Chennai Mathematical Institute

Asymptotic Convexity in Mean Curvature Flow

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Plan

1. Convexity
2. Asymptotic convexity
3. Maximum principles in mean convex setting
4. The L^p estimates
5. Stampacchia iteration

Notion of Convexity

- A hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$ is said to be uniformly convex if second fundamental form A is strictly positive. So the principal curvatures $\kappa_1, \ldots, \kappa_n$ are strictly positive.
- A hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$ is said to be mean convex if the mean curvature satisfies $H = \mathrm{tr}(A) = \kappa_1 + \cdots + \kappa_n > 0$.

- Recall the formula of k-th elementary symmetric polynomial.

Definition

For any $k = 1, \dots, n$, the **k-th elementary symmetric polynomial** $S_k : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$S_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. We adapt the convention that $S_0 \equiv 1$ and $S_k \equiv 0$ for $k > n$.

Notion of Convexity

- Associated to each k we can also define the domain of positivity of first k elementary symmetric polynomials Γ_k given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \dots, S_k(\lambda) > 0\}.$$

- Notice that Γ_k are cones in Euclidean space and have the inclusion property

$$\Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1.$$

- In this formulation, a hypersurface is mean convex if the vector $(\kappa_1, \dots, \kappa_n) \in \Gamma_1$.

A result from convex geometry

Lemma

Let $A = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_n > 0\}$ denote the positive cone. The sets Γ_k coincide with the connected component of the domain $\{\lambda \in \mathbb{R}^{n+1} : S_k(\lambda) > 0\}$ containing the positive cone A. Further, the cone Γ_n coincides with the positive cone A.

Hence,

$$\kappa_1 > 0, \dots, \kappa_n > 0 \iff (\kappa_1, \dots, \kappa_n) \in \Gamma_n$$

so uniform convexity is equivalent to $(\kappa_1, \dots, \kappa_n) \in \Gamma_n$.

Main theorem

Theorem

Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth solution of the mean curvature flow with $n \geq 2$ such that $X(M^n, 0) = \mathcal{M}_0$ is compact and of positive mean curvature. Then, for any $\eta > 0$ there exists a constant $C_\eta > 0$ depending only on n, η and $\mathcal{M}_0$ such that

$$S_k \geq -\eta H^k - C_{\eta,k} \tag{1.1}$$

on $\mathcal{M}_t$ for any $t \in [0, T)$.

This means that the negative part of S_k cannot grow faster than H^k.

Heuristically we want to do the following. Given the inequality

$$S_k \geq -\eta H^k - C_{\eta,k} \tag{2.1}$$

we can divide the equation by H^k to obtain,

$$\frac{S_k}{H^k} \geq -\eta - \frac{C_{\eta,k}}{H^k}.$$

Now as $t \to T$, $H^k \to \infty$ which implies

$$\frac{S_k}{H^k} \geq -2\eta$$

for t sufficiently close to T. We will do this rigorously after proving the main theorem.

We will restrict ourselves to the case $k = 2$. The higher estimates can be done similarly using a different function.

Goal

We want to prove that for any given $\eta > 0$, there exists a positive constant C_η such that

$$S_2 \geq -\eta H^2 - C_\eta.$$

Candidate function

Let $\sigma \in (0, 2)$. Consider a function $g_{\sigma,\eta} : \mathcal{M} \times [0, T) \to \mathbb{R}$ defined as

$$g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1 + \eta) \right) H^\sigma = \frac{|A|^2 - (1 + \eta)H^2}{H^{2-\sigma}}.$$

If we expand $|A|^2 = \sum \kappa_i^2$ and $H^2 = \sum \kappa_i^2 + 2S_2$, the function $g_{\sigma,\eta}$ can be written as

$$g_{\sigma,\eta} = \frac{-2S_2 - \eta H^2}{H^{2-\sigma}}.$$

Uniform bound on $g_{\sigma,\eta}$

We want a uniform bound on $g_{\sigma,\eta}$. Suppose $g_{\sigma,\eta} \leq K$. Then

$$\begin{aligned} \frac{|A|^2 - (1+\eta)H^2}{H^{2-\sigma}} &\leq K \\ |A|^2 - (1+\eta)H^2 &\leq KH^{2-\sigma} \\ -2S_2 - \eta H^2 &\leq KH^{2-\sigma} \\ -2S_2 &\leq KH^{2-\sigma} + \eta H^2. \end{aligned}$$

Now by Young's inequality

$$KH^{2-\sigma} \leq \eta H^2 + C_1$$

which after substitution gives the desired result.

