2. Convexity estimates

As observed in the previous chapter, a convex hypersurface evolving under (rescaled) mean curvature flow converges smoothly to a round sphere. This convergence cannot be expected for general mean-convex hypersurfaces as singularities might appear. Huisken and Sinestrari proved in [HS99a, HS99b] that mean convex hypersurface are asymptotically convex i.e. blowing the flow near singularity gives a convex ancient solution.

2.1. Elementary symmetric polynomials and cones

The mean curvature of a hypersurface at a point is the sum of principal curvatures which is a symmetric function. Similarly, Gauss curvature is the product of the principal curvatures. The study of elementary symmetric functions of principal curvatures will be crucial to analyze the convexity of singularities. We begin by recalling the definition of elementary symmetric polynomials.

Definition 2.1.1. For any $k = 1, \ldots, n$, the **k-th elementary symmetric polynomial** $S_k : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$S_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ with the convention $S_0 \equiv 1$.

Associated to each k we can also define the domain of positivity of first k elementary symmetric polynomials Γ_k given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \ldots, S_k(\lambda) > 0\}$$

It is easy to see that Γ_k are cones in the Euclidean space and satisfy $\Gamma_{k+1} \subset \Gamma_k$. In this formulation a hypersurface is mean-convex if the vector $(\kappa_1, \ldots, \kappa_n)$ is in Γ_1. The following proposition was proved in [HS99a] regarding the cones Γ_k.

Proposition 2.1.1. Let $A = \{x \in \mathbb{R}^n : x_1 > 0, \ldots, x_n > 0\}$ denote the positive cone. The sets Γ_k coincide with the connected component of the domain $\{\lambda \in \mathbb{R}^{n+1} : S_k(\lambda) > 0\}$ containing the positive cone A. Further, the cone Γ_n coincides with the positive cone A.

This establishes a hierarchy of convexity with the last one being uniformly convex where the principal curvature vector $(\kappa_1, \ldots, \kappa_n) \in \Gamma_n$ for all points in the hypersurface. The main result of the chapter is the following theorem.

Theorem 2.1.2. Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth solution of the mean curvature flow with $n \geq 2$ such that $X(M^n, 0) = \mathcal{M}_0$ is compact and of positive mean curvature. Then, for any $\eta > 0$ there exists a constant $C_\eta > 0$ depending only on n, η and $\mathcal{M}_0$ such that

$$S_k \geq -\eta H^k - C_{\eta,k} \tag{2.1.1}$$

on $\mathcal{M}_t$ for any $t \in [0, T)$.

This means that the negative part of S_k cannot grow faster that H^k. We will only prove the theorem for $k = 2$ adapted from [HS99b]. A complete proof is done using induction in [HS99a].

2.2. Estimate of S_2

For any $\eta \in \mathbb{R}$ and $\sigma \in [0, 2]$ let

$$g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1+\eta)\right) H^\sigma = \frac{|A|^2 - (1+\eta)H^2}{H^{2-\sigma}} = \frac{-2S_2 - \eta H^2}{H^{2-\sigma}}.$$

Our aim is to derive a uniform bound of $g_{\sigma,\eta}$ which using Young's inequality will imply the desired estimate. The proof of Theorem 2.1.2 for $k = 2$ is divided into two parts. The first part is obtaining an L^p estimate of $g_{\sigma,\eta}$ and the second part is utilizing Stampacchia lemma using Michael-Simon inequality in order to get an L^∞ bound. In order to prove the first part we derive the evolution equation of $g_{\sigma,\eta}$ using the product rule but before that we need the following lemmas.

Lemma 2.2.1. The following equality holds:

$$|\nabla A \cdot H - \nabla H \otimes A|^2 = |\nabla A|^2 H^2 + |A|^2 |\nabla H|^2 - \left\langle \nabla |A|^2, \nabla H \right\rangle H. \tag{2.2.1}$$

Proof. Computing the norm,

$$\begin{aligned}
|\nabla A \cdot H - \nabla H \otimes A|^2 &= \langle \nabla A \cdot H - \nabla H \otimes A, \nabla A \cdot H - \nabla H \otimes A \rangle \\
&= |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - 2H \langle \nabla A, \nabla H \otimes A \rangle \\
&= |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - \left\langle \nabla |A|^2, \nabla H \right\rangle H.
\end{aligned}$$

□

Lemma 2.2.2. The quantity $\frac{|A|^2}{H^2}$ satisfies the differential equation

$$\frac{\partial}{\partial t}\frac{|A|^2}{H^2} = \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A \cdot H - \nabla H \otimes A|^2. \tag{2.2.2}$$

