Lecture 14

(repetition from last time)

$$\zeta = \underline{c_B^T B^{-1} b}^{\,*} - \left((B^{-1}N)^T c_B - c_N \right)^T x_N$$

$$x_B = B^{-1} b - (B^{-1}N)\, x_N$$

In our example we had:

$$\zeta = \underline{2{,}5}^{*} - \frac{1}{2}\begin{pmatrix} 7 & -1 & 5 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 3/2 & 1/2 & 1/2 \\ -5 & 0 & -2 \\ 1/2 & 1/2 & -3/2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Now we will describe SM with algorithmic steps using matrix notation!

Primal (P) and dual (D) SM in matrix notation:

Primal

$$\overset{\text{vars}}{x_1 \dots x_n},\ \overset{\text{slack}}{w_1 \dots w_m} \rightarrow \overset{\text{vars}}{x_1 \dots x_n},\ \overset{\text{slack}}{x_{n+1} \dots x_{n+m}}$$

Dual

$$\overset{\text{Vars}}{z_1 \dots z_n},\ \overset{\text{slack}}{y_1 \dots y_m} \rightarrow \overset{\text{Vars}}{z_1 \dots z_n},\ \overset{\text{slack}}{z_{n+1} \dots z_{n+m}}$$

We will use x's and z's in the upcoming description.

Initial dictionaries :

for (P) :

$\mathcal{B} = \{n+1, \dots, n+m\}$, $N = \{1, \dots, n\}$

$x_{\mathcal{B}}^{T} = (x_{n+1}, \dots, x_{n+m})$ This is a row vector, so we put T here.

$x_N^T = (x_1, \dots, x_n)$

c_N = original vector c.

$x_{\mathcal{B}}$ = vector of BV for (P). That is,
$x_{\mathcal{B}} = x_i$, where $i \in \mathcal{B}$.

x_N = ——— " ——— NBV for (P). That is, $x_N = x_i$, where $i \in N$.

BUT! $z_B = (z_i, i \in B)$: NBV's of (D).

$z_N = (z_i, i \in N)$: BV's of (D).

Remember the complement slackness theorem. Each BV on one side is a NBV on the other side.

for (P):

initial dict : $c_B = 0$

$$y = c_B^T \overset{I^m}{B^{-1}} b - ((\overset{I^m}{B^{-1}} N)^T c_B - c_N)^T x_N$$

$$x_B = \underset{I^m}{B^{-1}} b - (\underset{I^m}{B^{-1}} N)^T x_N$$

for (D):

$$-\xi = -\overbrace{\underbrace{c_B^T}_{\text{initial dict: } c_B = 0}}^{}\overbrace{B^{-1}b}^{I^m} - \overbrace{(B^{-1}b)^T}^{I^m} z_B$$

$$z_N = \underbrace{(B^{-1}N)^T}_{I^m} \underbrace{c_B}_{=0} - c_N + \underbrace{(B^{-1}N)^T}_{I^m} z_B$$

Given situation for (P) (after zero or more pivot steps).

- Partition of index set $\{1..n+m\}$ into a collection of B (of m basis indices) and N (of non-basis indices).

Assume:

$$B = \{n+1, \ldots, n+m\}$$

$$N = \{1, \ldots, n\}$$

That is:

$$A = (N \;\; B)$$

- Corresponding primal solution is called $x_B^* \geq 0$, $x_N^* = 0$.
- ——— " ——— dual ——— " ———

  z_N^*, $z_B^* = 0$.

Descr. of SM

Step I - check of optimality

If $z_N^* \geq 0 \longrightarrow$ STOP! (we are optimal)

The dict. is feasible for (P) and (D)

$\Rightarrow$ optimal for (P) and (D).

If not, then go to step II.

Step II – choose NBV (entering)

Choose $j \in N$, $z_j^* < 0$, NBV j will become BV.

Step III – Compute the primal step direction (Δx_B)

Let $e^j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}_n$ jth position denote the jth unit vector.

Goal: increase x_j:

$$x_N = te^j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ 0 \end{pmatrix}$$

jth position

It was:

$$x_B = \underbrace{B^{-1}b}_{x_B^*} - B^{-1}N x_N = x_B^* - B^{-1}N t e^j$$

Note!

$$\begin{array}{cc} m \text{ cols} & \begin{pmatrix} \cdots & \cdots \\ B^{-1} & N \\ \cdots & \cdots \end{pmatrix} \\ & n \text{ rows} \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{t times the jth column of } B^{-1}N$$

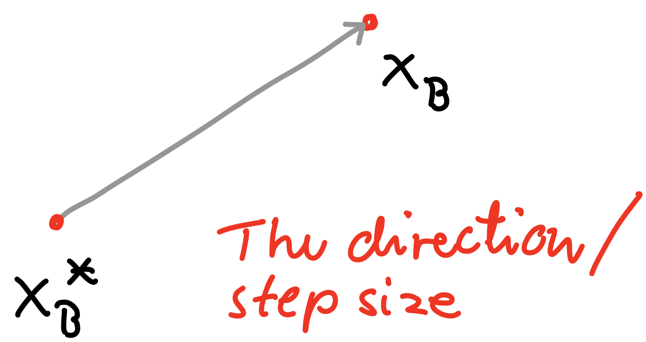
This is our pivot col.

