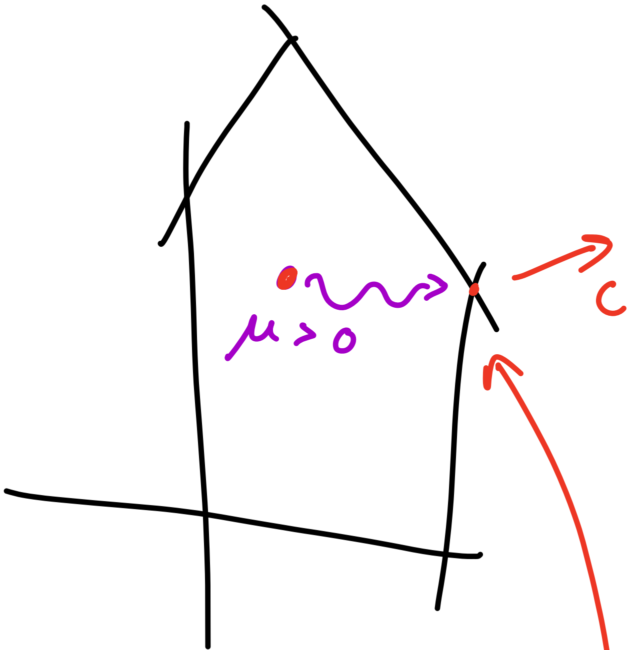
Lecture 23

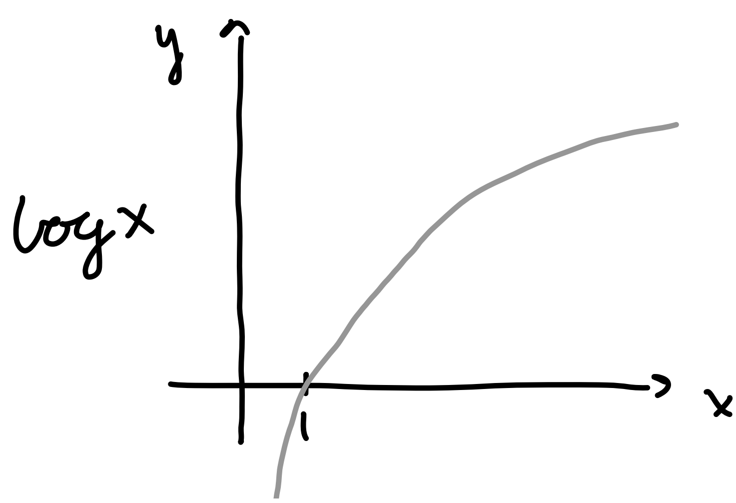


when $\mu = 0$ we have found our solution

Another approach: by using a so-called barrier function.

Consider:

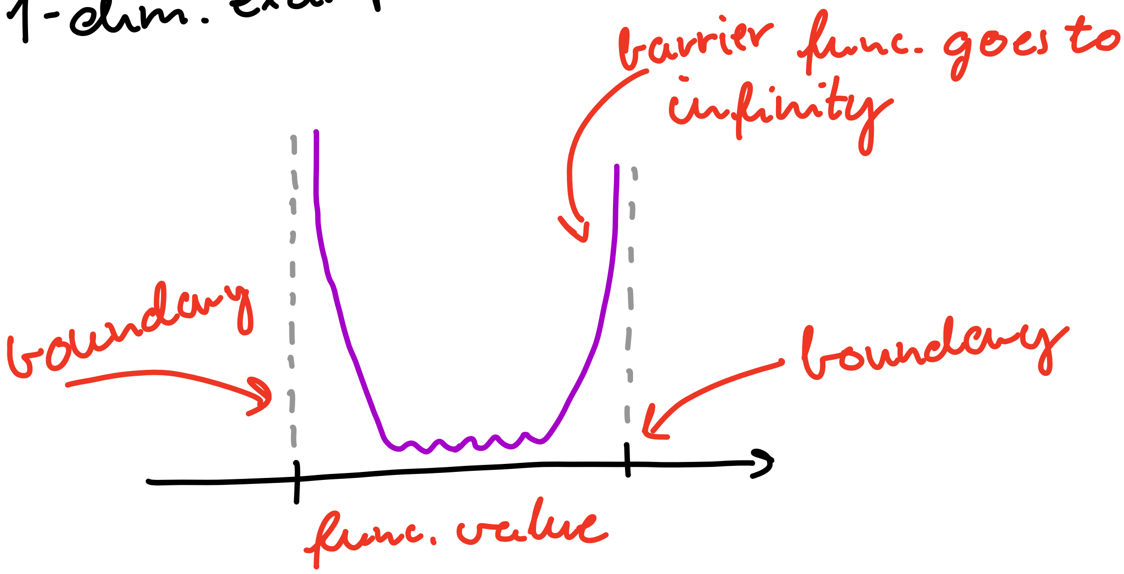
$$\boxed{\begin{aligned} &\max \quad c^T x + \mu \sum_{j=1}^{n} \log x_j + \sum_{i=1}^{n} \log w_i \qquad (P^{\mu}_{\log}) \\ &\text{s.t.} \quad Ax + w = b \\ &(\text{with } \mu > 0) \end{aligned}}$$



$x_j > 0$, $w_i > 0$ (meaning that no inequality constraint is fulfilled by equality). Hence, we are not on any boundary of the feasible set.

If $(P^{\mu}_{\log})$ has a solution (x^{μ}, w^{μ}), then $x_j^{\mu} > 0$, $w_i^{\mu} > 0$, $j = 1 \dots n$, $i = 1 \dots m$

1-dim. example:



Connection between $(P^{\mu}_{\log})$ and $(*)$

central path from last time.

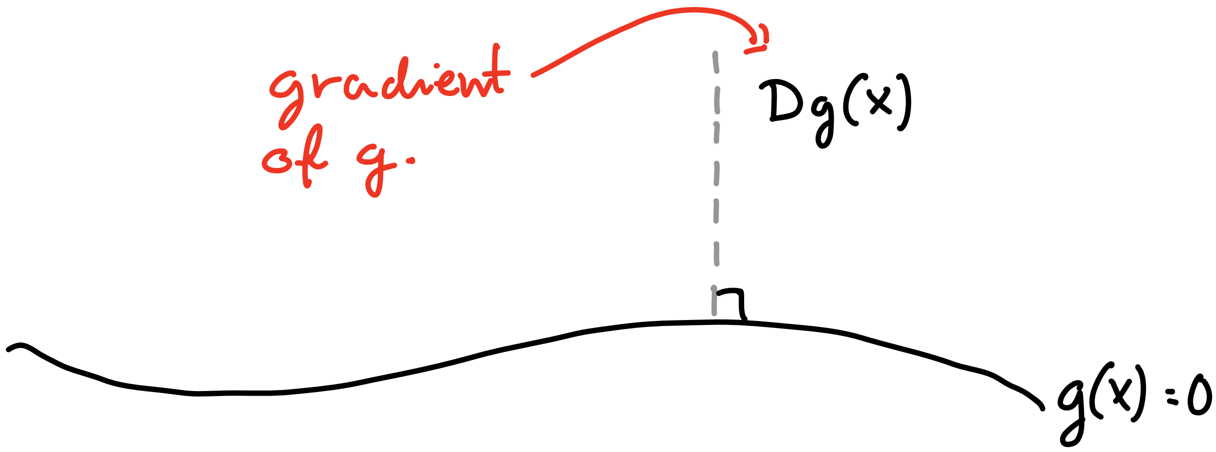
Small excursion to non-lin. programing (NLP).

$$\max. \; f(x)$$
$$\text{s.t.} \quad g(x) = 0$$

only one constraint. (in general, finitely many)