Uniform bound on $g_{\sigma,\eta}$

In order to get a uniform bound, we first calculate the time evolution of $g_{\sigma,\eta}$ in hopes of some application of the maximum principle.

Lemma

The evolution equation of $g_{\sigma,\eta}$ is given by

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} =& \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left\langle \nabla H, \nabla g_{\sigma,\eta}\right\rangle - \frac{\sigma(1-\sigma)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ & - \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta}. \end{aligned} \tag{2.2}$$

We can't directly apply the maximum principle here because of the presence of the last term $\sigma|A|^2 g_{\sigma,\eta}$. However, when $\sigma = 0$, we can conclude that

$$\frac{|A|^2}{H^2} \leq c_0(\mathcal{M}_0).$$

Together with algebraic identity $H^2 \leq n|A|^2$, this implies that

$$\frac{1}{n}H^2 \leq |A|^2 \leq c_0 H^2.$$

Hence, for mean-convex hypersurfaces the blow-up rate of $|A|^2$ and H^2 is the same.

This also implies that

$$g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1+\eta)\right) H^\sigma \leq c_0 H^\sigma.$$

After a failed attempt at maximum principle, our next hope is to get an integral estimate of $g_{\sigma,\eta}$.

Let $(g_{\sigma,\eta})_+ = \max\{(g_{\sigma,\eta})_+, 0\}$ be the positive part of $(g_{\sigma,\eta})$.

L^p estimates

For any $\eta \in (0,1)$ there exists constants a and b such that the $L^p(\mathcal{M})$ norm of $(g_{\sigma,\eta})_+$ is a non-decreasing function of t if the following holds

$$p \geq a, \qquad \sigma \leq (bp)^{-\frac{1}{2}}.$$

The time derivative of integral of g_+^p satisfies the inequality :

Lemma

There exists constant c_2, c_3 such that

$$\frac{d}{dt}\int_{\mathcal{M}_t} g_+^p d\mu \leq -\frac{p(p-1)}{2}\int_{\mathcal{M}_t} g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$-p\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu + p\sigma \int_{\mathcal{M}_t} |A|^2 g_+^p d\mu \tag{3.1}$$

for any $p \geq c_2$.

Again there is one bad term but we will see that it can be absorbed.

Proof. Differentiating with respect to time and using the evolution equation for $p \geq 2$

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{M}_t} g_+^p d\mu &= \int \left(p g_+^{p-1}\partial_t g - H^2 g_+^p \right) d\mu \\
&\leq \int p g_+^{p-1}\left(\Delta g + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g\rangle - \sigma(1-\sigma)\frac{|\nabla H|^2}{H^2} \right) d\mu \\
&\quad - \int p g_+^{p-1}\left(\frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 \right) d\mu + \int p\sigma |A|^2 g_+^p d\mu
\end{aligned} \tag{3.2}$$

where we are ignoring $-H^2 g_+^p$ term.
For the term with $g_+^{p-1}\Delta g$, we can use integration by parts to get

$$\begin{aligned}
\int p g_+^{p-1}\Delta g d\mu &= -p\int \left\langle \nabla g_+^{p-1}, \nabla g \right\rangle d\mu \\
&= -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu
\end{aligned} \tag{3.3}$$

For the next term we need the following lemma :

Lemma

If (x,t) is such that $g(x,t) = g_+(x,t) \geq 0$, so $(1+\eta)H^2 \leq |A|^2 \leq c_0 H^2$. Then

$$|\nabla A \cdot H - \nabla H \otimes A|^2 \geq \frac{\eta^2}{4n(n-1)^2 c_0} H^2 |\nabla H|^2.$$

From which we can write for any $c_1 \geq 4n(n-1)^2 c_0 \eta^{-2}$ or $\frac{1}{c_1} \leq \frac{\eta^2}{4n(n-1)^2 c_0}$,

$$\begin{aligned} \frac{g_+^{p-1}}{H^{4-\sigma}} |\nabla A \cdot H - \nabla H \otimes A|^2 &\geq \frac{g_+^{p-1}}{c_1 H^{2-\sigma}} |\nabla H|^2 \\ &\geq \frac{g_+^{p-1}}{2c_1 H^{2-\sigma}} |\nabla H|^2 + \frac{1}{2c_0 c_1} \frac{g_+^p}{H^2} |\nabla H|^2 \qquad (3.4) \end{aligned}$$

