# Short-Time Existence and Uniqueness of Ricci Flow

The main result of this note is the following short-time existence and uniqueness theorem for the initial value problem of Ricci flow.

**Theorem.** Let $(M, g_0)$ be a compact Riemannian manifold without boundary. Then there exist a positive number $T > 0$ and a smooth family of Riemannian metrics $\{g(t)\}_{t\in[0,T)}$ on $M$ such that $g(0) = g_0$ and that

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}_{g(t)}.$$

If $\{g_1(t)\}_{t\in[0,T_1)}$ and $\{g_2(t)\}_{t\in[0,T_2)}$ are two solutions to the above equation with the same initial value $g_1(0) = g_2(0) = g_0$, then

$$g_1(t) = g_2(t),\ \forall t \in [0, T_1) \cap [0, T_2).$$

The standard PDE theory does not apply to Ricci flow since the nonlinear operator $g \mapsto -2\mathrm{Ric}_g$ is not elliptic. In 1982, Hamilton proved the above theorem by employing the Nash-Moser inverse function theorem. This note will describe another proof given by DeTurck, which modifies the flow by a family of diffeomorphisms and avoids the use of the Nash-Moser theorem.

Differential Operators on Vector Bundles

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**Definition.** Let $M$ be a smooth manifold without boundary and $E, \hat{E}$ be smooth vector bundles over $M$. Choose any connections on $TM$ and $E$, and let $D$ be the induced connections on $T^*M^{\otimes k} \otimes E$ for any $k \in \mathbb{N}$. Assume that $E^+$ is an open subset of $E$ and that $\Gamma(E^+) = \{u \in \Gamma(E) \mid u_p \in E^+, \forall p \in M\}$. A map $P : \Gamma(E^+) \to \Gamma(\hat{E})$ is call a (nonlinear) differential operator of second order if for any $u \in \Gamma(E^+)$ and $p \in M$,

$$(Pu)|_p = F(D^2 u|_p, Du|_p, u|_p),$$

where $F : (T^*M \otimes T^*M \otimes E) \oplus (T^*M \otimes E) \oplus E^+ \to \hat{E}$ is a smooth map such that the restriction $F_p$ of $F$ to each fibre satisfies

$$F_p((T_p^*M \otimes T_p^*M \otimes E_p) \oplus (T_p^*M \otimes E_p) \oplus (E^+ \cap E_p)) \subseteq \hat{E}_p, \ \forall p \in M.$$

**Definition.** Let $E, \hat{E}$ be smooth vector bundles over a smooth manifold $M$ without boundary with connections $D$ on $TM$ and $E$. A map $L : \Gamma(E) \to \Gamma(\hat{E})$ is call a linear differential operator of second order if for any $u \in \Gamma(E)$,

$$Lu = \langle \lambda_2^L, D^2 u \rangle + \langle \lambda_1^L, Du \rangle + \langle \lambda_0^L, u \rangle,$$

where $\lambda_2^L \in \Gamma(TM \otimes TM \otimes E^* \otimes \hat{E})$, $\lambda_1^L \in \Gamma(TM \otimes E^* \otimes \hat{E})$, $\lambda_0^L \in \Gamma(E^* \otimes \hat{E})$ and $\langle , \rangle$ is the natural pairing of vector bundles. And the principal symbol $\hat{\sigma}$ of a linear differential operator $L$ in any direction $\eta \in T_p^* M$ is defined to be

$$\hat{\sigma}[L](\eta) = \langle \lambda_2^L|_p, \eta \otimes \eta \rangle \in E_p^* \otimes \hat{E}_p.$$

Now assume that $P : \Gamma(E^+) \to \Gamma(\hat{E})$ is a differential operator of second order and $u \in \Gamma(E^+)$. Define the linearise of $P$ at $u$ to be the following map

$$dP_u : \Gamma(E) \to \Gamma(\hat{E}), h \mapsto \lim_{s \to 0} \frac{P(u + sh) - P(u)}{s},$$

where the limit is taken at each point of $M$.

To see that $dP_u$ is linear, we write it as

$$
\begin{aligned}
dP_u(h)|_p &= \lim_{s\to 0} \frac{P(u+sh)|_p - P(u)|_p}{s} \\
= \lim_{s\to 0} & \frac{F(D^2(u+sh)|_p, D(u+sh)|_p, (u+sh)|_p) - F(D^2u|_p, Du|_p, u|_p)}{s} \\
=& d(F_p)_{(D^2u|_p, Du|_p, u|_p)}(D^2h|_p, Dh|_p, h|_p).
\end{aligned}
$$

Then there exist $\lambda_2^{dP_u} \in \Gamma(TM \otimes TM \otimes E^* \otimes \hat{E})$, $\lambda_1^{dP_u} \in \Gamma(TM \otimes E^* \otimes \hat{E})$ and $\lambda_0^{dP_u} \in \Gamma(E^* \otimes \hat{E})$ such that

$$dP_u(h) = \langle \lambda_2^{dP_u}, D^2h \rangle + \langle \lambda_1^{dP_u}, Dh \rangle + \langle \lambda_0^{dP_u}, h \rangle.$$

In the case $E = \hat{E}$, the highest term $\lambda_2^{dP_u}$ determines whether the differential operator is of elliptic type.

