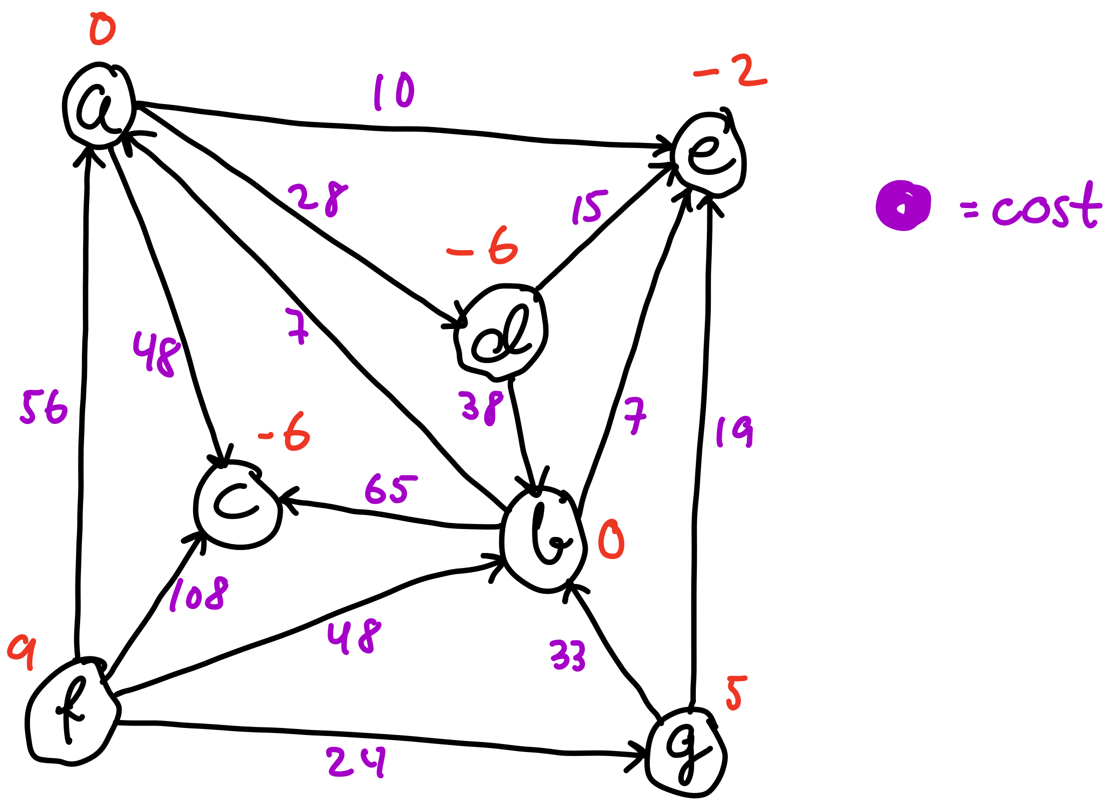


Network with supply/demand.
Neg. numbers = demand.

