Lecture 18

$Bx = b$, B is non-singular

$$Bx = L\overbrace{Ux}^{y} = b$$

upper triangular matrix

lower triangular matrix

→ 1. solve $Ly = b$

2. After getting y,

$Ux = y$

Back to the example! (lect. 17)

2. Solve $Ux = y$

example from the book!

$$\begin{pmatrix} 2 & 0 & 4 & 0 & -2 \\ 0 & 1 & -6 & 1 & 3 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -21 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ -25/2 \\ 7/2 \\ 23/2 \\ -7/2 \end{pmatrix}$$

Now start with the last line:

$$7x_5 = -\frac{7}{2} \implies x_5 = -\frac{1}{2}$$

$$x_4 - 21\left(-\frac{1}{2}\right) = \frac{23}{7} \implies x_4 = 1$$

$$\vdots$$

Solution $x^* = (-1 \;\; 0 \;\; 2 \;\; 1 \;\; -\frac{1}{2})$

- Using LU factorisation of B implies the solution of two "easier" systems $Ly = b$, $Ux = y$ instead of solving $Bx = b$ (since L and U are triangular).

- However the difficulty, efficiency of both approaches is similar

- We need perturbations of rows and/or columns in order to obtain non-zero diagonal elements. Since B is non-singular, each row/column contains at least one non-zero element.

Obviously, the more zero-elements matrix B has, the faster the solution is obtained.

Def. A matrix that contains zero-elements is called a sparse matrix.

Trick! Apply further row/column permutatations aiming at keeping L and U as sparse as possible.

Problem! How to find the best such permutation is a harder problem than solving the original LP (makes no sense to find it deterministically).

Use a heuristic, that is, a result/converg. cannot be proved, but works well in practice.

We will look at the "minimum degree ordering heuristic".

The effect of making L and U more sparse (L and U contains much more zero-elements).

This can have a dramatic effect for high dimensions.

Approximately: the cost of solving our original problem depends cubically on the dimension of B for Gaussian elimination or LU-factorization.

- But only linear for sufficiently sparse LU-matrices.

Idea for minimum degree ordering heuristic (MDOH).

- Choose in each step of Gaussian elimination/LU-factorisation within the uneliminated part of B the sparsest row (with the smallest number of non-zero elements) and
- From the non-zero elements of this row, the one whos column is the sparsest one.
- Then apply a permutation such that the chosen row/column define a diagonal element.

Example

Apply MDOH heuristic to B.

$$B = \begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ & \begin{pmatrix} 2 & 0 & 4 & 0 & -2 \\ 3 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & -2 \\ 0 & -1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 4 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

- Row 4 is the sparsest row
- Check columns 2 and 5 (non-zero elements). Sparsest column is 2.
- Permutate rows 4 and 1 and columns 2 and 1 s.t. -1 becomes

the (1,1)-diagonal element

$$
\begin{array}{c|ccccc}
 & 2 & 1 & 3 & 4 & 5 \\
\hline
4 & & & & & \\
2 & & & & & \\
3 & & & & & \\
1 & & & & & \\
5 & & & & &
\end{array}
\quad
\begin{pmatrix}
-1 & 0 & 0 & 0 & -6 \\
1 & 3 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 & -2 \\
0 & 2 & 4 & 0 & -2 \\
0 & 0 & 1 & 0 & 4
\end{pmatrix}
$$

Eliminate 1 by adding 2 + 4

and leave the 1 in its place.

Gauss elimination leads to :

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} 2 & 1 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 4 \\ 2 \\ 3 \\ 1 \\ 5 \end{array} &
\begin{pmatrix}
-1 & 0 & 0 & 0 & -6 \\
1 & 3 & 0 & 1 & -6 \\
0 & -1 & -1 & 0 & -2 \\
0 & 2 & 4 & 0 & -2 \\
0 & 0 & 1 & 0 & 4
\end{pmatrix}
\end{array}
$$

Uneliminated part

<u>Notation</u>: the # of non-zero elem. of a row or column in the uneliminated part is called the **degree** of a row/column.

- Row 5 has degree 2 (min.) in the uneliminated part (red square).