**Definition.** Assume that $P : \Gamma(E^+) \to \Gamma(E)$ is a differential operator of second order and $u \in \Gamma(E^+)$. The operator $P$ is said to be elliptic for $u$ at $p \in M$ if the eigenvalues of the principal symbol $\hat{\sigma}$ of $dP_u$ in any direction $\eta \in T_p^* M$,

$$\hat{\sigma}[dP_u](\eta) = \langle \lambda_2^{dP_u}|_p, \eta \otimes \eta \rangle \in E_p^* \otimes E_p,$$

have strictly positive real parts.

**Theorem.** Let $M$ be a smooth compact manifold without boundary and $E$ be a smooth vector bundle over $M$. Assume that $E^+$ is an open subset of $E$ and $u_0 \in \Gamma(E^+) \subseteq \Gamma(E)$. Suppose a differential operator $P : \Gamma(E^+) \to \Gamma(E)$ of second order is elliptic for $u_0$ on $M$. Then there exist a positive number $T > 0$ and a smooth family of sections of $E^+$,

$$u : [0, T) \times M \to E^+, (t, p) \mapsto u(t, p) = u(t)_p \in E_p \cap E^+,$$

such that $u(0) = u_0$ and

$$\frac{\partial u(t)}{\partial t} = P(u(t)).$$

If $u_i : [0, T_i) \times M \to E^+$ for $i = 1, 2$ are two solutions to the above equation with the same initial value $u_1(0) = u_2(0) = u_0$, then

$$u_1(t) = u_2(t),\ \forall t \in [0, T_1) \cap [0, T_2).$$

**Remark.** The choice of the connections does not influence the principal symbol or the ellipticity of a linear operator. In fact, assume that $D, \hat{D}$ are two connections of $E$. Then the difference $A = D - \hat{D}$ is a tensor field since

$$\begin{aligned} A(X, fs) =& D_X fs - \hat{D}_X fs \\ =& fD_X s + (Xf)s - f\hat{D}_X s - (Xf)s \\ =& fA(X, s), \end{aligned}$$

for any $X \in \mathfrak{X}(M), f \in C^\infty(M), s \in \Gamma(T^*M^{\otimes(k-1)}E)$ or $s \in \mathfrak{X}(M)$.

For any $X, Y \in \mathfrak{X}(M), u \in \Gamma(E)$,

$$\begin{aligned} &(D^2 u - \hat{D}^2 u)(X, Y) \\ =&D_Y D_X u - D_{D_Y X} u - \hat{D}_Y \hat{D}_X u + \hat{D}_{\hat{D}_Y X} u \\ =&A(Y, D_X u) + \hat{D}_Y(A(X, u)) - A(D_Y X, u) - \hat{D}_{A(Y,X)} u, \end{aligned}$$

which depends only on $u$ and the first derivate of $u$ linearly. If $L$ is a linear differential operator with

$$Lu = \langle \lambda_2^L, D^2 u\rangle + \langle \lambda_1^L, Du\rangle + \langle \lambda_0^L, u\rangle = \langle \hat{\lambda}_2^L, \hat{D}^2 u\rangle + \langle \hat{\lambda}_1^L, \hat{D} u\rangle + \langle \hat{\lambda}_0^L, u\rangle,$$

for any $u \in \Gamma(E)$, we have $\lambda_2^L = \hat{\lambda}_2^L$. So using different connections gives the same principal symbol.

