Introduction to Mean Curvature Flow

Australian Geometric PDEs Seminar

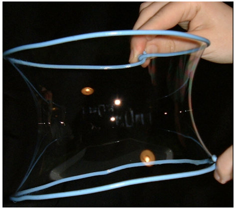
Marcus Flook

March 18, 2023

Australian
National
University

Motivation

A *minimal surface* is a surface which locally minimises its area. Such surfaces often arise in nature, most famously in soap films.



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Minimal surfaces have a mean curvature of zero everywhere.

A surface evolving in such a way as to decrease its area most efficiently is said to evolve by *mean curvature flow*.

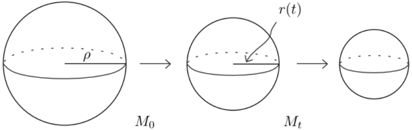

Image Credit: Klaus Ecker

The evolution of these surfaces satisfies what can be thought of as a *geometric heat flow*.

Quick Recap of Embedded Hypersurfaces in $\mathbb{R}^{n+1}$

For a more detailed explanation, see Appendix A of 'Regularity Theory of Mean Curvature Flow' by Klaus Ecker.

Suppose M^n is an n-dimensional manifold. A differentiable map $F : M^n \to \mathbb{R}^{n+1}$ is an *immersion* if, for any $x \in M^n$, the map $dF : T_x M^n \to T_{F(x)}\mathbb{R}^{n+1}$ is injective. If $F : M^n \to \mathbb{R}^{n+1}$ maps homeomorphically onto its image $F(M^n) = \mathcal{M}$ in $\mathbb{R}^{n+1}$, then F is called an *embedding*.

The coordinate tangent vectors $\partial_i F(x) := \frac{\partial F}{\partial x_i}(x)$ $(1 \leq i \leq n)$ form a basis of the tangent space $T_p\mathcal{M}$ at $p = F(x)$ for all $x \in M^n$.

Quick Recap of Embedded Hypersurfaces in $\mathbb{R}^{n+1}$

Suppose $X : M^n \to \mathbb{R}^{n+1}$ is an embedding and $\mathcal{M} = X(M^n)$. The *induced metric* on $\mathcal{M}$ is given by

$$g_{ij} = (\partial_i X \cdot \partial_j X),$$

for $1 \leq i, j \leq n$. The inverse metric is defined by

$$g^{ij} g_{jk} = \delta^i_k,$$

where δ^i_k is the Kronecker delta, and the area element of $\mathcal{M}$ by

$$\sqrt{g} = \sqrt{\det g_{ij}}.$$

Quick Recap of Embedded Hypersurfaces in $\mathbb{R}^{n+1}$

Suppose $f : \mathcal{M} \to \mathbb{R}$. Via the embedding map, we can think of f as a function on M^n. The *tangential gradient* of f is defined by

$$\nabla^{\mathcal{M}} f = g^{ij} \partial_i f \partial_j X.$$

We similarly define the Laplace-Beltrami operator of f on $\mathcal{M}$ by

$$\Delta_{\mathcal{M}} f = g^{ij}(\partial_i \partial_j f - \Gamma_{ij}{}^{k} \partial_k f),$$

where the Christoffel symbols $\Gamma_{ij}{}^{k}$ are given by

$$(\partial_i \partial_j X)^T = \Gamma_{ij}{}^{k} \partial_k X.$$

Let ν be a choice of unit normal field to $\mathcal{M}$. In particular,

$$\nu \cdot \partial_i X = 0$$

for all $1 \leq i \leq n$. Where it makes sense, we take ν to be the outward pointing normal vector. It should be noted that $\partial_i \nu$ is tangent to $\mathcal{M}$ as

$$0 = \partial_i |\nu|^2 = 2(\partial_i \nu \cdot \nu).$$

Quick Recap of Embedded Hypersurfaces in $\mathbb{R}^{n+1}$

The second fundamental form of $\mathcal{M}$ is defined by

$$\mathrm{II}_{ij} = (\partial_i \nu \cdot \partial_j X) = -(\nu \cdot \partial_i \partial_j X).$$

The mean curvature of $\mathcal{M}$ is given by

$$H = \sum_{i=1}^{n} \kappa_i = g^{ij}\mathrm{II}_{ij},$$

where κ_i are the principal curvatures of $\mathcal{M}$ (note that some authors $H = \frac{1}{n}\sum_{i=1}^{n} \kappa_i$).

