Lecture 20

Connected

Cyclic/acyclic

Tree : connected, acyclic.

Spanning tree

$(\tilde{N}, \tilde{A})$ is called a subnetwork of a given network A if $\tilde{N} \subset N$ and $\tilde{A} \subset A$. A subnetwork $(\tilde{N}, \tilde{A})$ is called spanning tree if it is a tree and $\hat{N} = N$.

- Constraints : $Ax = -b$, $x \geq 0$

Vector of suppl./demand

Graph in slide

- A solution of (1) is called a balanced flow (no corruption).

- A solution of (1) and (2) is called a feasible flow.

- Given a spanning tree, a balanced flow is called a tree solution if the balanced flow coeff. for any arc not belonging to this tree is zero.

Property : A tree with m nodes has $(m-1)$ arcs

- Coeff. matrix A : each column consist of one 1, one (-1) and the rest zero.

$\Rightarrow$ sum of all rows of A is a zero vector.

consequence of this is that A does not have rank m (different from a standard LP.

We cannot choose nonsingular (m,m)-matrix B (as in standard LP played the role of a basis matrix).

Tree property: $\operatorname{rank}(A) = m-1$, that is, one balanced constraint is redundant (we can delete it).

How to define basis/non-basis for a network flow problem? (needed for the SM).

7 nodes and 6 arcs in the example.

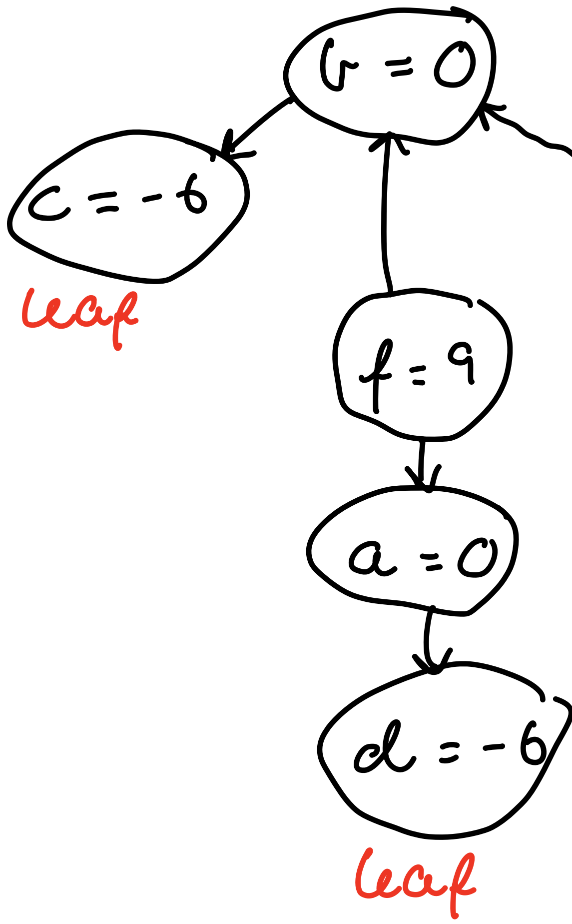
Each arc connects two nodes.

$\Rightarrow$ there must be at least two nodes with only one arc connection.

leaf nodes

Start at one of the leaf nodes to find a tree solution.

(from example in slide)



Getting from the leaf nodes:

c $X_{bc} = 6$ fulfill demand at c

d $\quad x_{ad} = 6$ $\qquad$ d

e $\quad x_{ge} = 2$ $\qquad$ e

a $\quad x_{fa} - x_{ad} = 0$

$x_{fa} = 6$

f $\quad x_{fb} + x_{fa} = 9$

$x_{fb} = 3$

b $\quad x_{gb} + x_{fb} - x_{bc} = 0$

$x_{gb} = 3$

Now we have fulfilled all balance constraints.

But we haven't considered the g node. It is redundant so we don't need to consider it.

Information we get from g is redundant. One can choose any internal node as redundant.

$$\left.\begin{aligned} X_{bc} &= 6 \\ X_{ge} &= 2 \\ X_{ad} &= 6 \\ X_{fa} &= 6 \\ X_{fb} &= 3 \\ X_{gb} &= 3 \end{aligned}\right\} \text{feasible solution}$$

Is it optimal ?

Coeff. matrix after deleting the g constraint is the following (after a useful permutation):

arcs belonging to the sp. tree

$$
\begin{array}{c|cccccc}
 & X_{ad} & X_{fa} & X_{fb} & X_{bc} & X_{gb} & X_{ge} \\
\hline
d & 1 & & & & & \\
a & -1 & 1 & & & & \\
f & & -1 & -1 & & & \\
c & & & & 1 & & \\
b & & & 1 & -1 & 1 & \\
e & & & & & & 1
\end{array}
$$

$\tilde{A}$

Clearly this is a singular matrix.

q is called the root node.

The remaining $(m-1, m-1)$-matrix is non-singular (all diagonal elements are non-zero).

of this section

The main theorem

An $(m-1, m-1)$-submatrix of the coeff. matrix A is a basis matrix iff. the arcs to which its belong form a spanning tree.

A spanning represents a basis. When we go from vertex to vertex we look at different spanning trees (with different obj. values). Basis variables are represented by arcs in a spanning tree.

Dual solution

(to check for optimality).

$$y_j - y_i + z_{ij} = c_{ij}$$

- If an arc (i,j) belongs to the spanning tree, then
  - x_{ij} is BV for (P)

  $\Rightarrow z_{ij}$ is NBV for (D)

  $\Rightarrow z_{ij} = 0$ (NBV's are always zero).

This implies $y_j - y_i = c_{ij}$

for the m nodes and the

m-1 arcs for the spanning tree.

$\Rightarrow$ m-1 equations with m variables.

- y_i is not uniquely determined. If y_i is a solution, then also $y_i + k$ is a solution for any fixed number $k \in \mathbb{R}$.

$\Rightarrow$ fix one comp. y_i $\Rightarrow$ we get $(m-1)$ equation with $(m-1)$ variables.