Magnetization dynamics of domain walls and skyrmions in micromagnetics

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2.1. Anisotropic energy

Magnetic materials may be anisotropic, meaning that certain directions of the magnetization in the material are more energetically favorable than others. This stands in contrast to isotropic materials where the energy is independent of the direction of the magnetization, assuming uniform magnetization. There are several possible causes for anisotropy in a material, the most common one being the crystalline structure in the material. One of the most important causes of magnetocrystalline anisotropy is the spin–orbit interaction. This interaction arises from the fact that in the restframe of the electron, the proton is orbiting the electron, thereby creating a varying electric field. Since the electric field is varying, Maxwell's equations say that there will also be a corresponding magnetic field. This magnetic field is proportional to the angular momentum $\vec{L}$ of the electron. The electron also has a magnetic moment $\vec{\mu}$, proportional to its spin $\vec{S}$, that will interact with the magnetic field. The energy is lowest when the magnetic moment and field are aligned, which we will see when discussing the Zeeman energy later in the text. Combining the spin–orbit interaction with the orientation of the atoms in the lattice it is possible that some directions in the material are magnetized easier than others. This is magnetocrystalline anisotropy. The shape of the magnetic particles may also cause anistropy in the material, which will be briefly discussed in the section on the demagnetization energy.

In anisotropic magnetic materials one defines different axes; the easy, intermediate and hard axes. If the magnetization is oriented along the easy axis, the anisotropic energy is at its minimum value. Similarly, if the magnetization is oriented along the hard axis the anisotropic energy is at its maximum value. The intermediate axis is the direction along which there is a saddlepoint in the anisotropic energy. In most materials the anisotropic energy is the same for both possible orientations of the magnetization along the axes. When this is the case, the terms in the anisotropic energy can only contain even powers of the components in the magnetization vector.

2.1.1. Uniaxial anisotropy

The simplest form of anisotropy is the case where there is only one axis that the anisotropic energy varies with. This is known as uniaxial anisotropy. This can often be found in crystal structures with a single principal axis, such as the hexagonal and simple tetragonal lattices. If we consider a uniform magnetization $\vec{M} = [M_x, M_y, M_z]$ and let one of the directions along the axis in the material that the anisotropic energy varies be the unit vector $\hat{n}$, the anisotropic energy must be on the form

$$E_A = V \sum_{i=0}^{\infty} K_i \left(\frac{\vec{M} \cdot \hat{n}}{M_s}\right)^{2i} \tag{2.1}$$

if there is to be no preferential direction with respect to energy along the axis. Here V is the volume, $M_s = |\vec{M}|$ the saturation magnetization and K_i are the anisotropy constants. By only considering the lowest order in the anisotropic energy it becomes

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$$E_A = KV\left(\frac{\vec{M} \cdot \hat{n}}{M_s}\right)^2 \qquad (2.2)$$

by ignoring the constant energy term as that only leads to a constant shift in the energy. If $K < 0$ the axis that $\hat{n}$ points along is then the easy axis, and if $K > 0$ the axis is the hard axis. Note that if we have a non-uniform magnetization we let

$$\vec{M} \to \vec{M}(\vec{r}),$$
$$V \to \int \mathrm{d}V.$$

2.1.2. Cubic anisotropy

Many magnetic materials, such as iron, have cubic crystal structures. This can give rise to cubic anisotropy in the magnetic material. To describe cubic anisotropy it can be useful to introduce directional cosines that give the component of a unitary vector along the x-, y- and z-axes. Using spherical coordinates, these directional cosines are defined to be $\alpha = \sin\theta\cos\phi$, $\beta = \sin\theta\sin\phi$, $\gamma = \cos\theta$. Note that the directional cosines can also be written in terms of the magnetization, so that $\alpha = M_x/M_s$, $\beta = M_y/M_s$ and $\gamma = M_z/M_s$. If there is to be no preferential direction with respect to energy along the crystal axes, the anisotropic energy can only contain even powers of α, β and γ. In addition, to have cubic symmetry in the anisotropic energy, the energy must be invariant under any interchanges among the directional cosines [1]. The first term in the anisotropic energy that depends on the directional cosines will therefore be proportional to $\alpha^2 + \beta^2 + \gamma^2$, but using the definitions of the directional cosines we find that

$$\alpha^2 + \beta^2 + \gamma^2 = \frac{M_x^2 + M_y^2 + M_z^2}{M_s^2} = \frac{|\vec{M}|^2}{|\vec{M}|^2} = 1. \qquad (2.3)$$

This term is therefore just a constant, so to find a term that varies with the directional cosines we must go to fourth order:

$$E_A = E_A^{(0)} + \int (K_1(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2) + \ldots)\,\mathrm{d}V. \qquad (2.4)$$

The sixth order term would be on the form $K_2 V \alpha^2\beta^2\gamma^2$ and so on. If one looks at the form of the energy in (2.4) one can determine the easy and hard axes in the material. The integrand has to be minimized along the easy axes, and maximized along the hard axes. If one only considers the fourth order terms in the energy, we have to minimize and maximize the function $K_1(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)$ with the constraint given by (2.3). Using Lagrange multipliers, this breaks down to solving

$$\nabla f(\alpha, \beta, \gamma) = \lambda \nabla g(\alpha, \beta, \gamma), \qquad (2.5)$$
$$g(\alpha, \beta, \gamma) = 0, \qquad (2.6)$$

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with

$$f(\alpha, \beta, \gamma) = K_1(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2), \tag{2.7}$$

$$g(\alpha, \beta, \gamma) = \alpha^2 + \beta^2 + \gamma^2 - 1, \tag{2.8}$$

$$\nabla = \left[\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}\right]. \tag{2.9}$$

This gives us the following set of equations to solve:

$$2K_1\alpha(\beta^2 + \gamma^2) = 2\alpha\lambda,$$
$$2K_1\beta(\alpha^2 + \gamma^2) = 2\beta\lambda,$$
$$2K_1\gamma(\alpha^2 + \beta^2) = 2\gamma\lambda,$$
$$\alpha^2 + \beta^2 + \gamma^2 - 1 = 0.$$

One can easily see that one solution of the equations is given by $\lambda = 0$, one of the directional cosines being ± 1 and the other two being 0. Inserting this solution into $f(\alpha, \beta, \gamma)$ we get $f = 0$.

Assuming only one of the directional cosines is 0, for example γ, one gets

$$K_1\beta^2 = \lambda,$$
$$K_1\alpha^2 = \lambda.$$

Using the constraint in (2.3) one finds $\lambda = K_1/4$, and both non-zero directional cosines being $\pm 1/2$. Inserting this solution into $f(\alpha, \beta, \gamma)$ we get $f = K_1/4$.

Assuming none of the directional cosines are 0, one gets

$$K_1(\beta^2 + \gamma^2) = \lambda,$$
$$K_1(\alpha^2 + \gamma^2) = \lambda,$$
$$K_1(\alpha^2 + \beta^2) = \lambda.$$

Using the constraint in (2.3) one finds $\lambda = 2K_1/3$, and each of the directional cosines being $\pm 1/\sqrt{3}$. Inserting this solution into $f(\alpha, \beta, \gamma)$ we get $f = K_1/3$. This means that if $K_1 > 0$ the energy is minimized by the first solution, making the x-, y- and z-axes the easy axes. The hard axes are then given by the third solution which corresponds to the four axes $x = \pm y = \pm z$, $x = \pm y = \mp z$. If $K_1 < 0$ the energy is maximized by the first solution and minimized by the third, so that the easy and hard axes switch from the case when $K_1 > 0$. The second solution denotes a local minimum if $K_1 < 0$ and a local maximum if $K_1 > 0$. The second solution does not give any intermediate axes, as there is no saddle point in the energy.

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2.2. Demagnetization energy

The magnetic dipoles inside a ferromagnet will generate a magnetic field, known as the demagnetizing field. This name stems from the behavior of the field, as it will try to reduce the total magnetic moment in the magnet. This is because a high magnetic moment in the magnet will generate a strong external magnetic field, which costs energy. Magnetic materials consist of magnetic dipoles. If the magnetization is uniform in the magnet there will be no poles inside the magnet, as the positive and negative poles cancel each other out. There will, however, be a positive pole at one surface of the magnet and a negative pole at the opposing surface that do not cancel out. This causes a large total magnetic moment in the magnet, as it is proportional to the length between the poles. To minimize the total magnetic moment and thereby the energy, the demagnetizing field will therefore create magnetic domains, where the different domains have uniform magnetization in different directions. There is a limit to how small the domains can be, as the exchange interaction will begin to dominate on short ranges, which we will discuss later.

The energy in a ferromagnet will depend on the orientation of the magnetic dipoles with respect to the demagnetizing field created by the other magnetic dipoles. This demagnetization energy is given by

$$E_D = -\frac{\mu_0}{2} \int \mathrm{d}^3 r \vec{M}(\vec{r}) \cdot \vec{H}_D(\vec{r}), \tag{2.10}$$

with $\vec{H}_D$ being the demagnetization field. From this equation it looks like the demagnetizing field will attempt to align the magnetization with itself, which is not the behavior that was discussed earlier. One can not make this conclusion yet, however, as the demagnetizing field is at this point an unknown function of the magnetization $\vec{M}$. To determine the demagnetization energy one needs to find the magnetic field generated by a magnetic dipole $\vec{M}$. The vector potential of a magnetic dipole is known to be

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{M} \times \vec{r}}{r^3}. \tag{2.11}$$

The magnetic induction $\vec{B}(\vec{r})$ is defined as

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r}). \tag{2.12}$$

Using the Einstein summation convention and the Levi–Civita tensor ε_{ijk} one can then find the magnetic induction generated by a single magnetic dipole:

$$\begin{aligned}
B_i &= \varepsilon_{ijk} \partial_j A_k \\
&= \varepsilon_{kij} \varepsilon_{klm} \frac{\mu_0}{4\pi} \partial_j \frac{M_l r_m}{r^3} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\mu_0}{4\pi} \left(\frac{M_l \delta_{jm}}{r^3} - \frac{3 M_l r_m r_j}{r^5} \right) \\
&= \frac{\mu_0}{4\pi} \left(\frac{3(\vec{M} \cdot \vec{r}) r_i}{r^5} - \frac{M_i}{r^3} \right).
\end{aligned}$$

This means that the magnetic field strength of the demagnetization field generated by the entire magnet at a position $\vec{r}$ is

$$\vec{H}_D = \frac{1}{4\pi} \int \mathrm{d}^3 r' \left(\frac{3(\vec{M}(\vec{r'}) \cdot (\vec{r} - \vec{r'}))(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^5} - \frac{\vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|^3} \right). \tag{2.13}$$
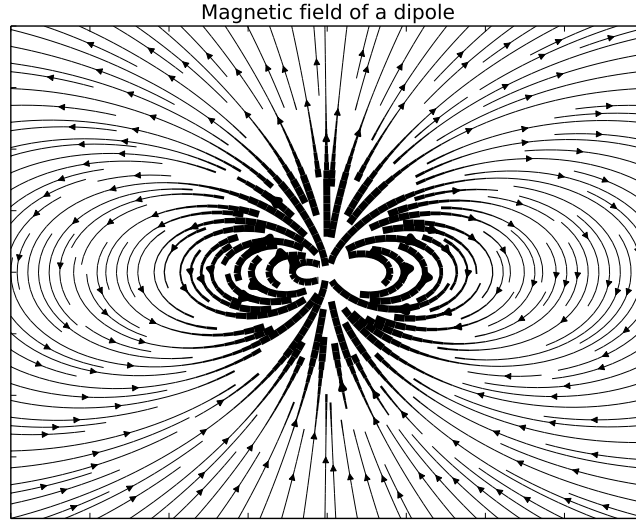
Magnetic field of a dipole

Figure 1. The magnetic field generated by a single magnetic dipole located at the origin pointing in the y-direction. Broad field lines indicate a strong magnetic field.

