c_N \right)^T x_N}^{\text{This is 0 (NBV's)}}$$

BV $x_B = B^{-1}b - (B^{-1}N)x_N$

Presented vertex: $\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$

Recall the notations for the dict. after zero or more pivot steps:

$$\zeta = \bar{\zeta} + \sum_{j \in N} c_j x_j$$

In matrix form :

$$\bar{\zeta} = c_B^T B^{-1} b$$

$$[\bar{c}_j] = c_N - (B^{-1} N)^T c_B$$

Vector with components $\bar{c}_j$, $j \in N$

$$x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \,, \quad i \in B$$

$$\left.\begin{aligned} [\bar{b}_i] &= B^{-1} b \\ [\bar{a}_{ij}] &= B^{-1} N \end{aligned}\right\} \text{This is from the BV above } i \in B,\ j \in N$$

For the dual dictionary:

Define variables $\begin{pmatrix} y \\ z \end{pmatrix} \begin{matrix} \in \mathbb{R}^m \\ \in \mathbb{R}^n \end{matrix}$

Recall, for (P) we had $\begin{pmatrix} x \\ w \end{pmatrix}$ and for (D) we had $\begin{pmatrix} z \\ y \end{pmatrix}$.

$$z_1 \dots z_n, \underbrace{z_{n+1} \dots z_{n+m}}_{\text{Corresponds to } x_1,\, x_2, \dots, x_{n+m} \text{ in (P).}}$$

Remember, everything above is only notation. There is nothing new here!

Example:

$$\left.\begin{aligned}
\max\quad & 5x_1 + 4x_2 + 3x_3 \\
\text{s.t.}\quad & 2x_1 + 3x_2 + x_3 \le 5 \\
& 4x_1 + x_2 + 2x_3 \le 11 \\
& 3x_1 + 4x_2 + 2x_3 \le 8 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}\right\}\ \mathrm{I}$$

Introduce slack vars:

$$\max\quad 5x_1 + 4x_2 + 3x_3$$

$$\left.\begin{aligned}
& 2x_1 + 3x_2 + x_3 + x_4 = 5 \\
& 4x_1 + x_2 + 2x_3 + x_5 = 11 \\
& 3x_1 + 4x_2 + 2x_3 + x_6 = 8 \\
& x_i \ge 0,\ i = 1 \ldots 6
\end{aligned}\right\}\ \mathrm{II}$$

Now write it in matrix notation :

$$\begin{pmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 8 \end{pmatrix}$$

This is our init. N

This is our initial B.

$$c^T x = \begin{pmatrix} 5 & 4 & 3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Initial vertex :

BV : $x_4 = 5$, $x_5 = 11$, $x_6 = 8$

NBV : $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

Initial index sets:

$$B = \{4, 5, 6\}, \quad N = \{1, 2, 3\}$$

$$\begin{array}{c} b \\ \begin{pmatrix} 5 \\ 11 \\ 8 \end{pmatrix} \end{array} = Ax = Bx_B + Nx_N =$$

$$= \begin{array}{c} B \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} x_B \\ \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \end{array} + \begin{array}{c} N \\ \begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{pmatrix} \end{array} \begin{array}{c} x_N \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{array}$$

Since $B = I^3$, $B = B^{-1}$

$$x_B = \underbrace{B^{-1}}_{I^3} b - \underbrace{B^{-1}}_{I^3} N x_N$$

This is from the general formula.

Note that the reason matrix notation is important is that it is more convenient when we work with LPs containing a lot of variables.

$$x_B = \begin{pmatrix} 5 \\ 11 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

After one pivot step, where we interchange x_1 and x_4, we get the following system:

BV: x_1, x_5, x_6, $B = \{1, 5, 6\}$

NBV: x_2, x_3, x_4, $N = \{2, 3, 4\}$

Note that this is a new B now

Now matrix B consists of the columns 1, 5 and 6:

$$Ax = \begin{pmatrix} 2x_1 + 3x_2 + x_3 + x_4 \\ 4x_1 + x_2 + 2x_3 + x_5 \\ 3x_1 + 4x_2 + 2x_3 + x_6 \end{pmatrix}$$

B now consist of the following columns.

We can rearrange it like this (commutative property):

$$= \begin{pmatrix} \underbrace{\begin{matrix} 2x_1 & & \\ 4x_1 & + x_5 & \\ 3x_1 & & + x_6 \end{matrix}}_{B} & \underbrace{\begin{matrix} +3x_2 + x_3 + x_4 \\ + x_2 + 2x_3 \\ +4x_2 + 2x_3 \end{matrix}}_{N} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}}_{B} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix}}_{x_B} + \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{pmatrix}}_{N} \cdot \underbrace{\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}}_{x_N}$$

The solution to this instance is:

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}$$

Entries in dict.:

$$\underset{i \in B}{[\bar{b}_i]} = x_B = B^{-1} b = \begin{pmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 11 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 5/2 \\ 1 \\ 1/2 \end{pmatrix} \quad \text{(feasible!)}$$

$$\underset{i \in B,\, j \in N}{[a_{ij}]} = B^{-1} N : \begin{pmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 & 1/2 & 1/2 \\ -5 & -2 & -2 \\ -1/2 & 1/2 & -3/2 \end{pmatrix}$$

$$[\bar{c}_j] = c_N - (B^{-1}N)^T c_B =$$

$$\underbrace{\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}}_{c_N} - \underbrace{\begin{pmatrix} 3/2 & -5 & -1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & -2 & -3/2 \end{pmatrix}}_{(B^{-1}N)^T} \underbrace{\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}}_{c_B} = -\frac{1}{2}\begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}$$

$$\bar{g} = c_B^T B^{-1} b = (5\ 0\ 0)\begin{pmatrix} 5/2 \\ 1 \\ 1/2 \end{pmatrix} = \frac{25}{2}$$

Obj. value in our given solution

$$\breve{x}_B = B^{-1}b = \begin{pmatrix} 5/2 \\ 1 \\ 1/2 \end{pmatrix}, \quad \breve{x}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x^T = (5/2, 0, 0, 0, 1, 1/2)$$

This is the vertex in our given solution