Proof. Computing the time derivative we get

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{|A|^2}{H^2} &= \frac{1}{H^2}\frac{\partial |A|^2}{\partial t} - 2\frac{|A|^2}{H^3}\frac{\partial H}{\partial t} \\
&= \frac{1}{H^2}\left(\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4\right) - 2\frac{|A|^2}{H^3}\left(\Delta H + |A|^2 H\right) \\
&= \frac{\Delta|A|^2}{H^2} - 2\frac{|\nabla A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3}.
\end{aligned}$$

Recall the division formula for Laplacian,

$$\Delta\left(\frac{u}{v}\right) = \frac{\Delta u}{v} - u\frac{\Delta v}{v^2} - \frac{2}{v^2}\langle \nabla u, \nabla v\rangle + 2\frac{u}{v^3}|\nabla v|^2.$$

Calculating the Laplace-Beltrami operator using this,

$$\begin{aligned}
\Delta\frac{|A|^2}{H^2} &= \frac{\Delta|A|^2}{H^2} - |A|^2\frac{\Delta H^2}{H^4} - \frac{2}{H^4}\left\langle \nabla|A|^2, \nabla H^2\right\rangle + \frac{2|A|^2}{H^6}|\nabla H^2|^2 \\
&= \frac{\Delta|A|^2}{H^2} - |A|^2\left(\frac{2H\Delta H + 2|\nabla H|^2}{H^4}\right) - \frac{2}{H^4}\left\langle \nabla|A|^2, 2H\nabla H\right\rangle + 8\frac{|A|^2}{H^6}|\nabla H|^2 \\
&= \frac{\Delta|A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3} + 6|A|^2\frac{|\nabla H|^2}{H^4} - \frac{4}{H^3}\left\langle \nabla|A|^2, \nabla H\right\rangle
\end{aligned}$$

which substituted in the time derivative gives

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{|A|^2}{H^2} &= \Delta\frac{|A|^2}{H^2} - 6|A|^2\frac{|\nabla H|^2}{H^4} + \frac{4}{H^3}\left\langle \nabla|A|^2, \nabla H\right\rangle - 2\frac{|\nabla A|^2}{H^2} \\
&= \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \frac{\nabla|A|^2}{H^2} - \frac{2}{H^3}|A|^2\nabla H\right\rangle \\
&\qquad - \frac{2}{H^4}\left(|A|^2|\nabla H|^2 + |\nabla A|^2H^2 - H\left\langle \nabla|A|^2, \nabla H\right\rangle\right) \\
&= \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A \cdot H - \nabla H \otimes A|^2.
\end{aligned}$$

□

Using this we compute the time derivative of $g_{\sigma,\eta}$.

Lemma 2.2.3. The evolution equation of $g_{\sigma,\eta}$ is given by

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} =& \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &- \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma |A|^2 g_{\sigma,\eta}. \end{aligned} \tag{2.2.3}$$

Proof. We can write $g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1+\eta)\right) H^\sigma$ so

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} =& \left\{\Delta \frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla \frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A \cdot H - \nabla H \otimes A|^2\right\} H^\sigma \\ &+ \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\left(\Delta H^\sigma - \sigma(\sigma-1)H^{\sigma-2}|\nabla H|^2 + \sigma|A|^2 H^\sigma\right) \\ =& \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left\langle \nabla H, \nabla \frac{|A|^2}{H^2}\right\rangle H^\sigma - \frac{\sigma(\sigma-1)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &- \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta} \\ =& \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left(\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma}{H} g_{\sigma,\eta}|\nabla H|^2\right) - \frac{\sigma(\sigma-1)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &- \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta} \\ =& \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &- \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta}.\end{aligned}$$

□

Applying the maximum principle on Lemma 2.2.2 gets that $\frac{|A|^2}{H^2}$ is uniformly bounded so there exists a positive constant depending only on $\mathcal{M}_0$ such that

$$|A|^2 \le \tilde{c_0} H^2 \quad \text{on} \quad \mathcal{M}_t,$$

for all time $t \in [0, T)$. This also implies $g_{\sigma,\eta} \le c_0 H^\sigma$ but as H blows up this isn't sufficient to prove the uniform bound. The following estimate of the good term in Eq. (2.2.3) will be required for the L^p estimate.

