Lecture 19

Network flow problems

What is a network?

Let $N = \{a_n, a_m\}$ be a set of nodes.

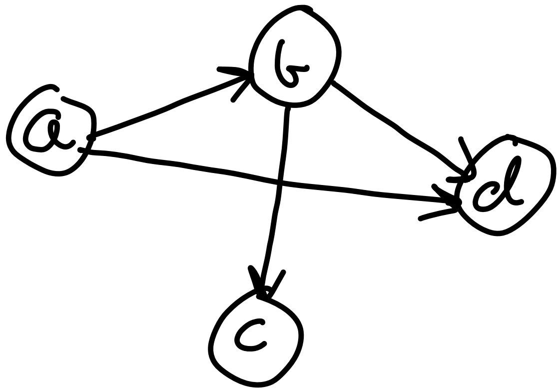
Let $A = \{(a_i, a_j),\ a_i, a_j \in N,\ i \neq j\}$ where arcs (a_i, a_j) are connecting a_i and a_j.

directed!

$a_i \longrightarrow a_j$ $\quad (a_i, a_j)$

$a_i \longleftarrow a_j$ $\quad (a_j, a_i)$

Example:



Here, the set $A = \{(a,b), (b,d),$
$(a,d), (b,c)\}$

A need not contain all possible arcs.

We will consider directed arcs:

$$a_1 \longrightarrow a_2 \qquad (a_1, a_2)$$

Material can be transported from a_1 to a_2 but not the other way around.

Assign to each arc (a_i, a_j) the cost c_{ij} for the transportation of 1 unit of material along (a_i, a_j), $c_{ij} \geq 0$.

- Supply/demand b_i at each node a_i:
  - b_i supply
  - $-b_i$ demand (neg. supply)

Goal: move the material from the nodes with supplies to the nodes with demands.

Assume that "$\sum$ of supplies = $\sum$ of demands"

(or else, take the difference out of the system).

$$\sum_{i \in N} b_i = 0$$

- Variables: x_{ij} (units of material transported along the arc a_{ij}).

$x_{ij} \geq 0, \forall (i,j) \in A$

- Costs: c_{ij} per unit of mat. along (a_i, a_j).

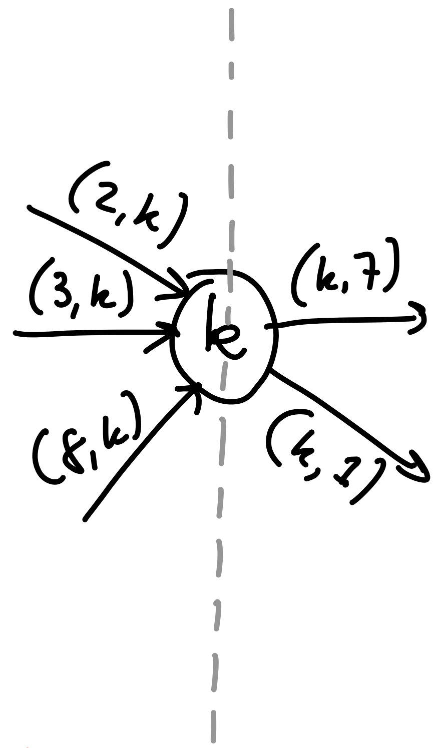
- Obj. func.: $\sum_{(i,j)\in A} c_{ij} x_{ij}$

- Constraints: at node k:

$$\sum_{i(i,k)\in A} x_{ik} = \sum_{j(k,j)\in A} x_{kj} - b_k$$

demand at k.

constr. at <u>each</u> node.



i is varying

Flow into k:

$$\sum_{i:(i,k)\in A} x_{ik} = x_{2k} + x_{3k} + x_{8k}$$

Flow out of k:

$$\sum_{j:(k,j)\in A} x_{kj} = x_{k1} + x_{k7}$$

demand on k:

$-b_k$

Constraints:

$$\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} = -b_k \,, \quad k \in N$$

Our optimization problem has the form of an LP:

$$\min.\ c^T x \quad \text{s.t.}\ Ax = -b \,, \quad x \geq 0$$

not in standard form

How does A look in this case?

$A = (m, n)$ - matrix

\# of arcs

\# of constraints

$$
A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}
\begin{array}{c}
\begin{array}{ccc} x_{ac} & x_{ad} & \cdots \end{array} \\
\begin{pmatrix}
-1 & -1 & \cdots \\
0 & 0 & \vdots \\
1 & 0 & \\
0 & 1 & \\
0 & 0 & \\
0 & 0 & \\
0 & 0 &
\end{pmatrix}
\end{array}
$$

sparse matrix!!

(-1) is the "from" node and 1 is the "to" node.

A is called the node-arc insidence matrix.