For the inner product term, let $p \geq \max 2, 1 + 4c_0c_1$ and use Cauchy-Schwarz inequality to get,

$$
\begin{aligned}
2(1-\sigma)p\frac{g_+^{p-1}}{H}\langle \nabla H, \nabla g\rangle &\leq 2p\frac{g_+^{p-1}}{H}|\nabla H||\nabla g| \\
&\leq p\left(\frac{g_+^{p}|\nabla H|^2}{2c_0c_1H^2} + 2c_0c_1g_+^{p-2}|\nabla g|^2\right) \\
&\leq p\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 - p\frac{g_+^{p-1}}{2c_1H^{2-\sigma}}|\nabla H|^2 \\
&\quad + \frac{p(p-1)}{2}g_+^{p-2}|\nabla g|^2 \qquad (3.5)
\end{aligned}
$$

Substituting this

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu \leq & -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu + p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu \\ & + \frac{p(p-1)}{2}\int g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ & - 2p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu + p\sigma\int |A|^2 g_+^p d\mu.\end{aligned}$$

where $c_3 = \frac{1}{2c_1}$.

Handling the bad term

We can estimate the positive term in the previous time derivative as follows

Lemma

There exists a constant c_4 such that

$$\frac{1}{c_4}\int |A|^2 g_+^p d\mu \leq \left(p + \frac{p}{\beta}\right)\int g_+^{p-2}|\nabla g|^2 + (1+\beta p)\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$+ \int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu$$

for any $\beta > 0, p > 2$.

Sketch of proof:

$$\begin{aligned}\Delta g &= \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 \\ &+ \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H - 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\ &+ \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle.\end{aligned}$$

Now we want to use Simon's identity

$$\Delta|A|^2 = 2\left\langle h_{ij}, \nabla_i\nabla_j H\right\rangle + 2|\nabla A|^2 + 2Z$$

and $Z = H\,\mathrm{tr}(A^3) - |A|^4$.

Handling the bad term

Multiplying by $g_+^p H^{-\sigma}$ and using Green's identity

$$
\begin{aligned}
-2\int \frac{g_+^p Z}{H^2} d\mu = p\int \frac{1}{H^\sigma} g_+^{p-1} |\nabla g|^2 d\mu - 2p \int \frac{g_+^{p-1}}{H^2} \langle h_{ij}, \nabla_i g \nabla_j H\rangle \, d\mu \\
+ 4\int \frac{g_+^p}{H^3} \langle h_{ij}, \nabla_i H \nabla_j H\rangle \, d\mu + 2\int \frac{g_+^p}{H^4} |\nabla A \cdot H - \nabla H \otimes A|^2 d\mu \\
+ p\int \left((2-\sigma)\frac{g_+^p}{H^{1+\sigma}} + 2(1+\eta)\frac{g_+^{p-1}}{H} \right) \langle \nabla g, \nabla H\rangle \, d\mu \\
- 2\int \left(\frac{g_+^{p+1}}{H^{2+\sigma}} + (2+\eta)\frac{g_+^p}{H^2} \right) |\nabla H|^2 d\mu \qquad (3.6)
\end{aligned}
$$

Handling the bad term

From Huisken's lemma $-2Z \geq \eta H^2|A|^2$ and using utilizing $g \leq c_0 H^\sigma$ (and $|A| \leq c_0 H$) with Cauchy-Schwarz inequality,

$$\eta \int g_+^p |A|^2 d\mu \leq c_0 p \int g_+^{p-2} |\nabla g|^2 d\mu + 4p(c_0+1) \int \frac{g_+^{p-1}}{H} |\nabla g||\nabla H| d\mu$$
$$+ 4c_0^2 \int \frac{g_+^{p-1}}{H^{2-\sigma}} |\nabla H|^2 d\mu + 2c_0 \int \frac{g_+^{p-1}}{H^{4-\sigma}} |\nabla A \cdot H - \nabla H \otimes A|^2 d\mu \tag{3.7}$$