**Example.** Let $(M, g)$ be a Riemannian manifold, $E = \hat{E} = \mathrm{Sym}^2(T^*M)$, $E^+ = \mathrm{Sym}^2_+(T^*M)$ and $P(g) = \mathrm{Ric}_g$. Choose any local smooth coordinate, the Ricci curvature can be written as

$$R_{jk} = \frac{1}{2} g^{ml}(\partial_j \partial_l g_{mk} + \partial_m \partial_k g_{jl} - \partial_m \partial_l g_{jk} - \partial_j \partial_k g_{ml}) + \cdots .$$

Using the Euclidean connection induced by coordinate, the linearise of $\mathrm{Ric}$ at $g$ is given by

$$
\begin{aligned}
&(d\mathrm{Ric}_g(h))_{jk} \\
=&\frac{1}{2}g^{ml}(\partial_j\partial_l h_{mk} + \partial_m\partial_k h_{jl} - \partial_m\partial_l h_{jk} - \partial_j\partial_k h_{ml}) + \cdots ,
\end{aligned}
$$

and the princial symbols in direction $\eta \in T_p^* M$ are

$$
\begin{aligned}
&(\hat{\sigma}[d\mathrm{Ric}_g](\eta)h)_{jk} \\
=&\frac{1}{2}g^{ml}(\eta_j\eta_l h_{mk} + \eta_m\eta_k h_{jl} - \eta_m\eta_l h_{jk} - \eta_j\eta_k h_{ml}).
\end{aligned}
$$

Equivalently,

$$\hat{\sigma}[d\mathrm{Ric}_g](\eta)h = \frac{(\eta^\sharp \rfloor h) \otimes \eta + \eta \otimes (\eta^\sharp \rfloor h) - |\eta|^2 h - (\mathrm{tr}_g h)\eta \otimes \eta}{2}.$$

It is easy to see that $\hat{\sigma}[d\mathrm{Ric}_g](\eta)$ has eigenvalue 0 on

$$\{\omega \otimes \eta + \eta \otimes \omega \mid \omega \in T_p^* M\}.$$

**Definition.** Let $M$ be any smooth manifold with or without boundary. For any $X \in \mathfrak{X}(M)$,

$$\mathcal{L}_X : \Gamma(TM^{\otimes k} \otimes T^*M^{\otimes l}) \to \Gamma(TM^{\otimes k} \otimes T^*M^{\otimes l})$$

is the unique family of linear differential operators of first order on the bundles of mixed tensors of any types $(k, l) \in \mathbb{N}^2$, satisfying

(a) $\mathcal{L}_X f = Xf = df(X)$,
(b) $\mathcal{L}_X Y = [X, Y]$,
(c) $\mathcal{L}_X(\mathrm{tr}\ A) = \mathrm{tr}(\mathcal{L}_X A)$,
(d) $\mathcal{L}_X(A \otimes B) = (\mathcal{L}_X A) \otimes B + A \otimes (\mathcal{L}_X B)$,

for any $A \in \Gamma(TM^{\otimes k} \otimes T^*M^{\otimes l})$, $B \in \Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q})$, $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$.

**Example.** Let $M$ be any smooth manifold with or without boundary and $D$ be any symmetric connection on $TM$. For any $X, V, W \in \mathfrak{X}(M)$ and $\omega \in \Gamma(T^*M \otimes T^*M)$,

$$\begin{aligned} &(\mathcal{L}_X\omega)(V, W) \\ =&X(\omega(V, W)) - \omega([X, V], W) - \omega(V, [X, W]) \\ =&(D_X\omega)(V, W) + \omega(D_X V, W) + \omega(V, D_X W) \\ &- \omega([X, V], W) - \omega(V, [X, W]) \\ =&(D_X\omega)(V, W) + \omega(D_V X, W) + \omega(V, D_W X). \end{aligned}$$

When $D = \nabla$ is the Levi-Civita connection of the Riemannian metric $\omega = g$,

$$\mathcal{L}_X g = 2\mathrm{Sym}(\nabla X^\flat).$$

Fix any compact smooth manifold $M$ without boundary and any symmetric connection $D$ on $TM$. Then we have the following two differential operators of first order

$$\tau : \Gamma(\mathrm{Sym}^2_+(T^*M)) \to \Gamma(T^*M \odot T^*M \otimes TM), g \mapsto D - \nabla^g,$$
$$V : \Gamma(\mathrm{Sym}^2_+(T^*M)) \to \Gamma(TM), g \mapsto \mathrm{tr}_g(\tau(g)),$$

where $\nabla^g$ is the Levi-Civita connection of $g$ and $\mathrm{tr}_g(\tau(g))$ be the trace of $\tau(g)$ on its first and second indices with respect to $g$.

Now we define a differential operator

$$P : \Gamma(\mathrm{Sym}^2_+(T^*M)) \to \Gamma(\mathrm{Sym}^2(T^*M)), g \mapsto -2\mathrm{Ric}_g - \mathcal{L}_{V(g)}g,$$

and show that $P$ is elliptic for any $g \in \Gamma(\mathrm{Sym}^2_+(T^*M))$. Then the Ricci-DeTurck flow

$$\frac{\partial \hat{g}(t)}{\partial t} = -2\mathrm{Ric}_{\hat{g}(t)} - \mathcal{L}_{V(\hat{g}(t))}\hat{g}(t)$$

enjoys short-time existence and uniqueness.