Lemma 2.2.4. [HS99b] If $(1+\eta)H^2 \le |A|^2 \le c_0 H^2$ for some $\eta, c_0 > 0$. Then

1. $-2Z \ge \eta H^2 |A|^2$
2. $|\nabla A \cdot H - \nabla H \otimes A|^2 \ge \frac{\eta^2}{4n(n-1)^2 c_0} H^2 |\nabla H|^2$

For the rest of proof we will restrict $\eta, \sigma \in (0,1)$ and c_i will denote a constant depending only on n, η and $\mathcal{M}_0$. For brevity, we will write $g = g_{\sigma,\eta}$ as long as σ, η is fixed. Let

$g_+ = \max\{g(x,t), 0\}$ denote the positive part of g. Then $g_+^p \in C^1(\mathcal{M} \times [0,T))$ for $p > 1$ and

$$\partial_t g_+^p = p g_+^{p-1} \partial_t g, \qquad \nabla(g_+^p) = p g_+^{p-1} \nabla g.$$

Lemma 2.2.5. There exists constant c_2, c_3 such that

$$\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu \leq -\frac{p(p-1)}{2}\int_{\mathcal{M}} g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int_{\mathcal{M}} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$- p\int_{\mathcal{M}} \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu + p\sigma \int_{\mathcal{M}} |A|^2 g_+^p d\mu \tag{2.2.4}$$

for any $p \geq c_2$.

Proof. Differentiating with respect to time and using Lemma 2.2.3 for $p \geq 2$

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu &= \int \left(p g_+^{p-1}\partial_t g - H^2 g_+^p\right) d\mu \\ &\leq \int p g_+^{p-1}\left(\Delta g + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g\rangle - \frac{2}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H h_{kl}|^2\right) d\mu \\ &\quad + \sigma|A|^2 g \end{aligned} \tag{2.2.5}$$

Using integration by parts,

$$\int p g_+^{p-1}\Delta g d\mu = -p\int \left\langle \nabla g_+^{p-1}, \nabla g\right\rangle d\mu \tag{2.2.6}$$
$$= -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu \tag{2.2.7}$$

Also from Lemma 2.2.4 we deduce that if $c_1 \geq 4n(n-1)^2 c_0 \eta^{-2}$

$$\begin{aligned}\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 &\geq \frac{g_+^{p-1}}{c_1 H^{2-\sigma}}|\nabla H|^2 \\ &\geq \frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2 + \frac{1}{2c_0 c_1}\frac{g_+^p}{H^2}|\nabla H|^2\end{aligned} \tag{2.2.8}$$

To handle the gradient term, let $p \geq \max\{2, 1+4c_0c_1\}$ to obtain

$$\begin{aligned}2(1-\sigma)p\frac{g_+^{p-1}}{H}\langle \nabla H, \nabla g\rangle &\leq 2p\frac{g_+^{p-1}}{H}|\nabla H||\nabla g| \\ &\leq \frac{p}{2c_0c_1}\frac{g_+^p}{H^2}|\nabla H|^2 + 2c_0c_1 p g_+^{p-2}|\nabla g|^2 \quad \text{[Peter-Paul inequality]} \\ &\leq p\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 - p\frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2 \\ &\quad + \frac{p(p-1)}{2}g_+^{p-2}|\nabla g|^2 \qquad \text{[Using Eq. (2.2.8)]}\end{aligned}$$

Substituting this back in Eq. (2.2.5) and using integration by parts from Eq. (2.2.7),

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu \leq & -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu + p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \\
& + \frac{p(p-1)}{2}\int g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\
& - 2p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu + p\sigma\int |A|^2 g_+^p d\mu
\end{aligned}
$$

which gives the desired inequality with $c_3 = \frac{1}{2c_1}$. □

To handle the bad positive term appearing in Eq. (2.2.4) we use the following lemma

Lemma 2.2.6. There exists a constant c_4 such that

$$
\begin{aligned}
\frac{1}{c_4}\int |A|^2 g_+^p d\mu \leq \left(p + \frac{p}{\beta}\right)\int g_+^{p-2}|\nabla g|^2 + (1+\beta p)\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\
+ \int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu
\end{aligned}
$$

for any $\beta > 0, p > 2$.

Proof. The Laplacian-Beltrami operator satisfies,

$$
\Delta(f^\sigma) = \sigma f^{\sigma-1}\Delta f + \sigma(\sigma-1)f^{\sigma-2}|\nabla f|^2
$$

We have an expression for the Laplacian of $\frac{|A|^2}{H^2}$ in Lemma 2.2.2 from which it follows that

$$
\begin{aligned}
\Delta g &= \Delta\left(\frac{|A|^2}{H^2}\right)H^\sigma + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\Delta H^\sigma + 2\left\langle \nabla\frac{|A|^2}{H^2}, \nabla H^\sigma\right\rangle \\
&= \left(\frac{\Delta|A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3} + 6|A|^2\frac{|\nabla H|^2}{H^4} - \frac{4}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle\right)H^\sigma \\
&\quad + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\left(\sigma H^{\sigma-1}\Delta H + \sigma(\sigma-1)H^{\sigma-2}|\nabla H|^2\right) \\
&\quad + 2\sigma H^{\sigma-1}\left\langle\frac{\nabla|A|^2}{H^2} - 2\frac{|A|^2}{H^3}\nabla H, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{|A|^2}{H^{3-\sigma}} - \sigma(1+\eta)H^{\sigma-1}\right)\Delta H + 6\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 - \frac{4}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle \\
&\quad + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2\sigma}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle - 4\sigma\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H + (6-4\sigma)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&\quad - \frac{2}{H^{4-\sigma}}H\left\langle\nabla|A|^2, \nabla H\right\rangle + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H + (6-4\sigma)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&\quad - \frac{2}{H^{4-\sigma}}\left(|\nabla A|^2H^2 + |A|^2|\nabla H|^2 - |\nabla A\cdot H - \nabla H\otimes A|^2\right) + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 \\
&\quad + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&\quad - 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle.
\end{aligned}
$$