Also, for any $\beta > 0$,

$$2\frac{g_+^{p-1}}{H} |\nabla H||\nabla g| \leq \frac{g_+^{p-2}}{\beta} |\nabla g|^2 + \beta \frac{g_+^p}{H^2} |\nabla H|^2$$
$$= \frac{g_+^{p-2}}{\beta} |\nabla g|^2 + c_0 \beta \frac{g_+^{p-1}}{H^{2-\sigma}} |\nabla H|^2 \tag{3.8}$$

The results so far:

Proposition

For any $\eta \in (0,1)$ there exists constants c_5, c_6 such that the $L^p(\mathcal{M})$ norm of $(g_{\sigma,\eta})_+$ is non-decreasing function of t if the following holds

$$p \geq c_5, \qquad \sigma \leq (c_6 p)^{-\frac{1}{2}}.$$

So we have L^p estimates of $g_{\sigma,\eta}$ provided σ is sufficiently small; specifically in the order of $p^{-\frac{1}{2}}$.

Stampacchia lemma

Let $\psi : [k_0, \infty) \to \mathbb{R}$ be a non-negative, non-increasing function which satisfies

$$\psi(h) \leq \frac{C}{(h-k)^\alpha}\psi(k)^\beta \text{ for all } h > k > k_0 \tag{4.1}$$

for some constants $C > 0$, $\alpha > 0$ and $\beta > 1$. Then

$$\psi(k_0 + d) = 0, \tag{4.2}$$

where $d^\alpha = C\psi(k_0)^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}$.

Step 1

Proof of main theorem: We want to prove a uniform estimate on $(g_{\sigma,\eta})_+$ using Stampacchia iteration. Let $k \geq k_0$, where

$$k_0 = \sup_{\sigma \in [0,1]} \sup_{\mathcal{M}_0} g_{\sigma,\eta}$$

Define $v = (g_{\sigma,\eta} - k)_+^{\frac{p}{2}}$ and $A(k,t) = \{x \in \mathcal{M}_t : v(x,t) > 0\}$. The function we will be applying Stampacchia lemma is

$$\psi(k) = \int_0^T \int_{A(k,t)} d\mu dt.$$

Differentiating v^2 with respect to time we get for p large enough

$$\frac{d}{dt}\int_{\mathcal{M}_t} v^2 d\mu + \int_{\mathcal{M}_t} |\nabla v|^2 d\mu \leq \sigma p \int_{\mathcal{M}_t} |A|^2 v^2 d\mu$$
$$\leq c_0 \sigma p \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu$$

Also from the Michael-Simon result, we have a Sobolev-type inequality given by

$$\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \leq C(n) \int_{\mathcal{M}_t} |\nabla v|^2 d\mu + C(n) \left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}} \left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \tag{4.3}$$

where $q = \frac{n}{n-2}$ if $n > 2$ and an arbitrary number greater than 1 if $n = 2$.

Step 1

We can estimate the H^n factor in the integral on $A(k,t)$ using the previous proposition and the equality

$$\begin{aligned}\int_{\mathcal{M}_t} H^n g^p_{\sigma,\eta}\,d\mu &= \int_{\mathcal{M}_t} H^n \left(\frac{|A|^2-(1+\eta)H^2}{H^2}\right)^p H^{p\sigma}\,d\mu \\ &= \int_{\mathcal{M}_t} g^p_{\sigma',\eta}\,d\mu\end{aligned}$$

where $\sigma' = \sigma + \frac{n}{p}$. Let

$$p \geq \max\{c_5, 4n^2 c_6\} \quad \text{and} \quad \sigma \leq (4c_6 p)^{-\frac{1}{2}}$$

so that

$$\sigma' = \sigma + \frac{n}{p} \leq \frac{1}{2\sqrt{c_6 p}} + \frac{1}{\sqrt{p}}\frac{n}{\sqrt{p}} \leq \frac{1}{\sqrt{c_6 p}}.$$

Step 1

From the previous result,

$$\begin{aligned}
\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}} &\leq \left(\int_{A(k,t)} H^n \left(\frac{g^p_{\sigma,\eta}}{k^p}\right) d\mu\right)^{\frac{2}{n}} \\
&= k^{-\frac{2p}{n}} \left(\int_{A(k,t)} g^p_{\sigma',\eta} d\mu\right)^{\frac{2}{n}} \\
&\leq k^{-\frac{2p}{n}} \left(\int_{\mathcal{M}_t} (g_{\sigma',\eta})^p_+ d\mu\right)^{\frac{2}{n}} \\
&\leq k^{-\frac{2p}{n}} \left(\int_{\mathcal{M}_0} (g_{\sigma',\eta})^p_+ d\mu\right)^{\frac{2}{n}} \\
&\leq \left(\frac{|\mathcal{M}_0| k_0}{k}\right)^{\frac{2p}{n}}
\end{aligned}$$