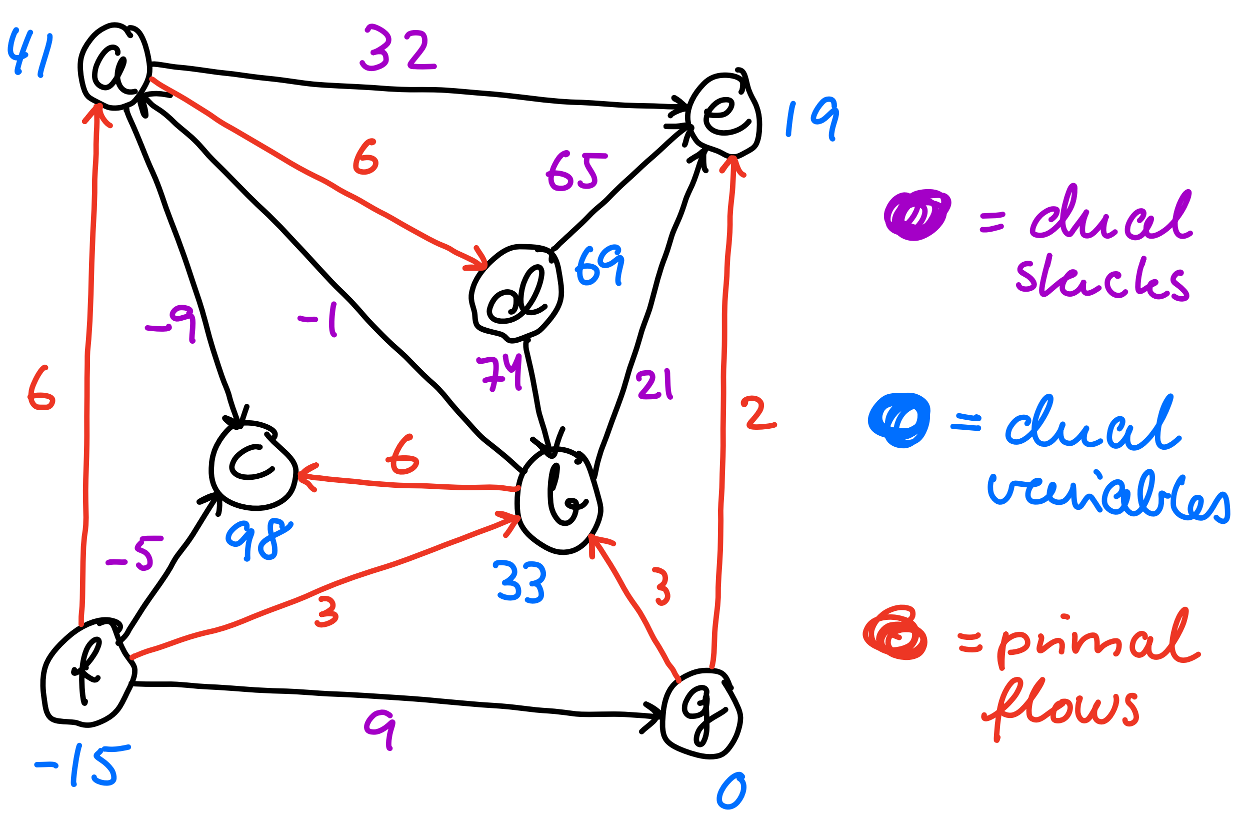
Objective: move supply over to demand nodes.

$$\underbrace{\sum_{i:(i,k)\in A} x_{ik}}_{\text{flow into } k} - \underbrace{\sum_{j:(k,j)\in A} x_{kj}}_{\text{flow out of } k} = \underbrace{-b_k}_{\text{demand at } k}$$

Net flow must be equal to the demand at k.

Obj. func. : $\text{min.} \underbrace{\sum_{(i,j)\in A} c_{ij} x_{ij}}_{\text{min. cost of transportation}}$

Initially, one row in the node-arc incidence matrix is redundant (meaning the sum of all rows in A is -1. A does <u>not</u> have rank m at this point, so we can delete one row.

