Lecture 15

(from last time)

$$x_B^* = \begin{pmatrix} x_1^* \\ x_4^* \\ x_5^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

Feasible in (P), but infeasible in (D), so we are not yet optimal!

$$z_N^* = \begin{pmatrix} z_3^* \\ z_2^* \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$$

B and N:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$$

Obtained mutually by neg. transp. coeff. matrix.

Dictionary of (P) $\quad$ Dictionary of (D)

"symmetry" between (P) and (D).

$$\zeta = \ldots\ldots\ldots$$

$$x_B = B^{-1}b - (B^{-1}N)x_N$$

$$-\xi = \ldots\ldots\ldots\ldots$$

$$z_N = (B^{-1}N)^T c_B - c_N + (B^{-1}N)^T z_B$$

$\boxed{\text{(A)} \quad -B^{-1}N \text{ neg. transposed of } (B^{-1}N)^T}$

<u>On the other hand</u>

Extended coeff. matr. of (P):

$$\begin{array}{c|cc} & n & m \\ \hline m & \multicolumn{2}{c}{\begin{pmatrix} A & I^m \end{pmatrix}} \end{array} \quad \text{— } (m, n+m)\text{-matrix}$$

———————— 〃 ———————— (D):

$$\begin{array}{c|cc} & m & n \\ \hline n & \multicolumn{2}{c}{\begin{pmatrix} A^T & I^n \end{pmatrix}} \end{array} \quad \text{— } (n, m+n)\text{-matrix}$$

This is <u>not</u> a pair of transposed matrices.

$$(A \ I^m)^T = \begin{pmatrix} A^T \\ I^m \end{pmatrix}$$

This is obviously not the matrix in (D).

- Basis matrix B of (P) is an (m,n)-matrix (we have m BVs):

$$(A \ I^m) = {}^{m}\Big(\underset{m}{B} \quad \underset{n}{N}\Big)$$

- Basis matrix B_D of (D) is an (n,n)-matrix (we have n BVs):

$$(A^T \ I^n) = {}^{n}\Big(\underset{n}{B_D} \quad \underset{m}{N_D}\Big)$$

Altogether:

- Extended coeff. matr. of (P) and (D) are NOT neg. transposed.
- B and B_D have different dimensions

BUT! (Δ) holds (also for B_D and N_D).
Why is that?

$$(P) \quad \max\ c^T x \ \text{ s.t. } Ax + w = b, \quad x \geq 0, w \geq 0$$

$$(D) \quad \min\ b^T y \ \text{ s.t. } -z + A^T w = c, \quad z \geq 0, y \geq 0$$

coeff. matr.: (P) $\bar{A} = (A \;\; I^m)$

(D) $\hat{A} = (-I^n \;\; A^T)$

$$\bar{A} = (\bar{N} \;\; \bar{B})$$

$$\hat{A} = (\hat{B} \;\; \hat{N})$$

In (P) dict.: $\bar{B}^{-1}\bar{N}$

In (D) dict.: $\hat{B}^{-1}\hat{N}$

To be shown (in order to get Ⓐ):

$$\boxed{\bar{B}^{-1}\bar{N} = -(\hat{B}^{-1}\hat{N})^T}$$

$$\bar{A}\hat{A}^T = \left(\underset{n}{\bar{N}}\ \underset{m}{\bar{B}}\right) \begin{array}{c} \\ n \\ m \\ \end{array}\!\!\begin{pmatrix} \hat{B}^T \\ \hat{N}^T \end{pmatrix}_{n} = \bar{N}\hat{B}^T + \bar{B}\hat{N}^T$$

$$\underbrace{(m,n)(n,n)}_{(m,n)} + \underbrace{(m,m)(m,n)}_{(m,n)}$$

equally

$$\bar{A}\hat{A}^T = (A\ I^m)\begin{pmatrix} -I^n \\ A \end{pmatrix} = -A + A = 0$$

$$\bar{N}\hat{B}^T + \bar{B}\hat{N}^T = 0$$

| $\cdot (\hat{B}^T)^{-1}$ from the right

| $\cdot \bar{B}^{-1}$ —— " —— left

$$\bar{B}^{-1}\bar{N} = -\hat{N}^T(\hat{B}^T)^{-1}$$

$$= \boxed{-(\hat{B}^{-1}\hat{N})^T}$$

This is what we wanted to show.

