Yesterday: how to calc. a feas. point if the starting point is not feasible.

Degeneracy: when there exists one or more zeros in b (rhs).

Some notation

Dictionary:

$$\zeta = \sum_{j=1}^{n} c_j x_j$$

$$x_{n+i} = b_i - \sum a_{ij} x_j \,, \quad 1 \le i \le m$$

$$x_j \ge 0, \quad 1 \le j \le n+m$$

+ notation.

w is a more convenien

$$\left(\begin{array}{c} x_1\\ \vdots\\ x_n\\ \hline w_1\\ \vdots\\ w_m\end{array}\right) \to \left(\begin{array}{c} x_1\\ \vdots\\ x_n\\ \hline x_{n+1}\\ \vdots\\ x_{n+m}\end{array}\right) \begin{array}{l}\left.\vphantom{\begin{array}{c}x_1\\ \vdots\\ x_n\end{array}}\right\}\text{NBV's}\\ \left.\vphantom{\begin{array}{c}x_1\\ \vdots\\ x_n\end{array}}\right\}\text{BV's}\end{array}$$

- Set of (indices of) NBV's : $\mathcal{N}$
- ———— ,, ——— BV's : $\mathcal{B}$
- Initially, $\mathcal{N} = \{1, \dots, n\}$

  $\mathcal{B} = \{n+1, \dots, n+m\}$

- Dictionary after zero or more steps:

$$\zeta = \bar{\zeta} + \sum_{j \in N} \bar{c}_j x_j$$

"bar": coefficients in this instance. fixed data.

$$x_i = \bar{b}_i - \sum \bar{a}_{ij} x_j, \quad i \in \mathcal{B}$$

$$x_j \geq 0, \quad j \in \mathcal{B} \cup \mathcal{N}$$

union

Degeneracy (read ch. 3)

$$\max c^T x \text{ s.t. } Ax \le b,\ x \ge 0$$

Up to now: $b > 0$. Let's now look at a situation where atleast one of $\{b_1, \dots, b_m\}$ is zero.

Def. A dict. is called degenerate if $\bar{b}_i = 0$ for some i from 1 to m.

Example:

$$\begin{array}{l} \varphi = 5 + x_3 - x_1 \\ \hline x_2 = 5 + 2x_3 - 3x_1 \\ x_4 = 7 \qquad\quad - 4x_1 \\ x_5 = \boxed{0} \qquad\quad + x_1 \end{array}$$

*

(unbounded problem)

degeneracy

solution is not optimal since we have a positive coefficient for x_3, but there are no x_3 with a neg. coefficient in the constraints, so there is no solution (x_3 can be augmented (increased) as much as we want. *

Remember, unbounded means there is no solution.

Example 2 :

$$\frac{y = 5 + x_3 - x_1}{\begin{aligned} x_2 &= 5 \boxed{-} 2x_3 - 3x_1 \\ x_4 &= 7 \qquad\quad - 4x_1 \\ x_5 &= \boxed{0} \qquad\quad + x_1 \end{aligned}}$$

we interchange x_3 and x_2.

$$x_3 = \frac{5}{2} - \frac{1}{2}x_2 - \frac{3}{2}x_1$$

New dict.:

$$\varphi = 5 + \left(\frac{5}{2} - \frac{1}{2}x_2 - \frac{3}{2}x_1\right) - x_1$$
$$= \frac{15}{2} \cdots$$

φ is improved!

Now we've seen two examples where the degeneracy (zero) has had no effect.

Example 3:

note the degeneracy

$$
\begin{aligned}
\max\quad & \tfrac{3}{4}x_1 - 20x_2 + \tfrac{1}{2}x_3 - 6x_4 \\
\text{s.t.}\quad & \tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \le 0 \\
& \tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 \le 0 \\
& \qquad\qquad\qquad x_3 \le 1 \\
& x_i \ge 0,\ i \in \{1,2,3,4\}
\end{aligned}
$$

Exercise: try to make a pivot step in pplex and see what happens.