In fact, $2\mathrm{Ric}_g + \mathcal{L}_{V(g)} g + \mathrm{tr}_g D^2 g$ depends only on $g$ and $Dg$ since

$$\begin{aligned}
2\mathrm{Ric}_{jk} =& g^{ml}(\partial_j\partial_l g_{mk} + \partial_m\partial_k g_{jl} - \partial_m\partial_l g_{jk} - \partial_j\partial_k g_{ml}),\\
g^{ml} g_{jk,ml} =& g^{ml}(\partial_m g_{jk,l} - \gamma^s_{mj} g_{sk,l} - \gamma^s_{mk} g_{js,l} - \gamma^s_{ml} g_{jk,s})\\
=& g^{ml}\partial_m\partial_l g_{jk},\\
V^i =& g^{ml}(\gamma^i_{ml} - \Gamma^i_{ml})\\
=& -\frac{1}{2} g^{ml} g^{ir}(\partial_m g_{rl} + \partial_l g_{rm} - \partial_r g_{ml}),\\
(\mathcal{L}_{V(g)} g)_{jk} =& V^i g_{jk,i} + g_{ik} V^i_{,j} + g_{ij} V^i_{,k}\\
=& g_{ik}\partial_j V^i + g_{ij}\partial_k V^i\\
=& -\frac{1}{2} g_{ik} g^{ml} g^{ir}(\partial_j\partial_m g_{rl} + \partial_j\partial_l g_{rm} - \partial_j\partial_r g_{ml})\\
& -\frac{1}{2} g_{ij} g^{ml} g^{ir}(\partial_k\partial_m g_{rl} + \partial_k\partial_l g_{rm} - \partial_k\partial_r g_{ml})\\
=& -g^{ml}(\partial_j\partial_l g_{mk} + \partial_m\partial_k g_{jl} - \partial_j\partial_k g_{ml}).
\end{aligned}$$

Let $M$ be a smooth manifold. A smooth time-dependent vector field on $M$ is a smooth map $V : J \times M \to TM$, where $J \subseteq \mathbb{R}$ is an interval, such that $V(t,p) \in T_pM$ for each $(t,p) \in J \times M$. This means that for each $t \in J$, the map $V_t : M \to TM$ defined by $V_t(p) = V(t,p)$ is a smooth vector field on $M$. If $V$ is a smooth time-dependent vector field on $M$, an integral curve of $V$ is a smooth curve $\gamma : J_0 \to M$, where $J_0$ is an interval contained in $J$, such that $\gamma'(t) = V(t,\gamma(t))$ for all $t \in J_0$.

A smooth time-dependent tensor field on $M$ is a smooth map $A : J \times M \to T^*M^{\otimes k}$ satisfying $A(t, p) \in T_p^*M^{\otimes k}$ for each $(t, p) \in J \times M$. Then for each $t \in J$, the map $A_t : M \to T_p^*M^{\otimes k}$ defined by $A_t(p) = A(t, p)$ is a smooth tensor field $A_t \in \Gamma(T_p^*M^{\otimes k})$ on $M$.

**Theorem.** Let $M$ be a smooth compact manifold without boundary, let $J \subseteq \mathbb{R}$ be an open interval, and let $V : J \times M \to TM$ be a smooth time-dependent vector field on $M$. There exists a smooth map

$$\theta : J \times J \times M \to M, (t, s, p) \mapsto \theta(t, s, p) = \theta^{(s,p)}(t) = \theta_{t,s}(p),$$

called the time-dependent flow of $V$, with the following properties:
(a) For each $t_0 \in J$ and $p \in M$, the smooth curve $\theta^{(t_0,p)} : J \to M$ defined by $\theta^{(t_0,p)}(t) = \theta(t, t_0, p)$ is the unique maximal integral curve of $V$ with initial condition $\theta^{(t_0,p)}(t_0) = p$.
(b) For each $t_0, t_1 \in J$ and $p \in M$, if $q = \theta(t_1, t_0, p)$, then $\theta^{(t_1,q)} = \theta^{(t_0,p)}$.
(c) For each $t_0, t_1 \in J$, the map $\theta_{t_1,t_0} : M \to M$ defined by $\theta_{t_1,t_0}(p) = \theta(t_1, t_0, p)$ is a diffeomorphism of $M$ with inverse $\theta_{t_0,t_1}$.
(d) For each $t_0, t_1, t_2 \in J$, $\theta_{t_2,t_1} \circ \theta_{t_1,t_0} = \theta_{t_2,t_0}$.

