Introduction to Mean Curvature Flow 2

Australian Geometric PDEs Seminar

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March 18, 2023

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Mean curvature flow

A smooth one-parameter family of embeddings $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfies *mean curvature flow* if

$$\frac{\partial X}{\partial t}(x, t) = -H(x, t)\nu(x, t),$$

where $H(x, t)$ is the mean curvature and $\nu(x, t)$ is the outward pointing normal vector, for all $(x, t) \in M^n \times [0, T)$.

In what follows, let $\mathcal{M}_t := X_t(M^n)$.

Last Time - Examples

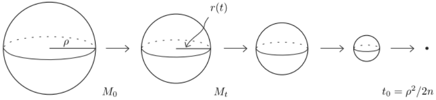

Image Credit: Klaus Ecker

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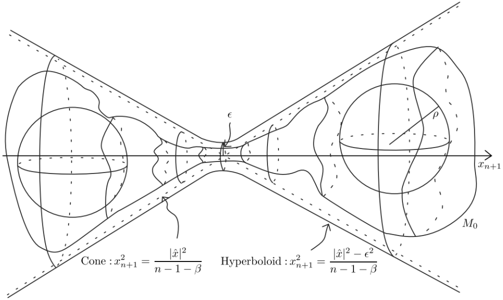

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Last Time - Evolution Equations

Suppose $\{X_t\}_{t \in I}$ is a family of embeddings satisfying mean curvature flow. Then the following evolution equations are satisfied.

$$\partial_t g_{ij} = -2H\mathrm{II}_{ij}$$

$$\partial_t g^{ij} = 2H\mathrm{II}^{ij}$$

$$\partial_t \sqrt{\det g_{ij}} = -H^2 \sqrt{\det g_{ij}}$$

$$\partial_t \mathrm{II}_{ij} = \nabla_i^{\mathcal{M}_t} \nabla_j^{\mathcal{M}_t} H - H\mathrm{II}_{i\ell}\mathrm{II}_j^{\ell}$$

$$\partial_t \nu = \nabla^{\mathcal{M}_t} H$$

$$(\partial_t - \Delta_{\mathcal{M}_t})H = |\mathrm{II}|^2 H$$

Today

Goals for today:

- Proving a weak maximum principle with ODE comparison.
- Proving preservation of mean convexity.
- Proving short time existence of MCF.

Weak Maximum Principle

Theorem

Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a one-parameter family of immersions of a compact manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T)) \cap C^0(M^n \times [0,T))$ satisfies

$$\partial_t u \geq \Delta_{\mathcal{M}_t} u + \nabla_b^{\mathcal{M}_t} u + F(u),$$

for some time-dependent vector field b and some locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$. If $u \geq \phi_0$ at $t = 0$ for some $\phi_0 \in \mathbb{R}$, then $u(x,t) \geq \phi(t)$ for all $x \in M^n$ and $0 \leq t < T$, where ϕ is the solution to the ODE

$$\begin{cases} \frac{d\phi}{dt} = F(\phi) & \text{in } (0,T), \\ \phi(0) = \phi_0. \end{cases}$$

A quick lemma

Lemma

Let (M^n, g) be a Riemannian manifold equipped with its Levi-Civita connection ∇. If $f \in C^2(M^n)$ attains a local minimum at $x_0 \in M^n$, then $0 = \nabla f(x_0)$ and $\nabla^2 f(x_0) \geq 0$.

Proof.

Consider some $v \in T_{x_0}M^n$ and let $\gamma : (-\varepsilon, \varepsilon) \to M^n$ be the geodesic through $x_0 = \gamma(0)$ with $\gamma'(0) = v$. Then $f \circ \gamma$ attains a local minimum at x_0, and so

$$0 = \frac{d}{dt}\bigg|_{t=0} (f \circ \gamma)(t) = \nabla_v f \qquad \text{and} \qquad 0 \leq \frac{d^2}{dt^2}\bigg|_{t=0} (f \circ \gamma)(t) = \nabla_v \nabla_v f.$$

$\square$

Weak Maximum Principle

Proof.

