Duality

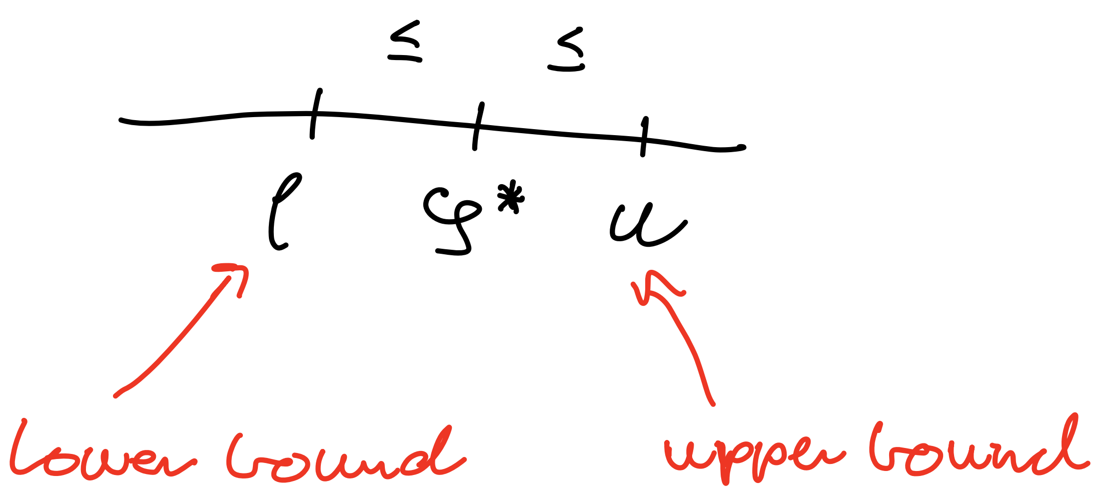
Example for motivation

(P) = (LP)

$$\underbrace{\max \quad 4x_1 + x_2 + 3x_3}$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + 4x_2 && \leq 1 \\ & 3x_1 - x_2 + x_3 && \leq 3 \\ & x_i \geq 0, \ i = 1, 2, 3 \end{aligned}$$

In practice one is interested in an upper and a lower bound for the optimal obj. f. value z^*.



Choose any feasible point:

$\begin{pmatrix}1\\0\\0\end{pmatrix}$ is feasible. $c^T x = 4\cdot 1 + 1\cdot 0 + 3\cdot 0$

$= 4$

$\Longrightarrow 4 \leq z^*$

4 is a lower bound

another point:

$$\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \text{ is feasible.}$$

a better lower bound

$$c^T x = 9 \leq g^*$$

How to get an upper bound:

combine linearly the constraints

"$2 \cdot (1) + 3(2)$":

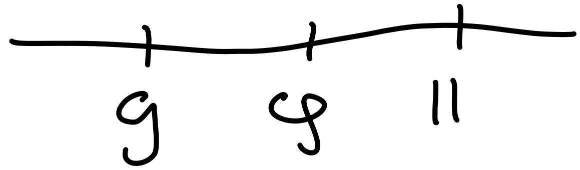
$$2(x_1 + 4x_2) \leq 2 \cdot 1$$

$$3(3x_1 - x_2 + x_3) \leq 3 \cdot 3$$

$$11x_1 + 5x_2 + 3x_3 \le 11$$

Since $x_i \ge 0$, $i = 1, 2, 3$

$$\underbrace{4x_1 + x_2 + 3x_3}_{8} \le 11x_1 + 5x_2 + 3x_3 \le 11$$



$(2\cdot(1) + 3\cdot(2))$

choice of the linear comb. was done in such a way that we get an upper bound (coeff are greater and eq than those in 8).

However, was this (11) the lightest/best combination to get an upper bound?

More generally: search for coeff. y_1 y_2 s.t. $y_1, y_2 \geq 0$:

$$y_1(x_1 + 4x_2) \leq y_1 \cdot 1$$

$$+ \; y_2(3x_1 + x_2 + x_3) \leq y_2 3$$

"$y_1(1) + y_2(2)$":

$$(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 \cdot x_3$$

$$\leq y_1 + 3y_2$$

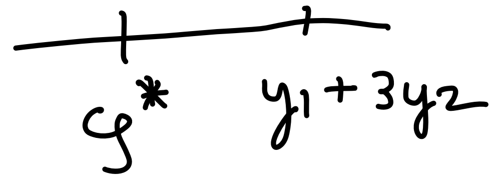
≥ 4 ≥ 1 ≥ 3

coeff. of y.