**Proposition.** Let $M$ be a smooth compact manifold without boundary and $J \subseteq \mathbb{R}$ be an open interval. Suppose $V : J \times M \to TM$ is a smooth time-dependent vector field on $M$, $\theta$ is its time-dependent flow, and $A : J \times M \to T^*M^{\otimes k}$ is a smooth time-dependent tensor field on $M$. Then for any $(t_1, t_0, p) \in J \times J \times M$,

$$\left.\frac{d}{dt}\right|_{t=t_1} (\theta^*_{t,t_0} A_t)_p = \left(\theta^*_{t_1,t_0}\left(\mathcal{L}_{V_{t_1}} A_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1} A_t\right)\right)_p .$$

Suppose that $(M, g_0)$ is a compact Riemannian manifold without boundary and $D$ is any fixed symmetric connection of $TM$. Then there exist a positive number $T > 0$ and a smooth family of Riemannian metrics $\{\hat{g}(t)\}_{t\in[0,T)}$ on $M$ such that $\hat{g}(0) = g_0$ and that

$$\frac{\partial \hat{g}(t)}{\partial t} = -2\mathrm{Ric}_{\hat{g}(t)} - \mathcal{L}_{V(\hat{g}(t))}\hat{g}(t)$$

on $M \times [0, T)$, where
$V(\hat{g}(t)) = \mathrm{tr}_{\hat{g}(t)}(\tau(\hat{g}(t))) = \mathrm{tr}_{\hat{g}(t)}(D - \nabla^{\hat{g}(t)})$ is defined as before.
Let $\theta$ be the time-dependent flow of $V(\hat{g}(t))$ and $\varphi_t = \theta_{t,0}$. By the fundamental theorem about time-dependent vector fields,

$$\frac{\partial}{\partial t}\varphi_t(p) = \frac{\partial}{\partial t}\theta_{t,0}(p) = V(\hat{g}(t))|_{\varphi_t(p)}.$$

Then $g(t) = \varphi_t^*(\hat{g}(t))$ is the solution to Ricci flow with initial value $g_0$ since

$$\begin{aligned}
&\frac{\partial g(t)}{\partial t} + 2\mathrm{Ric}_{g(t)} \\
=&\frac{\partial}{\partial t}(\theta_{t,0}^*(\hat{g}(t))) + 2\mathrm{Ric}_{\theta_{t,0}^*(\hat{g}(t))} \\
=&\theta_{t,0}^*\left(\mathcal{L}_{V(\hat{g}(t))}\hat{g}(t) + \frac{\partial \hat{g}(t)}{\partial t}\right) + 2\theta_{t,0}^*(\mathrm{Ric}_{\hat{g}(t)}) \\
=&\theta_{t,0}^*\left(\mathcal{L}_{V(\hat{g}(t))}\hat{g}(t) + \frac{\partial \hat{g}(t)}{\partial t} + 2\mathrm{Ric}_{\hat{g}(t)}\right) \\
=&0,
\end{aligned}$$

and

$$g(0) = \varphi_0^*(\hat{g}(0)) = \theta_{0,0}^*(\hat{g}(0)) = \mathrm{Id}_M^*(\hat{g}(0)) = \hat{g}(0) = g_0.$$

Now we have shown the short-time existence of Ricci flow by constructing a solution to Ricci flow from a solution to Ricci-DeTurck flow. To show the uniqueness of Ricci flow, we need to recover Ricci-DeTurck flow from Ricci flow. By above computation, it suffices to find a smooth family of diffeomorphisms $\{\varphi_t\}$ of $M$, satisfying

$$\frac{\partial}{\partial t}\varphi_t(p) = V((\varphi_t^{-1})^* g(t))|_{\varphi_t(p)},$$

where $\{g(t)\}$ is a given solution to Ricci flow. To understand the PDE about $\{\varphi_t\}$ well, we will introduce the pull-back bundle and show that $\{\varphi_t\}$ is a solution to harmonic map heat flow.

Suppose that $M$ and $N$ are two smooth manifolds without boundaries and that $\pi : E \to N$ is a smooth vector bundle over $N$. For any smooth map $f : M \to N$, define the pull-back bundle $f^*E$ to be

$$f^*E = \{(p, e) \in M \times E \mid f(p) = \pi(e)\} = \coprod_{p \in M} \{p\} \times E_{f(p)},$$

and the bundle projection $f^*\pi : f^*E \to M$ to be the restriction of the projection $M \times E \to M$ to $f^*E$.