Mean curvature flow

A smooth one-parameter family of embeddings $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfies *mean curvature flow* if

$$\frac{\partial X}{\partial t}(x, t) = -H(x, t)\nu(x, t),$$

where $H(x, t)$ is the mean curvature and $\nu(x, t)$ is the outward pointing normal vector, for all $(x, t) \in M^n \times [0, T)$.

In what follows, let $\mathcal{M}_t := X_t(M^n)$.

Example: Sphere

Let $\mathbb{S}_r^n \subset \mathbb{R}^{n+1}$ denote the n-sphere of radius $r > 0$ centred at the origin. Let $\mathcal{M}_0 = \mathbb{S}_\rho^n$ for some $\rho > 0$. Looking for a solution to mean curvature flow, it is reasonable to expect that the solution remains round, so we assume a solution of the form

$$\mathcal{M}_t = \mathbb{S}^n_{r(t)}.$$

We parameterise the sphere with the embeddings $X_t : \mathbb{S}_1^n \hookrightarrow \mathbb{R}^{n+1}$ defined by

$$X_t(\mathbf{x}) = r(t)\mathbf{x}.$$

The normal vector is then given by $\nu(\mathbf{x}, t) = \mathbf{x}$ and the mean curvature is given by $H(\mathbf{x}, t) = \dfrac{n}{r(t)}$. Then

$$\frac{\partial X_t}{\partial t} = \frac{\partial r}{\partial t}(t)\mathbf{x} = -\frac{n}{r(t)}\mathbf{x} = -H(\mathbf{x}, t)\nu(\mathbf{x}, t).$$

Example: Sphere

As such, X_t satisfies mean curvature flow if and only if

$$\frac{\partial r}{\partial t}(t) = -\frac{n}{r(t)}.$$

This gives us an IVP with initial condition $r(0) = \rho$, the solution of which is given by

$$r(t) = \sqrt{\rho^2 - 2nt}.$$

Example: Sphere

Thus, the *shrinking sphere*

$$X_t(\mathbf{x}) = \sqrt{\rho^2 - 2nt}\mathbf{x},$$

where $(\mathbf{x}, t) \in \mathbb{S}_1^n \times [0, \rho^2/2n)$, is a solution of mean curvature flow.

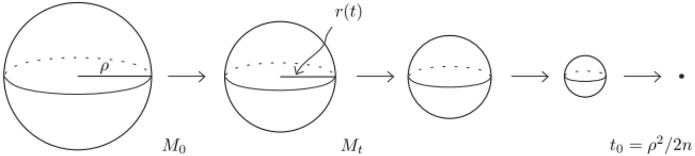

Image Credit: Klaus Ecker

Remark: This extends to a solution for times $t \in (-\infty, \rho^2/2n)$ (ancient solution).

Example: Cylinder

Suppose $\mathcal{M}_0 = \mathbb{S}^{n-m}_\rho \times \mathbb{R}^m$ for some $\rho > 0$ (i.e. cylinder). As before, it is reasonable to assume that a cylindrical solution remains cylindrical and so we search for a solution of the form

$$\mathcal{M}_t = \mathbb{S}^{n-m}_{r(t)} \times \mathbb{R}^m.$$

We parameterise the cylinder with the embeddings $X_t : \mathbb{S}^{n-m}_1 \times \mathbb{R}^m \hookrightarrow \mathbb{R}^{n+1}$ defined by

$$X_t(\mathbf{x}, \mathbf{y}) = (r(t)\mathbf{x}, \mathbf{y})$$

where $\mathbf{x} \in \mathbb{S}^{n-m}_1$ and $\mathbf{y} \in \mathbb{R}^m$. The normal vector is given by $\nu(\mathbf{x}, \mathbf{y}, t) = (\mathbf{x}, 0)$ and $H(\mathbf{x}, \mathbf{y}, t) = \dfrac{n-m}{r(t)}$.

Example: Cylinder

Therefore,

$$\frac{\partial X}{\partial t}(t) = \frac{\partial r}{\partial t}(t)(\mathbf{x}, 0) = -\frac{n-m}{r(t)}(\mathbf{x}, 0) = -H(\mathbf{x}, \mathbf{y}, t)\nu(\mathbf{x}, \mathbf{y}, t).$$

As such, X_t satisfies mean curvature flow if and only if

$$\frac{\partial r}{\partial t}(t) = -\frac{n-m}{r(t)}.$$

This gives us an IVP with initial condition $r(0) = \rho$, the solution of which is given by

$$r(t) = \sqrt{\rho^2 - 2(n-m)t}.$$

Example: Cylinder

Thus, the shrinking cylinder

$$X_t(\mathbf{x}, \mathbf{y}) = \left(\sqrt{\rho^2 - 2(n-m)t}\,\mathbf{x}, \mathbf{y}\right),$$

where $(\mathbf{x}, \mathbf{y}, t) \in \mathbb{S}_1^{n-m} \times \mathbb{R}^m \times \left[0, \frac{\rho^2}{2(n-m)}\right)$, is a solution to mean curvature flow.