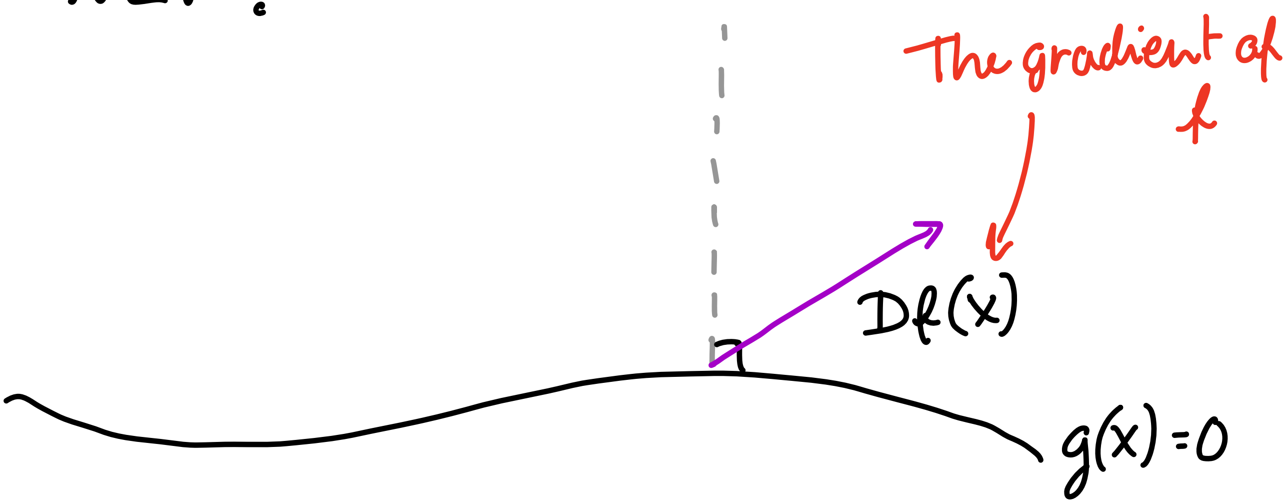
$$\mathbb{R}^2 : Dg(x) = \begin{pmatrix} \frac{\partial(g)}{\partial x_1} \\ \frac{\partial(g)}{\partial x_2} \end{pmatrix}$$

gradient of g at $\bar{x}$.



The line is our feasible set.

When can $\bar{x}$ be a solution of NLP?



If $Df(x)$ is not a multiple of $Dg(\bar{x})$ then $\bar{x}$ cannot be a solution of (NLP) since f increases along $\{x \mid g(x) > 0\}$.

$\bar{x}$ can only be the solution of (NLP) if $Df(\bar{x})$ is a multiple of $Dg(\bar{x})$:

$$Df(\bar{x}) = y\, Dg(\bar{x})$$

Lagrange multiplier

$$\left.\begin{aligned} Df(\bar{x}) &= y\,Dg(\bar{x}) \\ g(x) &= 0 \end{aligned}\right\}$$ Kansh-Kuhn-Tucker system (KKT system)

Now, apply the KKT system to $(P^{\mu}_{\log})$:

$$\max \quad \overbrace{c^T x + \mu \sum_{j=1}^{n} \log x_j + \mu \sum_{i=1}^{m} \log w_i}^{f(x,w)}$$

$$\text{s.t.} \quad \underbrace{Ax + w - b = 0}_{g(x)}$$

KKT system:

$$Df - y^T Dg = 0^{*}$$

$$g = 0$$

(Skipped the arguments here (x, w))

