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First and second variation and Bochner-Weitzenbock formulae and vanishing theorems in geometry and mathematical physics

1) Introduction

Information on the topology of a given Riemannian manifold M can often be obtained by studying certain canonical representatives of objects in its topological classes — such as cohomology, homology and homotopy classes and homotopy classes of maps.

The basic program goes as follows: the canonical representative ϕ is obtained by minimizing some "action" functional S, the Euler-Lagrange or first variation equation is typically $\Delta\phi = 0$ where Δ is a suitable Laplacian operator on the space over which S is being varied. Any ϕ satisfying this equation is then a critical point of S but is not necessarily minimizing, of course. The operator Δ can often be written

$$\Delta = \nabla^*\nabla + \mathcal{R}$$

which is the Bochner-Weitzenbock formula.

$\nabla^*\nabla$ is a non-negative operator and $\mathcal{R}$ is zero-order and depends on certain geometrical properties, such as the curvature, of M. Thus if the geometry of M results in $\mathcal{R} > 0$ there can be no solutions to $\Delta\phi = 0$, i.e. no critical points of S.

If ϕ is minimizing it is certainly stable, i.e. the second variation of S, $\left.\frac{d^2S}{dt^2}\right|_{t=0}$ is ≥ 0 for all 1-parameter variations ϕ_t of ϕ. The second variation formula is typically

$$\left.\frac{d^2S}{dt^2}\right|_{t=0} = \langle \mathcal{S}(X), X\rangle \triangleq \int_M (\mathcal{S}(X), X)$$

where $X = \left.\frac{\partial\phi}{\partial t}\right|_{t=0}$ is the "variation vector field".

The Jacobi operator $\mathcal{S}$ is elliptic, self-adjoint with

$$\mathcal{S} = \nabla^*\nabla - \mathcal{Q}$$

Again $\mathcal{Q}$ is zero order and depends on geometrical properties of M. ($\mathcal{Q}$ is closely related to $\mathcal{R}$.)

Thus if there exists a variation vector field X with, for example, $\nabla X = 0$ and if $Q > 0$ then $\left.\frac{d^2 S}{dt^2}\right|_{t=0} < 0$ for this variation and thus ϕ is not stable. This can sometimes give non-existence of minimizing representatives and hence vanishing of the topological class.

2) Caveat

This program obviously depends on proofs of the existence and regularity of the canonical representatives for a given nonvanishing class. This is the hard part which I shall largely ignore.
Also I shall require M to be compact without boundary.
Many of the results extend when suitably modified to $\partial M \neq \phi$ or M complete, noncompact, as recently shown by Yau and co-workers. I shall also largely ignore complex manifolds.

3) Harmonic forms

Let M be a compact Riemannian manifold, $\partial M = \phi$, $\dim M = n$.
Let $\mathcal{E}^p(M)$ be the vector space of smooth p-forms on M. The inner product on $T_x M$ (metric) naturally extends to an inner product on each $\Lambda^p(T_x^* M)$, the p-covectors at x; denoted $(\ ,\ )$.
The global inner product on $\mathcal{E}^p(M)$ is defined $\langle \phi, \psi \rangle = \int_M (\phi, \psi)$.

If d is the exterior derivative $\quad d : \mathcal{E}^{p-1}(M) \to \mathcal{E}^p(M)$
then we define $\quad \delta : \mathcal{E}^p(M) \to \mathcal{E}^{p-1}(M)$

3.1) by $\quad \delta = (-1)^{n(p+1)+1} * d *$

where $* : \mathcal{E}^p(M) \to \mathcal{E}^{n-p}(M)$ is the Hodge star operator (only defined up to sign if M is not oriented).
Then $d^2 = \delta^2 = 0$ and

$$\langle \phi, d\psi \rangle = \langle \delta\phi, \psi \rangle \quad \text{for all } \phi \in \mathcal{E}^p(M),\ \psi \in \mathcal{E}^{p-1}(M)$$

ie δ is the adjoint of d with respect to the global inner product.
Thus the orthogonal complement of $\operatorname{im} d$ is $\ker \delta$ and vice versa – since if $\langle \psi, d\phi \rangle = 0 \quad \forall \phi \in \mathcal{E}^{p-1}(M)$
then $\langle \delta\psi, \phi \rangle = 0 \quad \forall \phi \in \mathcal{E}^{p-1}(M)$
ie $\delta\psi = 0$.

Thus we decompose $\mathcal{E}^p(M)$ orthogonally as
(this requires an existence and regularity result)

$$\mathcal{E}^p(M) = \operatorname{im} d \oplus \ker \delta = \operatorname{im} d \oplus \operatorname{im}\delta \oplus \mathcal{H}_1$$
$$= \operatorname{im}\delta \oplus \ker d = \operatorname{im}\delta \oplus \operatorname{im} d \oplus \mathcal{H}_2$$

where $\mathcal{H}_1$ is the orthogonal complement of $\operatorname{im}\delta$ in $\ker\delta$
$\quad\cdots\quad \mathcal{H}_2 \qquad\qquad \operatorname{im} d$ in $\ker d$

Thus $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H} = \ker\delta \cap \ker d$

3.2) ie $\quad \phi \in \mathcal{H} \iff d\phi = \delta\phi = 0$

and we have the <u>Hodge decomposition</u>

3.3) $\quad \mathcal{E}^p(M) = \operatorname{im} d \oplus \operatorname{im}\delta \oplus \mathcal{H} \qquad$ (orthogonal)

We define the <u>Hodge – de Rham laplacian</u> Δ

3.4) $\quad \Delta = d\delta + \delta d$

and note that $\mathcal{H}$, the space of <u>harmonic p-forms</u>
$$\mathcal{H} = \{\phi : \Delta\phi = 0\} = \ker(\Delta)$$

– since $\langle \Delta\phi, \phi\rangle = \langle (d\delta + \delta d)\phi, \phi\rangle = \langle \delta\phi, \delta\phi\rangle + \langle d\phi, d\phi\rangle$
$$= \|\delta\phi\|^2 + \|d\phi\|^2$$

Let $[c] \in H^p(M, \mathbb{R})$ be a p-dimensional real cohomology class.
By de Rham's theorem $[c]$ is an affine subspace of $\mathcal{E}^p(M)$:

3.5) $\quad [c] = \{\phi \in \mathcal{E}^p(M) : d\phi = 0$ and $\phi - \phi_0 = d\psi$ for any $\phi_0 \in [c]$,
for some $\psi \in \mathcal{E}^{p-1}(M)\}$

Then since $\ker d = \mathcal{H} \oplus \operatorname{im} d$
we have $\quad H^p(M, \mathbb{R}) \simeq \mathcal{H} \quad$ ie

.6) <u>Hodge theorem</u> Each cohomology class is represented by a unique harmonic form.

To get the flavour of the ensuing variational problems we shall look at harmonic forms from this point of view:

define

3.7) $$S(\phi) = \|\phi\|^2 = \langle\phi,\phi\rangle = \int_m (\phi,\phi)$$

and minimize S over $\phi \in [c]$.

There is an existence (the solution exists in a weak sense)

and regularity (the solution is in fact smooth) result here.

(In fact we know that the minimum is achieved by the unique harmonic form ϕ_0 in $[c]$, since $\ker d = \operatorname{im} d \oplus \mathcal{H}$ is orthogonal.)

Let ϕ_t $t \in (-\epsilon, \epsilon)$ be a 1-parameter variation of ϕ in $[c]$ — ie ϕ_t is a smooth curve in $[c]$ such that $\phi_0 = \phi$.

Now $T_\phi [c] \simeq \operatorname{im} d$ so $\dot\phi_t = \frac{\partial}{\partial t}\phi_t\Big|_{t=0} = d\alpha$

where $\alpha \in \mathcal{E}^{p-1}(m)$ is arbitrary.

Thus
$$\frac{dS(\phi_t)}{dt}\Big|_{t=0} = \frac{d}{dt}\int_m (\phi_t,\phi_t)\Big|_{t=0} = \int_m \frac{\partial}{\partial t}(\phi_t,\phi_t)\Big|_{t=0}$$
$$= 2\int_m(\dot\phi_t,\phi)\Big|_{t=0} = 2\langle\dot\phi,\phi\rangle = 2\langle d\alpha,\phi\rangle$$
$$= 2\langle\delta\phi,\alpha\rangle$$

Thus if ϕ is a critical point of S, so that $\frac{dS(\phi_t)}{dt}\Big|_{t=0} = 0$ for all 1-parameter variations of ϕ,

then $\langle\delta\phi,\alpha\rangle = 0 \quad \forall \alpha \in \mathcal{E}^{p-1}(m)$

ie $\delta\phi = 0$

is the first variation formula.

Since $\phi \in [c]$ we have $d\phi = 0$,

thus equivalently the first variation formula is:

3.8) $$\Delta\phi = 0$$

3.9) Now $\frac{d^2S(\phi_t)}{dt^2}\Big|_{t=0} = 2\langle\dot\phi,\dot\phi\rangle = 2\|\dot\phi\|^2 \geqslant 0$

and is $= 0$ iff $\dot\phi \equiv 0$

Thus a harmonic form is always stable, the second variation tells us nothing.

4) Bochner – Weitzenbock formulae

This derivation is based on ideas in M. Gromov – H.B. Lawson Jr "Spin and scalar curvature in the presence of a fundamental group I" Annals 111 (1980) 209 – 230.

The laplacian is $\Delta = d\delta + \delta d$

Since ∇ is the Riemannian connection on M

4.1) $d = \theta^i \wedge \nabla_{e_i}$ where $\{e_i\}_{i=1,\dots,n}$ is a local frame field and $\{\theta^i\}_{i=1,\dots,n}$ is the dual local coframe field.

ie

d is the composition of $\wedge$ and ∇

$$\begin{tikzcd} \mathcal{E}^p(m) \arrow[r, "\nabla"] \arrow[rr, bend left, "d"] & \mathcal{E}^1(m) \otimes \mathcal{E}^p(m) \arrow[r, "\wedge"] & \mathcal{E}^{p+1}(m) \end{tikzcd}$$

4.2) Similarly $\delta = -e_i \lrcorner \nabla_{e_i}$ where the $\{e_i\}$'s are required to be orthonormal.

$$\delta = (-\lrcorner)\circ\nabla$$

$$\begin{tikzcd} \mathcal{E}^p(m) \arrow[r, "\nabla"] \arrow[rr, bend left, "\delta"] & \mathcal{E}^1(m) \otimes \mathcal{E}^p(m) \arrow[r, "\sim, -\lrcorner"] & \mathcal{E}^{p-1}(m) \end{tikzcd}$$

where $\sim$ is the isomorphism $\mathcal{E}^1(m) \leftrightarrow \mathcal{E}_1(m)$ (1 forms $\leftrightarrow$ vector field) defined by the metric and $\lrcorner$ is the interior product, defined by

4.3) $X \lrcorner \phi\ (\xi) = \phi\,(X \wedge \xi)$

$\forall X \in T_x M,\ \phi \in \Lambda^p T_x^* M,\ \xi \in \Lambda^{p-1} T_x M$

Since $d^2 = \delta^2 = 0$ $\qquad \Delta = (d + \delta)^2$

Define $\cdot : \mathcal{E}_1(M) \times \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M) \oplus \mathcal{E}^{p-1}(M)$

4.4) by $$X \cdot \phi = \tilde{X} \wedge \phi - X \lrcorner \phi$$

We have the basic anticommutator result

4.5) $$\{ \tilde{X}\wedge, Y \lrcorner \} \triangleq \tilde{X} \wedge Y \lrcorner + Y \lrcorner \tilde{X} \wedge = (X, Y)$$

ie $\{ \theta^j \wedge, e_i \lrcorner \} = \delta_{ij}$

we also have $\{ \theta^j \wedge, \theta^i \wedge \} = \{ e_i \lrcorner, e_j \lrcorner \} = 0$

In physics this is written $\{ a_j^+, a_i \} = \delta_{ij}$

and $\{ a_j^+, a_i^+ \} = \{ a_j, a_i \} = 0$

and a_j^+ (a_i) are called fermionic creation (annihilation) operators.

4.6) Thus $\{ X \cdot, Y \cdot \} = -2(X, Y)$.