In Figure 1 one can see that the demagnetizing field has a tendency to point in a different direction than the magnetization. At a position that is perpendicular to the magnetization the field is completely anti-aligned with the magnetization. Inserting (2.13) into (2.10) one finds that the energy is

$$E_D = \frac{\mu_0}{8\pi} \int \mathrm{d}^3 r \int \mathrm{d}^3 r' \left(\frac{\vec{M}(\vec{r}) \cdot \vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|^3} - \frac{3(\vec{M}(\vec{r}) \cdot (\vec{r} - \vec{r'}))(\vec{M}(\vec{r'}) \cdot (\vec{r} - \vec{r'}))}{|\vec{r} - \vec{r'}|^5} \right). \tag{2.14}$$

The energy is small when the total magnetization in the magnet is small, the demagnetizing field will therefore try to reduce the total magnetic moment in the magnet as discussed earlier.

This integrand in (2.14) can be rewritten in a tensor form [2]. Using the relation

$$\partial_i \frac{1}{|\vec{r} - \vec{r'}|} = -\frac{r_i - r_i'}{|\vec{r} - \vec{r'}|^3} \tag{2.15}$$

it is easy to see that

$$\partial_i \partial_j \frac{1}{|\vec{r} - \vec{r'}|} = \partial_i (-\frac{r_j - r'_j}{|\vec{r} - \vec{r'}|^3})$$
$$= -\frac{\delta_{ij}}{|\vec{r} - \vec{r'}|^3} + 3\frac{(r_i - r'_i)(r_j - r'_j)}{|\vec{r} - \vec{r'}|^5}.$$

Defining the continuous demagnetization tensor to be

$$N_{ij}(\vec{r} - \vec{r'}) = -\frac{1}{4\pi}\partial_i \partial_j \frac{1}{|\vec{r} - \vec{r'}|}, \tag{2.16}$$

the demagnetization energy and field can then be written as

$$E_D = \frac{\mu_0}{2} \int d^3r \int d^3r' \vec{M}(\vec{r}) N(\vec{r} - \vec{r'}) \vec{M}(\vec{r'}), \tag{2.17}$$

$$\vec{H}_D(\vec{r}) = - \int d^3r' N(\vec{r} - \vec{r'}) \vec{M}(\vec{r'}). \tag{2.18}$$

Something worth noting is that if the demagnetization tensor $\bar{N}(\vec{r} - \vec{r'})$ is strictly diagonal, the energy can be written as

$$E_D = \mu_0 (N_{xx} M_x^2 + N_{yy} M_y^2 + N_{zz} M_z^2). \tag{2.19}$$

This is the case for ellipsoidal bodies [2]. This form of the demagnetization energy is similar to that of the uniaxial anisotropic energy in (2.2). This means that ellipsoidal magnetic particles cause shape anisotropy, assuming the magnetic particle is not spherical ($N_{xx} = N_{yy} = N_{zz}$).

2.3. Zeeman energy

The Zeeman energy describes the energy from the interaction between the magnetization of the magnet and an external field $\vec{H}_Z$. The Zeeman energy is on the same form as the demagnetization energy in (2.10), with the internally generated demagnetization field replaced by the external magnetic field:

$$E_Z = -\mu_0 \int d^3r \vec{M}(\vec{r}) \cdot \vec{H}_Z(\vec{r}). \tag{2.20}$$

The factor $1/2$ in the demagnetization energy is to account for double counting of the dipole interactions, and is therefore not included in the Zeeman energy as the magnetic moments only interact with an external field. The Zeeman energy is at its minimum when the magnetization is aligned with the external field.

2.4. Exchange energy

Spins on a lattice will have an exchange energy with their nearest neighbors, as given by the Heisenberg Hamiltonian

$$H = -J \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j = -\frac{JS^2}{M_s^2} \sum_{<i,j>} \vec{M}_i \cdot \vec{M}_j, \tag{2.21}$$

with J being the exchange integral and $\vec{S}_{i,j}$ the dimensionless spins. For a ferromagnet the lowest exchange energy is when the spins are aligned, meaning $J > 0$. In the Hamiltonian we sum over each nearest neighbor pair $< i,j >$. To perform the sum we look at the magnetization in the continuum limit, with a slowly varying magnetization. It is then possible to do a Taylor expansion around the magnetization $\vec{M}_i$. If one only looks at a Taylor expansion in the x-direction, it can be written as

$$\vec{M}_{i+1} \approx \vec{M}_i + a\frac{\partial \vec{M}_i}{\partial x} + \frac{a^2}{2}\frac{\partial^2 \vec{M}_i}{\partial^2 x}, \tag{2.22}$$

with a being the lattice constant and $\vec{M}_{i+1}$ the magnetization at the neighboring site to $\vec{M}_i$ along the x-axis. Similar expansions can be done in the y- and z-directions. Using this we can take a look at the contribution to the energy from $\vec{M}_i$ and its nearest neighbors in (2.21). For simplicity we consider a cubic lattice with a lattice constant a. Each lattice site has six nearest neighbors in three dimensions. Applying (2.22) to find the contributions to the energy from the nearest neighbors in the x-direction located at $\pm a\hat{x}$ relative to $\vec{M}_i$, we find

$$\vec{M}_i \cdot \vec{M}_{i-1} + \vec{M}_i \cdot \vec{M}_{i+1} = 2\vec{M}_i \cdot \vec{M}_i + (a-a)\vec{M}_i \cdot \frac{\partial \vec{M}_i}{\partial x} + a^2 \vec{M}_i \cdot \frac{\partial^2 \vec{M}_i}{\partial^2 x}$$
$$= 2M_s^2 + a^2 \vec{M}_i \cdot \frac{\partial^2 \vec{M}_i}{\partial^2 x}.$$

Doing the same calculation for the nearest neighbors in the y- and z-directions we get that the exchange energy density from the magnetization at one site and its nearest neighbors is

$$\epsilon_i = -\frac{JS^2}{a^3}(6 + \frac{a^2}{M_s^2}\vec{M}_i\nabla^2\vec{M}_i). \tag{2.23}$$

To find the total exchange energy in the magnet we integrate the energy density over the volume of the magnet, so that

$$E_E = -\int \mathrm{d}^3 r \frac{JS^2}{2aM_s^2}\vec{M}(\vec{r})\nabla^2\vec{M}(\vec{r}). \tag{2.24}$$

The constant term in the energy has been ignored, and the factor $1/2$ has been introduced to account for the double counting of the interaction pairs. The energy in (2.24) can be

rewritten to a more common form. Introducing the index notation and using the Einstein summation convention, we integrate (2.24) by parts:

$$\int \mathrm{d}^3 r \vec{M}(\vec{r}) \cdot \partial_i \partial_i \vec{M}(\vec{r}) = \int \mathrm{d}S \vec{M}(\vec{r}) \cdot \frac{\partial}{\partial \hat{n}} \vec{M}(\vec{r}) - \int \mathrm{d}^3 r \partial_i \vec{M}(\vec{r}) \cdot \partial_i \vec{M}(\vec{r}). \qquad (2.25)$$

As $\vec{M}(\vec{r})$ is a vector of constant length, any change $\delta \vec{M}(\vec{r})$ must be perpendicular to $\vec{M}(\vec{r})$. This means that $\vec{M}(\vec{r}) \cdot \frac{\partial}{\partial \hat{n}} \vec{M}(\vec{r}) = 0$, and the first term on the right hand side vanishes. The energy in (2.24) then becomes

$$E_E = \int \mathrm{d}^3 r \frac{A}{M_s^2} ((\frac{\partial}{\partial x} \vec{M}(\vec{r}))^2 + (\frac{\partial}{\partial y} \vec{M}(\vec{r}))^2 + (\frac{\partial}{\partial z} \vec{M}(\vec{r}))^2), \qquad (2.26)$$

where we have introduced the exchange constant A. As one can see the energy depends on the variation in the magnetization in all directions, and is independent of which direction the magnetization changes, it is only dependent on the magnitude by which it changes. Because the exchange constant A is positive in a ferromagnet, the exchange energy is higher the more the magnetization varies. This can also easily be seen from (2.21), it costs energy when the spins are not aligned when $J > 0$, which is the case in a ferromagnet. To minimize the exchange energy the magnetization therefore has to vary slowly. On short ranges this effect is much stronger than the demagnetization and magnetocrystalline anisotropy. This is why neighbouring spins have a tendency to align in a ferromagnet, while on longer ranges when the demagnetization effect becomes stronger than the exchange interaction the spins tend to be anti-aligned.

3. Landau–Lifshitz–Gilbert equation

The time evolution of the magnetization in a magnet is described by the Landau–Lifshitz–Gilbert (LLG) equation. To get an understanding of the origin of the equation, it is useful to look at the time evolution of spin, as the magnetic moment and therefore the magnetization is proportional to it.

3.1. Larmor precession

The time evolution of an angular momentum $\vec{L}$ can be described by a variation on the classical Newton's second law for rotation:

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{T}, \qquad (3.1)$$

where $\vec{T}$ is the torque acting on the angular momentum. Spin is often compared to an intrinsic angular momentum of the electron, and will satisfy the same equation, with the exception that the vectors become operators as spin is a quantum mechanical effect. We are interested in looking at the magnetization on a microscopic scale, which would correspond to a semi-classical limit, so the quantum mechanical operators are replaced by their expectation values.

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A magnetic moment $\vec{\mu}$ will precess around an external magnetic field $\vec{H}$. This is known as Larmor precession, and is given by the torque

$$\vec{T} = \vec{\mu} \times \vec{H} = -\gamma \vec{S} \times \vec{H}, \tag{3.2}$$

where we have introduced the gyromagnetic ratio γ. The gyromagnetic ratio gives the ratio of the magnetic moment relative to the spin of a particle. The minus sign is introduced because the magnetic moment and the spin of an electron point in opposite directions. For the electron the gyromagnetic ratio is given by

$$\gamma = \frac{g_e \mu_B}{\hbar}, \tag{3.3}$$

with the g-factor $g_e \approx 2$ and μ_B being the Bohr magneton

$$\mu_B = \frac{e\hbar}{2m_e}. \tag{3.4}$$

By inserting (3.2) into (3.1) and replacing the angular momentum with spin, we get

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = -\gamma \vec{S} \times \vec{H}, \tag{3.5}$$

or by replacing the spin with the magnetization using that $\vec{S}$ is proportional to $\vec{M}$:

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = -\gamma \vec{M} \times \vec{H}. \tag{3.6}$$

3.2. Effective field

If the magnetic moments were free, meaning they did not interact with each other, the field $\vec{H}$ in (3.6) would just be the magnetic field in the material. The magnetic field is the sum of any external field and the internal demagnetizing field discussed earlier. The magnetic moments are not free, however, as we have effects such as exchange interaction between the magnetic moments and magnetic anisotropy. Landau and Lifshitz introduced an effective field $\vec{H}_{eff}$ that replaced the magnetic field $\vec{H}$ in (3.6) that would minimize the energy in the system at equilibrium [3]. The terms in the micromagnetic energy discussed earlier were the anisotropic energy, demagnetization energy, the Zeeman energy and the exchange energy. Using the results derived earlier, we can write the total energy in the system as

$$E(\vec{M}) = \int \mathrm{d}^3 r (\epsilon_A(\vec{M}) + \epsilon_D(\vec{M}) + \epsilon_Z(\vec{M}) + \epsilon_E(\vec{M})), \tag{3.7}$$

where the ϵ's are the energy densities of the different terms. In equilibrium we must require that $\delta E = 0$, with δE being defined by

$$\delta E = \int_a^b \frac{\delta E[\vec{M}]}{\delta \vec{M}(\vec{r})} \cdot \delta \vec{M}(\vec{r}) \, \mathrm{d}\vec{r}. \tag{3.8}$$

The functional derivative $\frac{\delta E[\vec{M}]}{\delta \vec{M}(\vec{r})}$ is just the left hand side of the Euler-Lagrange equation, so that

$$\frac{\delta E[\vec{M}]}{\delta \vec{M}(\vec{r})} = \frac{\partial E[\vec{M}]}{\partial \vec{M}(\vec{r})} - \frac{\partial}{\partial x}\frac{\partial E[\vec{M}]}{\partial(\frac{\partial \vec{M}}{\partial x})} - \frac{\partial}{\partial y}\frac{\partial E[\vec{M}]}{\partial(\frac{\partial \vec{M}}{\partial y})} - \frac{\partial}{\partial z}\frac{\partial E[\vec{M}]}{\partial(\frac{\partial \vec{M}}{\partial z})}. \tag{3.9}$$