$*$

$$\left.\begin{aligned} \frac{\partial}{\partial x_1} &: \quad c_1 + \mu \frac{1}{x_1} - \sum y_i a_{i1} = 0 \\ \frac{\partial}{\partial x_j} &: \quad c_j + \mu \frac{1}{x_j} - \sum_{i=1}^{m} y_i \underbrace{a_{ij}}_{\text{entries of the matrix } A.} = 0 \end{aligned}\right\} (1)$$

$$\vdots$$

$$\frac{\partial}{\partial w_i}: \quad \mu \frac{1}{w_i} - y_i = 0 \qquad (2)$$

$$Ax + w - b = 0 \qquad (3)$$

Notation: $z_j = \dfrac{\mu}{x_j}$, $j = 1 \ldots m$, the KKT system reads as follows:

$$(1) \qquad c + z - A^T y = 0$$

$$\Rightarrow A^T y - z = c$$

$$(2) \qquad \begin{aligned} z_j x_j &= \mu, \quad j = 1 \ldots n \\ w_i y_i &= \mu, \quad i = 1 \ldots m \end{aligned}$$

(3) $Ax + w = b$

This is now equal to the central path.

$\Rightarrow$ Primal-dual central path is also the solution path of the corresponding barrier problem. $(P^{\mu}_{\log})$

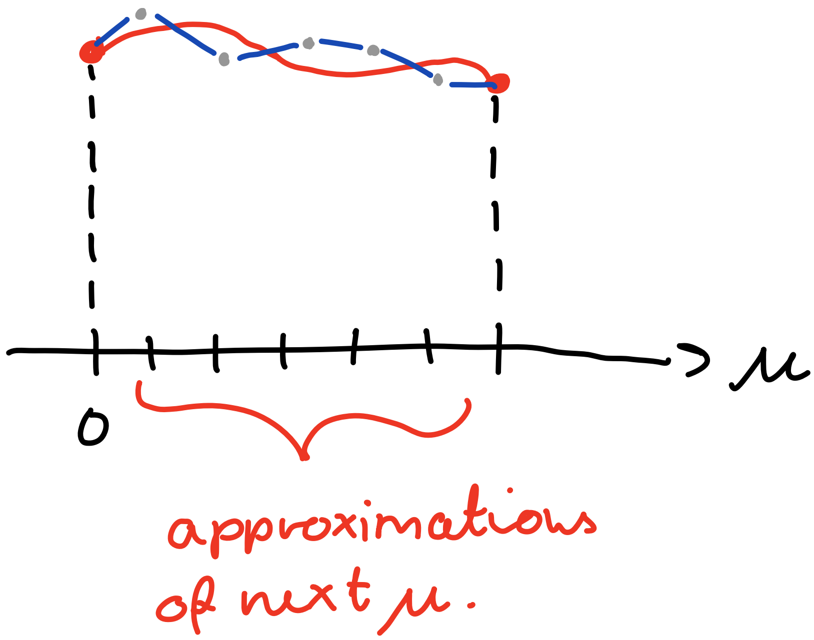
- Solution of $(P^{\mu}_{\log})$ depends on the barrier parameter $\mu > 0$.

- The solution is uniquely determ. if the feasible sets of (P) and (D) have non-empty interior.

- For $\mu \downarrow 0$ we obtain our original problem.

Path-following method for approximating the primal-dual central path with $\mu \downarrow 0$.

converging to zero

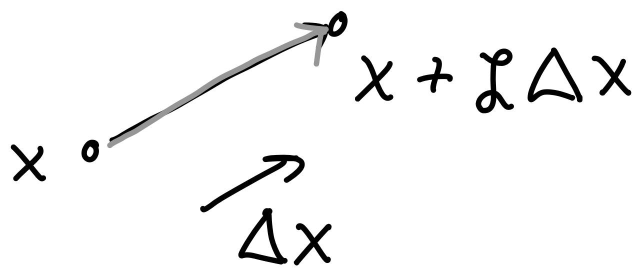


= one iteration step

Each iteration step consists of the following substeps:

- Calc. a new value for μ (smaller than the current one)
- Calc. step direction $(\Delta x, \Delta y, \Delta w, \Delta z)$ pointing approx. to the point $(x_\mu, y_\mu, w_\mu, z_\mu)$

- Calc. the step size θ



- Update: $(x, w, y, z) + \theta(\Delta x, \Delta w, \Delta y, \Delta z)$

For the exam: focus only on calculating the central path (see lecture 22).