An auxiliary problem is introduced when we don't have a feasible basis.

$$\max.\ y = x_1 + 2x_2$$

s.t.

$$x_1 + 3x_2 + x_3 = 4$$

$$2x_2 + x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

$x = (0,0,0)$

An initial feasible solution is <u>not</u> possible here because of the constraints.

we introduce aux. vars as "corrections" of infeasibility:

$$x_4 = 4 - x_1 - 3x_2 - x_3$$

$$x_5 = 2 \quad - 2x_2 - x_3$$

Now we solve aux. problem:

$$\max. \ \varphi = -x_4 - x_5$$

$$\begin{aligned}
\text{s.t.} \quad x_1 + 3x_2 + x_3 + x_4 \quad &= 4 \\
2x_2 + x_3 \quad + x_5 &= 2 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}$$

Now express the obj. func.
using the NBV's x_1, x_2 and x_3:

x_4 and x_5 form a basis:

$$x_4 = 4 - x_1 - 3x_2 - x_3$$

$$x_5 = 2 \quad\quad - 2x_2 - x_3$$

Obj. func. then becomes:

$$g = -x_4 - x_5$$

$$= -4 + x_1 + 3x_2 + x_3 - 2 + 2x_3 + x_3$$

$$\boxed{= -6 + x_1 + 5x_2 + 2x_3}$$

Now, the initial simplex table becomes:

$$\begin{array}{l}
\varphi = -6 + x_1 + 5x_2 + 2x_3 \\
\hline
x_4 = 4 - x_1 - 3x_2 - x_3 \\
x_5 = 2 \quad\quad\;\; - 2x_2 - x_3
\end{array}$$

$$x_4 = 4 - x_1 - 3x_2 - x_3$$

$$\boxed{\Rightarrow x_1 = 4 - x_4 - 3x_2 - x_3}$$

$$\varphi = -6 + 4 - x_4 - 3x_2 - x_3 + 5x_2 + 2x_3$$
$$= -2 - x_4 + 2x_2 + x_3$$

$$
\begin{array}{rl}
g = & -2 - x_4 + 2x_2 + x_3 \\
\hline
x_1 = & 4 - x_4 - 3x_2 - x_3 \\
x_5 = & 2 - 2x_2 - x_3
\end{array}
$$

$$x_5 = 2 - 2x_2 - x_3 \Rightarrow \boxed{x_3 = 2 - 2x_2 - x_5}$$

$$g = -2 + 2 - x_4 + 2x_2 - 2x_2 - x_5$$

$$
\boxed{
\begin{array}{rl}
= & -x_4 - x_5 \\
x_1 = & 2 - x_2 - x_4 + x_5 \\
x_3 = & 2 - 2x_2 - x_5
\end{array}
}
$$

final tableau of the auxiliary LP.

Now, $x^0 = (2, 0, 2, 0, 0)$ is the optimal solution in the aux. problem and the initial basis in the original LP.

As a recap, here is the original LP:

$$\text{max. } y = x_1 + 2x_2$$

$$\begin{aligned} \text{s.t.} \quad x_1 + 3x_2 + x_3 &= 4 \\ 2x_2 + x_3 &= 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$
\begin{aligned}
y &= x_1 + 2x_2 \\
&= 2 + x_2 \\
\hline
x_1 &= 2 - x_2 \\
x_3 &= 2 - 2x_2
\end{aligned}
$$

This is the orig. obj. f.

These come from the aux. tableau (w/ "correction" vars removed)

$$
\begin{aligned}
y &= 2 + x_2 \\
\hline
x_1 &= 2 - x_2 \\
x_3 &= 2 - 2x_2
\end{aligned}
$$

$$x_3 = 2 - 2x_2 \Rightarrow 2x_2 = 2 - x_3 \quad \Big| \div 2$$

$$\Rightarrow x_2 = 1 - \frac{1}{2}x_3$$

$$\varphi = 3 - \frac{1}{2}x_3$$

$$x_1 = 1 + \frac{1}{2}x_3$$

$$x_2 = 1 - \frac{1}{2}x_3$$

$$x^0 = (1, 1, 0), \quad c^T x = \underline{\underline{2 + 1 = 3}}$$