Case 1: $F \equiv 0$.

Suppose $\partial_t u \geq \Delta_{\mathcal{M}_t} u + \nabla_b^{\mathcal{M}_t} u$ and $u(\cdot, 0) \geq 0$. Define $u_\varepsilon(x,t) := u(x,t) + \varepsilon(t+1)$ for some $\varepsilon > 0$. Then u_ε satisfies

$$\begin{cases} \partial_t u_\varepsilon & \geq \Delta_{\mathcal{M}_t} u_\varepsilon + \nabla_b^{\mathcal{M}_t} u_\varepsilon + \varepsilon, \\ u_\varepsilon(\cdot, 0) & \geq \varepsilon. \end{cases}$$

Suppose $u_\varepsilon(x_0, t_0) = 0$, with t_0 the smallest such t. Then

$$\partial_t u_\varepsilon(x_0, t_0) \geq \Delta_{\mathcal{M}_t} u_\varepsilon(x_0, t_0) + \nabla_b^{\mathcal{M}_t} u_\varepsilon + \varepsilon.$$

$\square$

Proof.

Case 1: $F \equiv 0$.

Suppose $\partial_t u \geq \Delta_{\mathcal{M}_t} u + \nabla_b^{\mathcal{M}_t} u$ and $u(\cdot, 0) \geq 0$. Define $u_\varepsilon(x,t) := u(x,t) + \varepsilon(t+1)$ for some $\varepsilon > 0$. Then u_ε satisfies

$$\begin{cases} \partial_t u_\varepsilon & \geq \Delta_{\mathcal{M}_t} u_\varepsilon + \nabla_b^{\mathcal{M}_t} u_\varepsilon + \varepsilon, \\ u_\varepsilon(\cdot, 0) & \geq \varepsilon. \end{cases}$$

Suppose $u_\varepsilon(x_0, t_0) = 0$, with t_0 the smallest such t. Then

$$0 \geq \underbrace{\partial_t u_\varepsilon(x_0, t_0)}_{\leq 0} \geq \underbrace{\Delta_{\mathcal{M}_t} u_\varepsilon(x_0, t_0)}_{\geq 0} + \underbrace{\nabla_b^{\mathcal{M}_t} u_\varepsilon}_{=0} + \varepsilon \geq \varepsilon,$$

which is a contradiction. Therefore, $u_\varepsilon > 0$ for all ε and so $u \geq 0$. $\square$

Weak Maximum Principle

Proof.

Case 2: $F = -cu$.

Suppose $\partial_t u \geq \Delta_{\mathcal{M}_t} u + \nabla_b^{\mathcal{M}_t} u - cu$ and $u(\cdot, 0) \geq 0$. Define $\bar{u} := e^{ct} u$. Then $\bar{u}$ satisfies

$$\begin{aligned}
\partial_t \bar{u} &= e^{ct} \partial_t u + c e^{ct} u \\
&\geq e^{ct} (\Delta_{\mathcal{M}_t} u + \nabla_b^{\mathcal{M}_t} u - cu) + c e^{ct} u \\
&= \Delta_{\mathcal{M}_t} \bar{u} + \nabla_b^{\mathcal{M}_t} \bar{u}.
\end{aligned}$$

By Case 1, $\bar{u} \geq 0$ for all $t \in [0, T)$. As such, $u \geq 0$ for all $t \in [0, T)$. □

Proof.

Case 3: F is locally Lipschitz.