As $|\vec{M}| = M_s$ is a constant, $\delta \vec{M}$ must be perpendicular to $\vec{M}$. Because of this, $\frac{\delta E[\vec{M}]}{\delta \vec{M}(\vec{r})}$ must be parallel to $\vec{M}$ for δE to be 0 for arbitrary choices of a and b. In (3.6) one can see that the system is in equilibrium if the effective field $\vec{H}_{eff}$ is parallel to $\vec{M}$, therefore the effective field must also be parallel to $\frac{\delta E[\vec{M}]}{\delta \vec{M}(\vec{r})}$. It is reasonable to assume that one of the terms in the effective field will just be the external magnetic field $\vec{H}_Z$. We can therefore look at ϵ_Z, the energy density in (2.20), to find the proportionality constant. It is easy to see that by letting

$$\vec{H}_{eff} = -\frac{1}{\mu_0}\frac{\delta \epsilon[\vec{M}]}{\delta \vec{M}(\vec{r})}, \tag{3.10}$$

one of the terms in the effective field will be the external field $\vec{H}_Z$ (with $\epsilon = \frac{\partial E}{\partial V}$). By using this definition and the definition of the functional derivative in (3.9), one then finds that the effective field is

$$\vec{H}_{eff}(\vec{r}) = \vec{H}_Z(\vec{r}) + \vec{H}_D(\vec{r}) + \frac{2A}{\mu_0 M_s^2}\nabla^2 \vec{M}(\vec{r}) - \frac{1}{\mu_0}\frac{\delta \epsilon_A[\vec{M}]}{\delta \vec{M}(\vec{r})}. \tag{3.11}$$

The last term depends on what kind of anisotropy there is in the material. For a uniaxial anisotropy as in (2.1), the effective field term becomes

$$-\frac{1}{\mu_0}\frac{\delta \epsilon_A[\vec{M}]}{\delta \vec{M}(\vec{r})} = -\frac{2K_1}{M_s^2}(\vec{M}(\vec{r})\cdot \hat{n})\hat{n} - \frac{4K_2}{M_s^4}(\vec{M}(\vec{r})\cdot \hat{n})^3 \hat{n} - \ldots \tag{3.12}$$

up to fourth order in the directional cosine. For a cubic anisotropy the energy density ϵ_A becomes the integrand in (2.4) instead.

Looking at the effective field in (3.11) we see that it consists of the external magnetic field and the demagnetizing field, as one would expect, and two effective field terms that originate from quantum mechanical effects. The exchange interaction between the magnetic moments cause an effective field that points in the direction that is the average rate of change in the magnetization $\vec{M}(\vec{r})$ at a point $\vec{r}$. If the magnetization is uniform

or varies very slowly in space, the effective field from the exchange interaction is very small or non-existent. If the magnetization varies at a significant rate, the effective field from the exchange interaction will point in a direction that will try to decrease this rate of change, thereby minimizing the exchange energy. If one studies the effective field from the anisotropic energy, one finds that the field will try to align the magnetization with the easy axis, or move the magnetization away from the hard axis, depending on the axes in the material. For a uniaxial anisotropy, the effective field is described by (3.12). We discussed earlier that the direction $\hat{n}$ was parallel to the easy axis if the anisotropy constants K_i were < 0, and it was parallel to the hard axis if the anisotropy constants were > 0. Considering the case of $\hat{n}$ being parallel to te easy axis first, we see that the effective field will point in the direction that the component of $\vec{M}(\vec{r})$ has along the easy axis. This is regardless of which of the two directions the component points in, as we only have odd powers of $(\vec{M}(\vec{r}) \cdot \hat{n})$ in the field. If $\hat{n}$ is parallel to the hard axis, the constants K_i are positive, and the field points in the opposite direction that the component of $\vec{M}(\vec{r})$ has along the hard axis. The effective field then tries to reduce the component along the hard axis, to minimize the anisotropic energy.

3.3. Damping and the Landau–Lifshitz equation

The time evolution in (3.6) only describes the precession of the magnetization around a field $\vec{H}$. This system is not in equilibrium unless the magnetization is parallel to the field. In a physical system the magnetization will eventually relax to equilibrium. To model this, Landau and Lifshitz introduced a damping term that is perpendicular to the magnetization and the direction of the precession around the field [3]. The damping term will point in a direction as to attempt to align the magnetization with the field, (3.6) therefore becomes

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = -\gamma(\vec{M} \times \vec{H}_{eff} + \frac{\alpha}{M_s}\vec{M} \times (\vec{M} \times \vec{H}_{eff})), \tag{3.13}$$

with $\alpha > 0$ being a dimensionless damping constant. This is known as the Landau–Lifshitz equation.

3.4. The Gilbert damping term

The Landau-Lifshitz equation agrees with experiments for low damping constants α, but it does not perform well when the damping constants get large. To improve the model of damping in (3.13), Gilbert proposed a damping term on a different form. By comparing the damping of the magnetization to dampings in other physical systems, Gilbert used a Rayleigh dissipation functional that depended on the time derivative of the magnetization to model the dissipative force [4]. This is analogous to a model for friction where the Rayleigh dissipation functional depends on the time derivative of the position, which is just the velocity. By using this new damping term that is a function of $\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$, the Gilbert form of the Landau–Lifshitz equation becomes

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = -\gamma\vec{M} \times (\vec{H}_{eff} - \eta\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}), \tag{3.14}$$

with η being a damping parameter. This equation is implicit in $\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$. It can be shown that (3.14) and (3.13) are mathematically equivalent by rewriting it to an explicit form in $\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$. To rewrite (3.14) we must find an expression for $\vec{M} \times \frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$ that does not involve a cross-product. To do this we can just take the cross product of (3.14) with $\vec{M}$ from the left:

$$\vec{M} \times \frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = -\gamma \vec{M} \times (\vec{M} \times \vec{H}_{eff}) + \gamma \eta \vec{M} \times (\vec{M} \times \frac{\mathrm{d}\vec{M}}{\mathrm{d}t}). \qquad (3.15)$$

The first term on the right hand side is proportional to one of the terms in (3.13), so we leave that be. The triple cross product involving $\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$ can be rewritten using the relation $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$:

$$\vec{M} \times (\vec{M} \times \frac{\mathrm{d}\vec{M}}{\mathrm{d}t}) = (\vec{M} \cdot \frac{\mathrm{d}\vec{M}}{\mathrm{d}t})\vec{M} - (\vec{M} \cdot \vec{M})\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}. \qquad (3.16)$$

The first term on the right hand side can be found by taking the scalar product of $\vec{M}$ and (3.14). It is easy to see that this is 0, as the entire right hand side of (3.14) is perpendicular to $\vec{M}$. It is also worth noting that when $\vec{M} \cdot \frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = 0$ the modulus $|\vec{M}| = M_s$ is a constant, as

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_s^2) = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{M} \cdot \vec{M}) = 2\vec{M} \cdot \frac{\mathrm{d}\vec{M}}{\mathrm{d}t}. \qquad (3.17)$$

This is in agreement with what we have discussed earlier. The second term on the right hand side in (3.16) is just $-M_s^2 \frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$. Combining the results of (3.15) and (3.16) and inserting it into (3.14), we can simplify the expression to

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = -\frac{\gamma}{1 + \gamma^2 \eta^2 M_s^2}(\vec{M} \times \vec{H}_{eff} + \gamma \eta \vec{M} \times (\vec{M} \times \vec{H}_{eff})). \qquad (3.18)$$

Comparing this to the Landau–Lifshitz equation in (3.13) we see that by letting

$$\eta = \frac{\alpha}{\gamma M_s}, \qquad (3.19)$$

$$\tilde{\gamma} = \frac{\gamma}{1 + \alpha^2}, \qquad (3.20)$$

(3.18) can be written on the form of the Landau–Lifshitz equation:

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = -\tilde{\gamma}(\vec{M} \times \vec{H}_{eff} + \frac{\alpha}{M_s} \vec{M} \times (\vec{M} \times \vec{H}_{eff})). \qquad (3.21)$$

This is the explicit Landau–Lifshitz–Gilbert equation. The difference between the Landau–Lifshitz and the Landau–Lifshitz–Gilbert equation is that the gyromagnetic ratio in the

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LLG equation is redefined to include the dimensionless damping parameter α. When $\alpha \ll 1$, the Landau–Lifshitz equation is a good approximation of the LLG equation, but as the damping parameter becomes larger the correction in the gyromagnetic ratio becomes significant. The LLG equation agrees better with experiments where the damping constant α is large [5].

4. Domain walls

To minimize the energy a ferromagnet of a significant size will divide itself into domains. A domain is a region where the magnetization is uniform, and the domains in a magnet are separated by something called domain walls. These are regions where the magnetization in the material switches orientation between two domains with different direction in the magnetization. There are many different domain wall structures, but what they all have in common is that in the static case they have to satisfy (3.21) with the left hand side being zero. It is easily seen that this is satisfied when the local magnetization is parallel to the effective field $\vec{H}_{eff}$ at that point. From the derivation of the effective field, we remember that $\vec{H}_{eff}$ and $\vec{M}$ were parallel when the system was at an equilibrium, meaning $\delta E = 0$. This will be an easier problem to solve, as we can write the energy in terms of scalar quantities such as the components of the magnetization. To study the nature of the domain walls we will at first only consider internal effects, and not consider the case where an external magnetic field is applied. A static domain wall is therefore a magnetization structure that minimizes the exchange energy, the anisotropic energy and the demagnetization energy simultaneously.

4.1. The charge avoidance principle

In general, it's very hard to determine what configuration in the magnetization that will lead to a minimum in the demagnetization energy. For some magnetic particles we saw that it was possible to treat the demagnetization energy as an effective shape anisotropy, which makes the form of the energy easier to minimize, but this is not always possible. To make this task a little easier, it is useful to introduce magnetic charges. It is important to note that this is purely a mathematical concept, as magnetic monopoles have never been detected. This method is just a comparison to electric charges and fields. The Maxwell equations for the magnetic field in the magnetostatic limit are

$$\nabla \times \vec{H} = 0, \tag{4.1}$$

$$\nabla \cdot \vec{B} = 0. \tag{4.2}$$

Due to the form of the first equation it is possible to write the magnetic field $\vec{H}$ as a gradient of a potential: $\vec{H} = -\nabla U_M$. Using the second equation and the relation between the magnetic induction $\vec{B}$ and magnetic field $\vec{H}$,

$$\vec{B} = \mu_0(\vec{H} + \vec{M}), \tag{4.3}$$

one ends up with the Poisson equation for the magnetic potential U_M:

$$\nabla^2 U_M = \nabla \cdot \vec{M}. \tag{4.4}$$

Comparing this to the Poisson equation for the electric potential U_E,

$$\nabla^2 U_E = -\frac{\rho_f}{\epsilon}, \tag{4.5}$$

we see that $-\nabla \cdot \vec{M}$ behaves as a free magnetic charge density, or a volume charge density. We therefore define

$$\rho_M = -\nabla \cdot \vec{M}. \tag{4.6}$$

At the boundary of a magnet there is a discontinuty in the magnetization $\vec{M}$, as it drops to zero outside the magnet. If we apply the divergence theorem to (4.6) where we integrate over a Gaussian pillbox that is infinitesimally thin and has one surface on the outside of the magnet and one on the inside, we find that

$$\begin{aligned}
\int_\Omega \mathrm{d}V \rho_M &= -\int_\Omega \mathrm{d}V \nabla \cdot \vec{M} \\
&= -\int_{\partial\Omega} \mathrm{d}\vec{S} \cdot \vec{M} \\
&= -A(\vec{M}_{out} \cdot \hat{n} - \vec{M}_{in} \cdot \hat{n}) \\
&= A\vec{M} \cdot \hat{n}.
\end{aligned}$$

Here A is the area of the surfaces of the pillbox parallel to the surface of the magnet, and $\hat{n}$ is the unit vector pointing out of the surface of the magnet. The integral of ρ_M over the volume of the magnet is just the total magnetic charge Q_M, we can therefore define a magnetic surface charge density

$$\sigma_M = \frac{Q_M}{A} = \vec{M} \cdot \hat{n}. \tag{4.7}$$

Both the magnetic volume charge density $\rho_M = -\nabla \cdot \vec{M}$ and surface charge density $\sigma_M = \vec{M} \cdot \hat{n}$ will act as sources in the magnetic potential U_M. To minimize the demagnetizing field $\vec{H}_D$ and thereby the demagnetization energy E_D, it is therefore desireable to avoid these magnetic charges in the material as much as possible. This is the charge avoidance principle [6]. We see that physically this means that it costs energy if the magnetization of a magnet has a component perpendicular to its surface, or the divergence of the magnetization is different from zero inside the magnet.