Image Credit: Klaus Ecker

Example: Cylindrical Solutions

More generally, suppose $\{\mathcal{M}_t^{n-m}\}_{t\in I}$ is a solution to MCF in $\mathbb{R}^{n-m+1}$. Then the family of product hypersurfaces

$$\mathcal{M}_t^n := \mathcal{M}_t^{n-m} \times \mathbb{R}^m \subset \mathbb{R}^{n+1}$$

is a solution to MCF in $\mathbb{R}^{n+1}$.

In particular, taking $n = 2$ and $m = 1$ we can see all the solutions to curve shortening flow creating 'cylindrical' solutions in higher dimensions.

Avoidance Principles

Theorem (Comparison Principle)

Any two smooth compact solutions of mean curvature flow which are initially disjoint stay disjoint.

Theorem (Hyperboloid/Cone Comparison)

Let $(\mathcal{M}_t)_{t>0}$ be a solution of mean curvature flow. Let $\tilde{x} := (x_1, \ldots, x_n)$. If, for $0 \leq \beta \leq n$ and some $\varepsilon > 0$, the initial hypersurface satisfies

$$\mathcal{M}_0 \subset \{x \in \mathbb{R}^{n+1}, (n-1-\beta)x_{n+1}^2 \geq |\tilde{x}|^2 - \varepsilon^2\}$$

then

$$\mathcal{M}_t \subset \{x\mathbb{R}^{n+1}, (n-1-\beta)x_{n+1}^2 \geq |\tilde{x}|^2 - \varepsilon^2 + 2\beta t\}$$

for $t < \dfrac{\varepsilon^2}{2\beta}$ as long as the solution stays smooth for this time.

Example: Neck Pinch

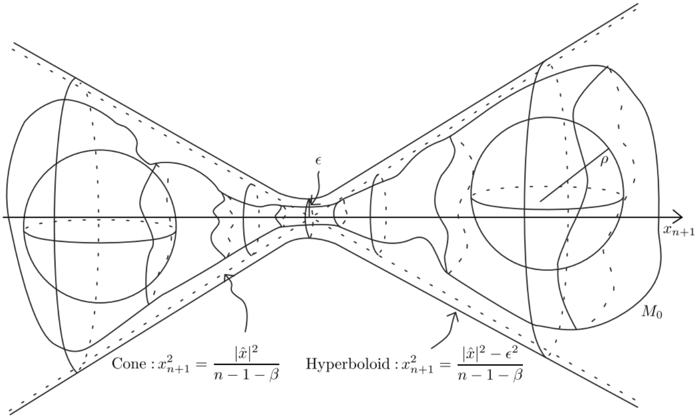

Image Credit: Klaus Ecker

Diffeomorphism invariance of solution

Suppose $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is a solution to

$$\partial_t X = -H\nu.$$

Let $X_t(x) := X(x, t)$. If $\phi : M^n \to M^n$ is a diffeomorphism, then $X_t \circ \phi : M^n \to \mathbb{R}^{n+1}$ is also a solution.

This is indicative of the fact that the curvature flow depends only on the geometry of the hypersurface, not how it is parameterised. This implies a degeneracy in the tangential directions.

Mean curvature flow as a geometric heat flow

Mean curvature flow can be viewed as a geometric heat flow in the following sense. If $X : M^n \to \mathbb{R}^{n+1}$ is an embedding where $X(M^n) = \mathcal{M}$, then

$$\Delta_{\mathcal{M}} X = -H\nu,$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator.