For any connection $\hat{D}$ on $E$, we can define a connection

$$D : \mathfrak{X}(M) \times \Gamma(f^*E) \to \Gamma(f^*E)$$

as follows.
For any $p \in M$, let $\{e_i\}$ be any smooth frame of $E$ on a neighborhood of $f(p)$. Define

$$(D_X s)_p = ds^i(X_p)e_i + s^i \hat{D}_{df_p(X_p)} e_i,$$

for any $X \in \mathfrak{X}(M)$ and $s = s^i e_i \in \Gamma(E)$.

Now we assume that $E = TN$. For any $X \in \mathfrak{X}(M)$ and $p \in M$, we obtain a vector $df_p(X_p) \in T_{f(p)}N$ by applying the diferential of $f$ to $X_p$. However, this does not in general define a vector field on $N$. For example, if $f$ is not surjective, there is no way to decide what vector to assign to a point $q \in N \setminus f(M)$. If $f$ is not injective, then for some points of $N$ there may be several different vectors obtained by applying $df$ to $X$ at different points of $M$. But now we can consider $df(X)$ as a section of $f^*TN$ and we see that $df \in \Gamma(T^*M \otimes f^*TN)$.

If we choose symmetric connections on $TM$ and $TN$, we can define the second fundamental form $\mathrm{II}_f \in \Gamma(T^*M^{\otimes 2} \otimes f^*TN)$ of $f$ to be

$$\mathrm{II}_f(X,Y) = (\nabla df)(X,Y) = \nabla_Y(df(X)) - df(\nabla_Y X), X, Y \in \mathfrak{X}(M),$$

where $\nabla$ is the induced connection on $TM$ or $T^*M^{\otimes k} \otimes f^*TN$ for any $k \in \mathbb{N}$. If $g$ is a Riemannian metric of $M$, we can define $\Delta_g f = \mathrm{tr}_g \mathrm{II}_f \in \Gamma(f^*TN)$.

**Propositon.** Suppose that $M, N, P$ are smooth manifolds without boundaries with symmetric connections and $f : M \to N$ and $\hat{f} : N \to P$ are smooth maps. Then

$$\mathrm{II}_{\hat{f}\circ f} = \mathrm{II}_{\hat{f}} \circ (df \otimes df) + d\hat{f} \circ \mathrm{II}_f,$$

or precisely,

$$\mathrm{II}_{\hat{f}\circ f}(X, Y)_p = \mathrm{II}_{\hat{f}}(df_p(X_p), df_p(Y_p)) + d\hat{f}_{f(p)}(\mathrm{II}_f(X, Y)_p),$$

for $p \in M$, $X, Y \in \mathfrak{X}(M)$. If $g$ is a Riemannian metric of $N$ and $f$ is a diffeomorphism,

$$\Delta_{f^*g}(\hat{f} \circ f) = (\Delta_g \hat{f}) \circ f + d\hat{f}(\Delta_{f^*g} f).$$

When $f : (M, \nabla) \to (M, D)$ is the identity map, the second fundamental form of $f$ is the difference tensor $D - \nabla$. And if $\nabla$ is the Levi-Civita connection of a Riemannian metric $g$ on $M$, then

$$\tau(g) = \mathrm{II}_f, \quad V(g) = \Delta_g f.$$

Now we can return to the equation

$$\frac{\partial}{\partial t}\varphi_t(p) = V(\hat{g}(t))|_{\varphi_t(p)} = V((\varphi_t^{-1})^* g(t))|_{\varphi_t(p)},$$

where $g(t)$ is given, $\hat{g}(t)$ and $\varphi_t : (M, g(t)) \to (M, \hat{g}(t))$ are unknown with

$$\varphi_t^* \hat{g}(t) = g(t),$$

which implies $\mathrm{II}_{\varphi_t} = 0$.

Let $f_t : (M, \hat{g}(t)) \to (M, D)$ be the identity map of $M$, we know that $V(\hat{g}(t)) = \Delta_{\hat{g}(t)} f_t$. Apply the proposition about the second fundamental form of the composition to $\varphi_t : (M, g(t)) \to (M, \hat{g}(t))$ and $f_t : (M, \hat{g}(t)) \to (M, D)$,

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_t =& V(\hat{g}(t)) \circ \varphi_t = (\Delta_{\hat{g}(t)} f_t) \circ \varphi_t \\ =& \Delta_{\varphi_t^* \hat{g}(t)}(f_t \circ \varphi_t) - df_t(\Delta_{\varphi_t^* \hat{g}(t)} \varphi_t) \\ =& \Delta_{g(t)}(f_t \circ \varphi_t).\end{aligned}$$

So $\psi_t = f_t \circ \varphi_t : (M, g(t)) \to (M, D)$ satisfies the harmonic map heat flow

$$\frac{\partial}{\partial t}\psi_t = \Delta_{g(t)} \psi_t,$$

with initial value $\psi_0 = \varphi_0 = \mathrm{Id}_M$.