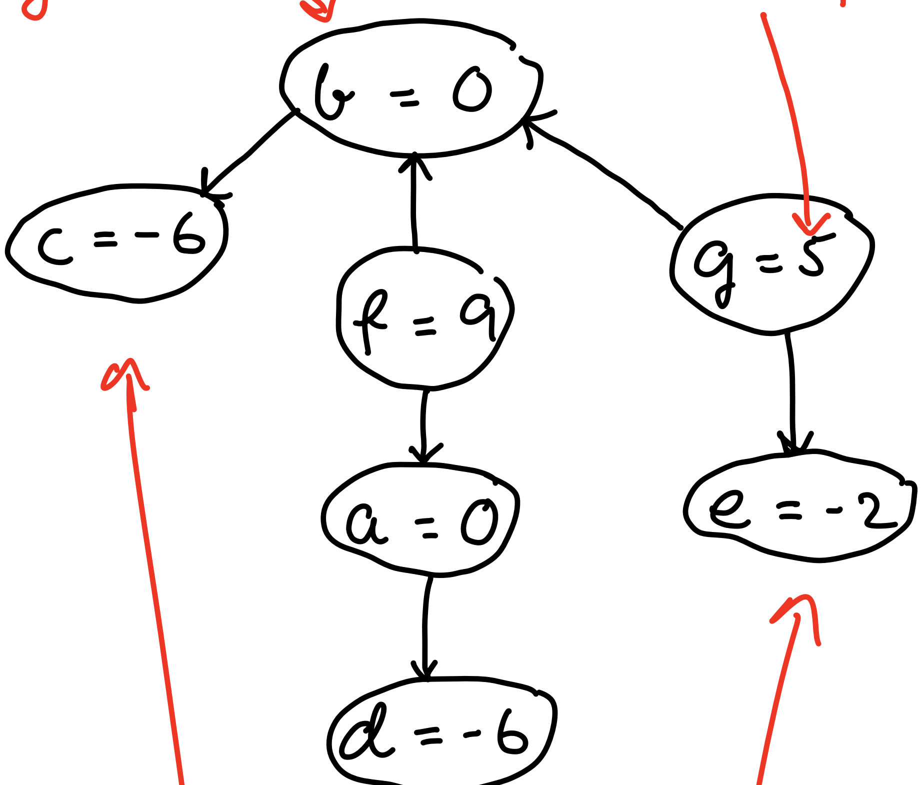


7 nodes and 6 arcs in the spanning tree.

At least two nodes must have only one arc connection.

Note that we can start with any node

means there is a demand of 5 at g.



c, d and e are the leaf nodes in this case.

Step 1

start at one of the leaves to find a tree solution.

(c) $\boxed{x_{bc} = 6}$ ← The only way to fulfill demand at c is to pass 6 along the x_{bc} arc.

(d) * $\boxed{x_{ad} = 6}$

(e) $\boxed{x_{ge} = 2}$

(a) $\boxed{x_{fa} - x_{ad} = 0}$ ← What comes in to a must go out of a.

$\Rightarrow \boxed{x_{fa} = 6}$ *

(f) $x_{fb} + x_{fa} = 9$ ← f has two outgoing arcs whis needs to have sum of 9.

$\Rightarrow \boxed{x_{fb} = 3}$

$$ⓖ \quad \overbrace{x_{gb} + x_{fb}}^{\text{entering}} - \overbrace{x_{bc}}^{\text{leaving}} = 0$$

$$\Rightarrow \boxed{x_{gb} = 3}$$

We have now covered all 6 arcs (note that we did not have to look at g). It is the redundant node in this case.

Note that any point that is <u>not</u> a leaf can be a redundant node.

Our initial solution looks as follows:

$$\left.\begin{aligned} x_{bc} &= 6 \\ x_{ge} &= 2 \\ x_{ad} &= 6 \\ x_{fa} &= 6 \\ x_{fb} &= 3 \\ x_{gb} &= 3 \end{aligned}\right\} \text{feasible!}$$

(all others are zero)

Is it optimal?

We can check the corresponding dual solution to determine this. (more on this later)

Coeff. matrix after deleting (g):
(notice also the permutation)

$$\tilde{A} = \begin{array}{c|cccccc}
 & \multicolumn{6}{c}{\text{arcs of our spanning tr.}} \\
 & x_{ad} & x_{fa} & x_{fb} & x_{bc} & x_{gb} & x_{ge} \\
\hline
d & 1 & & & & & \\
a & -1 & 1 & & & & \\
f & & -1 & -1 & & & \\
c & & & & 1 & & \\
b & & & 1 & -1 & 1 & \\
e & & & & & & 1
\end{array}$$

The redundant node (g) is called the root node.

The remaining (m-1, m-) - matrix is non-singular.

Now we look at (D) to determine if we are optimal.

- If (i,j) belongs to the spanning tree then:
  - X_{ij} is BV for (P). (red arcs)
  - Z_{ij} is NBV for (D) $\Rightarrow Z_{ij} = 0$

$y_j - y_i = C_{ij}$ for m nodes and $m-1$ arcs of the spanning tree.