Suppose $\partial_t u \geq \Delta_{\mathcal{M}_t} u + \nabla_b^{\mathcal{M}_t} u + F(u)$ and $\min u(\cdot, 0) \geq \phi_0$. Suppose $\phi(t)$ satisfies the IVP

$$\begin{cases} \frac{d\phi}{dt} = F(\phi), \\ \phi(0) = \phi_0. \end{cases}$$

Then

$$\begin{aligned} \partial_t(u - \phi) &\geq \Delta_{\mathcal{M}_t} u + \nabla_b^{\mathcal{M}_t} u + F(u) - F(\phi) \\ &= \Delta_{\mathcal{M}_t}(u - \phi) + \nabla_b^{\mathcal{M}_t}(u - \phi) + F(u) - F(\phi) \\ &\geq \Delta_{\mathcal{M}_t}(u - \phi) + \nabla_b^{\mathcal{M}_t}(u - \phi) - C_{T'}(u - \phi) \end{aligned}$$

on $M \times [0, T']$ for any $T' < T$. By Case 2, $u - \phi \geq 0$ on $M \times [0, T']$ for all $T' < T$. Therefore, $u(\cdot, t) \geq \phi(t)$ for all $t \in [0, T)$. $\square$

Mean Convexity

Definition

We shall call a hypersurface $X : M^n \to \mathbb{R}^{n+1}$ mean convex if it admits a unit normal field with respect to which its mean curvature is non-negative and strictly mean convex if it admits a unit normal field with respect to which its mean curvature is positive.

Preservation of Mean Convexity

Recall that the evolution of mean curvature H is given by

$$(\partial_t - \Delta_{\mathcal{M}_t})H = |\mathrm{II}|^2 H \geq \frac{1}{n}H^3,$$

the inequality holding by Cauchy-Schwarz. The function $F(x) = \frac{1}{n}x^3$ is locally Lipschitz and the associated ODE, with initial condition $\phi(0) = H_{\text{min}}$, is solved by

$$\phi(t) = \frac{H_{\text{min}}}{\sqrt{1 - \frac{2}{n}tH_{\text{min}}^2}}.$$

Preservation of Mean Convexity

The weak maximum principle then implies that if $H(\cdot, 0) \geq H_{\text{min}} > 0$, then

$$H(\cdot, t) \geq \frac{H_{\text{min}}}{\sqrt{1 - \frac{2}{n} t H_{\text{min}}^2}}$$

for all $t > 0$. As such, mean convexity is conserved.

Short Time Existence

Theorem (Short Time Existence)

Given $X_0 : M^n \to \mathbb{R}^{n+1}$ *smooth, closed, embedded hypersurface, there exists a unique solution of*

$$\begin{cases} \partial_t X(x,t) & = -H(x,t)\nu(x,t), \\ X(x,0) & = X_0(x) \end{cases}$$

defined on some positive time interval.

Remark: Stronger results for short time existence do hold. For example, the embedding condition can be weakened to immersion.

Short Time Existence

Theorem (Short Time Existence)

Given $X_0 : M^n \to \mathbb{R}^{n+1}$ smooth, closed, embedded hypersurface, there exists a unique solution of

$$\begin{cases} \partial_t X(x,t) & = -H(x,t)\nu(x,t), \\ X(x,0) & = X_0(x) \end{cases}$$

defined on some positive time interval.

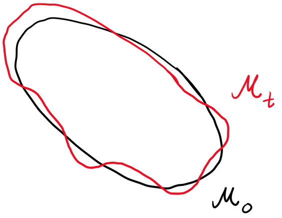
Main idea: MCF is only degenerate parabolic, we aim to 'fix' this by considering $\mathcal{M}_t$ as a graph over $\mathcal{M}_0$.

Proving STE.

Write $\mathcal{M}_t$ as a graph over $\mathcal{M}_0 = X_0(M^n)$.
More precisely, suppose $\nu_0(p)$ is a unit normal vector to $X_0(p)$ and define

$$\mathcal{M}_t = X(p,t) := X_0(p) + f(p,t)\nu_0(p)$$

where $f(\cdot, 0) \equiv 0$.



We aim to solve (MCF*),

$$\begin{cases} (\partial_t f(p,t)\nu_0(p)\cdot\nu(p,t)) = H(p,t), \\ f(\cdot,0) \equiv 0. \end{cases} \iff \begin{cases} \partial_t f(p,t) = \frac{H(p,t)}{(\nu_0(p)\cdot\nu(p,t))}, \\ f(\cdot,0)\equiv 0. \end{cases}$$

We want to express the right hand side of this PDE in terms of f.