Remember:

$$(CD)^T = D^T C^T$$

Notes on sensitivity and parametric analysis (ch. 7)

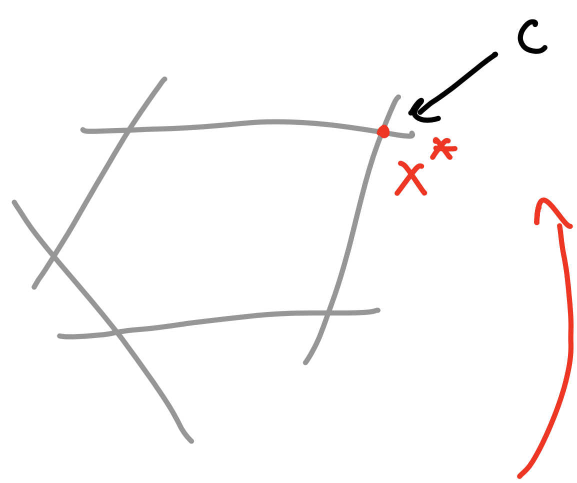
Why sensitivity (post-optimality) analysis:

LP has:

- Uncertain/stochastic data
- Perturbed data (noise)
- Can have continously changing data
- Uncertain/unexact data

- Need to consider certain scenarios
- Inexact calculations

After getting a solution: what would happen to this solution if data (A, b, c) will be changed a little bit?



How much can we change/perturb c such that x^* is still a solution?

Suppose that we have an optimal solution:

$$\frac{\varphi = \bar{\varphi} - z_N^{*} x_N}{x_B = x_B^{*} - B^{-1} N x_N}$$

- Does the optimality of the partition of B, N change for certain perturbations for (A, b, c)?
- How far can we change c (to $\tilde{c}$) such that the partition B, N still represents the optimal solution?

Change of c to $\tilde{c}$

$$\left.\begin{array}{ll} x_B^* = B^{-1}b & \text{— remain unchanged} \\ z_N^* = (B^{-1}N)^T \underline{c_B} - \underline{c_N} & \text{— changing} \\ y^* = \underline{c_B^T} B^{-1} b & \text{— changing} \end{array}\right\} *$$

$$\text{If } \overbrace{\tilde{z}_N^* = (B^{-1}N)^T \tilde{c}_B - \tilde{c}_N \geq 0}^{2*}$$

$\Rightarrow x_B^*$ and $\tilde{z}_N^*$ are still optimal

note that this is not changing (no ~).

If 2* is <u>not</u> fulfilled (not feas.) then a pivot step is necessary with a starting point $(x_B^*, \tilde{z}_N^*)$.

Choosing the perturbation of a currently opt. solution as a starting point for a slightly perturbed problem (here, $c \to c^*$) is called "warm start".

In many occasions warm start delivers a solution faster than a procedure with a standard solution.

Change of b to $\tilde{b}$

$(*)$: x_B^* and y^* — changing

$-z_N^*$ — remains unchanged

feasible for (D) $\longrightarrow$ warm start with dual SM.

General change: $(A, b, c) \to (\tilde{A}, \tilde{b}, \tilde{c})$: everything changes in $(*)$, then perhaps Phase I is necessary.

However: warm start gave much better results than choosing std. starting vertices (e.g. origin).

What does $c \to \tilde{c}$ mean exactly?

change c to $c + t \cdot \Delta c$

perturbation

How big can t be chosen?

(perturbation of c)

$\xrightarrow{(*)}$ x_B remains unchanged

$$z_N^* = (B^{-1}N)^T c_B - c_N$$

After perturbation:

$$\tilde{z}_N = (B^{-1}N)^T (c_B + t\Delta c_B) - (c_N + t\Delta c_N)$$

Change in z_N:

$$t\Delta z_N = t\left[(B^{-1}N)^T \Delta c_B - \Delta c_N\right]$$

$\longrightarrow$ Current dual solution remain feasible (and therefore optimal as long as $\tilde{z}_N \geq 0$).

Example:

$$\begin{array}{llllll}
\max & 5x_1 & + 4x_2 & + 3x_3 & & \\
\text{s.t.} & 2x_1 & + 3x_2 & + x_3 & \leq & 5 \\
 & 4x_1 & + x_2 & + 2x_3 & \leq & 11 \\
 & 3x_1 & + 4x_2 & + 2x_3 & \leq & 8 \\
 & x_1, & x_2, & x_3 & \geq & 0
\end{array}$$

Optimal dict.:

$$\begin{array}{l}
\zeta = 13 - 3x_2 - x_4 - x_6 \\
\hline
x_3 = 1 + x_2 + 3x_4 - 2x_6 \\
x_1 = 2 - 2x_2 - 2x_4 + x_6 \\
x_5 = 1 + 5x_2 + 2x_4
\end{array}$$

$\mathcal{B} = \{3, 1, 5\}$, $\mathcal{N} = \{2, 4, 6\}$

$c^T = (5 \; 4 \; 3 \; 0 \; 0 \; 0)$