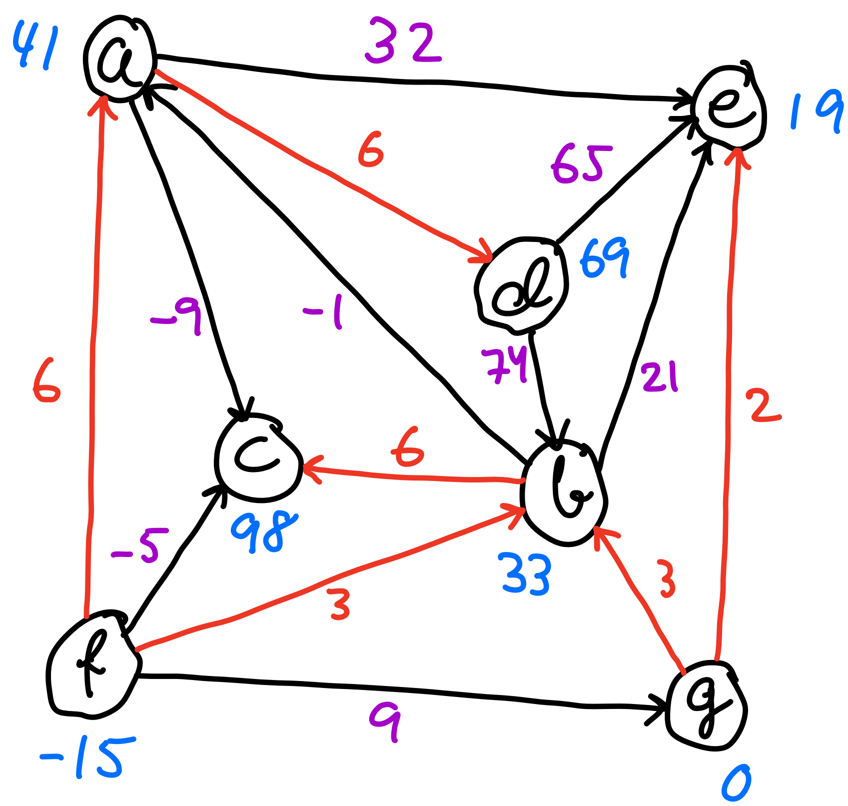
Calculate (D).

We look at the constraints of the dual problem.

$$y_j - y_i + z_{ij} = c_{ij} \quad (i,j) \in A$$

* Start with the root node (in our case 9).

Let $y_j = 0$. ← This is our 9.



NBV for (D), hence zero.

* arc (g, e): $y_e - y_g + z_{eg} = C_{ge}$

$$y_e - y_g = 19 - 0 = 19$$

$$\boxed{\Rightarrow y_e = 19}$$

arc (g, b): $y_b - y_g = 33 - 0 = 33$

$$\boxed{\Rightarrow y_b = 33}$$

arc (b, c): $y_c - y_b = 65$

$$\boxed{\Rightarrow y_c = 98}$$

arc (f, b): $y_b - y_f = 48$

$$\boxed{\Rightarrow y_f = -15}$$

arc (f, a): $y_a - y_f = 56$

$$\boxed{\Rightarrow y_a = 41}$$

arc (a, d): $y_d - y_a = 28$

$$\boxed{= y_d = 69}$$

(y_i) : part of the dual solution

Slack variables:

$$z_{ij} = \begin{cases} 0 & \text{if } (i,j) \text{ belongs to sp. tr.} \\ z_{ij} = y_i - y_j + c_{ij} \end{cases}$$

must be non-negative or else (D) is infeas.