4.2. Bloch walls

To simplify the calculation of the magnetization structure in domain walls we only consider the exchange and anisotropic energies. This has some merit as we can either assume

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that the demagnetization energy will be on the form of a uniaxial anisotropy, or we can evaluate how the solution does with regards to the charge avoidance principle.

The first case we consider is a 1D chain of magnetic moments along the x-axis, with an easy axis along the z-axis and the magnetization being parallel to the easy axis at $x = \pm\infty$. The energy then can be written as

$$E = \int \mathrm{d}V (\frac{A}{M_s^2}(\frac{\partial}{\partial x}\vec{M}(x))^2 - K\cos^2\theta(x)). \tag{4.8}$$

K is strictly positive in this case. The magnetization vector can be written in terms of spherical coordinates, so that $\vec{M}(x) = M_s(\sin\theta(x)\cos\phi(x), \sin\theta(x)\sin\phi(x), \cos\theta(x))$. We see that if we let the x-component of $\vec{M}$ be zero, there will be no magnetic volume charges as $\nabla \cdot \vec{M}(x) = 0$. We therefore restrict the magnetization to be in the yz-plane by letting $\phi = \pi/2$. The first term in the integral can then be written as

$$\frac{A}{M_s^2}(\frac{\partial}{\partial x}\vec{M}(\vec{r}))^2 = A(\cos\theta\frac{\partial\theta}{\partial x}\hat{y} - \sin\theta\frac{\partial\theta}{\partial x}\hat{z})^2 = A(\frac{\partial\theta}{\partial x})^2. \tag{4.9}$$

The energy then becomes

$$E = \int \mathrm{d}A \int \mathrm{d}x (A(\frac{\partial\theta}{\partial x})^2 - K\cos^2\theta(x)) = \int \mathrm{d}A \int \mathrm{d}x \epsilon(x, \theta, \frac{\partial\theta}{\partial x}). \tag{4.10}$$

For us to have equilibrium, $\delta E = 0$, the integrand ϵ must satisfy the Euler–Lagrange equation:

$$\frac{\partial\epsilon}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\epsilon}{\partial(\frac{\partial\theta}{\partial x})} = 0. \tag{4.11}$$

Plugging in ϵ, we get the second order differential equation

$$K\sin\theta(x)\cos\theta(x) = A\frac{\partial^2\theta}{\partial x^2}. \tag{4.12}$$

From this equation one can see both the domain walls and the domains. The domains are regions where the magnetization is uniform, meaning θ is a constant. It can easily be seen that $\theta = n\pi/2$ ($n = 0, 1, 2, \ldots$) satisfies this equation, but one should note that only the solutions $\theta = n\pi$ are stable, as that is the solution that minimizes the anisotropic energy when we have chosen the easy axis to be in the z-direction. For the domain walls the magnetization is not uniform, meaning we also have to consider the differentiation term. To solve this equation we multiply (4.12) by $\frac{\partial\theta}{\partial x}$,

$$K\sin\theta(x)\cos\theta(x)\frac{\partial\theta}{\partial x} = A\frac{\partial^2\theta}{\partial x^2}\frac{\partial\theta}{\partial x}, \tag{4.13}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}(K\sin^2\theta - A(\frac{\partial\theta}{\partial x})^2) = 0. \tag{4.14}$$

The expression inside the parantheses must therefore be a constant, and to determine what that constant is we have to use some boundary conditions. We want to describe the domain wall between two domains, meaning the domains are located at $x = \pm\infty$. Inside the domains there is no variation in the magnetization, so θ is a constant. The stable value of θ is either 0 or π, as discussed earlier. This means that the expression inside the parantheses is 0. We can then integrate and make the equation a first order differential equation:

$$K\sin^2\theta(x) = A(\frac{\partial\theta(x)}{\partial x})^2$$
$$\implies \frac{\mathrm{d}\theta}{\sin\theta} = \pm\sqrt{\frac{K}{A}}\,\mathrm{d}x.$$

For simplicity we choose the positive solution, which would correspond to $\theta = 0$ at $x = -\infty$. The negative solution will lead to the same answer as the positive, only with the domains switched, so that $\theta = \pi$ at $x = -\infty$. We note that both sides of the equation are dimensionless, and use that information to define a characteristic length δ_{DW} of the system, with

$$\delta_{DW} = \sqrt{\frac{A}{K}}. \tag{4.15}$$

Using that

$$\int\frac{d\theta}{\sin\theta} = \ln\tan\frac{\theta}{2} + C,$$

we find that

$$\ln\tan\frac{\theta}{2} - \ln\tan\frac{\theta_0}{2} = \frac{x - x_0}{\delta_{DW}}. \tag{4.16}$$

To get the final expression for $\theta(x)$, we must apply a last condition. As we want to describe a domain wall, we choose the domains on the different sides of the wall to be different. This is not a necessity, as one can also have two parallel domains separated by a 360° domain wall. We've seen that the stable domains occur for $\theta = 0$ and $\theta = \pi$. It is then reasonable to assume that if the function $\theta(x)$ is smooth, it will take the value $\theta = \pi/2$ at some point in the domain wall. If we let that point be located at $x = X$, we find that

$$\theta(x) = 2\arctan(\exp(\frac{x - X}{\delta_{DW}})). \tag{4.17}$$

Using the relations

$$\cos(2\arctan(x)) = \frac{1 - x^2}{1 + x^2},$$

$$\sin(2\arctan(x)) = \frac{2x}{1 + x^2},$$

it is easy to show that the magnetization components become

$$M_z = -M_s \tanh \frac{x - X}{\delta_{DW}}, \qquad (4.18)$$

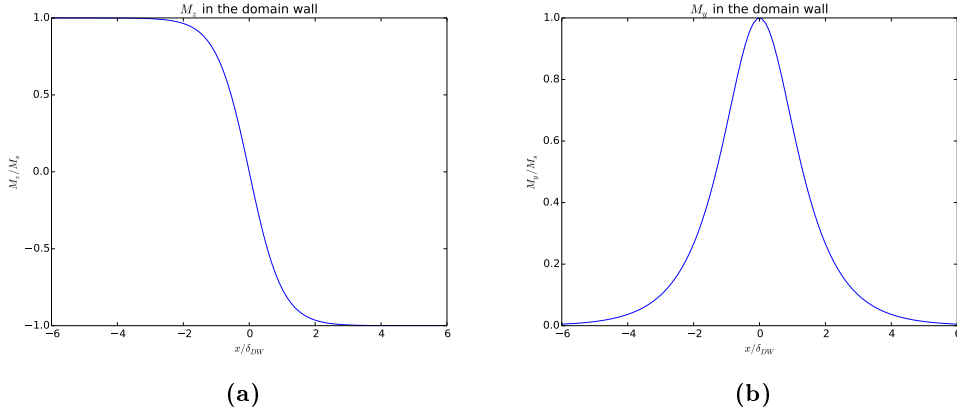$$M_y = \frac{M_s}{\cosh \frac{x - X}{\delta_{DW}}}. \qquad (4.19)$$

(a)   (b)

Figure 2. The magnetization component in the z-direction (a) and in the y-direction (b) inside a Bloch domain wall with $X = 0$.

The magnetization components are shown as a function of x/δ_{DW} in Figure 2. This type of domain wall where the magnetization rotates out of the plane that is parallel to the magnetization in the domains is known as a Bloch wall [7]. In the derivation of the expression for the Bloch wall we chose this direction of rotation as this magnetization structure did not create any magnetic volume charges $-\nabla \cdot \vec{M}$. To minimize the demagnetization energy we want to avoid both magnetic volume and surface charges. We see that surface charges can be created by the Bloch domain wall if the plane the domains are in is parallel and close to the surface of the magnet. This is because the Bloch domain wall rotates out of that plane, giving a component that is normal to the magnet's surface. If the domain wall is far away from a surface, however, there will not be any creation of a surface charge. Bloch domain walls are the most common domain walls in bulk materials. This is because of the thickness of the material makes it more energetically favorable to avoid volume charges than surface charges.

4.3. Néel walls

If the magnetic material has a large surface area, such as thin films, it may be more energetically favorable to avoid surface charges than to avoid volume charges. To have a domain wall that does this, we must require that the magnetization in the domain wall rotates in the plane of the domains, unlike the Bloch domain wall where the magnetization rotates out of that plane. If we look at a similar example as in the derivation of the Bloch domain wall, we must have a magnetization that rotates in the xz-plane rather than the yz-plane, as we let the y-direction be the direction normal to the magnet's surface. The easy axis in the material is kept to be along the z-axis. The energy expressions and the equation for $\theta(x)$ are exactly the same as for the Bloch domain wall, the only difference is that instead of forcing ϕ to be $\pi/2$ to avoid magnetic volume charges, we force ϕ to be 0 to let the M_y component be 0 and thereby avoiding generating magnetic surface charges. The function $\theta(x)$ is therefore exactly the same as for the Bloch domain wall, given by (4.17). The magnetization vector changes, however, from being $\vec{M} = M_s(0, \sin\theta, \cos\theta)$ for the Bloch domain wall to being $\vec{M} = M_s(\sin\theta, 0, \cos\theta)$ for our current domain wall. In other words, the magnetization component in the x- and y-directions just switch. The magnetization therefore looks the same as in Figure 2, with $M_y \to M_x$ and $M_y = 0$. This type of domain wall where the magnetization rotates in the plane of the domain magnetizations is known as a Néel domain wall. This type of domain wall does not generate any surface charges, assuming the domains are parallel to the surface of the magnet, but generates magnetic volume charges as the divergence of the magnetization is not zero. We can calculate the magnetic volume charge that the Néel wall generates:

$$
\begin{aligned}
\rho_M &= -\nabla \cdot \vec{M}(x) \\
&= -M_s \frac{\partial}{\partial x} \sin\theta(x) \\
&= \frac{M_s}{\delta_{DW}} \frac{\tanh \frac{x-X}{\delta_{DW}}}{\cosh \frac{x-X}{\delta_{DW}}}.
\end{aligned}
$$

This magnetic volume charge distribution is shown in Figure 3. It is antisymmetric, and the total magnetic volume charge is zero. This magnetic volume charge therefore acts as a magnetic dipole.

4.4. Domain wall width and energy

To find the width of the domain wall separating the two domains, we need to define where the domain begins, as M_z only approaches $\pm M_s$ asymptotically. One way to do that is to find the value of x where M_z has reached a certain percentage of M_s, and then let the domain wall width be twice that value as the domain wall is antisymmetric around $x = 0$. This can be done by inverting (4.18). We then find that the width of the domain wall Δ_{DW} is

$$
\Delta_{DW} = 2\delta_{DW} \ln \sqrt{\frac{M_s + |M_z|}{M_s - |M_z|}}. \tag{4.20}
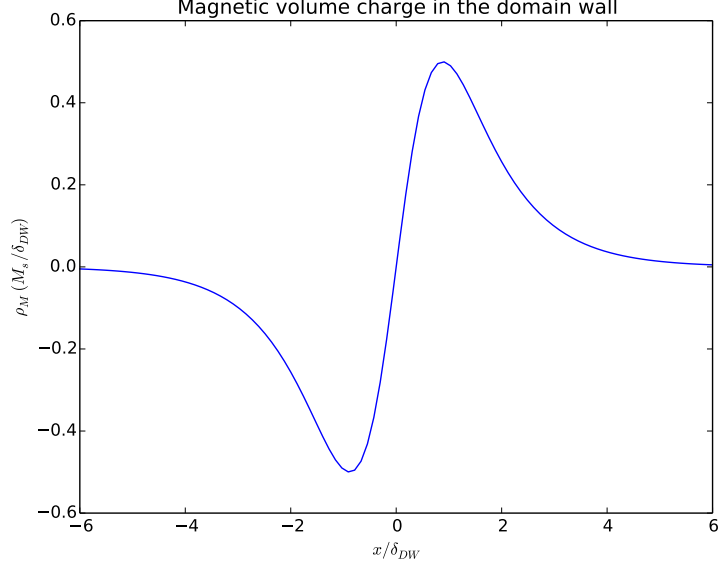$$

Figure 3. The distribution of the magnetic volume charge inside a Néel domain wall.

