$$
\begin{aligned}
\max \quad & -x_1 - x_2 \\
\text{s.t.} \quad & -2x_1 - x_2 \le 4 \\
& -2x_1 + 4x_2 \le -8 \\
& -x_1 + 3x_2 \le -7 \\
& x_1, x_2 \ge 0
\end{aligned}
\qquad (P)
$$

$$
\begin{array}{l}
\varphi = \quad -x_1 - x_2 \\
\hline
w_1 = 4 + 2x_1 + x_2 \\
w_2 = -8 + 2x_1 - 4x_2 \\
w_3 = -7 + x_1 - 3x_2
\end{array}
$$

The initial vertex (P) which is represented by this dictionary:

$$
x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 4 \\ \boxed{\begin{matrix} -8 \\ -7 \end{matrix}} \end{pmatrix}
$$

not feasible!

Let us see if the dual vertex (D) is feasible:

$$\begin{pmatrix} 0 & -1 & -1 \\ 4 & 2 & 1 \\ -8 & 2 & -4 \\ -7 & 1 & -3 \end{pmatrix}^{T} \cdot (-1) \Rightarrow \begin{pmatrix} 0 & -4 & 8 & 7 \\ 1 & -2 & -2 & -1 \\ 1 & -1 & 4 & 3 \end{pmatrix}$$

As we can see, (D) is feasible!

Now we can solve (D) instead of (P).

$$
\begin{aligned}
\min \quad & 4y_1 - 8y_2 - 7y_3 \\
\text{s.t.} \quad & -2y_1 - 2y_2 - y_3 \geq -1 \\
& -y_1 + 4y_2 + 3y_3 \geq -1 \qquad (D) \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

## The Dual Simplex method

we need to use this version of the SM to solve the dual problem.

$$
\begin{array}{rl}
\varsigma = & -x_1 - x_2 \\
\hline
w_1 = & 4 + 2x_1 + x_2 \\
w_2 = & -8 + 2x_1 - 4x_2 \\
w_3 = & -7 + x_1 - 3x_2
\end{array}
\qquad (P)
$$

$$
\begin{array}{rl}
-\dot{\varsigma} = & -4y_1 + 8y_2 + 7y_3 \\
\hline
z_1 = & 1 - 2y_1 - 2y_2 - y_3 \\
z_2 = & 1 - y_1 + 4y_2 + 3y_3
\end{array}
\qquad (D)
$$

Note! we can of course convert (D) to standard form and solve it using the regular SM...

...But! we can skip that and use only the dictionary for (P).

We will only use the dict. for (P) when solving (D).

When solving (D):

- Assign the corr. vars of (D) in (P).

$$
\begin{array}{lrcrcrcr}
 & & & & & z_1 & & z_2 \\
 & \zeta & = & & - & x_1 & - & x_2 \\
\hline
y_1 & w_1 & = & 4 & + & 2x_1 & + & x_2 \\
y_2 & w_2 & = & -8 & + & 2x_1 & - & 4x_2 \\
y_3 & w_3 & = & -7 & + & x_1 & - & 3x_2
\end{array}
$$

- For choosing a pivot col. in the dict. of (D), choose a row in Dict(P) with a <u>negative</u> constant ($y_2$-row).

Positive coeff. in $\xi$

- Choosing a pivot row in Dict(D): choose a column in Dict(P) with <u>positive</u> coeff. and corresponding quotient rule.

neg. in dict(D)

After exch. of $y_2$ and $z_1$ in dict(P):

$$\begin{array}{llcrr}
 & & & y_2 & z_2 \\
 & \varphi = & -4 & -\tfrac{1}{2} w_2 & -3x_2 \\ \hline
y_1 & w_1 = & 12 & + w_2 & + 5x_2 \\
z_1 & x_1 = & 4 & + \tfrac{1}{2} w_2 & + 2x_2 \\
y_3 & w_3 = & -3 & + \tfrac{1}{2} w_2 & - x_2
\end{array} \qquad (P)$$

still infeasible for (P).

$\Rightarrow$ hence (D) is not optimal.

According to the "new" rules:

- Neg. const.: $w_3$ ($y_3$ in (D))
- Pos. coeff. in $w_3$ ($y_3$ in (D)): choose $w_2$ ($y_2$)
- Interchange $w_3$ and $w_2$!

Then we get the following:

$$
\begin{array}{rcl}
\varphi & = & \boxed{-7 - w_3 - 4x_2} \\
\hline
w_1 & = & 18 + 2w_3 + 7x_2 \\
x_1 & = & 7 + w_3 + 3x_2 \\
w_2 & = & 6 + 2w_3 + 2x_2
\end{array}
$$

We now have an opt. point!

In (P): $x = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} 18 \\ 6 \\ 0 \end{pmatrix}$

In (D): $y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $z = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Find these

$\Rightarrow$ optimal for (P) and optimal for (D).

