Lecture 22

dual network SM!

$\left.\begin{array}{r}\text{unfeasible (P)}\\ \text{feasible (D)}\end{array}\right\} \longrightarrow$ dual SM

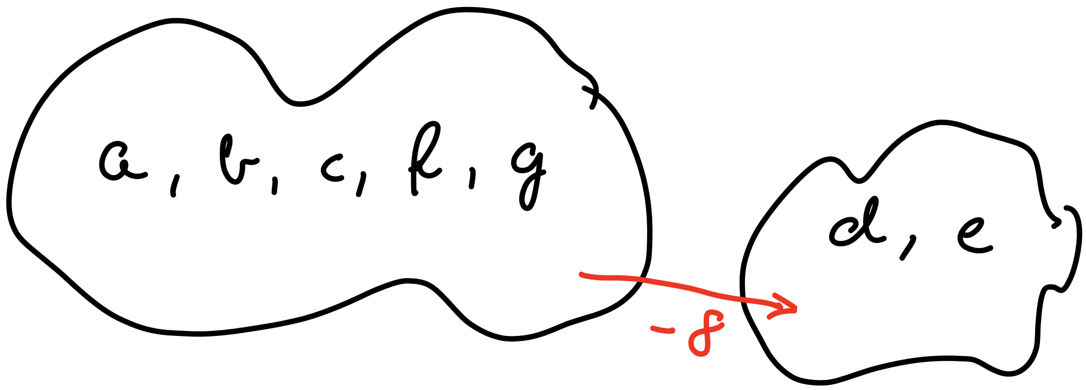
Choose a neg. primal variable:

$x_{db} = -f$ ⟵ take this out of the basis

We then get two disjoint sets:

$\{d, e\}$ $\{a, b, c, f, g\}$

Now connect the sets again.



4 possibilities for the new entering variable:

(a,e), (a, d), (b, e), (g, e)

As in the standard network SM, the dual slack vars of (a,e), (a,d), (b,e) and (g,e), change by the same amount and the slack var of the entering arc becomes 0. ← Balanced constr.

We choose the arc with the smallest slack variable as entering variable. In our case this is $z_{ge} = 9$.

	current slack	new slack	
(a,e)	40	31	(40 - 9)
(a,d)	73	64	(73 - 9)
(b,e)	30	21	(30 - 9)
(g,e)	9	0	(9 - 9)

We shift all values with the amount of the entering variable.

The new primal/dual solutions are calculated analogously to the standard network SM.
(fulfill balance constraints for primal solution and dual constr. for the dual solution).

Solution after first pivot step:

not yet optimal: $x_{de} = -6$, $x_{gb} = -3$

so we need to make additional pivot steps !!

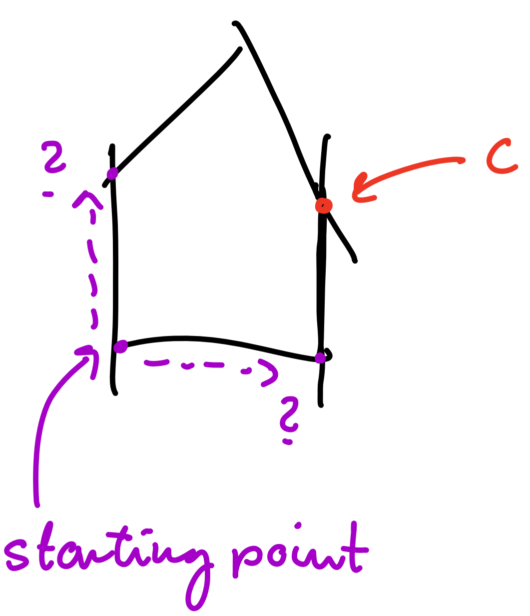
Integrality theorem

For network flow programs with integer data (all entries of data of A, b, c are integers), every basis feasible solution and in particular any basis optimal solution assigns an integer flow to each arc.