If we let the domains begin when $|M_z| = 0.99 \cdot M_s$, we get that the domain wall width is $\Delta_{DW} \approx 5.3 \cdot \delta_{DW}$. This is true for both the Bloch domain walls and Néel domain walls we have derived earlier. Remembering that the characteristic length δ_{DW} was defined as $\delta_{DW} = \sqrt{A/K}$, we see that the width of the domain wall is determined by what is the more dominating term in the energy; the exchange interaction or the anisotropy. If the exchange interaction influences the energy much more than the anisotropy, the domain wall becomes very wide, as the magnetization wants to change very slowly to reduce the exchange energy. If the material is isotropic, meaning $K \to 0$, the width of the domain wall becomes infinite. This is because when there is no anisotropy the stable energy state is when all spins are aligned. Anisotropy is therefore necessary for the existence of domain walls in this model where the demagnetization energy is not explicitly accounted for.

Even though domain walls are stable states, they have an energy cost from the exchange interaction and the anisotropic energy compared to uniform domains. This energy cost can be found by first rewriting (4.8) to remove the constant energy in the domains, and then plug in our domain wall solutions to do the integral over the domain wall. The integrand is $-K$ in our domains ($\theta = 0$ and $\theta = \pi$), so we add a constant term of K to the integrand to let the domains have no energy contributions from anisotropy or the exchange interaction. We then integrate to find the energy per domain wall area:

$$\frac{\mathrm{d}E_{DW}}{\mathrm{d}A} = \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{A}{M_s^2} \left(\frac{\partial}{\partial x} \vec{M}(x) \right)^2 - K\cos^2\theta(x) + K \right)$$
$$= \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{A}{\delta_{DW}^2} \frac{1}{\cosh^2 \frac{x}{\delta_{DW}}} + K \frac{1}{\cosh^2 \frac{x}{\delta_{DW}}} \right)$$
$$= 4K\delta_{DW} = 4\sqrt{AK}.$$

5. Domain wall dynamics

Domain walls can be moved by applying an electric current to the magnetic material. This is done either by a momentum transfer or spin transfer between the local electrons and the conduction electrons. A momentum transfer would correspond to actual movement of the electrons in the domain wall, so that the domain wall moves with them, while a spin transfer corresponds to a shift in the local magnetization due to the exchange interaction with the conduction electrons. Here only the domain wall moves, and the position of the electrons in the domain wall is unchanged. For domain walls where the magnetization doesn't change abruptly, so that the spin of the conduction electron can follow the local magnetization more or less adiabatically, the spin transfer effect will be the dominant cause of the domain wall motion [8].

5.1. Spin-transfer torque

The adiabatic spin-transfer torque is a consequence of conservation of angular momentum, or spin. As the spin of the conduction electron follows the variation in the magnetization in the domain wall, there will be a torque acting on the local magnetization in the opposite direction of the rotation of the itinerant spin. This is simply Newton's third law; for the torque acting on the itinerant electron moving through the local magnetization, there will be an equal and opposite torque acting on the local magnetization. When this process is adiabatic the total spin of the local and itinerant electrons is conserved. There is also a non-adiabatic spin-transfer torque, which occurs when the itinerant electron does not move completely adiabatically through the magnet.

To model the spin-transfer torque acting on the local magnetization in the magnet, we mainly follow the derivation by Zhang and Li [9]. The spin density of the itinerant electrons in the applied current can in general be written as

$$\vec{m} = \langle \vec{s} \rangle = \frac{m_0}{M_s} \vec{M} + \delta\vec{m}. \tag{5.1}$$

Here we have split the magnetization into two components, the first of which is parallel to the local magnetization $\vec{M}$, and the second component $\delta\vec{m}$ which is perpendicular to the local magnetization. It is assumed that the component parallel to the local magnetization is much larger than the perpendicular component, as the magnetization of the itinerant electrons follows the local magnetization mostly adiabatically. One should also note that $m_0/M_s \ll 1$ as the itinerant current is not near being saturated. The time evolution of the itinerant spins are given by the continuity equation

$$\frac{\partial \vec{s}}{\partial t} + \nabla \hat{J} = \frac{1}{i\hbar} \left[\vec{s}, H_{sd} \right] - \Gamma(\vec{s}), \tag{5.2}$$

with $\hat{J}$ being the spin current operator and $\Gamma(\vec{s})$ describes the relaxation of the non-adiabatic spins. The s-d interaction Hamiltonian H_{sd} describes the exchange energy between the itinerant spin $\vec{s}$ and the local spin $\vec{S}$,

$$H_{sd} = -J\vec{s} \cdot \vec{S} = -J\vec{s} \cdot \left(-\frac{S}{M_s} \vec{M} \right). \tag{5.3}$$

The commutator in (5.2) can be approximated by

$$\frac{1}{i\hbar} \left[\vec{s}, H_{sd} \right] = -\frac{1}{i\hbar} \left[H_{sd}, \vec{s} \right] \approx -\frac{1}{i\hbar} \sum_{i,j} \frac{\mathrm{d} H_{sd}}{\mathrm{d} s_i} \left[s_i, s_j \right] \tag{5.4}$$

in the semi-classical limit [2]. Using that the dimensionless spins can be written on the form $s_i = \sigma_i/2$, with σ_i being the Pauli matrices, we use the commutation relation between the Pauli matrices to find that

$$[s_i, s_j] = \sum_k i\varepsilon_{ijk} s_k. \tag{5.5}$$

Combining this with $\frac{\mathrm{d} H_{sd}}{\mathrm{d} s_i} = -JS_i$, we get that

$$\frac{1}{i\hbar} \left[\vec{s}, H_{sd} \right] \approx -\frac{1}{i\hbar} \sum_{i,j,k} (-JS_i) i\varepsilon_{ijk} s_k = -\frac{J}{\hbar} \vec{S} \times \vec{s}. \tag{5.6}$$

Using this result, and letting the operators become their expectation values, we get the continuity equation for $\vec{m}$:

$$\frac{\partial \vec{m}}{\partial t} + \nabla \langle \hat{J} \rangle = -\frac{JS}{\hbar M_s} \vec{m} \times \vec{M} - \langle \Gamma(\vec{m}) \rangle. \tag{5.7}$$

The first term on the right hand side is a torque acting on $\vec{m}$ from the interaction between $\vec{m}$ and $\vec{M}$. By Newton's third law there will be an equal and opposite torque acting on the local magnetization $\vec{M}$, so that the spin-transfer torque becomes

$$\vec{T}_{STT} = -\frac{JS}{\hbar M_s} \vec{M} \times \vec{m} = -\frac{JS}{\hbar M_s} \vec{M} \times \delta\vec{m} \tag{5.8}$$

To determine the spin-transfer torque we must find an expression for $\delta\vec{m}$. This is done by solving (5.7). To do that one has to make some basic assumptions. We assume that the relaxation of the non-adiabatic spins is proportional to the non-adiabatic component

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of $\vec{m}$. By letting the average spin-flip relaxation time be defined by τ_{sf}, the last term in
(5.7) becomes $\langle \Gamma(\vec{m}) \rangle \approx \delta\vec{m}/\tau_{sf}$.

The spin current density is a tensor of rank 2, defined by the vectors that describe the
charge flow and the spin of the conduction electrons. As the spin and magnetization of
an electron are anti-parallel, the adiabatic spin current density can then be written as

$$\langle \hat{J}_{\text{adiabatic}} \rangle = -\frac{\mu_B P}{e} \vec{j_e} \otimes \frac{\vec{M}}{M_s}. \tag{5.9}$$

The factor P describes the portion of the conduction electrons that are polarized in the
direction of the local electrons, and e is the elementary charge. The factors μ_B and e
are included to get the correct units. If one assumes a uniform charge current density
$\vec{j_e}$, and that the non-adiabatic spin current is negligible (which is alright if the domain
wall width is much greater than the transport length scale, see [9]), the spin current term
becomes

$$\nabla \langle \hat{J} \rangle = \nabla(-\frac{\mu_B P}{e} \vec{j_e} \otimes \frac{\vec{M}}{M_s}) = -\frac{\mu_B P}{e M_s}(\vec{j_e} \cdot \nabla)\vec{M}. \tag{5.10}$$

Remembering that we have an adiabatic component of $\vec{m}$ that is much greater than the
non-adiabatic component $\delta\vec{m}$, we ignore the term $\frac{\partial \delta\vec{m}}{\partial t}$, which is necessary to obtain an
analytical solution of $\delta\vec{m}$. Combining all of these assumptions and inserting them into
(5.7), the equation becomes

$$\frac{m_0}{M_s}\frac{\partial \vec{M}}{\partial t} - \frac{\mu_B P}{e M_s}(\vec{j_e} \cdot \nabla)\vec{M} = -\frac{1}{\tau_{ex} M_s}\delta\vec{m} \times \vec{M} - \frac{\delta\vec{m}}{\tau_{sf}}, \tag{5.11}$$

where we have introduced the spin exchange relaxation timescale $\tau_{ex} = \hbar/JS$. This
equation can be solved explicitly for $\delta\vec{m}$ in the same manner that we transformed the
LLG equation from the implicit form to the explicit Landau–Lifshitz form. To do that
we took the cross product of the equation with $\vec{M}$ from the left. Doing that, we find that
the term $\delta\vec{m} \times \vec{M}$ can be written linearly in $\delta\vec{m}$ as

$$\delta\vec{m} \times \vec{M} = \tau_{sf}(\frac{m_0}{M_s}\vec{M} \times \frac{\partial \vec{M}}{\partial t} - \frac{\mu_B P}{e M_s}\vec{M} \times ((\vec{j_e} \cdot \nabla)\vec{M}) + \frac{M_s}{\tau_{ex}}\delta\vec{m}), \tag{5.12}$$

as $\vec{M} \times (\delta\vec{m} \times \vec{M}) = M_s^2 \delta\vec{M}$. Inserting this result back into (5.11) and solving for $\delta\vec{m}$,
we find the explicit expression

$$\delta\vec{m} = \frac{\tau_{ex}}{1 + \xi^2}(-\frac{m_0 \xi}{M_s}\frac{\partial \vec{M}}{\partial t} + \frac{\xi \mu_B P}{e M_s}(\vec{j_e} \cdot \nabla)\vec{M} - \frac{m_0}{M_s^2}\vec{M} \times \frac{\partial \vec{M}}{\partial t} + \frac{\mu_B P}{e M_s^2}\vec{M} \times ((\vec{j_e} \cdot \nabla)\vec{M})), \tag{5.13}$$

with ξ being defined as the ratio $\xi = \tau_{ex}/\tau_{sf}$. Inserting this into the expression for
the spin-transfer torque given in (5.8), we find that the spin-transfer torque $\vec{T}_{STT}$ as a
function of the local magnetization $\vec{M}(\vec{r},t)$ and the charge current density $\vec{j_e}$ is

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$$\vec{T}_{STT} = \frac{1}{1+\xi^2}\left(\frac{m_0\xi}{M_s}\vec{M}\times\frac{\partial\vec{M}}{\partial t} - \frac{\xi\mu_B P}{eM_s^2}\vec{M}\times(\vec{j}_e\cdot\nabla)\vec{M} - \frac{m_0}{M_s}\frac{\partial\vec{M}}{\partial t} - \frac{\mu_B P}{eM_s^3}\vec{M}\times(\vec{M}\times(\vec{j}_e\cdot\nabla)\vec{M})\right).$$
$$(5.14)$$