Now similar to time derivative in Lemma 2.2.3, we calculate inner product of ∇g with ∇H,

$$
\begin{aligned}
\langle\nabla g, \nabla H\rangle &= \left\langle\nabla\frac{|A|^2}{H^2}, \nabla H\right\rangle H^\sigma + \sigma\left(\frac{|A|^2}{H^2} - (1+\eta)\right)H^{\sigma-1}|\nabla H|^2 \\
&= \left\langle\frac{\nabla|A|^2}{H^2}, \nabla H\right\rangle H^\sigma - 2\frac{|A|^2}{H^{3-\sigma}}|\nabla H|^2 + \sigma\frac{g}{H}|\nabla H|^2.
\end{aligned}
$$

Using Simon's identity [?] and the previous expression to eliminate the last mixed inner product term

$$\begin{aligned}
\Delta g =& \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&- 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 \\
&+ \frac{2(\sigma-1)}{H}\left(\langle\nabla g, \nabla H\rangle + 2\frac{|A|^2}{H^{3-\sigma}}|\nabla H|^2 - \sigma\frac{g}{H}|\nabla H|^2\right) \\
=& \frac{2\langle h_{ij}, \nabla_i\nabla_j H\rangle + 2Z}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&- \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H}\langle\nabla g, \nabla H\rangle
\end{aligned} \tag{2.2.9}$$

Recall Green's identity for compact manifold without boundary,

$$\int_M u\Delta v = -\int_M \langle\nabla u, \nabla v\rangle\,.$$

Multiplying Eq. (2.2.9) by $g_+^p H^{-\sigma}$ and using Green's identity the left-hand side evaluates to

$$\begin{aligned}
A = \int g_+^p H^{-\sigma}\Delta g d\mu &= -\int \left\langle\nabla(g_+^p H^{-\sigma}), \nabla g\right\rangle d\mu \\
&= -p\int \frac{1}{H^\sigma}g_+^{p-1}|\nabla g|^2 d\mu + \sigma\int\frac{g_+^p}{H^{1+\sigma}}\langle\nabla g, \nabla H\rangle d\mu
\end{aligned} \tag{2.2.10}$$

while the right-hand side is

$$\begin{aligned}
B =& 2\int\frac{\langle h_{ij}, \nabla_i\nabla_j H\rangle g_+^p}{H^2}d\mu + 2\int\frac{g_+^p Z}{H^2}d\mu + 2\int\frac{g_+^p}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \\
&+ (\sigma-2)\int\frac{g_+^{p+1}}{H^{1+\sigma}}\Delta H d\mu - 2(1+\eta)\int\frac{g_+^p}{H}\Delta H d\mu - \sigma(\sigma-1)\int\frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu \\
&+ 2(\sigma-1)\int\frac{g_+^{p+1}}{H^{1+\sigma}}\langle\nabla g, \nabla H\rangle d\mu
\end{aligned} \tag{2.2.11}$$

For the first term of Eq. (2.2.11) we can use divergence-type theorem for tensors to get,

$$\begin{aligned}
2\int\frac{\langle h_{ij}, \nabla_i\nabla_j H\rangle g_+^p}{H^2}d\mu &= -2\int\left\langle \mathrm{tr}_{ik}\left(\nabla_k\left(\frac{g_+^p h_{ij}}{H^2}\right)\right), \nabla_j H\right\rangle d\mu \\
&= -2p\int\frac{g_+^{p-1}}{H^2}\left\langle\nabla^i g\otimes h_{ij}, \nabla_j H\right\rangle d\mu \\
&\quad + 4\int\frac{g_+^p}{H^3}\left\langle\nabla^i H\otimes h_{ij}, \nabla_j H\right\rangle d\mu - 2\int\frac{g_+^p}{H^2}\left\langle\nabla^i h_{ij}, \nabla_j H\right\rangle d\mu
\end{aligned} \tag{2.2.12}$$