Thus this definition of $\cdot$ makes $\mathcal{E}^*(M)$ a module over $Cl(M)$, the sections of the Clifford algebra bundle of M

$\cdot$ has two useful properties:

4.7 (i)) multiplication by a unit vector is orthogonal:

$$(e_i \cdot \phi, e_i \cdot \phi) \text{ (no sum } i) = (\theta^i \wedge \phi, \theta^i \wedge \phi) + (e_i \lrcorner \phi, e_i \lrcorner \phi)$$
$$= |\phi|^2_{\alpha\ (i \notin \alpha)} + |\phi_\alpha|^2_{i \in \alpha} = |\phi|^2$$

4.7 ii) covariant differentiation is a derivation:

$$\nabla_X(e_i \cdot \phi) = \nabla_X e_i \cdot \phi + e_i \cdot \nabla_X \phi \qquad \text{for all } X \in \mathcal{X}(M)$$

We define the Dirac operator $\not{D} : \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M) \oplus \mathcal{E}^{p-1}(M)$

4.8) $$\not{D} = e_i \cdot \nabla_{e_i}$$

$\not{D}$ is a self-adjoint, elliptic first order operator.

4.9) Clearly $$\not{D} = d + \delta$$

4.10) Thus $$\not{D}^2 = \Delta$$

Thus $\Delta = \not{D}^2 = (e_i \cdot \nabla_{e_i})(e_j \cdot \nabla_{e_j})$

$= e_i \cdot e_j \cdot \nabla_{e_i, e_j}$ where we calculate in an "adapted frame" such that, at the point in question, $\nabla_{e_i} e_j = 0$.

4.11) and $\nabla_{e_i,e_j} \triangleq \nabla_{e_i}\nabla_{e_j} - \nabla_{\nabla_{e_i} e_j}$ is the covariant hessian.

Thus $\Delta = -\nabla_{e_i,e_i} + \frac{1}{2} e_i \cdot e_j \cdot R(e_i, e_j)$

4.12) where $R(e_i,e_j) = \nabla_{e_i,e_j} - \nabla_{e_j,e_i}$ is the curvature tensor of M

Thus

4.13) Bochner – Weitzenbock formula $\boxed{\Delta = \nabla^*\nabla + \mathcal{R}}$

4.14) with $\nabla^*\nabla \triangleq -\nabla_{e_i,e_i}$ the Bochner or "rough" laplacian

4.15) and $\mathcal{R} = \frac{1}{2} e_i \cdot e_j \cdot R(e_i, e_j)$

$\nabla^*\nabla$ is in fact ∇^* of ∇ where ∇^* is the (global) adjoint of ∇

$\nabla^* : \mathcal{E}^1(M) \otimes \mathcal{E}^p(M) \to \mathcal{E}^p(M)$

$\langle \nabla^*\psi, \phi \rangle = \langle \psi, \nabla\phi \rangle \quad \forall \psi \in \mathcal{E}^1(M) \otimes \mathcal{E}^p(M)$

$\forall \phi \in \mathcal{E}^p(M)$

4.16) ($\nabla^* = -\text{trace} \circ \nabla$ in fact)

For $i \neq j$ $\quad e_i \cdot e_j \cdot = \theta^i \wedge \theta^j \wedge + e_i \lrcorner\, e_j \lrcorner + (\theta^j \wedge e_i \lrcorner - \theta^i \wedge e_j \lrcorner)$

and $R(e_i,e_j) = R_{ijkl}\, \theta^l \wedge e_k \lrcorner$ on forms

where $R_{ijkl} = (R(e_i,e_j)e_k, e_l)$

So

4.17) $\boxed{\mathcal{R} = R(e_i,e_j)\,\theta^j \wedge e_i \lrcorner}$ on $\mathcal{E}^*(M)$

(by the first Bianchi identity)

4.18) $\mathcal{R}$ on 1-forms $\quad \mathcal{R}(\phi) = \mathrm{Ric}(\phi) \quad \forall \phi \in \mathcal{E}^1(M)$

since $R(e_i,e_j)\theta^j \wedge e_i \lrcorner\, \phi = \phi(e_i)\, R(e_i,e_j)\theta^j = \phi(e_i)\, \mathrm{Ric}(e_i,e_k)\theta^k \triangleq \mathrm{Ric}(\phi)$

4.19) Bochner's theorem M compact Riemannian, Ric > 0. Then there are no harmonic 1-forms on M, and thus (by 3.6) $b_1(M) = 0$ (1st Betti no.)

Proof $\langle \Delta\phi, \phi \rangle = \langle \nabla^*\nabla\phi, \phi \rangle + \langle \mathcal{R}\phi, \phi \rangle = \|\nabla\phi\|^2 + \langle \mathrm{Ric}(\phi,\phi) \rangle > 0$ ∎

4.20) If M is compact Riemannian with $Ric \geq 0$ then we see that any harmonic 1-forms ϕ are parallel ($\nabla\phi \equiv 0$) and satisfy $Ric(\phi) \equiv 0$.

4.21) <u>$\mathcal{R}$ on p-forms</u>

S. Gallot & D. Meyer "Opérateur de courbure et laplacien des formes differentielles d'une variété Riemannienne" J Math. pures et appl. <u>54</u> (1975) 259–284.

By raising and lowering indices using the metric we can regard the curvature tensor $R \in \mathcal{E}^2(M) \otimes \mathrm{End}(TM)$
as $R \in \mathcal{E}^2(M) \odot \mathcal{E}^2(M)$ or $R \in \mathcal{E}_2(M) \otimes \mathcal{E}^2(M)$ etc
We define the <u>curvature operator</u>

$$\rho \in \mathcal{E}_2(M) \otimes \mathcal{E}^2(M) \quad \text{or} \quad \rho : \mathcal{E}^2(M) \to \mathcal{E}^2(M)$$

by $\rho = -R$
ie $\left(\rho(\theta^i \wedge \theta^j), \theta^k \wedge \theta^l\right) = \left(R(e_i, e_j)e_l, e_k\right) = R_{ijlk}$
{note the sign change}.

4.22) $$\left(\mathcal{R}\phi, \psi\right) = \left(\rho({}_\alpha\phi), {}_\alpha\psi\right) \qquad \text{for all } \phi, \psi \in \mathcal{E}^p(M)$$

where the sum on the right is over all ordered multi-p-indices α and where

$$ {}_\alpha\phi \triangleq \left(\theta^j \wedge e_i \lrcorner\, \phi, \theta^\alpha\right)\theta^i \wedge \theta^j \qquad (\text{sum over all } i \,\&\, j)$$

$$= \phi_{\alpha - j + i}\, \theta^i \wedge \theta^j \quad \text{in an obvious notation.}$$

4.23) Also $$\sum_\alpha |{}_\alpha\phi|^2 = p(n-p)|\phi|^2$$

We say $\rho \geq k$ if $\left(\rho(\phi), \phi\right) \geq k|\phi|^2 \quad \forall \phi \in \mathcal{E}^2(M)$

4.24) Thus $\rho \geq k \Rightarrow (\mathcal{R}\phi, \phi) \geq kp(n-p)|\phi|^2 \quad \forall \phi \in \mathcal{E}^p(M)$
& hence, by ~~Hodge~~ theory

4.25) <u>Theorem</u>. $\rho > 0 \Rightarrow M$ is a real homology sphere
Also $\rho \geq 0 \Rightarrow$ any harmonic forms are parallel ($\nabla\phi \equiv 0$) and satisfy $\mathcal{R}\phi \equiv 0$.

4.26) The curvature operator can be pinched in terms of sectional curvatures. Note that

$$(\rho(\theta^i \wedge \theta^j), \theta^i \wedge \theta^j) = R_{ijji} = \sigma(e_i, e_j) \qquad \underline{\text{no sum}}$$

i.e. the curvature operator acting on <u>simple</u> unit 2-vectors is just a sectional curvature.

For $n > 1$ $\quad \mathbb{P}^n(\mathbb{C})$ has $\frac{1}{4} \leq \sigma \leq 1$

but ρ has a zero eigenvalue.

5) <u>Bochner-Weitzenbock for spinors</u>

In fact much of the above section works whenever we have a bundle of modules $S(M)$ over the bundle of algebras $Cl(M)$ and $S(M)$ has a fibre metric and compatible covariant derivative such that 4.7 (i) and (ii) hold.

The standard example is the spin-bundle over a spin-manifold M if $w_2(M) = 0$ (the second Stiefel-Whitney class)

If $\dim M = 2n$ then $\dim(Cl^{2n}) = 2^{2n}$

and $Cl^{2n} \otimes \mathbb{C} = End(S)$ with $\dim_{\mathbb{C}} S = 2^n$

If $\omega = i^n e_1 \cdot e_2 \cdot \ldots \cdot e_{2n} \in Cl^{2n}$

then $\omega^2 = 1$ and S can be decomposed

$$S = S^+ + S^-$$

(left-handed and right-handed spinors in physics)

S^+ and S^- are irreducible representation spaces for Cl^{2n}.

$Spin(2n)$ is the double cover of $SO(2n)$

A spin structure on M is a principal $Spin(2n)$-bundle over M, $Spin(M)$ and a Spin-equivariant bundle map of this bundle onto the bundle $SO(M)$ of oriented orthonormal frames. In this case we can construct the spin bundle $S(M)$ over M

as $S(M) \triangleq Spin(M) \times_{Spin(2n)} S$

the associated bundle with standard fibre S. The connection lifts to $Spin(M)$ and $S(M)$ has a fibre metric and covariant derivative satisfying 4.7 (i) and (ii).

Moreover $S(M) = S^+(M) \oplus S^-(M)$

and $e_i \cdot : S^+(M) \to S^-(M)$ and vice versa.

thus defining

5.1) $$D\!\!\!/ = e_i \cdot \nabla_{e_i} \quad \text{as before}$$

and

5.2) $$D\!\!\!/^+ = D\!\!\!/\big|_{S^+(M)} \qquad D\!\!\!/^- = D\!\!\!/\big|_{S^-(M)}$$

then

$$D\!\!\!/^+ : S^+(M) \to S^-(M)$$

and $$D\!\!\!/^- : S^-(M) \to S^+(M)$$

are adjoint first order elliptic operators

5.3) The index of $D\!\!\!/^+ = \dim \ker(D\!\!\!/^+) - \dim \ker(D\!\!\!/^-)$

5.4) By the Atiyah-Singer index theorem $\operatorname{index}(D\!\!\!/^+) = \hat{A}(M)$ the $\hat{A}$-genus of M.

From 5.1, calculating as in §4, we find

5.5) $$D\!\!\!/^2 = \nabla^*\nabla + \mathcal{R} \quad \text{on } S(M)$$

with $\nabla^*\nabla = -\nabla_{e_i,e_i}$

and $\mathcal{R} = \frac{1}{2} e_i \cdot e_j \cdot R(e_i,e_j)$ as before

By looking at the action of $Spin(2n)$ on S we find that

5.6) $$R(e_i,e_j) = \tfrac{1}{4} R_{ijkl}\, e_k \cdot e_l \cdot \quad \text{on } S(M)$$

5.7) $$\mathcal{R} = \tfrac{1}{8} R_{ijkl}\, e_i \cdot e_j \cdot e_k \cdot e_l \cdot \quad \text{on } S(M)$$

Then $(\mathcal{R}\phi, \psi) = \frac{1}{8} R_{ijkl} (e_i e_j e_k e_l \phi, \psi)$ $\quad \forall \phi, \psi \in \Gamma(S(M))$

Using 4.6, 4.7 (i) and the symmetries of R we deduce

5.8) $\boxed{\mathcal{R}\phi = \frac{1}{4} K \phi} \; \{\forall \phi \in S(M)\}$ where $K = R_{ijji}$ is the <u>scalar curvature</u>

5.9) <u>Lichnerowicz' theorem</u> M compact, oriented, Riemannian spin manifold with $\mathcal{K} > 0$. Then M has no harmonic spinors. Then by the Atiyah-Singer theorem $\hat{A}(M) = 0$.

5.10) If we only know that $\mathcal{K} \geq 0$ then for there to be any harmonic spinors they must be parallel ($\nabla\phi = 0$) and satisfy $\mathcal{K}\phi = 0$ ie (using Aronszajn thm)
$\mathcal{K} \equiv 0$.

