Lecture 5.

<u>Ex.</u> Raw mat. r_1 and r_2 costing both 1 per unit. Stock of r_1 is 10 units, stock of r_2 is 8 units.
Product p_1 require 2 units of r_1 and 1 unit of r_2.
Product p_2 require 1 unit of r_1 and 2 units of r_2.
p_1 sells at 6 per unit, p_2 sells at 5 per unit.

LP model of the above problem in std. form:

x = # of produced units of p_1.
y = —— " —— p_2.

$$\max. \ \varphi = 3x + 2y^{*} \quad \text{Obj. f.}$$

s.t.

$$2x + y \leq 10 \quad ① \quad \text{stock of } r_1$$

$$x + 2y \leq 8 \quad ② \quad \text{—— " —— } r_2$$

$$x, y \geq 0$$

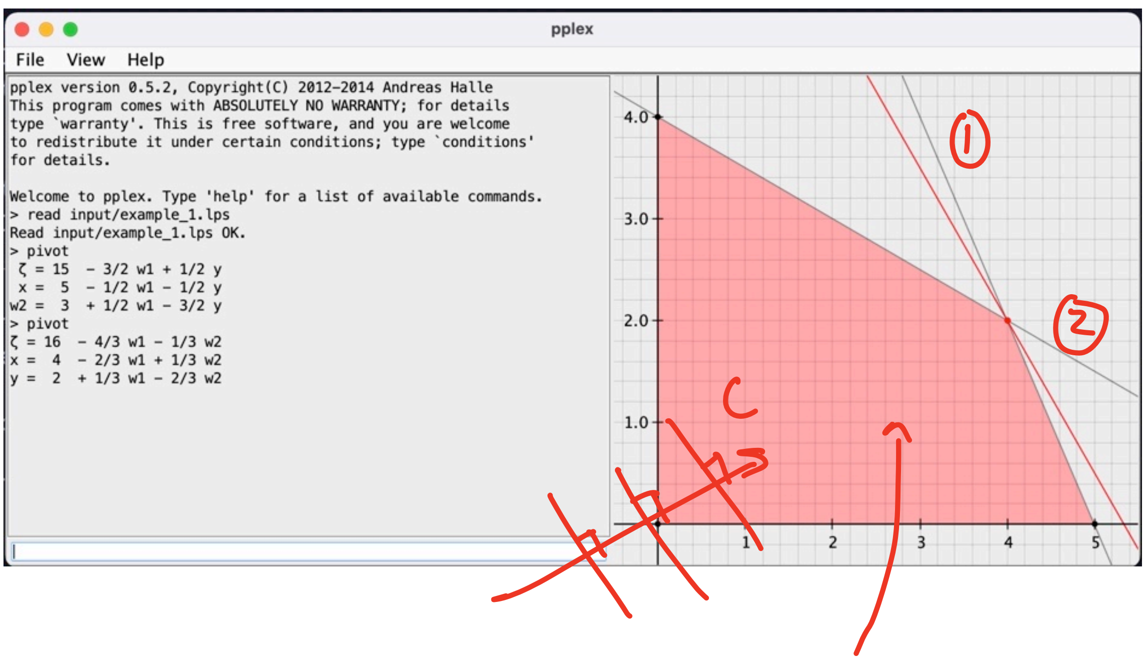
Cost of p_1 : $3x$ ($2\,r_1$ and $1\,r_2$)

Cost of p_2 : $3y$ ($1\,r_1$ and $2\,r_2$)

Profit: $6x - 3x + 5y - 3y$

Obj. f. : $3x + 2y$ *

Graphical representation:



Polyhedron.
Intersection of fin. many half-spaces.

$$2x + y = 10 \Rightarrow y = 10 - 2x$$

$$x + 2(10 - 2x) = 8$$

$$20 - 3x = 8 \Rightarrow \begin{cases} x = 4 \\ y = 2 \end{cases}$$

Solution:

$$\begin{pmatrix} x^0 \\ y^0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad 3x^0 + 2y^0 = 16$$

Solution by using the simplex method:

$$\frac{\varphi = 3x + 2y}{\begin{aligned} w_1 &= 10 - 2x - y \\ w_2 &= 8 - x - 2y \end{aligned}}$$

if $x > 0$ (and $y = 0$):