The mean curvature flow equation then becomes

$$\partial_t X(x,t) = \Delta_{\mathcal{M}_t} X(x,t).$$

Mean curvature flow as a geometric heat flow

Computing the Laplace-Beltrami operator of the embedding makes evident the degeneracy in the MCF problem.

$$\begin{aligned}
\Delta_{\mathcal{M}_t}(X \cdot e_m) &= g^{ij}(\partial_i\partial_j(X \cdot e_m) - \Gamma_{ij}{}^k\partial_k(X \cdot e_m)) \\
&= g^{ij}((\partial_i\partial_j X) \cdot e_m - \Gamma_{ij}{}^k(\partial_k X) \cdot e_m) \\
&= g^{ij}((\partial_i\partial_j X) - (\partial_i\partial_j X)^T) \cdot e_m \\
&= g^{ij}(\partial_i\partial_j X)^\perp \cdot e_m \\
&= g^{ij}(\partial_i\partial_j X \cdot \nu)\nu \cdot e_m \\
&= -g^{ij}\mathrm{II}_{ij}(\nu \cdot e_m) \\
\Delta_{\mathcal{M}_t}(X \cdot e_m) &= -H(\nu \cdot e_m).
\end{aligned}$$

In particular, we have a degeneracy of the Laplace-Beltrami operator in the tangential directions, so we have a *weakly parabolic PDE*. This directly corresponds with the diffeomorphism invariance of MCF.

Example: Graphical Hypersurfaces

Consider

$$\mathcal{M}_t := \{(\mathbf{x}, u(\mathbf{x}, t)) : \mathbf{x} \in \mathbb{R}^n, t \in I\}$$

for some smooth $u : \mathbb{R}^n \times I \to \mathbb{R}$.This graphical hypersurface satisfies mean curvature flow if and only if

$$\frac{\partial u}{\partial t} = \sqrt{1 + |\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

Note that this PDE is quasilinear and *strictly* parabolic.

Moral: Considering a hypersurface locally as the graph of a function breaks the diffeomorphism invariance of mean curvature flow.

Evolution Equations - Induced Metric

Suppose that $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is a smooth one-parameter family of embeddings satisfying mean curvature flow. As the embedding is evolving, we need to understand how the geometric quantities of the hypersurface are evolving.

$$
\begin{aligned}
\partial_t g_{ij} &= \partial_t(\partial_i X \cdot \partial_j X) \\
&= (\partial_i \partial_t X \cdot \partial_j X) + (\partial_i X \cdot \partial_j \partial_t X) \\
&= -(\partial_i (H\nu) \cdot \partial_j X) - (\partial_i X \cdot \partial_j (H\nu)) \\
&= -([(\partial_i H)\nu + H(\partial_i \nu)] \cdot \partial_j X) - (\partial_i X \cdot [(\partial_j H)\nu + H(\partial_j \nu)]) \\
\partial_t g_{ij} &= -2H\mathrm{II}_{ij}.
\end{aligned}
$$

Evolution Equations - Area Element

Suppose that $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is a smooth one-parameter family of embeddings satisfying mean curvature flow. Recall that $\partial_t \det g_{ij} = (\det g_{ij})\mathrm{tr}_g(\partial_t g_{ij})$.

$$\begin{aligned}
\partial_t \sqrt{\det g_{ij}} &= \frac{1}{2\sqrt{\det g_{ij}}}\partial_t(\det g_{ij}) \\
&= \frac{1}{2\sqrt{\det g_{ij}}}(\det g_{ij})\mathrm{tr}_g \partial_t g_{ij} \\
&= \frac{\sqrt{\det g_{ij}}}{2} g^{ij}(-2H\mathrm{II}_{ij}) \\
&= -H^2\sqrt{\det g_{ij}}
\end{aligned}$$

Evolution Equations - Area Element

Recalling that

$$d\mu_t(x) = \sqrt{\det g_{ij}(x,t)}\, d\mu_{M^n}(x),$$

we obtain the following.

Corollary

Suppose M^n is a closed manifold and $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ is a smooth one-parameter family of embeddings satisfying MCF. Then

$$\frac{d}{dt} Area(\mathcal{M}_t) = \int_{\mathcal{M}_t} \frac{d}{dt} d\mu_t = -\int_{\mathcal{M}_t} H^2\, d\mu_t \leq 0.$$

Indeed, it can be shown that mean curvature flow is the L^2 gradient flow of the area functional.

Evolution Equations - Normal Vector

Suppose $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is a family of embeddings which undergo MCF. To compute the time evolution of the normal vector, recognise that $2(\partial_t \nu, \nu) = \partial_t |\nu|^2 = 0$ and so $\partial_t \nu$ is tangent. As such, $\partial_t \nu = g^{ij}(\partial_t \nu \cdot \partial_i X)\partial_j X$.

$$\begin{aligned} 0 = \partial_t(\nu \cdot \partial_i X) &= (\partial_t \nu \cdot \partial_i X) + (\nu \cdot \partial_i(-H\nu)) \\ &= (\partial_t \nu \cdot \partial_i X) - H(\nu \cdot \partial_i \nu) - \partial_i H |\nu|^2 \\ \therefore (\partial_t \nu \cdot \partial_i X) &= \partial_i H. \end{aligned}$$