$$z_{be} = y_b - y_e + c_{be} = 33 - 19 + 7 = 21 \quad \checkmark$$

$$z_{ac} = y_a - y_c + c_{ac} = 41 - 98 + 48 = -9 \quad \times$$

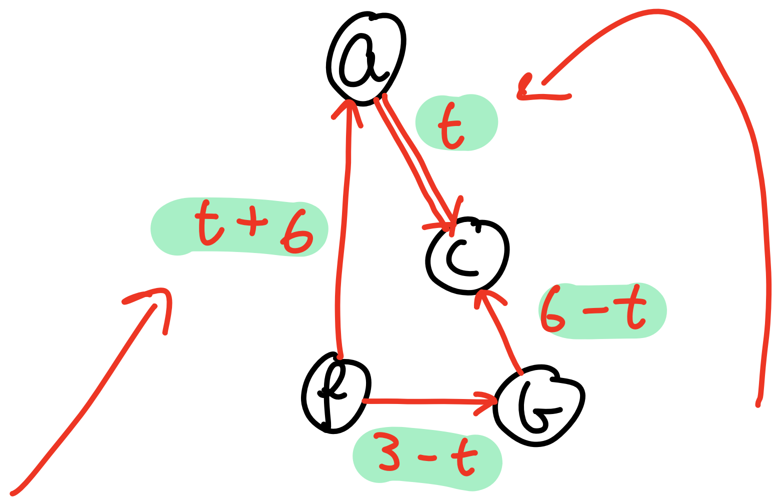
Current spanning tree is not optimal since (D) is not feasible!!

Pivot step in the network SM

Choose an infeasible dual slack variable representing an NBV arc which will become our new BV.

New BV: we choose $z_{ac} = -9$ (the "most" infeasible NBV arc.

New NBV: assign a positive value $t > 0$ to the new BV (a,c)
$\Rightarrow$ balance constr. along the cycle $(a,c)-(c,b)-(f,b)-(f,a)$ change accordingly.



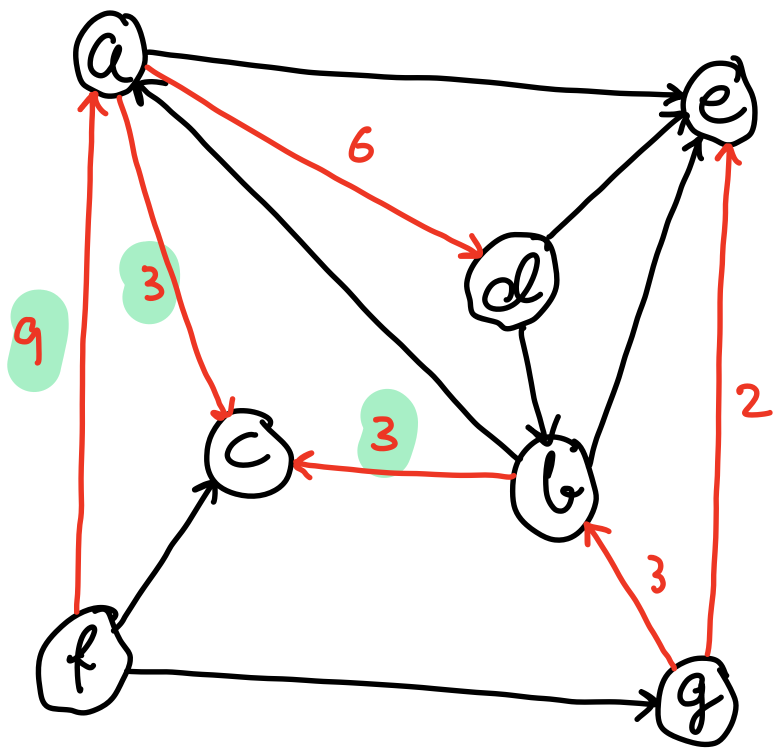
A new t goes out of a so a new additional t must go in.

How big can t be without breaking the balance constraints?