Each column of A has one 1, one (-1) and all other elements are zero.

$$-b = (\underbrace{0 \quad 0}_{\text{No supply/demand}} \quad \underbrace{6 \quad 6 \quad 2}_{\text{demand}} \quad \underbrace{-9 \quad -5}_{\text{supply}})$$

(P) $\min c^T x$ s.t. $Ax = -b, \geq 0$

(D) $\max b^T y$ s.t. $A^T y \leq c$

($y \in \mathbb{R}^m$ are free variables)

can have neg. values.

Introduce slack variables for (D):

$\max -b^T y$ s.t. $A^T y + z = c$, $z \geq 0$

($z \in \mathbb{R}^n$)

Each row of A^T contain exactly one 1, one (-1) and all other are zero.

Each constr. of (D) can be written:

$$-y_i + y_j + z_{ij} = c_{ij}$$

We don't need A

$$\text{(D)} \quad \max c^T x$$

$$\text{s.t.} \quad \left.\begin{aligned} -y_i + y_j + z_{ij} &= c_{ij} \\ z_{ij} &\geq 0 \end{aligned}\right\} \forall (i,j) \in A$$

The strong duality theorem implies that at optimal solutions x^* of (P) and (y^*, z^*) of (D):

$$c^T x^* = -b^T y^*$$

$$-b^T y^* = \underbrace{(x^*)^T A^T}_{=-b^T} y^*$$

$$= (x^*)^T (\overbrace{A^T y^* - c}^{z^*} \underbrace{+ c}_{0})$$

$$= -(x^*)^T z^* + (x^*)^T c$$

Since $x^* \geq 0$ and $z^* \geq 0$

$$\Rightarrow \boxed{x^*_{ij} z^*_{ij} = 0 \quad \forall (i,j) \in A}$$

complementarity condition at optimal solutions x^* of (P) and (y^*, z^*) of (D).

The constraints of (D):

$y_i + y_j + z_{ij} = c_{ij}$ implies that the solutions y_i is <u>not</u> uniquely determined.

One can add a constant to all y_i's. That is if $(y_i)_{i \in N}$ is feasible, then also $(y_i + k)_{i \in N}$ is feasible for all k.

Obj. func. value:

$$-\sum_{i \in N} b_i y_i$$

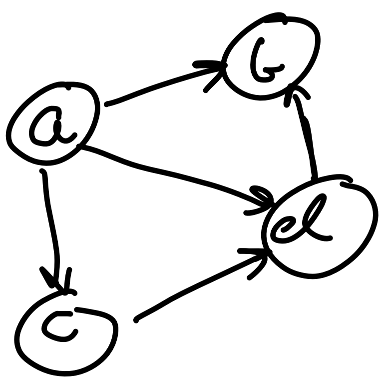
y_i substituted by $y_i + k$

$$-\sum b_i (y_i + k)$$

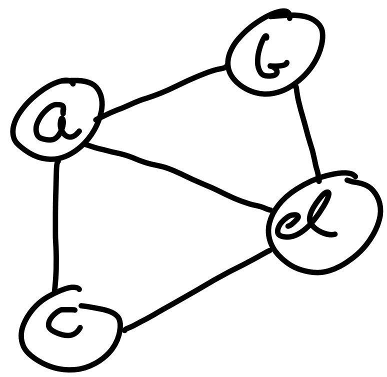
$$= -\sum_{i \in N} b_i y_i - k \sum_{i \in N} b_i$$

sum of b_i's are 0 because sum of suppl. = sum of demands.

Spanning trees and bases



Network



Underlying undirected graph.

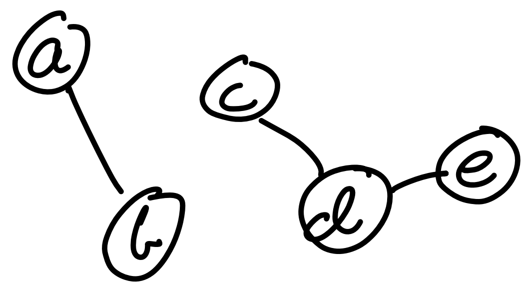
◦ An ordered list of nodes:

(need not be different nodes)

$n_1 - n_2 - n_3 - \cdots - n_k$

Is called a path if there are (indirected) arcs $n_i - n_{i+1}$

- A network is called connected if for any pair of nodes there is a path connecting these two nodes.



- A cycle is a path where the first node and the last node is the same.

- A network is called cyclic if there exists a cycle. Or acyclic if there are no cycles.

- A network is called a tree if it is connected and acyclic.

- $(\tilde{N}, \tilde{A})$ is called a subnetwork of (N, A) if $\tilde{N} \subset N$ and $\tilde{A} \subset A$.