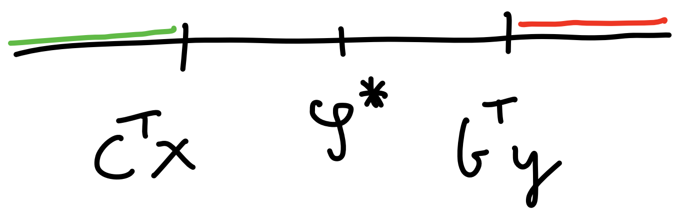
from last week:

$$c^T x \leq b^T y$$




$$\boxed{c^T x^* = b^T y^*}$$

How to get the solution of the dual problem from the primal?

(P) and (D) from the previous ex.:

$$\begin{array}{llr}
\max. & 4x_1 + x_2 + 3x_3 & \\
\text{s.t.} & x_1 + 4x_2 & \leq 1 \\
& 3x_1 - x_2 + x_3 & \leq 3 \\
& x_1, x_2, x_3 & \geq 0
\end{array} \qquad (P)$$

$$\begin{array}{llr}
-\max. & -y_1 - 3y_2 & \\
\text{s.t.} & -y_1 - 3y_2 & \leq -4 \\
& -4y_1 + y_2 & \leq -1 \\
& -y_2 & \leq -3 \\
& y_1, y_2 & \geq 0
\end{array} \qquad (D)$$

In std. form!!

Initial dictionaries for (P) and (D):

$$\begin{array}{lcl} \zeta & = & 4x_1 + x_2 + 3x_3 \\ \hline w_1 & = & 1 - x_1 + 4x_2 \\ w_2 & = & 3 - 3x_1 + x_2 - x_3 \end{array} \qquad (P)$$

$$\begin{array}{lcl} \breve{\zeta} & = & -y_1 - 3y_2 \\ \hline z_1 & = & -4 + y_1 + 3y_2 \\ z_2 & = & -1 + 4y_1 - y_2 \\ z_3 & = & -3 \qquad\quad + y_2 \end{array} \qquad (D)$$

Coefficients of the dictionaries are as follows:

(notice c is also included here)

$$\begin{pmatrix} 0 & 4 & 1 & 3 \\ 1 & -1 & -4 & 0 \\ 3 & -3 & 1 & -1 \end{pmatrix} \quad (P)$$

$$\begin{pmatrix} 0 & -1 & -3 \\ -4 & 1 & 3 \\ -1 & 4 & -1 \\ -3 & 0 & 1 \end{pmatrix} \quad (D)$$

Note! If you multiply (P) by (-1) and transpose it you get (D).

"negative transposed matrix"

For (D) the starting point

$\begin{pmatrix} -4 \\ -1 \\ -3 \end{pmatrix}$ is not feasible. (because of neg. constraints)

When would starting point be feasible?

Only if the corresponding dict for (P) is optimal (neg. coeff. in c).

"optimality on one side implies feasibility on the other side".

Let us do a pivot step in (P):

$$\begin{pmatrix} 0 & 4 & 1 & 3 \\ 1 & -1 & -4 & 0 \\ 3 & -3 & 1 & -1 \end{pmatrix}$$

Interchange $x_3$ and $w_2$.

$$\begin{array}{l} \zeta = 9 - 5x_1 + 4x_2 - 3w_2 \\ \hline w_1 = 1 - x_1 - 4x_2 \\ x_3 = 3 - 3x_1 + x_2 - w_2 \end{array}$$

we're now in the second vertex.
How to get the corresponding dict. for (D)?

We can transpose the matrix for (P) and multiply by (-1) to get the corresponding (D):

$$
\begin{array}{rcrrr}
-\xi & = & -9 & -\,y_1 & -\,3z_3 \\
\hline
z_1 & = & 5 & +\,y_1 & +\,3z_3 \\
z_2 & = & -4 & +\,4y_1 & -\,z_3 \\
y_2 & = & 3 & & +\,z_3
\end{array}
$$

Notice this is still not optimal for (D) because of the negativity.

Make an analogous step for (D):

Interchange $z_3$ (that corresponds to $x_3$) and $y_2$ (that corr. to $w_2$).

Note! this is not according to the rules we have learned up until now. (here the starting point is not feasible for (D)).

Relation between variables of (P) and (D):

| | Primal dict. | | | Dual dict. | |
|---|---|---|---|---|---|
| Primal vars | $x_1$ | NBV | BV | $z_1$ | Dual slack |
| | $x_2$ | NBV | BV | $z_2$ | |
| | $x_3$ | BV | NBV | $z_3$ | |
| Primal slack | $w_1$ | BV | NBV | $y_1$ | Dual variables |
| | $w_2$ | NBV | BV | $y_2$ | |

Rows become colums and columns become rows.

- # of vars in (P)/(D) is equal to the # of constraints in (D)/(P).
- $x_i z_i = 0$ (one BV, one NBV. Prod. is 0.)
- $y_i w_i = 0$ (——— " ———)

"corresponding slackness condition"

Back to our problem!