Using Codazzi equation $\nabla^i h_{ij} = \nabla_j h^i_i$ for the last term,

$$2\int \frac{\langle h_{ij}, \nabla_i\nabla_j H\rangle\, g_+^p}{H^2} d\mu = -2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g \nabla_j H\rangle\, d\mu \\ + 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H \nabla_j H\rangle\, d\mu - 2\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu \qquad (2.2.13)$$

Applying Green's formula on ΔH terms in Eq. (2.2.11) and putting together Eq. (2.2.10), Eq. (2.2.11) and Eq. (2.2.13)

$$\begin{aligned}
&- p\int \frac{1}{H^\sigma} g_+^{p-1}|\nabla g|^2 d\mu + \underbrace{\sigma\int \frac{g_+^p}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1} \\
&= -2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g \nabla_j H\rangle\, d\mu + 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H\nabla_j H\rangle\, d\mu - \underbrace{2\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu}_{2} \\
&+ 2\int \frac{g_+^p Z}{H^2} d\mu + 2\int \frac{g_+^p}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu - \underbrace{(\sigma-2)(p+1)\int \frac{g_+^p}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1} \\
&+ \underbrace{(\sigma-2)(1+\sigma)\int \frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu}_{3} + 2(1+\eta)p\int \frac{g_+^{p-1}}{H}\langle \nabla g, \nabla H\rangle\, d\mu \\
&\underbrace{- 2(1+\eta)\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu}_{2} \underbrace{- \sigma(\sigma-1)\int \frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu}_{3} + \underbrace{2(\sigma-1)\int \frac{g_+^{p+1}}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1}
\end{aligned}$$

clubbing the terms with same-numbered under bracket,

$$\begin{aligned}
-2\int \frac{g_+^p Z}{H^2} d\mu = p\int \frac{1}{H^\sigma} g_+^{p-1}|\nabla g|^2 d\mu - 2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g\nabla_j H\rangle\, d\mu \\
+ 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H\nabla_j H\rangle\, d\mu + 2\int \frac{g_+^p}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \\
+ p\int \left((2-\sigma)\frac{g_+^p}{H^{1+\sigma}} + 2(1+\eta)\frac{g_+^{p-1}}{H} \right)\langle \nabla g, \nabla H\rangle\, d\mu \\
- 2\int \left(\frac{g_+^{p+1}}{H^{2+\sigma}} + (2+\eta)\frac{g_+^p}{H^2} \right)|\nabla H|^2 d\mu \qquad (2.2.14)
\end{aligned}$$

From Lemma 2.2.4 $-2Z \geq \eta H^2|A|^2$ and using utilizing $g \leq c_0 H^\sigma$ (and $|A| \leq c_0 H$) with Cauchy-Schwarz inequality in Eq. (2.2.14),

$$
\begin{aligned}
\eta \int g_+^p |A|^2 d\mu \leq c_0 p \int g_+^{p-2} |\nabla g|^2 d\mu + 4p(c_0+1) \int \frac{g_+^{p-1}}{H} |\nabla g||\nabla H| d\mu \\
+ 4c_0^2 \int \frac{g_+^{p-1}}{H^{2-\sigma}} |\nabla H|^2 d\mu + 2c_0 \int \frac{g_+^{p-1}}{H^{4-\sigma}} |\nabla A \cdot H - \nabla H \otimes A|^2 d\mu
\end{aligned}
\tag{2.2.15}
$$

Also, for any $\beta > 0$,

$$
\begin{aligned}
2\frac{g_+^{p-1}}{H} |\nabla H||\nabla g| &\leq \frac{g_+^{p-2}}{\beta} |\nabla g|^2 + \beta \frac{g_+^p}{H^2} |\nabla H|^2 \\
&= \frac{g_+^{p-2}}{\beta} |\nabla g|^2 + c_0 \beta \frac{g_+^{p-1}}{H^{2-\sigma}} |\nabla H|^2
\end{aligned}
\tag{2.2.16}
$$

Combining Eq. (2.2.14), Eq. (2.2.15) and Eq. (2.2.16) proves the lemma. □

Proposition 2.2.7. For any $\eta \in (0,1)$ there exists constants c_5, c_6 such that the $L^p(\mathcal{M})$ norm of $(g_{\sigma,\eta})_+$ is non-decreasing function of t if the following holds

$$
p \geq c_5, \qquad \sigma \leq (c_6 p)^{-\frac{1}{2}}.
$$

Proof. Choose $\beta \sim p^{-\frac{1}{2}}$ and $\sigma \sim cp^{-\frac{1}{2}}$ in the previous lemma. □

Lemma 2.2.8 (Stampacchia lemma)**.** Let $\psi : [k_0, \infty) \to \mathbb{R}$ be a non-negative, non-increasing function which satisfies

$$
\psi(h) \leq \frac{C}{(h-k)^\alpha} \psi(k)^\beta \text{ for all } h > k > k_0 \tag{2.2.17}
$$

for some constants $C > 0$, $\alpha > 0$ and $\beta > 1$. Then

$$
\psi(k_0 + d) = 0, \tag{2.2.18}
$$

where $d^\alpha = C\psi(k_0)^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}$.