$w_1 = 10 - 2x \Rightarrow x \leq 5$ ✓

$w_2 = 8 - x \Rightarrow x \leq 8$

$\min.\{5, 8\} = \underline{5}$

$$\underset{\text{new BV}}{x}\ \text{exchange } x \text{ and } \underset{\text{new NBV}}{w_1}$$

exchange x and w_1

new BV

new NBV

$$
\begin{aligned}
\varsigma &= 3(10 - w_1) + y \\
&= 30 - 3w_1 + y \\
&= 15 - \tfrac{3}{2}w_1 + \tfrac{1}{2}y \qquad \Big|\cdot \tfrac{1}{2}
\end{aligned}
$$

$\frac{3}{2}$??

$$
\begin{aligned}
x &= 5 - \tfrac{1}{2}w_1 - \tfrac{1}{2}y \qquad \Big|\cdot \tfrac{1}{2} \\
w_2 &= 3 + \tfrac{1}{2}w_1 - \tfrac{3}{2}y \qquad \Big|\cdot \tfrac{1}{4}
\end{aligned}
$$

y has a positive coeff. so continue!

If $y > 0$ (and $w_1 = 0$)

$$x = 5 - \tfrac{1}{2}y \Rightarrow y \le 10$$

$$w_2 = 3 - \tfrac{3}{2}y \Rightarrow y \le 2 \ \checkmark$$

$\min\{10, 2\} = \underline{2}$

new BV

new NBV

exchange y and w_2

$$y = 15 - \frac{3}{2}w_1 + \frac{1}{2}\left(2 - \frac{2}{3}w_2\right)$$
$$= 16 - \frac{3}{2}w_1 - \frac{1}{3}w_2$$

$$x = 5 - \frac{1}{2}\left(2 - \frac{2}{3}w_2\right)$$
$$= 4 - \frac{2}{3}w_1 + \frac{1}{3}w_2$$
$$y = 2 + \frac{1}{3}w_1 - \frac{2}{3}w_2$$

TODO: Redo this!

$$x^\circ = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad 3x^\circ + 2y^\circ = 16$$

$$\max.\ 3x + 2y \quad \text{s.t.} \quad \begin{aligned} 2x + y &\le 10 \\ x + 2y &\le 8 \\ x, y &\ge 0 \end{aligned}$$

<u>Alt</u>. Now you want to sell the raw materials (or use in a different way). The whole stock of both r_1 and r_2 will be used. The profit for p_1, p_2 defines an upper bound.
?

Variables : selling price per unit of

r_1 (r_2) : u (v)

"Best price" :

Profit for p_1

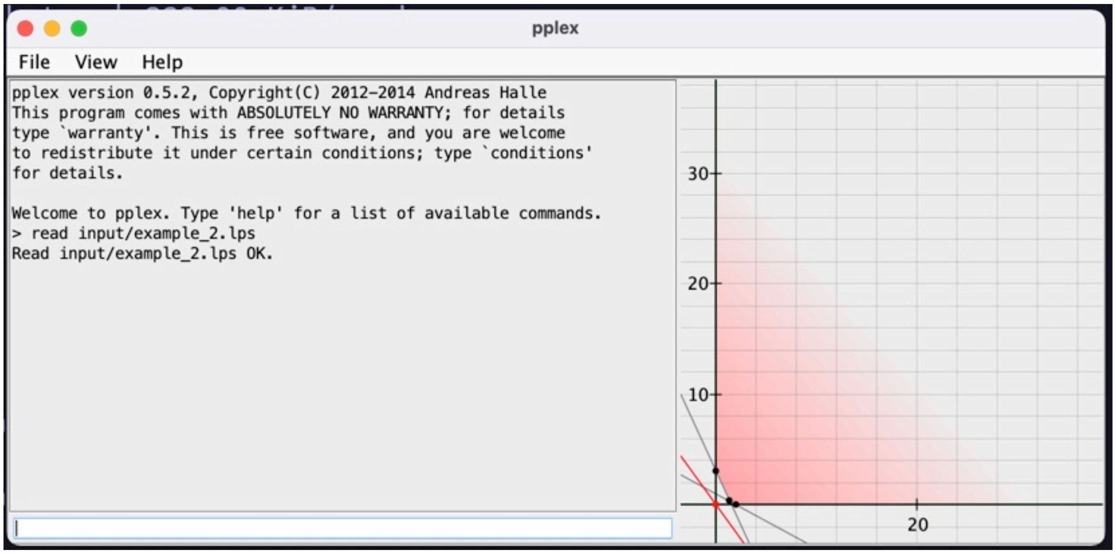
$$\min.\ 10u + 8v \quad \text{s.t.} \quad 2u + v \geq 3 \quad \text{—"— } p_2$$

$$u + 2v \geq 2$$

$$u, v \geq 0$$

$$\min \begin{pmatrix} 10 & 8 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{s.t.} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 1/3 \end{pmatrix}, \quad \varphi^0 = 16$$



Auxilary problem

$$\max \; 2x_1 - 6x_2$$

s.t.

$$-x_1 - x_2 - x_3 \le -2$$

$$2x_1 - x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \ge 0$$

here, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is <u>not</u> a feasible point!

