Notes on Linear Algebra

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1 Introduction

This document contains my own notes from the course INF270 at the University of Bergen (Autumn, 2021 semester). It will be updated regularly with useful information as the semester progresses. The first section contains some preliminary notes on linear algebra, which is not strictly related to the course, but bight come in handy as a reference when doing linear programming exercises.

2 Some notes on linear algebra

2.1 Vectors

We work exclusively with vectors in $\mathbb{R}^n$, and so for us, a vector is an ordered n-tuple $\mathbf{v} = v_1, v_2, \ldots, v_n \in Rn$ of real numbers. For **u,** $\mathbf{v} \in \mathbb{R}^n$ we define the sum component-wise:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

The multiplication of $\mathbf{v} \in \mathbb{R}^n$ by a real number t is also given component-wise, by

$$t\mathbf{v} = (tv_1, tv_2, \ldots, tv_n).$$

Linear subspace. A linear subspace (or vector subspace) of $\mathbb{R}^n$ is a set $V \in \mathbb{R}^n$ that contains $\mathbf{0}$ and is closed under addition and multiplication by a real number; that is, if $\mathbf{u},\mathbf{v} \in V$ and $t \in \mathbb{R}$, we have $\mathbf{u} + \mathbf{v} \in V$ and $\mathbf{tv} \in V$. For example, the linear subspaces of $\mathbb{R}^3$ are $\mathbf{0}$, lines passing through $\mathbf{0}$, planes passing through $\mathbf{0}$, and $\mathbb{R}^3$ itself.

Affine subspace. TODO:

Linear combination. A linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ is any vector of the form $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \ldots, t_m\mathbf{v}_m$ where $t_1 \ldots t_m$ are real numbers.

2.2 Matrices

A matrix is **multiplied** by a number t by multiplying each entry by t. Two $m * n$ matrices A and B are **added** by adding the corresponding entries. That is, if we set $C = A + B$, we have $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Note that two matrices needs to have the same dimensions in order for us to do addition and subtraction.

Matrix multiplication. A product AB, where A and B are matrices, is only defined if the number of columns of A is the same as the number of rows of B. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product $C = AB$ is an $m \times p$ matrix given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Example:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 5 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ -4 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1*(-1)+0*0+3*(-4) & 1*2+0*3+3*1 \\ (-1)*(-1)+2*0+5*(-4) & (-1)*2+2*3+5*1 \end{pmatrix} = \begin{pmatrix} -13 & 5 \\ -19 & 9 \end{pmatrix}$$

We also multiply **matrices and vectors**. In such context, a vector $x \in \mathbb{R}^n$ is usually considered as an $n \times 1$ matrix (i.e. a vector with only one column). The product of the matrix A and the vector x is a new vector. The matrix notation $\mathbf{Ax = b}$ is used for a system of linear equations.

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, x = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

$$\mathbf{Ax} = \begin{pmatrix} 1*10+2*2 \\ 3*10+4*2 \end{pmatrix} = \begin{pmatrix} 14 \\ 38 \end{pmatrix}$$

Transpose. If A is an $m \times n$ matrix, then A^T denotes the $n \times m$ matrix having the element a_{ji} in the ith row and the jth column. The matrix A^T is called the transpose of the matrix A. Note also the formula for transposing the matrix product; $AB^T = B^T A^T$.

Example:

$$A = \begin{pmatrix} 2 & 5 & 1 \\ 4 & 1 & 8 \end{pmatrix}, A^T = \begin{pmatrix} 2 & 4 \\ 5 & 1 \\ 1 & 8 \end{pmatrix}$$

Rank. An important result of linear algebra tells us that for every matrix A the row space and the column space have the same dimension, and this dimension is called the rank of A. In particular, an $m \times n$ matrix A has rank m if and only if the rows of A are **linearly independent** (which can happen only if $m \leq n$). An $n \times n$ matrix is called non-singular if it has rank n; otherwise, it is singular.

Example:

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & 2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{pmatrix}, rref(A) = \begin{pmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, dim(C(A)) = 3$$

The **rank** is simply the dimension of the column space of a matrix. As we see in the example of A, it has three pivot columns, $\{a_1, a_2, a_4\}$. These columns form the basis of the columns space of A, since they are **linearly independent** and any other vector in the matrix can be represented using a linear combination of these.

Inverse. Let A be a square matrix. A matrix B is called an inverse of A if $AB = I_n$, where I_n is the **identity matrix**. An inverse to A exists if and only if A is **non-singular**.

Example:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A^{-1} = 1/(ad-bc) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Row operations and Gaussian elimination. By an **elementary row operation** we mean one of the following two operations on a given matrix A:

1. Multiplying all entries in some row of A by a non-zero real number t.

2. Replacing the ith row of A by the sum of the ith row and jth row for some $i \neq j$.

Gaussian elimination is a a systematic procedure that, given an input matrix A, performs a sequence of elementary row operations on it so that it is converted to a row echelon form. The matrix A above has been reduced to row echelon form.

Determinants. Every square matrix A is assigned a number $det(A)$ called the determinant of A. The determinant of A is defined by the formula

Scalar product, Euclidean norm, orthogonality. The (standard) **scalar product** of two vectors $x, y \in \mathbb{R}^n$ is the number $x_1y_1 + x_2y_2 \ldots x_ny_n$. We often write the scalar product as x^Ty although formally, x^T y is a 1×1 matrix whose single entry is the scalar product. The **Euclidean norm** of a vector $x \in \mathbb{R}$