Step 1

We can fix $k_1 > k_0$ such that for any $k \geq k_1$ the term $\int_{A(k,t)} H^n d\mu$ in Michael-Simon inequality is less than $\frac{1}{2C(n)}$. For such k, eliminating the gradient term,

$$\frac{d}{dt}\int_{\mathcal{M}_t} v^2 d\mu + \frac{1}{2C(n)}\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \leq c_0 \sigma p \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu. \tag{4.4}$$

Let $t_0 \in [0,T]$ be the time when $\sup_{t\in[0,T)} \int_{\mathcal{M}_t} v^2 d\mu$ is attained (we let $t_0 = T$ if it is not attained in the interior). Integrating from 0 to t_0,

$$\int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{2C(n)} \int_0^{t_0} \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q}} dt \leq c_0 \sigma p \int_0^{t_0} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \quad (4.5)$$

Now integrating this from t_0 to $T-\epsilon$,

$$\int_{\mathcal{M}_{T-\epsilon}} v^2 d\mu - \int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{2C(n)} \int_{t_0}^{T-\epsilon} \left(\int_{\mathcal{M}_t} v^{2q} \right)^{\frac{1}{q}} dt \leq c_0 \sigma p \int_{t_0}^{T-\epsilon} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt. \quad (4.6)$$

Throwing away $\int_{\mathcal{M}_{T-\epsilon}} v^2 d\mu$ term and adding to half of this to the previous inequality,

$$\frac{1}{2} \int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{4C(n)} \int_0^{T-\epsilon} \left(\int_{\mathcal{M}_t} v^{2q} \right)^{\frac{1}{q}} dt \leq c_0 \sigma p \int_0^{T-\epsilon} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt$$

Step 1

This is same as

$$\sup_{[0,T)} \int_{\mathcal{M}_t} v^2 d\mu + \int_0^T \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q}} dt \leq 2\max\{1, 2C(n)\} c_0 \sigma p \int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt. \tag{4.7}$$

This finishes step 1.

Step 2

Recall the interpolation inequality for L^p spaces for any $f \in L^q \cap L^r$,

$$||f||_{q_0} \leq ||f||_q^{\alpha} ||f||_r^{1-\alpha}$$

where $\frac{1}{q_0} = \frac{\alpha}{q} + \frac{1-\alpha}{q}$ and $1 < q_0 < q$. Setting $r = 1, \alpha = \frac{1}{q_0}$ and $f = v^2$ we get

$$\left(\int_{\mathcal{M}_t} v^{2q_0} d\mu \right)^{\frac{1}{q_0}} \leq \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q_0 q}} \left(\int_{\mathcal{M}_t} v^2 d\mu \right)^{1-\frac{1}{q_0}} . \tag{4.8}$$

Step 2

Integrating this in time and using Young's inequality,

$$
\begin{aligned}
\left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu dt\right)^{\frac{1}{q_0}} &\leq \left(\sup_{[0,T)} \int_{A(k,t)} v^2 d\mu\right)^{1-\frac{1}{q_0}} \left(\int_0^T \left(\int_{A(k,t)} v^{2q} d\mu\right)^{\frac{1}{q}} dt\right)^{\frac{1}{q_0}} \\
&\leq \frac{\sup_{[0,T)} \int_{A(k,t)} v^2 d\mu}{\frac{q_0}{q_0-1}} + \frac{\int_0^T \left(\int_{A(k,t)} v^{2q} d\mu\right)^{\frac{1}{q}} dt}{q_0} \\
&\leq \sup_{[0,T)} \int_{A(k,t)} v^2 d\mu + \int_0^T \left(\int_{A(k,t)} v^{2q} d\mu\right)^{\frac{1}{q}} dt \\
&\leq c_8 \sigma p \int_0^T \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt
\end{aligned}
$$

where $c_8 = 2\max\{1, 2C(n)\}c_0$.