6) <u>Bochner - Weitzenbock formula for section-valued p-forms</u>

Most of §4 goes through almost unchanged for section-valued forms:

a vector bundle V over M is called a <u>Riemannian vector bundle</u> if it is equipped with a fibre metric and compatible covariant derivative (parallel transport is an isometry in the fibres). Tensoring V with the pth exterior power of the cotangent bundle we get $\Lambda^p T^*(M) \otimes V$ whose sections ϕ are called <u>section-valued p-forms</u>

$$\phi \in \mathcal{E}^p(M, V) \triangleq \Gamma\left(\Lambda^p T^*M \otimes V\right)$$

The <u>exterior covariant derivative</u> $d^\nabla : \mathcal{E}^p(M,V) \to \mathcal{E}^{p+1}(M,V)$ is defined as the obvious generalization of d:

there is a 1-1 correspondence between section-valued p-forms on M $\mathcal{E}^p(M,V)$, and horizontal, equivariant p-forms on the associated principal bundle $P(M,G)$ with values in the standard fibre $\bar{V}$ of V. $V = P \times_{\varrho(G)} \bar{V}$. This is called "taking components". We denote this correspondence by $\phi \to \underset{\sim}{\phi}$ and in the reverse direction by $\phi \to \tilde{\phi}$.
Then

6.1) $$d^\nabla \phi = \widetilde{D\underset{\sim}{\phi}}$$

where $D = h \circ d$ (the horizontal component of the exterior derivative) is the exterior covariant derivative of vector-valued p-forms on $P(M,G)$.

The action of d^∇ on section valued forms can be easily calculated from the usual formulae for the action of d on forms, simply by replacing derivatives by covariant derivatives,

e.g.

6.2) $$d\phi(X,Y) = X\phi(Y) - Y\phi(X) - \phi([X,Y])$$
$$\forall \phi \in \mathcal{E}^1(M), \quad X, Y \in \mathcal{X}(M)$$

6.3) $$d^\nabla \phi(X,Y) = \nabla_X \phi(Y) - \nabla_Y \phi(X) - \phi([X,Y])$$
$$\forall \phi \in \mathcal{E}^1(M,V)$$

where ∇ is the covariant derivative of V.

We shall denote all covariant derivatives by ∇ — when we need to distinguish between the covariant derivative of V and that of TM etc we shall call the former $\nabla^{(P)}$ and the latter $\nabla^{(M)}$.

Take M to be Riemannian, $\dim M = n$.

The ~~Hodge~~ $*$ operator $\quad * : \mathcal{E}^p(M) \to \mathcal{E}^{n-p}(M)$
extends by trivial action on fibres to
$$* : \mathcal{E}^p(M,V) \to \mathcal{E}^{n-p}(M,V)$$

As before define

6.4) $$\delta^\nabla = (-1)^{n(p+1)+1} * d^\nabla *$$
$$\delta^\nabla : \mathcal{E}^p(M,V) \to \mathcal{E}^{p-1}(M,V)$$

Then δ^∇ is the adjoint of d with respect to the global inner product defined by composition of the inner product on $\Lambda^p(M)$ with the fibre metric
$$\langle\ ,\ \rangle = \int_M (\ ,\ )$$

ie

6.5) $$\langle \psi, d^\nabla \phi \rangle = \langle \delta^\nabla \psi, \phi \rangle \qquad \forall \phi \in \mathcal{E}^{p-1}(M,V), \ \psi \in \mathcal{E}^p(M,V)$$

Also

6.6) $$d^{\nabla 2} = \Omega \wedge$$
$$\delta^{\nabla 2} = \Omega \lrcorner$$

where $\Omega \in \mathcal{E}^2(M, \mathrm{End}\, V)$ is the curvature of the connection on V

Again we can decompose orthogonally

(6.7) $\quad \mathcal{E}^p(M,V) = \operatorname{im} d^\nabla \oplus \ker \delta^\nabla$

We define the Hodge-de Rham laplacian Δ^∇

(6.8) $\quad \Delta^\nabla = d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$

and note that, as before, $\mathcal{H}$, the space of harmonic section-valued p-forms

(6.9) $\quad \mathcal{H} = \ker(\Delta^\nabla) = \{ \phi : \Delta^\nabla \phi = 0 \}$

is

(6.10) $\quad \mathcal{H} = \ker d^\nabla \cap \ker \delta^\nabla$

as

$$\langle \Delta^\nabla \phi, \phi \rangle = \langle (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)\phi, \phi \rangle = \langle \delta^\nabla\phi, \delta^\nabla\phi \rangle + \langle d^\nabla\phi, d^\nabla\phi \rangle$$
$$= \| \delta^\nabla \phi \|^2 + \| d^\nabla \phi \|^2$$

Since ${d^\nabla}^2 \neq 0$ the Hodge theorem (3.6) does not extend to this setting.

However §4 goes through almost unchanged.
Define a covariant derivative

$$\nabla : \mathcal{E}^p(M,V) \to \mathcal{E}^1(M) \otimes \mathcal{E}^p(M,V)$$

by

(6.11) $\quad \nabla = \nabla^{(M)} \otimes \nabla^{(P)}$

then we have

(6.12) $\quad d^\nabla = \theta^i \wedge \nabla_{e_i}$
$\quad \delta^\nabla = - e_i \lrcorner \nabla_{e_i}$

and we define

$$\cdot : \mathcal{E}_1(M) \times \mathcal{E}^p(M,V) \to \mathcal{E}^{p+1}(M) \oplus \mathcal{E}^{p-1}(M)$$

(6.13) $\quad X \cdot \phi = \widetilde{X} \wedge \phi - X \lrcorner \phi$

all as before.

This makes $\mathcal{E}^*(M,V)$ a module over $Cl(M)$.
• has the properties 4.7 (i) and (ii),
we define the Dirac operator $D\!\!\!/^{\nabla} : \mathcal{E}^*(M,V) \to \mathcal{E}^*(M,V)$

6.14)
$$D\!\!\!/^{\nabla} = e_i \cdot \nabla_{e_i}$$
$$D\!\!\!/^{\nabla} = d^{\nabla} + \delta^{\nabla}$$

6.15)
$$D\!\!\!/^{\nabla 2} = \Delta^{\nabla} + \Omega \wedge + \Omega \lrcorner$$
ie
$$\Delta^{\nabla} = \mathbb{P}(D\!\!\!/^{\nabla 2})$$
where $\mathbb{P}$ is the orthogonal projection onto the order-preserving part.

So

6.16)
$$D\!\!\!/^{\nabla 2} = \nabla^*\nabla + \tfrac{1}{2}\, e_i \cdot e_j \cdot R(e_i, e_j)$$
with
$$\nabla^*\nabla = -\nabla_{e_i,e_i} \qquad \nabla_{X,Y} \triangleq \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$$
$$R(e_i,e_j) = \nabla_{e_i,e_j} - \nabla_{e_j,e_i}$$
It is easy to see that
$$R = R^{(\rho)} + R^{(m)} = \Omega + R^{(m)}$$
$R^{(m)}$ the Riemannian curvature of M

Ω the curvature of V.

and, calculating as in §4

6.17)
$$\Delta^{\nabla} = \nabla^*\nabla + \mathcal{R}$$
Bochner-Weitzenbock formula

with

6.18)
$$\mathcal{R} = R(e_i,e_j)\, \theta^j \wedge e_i \lrcorner$$
$$\mathcal{R} = \mathcal{R}^{\rho} + \mathcal{R}^{m}$$
$$\mathcal{R}^{\rho} = \Omega(e_i,e_j)\, \theta^j \wedge e_i \lrcorner$$
$$\mathcal{R}^{m} = R^{(m)}(e_i,e_j)\, \theta^j \wedge e_i \lrcorner$$

Expressions 6.17) and 6.18) are very useful in the following.

7) Bochner-Weitzenbock formula for section-valued spinors

Extending §5, if

$$S(V) = S(M) \otimes V$$

is a twisted spin bundle, where $S(M)$ is the spin bundle of M and V is a Riemannian (or hermitian) vector bundle we have the obvious extensions

7.1) $$\not{D}^\nabla = e_i \cdot \nabla_{e_i} \qquad \left(\nabla = \nabla^{(P)} \otimes \nabla^{(m)}\right)$$

$$\not{D}^\nabla : \Gamma(S(V)) \to \Gamma(S(V))$$

7.2) $$\not{D}^{\nabla 2} = \nabla^* \nabla + \mathcal{R}$$
7.3) $$\mathcal{R} = \tfrac{1}{2}\, e_i \cdot e_j \cdot R(e_i, e_j) = \mathcal{R}^{(P)} + \mathcal{R}^{(m)}$$
$$R = \Omega + R^{(m)}$$

7.4) $$\mathcal{R}^{(m)} = \tfrac{1}{4} K$$ as before, K the scalar curvature.

Again, if M is even dimensional, $\dim M = 2n$
then $\dim S = 2^n$
and

$$S(V) = S^+(V) \oplus S^-(V)$$
$$e_i : S^+(V) \to S^-(V) \quad \text{and vice versa}$$

$$\not{D}^{\nabla +} \triangleq \not{D}^\nabla\big|_{S^+(V)} : S^+(V) \to S^-(V)$$
$$\not{D}^{\nabla -} \triangleq \not{D}^\nabla\big|_{S^-(V)} : S^-(V) \to S^+(V)$$
are adjoint

$$\operatorname{index}(\not{D}^{\nabla +}) = \dim \ker(\not{D}^{\nabla +}) - \dim \ker(\not{D}^{\nabla -})$$

7.5) Atiyah-Singer index theorem $\operatorname{index}(\not{D}^{\nabla +}) = \{ ch(V) \cdot \hat{A}(M) \}[M]$
where $ch(V)$ is the Chern character of V
and $\hat{A}(M)$ is the total $\hat{A}$-class of M.

Thus by 7.2) and 7.4)

7.6) $\frac{1}{4}K + \mathcal{R}^{\rho} > 0 \Rightarrow$ there are no harmonic section-valued spinors {sections of the twisted spin-bundle $S(V)$ }

Solutions to

7.7) $$(\not{D}^{\rho} - m)\phi = 0$$

$\phi \in S(V)$, m constant ;
i.e. eigen-(section-valued) spinors of the Dirac operator, are called <u>fermions of mass m</u> in physics.
Thus the conclusion of 7.6) is often stated as
$\Rightarrow$ there are no massless fermions.

<u>Gromov-Lawson</u> defined, for M a compact spin manifold of even dimension, a topological invariant $\hat{\mathfrak{A}}(M)$ which they called the "higher $\hat{A}$-genus". Their constructions closely followed those of G. Lusztig "Novikov's higher signature and families of elliptic operators", J. Diff. Geom. 7, (1971) 229-256.
They considered a parametrised family $S^f(V) = S(M) \otimes V^f$ of twisted spin bundles, where the family V^f were flat line bundles (so that $\mathcal{R}^{\rho} \equiv 0$) and the associated family of Dirac operators $\not{D}_f^+$.
They showed that

7.8) $$\operatorname{index}(\not{D}^+_f) = \hat{\mathfrak{A}}(M)$$

Thus if $K > 0$ there are no harmonic section-valued spinors so $\operatorname{index}(\not{D}^+_f) = 0$

7.9) <u>Theorem</u> (<u>Gromov-Lawson</u>) Let M be a compact spin manifold of even dimension. If M admits a metric of positive scalar curvature then $\hat{\mathfrak{A}}(M) = 0$

Using some previous results of J. Kazdan and F. Warner "Prescribing curvatures" Proc. Symp. Pure Math 27 part 2 A.M.S (1975) 309-319. and J. Cheeger and D. Gromoll "The splitting theorem for manifolds of non-negative Ricci curvature" J. Diff. Geom. 6 (1971) 119-128. they obtained the

7.10) Corollary (Gromov-Lawson) Any Riemannian metric of non-negative scalar curvature on the torus T^n is flat.

The case $n \leq 7$ had been previously established by R. Schoen and S.T. Yau "The structure of manifold with positive scalar curvature" Manus. Math. 28 (1979) 159-183, using regularity of stable minimal hypersurfaces.

7.11) Significance of twisted spin-bundles

A minor revolution in our understanding of the fundamental forces of nature has occurred in the last 10-15 years. Amazingly the current very successful theories, when viewed in their natural geometrical setting, are very similar to the earlier highly successful theories :

In the 1850's Maxwell postulated the existence of a complex hermitian line-bundle over space-time and explained the forces of electricity and magnetism (and hence the phenomenon of light and electro-magnetic radiation) as the curvature of the line bundle.