We complete the proof of Theorem 2.1.2 using Stampacchia lemma which gives an L^∞ bound from the L^p bounds.

Proof. Let $k \geq k_0$, where

$$
k_0 = \sup_{\sigma \in [0,1]} \sup_{\mathcal{M}_0} g_{\sigma,\eta}
$$

Define $v = (g_{\sigma,\eta} - k)_+^{\frac{p}{2}}$ and $A(k,t) = \{x \in \mathcal{M}_t : v(x,t) > 0\}$. Differentiating v with respect to time we get for p large enough (similar to Lemma 2.2.5)

$$\frac{d}{dt}\int_{\mathcal{M}_t} v^2 d\mu + \int_{\mathcal{M}_t} |\nabla v|^2 d\mu \le \sigma p \int_{\mathcal{M}_t} |A|^2 v^2 d\mu \le c_0 \sigma p \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu \tag{2.2.19}$$

Also from the Michael-Simon result in [MS73], we have a Sobolev-type inequality given by

$$\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \le C(n)\int_{\mathcal{M}_t} |\nabla v|^2 d\mu + C(n)\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}} \left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \tag{2.2.20}$$

where $q = \frac{n}{n-2}$ if $n > 2$ and an arbitrary number greater than 1 if $n = 2$. We can estimate the H^n factor in the integral on $A(k,t)$ using the previous proposition and the equality

$$\int_{\mathcal{M}_t} H^n g_{\sigma,\eta}^p d\mu = \int_{\mathcal{M}_t} g_{\sigma',\eta}^p d\mu$$

where $\sigma' = \sigma + \frac{n}{p}$. Let

$$p \ge \max\{c_5, 4n^2 c_6\} \quad \text{and} \quad \sigma \le (4c_6 p^{-\frac{1}{2}})$$

so that

$$\sigma' = \sigma + \frac{n}{p} \le \frac{1}{2\sqrt{c_6 p}} + \frac{1}{\sqrt{p}}\frac{n}{\sqrt{p}} \le \frac{1}{\sqrt{c_6 p}}$$

which allows us to use Proposition 2.2.7,

$$\begin{aligned}
\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}} &\le \left(\int_{A(k,t)} H^n \left(\frac{g_{\sigma,\eta}^p}{k}\right) d\mu\right)^{\frac{2}{n}} \\
&= k^{-\frac{2p}{n}} \left(\int_{A(k,t)} g_{\sigma',\eta}^p d\mu\right)^{\frac{2}{n}} \\
&\le k^{-\frac{2p}{n}} \left(\int_{\mathcal{M}_t} (g_{\sigma',\eta})_+^p d\mu\right)^{\frac{2}{n}} \\
&\le k^{-\frac{2p}{n}} \left(\int_{\mathcal{M}_0} (g_{\sigma',\eta})_+^p d\mu\right)^{\frac{2}{n}} \\
&\le \left(\frac{|\mathcal{M}_0| k_0}{k}\right)^{\frac{2p}{n}}
\end{aligned}$$

We can fix $k_1 > k_0$ such that for any $k \ge k_1$ the term $\int_{A(k,t)} H^n d\mu$ in Eq. (2.2.20) is less than $\frac{1}{2C(n)}$. For such k, using Eq. (2.2.19) with Eq. (2.2.20) to eliminate the gradient term,

$$\frac{d}{dt}\int_{\mathcal{M}_t} v^2 d\mu + \frac{1}{2C(n)}\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \le c_0 \sigma p \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu. \tag{2.2.21}$$

Let $t_0 \in [0, T]$ be the time when $\sup_{t\in[0,T)} \int_{\mathcal{M}_t} v^2 d\mu$ is attained (we let $t_0 = T$ if it is not attained in the interior). Integrating Eq. (2.2.21) from 0 to t_0,

$$\int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{2C(n)} \int_0^{t_0} \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q}} dt \le c_0 \sigma p \int_0^{t_0} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \tag{2.2.22}$$

where we used the fact that $k > k_0 \ge \sup_{\mathcal{M}_0} g_{\sigma,\eta}$ so $\int_{\mathcal{M}_0} v^2 d\mu = 0$. Now integrating Eq. (2.2.21) from t_0 to T,

$$\int_{\mathcal{M}_T} v^2 d\mu - \int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{2C(n)} \int_{t_0}^{T} \left(\int_{\mathcal{M}_t} v^{2q} \right)^{\frac{1}{q}} dt \le c_0 \sigma p \int_{t_0}^{T} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt. \tag{2.2.23}$$