Step 3

Set $\psi(k) = \int_0^T \int_{A(k,t)} d\mu dt$. We will obtain bounds on ψ which along with the Stampacchia lemma will imply a uniform bound of $g_{\sigma,\eta}$. Now Hölder inequality yields,

$$\int_0^T \int_{A(k,t)} v^2 d\mu dt \leq \left(\int_0^T \int_{A(k,t)} 1 d\mu dt \right)^{1-\frac{1}{q_0}} \left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu dt \right)^{\frac{1}{q_0}} \tag{4.9}$$

$$\leq c_8 \sigma p \psi(k)^{1-\frac{1}{q_0}} \int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \tag{4.10}$$

Let $r > 1$ which will be chosen later. Applying Hölder again on the right side with weights r and $\frac{r}{r-1}$,

$$\int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \leq \left(\int_0^T \int_{A(k,t)} d\mu dt \right)^{1-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} H^{2r} g^{pr}_{\sigma,\eta} d\mu dt \right)^{\frac{1}{r}}$$

$$= \psi(k)^{1-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} g^{pr}_{\sigma'',\eta} d\mu dt \right)^{\frac{1}{r}}$$

where $\sigma'' = \sigma + \frac{2}{p}$.

For r large enough and p, σ^{-1} small enough there exists a constant $c_9 > 0$ independent of time such that

$$\int_0^T \int_{A(k,t)} H^2 g_{\sigma,\eta}^p \, d\mu dt \leq c_9^{\frac{1}{r}} \psi(k)^{1-\frac{1}{r}}. \tag{4.11}$$

Combining the two for all $h > k \geq k_1$, we have

$$\begin{aligned}(h-k)^p \psi(h) &= \int_0^T \int_{A(h,t)} (h-k)^p \, d\mu dt \\ &\leq \int_0^T \int_{A(k,t)} v^2 \, d\mu dt \\ &\leq c_8 \sigma p c_9^{\frac{1}{r}} \psi(k)^{2-\frac{1}{r}-\frac{1}{q_0}}.\end{aligned}$$

Let $\gamma = 2 - \frac{1}{r} - \frac{1}{q_0}$ and $c_{10} = c_8 c_9^{\frac{1}{r}}$. Fix $r > \frac{q_0}{q_0 - 1}$ (so $\gamma > 1$) and p large enough, σ small enough such that $\sigma p < 1$ then gives

$$\psi(h) \leq \frac{c_{10}}{(h-k)^p} \psi(k)^\gamma \tag{4.12}$$

Stampacchia lemma now implies $\psi(k) = 0$ for all $k \geq k_1 + d$ where $d^p = c_{10} 2^{\frac{\gamma p}{\gamma - 1} + 1} \psi(k_1)^{\gamma - 1}$. Hence,

$$g_{\sigma,\eta} \leq k_1 + d \leq K \doteqdot k_1 + c_{10} 2^{\frac{\gamma p}{\gamma - 1} + 1} (|\mathcal{M}_0| T)^{\gamma - 1}$$

or

$$|A|^2 - (1 + \eta) H^2 \leq K H^{2 - \sigma}.$$

Step 3

By Young's inequality there exists a constant C_η such that,

$$|A|^2 - H^2 \leq \eta H^2 + KH^{2-\sigma} \leq 2\eta H^2 + 2C_\eta.$$

Notice that $|A|^2 - H^2 = -\sum_{i \neq j} \kappa_i \kappa_j = -2S_2$ which implies the desired estimate. This completes the proof.

- The norm of the second fundamental form satisfies the evolution equation

$$\begin{aligned}\partial_t|A|^2 &= \Delta|A|^2 + -2|\nabla A|^2 + 2|A|^4 \\ &\leq \Delta|A|^2 + 2|A|^4.\end{aligned}$$

- If we consider the times when $|A|^2$ achieves the maximum in the previous equation we can derive

$$\max_{\mathcal{M}_t} |A|^2 \geq \frac{1}{2(T-t)}.$$

- If there exists a constant $C > 1$ such that we have the upper bound

$$\max_{\mathcal{M}_t} |A| \leq \frac{C}{\sqrt{2(T-t)}}$$

we say that the flow is developing at time T a **type I singularity**.

- If such a constant does not exist, we say the flow is developing a **type II singualrity**. In this case

$$\limsup_{t\to T} \max_{\mathcal{M}_t} |A|\sqrt{T-t} = \infty$$

- Type I singularities are better understood. In fact the limiting hypersurface of type I singularity after rescaling satisfy the equation

$$H = \langle X, \nu \rangle \, .$$

- Huisken classified compact hypersurfaces which limits of type I singularity. If M^n, $n \geq 2$, is compact with non-negative mean curvature H and satisfies the equation $H = \langle X, \nu \rangle$, then M^n is a sphere of radius $\sqrt{n}$.