Here we have used that $\vec{M}\times(\vec{M}\times\frac{\partial\vec{M}}{\partial t}) = -M_s^2\frac{\partial\vec{M}}{\partial t}$ as $\vec{M}\cdot\frac{\partial\vec{M}}{\partial t} = 0$, and that $\tau_{ex} = \hbar/JS$. Using this result we can extend the LLG equation in (3.14) to also include the spin-transfer torque effects from the charge current, by inserting $\vec{T}_{STT}$ into the right hand side. Looking at the implicit LLG equation we see that we already have terms that go as $\frac{\partial\vec{M}}{\partial t}$ and $\vec{M}\times\frac{\partial\vec{M}}{\partial t}$. These terms in the spin-transfer torque $\vec{T}_{STT}$ can therefore be incorporated into the LLG equation by redfining the gyromagnetic ratio γ and the damping parameter η. If one inserts the terms in $\vec{T}_{STT}$ that do not depend on the current into (3.14), one finds that by letting

$$\beta = \frac{m_0}{M_s}\frac{1}{1+\xi^2}, \tag{5.15}$$

$$\gamma' = \frac{\gamma}{1+\beta}, \tag{5.16}$$

$$\eta' = \eta + \frac{\xi\beta}{\gamma}, \tag{5.17}$$

the LLG equation keeps its original form. As $m_0 \ll M_s$, we also get $\beta \ll 1$, and so the corrections to γ and η are relatively small. The real impact from the spin-transfer torque comes from the terms dependent on the current. Including these terms the extended LLG equation becomes

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = -\gamma'\vec{M}\times(\vec{H}_{eff} - \eta'\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}) - \frac{b_J}{M_s^2}\vec{M}\times(\vec{M}\times(\hat{j}_e\cdot\nabla)\vec{M})) - \frac{c_J}{M_s}\vec{M}\times(\hat{j}_e\cdot\nabla)\vec{M}, \tag{5.18}$$

with $\hat{j}_e$ being the unit vector along the electring current, and b_J and c_J being defined by

$$b_J = \frac{1}{1+\xi^2}\frac{\mu_B j_e P}{eM_s}, \tag{5.19}$$

$$c_J = \xi b_J. \tag{5.20}$$

By looking at the directions of the b_J and c_J terms in (5.18), one sees that the b_J term is parallel to the direction along which the local magnetization varies, and that the c_J term is perpendicular to the plane that the local magnetization and the change in the local magnetization constitute. If one considers the magnetization of a Néel wall in the x-direction and a flow of electrons in the positive x-direction, the direction of the charge current becomes $\hat{j}_e = -\hat{x}$. The minus sign is because there is a flow of negative charges. If one lets the magnetization of the Néel wall point along $\hat{z}$ at $x = -\infty$, and along $-\hat{z}$ at $x = \infty$, and let the magnetization of the conduction electrons follow the local

magnetization mostly adiabatically, the b_J term will point in the opposite direction of $\frac{\partial \vec{M}}{\partial x}$ and the c_J term will point along $-\hat{y}$. The b_J term is therefore known as the adiabatic spin-transfer torque as it adjusts the local magnetization in the opposite direction of the adiabatic motion of the magnetization of the conduction electrons. Similarly the c_J term is known as the non-adiabatic spin-transfer torque as it adjusts the local magnetization perpendicularly to the adiabatic motion of the magnetization of the conduction electrons. The form of the extended LLG equation in (5.18) is implicit in $\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$. Like the implicit LLG equation, it can be transformed into an explicit form by taking the cross product of it with $\vec{M}$, to get a linear expression for $\vec{M} \times \frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$ in $\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}$. It is then easy to show that by choosing

$$\tilde{\eta}' = \frac{\alpha}{\gamma' M_s}, \tag{5.21}$$

$$\tilde{\gamma}' = \frac{\gamma'}{1 + \alpha^2}, \tag{5.22}$$

with α being a dimensionless damping constant like in the Landau–Lifshitz equation, we get the explicit form of the extended LLG equation:

$$\begin{aligned}
\frac{\partial \vec{M}}{\partial t} = &- \tilde{\gamma}' \vec{M} \times (\vec{H}_{eff} + \frac{\alpha}{M_s} \vec{M} \times \vec{H}_{eff}) \\
&- \frac{(1 + \alpha\xi)b_J}{M_s^2(1 + \alpha^2)} \vec{M} \times (\vec{M} \times (\hat{j}_e \cdot \nabla)\vec{M})) - \frac{(\xi - \alpha)b_J}{M_s(1 + \alpha^2)} \vec{M} \times (\hat{j}_e \cdot \nabla)\vec{M}. \quad (5.23)
\end{aligned}$$

From this equation one can easily see that there will always be an adiabatic spin-transfer torque when a charge current is propagating through a domain wall, but there will only be a non-adiabatic spin-transfer torque if $\xi \neq \alpha$. As $\xi = \tau_{ex}/\tau_{sf}$, with τ_{ex} being a time scale in the exchange interaction and τ_{sf} being an average relaxation time of the non-adiabatic electrons, ξ describes a dimensionless damping parameter for the conduction electrons [2]. There will therefore be a non-adiabatic spin-transfer torque when the damping of the local and the conduction electron spins are different. The non-adiabatic spin-transfer torque causes a rotation of the domain wall, as one of the spins will relax faster than the other.

5.2. Domain wall motion

5.2.1. Advection

In the simplest case when the damping of the itinerant and local spins are equal ($\xi = \alpha$), (5.23) can be reduced considerably to

$$\frac{\partial \vec{M}(\vec{r}, t)}{\partial t} - b_J(\hat{j}_e \cdot \nabla)M(\vec{r}, t) = 0. \tag{5.24}$$

To get this one assumes that the local magnetization is in equilibrium when the current is applied, so that the terms including the effective field vanish. One also assumes a homogeneous material so that $\vec{M} \cdot (\hat{j}_e \cdot \nabla)\vec{M} = 0$ as $|\vec{M}(\vec{r}, t)| = M_s$ is a constant,

meaning $\vec{M} \times (\vec{M} \times (\hat{j}_e \cdot \nabla)\vec{M})) = -M_s^2(\hat{j}_e \cdot \nabla)\vec{M}$. Comparing (5.24) to the advection equation for a vector field $\vec{M}(\vec{r}, t)$ with an incompressible flow $\vec{v}$,

$$\frac{\partial \vec{M}(\vec{r}, t)}{\partial t} + (\vec{v} \cdot \nabla)\vec{M}(\vec{r}, t) = 0, \tag{5.25}$$

one sees that the domain wall moves with a velocity $\vec{v} = -b_J \hat{j}_e$. As $\hat{j}_e$ points in the opposite direction of the electron flow, the domain wall moves in the same direction as the electrons with a speed b_J. The solution of the advection equation is $\vec{M}(\vec{r} - \vec{v} \cdot (t - t_0), t_0)$, with $\vec{M}(\vec{r}, t_0)$ having to satisfy the LLG equation per our assumptions. The simple case described by (5.24) is then just an advection of a domain wall, such as a Bloch or Néel wall.

5.2.2. Motion including the non-adiabatic spin-transfer torque

To solve the extended LLG equation, it will be useful to write the equation in spherical coordinates and use the solution for $\theta(x)$ we found earlier in (4.17). To do this, we write the vectors in terms of their spherical components. An infitesimal change in the magnetization can then be written as

$$d\vec{M} = dM_s \hat{r} + M_s \, d\theta \hat{\theta} + M_s \sin\theta \, d\phi \hat{\phi}, \tag{5.26}$$

which is easily seen from the geometry. Using that $\frac{dM_s}{dt} = 0$, we then find

$$\frac{\partial \vec{M}}{\partial t} = M_s \begin{pmatrix} 0 \\ \dot{\theta} \\ \sin\theta \dot{\phi} \end{pmatrix}. \tag{5.27}$$

Note here that the first component in the vector is the radial component, the second is the component along $\hat{\theta}$, and the last component is along $\hat{\phi}$. Using the definition of the effective field, we find in a similar manner that

$$\vec{H}_{eff} = -\frac{1}{\mu_0}\frac{\delta\epsilon}{\delta\vec{M}} = -\frac{1}{\mu_0}\begin{pmatrix} \frac{\delta\epsilon}{\delta M_s} \\ \frac{1}{M_s}\frac{\delta\epsilon}{\delta\theta} \\ \frac{1}{M_s \sin\theta}\frac{\delta\epsilon}{\delta\phi} \end{pmatrix}. \tag{5.28}$$

For the current terms in (5.23), we note that $(\hat{j}_e \cdot \nabla)$ is a scalar operator, so that $(\hat{j}_e \cdot \nabla)\vec{M} = \sum_i c_i \frac{d\vec{M}}{dx_i}$. Using (5.26) we then easily see that we can write

$$(\hat{j}_e \cdot \nabla)\vec{M} = M_s \begin{pmatrix} 0 \\ (\hat{j}_e \cdot \nabla)\theta \\ \sin\theta(\hat{j}_e \cdot \nabla)\phi \end{pmatrix}, \tag{5.29}$$

where we have also used that $\frac{dM_s}{dx_i} = 0$. The scalar operator $(\hat{j}_e \cdot \nabla)$ can be kept in cartesian coordinates, as we can have θ and ϕ as a function of x, y and z. Finally, we

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use again that $\vec{M} \times (\vec{M} \times (\hat{j}_e \cdot \nabla)\vec{M})) = -M_s^2 (\hat{j}_e \cdot \nabla)\vec{M}$. Combining all of this into (5.23) and calculating the cross products yields the extended LLG equation in spherical coordinates:

$$
\begin{pmatrix} 0 \\ \dot{\theta} \\ \sin\theta\dot{\phi} \end{pmatrix} = \frac{\tilde{\gamma}'}{\mu_0 M_s} \begin{pmatrix} 0 \\ -\frac{1}{\sin\theta}\frac{\delta\epsilon}{\delta\phi} \\ \frac{\delta\epsilon}{\delta\theta} \end{pmatrix} - \frac{\tilde{\gamma}'\alpha}{\mu_0 M_s} \begin{pmatrix} 0 \\ \frac{\delta\epsilon}{\delta\theta} \\ \frac{1}{\sin\theta}\frac{\delta\epsilon}{\delta\phi} \end{pmatrix} + \frac{(1+\xi\alpha)b_J}{1+\alpha^2} \begin{pmatrix} 0 \\ (\hat{j}_e \cdot \nabla)\theta \\ \sin\theta(\hat{j}_e \cdot \nabla)\phi \end{pmatrix}
$$
$$
+ \frac{(\xi-\alpha)b_J}{1+\alpha^2} \begin{pmatrix} 0 \\ \sin\theta(\hat{j}_e \cdot \nabla)\phi \\ -(\hat{j}_e \cdot \nabla)\theta \end{pmatrix}. \tag{5.30}
$$

The radial equation is trivial, but we have two equations in θ and ϕ. Now we use this equation to look at a similar case as the Néel wall described in the last section. We consider a material with an easy axis along the z-axis and a hard axis along the y-axis. The anisotropic energy density can then be written as

$$
\begin{aligned}
\epsilon_A &= K_\perp (\frac{\vec{M} \cdot \hat{y}}{M_s})^2 - K(\frac{\vec{M} \cdot \hat{z}}{M_s})^2 \\
&= K_\perp \sin^2\theta \sin^2\phi - K\cos^2\theta \\
&\rightarrow (K + K_\perp \sin^2\phi)\sin^2\theta, \tag{5.31}
\end{aligned}
$$

where we added a constant shift K in the energy density at the end to simplify the expression. The constants K and $K_\perp$ are strictly positive. Comparing this to the anisotropic energy density in the Bloch and Néel wall calculations previously, we see that they are the same when we let $K \rightarrow K + K_\perp \sin^2\phi$. For the Néel wall case we let $\phi = 0$ previously, but now instead we set $\phi = \phi(t)$. Spatially ϕ is a constant, but we allow it to vary in time due to the non-adiabatic spin-transfer torque from the applied current. As ϕ does not vary along the 1D chain of local magnetic moments, which we define to be along the x-axis as before, the exchange energy is unchanged from the example with the Bloch and Néel walls discussed earlier. If we let the demagnetization energy be incorporated into the anisotropic energy, the total energy density then becomes

$$
\epsilon = \epsilon_E + \epsilon_A = A(\frac{\partial\theta}{\partial x})^2 + K\sin^2\theta + K_\perp \sin^2\theta \sin^2\phi \tag{5.32}
$$

