Differential Geometry Background

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1 Goal

Understand the equation

$$\underbrace{\partial_t g_t}_{\text{talk 1}} = \underbrace{-2\,\mathrm{Ric}(g_t)}_{\text{talk 2}}$$

2 Manifolds

Let M be a smooth manifold. The tangent bundle,

$$TM = \sqcup_{x \in M} T_x M$$

where

$$T_x M = \{X : C^\infty(M) \to \mathbb{R}\}$$

is the set of derivations based at x: $X(fg) = X(f)g(x) + X(g)f(x)$.

In local coordinates (x^i) in a local chart U, φ a basis is

$$\{\partial_i\}_{i=1}^n$$

acting by $\partial_i f = \partial_i (f \circ \varphi^{-1})|_{\varphi(x)}$.

Given an curve

$$\gamma : (-\epsilon, \epsilon) \to M$$

with $\gamma(0) = x$ then we define

$$X_\gamma(f) = (f \circ \gamma)'(0)$$

so that velocity vectors to curves act as derivations by differentiating along curves with $X = \gamma'(0)$. That is,

$$D_X(f) = (f \circ \gamma)'(0)$$

where $\gamma(0) = x$ and $X = \gamma'(0)$.

The cotangent bundle is

$$T^*M = \sqcup_{x \in M} T_x^*M$$

with basis $\{dx^i\}$ dual to ∂_i.

Let

$$\mathfrak{X}(M) = \Gamma(TM)$$

denote the set of vector fields on M such that $X(x) \in T_xM$. The set of dual vector fields in $\Omega(M) = \Gamma(T^*M$.

3 Riemannian metric

Definition 3.1. A Riemannian metric on M is a smooth assignment

$$x \mapsto g_x : T_xM \times T_xM \to \mathbb{R}$$

of inner-products to each tangent space.

Thus g_x is bilinear for each x, $g_x(X, Y) = g_x(Y, X)$ and $g(X, X) \geq 0$. Smoothness means for any vector fields, the map

$$x \mapsto g_x(X(x), Y(x)) \in \mathbb{R}$$

is smooth. g is a *2-tensor field*.

In coordinates,

$$g = g_{ij} dx^i \otimes dx^j = g_{ji} dx^i dx^j$$

with the summation convention applying and where

$$g_{ij} = g(\partial_i, \partial_j), \quad dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i).$$

If $X = X^i \partial_i$ and $Y = Y^i \partial_i$ then

$$g(X, Y) = X^i Y^j g_{ij}$$

Denote the inverse matrix by g^{ij}. Then

$$g^{ij} g_{jk} = \delta_k^i.$$

Example 3.2. Let $M = \mathbb{R}^n$. Then $T_x\mathbb{R}^n \simeq \mathbb{R}^n$. Take

$$g = g_{\text{Euc}} = \langle \cdot, \cdot \rangle$$

so $g_{\text{Euc}}(X, Y) = X^i Y^j \delta_{ij}$, $g_{ij} = \delta_{ij}$,

$$g = \delta_{ij} dx^i dx^j = \sum_i (dx^i)^2.$$

Example 3.3. Let $M = \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. For $x \in \mathbb{S}^n$ write

$$T_x\mathbb{R}^{n+1} = T_x\,\mathbb{S}^n \oplus \mathbb{R}x$$

and define

$$g_{\text{rnd}} = (g_{\text{Euc}})_x|_{T_x\,\mathbb{S}^n \times T_x\,\mathbb{S}^n}$$

Similarly for any embedded submanifold the Euclidean metric induces a metric on the submanifold.

Example 3.4. Let $M = \mathbb{R}^{n-1} \times \mathbb{R}_{>0} = \mathbb{H}^n$ with $g_{\text{Hyp}} = \frac{1}{(x^{n+1})^2} g_{\text{Euc}}$.

Example 3.5. Let (M_i, g_i) $i = 1, 2$ be Riemannian manifolds. Consider $M_1 \times M_2$. The tangent space can be canonically identified with

$$T_{(x,y)}M_1 \times M_2 = T_x M_1 \oplus T_y M_2$$

The metric is

$$g_{(x,y)}(X_1 \oplus Y_1, X_2 \oplus Y_2) = (g_1)_x(X_1, X_2) + (g_2)_y(Y_1, Y_2)$$

with $X_1, X_2 \in TM_1$, $Y_1, Y_2 \in TM_2$.

Example 3.6. Let $N \subseteq (M, g)$ be an embedded submanifold. Then $TN \subseteq TM$ and define $\tilde{g} = g|_{TN}$.