In 1915 Einstein regarded space-time as a pseudo-Riemannian (Lorentzian) manifold and interpreted gravitational forces in terms of the pseudo-Riemannian curvature of space-time.

In the 1930's Dirac was able to find a relativistic quantum-mechanical theory of the interaction of electrons (and positrons) and electro-magnetic radiation by regarding electrons as sections of the spin-bundle of space-time $\otimes$ Maxwell's line bundle where the forces were again the curvature of the line bundle. The equation of motion of electrons is $D\!\!\!/ \phi = m\phi$, the Dirac equation, with m the electron mass.

This theory has become known as QED — quantum electro-dynamics. It is the most accurate physical theory ever, its accuracy is rivalled only by Einsteins' theory, General Relativity.

After decades of trying to find an acceptable theory of the strong nuclear interaction, in the early 1970's physicists started to take seriously a theory which has become known as QCD — quantum chromo-dynamics. This is a theory of the interaction of quarks, which are the constituents of baryons — neutrons, protons etc The quarks are regarded as sections of the spin bundle of space-time $\otimes$ a hermitian 3-plane ($\mathbb{C}^3$) bundle (the associated principal bundle has group SU(3)). Again the forces between quarks are simply given by the curvature of the bundle, with the equation of motion of the quarks

$$\not{D}^{a}\phi = m\phi$$

the Dirac equation again (in this slightly different setting), with m the mass of the quark.

QCD has had a number of successes and is taken very seriously as an acceptable theory of the strong interactions. However the non-linearities make it very difficult to accurately calculate its predictions.

8) Harmonic maps

Let ϕ be a smooth map between Riemannian (compact) manifolds M, N (only M needs to be compact actually)

$$\phi : M \to N$$

Consider the pull-back bundle $\phi^*(TN)$ over M.
$\phi^*(TN)$ is a Riemannian vector bundle over M if we pull back the Riemannian metric on TN to give a fibre metric and the Riemannian covariant derivative of N to give a connection in $\phi^*(TN)$

$$d\phi : T_x M \to T_{\phi(x)} N \qquad \forall x \in M$$

can be viewed as a <u>section-valued 1-form on M</u>
ie $d\phi \in \mathcal{E}^1(M, \phi^* TN)$
We have

8.1) $$d^\nabla d\phi = 0$$

ie

$$\nabla_X(\phi_* Y) - \nabla_Y(\phi_* X) - \phi_*([X,Y]) \overset{6.3}{=} d^\nabla d\phi\,(X,Y) = 0$$
$$\forall X, Y \in \mathcal{X}(M)$$

Now $d\phi = \phi^*(\mathrm{id})$
where $\mathrm{id} \in \mathcal{E}^1(N) \otimes \mathcal{E}_1(N)$ is the identity map on TN
and $d^\nabla \phi = \phi^*(d^\nabla(\mathrm{id}))$
But

$d^\nabla(\mathrm{id}) = \tau$ the torsion of N, which vanishes!
since N is Riemannian:

$$d^\nabla(\mathrm{id})\,(X,Y) \overset{6.3}{=} \nabla_X Y - \nabla_Y X - [X,Y]$$

and thus we have proved 8.1).

If $[M,N]$ is the homotopy classes of maps $M \to N$ and $[c] \in [M,N]$ is some class we seek a <u>canonical representative</u> $\phi : M \to N$ by minimizing some action functional over $[c]$.

The action integral for maps $\phi : M \to N$ is called the <u>energy</u> $E(\phi)$ of ϕ,

$$8.2) \qquad E(\phi) = \| d\phi \|^2 = \langle d\phi, d\phi \rangle = \int_M (d\phi, d\phi).$$

Again we will need <u>existence</u> and <u>regularity</u> proofs here — we need to complete the space of smooth maps in some norm (eg to the space of $W^{1,2}$ maps) and show the existence of a minimizing map in this space and then to prove that the minimizing map is sufficiently regular to be in the domain of the following differential operators. This can not always be done — there is no energy minimizing degree 1 map from S^n to S^n for $n \geq 3$.
See J. Eells and L. Lemaire "A report on harmonic maps" Bull. Lond. Math. Soc <u>10</u> (1978) 1–68.

A special case is when $\dim M = 1$. Then $\phi : M \to N$ is a smooth path (with parameterisation) in N and $E(\phi)$ is its energy. We can take $M = I$, some interval (a manifold with boundary) and consider minimizing E over paths with fixed end points. Critical points of E are exactly the <u>geodesics</u>.

<u>First variation formula for harmonic maps</u>

Let ϕ_t, $t \in (-\epsilon, \epsilon)$, $\phi_0 = \phi$ be a smooth 1-parameter variation of $\phi : M \to N$
ie

$$\phi_{(t)} : (-\epsilon, \epsilon) \times M \to N$$
$$\phi_{(t)}(t_0, x) = \phi_{t_0}(x)$$

3) We give $(-\epsilon, \epsilon) \times M$ the product connection and metric and consider the pull-back bundle $(\phi^* TN)$ over $(-\epsilon, \epsilon) \times M$
[see J. Cheeger & D.G. Ebin "Comparison theorems in Riemannian geometry" North Holland, Amsterdam (1975); pp 3–4]

By abuse of notation we denote $\frac{\partial}{\partial t}$ and $\phi_*(\frac{\partial}{\partial t})$ as X.
Let $\{e_i\}_{i=1,\dim M=n}$ be an orthonormal local frame field for TM near some point $x \in M$ and extend $\{e_i\}$ off M in $(-\epsilon,\epsilon)\times M$ by pushforward (or parallel transport in the t direction).
This results in, by 8.1),

8.4) $$\nabla_X \, d\phi(e_i) = \nabla_{e_i} X \qquad \text{as } [X, e_i] = 0$$

We call $d\phi(e_i) = \tau_i$ when convenient.

Then we calculate the first variation of energy as follows:

$$E = \|d\phi\|^2 = \int_M (d\phi, d\phi) = \int_M (d\phi(e_i), d\phi(e_i)) = \int_M (\tau_i, \tau_i)$$

$$\left.\frac{dE(\phi_t)}{dt}\right|_{t=0} = \frac{d}{dt}\int_M (d\phi, d\phi) = \int_M \frac{\partial}{\partial t}(d\phi, d\phi)$$

$$= \int_M X(d\phi, d\phi) = \int_M X(d\phi(e_i), d\phi(e_i))$$

$$= 2\int_M (\nabla_X \, d\phi(e_i), d\phi(e_i))$$

$$\overset{8.4}{=} 2\int_M (\nabla_{e_i} X, d\phi(e_i))$$

$$= 2\int_M (\nabla X, d\phi)$$

$$= 2\langle d\phi, \nabla X\rangle$$

8.5) $$\boxed{\left.\frac{dE}{dt}\right|_{t=0} = 2\langle \nabla^* d\phi, X\rangle}$$

with $\nabla^* = -\operatorname{tr}\circ\nabla$ the global adjoint of ∇, as in §4.
$\nabla^* = \delta^\nabla$ on $\mathcal{E}^1(M, \phi^*TN)$, ie on $\underline{\underline{1}}$-forms.
just as $\nabla = d^\nabla$ on $\mathcal{E}^0(M, \phi^*TN)$, ie on sections

This is the <u>first variation formula</u>.

8.6) We call $\tau = -\nabla^* d\phi = \{\nabla_{e_i}(d\phi)\}(e_i)$ the <u>tension field</u> of ϕ

Thus the energy is stationary under all 1-parameter variations of ϕ if ϕ satisfies the Euler-Lagrange equation

8.7)
$$\nabla^* d\phi = -\tau = 0$$
or
$$\delta^\nabla d\phi = 0$$

But by 8.1 $d^\nabla d\phi = 0$ (as the torsion of N vanishes)
thus 8.7 is equivalent to

8.8)
$$\Delta^\nabla d\phi = 0$$

ie $d\phi \in \mathcal{E}^1(M, \phi^* TN)$ is a harmonic section-valued 1-form.

8.9) We call ϕ <u>totally geodesic</u> if $\nabla d\phi = 0$

Since $d^\nabla = \theta^i \wedge \nabla_{e_i}$ and $\delta^\nabla = -e_i \lrcorner \nabla_{e_i}$
totally geodesic $\Rightarrow$ harmonic

If $\{e_i\}$ is an <u>adapted</u> orthonormal frame near $x \in M$, so that $\nabla_{e_i} e_j = 0$ at x,
then
$$\tau = -\nabla^* d\phi = (\nabla_{e_i} d\phi)(e_i) = \nabla_{e_i}(d\phi(e_i)) = \nabla_{e_i} \tau_i$$
$$= \nabla_{\tau_i} \tau_i \qquad \text{by abuse of notation}$$

8.10) Thus ϕ harmonic $\Leftrightarrow$ $\nabla_{\tau_i} \tau_i = 0$ (in an adapted frame)

If M is one-dimensional this says $\nabla_{\tau_1} \tau_1 = 0$ which is the usual <u>geodesic equation</u>.

Bochner – Weitzenbock formula for harmonic maps

We have $\phi : M \to N$ is harmonic $\Leftrightarrow \Delta^\nabla d\phi = 0$

By §6, 6.17

$$\Delta^\nabla = \nabla^*\nabla + \mathcal{R}$$

$$\nabla^*\nabla = -\nabla_{e_i,e_i} \quad \text{the } \underline{\text{Bochner (rough) laplacian}}$$

$$\mathcal{R} = R(e_i,e_j)\,\theta^j \wedge e_i \lrcorner$$

$$\mathcal{R} = \mathcal{R}^P + \mathcal{R}^m$$

$$\mathcal{R}^P = \Omega(e_i,e_j)\,\theta^j \wedge e_i \lrcorner$$

$$\mathcal{R}^m = R^{(m)}(e_i,e_j)\,\theta^j \wedge e_i \lrcorner$$

Since $d\phi \in \mathcal{E}^1(M, \phi^* TN)$

8.11)
$$\begin{aligned} \mathcal{R}^m(d\phi) &= R^{(M)}(e_i,e_j)\,\theta^j \wedge e_i \lrcorner\, d\phi \\ &= d\phi(e_i)\, Ric^{(m)}(e_i,e_j)\,\theta^j \\ &\stackrel{\Delta}{=} Ric^{(m)}(d\phi) \end{aligned}$$

$$\begin{aligned} \mathcal{R}^P(d\phi) &= \Omega(e_i,e_j)\,\theta^j \wedge e_i \lrcorner\, d\phi \\ &= \{\Omega(e_i,e_j)\, d\phi(e_i)\}\,\theta^j \\ &= \{R^N(d\phi(e_i), d\phi(e_j))\, d\phi(e_i)\}\,\theta^j \end{aligned}$$

8.12)
$$\mathcal{R}^P(d\phi) = -\, R^N(d\phi(\cdot), d\phi(e_i))\, d\phi(e_i)$$

since the curvature Ω of $\phi^* TN$ is simply $\phi^* R^N$.

8.13)
$$\boxed{(\mathcal{R}(d\phi), d\phi) = Ric^m(d\phi, d\phi) - (R^N(d\phi(e_j), d\phi(e_i))\, d\phi(e_i), d\phi(e_j))}$$

Thus we get the <u>vanishing theorems</u>

8.14) $Ric^m > 0$ and σ^N (sectional curvatures) $\leq 0 \Rightarrow d\phi \equiv 0$
ie any harmonic map is a constant map.