To apply the standard PDE theory on vector bundles to the harmonic map heat flow, we consider $\iota \circ \psi_t$ instead of $\psi_t$, where $\iota : M \to \mathbb{R}^q$ is a fixed smooth embedding. Then

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\iota \circ \psi_t) - \Delta_{g(t)}(\iota \circ \psi_t) \\
=& d\iota \left( \frac{\partial}{\partial t}\psi_t \right) - d\iota \left( \Delta_{g(t)}\psi_t \right) - \left( \Delta_{(\psi_t^{-1})^* g(t)} \iota \right) \circ \psi_t \\
=& - \left( \Delta_{(\psi_t^{-1})^* g(t)} \iota \right) \circ \psi_t,
\end{aligned}
$$

depends only on $D(\iota \circ \psi_t)$ and $\iota \circ \psi_t$, and therefore the harmonic map heat flow enjoys short-time existence and uniqueness.

Let $(M, g_0)$ be a compact Riemannian manifold without boundary. Assume that $\{g_1(t)\}_{t\in[0,T_1)}$ and $\{g_2(t)\}_{t\in[0,T_2)}$ are two solutions to

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}_{g(t)},$$

with the same initial value $g_1(0) = g_2(0) = g_0$. We claim that

$$g_1(t) = g_2(t),\ \forall t \in [0, T_1) \cap [0, T_2).$$

In order to prove this, we argue by contradiction. Suppose that $g_1(t) \neq g_2(t)$ for some $t \in [0, T_1) \cap [0, T_2)$. Define

$$T_* = \inf\{t \in [0, T_1) \cap [0, T_2) \mid g_1(t) \neq g_2(t)\} > 0.$$

By continuity, $g_1(T_*) = g_2(T_*)$. For $i = 1, 2$, let $\psi_t^i : (M, g_i(t)) \to (M, D)$ be the solution of the harmonic map heat flow

$$\frac{\partial}{\partial t}\psi_t^i = \Delta_{g_i(t)}\psi_t^i$$

with initial condition $\psi_{T_*}^i = \mathrm{Id}_M$. It follows from standard PDE theory that $\psi_t^1$ and $\psi_t^2$ are defined on some time interval $[T_*, T_* + \varepsilon)$ for some $\varepsilon > 0$. Moreover, if we choose $\varepsilon > 0$ small enough, then $\psi_t^1$ and $\psi_t^2$ are diffeomorphisms of $M$ for $t \in [T_*, T_* + \varepsilon)$.

For each $t \in [T_*, T_* + \varepsilon)$, we define two Riemannian metrics $\hat{g}_1(t)$ and $\hat{g}_2(t)$ on $M$ by

$$(\psi_t^1)^*(\hat{g}_1(t)) = g_1(t), \ (\psi_t^2)^*(\hat{g}_2(t)) = g_2(t).$$

Then $\hat{g}_1(t)$ and $\hat{g}_2(t)$ are solutions of the Ricci-DeTurck flow with $\hat{g}_1(T_*) = \hat{g}_2(T_*)$ since we have shown that

$$\frac{\partial}{\partial t}\psi_t^i = \Delta_{g(t)}\psi_t^i = \Delta_{(\psi_t^i)^*(\hat{g}_i(t))}\psi_t^i = (\Delta_{\hat{g}_i(t)} f_t^i) \circ \psi_t^i = V(\hat{g}_i(t)) \circ \psi_t^i,$$

with the identity map $f_t^i : (M, \hat{g}_i(t)) \to (M, D)$ and that

$$
\begin{aligned}
&\frac{\partial g_i(t)}{\partial t} + 2\mathrm{Ric}_{g_i(t)} \\
=&\frac{\partial}{\partial t}((\psi_t^i)^*(\hat{g}_i(t))) + 2\mathrm{Ric}_{(\psi_t^i)^*(\hat{g}_i(t))} \\
=&(\psi_t^i)^* \left(\mathcal{L}_{V(\hat{g}_i(t))}\hat{g}_i(t) + \frac{\partial \hat{g}_i(t)}{\partial t}\right) + 2(\psi_t^i)^*(\mathrm{Ric}_{\hat{g}_i(t)}) \\
=&(\psi_t^i)^* \left(\mathcal{L}_{V(\hat{g}_i(t))}\hat{g}_i(t) + \frac{\partial \hat{g}_i(t)}{\partial t} + 2\mathrm{Ric}_{\hat{g}_i(t)}\right).
\end{aligned}
$$

