Mandatory Assignment 2

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Problems

2.1

(a)

False. An LP can be defined as a system of linear equations with m rows (constraints) and n columns (variables). For the rows in an LP to be linearly independent it has to have rank m, which can only happen if $m \leq n$. If $m = n$ we get an LP defined by a square matrix with the same number of constraints in (P) as in (D).

(b)

True. The dual of the dual problem is the primal problem. Since it is feasible it is also bounded.

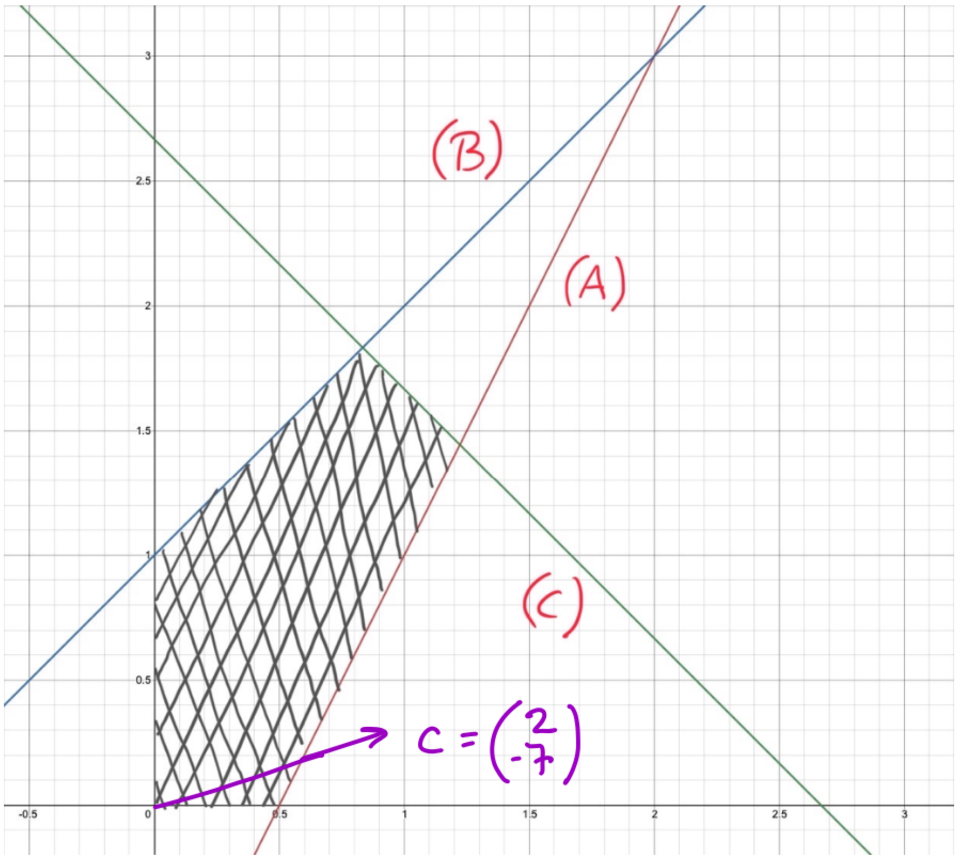
(c)

True. If (P) has no optimal solution this means that it is either infeasible or unbounded. This again means that (D) is either infeasible or unbounded (from the strong duality theorem).

2.2

(a)

$$
\begin{array}{lrcl}
\max & 2x_1 - 7x_2 & & \\
\text{s.t.} & 2x_1 - x_2 & \leq 1 & \text{(A)} \\
& -x_1 + x_2 & \leq 1 & \text{(B)} \\
& 3x_1 + 3x_2 & \leq 8 & \text{(C)}
\end{array}
$$



Optimal solution looks as follows:

$$
\begin{array}{l}
\zeta = 1 - w_1 - 6x_2 \\
\hline
x_1 = \frac{1}{2} - \frac{1}{2}w_1 + \frac{1}{2}x_2 \\
w_2 = \frac{3}{2} - \frac{1}{2}w_1 - \frac{1}{2}x_2 \\
w_3 = \frac{13}{2} + \frac{3}{2}w_1 - \frac{9}{2}x_2
\end{array}
$$

Where $x^o = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, c^T x^o = 1.$

(b)

$\mathcal{B} = \{1, 4, 5\}$, $\mathcal{N} = \{3, 2\}$

$$c = \begin{bmatrix} 2 & -7 & 0 & 0 & 0 \end{bmatrix}^T, \Delta c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$(B^{-1}N)^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{9}{2} \end{bmatrix}$$

$$\Delta c_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; \Delta c_{\mathcal{N}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Delta z_{\mathcal{N}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\tilde{z}_{\mathcal{N}} = z^*_{\mathcal{N}} + t\Delta z_{\mathcal{N}} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \geq 0$$

$$-2 \leq t \leq 12 = \begin{cases} 1 + \frac{1}{2}t \geq 0 \longrightarrow t \geq -2 \\ 6 - \frac{1}{2}t \geq 0 \longrightarrow t \leq 12 \end{cases}$$

$$t\Delta c_B = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ coefficient } c_1 = 2 \text{ can vary between } [0,\, 14].$$

(c)

If $c_1 = \gamma_{min} = 0$ then we are already optimal with $x^o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $c^T x^o = 0$, since x_2 has a negative coefficient and all basic variables are positive.

(d)

Here, we can convert the LP to the dual problem and then solve it as we did in 2 a). The dual problem looks as follows:

$$\begin{array}{l} \zeta = -1 - \frac{1}{2}y_4 - \frac{3}{2}y_2 - \frac{13}{2}y_3 \\ \hline y_1 = 1 + \frac{1}{2}y_4 + \frac{1}{2}y_2 - \frac{3}{2}y_3 \\ y_5 = 6 - \frac{1}{2}y_4 + \frac{1}{2}y_2 - \frac{9}{2}y_3 \end{array}$$

$$\mathcal{B} = \{1, 5\},\; \mathcal{N} = \{4, 2, 3\}$$

$$c = \begin{bmatrix} -1 & -1 & -8 & 0 & 0 \end{bmatrix}^T,\; \Delta c = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$$(B^{-1}N)^T = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{9}{2} \end{bmatrix}$$

$$\Delta c_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \Delta c_{\mathcal{N}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Delta z_{\mathcal{N}} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\tilde{z}_{\mathcal{N}} = z^*_{\mathcal{N}} + t\Delta z_{\mathcal{N}} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ \frac{13}{2} \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \geq 0$$

$\frac{13}{2} - t \geq 0 \longrightarrow t \leq \frac{13}{2}$???

Coefficient $b_3 = 8$ can vary between [0, 6.5]. ???

(e)

Since the optimal solution lies in the point where the line $2x_1 - x_2 \leq 1$ intersects with the x_1-axis, $x^o = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$, the objective value will increase or decrease if we translate (move) said line (which is what happens when we alter the right-hand side of the line equation). The objective value would not change if we changed b_2 (at least not until it was moved to the right of the vertex of the objective function, in which case the LP would become infeasible).