8.15) $Ric^m \geq 0$ and $\sigma^N < 0$. Then any harmonic map is a constant or the image of M ($\phi(M)$) is 1-dimensional, in which case it is a geodesic (up to parametrisation). ($\nabla d\phi = 0$)

8.14 a) <u>Corollary</u> Using an existence theorem (Eells – Sampson) for harmonic maps $M \to N$ when $\sigma^N \leq 0$ and N compact then 8.14) $\Rightarrow$ such an N is a $K(\pi, 1)$

This is a well-known consequence of the Cartan–Hadamard theorem. (see Cheeger & Ebin)

8.16) $Ric^M \geq 0$ and $\sigma^N \leq 0$. Then any harmonic map is totally geodesic and satisfies $Ric^M(d\phi) \equiv 0$ and $R^N(d\phi(\cdot), d\phi(e_i))d\phi(e_i) \equiv 0$

Second variation formula for harmonic maps

As for the first variation, let $\phi_{t,s}$ $(t,s) \in I_\epsilon \times I_\epsilon$, $I_\epsilon = (-\epsilon, \epsilon)$ be a smooth 2-parameter variation of $\phi : M \to N$, $\phi_{0,0} = \phi$.

$$\phi_{(t,s)} : I_\epsilon \times I_\epsilon \times M \to N$$
$$\phi_{(t,s)}(t_0, s_0, x) = \phi_{t_0 s_0}(x).$$

We calculate as in 8.3), denoting $\frac{\partial}{\partial t}$ and $\phi_*(\frac{\partial}{\partial t}) = X$, $\frac{\partial}{\partial s}$ and $\phi_*(\frac{\partial}{\partial s}) = Y$

$$\frac{\partial^2 E(\phi_{ts})}{\partial t \partial s}\bigg|_{t,s=0} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\int_M (d\phi, d\phi) = \int_M \frac{\partial}{\partial t}\frac{\partial}{\partial s}(d\phi, d\phi)$$

$$= \int_M XY(d\phi, d\phi) = \int_M XY(d\phi(e_i), d\phi(e_i))$$

$$= 2\int_M X(\nabla_{e_i} Y, d\phi(e_i))$$

$$= 2\int_M (\nabla_{e_i} X, \nabla_{e_i} Y) + 2\int_M (\nabla_X \nabla_{e_i} Y, d\phi(e_i))$$

$$= 2\int_M (\nabla_{e_i} X, \nabla_{e_i} Y) + 2\int_M (R^N(X, e_i)Y, d\phi(e_i)) + 2\int_M (\nabla_{e_i}\nabla_X Y, d\phi(e_i))$$

$$= 2\langle \nabla X, \nabla Y\rangle - 2\langle R^N(X, d\phi(e_i))d\phi(e_i), Y\rangle + 2\langle \nabla \nabla_X Y, d\phi\rangle$$

Thus if ϕ is harmonic we have the second variation formula

$$8.17) \quad \frac{1}{2}\frac{\partial^2 E}{\partial t \partial s} = \langle \nabla X, \nabla Y\rangle - \langle R^N(X, d\phi(e_i))d\phi(e_i), Y\rangle$$
$$= \langle (\nabla^*\nabla - R^N(\cdot, d\phi)d\phi)X, Y\rangle$$
$$= \langle S(X), Y\rangle$$

where the Jacobi operator, S is defined

$$8.18) \quad \boxed{S = \nabla^*\nabla - R^N(\cdot, d\phi)d\phi}$$

8.19) The <u>stability condition</u> for harmonic maps, which of course must be satisfied for any minimizing map, is

$$\frac{1}{2}\frac{\partial^2 E}{\partial t \partial s}\Big|_{t,s=0} = \langle S(X), Y\rangle = I^E(X,Y) \geq 0 \quad \forall X, Y \in \Gamma(\phi^* T$$

The symmetric bilinear form I^E is called the (energy) <u>index form</u> (for harmonic maps). Solutions $X \in \Gamma(\phi^* TN)$ to

$$S(X) = 0$$

are called <u>Jacobi field</u>, the nullity of I^E is the dimension of the space of Jacobi fields. The <u>index</u> of I^E is the maximal dimension of a subspace of $\Gamma(\phi^* TN)$ on which I^E is negative definite. S is <u>self-adjoint elliptic</u> second-order and standard elliptic theory says that the eigenspaces of S are finite dimensional, and that the index is finite.

In particular

8.20) $$\frac{1}{2}\frac{\partial^2 E}{\partial t^2}\Big|_{t=0} = \langle S(X), X\rangle = \|\nabla X\|^2 - \langle R^N(X, \tau_i)\tau_i, X\rangle$$

Thus

8.21) <u>Theorem</u> Let ϕ be harmonic $\phi: M \to N$ with $\sigma^N > 0$. If there exists a variation vector field $X \in \Gamma(\phi^*(TN))$ which is parallel, $\nabla X = 0$, then ϕ is not stable.

9) <u>A simple application in dimension $M = 1$. The Synge lemma</u>

If $\dim M = 1$ a map $\phi : M \to N$ is a parameterised path in N. Instead of varying the energy of the path we could have been varying the length. $L(\phi) = \int_M |d\phi|$.

The length of the path is independent of its parameterisation (unlike its energy). Physicists would call the diffeomorphism group of reparameterisations the "gauge group" of the problem, and would talk of the "gauge invariance" of the action L. Variation vector fields X which are tangential : $X_{\phi(x)} \propto d\phi(e_x) = \dot{\phi}_x$ do not change the length to any order. Tangential vector fields are given by $\mathrm{im}(d\phi) \subset \phi^*(TN)$. They are called <u>infinitesimal gauge transformations</u> in physics. If we restrict to <u>normal variation vector field</u> then the first and second variation formulas for length and energy are the same.

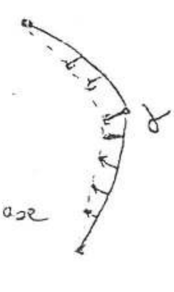
9.1) <u>Synge Lemma</u> If N is an even dimensional complete oriented compact Riemannian manifold of positive sectional curvature ($\sigma^N > 0$) then N is simply connected.

<u>Proof</u>: Assume false. Let $[c]$ be a nontrivial free homotopy class of loops. <u>Let γ be a shortest closed path</u> in $[c]$.

We prove the <u>existence</u> of γ as follows: it is not difficult to show (see Cheeger & Ebin) that locally geodesics minimize length (i.e. if they are short enough). Thus we can choose a minimizing sequence of loops which can be taken to be piecewise geodesic. By compactness of M a subsequence of the end-points of the geodesic segments converge.

Thus γ exists and is piecewise geodesic.

We prove the <u>regularity</u> of γ as follows:



if γ has any kinks then there exists a 1-parameter variation which by the <u>first variation formula</u> would decrease length. Thus γ is a smooth closed geodesic.

given this smooth closed geodesic γ then because N is even-dimen
there is a normal variation vector field X on γ with $\nabla X \equiv 0$.

— as N is even dimensional the normal space P in $T_x N$ to $\dot{\gamma}$ is odd dimensional. Since γ is smooth, parallel transport around γ is an isometry $P \to P$. This isometry must have a fixed vector $X_x \in P$, and parallel transport of X_x around γ provides the required normal variation vector field X.

Then the second variation formula for length (or energy) 8.20) applied to this variation X gives

$$\frac{1}{2}\frac{\partial^2 \mathcal{E}}{\partial t^2}\bigg|_{t=0} = \|\nabla X\|^2 - \langle R^N(X,\dot{\gamma})\dot{\gamma}, X\rangle$$

$$= -\langle R^N(X,\dot{\gamma})\dot{\gamma}, X\rangle$$

$$< 0 \quad \text{since} \quad \sigma^N > 0$$

which is the required contradiction.

$\square$

10) Area of submanifolds M of a Riemannian N

Another useful variational problem is $S = |M| =$ the p-dimension "area" of a p-dimensional submanifold M of an n-dimensional Riemannian manifold N. Critical points of area are called minimal submanifolds. To prove existence we complete the space of smooth submanifolds in a suitable norm to, for example, the space of p-dimensional integral currents in M [H. Federer "Geometric measure theory" Springer-Verlag, Berlin (1969 ? or p-dimensional integral varifolds. There is an existence theorem: every (singular) homology class in $H_p(N, \mathbb{Z})$ is representable by a stable p-dim. integral varifold.
[L. Simon "First and second variation in geometry and topology" U. Melb. Math. Res. Report no 23 (1979).]
Unfortunately, there are known counter-examples to regularity.

<u>First variation formula</u> <u>for minimal submanifolds</u> is

$$(0.1)\qquad \left.\frac{d|M|}{dt}\right|_{t=0} = \langle \mathrm{div}_M X \rangle = \langle id_M , \nabla X \rangle = \langle \nabla^* id_M , X \rangle = -\langle H, X \rangle$$

where $H \in \Gamma(NM)$ NM the normal bundle of M in N is the <u>mean curvature</u>

$\mathrm{div}_M X = (\nabla_{e_i} X , e_i)$ for $\{e_i\}$ an orthonormal frame field for M
$id_M = \theta^i \otimes e_i \in \mathcal{E}^1(M, TN)$
∇ is the connection of N here, or of $TM \otimes NM$ where appropriate.

$$(0.2)\qquad H = (\nabla_{e_i} e_i)^{NM} = \mathrm{trace}\,(\alpha)$$

with α the second fundamental form of M.

Minimal hypersurfaces (codimension $= 1$) are known to be smooth up to dimension $N = 7$.

The <u>second variation</u> <u>formula</u> for minimal submanifolds is

$$(0.3)\qquad \left.\frac{\partial^2 |M|}{\partial t \partial s}\right|_{t,s=0} = \langle \nabla^* \nabla X - \mathrm{Ric}_M (X) - ((\alpha \cdot X), \alpha), Y \rangle$$

for $X \in \Gamma(NM)$, a normal variation,
where $\mathrm{Ric}_M (X,Y) = (R^N(X, e_i)e_i , Y)$
and the $\cdot$ and $(,)$ are the natural partial inner products.

Thus the <u>Jacobi operator</u> is

$$(0.4)\qquad S(X) = \nabla^* \nabla X - \mathrm{Ric}_M(X) - (\alpha \cdot X, \alpha)$$

This is also self-adjoint elliptic on $\Gamma(NM)$

R. Schoen & S.T. Yau "On the structure of manifolds with positive scalar curvature" Manus. math. <u>28</u> (159–183)(1979). use the second variation formula, the existence theorem for minimizing representatives of homology classes, and the regularity of minimal hypersurfaces in dimensions ≤ 7

to obtain useful information on manifolds of dimension ≤ 7 which admit metrics of positive scalar curvature.

10.5) Claim M minimal $\iff$ the inclusion map i is harmonic

Proof: By 10.1) since $id_M = di$ the derivative of the inclusion map. — ie $H = \tau$ the tension field of the inclusion map.

10.6) Theorem [E.A. Ruh & J. Vilms Trans. A.M.S. 149, 569–573 (1970)]
If M is a submanifold of $N = \mathbb{R}^n$ with mean curvature H
and $g : M \to G(p,n)$ ($\dim M = p$, $\dim N = n$)
($G(n,p)$ the Grassmann manifold)
is the Gauss map of M
then $\boxed{g \text{ is harmonic} \iff \nabla H = 0}$

Proof: There is a natural identification of

$$T_{g(x)}\, G(n,p) \sim T^*_x M \otimes NM_x$$

ie

$$\Gamma\left(g^* TG(n,p)\right) \simeq \mathcal{E}^1(M) \otimes NM$$
$$\simeq \mathcal{E}^1(M, NM))$$

Under this identification it is easy to see that

$$\alpha = dg \quad \in \mathcal{E}^1(M, \mathcal{E}^1(M, NM)) \simeq \mathcal{E}^1(M) \otimes \mathcal{E}^1(M) \otimes NM$$

The connection ∇ on $g^* TG(n,p)$ is, in fact, the induced connection on $T^*M \otimes NM$.

We know $d^\nabla \alpha = d^\nabla dg = 0$: this is the Codazzi equation
Then by 8.7 g is harmonic $\iff \nabla^* dg = \nabla^* \alpha = 0$
But $\nabla^* \alpha = -\nabla H$ (the contracted Codazzi equation).
Thus the result.

□

10.7) <u>Corollary</u> M minimal in $\mathbb{R}^n \Rightarrow$ Gauss map harmonic.

This result has been recently used by S Hildebrandt, J. Jost and K-O. Widman "Harmonic mappings and minimal submanifolds Invent. math. <u>62</u>, 269–298 (1980) to prove a Bernstein type theorem.

10.8) In general if $\phi \in \mathcal{E}^p(M) \odot \mathcal{E}^r(M) \otimes V$ then ϕ is harmonic $\Leftrightarrow$ its irreducible components under the action of $O(n)$ (or $SO(n)$ if M is oriented) are harmonic ($n = \dim M$).