Answer: 3

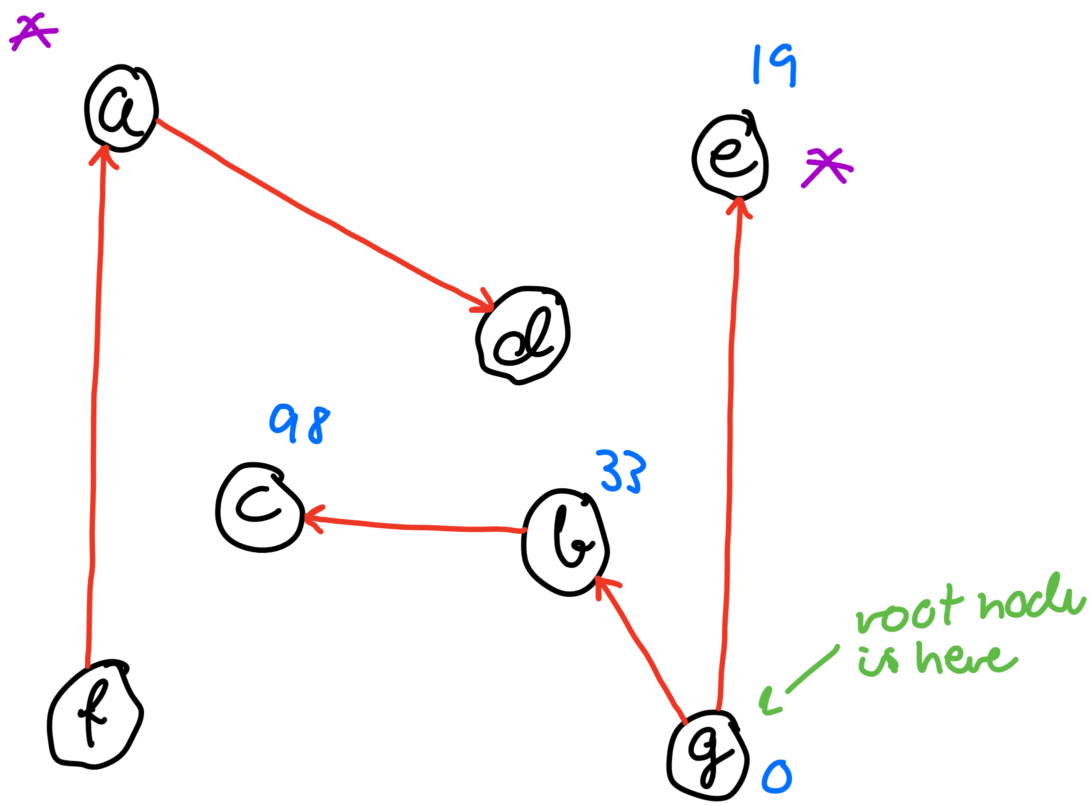
$x_{fG} = 0$ for $t = 3 \Rightarrow (f, G)$ leaves the basis.

New solution:



Optimal?

check for feas. of (D).



Deleted the leaving arc (f,b).

Do not include the entering arc (a,c). We obtain two disjoint sets.*

How to update the dual variables?

calc. starts with the root node

$\Rightarrow$ the dual solutions in the part containing the root node will <u>not</u> change (c, b, g, e).

Cac

(a, c) : $y_c - y_a = 48$

$\Rightarrow \boxed{y_a = 50}$

(f, a) : $y_a - y_f = 56$

$\boxed{\Rightarrow y_f = -6}$

(a, d) : $y_d - y_a = 28$

$\boxed{\Rightarrow y_d = 78}$

Previous dual solution:

$$\left.\begin{aligned} y_a &= 41 \\ y_f &= -15 \\ y_d &= 69 \end{aligned}\right\} + 9 \text{ gives us the new values.}$$

<u>Rule:</u>

We choose the infeasible dual slack variable $z_{ac} = -9$ $\Rightarrow$ (a,c) became new BV $\Rightarrow$ New value of z_{ac} after this iteration becomes zero.

After iteration: $0 = y_a - y_c + c_{ac}$

Before iteration: $-9 = \tilde{y}_a - \tilde{y}_c + c_{ac}$

belongs to root node.

$$\boxed{y_a = \tilde{y}_a + 9}$$

Analogously for all nodes not belonging to the root node part (a, f, d).

Slack variables: (for the NBV arcs)

$$z_{ae} - y_a + y_e = c_{ae}$$

$\tilde{y}_a + q$ not changing

$$z_{ae} = \tilde{z}_{ae} + q$$

41 32

Optimal?

No, since (a,b) in the new sol. $= -10$

Do two more iterations to get
the optimal sp. tree.