We need to solve an auxilary problem to get a feasible point.

<u>Trick:</u> solve an aux. problem by using an additional variable x_0.

Phase I. (AP)

$$\begin{array}{lr}
\max & \boxed{-x_0} \\
-x_1 - x_2 - x_3 & -x_0 \le -2 \\
2x_1 - x_2 + x_3 & -x_0 = 1 \\
x_1, x_2, x_3, x_0 & \ge 0
\end{array}$$

Introduce slack vars and create an initial infeasible solution.

$$\begin{aligned}
\mathcal{L} &= -x_0 \\
\hline
w_1 &= \boxed{-2} + x_1 + x_2 + x_3 + x_0 \\
w_2 &= 1 - 2x_1 + x_2 - x_3 + x_0
\end{aligned}$$

obviously infeasible

Now, interchange x_0 and the "most feasible" slack variable. In this case, $\boxed{w_2 = -2}$.

$$x_0 = 2 - x_1 - x_2 - x_3 + w_1$$

New dictionary:

$$\begin{array}{rcl}
\mathcal{L} & = & -2 + x_1 + x_2 + x_3 - w_1 \\
\hline
x_0 & = & 2 - x_1 - x_2 - x_3 + w_1 \\
w_2 & = & 3 - 3x_1 \qquad - 2x_3 + w_1
\end{array}$$

x_0 added

$$\left.\begin{array}{l} x_0 = 2 \\ w_2 = 3 \end{array}\right\} \text{ feasible dictionary}$$

Now, apply the simplex method as usual – Interchange x_2 and x_0:

$$x_2 = 2 - x_1 - x_0 - x_3 + w_1$$

New dictionary:

$$\begin{array}{rl} \mathcal{L} = & \quad\quad\quad - x_0 \\ \hline x_2 = & 2 - x_1 - x_0 - x_3 + w_1 \\ w_2 = & 3 - 3x_1 \quad\quad - 2x_3 + w_1 \end{array}$$

$x_1 = x_2 = x_3 = w_1 = 0$

$x_2 = 2$

$w_2 = 3$

Aux. problem

Optional solution for AP:

- If $\xi^\circ = 0$, then LP has a feasible point.
- If $\xi^\circ \neq 0$, then LP has <u>no</u> feasible point.

Phase II.

Trick: take the original obj. func. and drop x_0 from the constraints (since $x_0 = 0$):

$$\varphi = 2x_1 - 6x_2$$

$$\varphi = -12 + 8x_1 + 6x_3 - 6w_1$$

$$x_2 = 2 - x_1 - x_3 + w_1$$

$$w_2 = 3 - 3x_1 - 2x_3 + w_1$$

$$x_1 = x_3 = w_1 = 0$$

$$x_2 = 2, \quad w_2 = 3$$

Interchange x_1 and w_2:

$$x_1 = 1 - \frac{1}{3}w_2 - \frac{2}{3}x_3 + \frac{1}{3}w_1$$

$$\zeta = -4 - \frac{8}{3}w_2 - \frac{2}{3}x_3 - \frac{10}{3}w_1$$

$$x_2 = 1 + \frac{1}{3}w_2 - \frac{1}{3}x_3 + \frac{2}{3}w_1$$

$$x_1 = 1 - \frac{1}{3}w_2 - \frac{2}{3}x_3 - \frac{1}{3}w_1$$

$w_2 = x_3 = w_1 = 0$

$x_2 = 1, \; x_1 = 1$

Still not optimal. Interchange x_3 and x_1:

$$x_3 = \frac{3}{2} - \frac{1}{2}w_2 - \frac{3}{2}x_1 + \frac{1}{2}w_1$$

New dictionary:

$$\zeta = -3 - 3w_2 - x_1 - 3w_1$$

$$x_2 = \frac{1}{2} + \frac{1}{2}w_2 + \frac{1}{2}x_1 + \frac{1}{2}w_1$$

$$x_3 = \frac{3}{2} - \frac{1}{2}w_2 - \frac{3}{2}x_1 + \frac{1}{2}w_1$$

Optimal solution found!

$$x^0 = \begin{pmatrix} 0 \\ 1/2 \\ 3/2 \end{pmatrix}, \quad c^T x^0 = -3$$