Throwing away $\int_{\mathcal{M}_T} v^2 d\mu$ term and adding Eq. (2.2.22) to half of Eq. (2.2.23),

$$\frac{1}{2} \int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{4C(n)} \int_0^{T} \left(\int_{\mathcal{M}_t} v^{2q} \right)^{\frac{1}{q}} dt \le c_0 \sigma p \int_0^{T} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt$$

which is same as

$$\sup_{[0,T)} \int_{\mathcal{M}_t} v^2 d\mu + \int_0^{T} \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q}} dt \le 2 \max\{1, 2C(n)\} c_0 \sigma p \int_0^{T} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt. \tag{2.2.24}$$

Recall the interpolation inequality for L^p spaces for any $f \in L^q \cap L^r$,

$$||f||_{q_0} \le ||f||_q^{\alpha} ||f||_r^{1-\alpha}$$

where $\frac{1}{q_0} = \frac{\alpha}{q} + \frac{1-\alpha}{q}$ and $1 < q_0 < q$. Setting $r = 1, \alpha = \frac{1}{q_0}$ and $f = v^2$ we get

$$\left(\int_{\mathcal{M}_t} v^{2q_0} d\mu \right)^{\frac{1}{q_0}} \le \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q_0 q}} \left(\int_{\mathcal{M}_t} v^2 d\mu \right)^{1-\frac{1}{q_0}}. \tag{2.2.25}$$

Integrating this in time and using Young's inequality,

$$\begin{aligned}
\left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu dt \right)^{\frac{1}{q_0}} &\le \left(\sup_{[0,T)} \int_{A(k,t)} v^2 d\mu \right)^{1-\frac{1}{q_0}} \left(\int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q_0}} \\
&\le \frac{\sup_{[0,T)} \int_{A(k,t)} v^2 d\mu}{\frac{q_0}{q_0-1}} + \frac{\int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt}{q_0} \\
&\le \sup_{[0,T)} \int_{A(k,t)} v^2 d\mu + \int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt \\
&\le c_8 \sigma p \int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt
\end{aligned}$$

where $c_8 = 2\max\{1, 2C(n)\}c_0$. Set $\psi(k) = \int_0^T \int_{A(k,t)} d\mu dt$. We will obtain bounds on ψ which along with the Stampacchia lemma will imply a uniform bound of $g_{\sigma,\eta}$. Now Eq. (2.2.24) and Hölder inequality yields,

$$\int_0^T \int_{A(k,t)} v^2 d\mu dt \leq \left(\int_0^T \int_{A(k,t)} 1 d\mu dt\right)^{1-\frac{1}{q_0}} \left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu dt\right)^{\frac{1}{q_0}} \tag{2.2.26}$$

$$\leq c_8 \sigma p \psi(k)^{1-\frac{1}{q_0}} \int_0^T \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt \tag{2.2.27}$$

Let $r > 1$ which will be chosen later. Applying Hölder again on the right side with weights r and $\frac{r}{r-1}$,

$$\int_0^T \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt \leq \left(\int_0^T \int_{A(k,t)} d\mu dt\right)^{1-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} H^{2r} g_{\sigma,\eta}^{pr} d\mu dt\right)^{\frac{1}{r}}$$

$$= \psi(k)^{1-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} g_{\sigma'',\eta}^{pr} d\mu dt\right)^{\frac{1}{r}}$$

where $\sigma'' = \sigma + \frac{2}{p}$. For r large enough and p, σ^{-1} small enough from Proposition 2.2.7 there exists a constant $c_9 > 0$ independent of time such that

$$\int_0^T \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu dt \leq c_9^{\frac{1}{r}} \psi(k)^{1-\frac{1}{r}}. \tag{2.2.28}$$

Combining Eq. (2.2.27) and Eq. (2.2.28) for all $h > k \geq k_1$, we have

$$\begin{aligned} (h-k)^p \psi(h) &= \int_0^T \int_{A(h,t)} (h-k)^p d\mu dt \\ &\leq \int_0^T \int_{A(k,t)} v^2 d\mu dt \\ &\leq c_8 \sigma p c_9^{\frac{1}{r}} \psi(k)^{2-\frac{1}{r}-\frac{1}{q_0}}. \end{aligned}$$