Comparing this comb. with φ delivers upper bound for φ^*.

$$\varphi = 4x_1 + x_2 + 3x_3 \leq$$
$$(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \leq$$
$$y_1 + 3y_2 .$$

Wanted: upper bound should be nearest to φ^* as possible.



$$\left.\begin{array}{lrcl} \min & y_1 + 3y_2 & & \\ \text{s.t.} & y_1 + 3y_2 & \geq & 4 \\ & 4y_1 - y_2 & \geq & 1 \\ & y_2 & \geq & 3 \\ & y_1, y_2 \geq 0 & & \end{array}\right\} \begin{array}{l}(D) \\ \text{dual pr.}\end{array}$$

How to get the dual problem (D) from (P): (using a formula)

$$
\begin{aligned}
\max.\quad & 4x_1 + 1x_2 + 3x_3 \\
\text{s.t.}\quad & 1x_1 + 4x_2 + 0x_3 \le 1 \\
& 3x_1 - 1x_2 + 1x_3 \le 3 \\
& x_i \ge 0 \quad i = 1, 2, 3
\end{aligned}
\qquad (P)
$$

$$
\begin{aligned}
\min.\quad & 1y_1 + 3y_2 \\
\text{s.t.}\quad & 1y_1 + 3y_2 \ge 4 \\
& 4y_1 + 1y_2 \ge 1 \\
& 0y_1 + 1y_2 \ge 3 \\
& y_i \ge 0, \; i = 1, 2
\end{aligned}
\qquad (D)
$$

In general:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned} \qquad (P)$$

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned} \qquad (D)$$

notice A becomes transposed

$$\max \overset{c^T}{\begin{pmatrix} 4 & 1 & 3 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(P)

s.t.

$$\underset{A}{\begin{pmatrix} 1 & 4 & 0 \\ 3 & -1 & 1 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \underset{b}{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

$$\min \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

s.t.

$$\underset{A^T}{\begin{pmatrix} 1 & 3 \\ 4 & -1 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \geq \underset{c}{\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}}$$

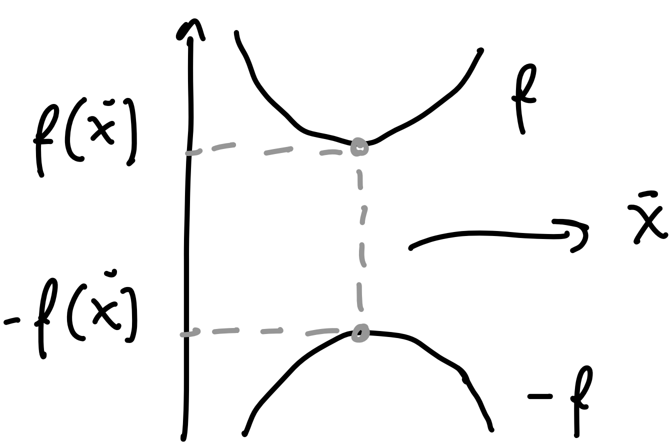
(D)

Note that the dual of the dual is the original problem.

The dual problem into its std. form:

$$
\begin{aligned}
(D)\quad & \min.\ b^T y \\
\text{s.t.}\quad & A^T y \geq c \\
& y \geq 0
\end{aligned}
\quad \xrightarrow{\text{std. form}} \quad
\begin{aligned}
& -\max -b^T y \\
\text{s.t.}\quad & -A^T y \leq c \\
& y \geq 0
\end{aligned}
$$

now the dual prob. is in std. form.



what is the dual problem $(\bar{D})$ of

(D) ?

$$\begin{aligned} -\min\ & -c^T x \\ \text{s.t.}\ & -Ax \le -b \\ & x \ge 0 \end{aligned} \xrightarrow{\text{std. form}} \begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

our original problem (P). $\longrightarrow$ $*$ holds.

General rules for (P) and (D):

\# of variables of (P) = \# of constr. of (D)

\# of constraints of (P) = \# of variables of (D)

Using SM, you will get both the solution to (P) and (D).

(P)	(D)
max. in (P)	min in (D)
$\leq$ - constr.	$\geq$ - constr.
$\varphi = c^T x$	c becomes RHS of the constraints.
RHS of the constr. in (P)	$b^T y$ becomes obj. f. of (D).
A	A^T

Example:

$$\begin{aligned}\max.\ & 3x_1 + 2x_2 \\ \text{s.t.}\ & x_1 + 2x_2 \le 4 \\ & 2x_1 + x_2 \le 4 \\ & x_i \ge 0,\ i = 1,2\end{aligned} \qquad (P)$$

