The Simplex method

(Ex. from the book)

max. $5x_1 + 4x_2 + 3x_3$

s.t.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\le 5 \\ 4x_1 + x_2 + 2x_3 &\le 11 \qquad (1) \\ 3x_1 + 4x_2 + 2x_3 &\le 8 \\ x_i \ge 0, \quad i &= \{1, 2, 3\} \end{aligned}$$

1. Introduce slack variables:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + \boxed{w_1} &= 5 \\ 4x_1 + x_2 + 2x_3 + \boxed{w_2} &= 11 \qquad (2) \\ 3x_1 + 4x_2 + 2x_3 + \boxed{w_3} &= 8 \end{aligned}$$

An equivalent formulation (dict.):

$$\max.\ \zeta = 5x_1 + 4x_2 + 3x_3$$

$$w_1 = 5 - 2x_1 - 3x_2 - x_3$$

$$w_2 = 11 - 4x_1 - x_2 - 2x_3$$

$$w_3 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$x_1, x_2, x_3 \geq 0$$

The simplex method is an iterative algorithm starting at a vertex:

$$(x_1^1, x_2^1, x_3^1, w_1^1, w_2^1, w_3^1)$$

and searching for a vertex:

$(x_1^2, x_2^2, x_3^2, w_1^2, w_2^2, w_3^2)$ with a greater obj. f. $c^T x^1 < c^T x^2$

If there is no "better" vertex we have an optimal solution x^k.

That is: $c^T x^k \leq c^T x \quad \forall x \in M$

??

The dict. above (1) represents the following vertex:

$w_1 = 5 \quad x_1 = 0$

$w_2 = 11 \quad x_2 = 0 \quad \Longrightarrow \quad \zeta = 0$

$w_3 = 8 \quad x_3 = 0$

BV NBV

Initially the obj. f. is zero.

The point $(\underbrace{0, 0, 0}_{x}, \underbrace{5, 11, 8}_{w})$ is called a basic solution and represents a vertex of the feasible set.

Can the value of ζ be improved? Assign a positive value to one of the NBVs with a positive coeff. in the obj. f.

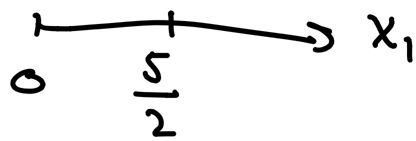
If $x_1 > 0$ (and $x_2 = x_3 = 0$):

$$w_1 = 5 - 2x_1 \geq 0 \Rightarrow \frac{5}{2} \geq x_1$$

$$w_2 = 11 - 4x_1 \geq 0 \Rightarrow \frac{11}{4} \geq x_1$$

$$w_3 = 8 - 3x_1 \geq 0 \Rightarrow \frac{8}{3} \geq x_1$$

The minimum of $\left\{ \frac{5}{2}, \frac{11}{4}, \frac{8}{3} \right\} = \frac{5}{2}$



The pivot - going from a vertex to the next.

$$x_1 = \frac{5}{2} \qquad w_1 = 0$$

new NBV

new BV

x_1 goes from NBV to BV. x_1 is now called the entering variable. w_1 is now the leaving variable.

separate the new BV x_1 in the first constraint:

$$x_1 = \frac{5}{2} - w_1 - 3x_2 - x_3 \quad (*)$$

substitute $(*)$ in the other constraints:

$$\begin{aligned}
w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\
&= 11 - 4\left(\frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - x_2 - 2x_3 \\
&= \boxed{1 + 2w_1 + 5x_2}
\end{aligned}$$

$$\begin{aligned}
w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
&= 8 - 3\left(\frac{5}{2} - w_1 - 3x_2 - x_3\right) - 4x_2 - 2x_3 \\
&= \boxed{\frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3}
\end{aligned}$$

$$\zeta = 5x_1 + 4x_2 + 3x_3$$

$$= 5\left(\frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) + 4x_2 + 3x_3$$

$$= \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

New dict.:

$$\begin{array}{rcl}
\zeta &=& \frac{25}{2} \overbrace{- \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3}^{\text{NBV}} \\
\hline
x_1 &=& \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
w_2 &=& 1 + 2w_1 + 5x_2 \\
w_3 &=& \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3
\end{array} \tag{11}$$

BV

(II) Represents :

$$x_1 = \frac{5}{2} \quad w_1 = 0$$

$$w_2 = 1 \quad x_2 = 0 \quad \Longrightarrow \quad \varphi = 25$$

$$w_3 = \frac{1}{2} \quad x_3 = 0$$

New and improved obj. v.

Now, we see that w_1 and x_2 have negative coefficients, so increasing them will decrease φ.

Hence, we choose the column with a positive coefficient (x_3 - column).

pivot column

Our goal now is to choose x_3 as a new BV and exchange it with an existing BV.

If $x_3 > 0$ ($w_1 = x_2 = 0$ remain)

$$x_1 = \frac{5}{2} - \frac{1}{2}x_3 \geq 0 \longrightarrow 5 \geq x_3$$

$$w_2 = 1$$

$$\boxed{w_3 = \frac{1}{2} - \frac{1}{2}x_3 \geq 0 \longrightarrow 1 \geq x_3}$$

w_3 is now the pivot row.
(WHY?)

Now we exchange x_3 and w_3.

Separate the new BV x_3 in the pivot row:

$$x_3 = 1 + 3w_1 + x_2 - 2w_3$$

$$\zeta = \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}\left(1 + 3w_1 + x_2 - 2w_3\right)$$

$$= 13 - w_1 - 3x_2 - w_3$$

$$x_1 = 2 - 2w_1 - 2x_2 + w_3$$

$$\zeta = 13 \overbrace{- w_1 - 3x_2 - w_3}^{\text{NBV}}$$

$$\left.\begin{array}{cl}
\hline
\underbrace{\begin{array}{c} x_1 \\ w_2 \\ x_3 \end{array}}_{\text{BV}} & \begin{array}{l} = 2 - 2w_1 - 2x_2 + w_3 \\ = 1 + 2w_1 - 5x_2 \\ = 1 + 3w_1 + x_2 - 2w_3 \end{array}
\end{array}\right\} \text{(III)}$$

(III) Represents the vertex:

$x_1 = 2 \qquad w_1 = 0$

$w_2 = 1 \qquad x_2 = 0$

$x_3 = 1 \qquad w_3 = 0$

BV $\qquad$ NBV

Now any increase in w_1, x_2 or w_3 would decrease φ.

The vertex $x_1 = 2$ $x_2 = 0$ $x_3 = 1$ is optimal!

$$x^\circ = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad c^T x^\circ = 13$$