11) <u>Yang – Mills fields</u>

Of particular importance in physics is the Yang-Mills variational problem. Here the action, S, is given by

11.1) $$S(\omega) = \|\Omega\|^2 = \int_M (\Omega, \Omega)$$

where $\Omega \in \mathcal{E}^2(M, \mathrm{Ad}P)$ is the curvature of the connection ω. of the principal bundle $P(M, G)$, and $\mathrm{Ad}P$ is the bundle of Lie-algebras

11.2) $$\mathrm{Ad}P = P \times_{\mathrm{Ad}(G)} \mathfrak{g}$$
with $\mathfrak{g}$ the Lie-algebra of G.

The inner-product on $\mathrm{Ad}P$ is defined using an Ad_G invariant inner-product on $\mathfrak{g}$.

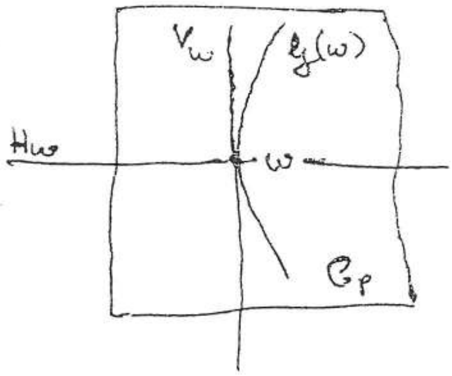
The space $\mathcal{C}_P = \{\omega : \omega \text{ is a connection on } P(M,G)\}$ is an affine space:

1.3) $$T_\omega \mathcal{C}_P \simeq \mathcal{E}^1(M, \mathrm{Ad}P)$$
since the difference of any two connections is a horizontal Ad-equivariant $\mathfrak{g}$-valued 1-form on P (see 6.1 above)

A <u>gauge transformation</u> of $P(M,G)$ is an equivariant bundle automorphism projecting to the identity of M, the <u>group of gauge transformations</u> $\mathcal{G}(P)$, is naturally isomorphic to $\Gamma(\mathrm{ad}\, P)$

11.4) $$\mathcal{G}(P) \simeq \Gamma(\mathrm{ad}\, P)$$
$$\mathrm{ad}\, P \triangleq P \times_{\mathrm{ad}_G} G \quad \text{is a bundle of Lie groups.}$$

$\mathcal{G}(P)$ acts naturally on $\mathcal{C}_P$
(just push-forward the horizontal subspaces defining ω by the gauge transformation)



It is not difficult to see that V_ω, the tangent space at $\omega \in \mathcal{C}_P$ to the action of $\mathcal{G}(P)$ is just

11.5) $$V_\omega \simeq \operatorname{im} d^\nabla \subset \mathcal{E}^1(M, \mathrm{Ad}\, P)$$

The orthogonal complement to V_ω in $T_\omega(\mathcal{C}_P) \simeq \mathcal{E}^1(M, \mathrm{Ad}\, P)$ is then

11.6) $$H_\omega \simeq \ker \delta^\nabla$$

<u>First variation for Yang-Mills fields</u>

The curvature Ω is given in terms of the connection ω, by

11.7) $$\underset{\sim}{\Omega} = D\omega = d\omega + \omega \wedge \omega$$
is a horizontal equivariant $\mathfrak{g}$-valued 2-form on $P(M,G)$.

If ω_t is any 1-parameter variation of ω we know by 11.3) that $\left.\frac{\partial\omega}{\partial t}\right|_{t=0} \triangleq \dot\omega \in \mathcal{E}^1(M, \mathrm{Ad}\, P)$

It is not difficult to show that $\forall B \in \mathcal{E}^1(M, \mathrm{Ad}\, P)$

11.8) $$\underset{\sim}{d}^\nabla B = d\underset{\sim}{B} + \omega \wedge \underset{\sim}{B} + \underset{\sim}{B} \wedge \omega$$

Thus

$$\frac{dS}{dt}\Big|_{t=0} = \frac{d}{dt}\|\Omega\|^2 = \frac{d}{dt}\int_M (\Omega,\Omega) = \int_M \frac{\partial}{\partial t}(\Omega,\Omega)$$

$$= 2\int_M \left(\frac{\partial\Omega}{\partial t}, \Omega\right)$$

and $\frac{\partial \underset{\sim}{\Omega}}{\partial t}\Big|_{t=0} = \frac{\partial}{\partial t}(d\omega + \omega\wedge\omega) = d\dot{\omega} + \dot{\omega}\wedge\omega + \omega\wedge\dot{\omega} = D\dot{\omega}$

so $\frac{\partial\Omega}{\partial t}\Big|_{t=0} = d^\nabla\dot{\omega}$

11.9) Thus $\frac{dS}{dt} = 2\langle\Omega, d^\nabla\dot{\omega}\rangle = 2\langle\delta^\nabla\Omega, \dot{\omega}\rangle$

is the <u>first variation formula</u>

Thus S is stationary for all 1-parameter variations of ω $\Leftrightarrow$

11.10) $\delta^\nabla\Omega = 0$

But $d^\nabla\Omega = 0$ is the Bianchi identity.

11.11) Thus $\boxed{S \text{ stationary} \Leftrightarrow \Delta^\nabla\Omega = 0}$ ie $\Omega\in\mathcal{E}^2(M, \mathrm{Ad}P)$ is harmonic

11.12) As before Ω is called <u>parallel</u> $\Leftrightarrow \nabla\Omega \equiv 0$.

<u>Bochner – Weitzenbock formula</u>

Section 6 applies and we have

$$\Delta^\nabla = \nabla^*\nabla + \mathcal{R}$$

$\nabla^*\nabla = -\nabla_{e_i,e_i}$ the Bochner (rough) Laplacian

$$\mathcal{R} = R(e_i,e_j)\,\theta^j\wedge e_i\lrcorner$$
$$= \mathcal{R}^l + \mathcal{R}^m$$

J-P Bourguignon & H.B. Lawson Jr "Stability and isolation phenomena for Yang-Mills fields" Comm. Math. Phys (1981?) use this, and certain special results in dim $M = 4$, to prove some vanishing theorems

Second variation for Yang – Mills fields

If ω_{ts} is a 2-parameter variation of ω, and we denote $\left.\frac{\partial\omega}{\partial t}\right|_{t=0} = B_1$ $\left.\frac{\partial\omega}{\partial s}\right|_{s=0} = B_2$ $\left.\frac{\partial^2\omega}{\partial t\partial s}\right|_{s,t,=0} = B_3$ $\in \mathcal{E}^1(M,$

we calculate

$$\left.\frac{\partial^2 S}{\partial t\partial s}\right|_{t,s=0} = \int_M \frac{\partial^2}{\partial t\partial s}(\Omega,\Omega)\Big|_{s,t=0}$$

$$\left.\frac{\partial^2 \underset{\sim}{\Omega}}{\partial t\partial s}\right|_{s,t=0} = dB_3 + \omega\wedge B_3 + B_3\wedge\omega + \underset{\sim}{B}_1\wedge\underset{\sim}{B}_2 + \underset{\sim}{B}_2\wedge\underset{\sim}{B}_1$$
$$= DB_3 + \underset{\sim}{B}_1\wedge\underset{\sim}{B}_2 + \underset{\sim}{B}_2\wedge\underset{\sim}{B}_1$$

Thus
$$\frac{1}{2}\left.\frac{\partial^2 S}{\partial t\partial s}\right|_{t,s=0} = \langle d^\nabla B_1, d^\nabla B_2\rangle + \langle\Omega, B_1\wedge B_2 + B_2\wedge B_1\rangle + \langle\Omega, d^\nabla B_3$$

If Ω is harmonic this is
$$= \langle d^\nabla B_1, d^\nabla B_2\rangle + \langle\Omega, B_1\wedge B_2 + B_2\wedge B_1\rangle$$

A calculation shows that $(\Omega, B_1\wedge B_2 + B_2\wedge B_1) = (Q^\rho(B_1), B_2)$

Thus we have the

11.13) second variational formula
$$\frac{1}{2}\left.\frac{\partial^2 S}{\partial t\partial s}\right|_{s,t=0} = \langle(\delta^\nabla d^\nabla + Q^\rho)B_1, B_2\rangle$$

If we restrict to "normal variation vector fields"

ie $B_i \in H_\omega = \ker(\delta^\nabla)$

we have

11.14) $$\frac{1}{2}\left.\frac{\partial^2 S}{\partial t\partial s}\right|_{s,t=0} = \langle(\Delta^\nabla + Q^\rho)B_1, B_2\rangle = \langle S(B_1), B_2\rangle$$

11.15) The Jacobi operator $S = \Delta^\nabla + Q^\rho$ is self-adjoint, elliptic.

ω (and Ω) is called a <u>Yang-Mills field</u> if $\Delta^\nabla \Omega = 0$

11.16) <u>Corollary</u> If $\Omega \in \mathcal{E}^2(M, \mathrm{Ad}\,P)$ is a Yang-Mills field for some principal bundle $P(M,G)$ over a Riemannian manifold M, and if $\mathrm{Ric}^M > 0$ and if there exists a $B \in H_\omega = \ker \delta^\nabla \subset \mathcal{E}^1(M, \mathrm{Ad}\,P)$ which is parallel, $\nabla B = 0$, then Ω is not stable.

<u>Proof</u>.
$$\frac{1}{2}\frac{\partial^2 S}{\partial t^2}\Big|_{t=0} = \langle \Delta^\nabla B + Q^P B, B\rangle$$
$$= \|\nabla B\|^2 + \langle \mathrm{Ric}^M(B,B)\rangle + 2\langle Q^P(B), B\rangle$$

Since $\nabla B = 0$ then $d^\nabla B = \delta^\nabla B = 0$ ie $\Delta^\nabla B = 0$
By Bochner-Weitzenbock, $\Delta^\nabla = \nabla^*\nabla + Q^M + Q^P$
so $Q^P(B) = -Q^M(B) = -\mathrm{Ric}^M(B)$

$$\Rightarrow \frac{1}{2}\frac{\partial^2 S}{\partial t^2}\Big|_{t=0} = -\langle \mathrm{Ric}^M(B,B)\rangle < 0$$

$\square$.

12) <u>Aside on the case M complete, non-compact</u>

In this case the Bochner-Weitzenbock formulas are still often useful — one uses them to prove something, such as $|\Omega|$ or $|d\phi|$, is a (non-negative) <u>subharmonic function</u> on M. If the action, S is finite then Ω or $d\phi$, as the case may be, is L^2 integrable. One then uses the

<u>Theorem</u> (S. T. Yau "Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry" Indiana J Math. ..)

Every non-negative L^2 subharmonic function on a complete Riemannian manifold is constant.