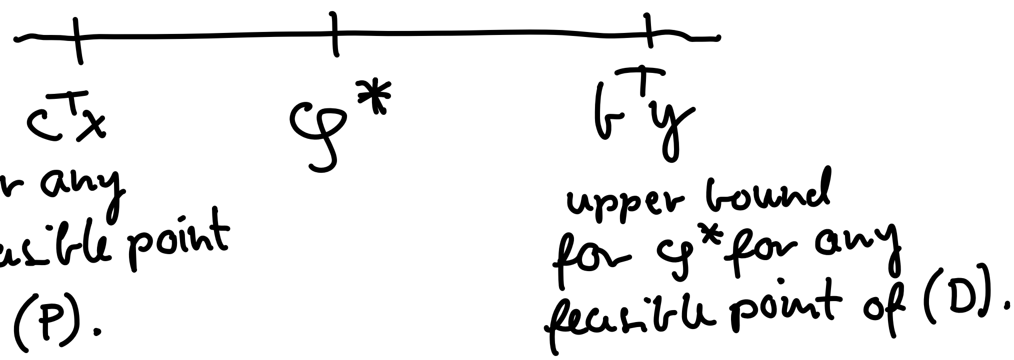
$$\begin{aligned}\min\ & 4y_1 + 4y_2 \\ \text{s.t.}\ & 1y_1 + 2y_2 \ge 3 \\ & 2y_1 + y_2 \ge 2 \\ & y_i \ge 0,\ i = 1,2\end{aligned} \qquad (D)$$

solve this at home!

Optimal dictionary:

$$\begin{array}{l} \zeta = 6\frac{2}{3} - 1\frac{1}{3}w_2 - \frac{1}{3}w_1 \\ \hline x_2 = \frac{4}{3} + \frac{1}{3}w_2 - \frac{2}{3}w_1 \\ x_1 = \frac{4}{3} - \frac{2}{3}w_2 + \frac{1}{3}w_1 \end{array}$$

$$x^* = \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix}, \quad \zeta^* = 6\frac{2}{3}$$



Weak duality theorem:

Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be feasible points of (P) and (D) respectively.

Then:

$$c^T x \leq b^T y$$

Proof:

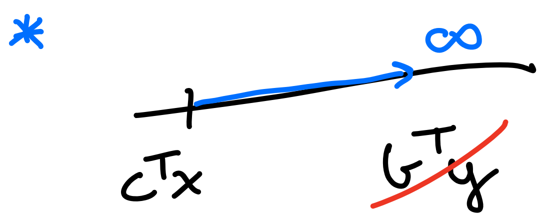
$$c^Tx \le y^TAx \le y^Tb$$

$(AB)^T = A^TB^T$

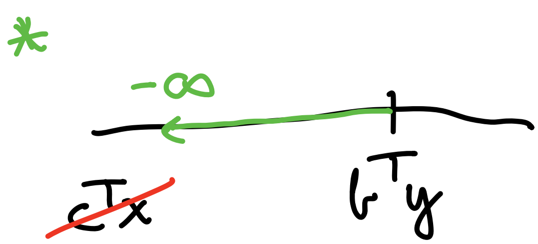
$A^Ty \ge c$

$y^TA \ge c^T$

- If (P) is unbounded (that is, $c^Tx \to \infty$ for feasible x of (P)) then (D) is infeasible (doesn't exist) *



- If (D) is unbounded, (P) is infeasible. *



Strong duality theorem

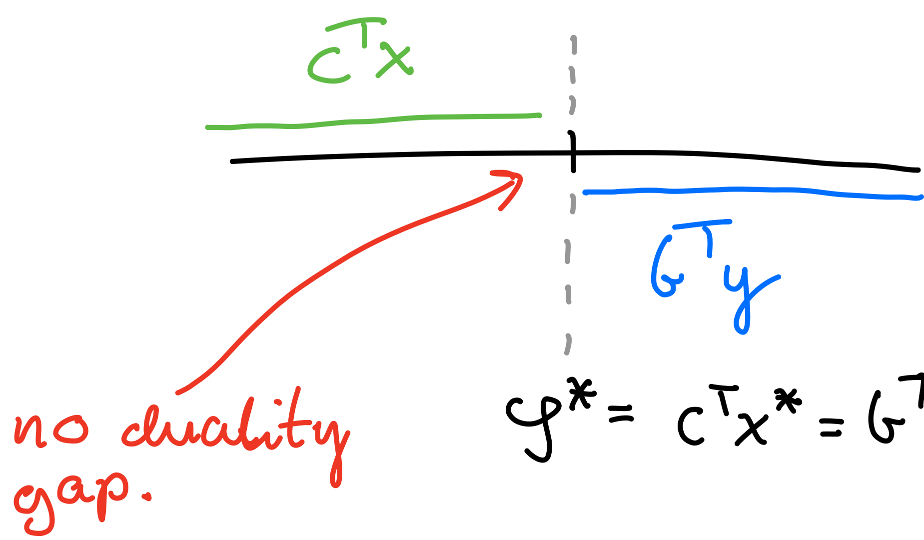
If (P) has an optimal solution $x^* \in \mathbb{R}^n$, then (D) has an optimal sol. $y^* \in \mathbb{R}^m$ with:

$$\boxed{c^T x^* = b^T y^*}.$$

- Both (P) and (D) have a corresponding solution (x^* and y^*, respectively) or none of them has a solution.

- We will see that SM delivers both solutions in the dictionary.

- If solutions of (P) and (D) exist then there is no duality gap, that is, (P) and (D) have the same optimal obj. func. value.



$$\zeta^* = c^T x^* = b^T y^*$$