Lecture 12

Example of a (P)/(D) - problem:
(resource allocation problem)

- m raw materials $i = 1 \dots m$, with market prices $\rho_i = 1 .. m$
- stock quantities for $i \leq b_i$.
- n products j with selling price $\sigma_j = 1 .. n$
- production for one unit of product j requires a_{ij} of raw material i.
- profit per unit j : $\sigma_j - \sum_{i=1}^{m} \rho_i a_{ij}$
- variables : x_j – produced units of j.

New situation

changing market price!

$$w_i = \overset{\text{old price}}{\rho_i} + \overset{\text{new variable}}{y_i}$$

- Producer sets x_j with the goal of profit maximation.
- Supplier of raw material i sets y_i (= price difference per unit of raw material i) with the goal of his profit maximization.
- Producer maximizes while supplier minimizes the producers profit.

Value of stock of raw material i after production.

$$w_i \Big(b_i - \sum_j a_{ij} x_j \Big)$$

new market price

originally in stock

needed for production

Profit calculation

$$\sum_j p_j x_j + \sum_i w_i \Big(b_i - \sum a_{ij} x_j \Big) - \sum g_i b_i =$$

selling price

substitute chance in market price.

$w_i = g_i + y_i$

get this!

$$\sum_j s_j x_j + \sum_i \left(\cancel{s_i b_i} - s_i \sum_j a_{ij} x_j \right) +$$

$$\sum_i y_i \left(b_i - \sum_j a_{ij} x_j \right) - \cancel{\sum_i s_i b_i}$$

can simplify

(we defined : $c_j = s_j - \sum_i a_{ij} s_j$)

we call it this

$$\sum_j c_j x_j + \sum_i y_i \left(b_i - \sum_j a_{ij} x_j \right) = \pi(x, y) \quad \times \text{①}$$

This is the same expression (taking c_j into play).

now, putting together the x_j's we get the following

$$\sum_j \left(c_j - \sum_i a_{ij} y_i\right) x_j + \sum_i y_i b_i \quad (2)$$

Original problem

general form.

(P) $\max\ c^T x$ s.t. $Ax \le b$, $x \ge 0$

In our example we then get:

$$\max_j \sum_j c_j x_j \quad \text{s.t.} \sum_j a_{ij} x_j \le b_i, \ i = 1..m$$
$$x_j \ge 0$$

(D) $\min\ b^T y$ s.t. $A^T y \ge c$, $y \ge 0$

and in our example:

$$\min \sum_i b_i y_i \quad \text{s.t.} \sum_i a_{ij} y_i \ge c_j, \ y_i \ge 0$$
$$i = 1 \ldots n$$

Return to π:

$$\textcircled{1} \quad \min_{y\geq 0}\left(\sum_j c_j x_j + \sum \overset{\geq 0}{y_i}\overbrace{\left(b_i - \sum_j a_{ij}x_j\right)}^{\geq 0 \text{ if } x \text{ is feas.}}\right)$$

$$= \begin{cases} \sum_j c_j x_j, & \text{if } x \text{ is feas.} \\ -\infty, & \text{else} \end{cases}$$

(P) is equivalent to:

$$\max_{\substack{Ax \leq b \\ x \geq 0}} c^T x = \max_{x \geq 0} \min_{y\geq 0} \sum_j c_j x_j + \sum_i y_i \left(b_i - \sum_j a_{ij} x_j\right)$$

$$\boxed{= \max_{x\geq 0} \min_{y \geq 0} \pi(x,y)}$$

Analogous for ②:

$$\max \sum_i y_i b_i + \sum_j \overbrace{\left(c_j - \sum_i a_{ij} y_i\right)}^{\leq 0 \text{ if } y \text{ feas.}} \overset{\geq 0}{x_j}$$

$$= \begin{cases} \sum y_i b_i, & \text{if } y \text{ is feas.} \\ \infty, & \text{else} \end{cases}$$

(D) is equivalent to:

$$\min b^T y = \min_{y \geq 0} \max_{x \geq 0} \sum_i y_i b_i + \sum_j \left(c_j - \sum_i a_{ij} y_i \right) x_j$$

$$\boxed{= \min_{y \geq 0} \max_{x \geq 0} \pi(x, y)}$$

This is the same number as for (P) above.

From the strong duality theorem.

Optimal obj. func. values!!