Interior point methods

Introduction to interior point approach. (17)

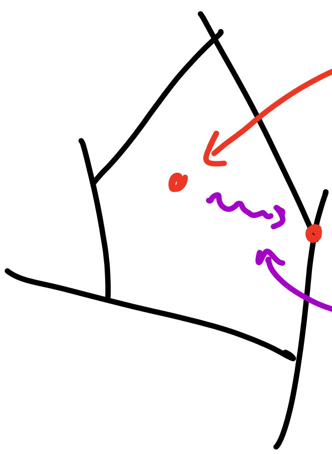
So far: Simplex method



Only use vertices (boundary points of the feas. set).

We are checking a seq. of. vertices of the feasible set until we find a solution (if it exists).

New idea:



Approach the solution (if it exist) following a path of interior points.

The main difference to the SM.

Important property

The SM has exponential complexity (see Klee-Minty problem).

Some interior point methods are polynomial, so they work better on very big problems.

But! for many problems the SM works better (paradoxon).

How to find the path in the interior point method?

Remember:

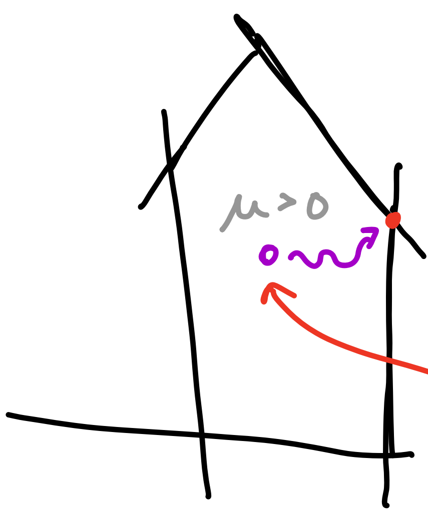
Primal/dual feasible and complementarity $\Big\} \Rightarrow$ optimal

$$
\begin{aligned}
Ax + w &= b \\
A^T y - z &= c \\
x_j z_j &= 0, \quad j = 1 \ldots n \\
y_i w_i &= 0, \quad i = 1 \ldots m \\
x, w, y, z &\geq 0
\end{aligned}
$$

A solution (x^*, w^*, y^*, z^*) of this system $(*)$, represent optimal solutions for (P) and (D).

Idea: Insert (as rhs) in the complementarity constraints in (*) the number $\mu > 0$.

$$\left.\begin{aligned} Ax + w &= b \\ A^T y - z &= c \\ x_j z_j &= \mu, \quad j = 1 \dots n \\ y_i w_i &= \mu, \quad i = 1 \dots m \\ x, w, y, z &\geq 0 \end{aligned}\right\} \quad (2*)$$



we start here, for some $\mu > 0$.

For $\mu = 0$ we get the original system $(*)$ for optimality.

<u>Hope</u>: find solutions $(x_\mu, w_\mu, y_\mu, z_\mu)$ for $\mu > 0$ s.t. for $\mu \downarrow 0$, the limit represents the optimal solution of (P) and (D).

<u>Existence of $x_\mu, w_\mu, y_\mu, z_\mu$:</u>

If the feasible sets of (P) and (D) have a non-empty interior (there are interior points), then for each $\mu > 0$ there exist a unique solution $(x_\mu, w_\mu, y_\mu, z_\mu)$ of $(2*)$. The set

$\{(x_\mu, w_\mu, y_\mu, z_\mu) \mid \mu > 0\}$ is called primal-dual central path.

Geometric idea:

follow the primal-dual central path to find the optimal solution of (P) and (D).

$$x_j z_j = \mu > 0 \longleftrightarrow x_j > 0,\ z_j > 0$$

$$y_i w_i = \mu > 0 \longleftrightarrow y_i > 0,\ w_i > 0$$

$$z_j > 0 \longrightarrow A^T y - z^{>0} = c$$

$$\longrightarrow \boxed{A^T y > c,\ y > c}$$

x and y* are interior points.