13) Derrick – Simons – Xin theorems

A) Yang – Mills fields on S^n

Let $P(M,G)$ be a principal G-bundle over $M = S^n$ and let $\Omega \in \mathcal{E}^2(M, \mathrm{Ad}\,P)$ be a Yang–Mills field

ie $\Delta^\nabla \Omega = 0$

Then for all vector fields $X \in \mathfrak{X}(M)$

$$X \lrcorner \Omega \in \mathcal{E}^1(M, \mathrm{Ad}\,P)$$

ie $X \lrcorner \Omega$ is a "variation vector field" for ω (or Ω)

$$\nabla^*\nabla(v \lrcorner \Omega) = -\nabla_{e_i}\nabla_{e_i}(v \lrcorner \Omega) \qquad \text{calculating in an adapted frame}$$

$$= -\nabla_{e_i}(\nabla_{e_i} v \lrcorner \Omega + v \lrcorner \nabla_{e_i}\Omega)$$

13.1) $$= (\nabla^*\nabla v) \lrcorner \Omega + v \lrcorner \nabla^*\nabla \Omega - 2(\nabla_{e_i} v) \lrcorner \nabla_{e_i}\Omega$$

If f is a linear function on $\mathbb{R}^{n+1}$ vanishing at the origin and if $v \in \mathfrak{X}(M)$ is given by

13.2) $$v = \nabla f|_{S^n}$$

13.3) Then $\nabla_{e_i} v = -f e_i$

and $\nabla^*\nabla v = v$

Thus $\nabla^*\nabla(v \lrcorner \Omega) = v \lrcorner \Omega + v \lrcorner \nabla^*\nabla\Omega - 2f(\delta^\nabla \Omega)$

So if Ω is harmonic

13.4) $$\nabla^*\nabla(v \lrcorner \Omega) = v \lrcorner \Omega + v \lrcorner \nabla^*\nabla\Omega$$

$$= v \lrcorner \Omega - v \lrcorner \mathcal{R}(\Omega)$$

13.5) $$\mathcal{R}^M(\Omega) = 2(n-2)\Omega$$

Proof: $$\mathcal{R}^M(\Omega) = \overset{M}{R}(e_i,e_j)\theta^j \wedge e_i \lrcorner \Omega = R^M(e_i,e_j)\Omega(e_i,e_k)\theta^j \wedge \theta^k$$

$$= \Omega(e_i,e_k) R^M(e_i,e_j)\theta^j \wedge \theta^k$$

$$= \Omega(e_i,e_k)\{R_{ijjl}\,\theta^l \wedge \theta^k + R_{ijkl}\,\theta^j \wedge \theta^l\}$$

$$= \Omega(e_i,e_k)\{(n-1)\delta_{il}\,\theta^l \wedge \theta^k + (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})\theta^j \wedge \theta^l\}$$

$$= \Omega(e_i,e_k)\{(n-1)\theta^i \wedge \theta^k - \theta^i \wedge \theta^k\}$$

$$= (n-2)\Omega(e_i,e_k)\,\theta^i \wedge \theta^k = 2(n-2)\Omega \qquad \square$$

13.6) $$v \lrcorner Q^P(\Omega) = 2\, Q^P(v \lrcorner \Omega)$$

<u>Proof</u>:
$$\begin{aligned} Q^P(\Omega) &= \Omega(e_i, e_j)\, \theta^j \wedge e_i \lrcorner \Omega \\ &= \Omega(e_i, e_j)\, \Omega(e_i, e_k)\, \theta^j \wedge \theta^k \\ &= [\Omega(e_i, e_j), \Omega(e_i, e_k)]\, \theta^j \wedge \theta^k \end{aligned}$$

as Ω acts by Lie bracket in $\mathrm{Ad}\,P$.

$$\begin{aligned} v \lrcorner Q^P(\Omega) &= [\Omega(e_i, v), \Omega(e_i, \cdot)] - [\Omega(e_i, \cdot), \Omega(e_i, v)] \\ &= 2\,[\Omega(e_i, v), \Omega(e_i, \cdot)] \\ Q^P(v \lrcorner \Omega) &= \Omega(e_i, e_j)\, \theta^j \wedge e_i \lrcorner v \lrcorner \Omega \\ &= [\Omega(e_i, \cdot), \Omega(v, e_i)] \\ &= [\Omega(e_i, v), \Omega(e_i, \cdot)] \end{aligned}$$

□.

13.7) $$S(v \lrcorner \Omega) = -(n-4)\, v \lrcorner \Omega$$

<u>Proof</u>:
$$\begin{aligned} S(v \lrcorner \Omega) &= \Delta^\nabla (v \lrcorner \Omega) + Q^P(v \lrcorner \Omega) \\ &= \nabla^* \nabla (v \lrcorner \Omega) + Q^m(v \lrcorner \Omega) + 2\,Q^P(v \lrcorner \Omega) \\ &\overset{13.4,5,6}{=} v \lrcorner \Omega - 2(n-2)\, v \lrcorner \Omega - 2\,Q^P(v \lrcorner \Omega) \\ &\qquad + (n-1)\, v \lrcorner \Omega + 2\,Q^P(v \lrcorner \Omega) \\ &= -(n-4)\, v \lrcorner \Omega. \end{aligned}$$

□

Thus

13.8) <u>J. Simons theorem</u> For $n > 4$ there are no stable Yang-Mills fields on S^n.

B) Harmonic maps from S^n

Let ϕ be a harmonic map from S^n into an arbitrary Riemannian manifold N

$$\phi : M \longrightarrow N \qquad (M = S^n).$$

Using the same v as above, $v \lrcorner d\phi = d\phi(v) \in \Gamma(\phi^* TN)$ is a "variation vector field", and the calculation above gives

13.9) $$\nabla^* \nabla (v \lrcorner d\phi) = v \lrcorner d\phi - v \lrcorner Q(d\phi)$$

13.10) $$Q^M(d\phi) = Ric^M(d\phi) = (n-1)\, d\phi$$

13.11) $$Q^P(d\phi) = - R^N(d\phi(\cdot), d\phi(e_i))\, d\phi(e_i)$$

Proof: $$\begin{aligned} Q^P(d\phi) &= \{ R^N(d\phi(e_i), d\phi(e_j))\, d\phi(e_i) \} \theta^j \\ &= - R^N(d\phi(\cdot), d\phi(e_i))\, d\phi(e_i) \end{aligned}$$ □

13.12) $$S(v \lrcorner d\phi) = -(n-2)\, v \lrcorner d\phi$$

Proof $$\begin{aligned} S(v \lrcorner d\phi) &\overset{8.18}{=} \nabla^* \nabla (v \lrcorner d\phi) - R^N(v \lrcorner d\phi, d\phi(e_i))\, d\phi(e_i) \\ &\overset{13.9,10,11}{=} v \lrcorner d\phi - (n-1)\, v \lrcorner d\phi + R^N(v \lrcorner d\phi, d\phi(e_i))\, d\phi(e_i) \\ &\qquad - R^N(v \lrcorner d\phi, d\phi(e_i))\, d\phi(e_i) \\ &= -(n-2)\, v \lrcorner d\phi \end{aligned}$$ □

Thus

13.13) Xin's theorem For $n > 2$ there are no stable harmonic maps from S^n to any Riemannian manifold N.

[Y.L. Xin "Some results on stable harmonic maps" Duke math j. 47 609–613 (1980).]

The vector fields $v \in \mathfrak{X}(M)$ are conformal vector fields. A physicist, Derrick, proved 1964 (?) a general theorem about nonexistence of "instanton solutions" in a general field theory setting, using a simple scaling argument (physicists work on $\mathbb{R}^n$ with suitable boundary conditions rather than S^n). Simon's and Xin's theorems seem to be the translation of Derricks theorem into this context.

14) Yang – Mills on S^4 and harmonic maps $S^2 \to$ Kähler N

The above argument gives no information in these cases, as the Yang–Mills action is conformally invariant in 4 dimensions and the energy of a map is conformally invariant in 2 dimensions. Nevertheless some progress can be made:

A) Self duality for Yang Mills

$$\Omega \in \mathcal{E}^2(M, \mathrm{Ad}P)$$

When $\dim M = 4$

$$* : \mathcal{E}^2(M, \mathrm{Ad}P) \to \mathcal{E}^2(M, \mathrm{Ad}P)$$

and $\quad *^2 = 1$

We decompose $\mathcal{E}^2(M, \mathrm{Ad}P)$ into the + and − eigenspaces of $*$ as

14.1) $$\Omega = \Omega^+ + \Omega^-$$

This is an orthogonal decomposition.

Then

14.2) $$S = \|\Omega^+\|^2 + \|\Omega^-\|^2$$

It turns out that

(14.3) $$\|\Omega^+\|^2 - \|\Omega^-\|^2 = \mathrm{const}\,\{c_2(P)[M]\} = k \text{ say}$$

is an invariant of the isomorphism class of P — $c_2(P)$ is the second Chern class of P and $|\Omega^+|^2 - |\Omega^-|^2$ is the representation by curvature polynomials given by Chern-Weil theory.

Thus $\quad S \geq |k|$

14.4) and $\quad S = |k| \iff \Omega^+$ or $\Omega^- = 0$

ie $\iff \Omega$ is self-dual or anti-self-dual

14.5) A self-dual Yang-Mills field trivially is harmonic:

$$\begin{aligned}\delta^\nabla \Omega^+ &= (-1)^{n(p+1)+1} * d^\nabla * \Omega^+ \\ &= (-1)^{n(p+1)+1} * d^\nabla \Omega^+ \\ &= 0 \quad \text{by the Bianchi identity.}\end{aligned}$$

14.6) By 14.4) any self-dual Yang-Mills field is stable

Since $*$ is parallel

14.7) $$\Delta^\nabla \Omega = 0 \iff \Delta^\nabla \Omega^+ = \Delta^\nabla \Omega^- = 0$$

B) <u>Self-duality for harmonic maps</u>

$$d\phi \in \mathcal{E}^1(M, \phi^* TN)$$

When $\dim M = 2$

$$* : \mathcal{E}^1(M, \phi^* TN) \to \mathcal{E}^1(M, \phi^* TN)$$

and $*^2 = -1$.

Suppose <u>N is a Kähler manifold</u>

Then we have $J : \phi^* TN \to \phi^* TN$. J the complex structure of N, and $J^2 = -1$, also $\nabla J = 0$.

Thus $J \circ * : \mathcal{E}^1(M, \phi^* TN) \to \mathcal{E}^1(M, \phi^* TN)$

with $(J \circ *)^2 = 1$.

We decompose $\mathcal{E}^1(M, \phi^* TN)$ into the $+$ and $-$ eigenspaces of $J \circ *$ (orthogonal decomposition)

So

14.8) $$d\phi = d\phi^+ + d\phi^-$$

$*$ is actually the complex structure on M of course, and this decomposition is the usual decomposition into holomorphic and anti-holomorphic parts:

$$d\phi^+ = \bar\partial \phi \qquad d\phi^- = \partial\phi.$$

ϕ is said to be <u>holomorphic</u> (anti-self-dual) if $d\phi^+ = 0$.

Then

14.9) $$E(\phi) = \| d\phi^+ \|^2 + \| d\phi^- \|^2$$

It turns out that

14.10) $$\| d\phi^+\|^2 - \| d\phi^-\|^2 = \text{const}\{(\phi^*\omega)[M]\} = k \text{ say}$$

is an invariant of the homotopy class of $\phi : M \to N$, ω is the Kähler class of N.

Thus $E \geq |k|$

14.11) and $E = |k| \iff d\phi^+$ or $d\phi^- = 0$

ie $\iff \phi$ is <u>holomorphic</u> or <u>conjugate holomorphic</u>

14.12) A holomorphic map is trivially harmonic and stable.

Since $*$ and J are parallel

14.13) $$\Delta^\nabla(d\phi) = 0 \iff \Delta^\nabla d\phi^+ = \Delta^\nabla d\phi^- = 0 .$$

c) <u>Self-duality for Riemannian 4-manifolds</u>

On a Riemannian 4-manifold the curvature tensor
$$R \in \mathcal{E}^2(M) \odot \mathcal{E}^2(M)$$
Thus we have $*$ acting on both factors, so R decomposes as

14.14) $$R = R^+_+ + R^+_- + R^-_+ + R^-_-$$

The notion of self-duality of A) above corresponds to
$$R = R^+_+ + R^+_- .$$
An alternative notion of self-duality (<u>double self-duality</u>) is that
$$R = R^+_+ + R^-_-$$
This is equivalent to M being an <u>Einstein manifold</u> as $R^+_- + R^-_+$ is the part of the curvature tensor coming from the traceless Ricci tensor, Ric_0.

It is easy to prove that

14.15) M Einstein $\Rightarrow$ R harmonic

Now the signature of M, $\sigma(M)$ is given by

14.16) $$\sigma(M) = \tfrac{1}{3}\, p_1\,[M] = \| W_+ \|^2 - \| W_- \|^2$$

where p_1 is the first Pontriagin class of M and W_+, W_- are orthogonal components of the Weyl conformal tensor.

By careful analysis of the Bochner-Weitzenbock formula in dimension 4, Bourguignon was able to prove

14.17) Bourguignon's theorem M compact, 4 dimensional, $\sigma(M) \neq 0$
Then R harmonic $\Rightarrow$ R Einstein.

[J. P. Bourguignon "Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein" Invent. Math. 63, 263–286 (1981).]

4.18) The hypothesis on the signature is necessary as A. Derdzinski ["Classification of certain compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor" Mat. Z. 172, 273–280 (1980)] has given examples of warped products of $S^1 \times S^3$ with harmonic curvature and non-parallel Ricci tensor. These metrics are conformally flat ($W \equiv 0$) with constant scalar curvature.

It is easy to show that, in general

14.19) $$W \equiv 0 \text{ and } \mathcal{K} = \text{constant} \Rightarrow R \text{ harmonic}.$$

D) Yang-Mills on S^4 (cont'd)

Let Ω be a Yang Mills field on S^4, $\Delta^\nabla \Omega = 0$.
Thus Ω^+ and Ω^- are harmonic (14.7).