BV	ζ	
$w_1\ w_2\ w_3$	0	
$x_1\ w_2\ w_3$	0	(ζ is not improving)
$x_1\ x_2\ w_3$	0	
$x_3\ x_2\ w_3$	0	
$x_3\ x_4\ w_3$	0	
$w_1\ x_1\ w_3$	0	
$w_1\ w_2\ w_3$	0	
	$\vdots$	

We will never leave! Cycling!

we're always in the same point.
How should the software handle this?

- We remain in the same point
- ζ stays the same
- Sometimes different vertices with the same value for ζ are presented

Example 4 (illustration of degeneracy):

$$\zeta = 5 + x_3 - x_1$$

$$x_2 = 5 + 2x_3 - 3x_1$$
$$x_4 = 7 - 4x_1$$
$$x_5 = \boxed{0}^{*} - x_3 + x_1$$

Problem arises if the pivot row has index i_0 and $\bar{b}_0 = 0^{*}$.

we interchange x_3 and x_5

$$x_3 = -x_5 + x_1$$

New dict.:

$$\zeta = 5 - x_5 \cancel{+ x_1} \cancel{- x_1}$$

$$= 5 - x_5$$

Notice that after the pivot step we got the same obj. value.

If there is a zero in the pivot row, the value of the obj. func. is **not** improving!

- Degeneracy (or degeneration) is common (that is, $b_{i_0} = 0$ for some $i \in \{1 \dots m\}$
- Cycling is rare. Note that degeneracy does not imply cycling
- Cycling can be avoided at some cost (we explain two possibilities.)

Theorem:

Either the simplex method terminates or there is a cycle.

Proof. Assume the contrary.
Assume it does not terminate and that it doesn't cycle. Since a dict. is uniquely determined by a partition of $n+m$ variables into m basis variables and n non-basis variables.

There are exactly $\binom{n+m}{m}$ such partitions.

upper bound for # dictionaries and vertices!

Since SM doesn't terminate, one of the already calculated dictionaries will be obtained after at most $\binom{n+m}{m}$ steps.

Furthermore, since the value of the obj. func. $\varphi = c^T x$ is not decreasing at the dictionaries (or the vertices) the method is obviously cycling. Contradiction Δ.

after at most $\binom{n+m}{m}$ steps

$$\varphi^1 \le \varphi^2 \le \varphi^3 \le \cdots \varphi^{\binom{n+m}{m}} \cdots \varphi^1$$

dictionary

$$\Rightarrow \varphi^1 = \varphi^2 = \cdots \varphi^{\binom{n+m}{m}}$$

This implies cycling.

How to prevent cycling ?

Technique I : Lexicographic SM

Introduction of m positive perturbations $\varepsilon_i > 0$, $i = 1 \dots m$ of the right hand sides (b) with:

$$0 < \varepsilon_m \ll \varepsilon_{m-1} \ll \dots \ll \varepsilon_1 \ll \text{all other}$$

data of the LP.

"Each ε_i acts on an entirely different scale from all other ε_i and the data of the problem. What we mean by this is that no lin. comb. of the ε_i use coeff. that may arise in the course of the SM can ever produce a number whos size is of the same order as the date in the problem. Similarly any ε_i can never escalate to a level of ε_j $(i \neq j)$."

Trick: we do not assign any values to ε_i (ε_i are considered parameters).

Example:

$$\begin{aligned}
\zeta &= 6x_1 + 4x_2 \\
\hline
w_1 &= 0 + 9x_1 + 4x_2 \\
w_2 &= 0 - 4x_1 - 2x_2 \\
w_3 &= 1 \qquad\quad - x_2
\end{aligned}$$

Now, introduce perturbations to the zeros.

$0 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1 <$ all other data.

$$\begin{array}{llllll}
\varphi = & & & & 6x_1 + 4x_2 \\
\hline
w_1 = 0 & + \boxed{\varepsilon_1} & & & + 9x_1 + 4x_2 \\
w_2 = 0 & & + \boxed{\varepsilon_2} & & + 4x_1 - 2x_2 \\
w_3 = 1 & & & + \boxed{\varepsilon_3} & - x_2
\end{array}$$

ε_i obtain colums like variables.

- Now, determine pivot column as usual : x_1
- Pivot row is w_2
- Interchange these as usual

$$X_1 = \frac{\varepsilon_2}{4} - \frac{1}{4}w_2 - \frac{1}{2}x_2$$

(continues next lecture)