This energy density is the same as the one we solved in the last section, when the substitution of K mentioned above is done. Because of the applied current, the angle θ is now also a function of time, meaning $\theta = \theta(x,t)$. If we assume that the width of the domain wall does not change in time, we can use (4.17) to find the shape of the domain wall. The temporal part of $\theta(x,t)$ is made by letting the domain wall center X be a time dependent position $X(t)$. As the length scale δ_{DW} depends on K, we define a new length scale λ where we make the substitution for K, so that

$$
\lambda = \sqrt{\frac{A}{K + K_\perp \sin^2\phi}}. \tag{5.33}
$$

Together, this gives us

$$\theta(x,t) = 2\arctan(\exp(\frac{x - X(t)}{\lambda})). \tag{5.34}$$

We now have the parameters that describe the motion of the domain wall: $X(t)$ and $\phi(t)$. The former describes a translational motion of the domain wall, and the latter describes a rotation. For low energies these parameters fully describe the domain wall dynamics, under the criterion that

$$K_\perp \ll K. \tag{5.35}$$

If this is not the case, then the $K_\perp$ term may lead to deformations of the domain wall, making our assumption of a static shape of the wall invalid. To describe the dynamics of a deformed wall one must use other parameters [10]. Assuming that we are in the situation where we can describe the motion of the domain wall with $X(t)$ and $\phi(t)$, we want to rewrite (5.30) using these parameters. First we can calculate $\frac{\delta\epsilon}{\delta\theta}$ and $\frac{\delta\epsilon}{\delta\phi}$ by using (5.32):

$$\frac{\delta\epsilon}{\delta\phi} = \frac{\partial\epsilon}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\epsilon}{\partial(\frac{\partial\phi}{\partial x})} = 2K_\perp \sin^2\theta \sin\phi\cos\phi, \tag{5.36}$$

$$\frac{\delta\epsilon}{\delta\theta} = 2(K + K_\perp \sin^2\phi)\sin\theta\cos\theta - 2A\frac{\partial^2\theta}{\partial x^2}. \tag{5.37}$$

We then use (5.34) to find

$$\frac{\partial\theta}{\partial x} = \frac{1}{\lambda\cosh\frac{x-X}{\lambda}} = \frac{\sin\theta}{\lambda}, \tag{5.38}$$

$$\frac{\partial^2\theta}{\partial x^2} = -\frac{\tanh\frac{x-X}{\lambda}}{\lambda^2\cosh\frac{x-X}{\lambda}} = \frac{\sin\theta\cos\theta}{\lambda^2}, \tag{5.39}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\dot{X}}{\lambda\cosh\frac{x-X}{\lambda}} = -\frac{\dot{X}}{\lambda}\sin\theta. \tag{5.40}$$

This means that $\frac{\delta\epsilon}{\delta\theta} = 0$. By using this we can look at how the different terms in the extended LLG equation acts on the domain wall. Looking at (5.30), we see that the equation for $\dot\theta$ is decided by the first and third term on the right hand side, as $\frac{\delta\epsilon}{\delta\theta} = (\hat{j}_e\cdot\nabla)\phi = 0$. The first term is the magnetostatic torque from the effective field, and the third is the adiabatic spin-transfer torque. The in-plane motion of the domain wall is therefore decided by the balance of these two torques; the adiabatic torque will cause a translational movement of the domain wall, which causes the local magnetization to be moved away from equilibrium. This in turn generates an effective field that will attempt to reverse the motion of the domain wall to return to equilibrium. The resulting torques are shown in Figure 4a. In the equation for $\dot\phi$ in (5.30), we only get a contribution from the second and fourth terms on the right hand side. These are the Gilbert damping term

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and the non-adiabatic spin-transfer torque term respectively, which is seen by comparing (5.30) to (5.18). The Gilbert-damping term rotates the Néel domain wall out of plane, which causes an increase in the anisotropic energy as the hard axis is perpendicular to the domain wall plane. The non-adiabatic spin-transfer torque acts as a counterpart to this rotation, and attempts to rotate the domain wall in the other direction, as seen in Figure 4b.
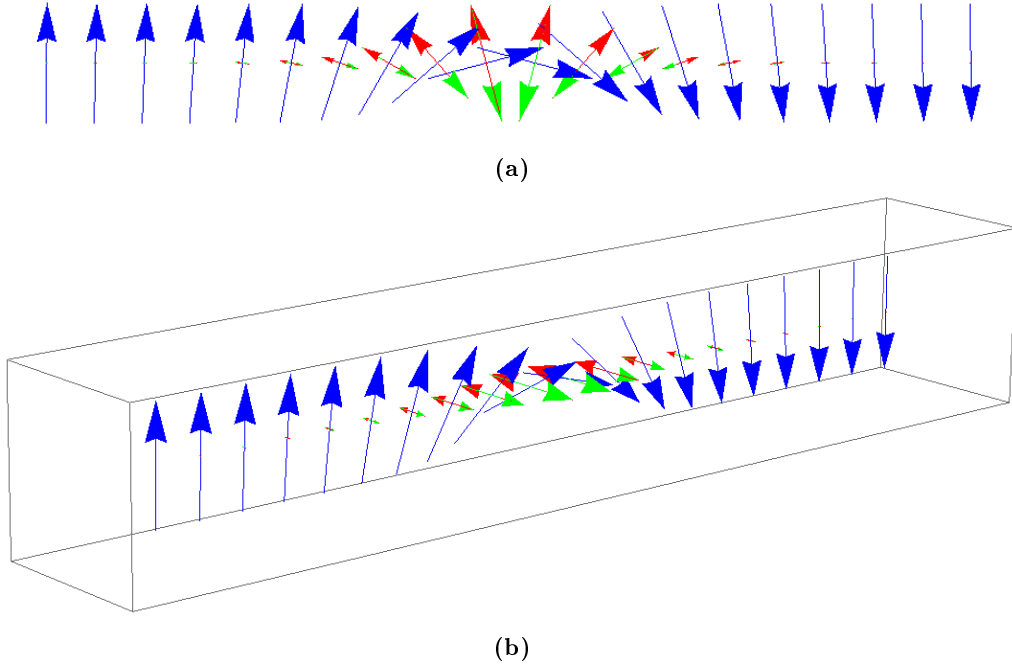

(a)

(b)

Figure 4. The different torques acting on the initial Néel domain wall. The blue vectors are the Néel domain wall magnetization in both figures. In (a) the red vectors represent the adiabatic spin-transfer torque and the green vectors the magnetostatic torque from the effective field. In (b) the red vectors represent the non-adiabatic spin-transfer torque and the green vectors the Gilbert damping torque. The magnitude of the torques are not to scale.

If we insert the calculated differentials into (5.30) it gives us the two equations

$$\begin{pmatrix} \sin\theta \dot{X} \\ \sin\theta \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\lambda\tilde{\gamma}'}{\mu_0 M_s} 2K_\perp \sin\theta \sin\phi \cos\phi + \frac{(1+\xi\alpha)b_J}{1+\alpha^2} \sin\theta \\ -\frac{\tilde{\gamma}'\alpha}{\mu_0 M_s} 2K_\perp \sin\theta \sin\phi \cos\phi + \frac{(\xi-\alpha)b_J}{\lambda(1+\alpha^2)} \sin\theta \end{pmatrix}.$$

We see that $\sin\theta$ is included in all the terms, meaning the domains where $\theta = 0$, $\theta = \pi$ automatically satisfy the equations. As $\sin\theta \neq 0$ except at $x = \pm\infty$, we can safely divide both the equations with $\sin\theta$ when considering the domain wall region:

$$\begin{pmatrix} \dot{X} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\lambda\tilde{\gamma}'}{\mu_0 M_s} K_\perp \sin 2\phi + \frac{(1+\xi\alpha)b_J}{1+\alpha^2} \\ -\frac{\tilde{\gamma}'\alpha}{\mu_0 M_s} K_\perp \sin 2\phi + \frac{(\xi-\alpha)b_J}{\lambda(1+\alpha^2)} \end{pmatrix}. \tag{5.41}$$

Note that λ is a function of ϕ, but due to the criterion in (5.35) λ varies very little as ϕ changes. It is therefore alright to assume that λ is a constant in the equations. As a cross-check of the validity of (5.41) we can see what happens if there is no applied current, meaning $b_J = 0$. We easily see that the domain wall is static if $\dot{X} = \dot{\phi} = 0$, which requires that $\sin 2\phi = 0$. This agrees with our solutions earlier, as we have a Néel wall when $\phi = 0, \pi$ and a Bloch wall when $\phi = \pi/2, 3\pi/2$. Because we are considering a system with a hard axis along the y-axis the stable solutions are Néel walls, so we set $\phi(0) = 0$ in our example. Then we can see that the initial speed of the domain wall is

$$\dot{X}(0) = \frac{1 + \xi\alpha}{1 + \alpha^2} b_J. \tag{5.42}$$

We also have $\dot{\phi}(0) \neq 0$ if $\xi \neq \alpha$, so that the wall rotates slightly. For small currents j_e, eventually $\phi(t)$ will obtain a value such that $\dot{\phi}(t) = 0$. If one then solves for $\sin 2\phi$ in the second line of (5.41) and substitutes into the first line, one then finds that the final domain wall speed becomes

$$\dot{X}(t \to \infty) = \frac{\xi}{\alpha} b_J. \tag{5.43}$$

The final speed of the domain wall is therefore decided by the non-adiabatic spin-transfer torque. Note that this relation is only true up to a certain critical current, as when the current passes a certain threshold $\dot{\phi} = 0$ is no longer a solution. As $\sin 2\phi$ can maximum be 1, we then find that the critical current j_c is given by the relations

$$b_{JC} = \frac{1}{1 + \xi^2} \frac{\mu_B P}{e M_s} j_c, \tag{5.44}$$

$$j_c = \frac{\alpha(1 + \xi^2)}{\xi - \alpha} \frac{\lambda e \gamma' K_\perp}{\mu_0 \mu_B P}. \tag{5.45}$$

Defining a time scale T for the system,

$$T = \frac{\mu_0 M_s}{\tilde{\gamma}' \alpha K_\perp}, \tag{5.46}$$

one can write the equations for $\dot{X}$ and $\dot{\phi}$ as

$$\begin{pmatrix} \dot{X} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\alpha T} \sin 2\phi + \frac{\lambda}{T} \frac{(1 + \xi\alpha)}{\xi - \alpha} \cdot \frac{j_e}{j_c} \\ -\frac{1}{T} \sin 2\phi + \frac{1}{T} \cdot \frac{j_e}{j_c} \end{pmatrix}. \tag{5.47}$$

It can be shown that the solution of $\phi(t)$ is

$$\phi(t) = \arctan(\frac{1 + \sqrt{(j_e/j_c)^2 - 1} \tan(\sqrt{(j_e/j_c)^2 - 1}(t/T + C))}{(j_e/j_c)}), \tag{5.48}$$

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with C being an integration constant. As $\sin 2\phi(t)$ is now oscillating in time without reaching a stable value, $\dot{X}$ will always be an oscillating function in time. From now on we will therefore consider the time averaged translational domain wall speed $\langle \dot{X} \rangle$. Using the expression for $\phi(t)$, we find that

$$\langle \sin 2\phi(t) \rangle = \frac{1}{\frac{\pi T}{\sqrt{(j_e/j_c)^2 - 1}}} \int_0^{\frac{\pi T}{\sqrt{(j_e/j_c)^2 - 1}}} dt \sin 2\phi(t) = \frac{j_e}{j_c} - \sqrt{(\frac{j_e}{j_c})^2 - 1}. \quad (5.49)$$

Inserting this result into the equation for $\dot{X}$, we find that when $j_e > j_c$

$$\langle \dot{X} \rangle = \frac{\lambda}{T} \left[\frac{1}{\alpha}(\frac{j_e}{j_c} - \sqrt{(\frac{j_e}{j_c})^2 - 1}) + \frac{1 + \xi\alpha}{\xi - \alpha} \cdot \frac{j_e}{j_c} \right]$$

$$= \frac{\xi - \alpha}{1 + \alpha^2} \left[\frac{1}{\alpha}(\frac{j_e}{j_c} - \sqrt{(\frac{j_e}{j_c})^2 - 1}) + \frac{1 + \xi\alpha}{\xi - \alpha} \cdot \frac{j_e}{j_c} \right] b_{JC}. \quad (5.50)$$
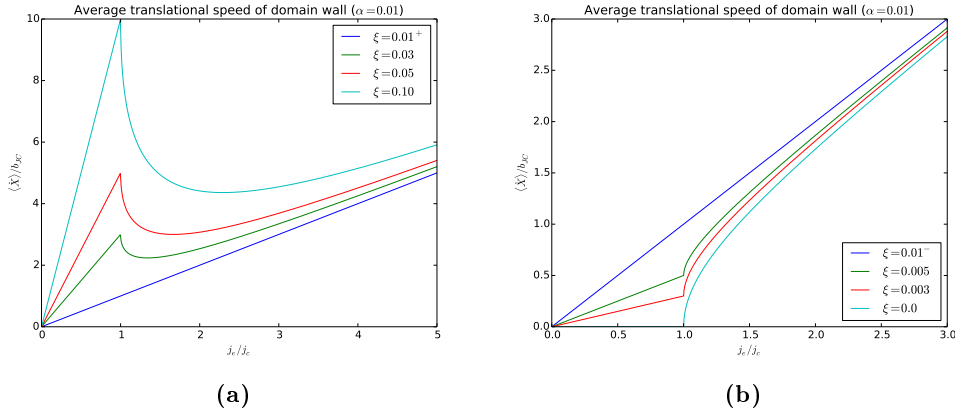

(a)   (b)

Figure 5. The translational speed of the domain wall with a high damping parameter ξ of the itinerant electrons (a) and with a small damping of the itinerant electrons (b) as a function of applied current.

