Lecture 1 (25.8)

General opt. problem

n # of variables

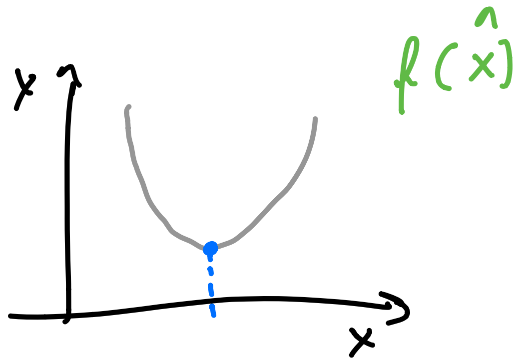
For a given function $f : \mathbb{R}^n \to \mathbb{R}$ and a given set M from $\mathbb{R}^n$ find $\hat{x} \in M$ which minimizes f on M, that is, $\hat{x}$ satisfies $f(\hat{x}) \leq f(x)$, $\forall x \in M$.

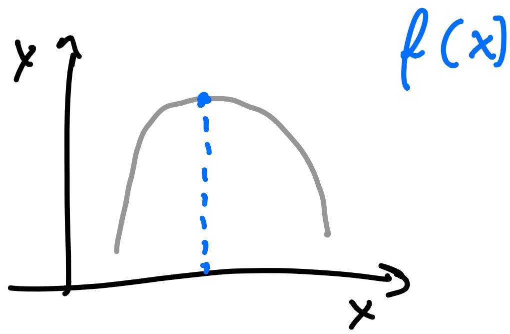
$$\min.\ f(x) \quad \text{s.t.}\quad x \in M$$

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ f = objective function

M = feasible set $\hat{x}$ = minimizer of f on M.

Any $x \in M$ is called a feasible set





If we can solve a min problem we can also solve the corresponding max problem.

Def.

A linear function is a function $f: \mathbb{R}^n \to \mathbb{R}^m$ iff:

$$\left.\begin{aligned} f(x+y) &= f(x) + f(y) & \forall (x,y) \in \mathbb{R}^n \\ f(\alpha x) &= \alpha f(x) & \forall \alpha \in \mathbb{R} \end{aligned}\right\}$$

alpha

Linear or not?

$f(x) = x$ Yes! $f(x) = 1$ No!

$f(x) = 3x_1 + 5x_2$ Yes!

symbols

x = column vector

x^T = row vector (x transposed)

A = (m, n) matrix

A^T = (n, m) matrix

Def. Linear program (LP)

$$\max.\ c^T x \quad \leftarrow c_1x_1 + c_2x_2 + \dots c_nx_n$$

s.t.

$$a_{i1}x + a_{i2}x_2 + \cdots a_{in}x_n \begin{Bmatrix} \le \\ \ge \\ = \end{Bmatrix} b_i$$

In addition some (or all) components x_i, $i \in I$ where $I \subset \{1 \dots n\}$ is an index set, may be restricted by sign constraints.

$$x_i \begin{Bmatrix} \le \\ \ge \end{Bmatrix} 0,\ i \in I$$

Free variables

Variables x_i, $i \notin I$ are called free variables.

Example

$$\max \quad -5x_1 + x_2 + 2x_3 + 2x_4$$

$$\begin{aligned} \text{s.t.} \quad & x_2 - 4x_3 && \le 5 \\ & 2x_1 + x_2 - x_3 - x_4 && \ge -3 \\ & x_1 + 3x_2 - 4x_3 - x_4 && = -1 \\ & x_3, x_1 \le 0 \quad x_2 \ge 0 \quad x_4 \ge 0 \end{aligned}$$

From the example above we see that:

$x \in \mathbb{R}^4$ ← 4 variables in the o.f.

$c = \begin{pmatrix} -5 \\ 1 \\ 2 \\ 2 \end{pmatrix}$ ← all coefficients of the vars.

lhs

rhs

$A = \begin{pmatrix} 0 & 1 & -4 & 0 \\ 2 & 1 & -1 & -1 \\ 1 & 3 & -4 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$

x_3 = free variable ← no sign constraint

$I = \{1, 2, 4\}$?

- Existence of feasible points
- Existence of solution
- Uniqueness of solution
- How does an optimal solution depend on problem data.
- Which alg. can be used for its sol.
  - Properties of these algorithms.