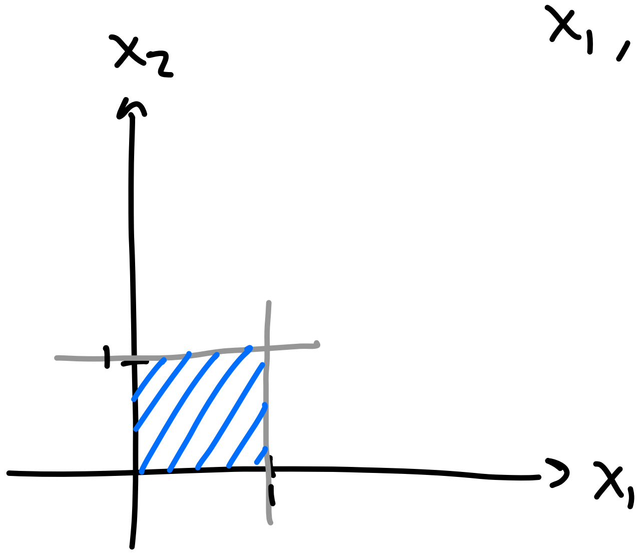
$$w_i > 0 \longrightarrow Ax + \underbrace{w}_{>0} \longrightarrow \boxed{Ax \le b,\ x > 0}$$

*

Since none of the constraints are fulfilled as equality.

Example: calulation of the central path.

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \qquad \begin{aligned} \max\ & x_1 \\ \text{s.t.}\ & x_1 \le 1 \\ & x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$



Calculation of the sentral path means we have to solve the system (2*).

$$
\begin{array}{llllllll}
(1) & x_1 & & + w_1 & & & & = 1 \\
(2) & & x_2 & & + w_2 & & & = 1 \\
(3) & y_1 & & - z_1 & & & & = 1 \\
(4) & & y_2 & & - z_2 & & & = 0 \\
(5) & & & x_1 z_1 & & & & = \mu \\
(6) & & & x_2 z_2 & & & & = \mu \\
(7) & & & y_1 w_1 & & & & = \mu \\
(8) & & & y_2 w_2 & & & & = \mu
\end{array}
$$

Solve this system to get $(x_\mu, w_\mu, y_\mu, z_\mu)$ for $\mu > 0$.

$$\left.\begin{aligned} (1):\ & x_1 = 1 - w_1 \\ (3):\ & y_1 = 1 + z_1 \end{aligned}\right\} \quad \text{((2) and (4) analogously)}$$

$$(5):\ (1 - w_1) z_1 = \mu$$

$$\Rightarrow {}^{*}z_1 = \frac{\mu}{1 - w_1}$$

$$(7):\ (1 + z_1) w_1 = \mu$$

$$\Rightarrow w_1 + \overset{*}{z_1} w_1 = \mu$$

$$w_1 + \frac{\overset{*}{\mu} w_1}{1 - w_1} = \mu \quad \Big| \cdot (1 - w_1)$$

$$\Rightarrow (1 - w_1) w_1 + \mu w_1 = \mu (1 - w_1)$$

$$\Rightarrow 0 = w_1^{2} + (-1-2\mu)w_1 + \mu$$

quadratic equation!

$$w_1 = \frac{1+2\mu}{2} \pm \frac{\sqrt{1+4\mu^2}}{2}$$

Let $a_\mu = \dfrac{1-2\mu+\sqrt{1+4\mu^2}}{2}$

The path $(x_\mu, w_\mu, y_\mu, z_\mu)$ is defined as follows:

$$x_\mu = \begin{pmatrix} a_\mu \\ 1/2 \end{pmatrix}, \quad w_\mu = \begin{pmatrix} 1-a_\mu \\ 1/2 \end{pmatrix}$$

$$y_\mu = \begin{pmatrix} \frac{\mu}{1-a\mu} \\ 2\mu \end{pmatrix}, \quad z_\mu = \begin{pmatrix} \frac{\mu}{a\mu} \\ 2\mu \end{pmatrix}$$

Higly non-linear (even for a small problem).