Therefore,

$$\partial_t \nu = g^{ij} \partial_i H \partial_j X = \nabla^{\mathcal{M}_t} H.$$

Evolution Equations - Inverse Metric

Suppose $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is a family of embeddings which undergo MCF. To compute the time evolution of the inverse metric, recall that $g^{ij}g_{jk} = \delta^i_k$. Therefore, $\partial_t(g^{ij}g_{jk}) = 0$. Expanding this yields

$$\begin{aligned} 0 = \partial_t(g^{ij}g_{jk}) &= (\partial_t g^{ij})g_{jk} + g^{ij}(\partial_t g_{jk}) \\ &= (\partial_t g^{ij})g_{jk} + g^{ij}(-2H\mathrm{II}_{jk}) \\ &= (\partial_t g^{ij})g_{jk} - 2H\mathrm{II}^i_k. \end{aligned}$$

Therefore

$$\begin{aligned} (\partial_t g^{ij})g_{jk} &= 2H\mathrm{II}^i_k \\ (\partial_t g^{ij})g_{jk}g^{ks} &= 2H\mathrm{II}^i_k g^{ks} \\ (\partial_t g^{ij})\delta^s_j &= 2H\mathrm{II}^{is} \\ \therefore \partial_t g^{is} &= 2H\mathrm{II}^{is}. \end{aligned}$$

Evolution Equations - Second Fundamental Form

To compute the time evolution of the second fundamental form II_{ij}, it will be convenient to work in geodesic normal coordinates such that $g_{ij} = \delta_{ij}$ and $(\partial_i \partial_j X)^T = 0$. Then

$$\begin{aligned}
\partial_t \mathrm{II}_{ij} &= -\partial_t(\nu \cdot \partial_i \partial_j X) \\
&= -(\partial_t \nu \cdot \partial_i \partial_j X) - (\nu \cdot \partial_i \partial_j(-H\nu)) \\
&= (\nu \cdot [(\partial_i \partial_j H)\nu + (\partial_i H)(\partial_j \nu) + (\partial_j H)(\partial_i \nu) + H \partial_i \partial_j \nu]) \\
&= \partial_i \partial_j H + H(\nu \cdot \partial_i \partial_j \nu) \\
&= \partial_i \partial_j H - H(\partial_i \nu \cdot \partial_j \nu).
\end{aligned}$$

Evolution Equations - Second Fundamental Form

$$\partial_t \mathrm{II}_{ij} = \partial_i \partial_j H - H(\partial_i \nu \cdot \partial_j \nu).$$

Recognise that $\partial_j \nu$ is tangent, and so $\partial_j \nu = g^{k\ell}(\partial_j \nu \cdot \partial_k X)\partial_\ell X = g^{k\ell}\mathrm{II}_{jk}\partial_\ell X = \mathrm{II}_j^\ell \partial_\ell X$.
Therefore,

$$\begin{aligned}\partial_t \mathrm{II}_{ij} &= \partial_i \partial_j H - H\mathrm{II}_j^\ell(\partial_i \nu \cdot \partial_\ell X) \\ &= \partial_i \partial_j H - H\mathrm{II}_j^\ell \mathrm{II}_{i\ell}.\end{aligned}$$

Recalling that we are working in geodesic normal coordinates, we obtain

$$\partial_t \mathrm{II}_{ij} = \nabla_i^{\mathcal{M}_t} \nabla_j^{\mathcal{M}_t} H - H\mathrm{II}_{i\ell}\mathrm{II}_j^\ell.$$

Evolution Equations - Mean Curvature

We are now ready to compute the evolution of the mean curvature. Recognising that $H = g^{ij}\mathrm{II}_{ij}$, the time evolution can be computed as

$$\begin{aligned}\partial_t H &= \partial_t(g^{ij}\mathrm{II}_{ij})\\ &= (\partial_t g^{ij})\mathrm{II}_{ij} + g^{ij}(\partial_t \mathrm{II}_{ij})\\ &= 2H|\mathrm{II}|^2 + g^{ij}(\nabla_i^{\mathcal{M}_t}\nabla_j^{\mathcal{M}_t}H - H\mathrm{II}_{i\ell}\mathrm{II}_j^{\ell})\\ \partial_t H &= H|\mathrm{II}|^2 + \Delta_{\mathcal{M}_t}H.\end{aligned}$$

As such,

$$(\partial_t - \Delta_{\mathcal{M}_t})H = |\mathrm{II}|^2 H.$$