By uniqueness of the Ricci-DeTurck flow, $\hat{g}_1(t) = \hat{g}_2(t)$ on for $[T_*, T_* + \varepsilon)$. Then the solutions to

$$\frac{\partial}{\partial t}\psi_t^1 = V(\hat{g}_1(t)) \circ \psi_t^1,\ \frac{\partial}{\partial t}\psi_t^2 = V(\hat{g}_2(t)) \circ \psi_t^2,$$

are the integral curves of the same time-dependent vector fields with the same initial condition $\psi_0^1 = \psi_0^2$. Therefore,

$$\psi_t^1 = \psi_t^2,\ g_1(t) = (\psi_t^1)^*(\hat{g}_1(t)) = (\psi_t^2)^*(\hat{g}_2(t)) = g_2(t),$$

for $t \in [T_*, T_* + \varepsilon)$, which contradicts the definition of $T_*$.

Let $M$ be a smooth manifold without boundary and $E$ be smooth vector bundles over $M$. Choose any connections on $TM$ and $E$, and let $D$ be the induced connections on $T^*M^{\otimes k} \otimes E$ for any $k \in \mathbb{N}$. Assume that $E^+$ is an open subset of $E$. Let $P : \Gamma(E^+) \to \Gamma(E)$ be the differential operator given by

$$(Pu)|_p = \langle a(u|_p), D^2u|_p \rangle + f(Du|_p, u|_p), \; p \in M,$$

where $a : E^+ \to TM \otimes TM \otimes E^* \otimes E$ and $f : (T^*M \otimes E) \oplus E^+ \to E^* \otimes E$ are smooth maps which map each fibre to the fibre of the same point. Then $P$ is elliptic at $u$ is equivalent to there exists some $c > 0$,

$$\langle \hat{\sigma}[dP_u](\eta)h, h\rangle_E = \langle \langle a(u|_p), \eta \otimes \eta \otimes h\rangle, h\rangle_E \geq c\langle h, h\rangle_E,$$

for any $p \in M$, $\eta \in T_p^*M$, $h \in E_p$ and any metric $\langle , \rangle_E$ of $E$.

**Theorem.** If $P$ is elliptic at $u_0 \in \Gamma(E^+)$, then there exist a positive number $T > 0$ and a smooth family of sections $\{u(t)\}_{t\in[0,T)}$ of $E^+$ on $M$ such that $u(0) = u_0$ and that

$$\frac{\partial u(t)}{\partial t} = P(u(t)).$$

**Proof.** Without loss of generality, assume that $u_0 = 0$.
For $s > 0$ and $0 < \alpha < 1/2$, define

$$U_s = \{u \in C^{2m+2,2\alpha,m+1,\alpha}(M \times [0,s], E) \mid u(0) = 0\},$$
$$V_s = C^{2m,2\alpha,m,\alpha}(M \times [0,s], E),$$
$$F_s : U_s \to V_s, u \mapsto \frac{\partial u(t)}{\partial t} - P(u(t)).$$

Define $\hat{u}(t) = tf(0)$. Then $F_s(\hat{u})|_{t=0} = 0$.
By continuity, $\varepsilon_s = \|F_s(\hat{u})\|_{V_s} \to 0$ for $s \to 0$.
Let $L_s$ be the linearise of $F_s$ at $\hat{u}$.
Then for $h \in U_s$,

$$L_s(h) = d(F_s)_{\hat{u}}(h) = \frac{\partial h(t)}{\partial t} - \langle a(\hat{u}), D^2 h \rangle - \langle b, Dh \rangle - \langle c, h \rangle,$$

where $b(t) \in \Gamma(TM \otimes E^* \otimes E)$ and $c(t) \in \Gamma(E^* \otimes E)$ depends on $a$ and $f$.

By parabolic Schauder estiamte and parabolic maximum principle,

$$\|h\|_{U_s} \le C(\|L_s h\|_{V_s} + \|h\|_\infty) \le C\|L_s h\|_{V_s}.$$

So $L_s$ has a trivial kernel.
Moreover, since the linear parabolic PDE is solvable, $L_s$ is surjective when $s$ is sufficiently small.
Thus, $L_s$ is invertible and $\|L_s\| \le C$.
By the inverse function theorem, $F_s$ is invertible on a neighborhood of $\hat{u}$ to a ball of radius $r$ centered at $F_s(\hat{u})$, where $r$ is independent of $s$.
When $\|F_s(\hat{u})\|_{V_s} = \varepsilon_s < r$, $0 \in B(F_s(\hat{u}), r)$ and we obtain the solution of $F_s(u) = 0$.