We now use $\boxed{v \lrcorner \Omega^+} \in \mathcal{E}^1(M, \mathrm{Ad}\,P)$
as the "variation vector field for Ω, with v the conformal vector fields on S^4 described in 13 A).

$$\nabla^* \nabla (v \lrcorner \Omega^+) = (\nabla^* \nabla v) \lrcorner \Omega^+ + v \lrcorner (\nabla^* \nabla \Omega^+)$$

as Ω^+ is harmonic

14.20) $$= v \lrcorner \Omega^+ - v \lrcorner Q^m(\Omega^+) - v \lrcorner Q^P(\Omega^+)$$

$$\begin{aligned} S^\nabla (v \lrcorner \Omega^+) &= \Delta^\nabla (v \lrcorner \Omega^+) + Q^P(v \lrcorner \Omega^+) \\ &= \nabla^* \nabla (v \lrcorner \Omega^+) + Q^m(v \lrcorner \Omega^+) + 2Q^P(v \lrcorner \Omega^+) \end{aligned}$$

14.21) $$\begin{aligned} &= v \lrcorner \Omega^+ - v \lrcorner Q^m(\Omega^+) + Q^m(v \lrcorner \Omega^+) \\ &\quad - v \lrcorner Q^P(\Omega^+) + 2Q^P(v \lrcorner \Omega^+) \end{aligned}$$

14.22) $$Q^m(\Omega^+) = 2(n-2)\,\Omega^+ \qquad (\text{as } 13.5)$$

14.23) $$Q^m(v \lrcorner \Omega^+) = (n-1)\, v \lrcorner \Omega^+$$

14.24) $$v \lrcorner Q^P(\Omega^+) = [\Omega(e_i, v), \Omega^+(e_i, \cdot)] - [\Omega(e_i, \cdot), \Omega^+(e_i, v)]$$

(as 13.6)

14.25) $$Q^P(v \lrcorner \Omega^+) = -[\Omega(e_i, \cdot), \Omega^+(e_i, v)] \qquad ''$$

Thus
$$S^\nabla (v \lrcorner \Omega^+) = -(n-4)\, v \lrcorner \Omega^+ - \{ [\Omega(e_i, \cdot), \Omega^+(e_i, v)] + [\Omega(e_i, v), \Omega^+(e_i, \cdot)] \}$$

So (as $n = 4$)

14.26) $$S^\nabla (v \lrcorner \Omega^+) = -\{ [\Omega^-(e_i, \cdot), \Omega^+(e_i, v)] + [\Omega^-(e_i, v), \Omega^+(e_i, \cdot)] \}$$

<u>Assume Ω is stable</u>

Then $(S^\nabla (v \lrcorner \Omega^+), v \lrcorner \Omega^+) \geq 0 \quad \forall v$

Choose $v = e_k$ and sum :

$$(S^\nabla(e_k \lrcorner \Omega^+), e_k \lrcorner \Omega^+) = -\Big(\{ [\Omega^-(e_i,e_j), \Omega^+(e_i,e_k)] + [\Omega^-(e_i,e_k), \Omega^+(e_i,e_j)] \} \cdot \Omega^+(e_k,e_j) \Big)$$

$= 0$ since the first term is symmetric in k and j while the second is skew.

Thus

14.27) $S^\nabla(e_k \lrcorner \Omega^+) = 0 \quad \forall k$

Thus the symmetric part of $[\Omega^-(e_i,\cdot), \Omega^+(e_i,\cdot)] = 0$

14.28) But it can be shown (Bourguignon–Lawson) that $[\Omega^-(e_i,\cdot), \Omega^+(e_i,\cdot)]$ is symmetric.

14.29) Thus $[\Omega^-(e_i,\cdot), \Omega^+(e_i,\cdot)] = 0 \quad \in \mathcal{E}^1(M) \odot \mathcal{E}^1(M) \otimes Ad\,P$

ie $c[\Omega^-, \Omega^+] = 0$

where $[\Omega^-, \Omega^+] \quad \in \mathcal{E}^2(M) \otimes \mathcal{E}^2(M) \otimes Ad\,P$

and c is the contraction.

Bourguignon–Lawson show that this implies that

14.30) $$[\Omega^-, \Omega^+] = 0$$

Now if G is small enough, ie <u>$G = SU(2)$, $SU(3)$ or $SU(2)\times U(1)$</u>

14.31) then $[\Omega^-, \Omega^+] = 0 \Rightarrow \Omega^-$ or Ω^+ is abelian in some open set.

Say Ω^+ is abelian in an open set

then $[\Omega^+, \Omega^+] = 0$ in this open set

But $[\Omega^+, \Omega^+]$ satisfies an elliptic system and hence, by the <u>Aronszajn unique continuation theorem</u>

14.32) $$[\Omega^+, \Omega^+] \equiv 0 \quad \text{on } M.$$

Thus

14.33) $$[\Omega, \Omega^+] \equiv 0 \quad \text{on } M$$

Then

$$\begin{aligned} \Delta^\nabla \Omega^+ &= \nabla^*\nabla \Omega^+ + \mathcal{R}^M(\Omega^+) + \mathcal{R}^P(\Omega^+) \\ &= \nabla^*\nabla \Omega^+ + 4\Omega^+ + [\Omega(e_i,e_j), \Omega^+(e_i,e_k)]\,\theta^j \wedge \theta^k \\ &= \nabla^*\nabla \Omega^+ + 4\Omega^+ \end{aligned}$$

14.34) Thus $\Omega^+ \equiv 0$ on M.

14.35) <u>Bourguignon–Lawson theorem</u>. Any stable Yang-Mills field on S^4 with group $G = SU(2)$, $SU(3)$ or $SU(2) \times U(1)$ is $\pm$ self-dual.

E) Harmonic maps $S^2 \to$ Kähler N.

Let ϕ be a stable harmonic map $\phi: S^2 \to N$ with N Kähler. Then $d\phi^+$ and $d\phi^-$ are harmonic.

We now use $\boxed{v \lrcorner d\phi^+} \in \phi^* TN$ as the variation vector field for ϕ, with v the conformal vector field on S^2 described in 13A).

14.36) $$\begin{aligned}\nabla^*\nabla(v \lrcorner d\phi^+) &= v \lrcorner d\phi^+ - v \lrcorner Q^m(d\phi^+) - v \lrcorner Q^P(d\phi^+) \\ &= v \lrcorner d\phi^+ - v \lrcorner (n-1)\, d\phi^+ \\ &\qquad + v \lrcorner R^N(d\phi(\cdot), d\phi(e_i))\, d\phi^+(e_i) \\ &= -(n-2)\, v \lrcorner d\phi^+ + R^N(d\phi(v), d\phi(e_i))\, d\phi^+(e_i)\end{aligned}$$

14.37) $$\nabla^*\nabla(v \lrcorner d\phi^+) = R^N(d\phi(v), d\phi(e_i))\, d\phi^+(e_i)$$

$$S(d\phi^+(v)) = \nabla^*\nabla(d\phi^+(v)) - R^N(d\phi^+(v), d\phi(e_i))\, d\phi(e_i)$$

4.38) $$S(d\phi^+(v)) = R^N(d\phi(v), d\phi(e_i))\, d\phi^+(e_i) - R^N(d\phi^+(v), d\phi(e_i))\, d\phi(e_i).$$

Now, by stability $(S(d\phi^+(v)), d\phi^+(v)) \geq 0 \quad \forall v$

So choose $v = e_j$ and sum

$$\Rightarrow \big(R^N(d\phi(e_j), d\phi(e_i))\, d\phi^+(e_i), d\phi^+(e_j)\big) - \big(R^N(d\phi^+(e_j), d\phi(e_i))\, d\phi(e_i), d\phi^+(e_j)\big) \geq 0$$

4.39) $$\Rightarrow \big(R^N(d\phi^-(e_j), d\phi^-(e_i))\, d\phi^+(e_i), d\phi^+(e_j)\big) - \big(R^N(d\phi^+(e_j), d\phi^-(e_i))\, d\phi^-(e_i), d\phi^+(e_j)\big) \geq 0$$

We write this as $T1 + T2 \geq 0$.

Now $d\phi^\pm = d\phi^\pm(e_1)\,\theta^1 + d\phi^\pm(e_2)\,\theta^2$

and $(J \circ *)\, d\phi^\pm = -J\, d\phi^\pm(e_2)\,\theta^1 + J\, d\phi^\pm(e_1)\,\theta^2$

Thus

4.40) $$\begin{aligned} d\phi^\pm(e_1) &= \mp J\, d\phi^\pm(e_2) \\ d\phi^\pm(e_2) &= \pm J\, d\phi^\pm(e_1)\end{aligned}$$

(using 14.40) we calculate (using properties of the curvature tens of Kähler manifolds) that

14.41) $$T1 = -2\left(R^N(d\phi^-(e_1), J\,d\phi^-(e_1))J\,d\phi^+(e_1),\ d\phi^+(e_1)\right)$$

This is a <u>holomorphic bisectional curvature</u> term, as the holomorphic bisectional curvatures $\sigma^{\text{holom}}(X,Y)$, for X, Y unit vectors, are defined

14.42) $$\sigma^{\text{holom}}(X,Y) = (R(X,JX)JY, Y)$$

Similarly we calculate that

14.43) $$T2 = -2\left\{\left(R^N(d\phi^+(e_1), d\phi^-(e_1))d\phi^-(e_1), d\phi^+(e_1)\right) + \left(R^N(d\phi^+(e_1), J\,d\phi^-(e_1))J\,d\phi^-(e_1), d\phi^+(e_1)\right)\right\}$$

This is a sum of two sectional curvature terms. In fact $T1 = T2$ as, in general,

14.44) $$\begin{aligned}(R(X,JX)JY,Y) &= -(R(JX,JY)X,Y) - (R(JY,X)JX,Y) \\ &\qquad \text{by the first Bianchi identity} \\ &= (R(Y,X)X,Y) + (R(Y,JX)JX,Y) \\ &\qquad \text{by the symmetries of } R.\end{aligned}$$

Thus for stability of ϕ we require that

14.45) $$\left(R^N(d\phi^-(e_1), J\,d\phi^-(e_1))J\,d\phi^+(e_1),\ d\phi^+(e_1)\right) \leq 0\,!$$

Thus if the holomorphic bisectional curvature of N is > 0 we require $d\phi^+$ or $d\phi^- = 0$ on an open set, and hence on M i.e.

14.46) <u>Siu – Yau's theorem</u> Any stable harmonic map from S^2 to a Kähler manifold N of positive holomorphic bisectional curvature is $\pm$ holomorphic.

[Y.T. Siu & S.T. Yau "Compact Kähler manifolds of positive bisectional curvature" Invent. math. 59 189–204 (1980).]

Note: Here we have given the variation vector field $d\phi^+(v)$ explicitly, we do not need the Siu-Yau existence theorem.

14.46) is the key step in Siu & Yau's proof of the Frankel conjecture: Every compact Kähler manifold of positive bisectional curvature is biholomorphic to the complex projective space.

14.47) Let N be a compact hermitian symmetric space. $N = G/H$. If

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} \qquad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h} \quad \text{etc}$$

is the usual decomposition of the Lie algebras

Then sectional curvature of N are given by

14.48) $$(R^N(X,Y)Y, X) = \| [X,Y] \|^2 \geq 0.$$

for $X, Y \in \mathfrak{m}$

Thus (with John Harnad)

14.49) Claim: Any stable harmonic map from S^2 to a compact hermitian symmetric space satisfies $[d\phi^+, d\phi^-] = 0$.

4.50) Physicists call harmonic maps (over space time) "nonlinear sigma models". Some years ago they noticed close analogies between nonlinear sigma models in 2-dimensions (harmonic maps from S^2) and Yang-Mills theory in 4-dimensions (Yang-Mills over S^4)

.51) We should mention A.M. Din and W.J. Zakrzewski "General classical solutions in the CP^{n-1} model" Nuclear Physics B174, (1980) 397-406. They study L^2 harmonic maps from S^2 to $\mathbb{P}^{n-1}(\mathbb{C})$, and express the general solution explicitly in terms of n rational analytic functions. All the stable maps are shown to be $\pm$ holomorphic (i.e. $\pm$ "instantons") [this follows from the Siu-Yau result 14.46), of course].