- We swap 5 and 2. Check columns 3 and 5. Column 3 has min. degree.

$$
\begin{array}{c}
\\ 4 \\ 5 \\ 3 \\ 1 \\ 2
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 2 & 3 & 1 & 4 & 5 \end{array} \\
\begin{pmatrix}
-1 & 0 & 0 & 0 & -6 \\
0 & 1 & 0 & 0 & 4 \\
0 & -1 & -1 & 0 & -2 \\
0 & 4 & 2 & 0 & -2 \\
1 & 0 & 3 & 1 & -6
\end{pmatrix}
\end{array}
$$

$$
\xRightarrow{\text{Gauss}}
\begin{array}{c}
\\ 4 \\ 5 \\ 3 \\ 1 \\ 2
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 2 & 3 & 1 & 4 & 5 \end{array} \\
\begin{pmatrix}
-1 & 0 & 0 & 0 & -6 \\
0 & 1 & 0 & 0 & 4 \\
0 & -1 & -1 & 0 & 2 \\
0 & 4 & 2 & 0 & -18 \\
1 & 0 & 3 & 1 & -6
\end{pmatrix}
\end{array}
$$

uneliminated part.

Row 3 has min. degree. Column 1 has min. degree. Since there are no zeros there are no necessary permutations.

$$
\xRightarrow{\text{Gauss}}
\begin{array}{c}
\\ 4 \\ 5 \\ 3 \\ 1 \\ 2
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 2 & 3 & 1 & 4 & 5 \end{array} \\
\left(\begin{array}{ccccc}
-1 & 0 & 0 & 0 & -6 \\
0 & 1 & 0 & 0 & 4 \\
0 & -1 & -1 & 0 & 2 \\
0 & 4 & 2 & 0 & -14 \\
1 & 0 & 3 & 1 & 0
\end{array}\right)
\end{array}
$$

Uneliminated part.

Row 1 has min. degree. Swap 4 an 5.

$$\Rightarrow \begin{array}{c} \\ 4 \\ 5 \\ 3 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 3 & 1 & 5 & 4 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 4 & 2 & -14 & 0 \\ 1 & 0 & 3 & 0 & 1 \end{pmatrix} \end{array}$$

Elimination is now complete since we have now non-zero elements in the diagonal.

For L and U we get the following:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \underline{-1} & 1 & 0 & 0 \\ 0 & \underline{4} & \underline{-2} & 1 & 0 \\ \underline{-1} & 0 & \underline{-3} & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} -1 & 0 & 0 & \underline{-6} & 0 \\ 0 & 1 & 0 & \underline{4} & 0 \\ 0 & 0 & -1 & \underline{2} & 0 \\ 0 & 0 & 0 & -14 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The number of non-zero non-diagonal elements in L and U :

$5 + 3 = \underline{8}$

We had 12 non-zero non-diagonal elements in L and U before our heuristic method so this is an improvement.

This can have a dramatic effect on large problems in terms of efficiency!!

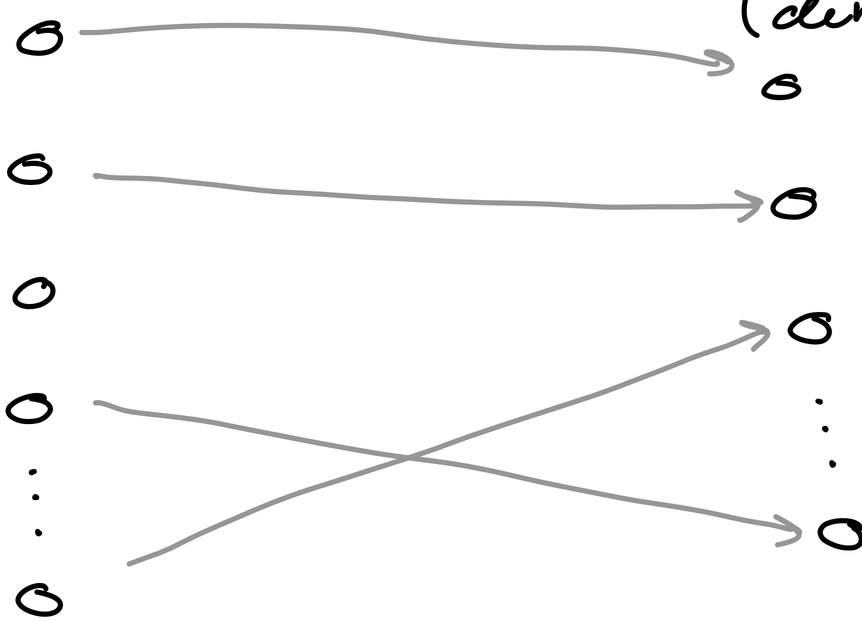
Note that a heuristic will not always provide a better result. It might sometimes even be worse.

Chapter 14 Network flow problems

We consider an LP problem with a particular structure.

General situation of a network:

Sources (suppliers) Locations (demand)



E.g. transportation, electricity, communictation network etc.

Possible objectives: min. of costs, time etc.

Here: min. cost network flow problem.