Vertex at this point:

$$\begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} \text{(P)} \qquad \begin{pmatrix} 5 \\ -4 \\ 0 \\ 0 \\ 3 \end{pmatrix} \text{(D)}$$

not yet feasible!

Now, let's do another pivot step in (P). Interchange $x_2$ and $w_1$.
(we get automatically the dict. for (D) by negative transposed).

$$\begin{array}{r l}
 & \zeta = 10 - 6\overset{z_1}{x_1} - \overset{y_1}{w_1} - 3\overset{y_2}{w_2} \\
\hline
z_2 & x_2 = \frac{1}{4} - \frac{1}{4}x_1 - \frac{1}{4}w_1 \\
z_3 & x_3 = 3\frac{1}{4} - 3\frac{1}{4}x_1 - \frac{1}{4}w_1 - w_2
\end{array}$$

Optimal dictionary!

notice also that (D) is optimal. You can see that by looking at the coeff. of obj. func.

From the optimal solution for (P):

$x^* = (0, \frac{1}{4}, 3\frac{1}{4})$, $w^* = (0,0)$

slack vars

$\zeta^* = \underline{10}$

——————— " ——————— (D):

$z^* = (6,0,0)$, $y^* = (1,3)^T$, $\bar{\zeta}^* = \underline{10}$

The dictionary is optimal for (P) and (D).

$\iff$ Both represented vertices are feasible for (P) and (D).

Complementary slackness condition (csc) holds:

- $x_i z_i = 0 \qquad x_i \geq 0, \; z_i \geq 0$
- $y_j w_j = 0 \qquad y_j \geq 0, \; w_j \geq 0$

**Dictionary from last week:**

$$\frac{\zeta = 6\frac{2}{3} - 1\frac{1}{3} w_2 - \frac{1}{3} w_1}{\begin{aligned} x_2 &= \frac{4}{3} + \frac{1}{3} w_2 - \frac{2}{3} w_1 \\ x_1 &= \frac{4}{3} - \frac{2}{3} w_2 + \frac{1}{3} w_1 \end{aligned}}$$

Optimal solution: $x^* = \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix}, \; w^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

What is the dual solution here?

$$\bar{\zeta} = -6\tfrac{2}{3} - \tfrac{4}{3}y_2 - \tfrac{4}{3}y_1$$

$$z_2 = 1\tfrac{1}{3} - \tfrac{1}{3}y_2 + \tfrac{2}{3}y_1$$

$$z_1 = \tfrac{1}{3} + \tfrac{2}{3}y_2 - \tfrac{1}{3}y_1$$

Optimal solution for (D):

$$y^* = \begin{pmatrix} 1\tfrac{1}{3} \\ \tfrac{1}{3} \end{pmatrix}, \quad z^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Complementary slackness condition:

$$x_i^* z_i^* = 0, \quad y_i^* w_i^* = 0, \quad i = 1, 2$$

$x^*$ optimal for (P) $\longrightarrow$ CSC + feasibility of $x^*$ $y^*$

$y^*$ optimal for (D) $\xleftarrow{?}$

## Complementary slackness Theorem

Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be feasible points for (P) and (D) respectively with corresponding slack variables vectors $w \in \mathbb{R}^m$ (for $x$) and $z \in \mathbb{R}^n$ (for $y$).

Then: $x$ is optimal for (P) and $y$ is optimal for (D) $\longleftrightarrow$ $x^T z = 0$ $(x_1 z_1 = 0 \ldots)$
$y^T w = 0$ $(y_1 w_1 = 0 \cdots)$

## Proof:

To be shown:

$x^T z = 0$, $y^T w = 0$

$x$ feasible for (P), $y$ feasible for (D)

$$\Longrightarrow \quad \begin{array}{l} x \text{ is optimal for (P)} \\ y \text{ ——— '' — (D)} \end{array}$$

We have $Ax \le b$ (for P)

$A^T y \ge c$ (for D)

$$z = A^T y - c \ge 0 \ , \quad w = b - Ax \ge 0$$

From the weak duality theorem:

$$c^T x \underset{1}{\le} \underbrace{y^T A}_{\ge c} x \underset{2}{\le} y^T b$$

for all feas. $x$ for (P)

——— '' — $y$ for (D).

$$\xrightarrow{①} \underbrace{(c^T - y^T A)}_{-z^T} x \le 0 \quad \Big\} \xrightarrow{z^T x = 0} c^T x = y^T A x$$

Analogously:

$$\xrightarrow{②} y^T (Ax - w) \le 0 \quad \Big\} \xrightarrow{y^T w = 0} y^T A x = y^T b$$

$$\xrightarrow{*} c^T x = b^T y$$

- This theorem represents an easy tool to check wether a given pair $x^*$ $y^*$ with corresponding slack vars $w^*$ $z^*$ are opt. solutions for (P) and (D).
- Sometimes it is easier to solve the dual prob. rather than the primal problem (dep. on # of vars / # of constraints.
- Initial vertex is infeasible for (P) but feasible for (D) then we can solve (D) instead of (P) and avoid phase I - aux.p.