Short Time Existence

Proving STE.

To express the RHS of the PDE in terms of f, we need to compute the induced metric, second fundamental form and the mean curvature of the perturbed hypersurface.
First, we compute the induced metric. Let $\tilde{g}_{ij}$ and $\tilde{\mathrm{II}}_{ij}$ denote the metric and second fundamental form of the initial hypersurface and let $f_i := \partial_i f$. Then,

$$\begin{aligned}
\partial_i X(p,t) &= \partial_i X_0 + f_i \nu_0 + f(\partial_i \nu_0) \\
&= \partial_i X_0 + f_i \nu_0 + f[\tilde{g}^{kn}](\partial_i \nu_0 \cdot \partial_n X_0)(\partial_k X_0) \\
&= \partial_i X_0 + f_i \nu_0 - f[\tilde{g}^{kn}][\tilde{\mathrm{II}}_{in}](\partial_k X_0).
\end{aligned}$$

To compute the induced metric of the perturbed metric, we need to evaluate

$$g_{ij}(p,t) = (\partial_i X(p,t) \cdot \partial_j X(p,t)).$$

Proving STE.

Let $\tilde{g}_{ij}$ and $\tilde{\mathrm{II}}_{ij}$ denote the metric and second fundamental form of the initial hypersurface and let $f_i := \partial_i f$. Then,

$$\partial_i X(p,t) = \partial_i X_0 + f_i \nu_0 - f[\tilde{g}^{kn}][\tilde{\mathrm{II}}_{in}](\partial_k X_0).$$

To compute the induced metric of the perturbed hypersurface, we obtain

$$\begin{aligned} g_{ij}(p,t) &= (\partial_i X(p,t) \cdot \partial_j X(p,t)) \\ &= (\partial_i X_0 + f_i \nu_0 - f[\tilde{g}^{kn}][\tilde{\mathrm{II}}_{in}](\partial_k X_0) \cdot \partial_j X_0 + f_j \nu_0 - f[\tilde{g}^{\ell m}][\tilde{\mathrm{II}}_{j\ell}](\partial_m X_0)) \\ &= f_i f_j + \underbrace{(\partial_i X_0 - f[\tilde{g}^{kn}][\tilde{\mathrm{II}}_{in}](\partial_k X_0) \cdot \partial_j X_0 - f[\tilde{g}^{\ell m}][\tilde{\mathrm{II}}_{j\ell}](\partial_m X_0))}_{=:(*)}. \end{aligned}$$

Short Time Existence

Proving STE.

$$g_{ij}(p,t) = f_i f_j + \underbrace{(\partial_i X_0 - f[\tilde{g}^{kn}][\tilde{\mathrm{II}}_{in}](\partial_k X_0) \cdot \partial_j X_0 - f[\tilde{g}^{\ell m}][\tilde{\mathrm{II}}_{j\ell}](\partial_m X_0))}_{=:(*)}$$

Let's evaluate $(*)$.

$$\begin{aligned}
(*) &= \tilde{g}_{ij} + f^2 \tilde{g}^{kn}\tilde{g}^{\ell m}\tilde{\mathrm{II}}_{in}\tilde{\mathrm{II}}_{j\ell}(\partial_k X_0 \cdot \partial_m X_0) - f\tilde{g}^{kn}\tilde{\mathrm{II}}_{in}(\partial_k X_0 \cdot \partial_j X_0) - f\tilde{g}^{\ell m}\tilde{\mathrm{II}}_{j\ell}(\partial_m X_0 \cdot \partial_i X_0) \\
&= \tilde{g}_{ij} + f^2 \tilde{g}^{kn}\tilde{g}^{\ell m}\tilde{\mathrm{II}}_{in}\tilde{\mathrm{II}}_{j\ell}\tilde{g}_{km} - f\tilde{g}^{kn}\tilde{\mathrm{II}}_{in}\tilde{g}_{kj} - f\tilde{g}^{\ell m}\tilde{\mathrm{II}}_{j\ell}\tilde{g}_{mi} \\
&= \tilde{g}_{ij} + f^2 \tilde{g}^{\ell n}\tilde{\mathrm{II}}_{in}\tilde{\mathrm{II}}_{j\ell} - 2f\tilde{\mathrm{II}}_{ij} \\
&= \tilde{g}_{ij} + f^2 \tilde{\mathrm{II}}_i^{\ell}\tilde{\mathrm{II}}_{j\ell} - 2f\tilde{\mathrm{II}}_{ij}
\end{aligned}$$