Strong Duality Theorem

The optimal obj. func. value of (P) and (D) are the same (if they exist):

$$\max_{x \geq 0} \min_{y \geq 0} \pi(x, y) = \min_{y \geq 0} \max_{x \geq 0} \pi(x, y)$$

"Lagrangian Duality"

The Simplex Method in matrix notation

(LP)

obj. func.

$$\max c^T x = \max \sum_{j=1}^{n} c_j x_j = \max c_1 x_1 + \ldots + c_n x_n$$

constraints:

$$\text{s.t.} \sum_{j=1}^{m} a_{ij} x_j \leq b_i \,, \quad i = 1 \ldots m \,, \quad x_j \geq 0$$

or, without summation:

$$= a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m$$

Slack variables (choose $x_{n+1} = w_1$)

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \; , \quad i = 1 \ldots m$$

Rewritten as :

$$\sum_{j=1}^{m} a_{ij} x_j + x_n + i = b_i \; , \quad i = 1 \ldots m$$

without sum

$$\begin{bmatrix} a_{11} x_1 + a_{1n} x_n + x_{n+1} & & & = b_1 \\ a_{21} x_1 + a_{2n} x_n & + x_{n+2} & & = b_2 \\ \vdots \qquad \vdots & \vdots & \vdots & \vdots \\ a_{m1} x_1 + a_{mn} x_n & & + x_{n+m} & = b_m \end{bmatrix}$$

Rewrite our original (LP) with equality constraints

$$\max \sum c_j x_j \quad \text{s.t.} \quad Ax = b, \; x \geq 0$$

with:

Identity matrix

$$\overset{*}{A} = m\left[\overbrace{\left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & 0 & 0 \\ a_{21} & \cdots & a_{2n} & 0 & 1 & 0 \\ \vdots & & \vdots & & & \\ a_{m1} & \cdots & a_{mn} & 0 & 0 & 1 \end{array}\right)}^{n+m}\right.$$

This is now $\mathbb{R}^{n+m}$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+m} \end{pmatrix}$$

$\mathbb{R}^m$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \qquad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$\mathbb{R}^{n+m}$

* This is <u>not</u> the same A as before

(LP) $\max\ c^T x$ s.t. $Ax = b,\ x \geq 0$

- Rank $A = m$, that is, the m rows are linearly independent
- Each dict. represents a split of $x \in \mathbb{R}^{n+m}$ into a subvector $x_B \in \mathbb{R}^m$ consisting of BV's and subvector $x_N \in \mathbb{R}^n$ " — NBV's.
- B = set of col. indices belonging to the BV's.
- N —— " —— NBV's.

Dictionary:

The previous value

$$\zeta = \bar{\zeta} + \sum_j \bar{c}_j (x_N)_j$$

$$(x_B)_i = \bar{b}_i - \sum_j \bar{a}_{ij} (x_N)_j$$

Each BV is a linear comb. of the NBV's and a constant ($\bar{b}_i$).

For a particular point we can rearrange the order of coeff. of. x such that:

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} \quad \begin{array}{l} - \; x_B \in \mathbb{R}^n \; (BV) \\ \; x_N \in \mathbb{R}^m \; (NBV) \end{array}$$

Analogously split A and $c \in \mathbb{R}^{n+m}$

$$A = \left(\begin{array}{c:c} B & N \end{array}\right) \quad \begin{array}{cc} \underbrace{\qquad}_{m} & \underbrace{\qquad}_{n} \end{array}, \qquad c = \begin{pmatrix} c_B \\ c_N \end{pmatrix} \begin{array}{l} \in \mathbb{R}^m \\ \in \mathbb{R}^n \end{array}$$

B : (m,m)-matrix

N : (m,n)-matrix

We can rewrite our constraints as:

$$Ax = (B \ N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} = Bx_B + Nx_N = b$$

(LP) at a fixed point:

$$
\begin{aligned}
\max \quad & c_B^T x_B + c_N^T x_N \\
\text{s.t.} \quad & B x_B + N x_N = b \\
& x_B \geq 0 \\
& x_N \geq 0
\end{aligned}
$$

At the inital vertex we get:

$B = I^m$ Identity matrix with m rows

rows and cols of I^m are linearly independent.

In general: matrix belonging to BV has maximal rank (non-singular)

$\Longrightarrow$ The inverse matrix B^{-1} exists (uniquely determined).

$$B \cdot B^{-1} = B^{-1} B = I^{m}$$