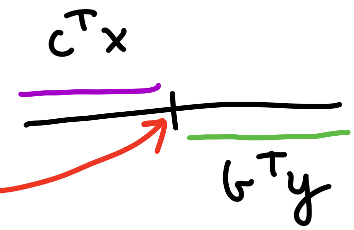
**Another check of optimality: complementary slackness condition!** csc!!

$$x^T z = (7 \; 0) \begin{pmatrix} 0 \\ 4 \end{pmatrix} = 7 \cdot 0 + 0 \cdot 4 = \underline{0}$$

$$y^T w = (0 \; 0 \; 1) \begin{pmatrix} 18 \\ 6 \\ 0 \end{pmatrix} = 0 \cdot 18 + 0 \cdot 6 + 1 \cdot 0 = \underline{0}$$

optimality means we can write:

$$b^T y^* = c^T x^*$$




Initial point (origin) is:

- $\left.\begin{cases}\text{feas. for (D)}\\ \text{infeas. for (P)}\end{cases}\right\}$ solve (D)

- $\left.\begin{cases}\text{feas. for (P)}\\ \text{infeas. for (D)}\end{cases}\right\}$ solve (P)

- $\left.\begin{cases}\text{feas. for (P)}\\ \text{feas. for (D)}\end{cases}\right\}$ both are optimal

- $\left.\begin{cases} \text{infeas. for } (P) \\ \text{infeas. for } (D) \end{cases}\right\}$ How to handle?

Example:

$$\begin{array}{lrcl} & & & (P) \\ \max & 2x_1 - x_2 & & \\ \text{s.t.} & x_1 - x_2 & \le & 1 \\ & -x_1 + x_2 & \le & -2 \\ & x_1, x_2 & \ge & 0 \end{array}$$

This tells us initial point for (P) is infeas.

In dict. form:

$$\begin{array}{rcl} \zeta & = & 2x_1 - x_2 \\ \hline w_1 & = & 1 - x_1 + x_2 \\ w_2 & = & -2 + x_1 - x_2 \end{array}$$

Initial point for (P):

$$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ \underline{-2} \end{pmatrix}$$ Not feasible!

Initial point for (D):

$$y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} \underline{-2} \\ 1 \end{pmatrix}$$ Not feasible!

We need Phase I either for (P) or (D) in order to calc. a feas. starting vertex.

$\swarrow$

## A dual based Ph. I alg.

(differs a bit from Ph. I from primal problems.)

<u>Ex.</u>

$$\begin{array}{lrcl} \max & -x_1 + 4x_2 & & \\ \text{s.t.} & -2x_1 - x_2 & \le & 4 \\ & -2x_1 + 4x_2 & \le & -8 \\ & -x_1 + 3x_2 & \le & -7 \\ & x_1, x_2 & \ge & 0 \end{array}$$

In dict. form:

$$
\begin{array}{rcl}
\zeta & = & \quad -x_1 + 4x_2 \\
\hline
w_1 & = & 4 + 2x_1 + x_2 \\
w_2 & = & -8 + 2x_1 - 4x_2 \\
w_3 & = & -7 + x_1 - 3x_2
\end{array}
\qquad (P)
$$

$$
x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 4 \\ \underline{-8} \\ \underline{-7} \end{pmatrix}
$$

$$
\begin{array}{rcl}
\check{\zeta} & = & \quad -4y_1 + 8y_2 + 7y_3 \\
\hline
z_1 & = & 1 - 2y_1 - 2y_2 - y_3 \\
z_2 & = & -4 - y_1 + 4y_2 + 3y_3
\end{array}
\qquad (D)
$$

$$
y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} 1 \\ \underline{-4} \end{pmatrix}
$$

$A^T y \geq c$ : feasibility of (D) depends on the coeff. of the obj. func. of (P): $\zeta = -x + 4x_2$

We want to change c in order to make (D) feasible.

Generally, dual-based ph. I alg. for calculating a feasible point

change of the obj. func. $\zeta$ in $\text{dict}_P$ such that the initial point becomes feasible for the auxiliary problem (AD).

New obj. func. should only have non-positive coeff.

(AD) $\eta = -x_1 - x_2$

(AP) $\max \; -x_1 - x_2$

s.t. [as before]

## Dict of Phase I:

$$
\begin{array}{l}
\eta = -x_1 - x_2 \\
\hline
w_1 = 4 + 2x_1 + x_2 \\
w_2 = -8 + 2x_1 - 4x_2 \\
w_3 = -7 + x_1 - 3x_2
\end{array}
$$

notice this has changed to negative.

Initial point for (P):

$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} 4 \\ -8 \\ -7 \end{pmatrix}$ Not feasible!

Initial point for (D):

$y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Feasible!

Now we can solve using dual SM and obtain an optimal dict. for $(AP_D)$.

Note that we are omitting steps here.

$$
* \left\{
\begin{array}{rcl}
\eta & = & -7 - w_3 - 4x_2 \\
\hline
w_1 & = & 18 - 2w_3 + 7x_2 \\
x_1 & = & 7 + w_3 + 3x_2 \\
w_2 & = & 6 + 2w_3 + 2x_2
\end{array}
\right.
$$

Note that it is not optimal for the original problem!!

## Phase II:

Resubstitute original obj. func. in (P).

$$\varsigma = -x_1 + 4x_2 \Rightarrow -\overbrace{\left(7 + w_3 + 3x_2\right)}^{x_1 \text{ in } *} + 4x_2$$

Now we get the dict.:

$$\begin{array}{rcl} \zeta &=& -7 - w_3 + x_2 \\ \hline w_1 &=& 18 - 2w_3 + 7x_2 \\ x_1 &=& 7 + w_3 + 3x_2 \\ w_2 &=& 6 + 2w_3 + 2x_2 \end{array}$$

note, the constr. are not changing

feasible point for (P):

$$x = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$$

?

still infeasible for (D).

Also, looking at the dict. for (P), we can see that the problem is unbounded.

Apply (primal) SM.

choose $x_2$ as a pivot column.

$\Longrightarrow$ unboundedness of (P).

$\Longrightarrow$ Implies infeasibility for (D).

(there does not exist any feas. point for (D).