Proving STE.

Therefore, the metric on the perturbed hypersurface is given by

$$g_{ij}(p,t) = \tilde{g}_{ij} + f_i f_j + f^2 \tilde{\mathrm{II}}_i^{\ell} \tilde{\mathrm{II}}_{j\ell} - 2f\tilde{\mathrm{II}}_{ij}.$$

As such,

$$g_{ij}(p,t) = \tilde{g}_{ij}(p) + \text{small perturbation},$$

provided $||f||_{C^1}$ sufficiently small.

Proving STE.

Recall that we want to express the right hand side of the PDE in terms of f. We need to compute the normal of the perturbed hypersurface. Recall that

$$\partial_i X = \partial_i X_0 + f_i \nu_0 - \tilde{\mathrm{II}}_i^k (\partial_k X_0).$$

Then the normal of the perturbed hypersurface is given by

$$\begin{aligned}\nu(p,t) &= \frac{\nu_0(p) - g^{ij}(p,t)(\nu_0 \cdot \partial_i X)(\partial_j X)}{|\nu_0(p) - g^{ij}(p,t)(\nu_0 \cdot \partial_i X)(\partial_j X)|} \\ &= \frac{\nu_0(p) - g^{ij}(p,t) f_i(p,t)(\partial_j X)}{|\nu_0(p) - g^{ij}(p,t) f_i(p,t)(\partial_j X)|} \\ \nu(p,t) &= \nu_0(p) + \text{small perturbations},\end{aligned}$$

provided $||f||_{C^1}$ sufficiently small.

Proving STE.

The normal of the perturbed hypersurface is given by

$$\nu(p,t) = \nu_0(p) + \text{small perturbations},$$

provided $||f||_{C^1}$ sufficiently small.

In particular, for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $||f||_{C^1} < \delta$, then $|(\nu(p,t), \nu_0(p)) - 1| < \varepsilon$.

Proving STE.

Recall that

$$\partial_i X = \partial_i X_0 + f_i \nu_0 - \tilde{\mathrm{II}}_i^k(\partial_k X_0).$$

The second fundamental form of the perturbed hypersurface is then given by

$$\begin{aligned}
\mathrm{II}_{ij} &= (\nu(p,t)\cdot \partial_i\partial_j X(p,t)) \\
&= (\nu \cdot (\partial_i \partial_f)\nu_0 + \partial_i\partial_j X_0 - \partial_i f \tilde{\mathrm{II}}_j^k \partial_k X_0 - \partial_j f \tilde{\mathrm{II}}_i^k \partial_k X_0 - f\partial_j \tilde{\mathrm{II}}_i^k \partial_k X_0 - f\tilde{\mathrm{II}}_i^k \partial_j \partial_k X_0) \\
&= (\nu(p,t)\cdot\nu_0(p))\partial_i\partial_j f(p,t) + P_{ij}(p, f(p,t), \partial f(p,t)),
\end{aligned}$$

where P_{ij} is smooth provided $||f||_{C^1}$ small.

Proving STE.