- We will focus on type II singularities and their asymptotic convexity.
- Suppose a maximal solution $X : M \times [0, T) \to \mathbb{R}^{n+1}$ develops a type II singularity. Choose a sequence of points $\{(x_m, t_m)\}$ in spacetime as follows. For each integer $m \geq 1$, let $t_m \in [0, T - \frac{1}{m}]$, $x_m \in M$ such that

$$H^2(x_m, t_m)\left(T - \frac{1}{m} - t_m\right) = \sup_{(x,t) \in M \times \left[0, T - \frac{1}{m}\right]} H^2(x, t)\left(T - \frac{1}{m} - t\right) \quad (5.1)$$

Set $L_m = H(x_m, t_m)$, $\alpha_m = -L_m^2 t_m$ and $\omega_m = L_m^2(T - \frac{1}{m} - t_m)$.

The blow-up process

$$H^2(x_m, t_m)\left(T - \frac{1}{m} - t_m\right) = \sup_{(x,t)\in M\times\left[0, T-\frac{1}{m}\right]} H^2(x,t)\left(T - \frac{1}{m} - t\right)$$

Set $L_m = H(x_m, t_m)$, $\alpha_m = -L_m^2 t_m$ and $\omega_m = L_m^2(T - \frac{1}{m} - t_m)$.

Lemma

For singularities of type II, the following holds as $m \to \infty$,

$$t_m \to T, \quad L_m \to \infty, \quad \alpha_m \to -\infty, \text{ and } \omega_m \to \infty.$$

Now we will rescale the hypersurfaces to analyze the limiting behavior. For each $m \geq 1$, define a family of immersions by

$$X_m(x,t) = L_m(X(x, L_m^{-2}t + t_m) - X(x_m, t_m)) \text{ for } t \in [\alpha_m, \omega_m].$$

Let A_m and H_m denote the fundamental form of the rescaled immersions. Then by the definition of L_m and X_m we have

$$X_m(x_k, 0) = 0 \quad \text{and} \quad H_m(x_m, 0) = 1.$$

Further, observe that

$$H_m^2(x,t) = L_m^{-2} H^2(x, L_m^{-2}t + t_m) \leq \frac{T - \frac{1}{m} - t_m}{T - \frac{1}{m} - t_m - L_m^{-2}t} = \frac{\omega_m}{\omega_m - t}.$$

From the previous lemma $\omega_m \to \infty$, so for any $\epsilon > 0$ and $\overline{\omega}$, there exists a m_0 such that

$$\max_{x \in M} H_m(x,t) \leq 1 + \epsilon$$

for any $m \geq m_0$ and $t \in [\alpha_{m_0}, \overline{\omega}]$. This curvature bound implies analogous bounds on the second fundamental form as well as its covariant derivatives. Further,

$$(S_k)_m = L_m^{-k} S_k \geq -\eta L_m^{-k} H^k - L_m^{-k} C_{\eta,k}$$

or

$$\begin{aligned}(S_k)_m &\geq -\eta H_m^k - L_m^{-k} C_{\eta,k} \\ &\geq -\eta (1+\epsilon)^k - L_m^{-k} C_{\eta,k}\end{aligned}$$

which can be made arbitrarily small in the limit $m \to \infty$.

Invoking Arzela-Ascoli theorem there exists a subsequence of X_m converging uniformly on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}$ to a limiting solution X_∞ of the mean curvature flow. This proves the asymptotic convexity of the flow in the following sense.

Theorem

Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth maximal solution of the mean curvature flow with $X(\cdot, 0) = \mathcal{M}_0$ compact and of positive mean curvature. Further, assume that the flow develops a singularity of type II. Then there exists a sequence of rescaled flow $X_k(\cdot, t)$ converging smoothly on every compact set to a mean curvature flow $X_\infty(\cdot, t)$ which is defined for $t \in (-\infty, \infty)$. Also, the limit hypersurface X_∞ is convex (not necessarily uniformly convex) for each $t \in (-\infty, \infty)$ and satisfies $0 < H_\infty \leq 1$ everywhere with equality at least at one point.