Let $\gamma = 2 - \frac{1}{r} - \frac{1}{q_0}$ and $c_{10} = c_8 c_9^{\frac{1}{r}}$. Fix $r > \frac{q_0}{q_0-1}$ (so $\gamma > 1$) and p large enough, σ small enough while satisfying the hypothesis of Proposition 2.2.7 such that $\sigma p < 1$ then gives

$$\psi(h) \leq \frac{c_{10}}{(h-k)^p} \psi(k)^\gamma \tag{2.2.29}$$

Stampacchia lemma now implies $\psi(k) = 0$ for all $k \geq k_1 + d$ where $d^p = c_{10} 2^{\frac{\gamma p}{\gamma-1}+1} \psi(k_1)^{\gamma-1}$. Hence,

$$g_{\sigma,\eta} \leq k_1 + d \leq K := k_1 + c_{10} 2^{\frac{\gamma p}{\gamma-1}+1} (|\mathcal{M}_0|T)^{\gamma-1}$$

or

$$|A|^2 - (1+\eta)H^2 \leq K H^{2-\sigma}$$

so by Young's inequality there exists a constant C_η such that,

$$|A|^2 - H^2 \leq \eta H^2 + KH^{2-\sigma} \leq 2\eta H^2 + 2C_\eta.$$

Notice that $|A|^2 - H^2 = -\sum_{i\neq j} \kappa_i\kappa_j = -2S_2$ which implies the desired estimate. □

2.3. Asymptotic convexity

As mentioned in Section 1.8.1, we classify the singularities based on the blow-up rate of $|A|^2$. Recall from maximum principle on Lemma 2.2.2 there exists a c_0 such that $|A|^2 \leq c_0 H^2$ and from algebra we get $H^2 \leq n|A|^2$ so $|A|^2$ and H^2 have same rate of growth. We will focus on the growth of H^2.

The estimates obtained in the previous section will be very useful to obtain an asymptotic analysis of type II singularities. Following [HS99b] suppose a maximal solution $X : M \times [0, T) \to \mathbb{R}^{n+1}$ develops a type II singularity. Choose a sequence of points $\{(x_k, t_k)\}$ in spacetime as follows. For each integer $k \geq 1$, let $t_k \in [0, T - \frac{1}{k}]$, $x_k \in M$ such that

$$H^2(x_k, t_k)\left(T - \frac{1}{k} - t_k\right) = \sup_{(x,t)\in M\times\left[0,T-\frac{1}{k}\right]} H^2(x, t)\left(T - \frac{1}{k} - t\right) \tag{2.3.1}$$

Set $L_k = H(x_k, t_k)$, $\alpha_k = -L_k^2 t_k$ and $\omega_k = L_k^2(T - \frac{1}{k} - t_k)$.

Lemma 2.3.1. For singularities of type II, the following holds as $k \to \infty$,

$$t_k \to T, \quad L_k \to \infty, \quad \alpha_k \to -\infty, \text{ and } \omega_k \to \infty.$$

Proof. Fix $M > 0$. As the singularity is of type II, there exists a $t_M \in [0, T)$ and $x_M \in \mathcal{M}$ such that $H^2(x_M, t_M)(T - t_m) > 2M$.

TO DO □

Now we will rescale the hypersurfaces to analyze the limiting behavior. For each $k \geq 1$, define a family of immersions by

$$X_k(x, t) = L_k(X(x, L_k^{-2}t + t_k) - X(x_k, t_k)) \text{ for } t \in [\alpha_k, \omega_k].$$

Let A_k and H_k denote the fundamental form of the rescaled immersions. Then by the definition of L_k and X_k we have

$$X_k(x_k, 0) = 0 \quad \text{and} \quad H_k(x_k, 0) = 1.$$

Further, observe that

$$H_k^2(x, t) = L_k^{-2}H^2(x, L_k^{-2}t + t_k) \leq \frac{T - \frac{1}{k} - t_k}{T - \frac{1}{k} - t_k - L_k^{-2}t} = \frac{\omega_k}{\omega_k - t}.$$

From the previous lemma $\omega_k \to \infty$, so for any $\epsilon > 0$ and $\overline{\omega}$, there exists a k_0 such that

$$\max_{x \in M} H_k(x,t) \leq 1 + \epsilon$$

for any $k \geq k_0$ and $t \in [\alpha_{k_0}, \overline{\omega}]$. This curvature bound implies analogous bounds on the second fundamental form as well as its covariant derivatives. Invoking Theorem A.2.1 there exists a subsequence of X_k converging uniformly on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}$ to a limiting solution X_∞ of the mean curvature flow. This proves the asymptotic convexity of the flow in the following sense.

Theorem 2.3.2. Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth maximal solution of the mean curvature flow with $X(\cdot, 0) = \mathcal{M}_0$ compact and of positive mean curvature. Further, assume that the flow develops a singularity of type II. Then there exists a sequence of rescaled flow $X_k(\cdot, t)$ converging smoothly on every compact set to a mean curvature flow $X_\infty(\cdot, t)$ which is defined for $t \in (-\infty, \infty)$. Also, the limit hypersurface X_∞ is convex (not necessarily uniformly convex) for each $t \in (-\infty, \infty)$ and satisfies $0 < H_\infty \leq 1$ everywhere with equality at least at one point.