4 Geometry

Definition 4.1. Let $\gamma : (a, b) \to M$ be a piecewise smooth curve. The length

$$L_g(\gamma) = \int_a^b \|\dot{\gamma}\| dt = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}, \dot{\gamma})} dt$$

The induced metric (distance function) is

$$d_g(x, y) = \inf\{L_g(\gamma) : \gamma(0) = x, \gamma(1) = y\}.$$

If (M, d_g) is complete we say (M, g) is complete.

Definition 4.2. If M is orienteable g induces a *volume form.* In coords,

$$d_g = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

For a compactly support function $f : M \to \mathbb{R}$ we can integrate, $\int_M f dg$. If $\operatorname{supp}(f) \subseteq U$ for a coordinate patch then,

$$\int_M f dg = \int_U \sqrt{\det(g_{ij})} dx^1 \ldots dx^n.$$

For arbitrary f defined $\int_M f dg$ via a partition of unity. If M is compact define the volume, $\mathrm{Vol}(M) = \int_M dg$.

Example 4.3. Let $M = \mathbb{S}^2(r)$ be the sphere of radius $r > 0$. In polar coords,

$$g_{\mathrm{rnd}} = r^2 d\theta^2 + r^2 \sin^2\theta d\theta^2$$

where

$$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$$

5 Isometries

Definition 5.1. An isometry is a diffeomorhpism $f : M_1 \to M_2$ such that

$$g_1 = f^* g_2$$

That is

$$g_1(X, Y) = g_2(df(X), df(Y))$$

Definition 5.2. Iso(M, g) $= \{$isometries $f : M \to M\}$ is a group under composition.

Example 5.3. The Euclidean group is

$$\mathrm{Iso}(\mathbb{R}^n, g_{\mathrm{Euc}}) = E(n) = \mathbb{R}^n \rtimes O(n)$$

Here $(v, A) \in E(n)$ acts by $x \mapsto Ax + v$.

Example 5.4. The isometries of the sphere are $\mathrm{Iso}(\mathbb{S}^n, g_{\mathrm{rnd}}) = O(n+1)$ since we have to take ambient isometries preserving the norm.

Theorem 5.5 (Myer-Steenrod)**.** *If* (M, g) *is connected and complete, then the isometry group* $\text{Iso}(M, g)$ *is a Lie group.*

Example 5.6. Let (M, g) be connected and complete and let $G < \text{Iso}(M, g)$ be a closed Lie subgroup acting freely on M, i.e. $f(p) = p$ for $p \in G$ if and only if $f = Id_M$. Then the orbit space

$$M/G = \{G(p) : p \in M\}$$

is a smooth manfiold with Riemannian metric $\tilde{g}$ such that

$$\pi : (M, g) \to (M/G, \tilde{g})$$

is a *local isometry*. We define the metric on M/G so that the quotient map $\pi : M \to M/G$ becomes a local isometry: $g(X, Y) = g(d\pi(X), d\pi(Y))$.

Remark 5.7. It's possible to have a discrete isometry group (e.g. $\mathbb{Z}/2\mathbb{Z}$) which is then a smooth 0-dimensional Lie group.

Example 5.8. The torus. The isometry group has rotational isometries and reflections which are in different connected components of the isometry group.

Let $M = \mathbb{R}^n$, $\mathbb{Z}^n \subseteq \text{Iso}(\mathbb{R}^n, g)$ be defning $z \in \mathbb{Z} \mapsto f_z$

$$f_z(x) = x + z.$$

Then ZZ^n acts freely and $\mathbb{R}^n/\mathbb{Z}^n \simeq \mathbb{T}^n = (\mathbb{S}^1)^n$ is the n-torus. The metric is

$$\delta_{ij} dx^i dx^j$$

where $(x^1, \ldots, x^n)$ are coordinates on $(\mathbb{S}^1)^n$.

6 Levi-Civita connection

Let $Y \in \Gamma(TM)$. We want a notion of *directional derivative* of Y in the direction $X \in T_xM$. In Euclidean space we may define

$$D_X Y = \frac{d}{dt}(Y(\gamma(t)))|_{t=0} = \lim_{t \to 0} \frac{Y(\gamma(t)) - Y(\gamma(0))}{t}$$

where $\gamma(0) = x$ and $\gamma'(0) = X$. The problem in directly generalising this is that we needed to use the linear (additive) structure on $\mathbb{R}^n$ to add the vectors $Y(\gamma(t))$ and $Y(\gamma(0))$ which lie in different tangent spaces. In Euclidean space we may identify all tangent spaces thus allowing the addition.

Properites we would like to generalise:

- The product rule,

$$D_X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle X, D_X Z \rangle$$

- Torsion free

$$D_{\partial_i} \partial_j = D_{\partial_j} \partial_j.$$

That is, second partial derivatives are symmetric $\partial_i \partial_j = \partial_j \partial_i$.