Old point minus the new point is the direction were going. Δ

$$x_B^* - x_B = tB^{-1}e^j$$

t = our step size, $B^{-1}e^j$ = our direction

Visually:



Step IV - Compute primal step length

$$x_B = x_B^* - tB^{-1}e^j = x_B^* - t\underbrace{\Delta x_B}_{\text{Neg. pivot col.}} \geq 0$$

$$x_B \geq t \Delta x_B$$

- If $(\Delta x_B)_i \leq 0 \Rightarrow (x_B^*)_i \geq t (\Delta x_B)_i$

- If $\underbrace{(\Delta x_B)_i > 0}_{\text{Look at the neg. coeff. of the pivot columns.}} \Rightarrow \dfrac{(x_B^*)_i}{(\Delta x_B)_i} \geq t$ This is what we want!

$$t = \overbrace{\left\{ i \text{ s.t. } i \in B \,/\, (\Delta x_B)_j > 0 \right\}}^{\min} \frac{(x_B^*)_i}{(\Delta x_B)_i}$$

Step V – Select leaving variable for interchange with x_j.

Choose a BV x_{i_0} s.t.:

$$\frac{(x_B^*)_i}{(\Delta x_B)_i} = t \quad \text{(as calc. in step IV)}$$

Step VI – Compute dual step direction Δz_N

Analogously to step III (and looking at the dual dict.)

$$\overbrace{\Delta z_N = -(B^{-1}N)^T}^{\text{The neg. transposed matrix!}} \underbrace{e^{6-n}}_{\in \mathbb{R}^m}$$

??

Step VII – compute dual step length

Analogously to step IV. We know that z_j is the leaving variable for (D)

$$\Longrightarrow \text{step length} : \; s = \frac{z_j^*}{\Delta z_j}$$

Step VIII - Updating current (P) and (D) solutions

$$x_j^* \leftarrow t$$

$$x_B^* \longleftarrow x_B^* - t\Delta x_B$$

new old

$$z_j^* \longleftarrow s$$

$$z_N^* \longleftarrow z_N^* - s\Delta z_N$$

Step VIIII - Update basis B

$$B \leftarrow B \setminus \{i_0\} \cup \{x_j\}$$

Example

$$\begin{aligned}
\max \quad & 4x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 - x_2 \leq 1 \\
& 2x_1 - x_2 \leq 3 \\
& x_2 \leq 5 \\
& x_1, x_2 \geq 0
\end{aligned}$$

$$\begin{aligned}
x_1 - x_2 + x_3 &= 1 \\
2x_1 - x_2 + x_4 &= 3 \\
x_2 + x_5 &= 5
\end{aligned}$$

$$
A = \begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left(\begin{array}{cc|ccc}
1 & -1 & 1 & 0 & 0 \\
2 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1
\end{array}\right) \\
\begin{array}{cc} \quad N \quad & \qquad \mathcal{B} \qquad \end{array}
\end{array}
$$

$$
\mathcal{B} = \{3, 4, 5\}, \quad N = \{1, 2\}
$$

$$
x_{\mathcal{B}}^{*} = b = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \begin{array}{l} x_3 \\ x_4 \\ x_5 \end{array}
\qquad
z_{\mathcal{B}}^{*} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} z_3 \\ z_4 \\ z_5 \end{array}
$$

$$
x_{N}^{*} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{array}{l} x_1 \\ x_2 \end{array}
\qquad
z_{N}^{*} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \begin{array}{l} z_1 \\ z_2 \end{array}
$$

First iteration (following the 9 steps):

SI

since z_N^* has neg. comp., x^* is not optimal for (P).

SII

Choose z^* (-4) $\longrightarrow$ entering var is x_1 (that is, x_1 col. becomes pivot col.)

SIII

neg. pivot col.

Compute (P) step direction.

$$\Delta x_B = B^{-1} N e_1^{*} = N = \begin{pmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

*Because here $j = 1$

S IIII

$$t = \min\left\{\frac{1}{1}, \frac{3}{2}\right\} = 1$$

step length

S V

entering

	x_1	x_2
x_3		
x_4		
x_5		

leaving

$i_0 = 3$ (x_3 row is pivot row)

Interchange x_1 and x_3!

S VI

$$\Delta z_N = -\left(B^{-1}N\right)^T e^{3-2} =$$

$$= -\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

S VII

$$s = \frac{z_j^*}{\Delta z_j} = \frac{-4}{-1} = 4$$

$j = 1$

S VIII

New BV

$$x_1^* \leftarrow t \implies x_1^* = 1$$

New NBV

$$x_B^* = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$$

Old BV t Δx_B (direction)

$$z_3^* \leftarrow s, \quad z_3^* = 4$$ New BV

$$z_N^* = \begin{pmatrix} -4 \\ -3 \end{pmatrix} - 4 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \end{pmatrix}$$ New NBV

old z_j^*

S $\underline{\text{VIIII}}$

New index sets:

$B = \{1, 4, 5\}$, $N = \{2, 3\}$