The second fundamental form of the perturbed hypersurface is given by

$$\mathrm{II}_{ij} = (\nu(p,t) \cdot \nu_0(p))\partial_i \partial_j f(p,t) + P_{ij}(p, f(p,t), \partial f(p,t)),$$

where P_{ij} is smooth provided $||f||_{C^1}$ small. The mean curvature of the perturbed hypersurface is then given by

$$\begin{aligned} H(p,t) &= g^{ij}(p,t)\mathrm{II}_{ij}(p,t) \\ &= (\nu(p,t) \cdot \nu_0(p))\, g^{ij}(p,t)\partial_i \partial_j f(p,t) + P(p, f(p,t), \partial f(p,t)), \end{aligned}$$

where P is smooth provided $||f||_{C^1}$ small.

Proving STE.

The mean curvature of the perturbed hypersurface is given by

$$H(p,t) = (\nu(p,t)\cdot\nu_0(p))g^{ij}(p,t)\partial_i\partial_j f(p,t) + P(p, f(p,t), \partial f(p,t)).$$

Define $Q(p,f,\partial f) := \frac{P}{(\nu_0\cdot\nu)}$, which is smooth provided $||f||_{C^1}$ small. Then (MCF*) can be expressed as

$$\begin{cases} \partial_t f(p,t) &= \frac{H(p,t)}{(\nu_0(p)\cdot\nu(p,t))}, \\ f(\cdot,0) &\equiv 0. \end{cases} \iff \begin{cases} \partial_t f(p,t) &= g^{ij}\partial_i\partial_j f + Q(p,f,\partial f), \\ f(\cdot,0) &\equiv 0. \end{cases}$$

This is a quasilinear, strictly parabolic PDE, so we obtain a unique solution on a short time interval by (relatively) standard theory.

Proving STE.

Note, however, that we have constructed a solution to (MCF*). Does this give a solution to (MCF)? Suppose $X : M^n \times [0, \varepsilon) \to \mathbb{R}^{n+1}$ satisfies the system

$$\begin{cases} \partial_t X(p,t) & = -H(p,t)\nu(p,t) + T(p,t), \end{cases}$$

where $T(p,t)$ is a tangential vector field (i.e. X satisfies (MCF*)). Suppose the family $\varphi_t \in \mathrm{Diff}(M^n)$ and make the Ansatz

$$\tilde{X}(p,t) := X(\varphi_t(p), t).$$

Proving STE.

Suppose $X : M^n \times [0, \varepsilon) \to \mathbb{R}^{n+1}$ satisfies the system

$$\begin{cases} \partial_t X(p,t) & = -H(p,t)\nu(p,t) + T(p,t), \end{cases}$$

where $T(p,t)$ is a tangential vector field. Then

$$\begin{aligned} \partial_t \tilde{X}(p,t) &= \partial_t X(\varphi_t(p),t) + dX(\varphi_t(p),t)\partial_t \varphi_t(p) \\ &= -H(\varphi_t(p),t)\nu(\varphi_t(p),t) + T(\varphi_t(p),t) + dX(\varphi_t(p),t)\partial_t \varphi_t(p) \\ &= -H(\varphi_t(p),t)\nu(\varphi_t(p),t) \end{aligned}$$

if and only if

$$\frac{\partial \varphi_t}{\partial t}(p) = -dX(\varphi_t(p),t)^{-1} T(\varphi_t(p),t).$$

Short Time Existence

Proving STE.

Suppose $X : M^n \times [0, \varepsilon) \to \mathbb{R}^{n+1}$ satisfies the system

$$\begin{cases} \partial_t X(p,t) & = -H(p,t)\nu(p,t) + T(p,t), \end{cases}$$

where $T(p,t)$ is a tangential vector field. Then

$$\partial_t \tilde{X}(p,t) = -H(\varphi_t(p),t)\nu(\varphi_t(p),t)$$

if and only if

$$\frac{\partial \varphi_t}{\partial t}(p) = -dX(\varphi_t(p),t)^{-1} T(\varphi_t(p),t).$$

By standard ODE theory, a solution to this exists and is unique. As such, we have shown the short time existence of MCF. □