Combining the results in (5.43) and (5.50) we can plot the translational speed of the domain wall as a function of the applied current for different values of ξ, as shown in Figure 5. In Figure 5a we see a drop in the speed when the applied current exceeds the critical current in the case that $\xi > \alpha$. This is known as a Walker breakdown [11]. Above the critical current there is no stable solution of ϕ, meaning there will be an energy loss to the rotational excitation of the domain wall motion as shown in Figure 6. In the limits $j_e/j_c < 1$ and $j_e/j_c \gg 1$ the translational speed $\dot{X}$ is a linear function of j_e/j_c. The slope of the function is much steeper for low charge currents, as in this region we do not get any permanent rotational motion of the domain wall. To get a high translational speed at an efficient energy cost through the applied current, it is best to apply a current

equal to the critical current. As the correlation between the translational speed of the domain wall and the current is linear, we want a high critical current. It is easily seen from (5.45) that the critical current diverges when $\xi \to \alpha$. If $\xi = \alpha$ there will not be any Walker breakdown. The cause for this is easily seen from Figure 4b, as when $\xi = \alpha$ the Gilbert damping torque and the non-adiabatic spin-transfer torque cancel each other out, meaning there is no rotational motion of the domain wall. In regards to materialistic properties, one can increase the critical current by making $\lambda K_\perp$ large. As

$$\lambda K_\perp = \sqrt{\frac{AK_\perp}{\frac{K}{K_\perp} + \sin^2 \phi}}, \tag{5.51}$$

this is not that easy due to the condition in (5.35). One can see, however, that the critical current is larger in strongly ferromagnetic materials with large coupling constants A.

When the itinerant spins have a low damping parameter compared to the local spins ($\xi < \alpha$), the translational speed of the domain wall increases when the applied current is greater than the critical current, as shown in 5b. It is worth noting that the domain wall does not get a final translational speed when the applied current is below the critical current, in the case when $\xi = 0$. There will be some initial movement as seen from (5.42), but this eventually drops off. The reason why there is an increase in the translational speed of the domain wall when $j_e/j_c > 1$ in the case of $\xi < \alpha$ is that the non-adiabatic spin-transfer torque could not compensate for the Gilbert damping when there was a stable solution of ϕ. Below j_e/j_c the added energy from the current was absorbed by the perpendicular anisotropic energy along the hard axis, but as there is a limit to how much anisotropic energy can be absorbed more energy is available to the translational motion of the domain wall when $j_e > j_c$. This is the opposite of the case when $\xi > \alpha$, where there was a decrease in the translational speed when $j_e > j_c$. As $\xi > \alpha$, the non-adiabatic spin-transfer torque was able to compensate the Gilbert damping and the anisotropic energy cost, therefore more energy was available to the translational motion.

5.2.3. Motion due to external field

Instead of applying a current to move domain walls, one can also apply an external magnetic field. Let us now consider the same system as before when we applied the current, but instead we apply a general magnetic field. This causes a change in the energy density, as we now also get a contribution from the Zeeman energy. From the expression of the Zeeman energy in (2.20), one sees that the total energy density of the system becomes

$$\epsilon = \epsilon_E + \epsilon_A + \epsilon_Z = A\left(\frac{\partial \theta}{\partial x}\right)^2 + K \sin^2 \theta + K_\perp \sin^2 \theta \sin^2 \phi - \mu_0 \vec{M} \cdot \vec{H}$$

$$= A\left(\frac{\partial \theta}{\partial x}\right)^2 + K \sin^2 \theta + K_\perp \sin^2 \theta \sin^2 \phi - \mu_0 M_s \left[(H_x \cos \phi + H_y \sin \phi) \sin \theta + H_z \cos \theta\right].$$
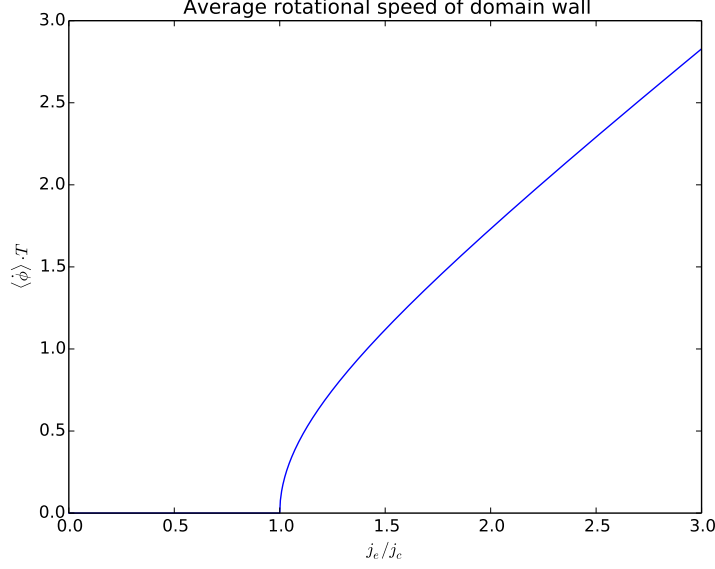
$$\tag{5.52}$$

This gives us

Figure 6. The rotational speed of the domain wall as a function of applied current.

$$\frac{\delta\epsilon}{\delta\theta} = \mu_0 M_s \left[H_z \sin\theta - (H_x \cos\phi + H_y \sin\phi) \cos\theta \right], \tag{5.53}$$

$$\frac{\delta\epsilon}{\delta\phi} = K_\perp \sin^2\theta \sin 2\phi + \mu_0 M_s \sin\theta (H_x \sin\phi - H_y \cos\phi). \tag{5.54}$$

If we insert these results into (5.30) and let $j_e = 0$, we get the equations of motion

$$\begin{pmatrix} \dot{X} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\lambda\tilde{\gamma}'}{\mu_0 M_s} K_\perp \sin 2\phi + \tilde{\gamma}'\lambda\alpha H_z - \lambda\tilde{\gamma}' \left[H_x(\alpha\frac{\cos\phi}{\tan\theta} - \frac{\sin\phi}{\sin\theta}) + H_y(\frac{\alpha\sin\phi}{\tan\theta} - \frac{\cos\phi}{\sin\theta}) \right] \\ -\frac{\alpha\tilde{\gamma}'}{\mu_0 M_s} K_\perp \sin 2\phi + \tilde{\gamma}' H_z - \tilde{\gamma}' \left[H_x(\frac{\cos\phi}{\tan\theta} + \frac{\alpha\sin\phi}{\sin\theta}) + H_y(\frac{\sin\phi}{\tan\theta} - \frac{\alpha\cos\phi}{\sin\theta}) \right] \end{pmatrix}. \tag{5.55}$$

As one can see, if there is an external magnetic field perpendicular to the easy axis (z-axis in this example), the equations of motion will depend on x as $\theta = \theta(x,t)$ and we will get a distortion of the domain wall. As discussed earlier, this motion cannot be described by the variables X and ϕ. We also see that if we set $H_x = H_y = 0$, so that $H_z = H$, the equations become independent of x and we get a motion of the domain wall where the shape is conserved. Doing this, (5.55) gets a similar form as the equations with an applied current in (5.47). By the same argument, there is a critical field strength H_c above which there is no stable solution of ϕ. This is the field strength H when $\dot{\phi} = 0$ and $\sin 2\phi = 1$,

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$$H_c = \frac{\alpha K_\perp}{\mu_0 M_s}. \tag{5.56}$$

We can then rewrite the equation to

$$\begin{pmatrix} \dot{X} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{T}\sin 2\phi + \frac{\lambda}{T}\alpha\frac{H}{H_c} \\ -\frac{1}{T}\sin 2\phi + \frac{1}{T}\frac{H}{H_c} \end{pmatrix}. \tag{5.57}$$

In the case that $\dot{\phi} = 0$ we substitute $\sin 2\phi$ from the second equation into the first, and find that

$$\dot{X}(H \le H_c) = \frac{\lambda}{T}(1+\alpha)\frac{H}{H_c}. \tag{5.58}$$

As the equation for $\dot{\phi}$ is on the exact same form as (5.47) with $j_e/j_c \to H/H_c$, we can use the result that $\langle \sin 2\phi \rangle = (H/H_c) - \sqrt{(H/H_c)^2 - 1}$ when $\dot{\phi} \ne 0$. The translational speed of the domain wall then becomes

$$\langle \dot{X} \rangle (H \ge H_c) = \frac{\lambda}{T}\left[(1+\alpha)\frac{H}{H_c} - \sqrt{(\frac{H}{H_c})^2 - 1}\right]. \tag{5.59}$$

This result is shown in Figure 7. We see that we also get a Walker breakdown in the case of an applied magnetic field. The rotational speed of the domain wall is entirely equivalent with that of the case with an applied current, as shown in Figure 6 (with $j_e/j_c \to H/H_c$). There is always a decrease in the translational velocity right after the Walker breakdown at $H = H_c$ as the Gilbert damping torque does not have a counterpart like in the motion due to a current, and there is a loss to the rotational motion once there is no longer a stable value for ϕ.

The reason behind the motion of the domain wall because of a magnetic field parallel to the easy axis is quite easy to understand phenemenologically. As the domains in the magnet are either parallel or anti-parallel to the applied magnetic field, the domain that is parallel will want to grow to minimize the energy. This comes at the cost of the anti-parallel domain, and the domain wall moves into the anti-parallel domain as a consequence.

6. Skyrmions

7. References

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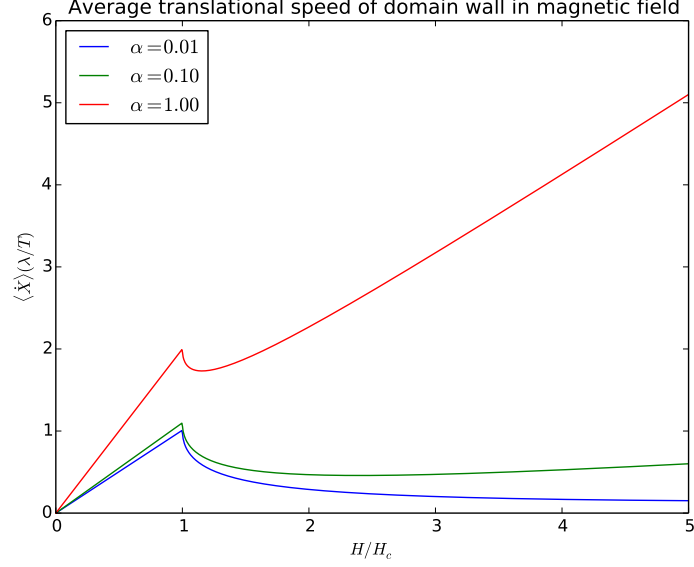
Figure 7. The translational speed of the domain wall as a function of applied magnetic field along the easy axis for different damping parameters α.

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