Functions: Summary of Key Definitions

Below is a summary of the formal definitions of functions and related concepts such as injective function, inverse function, etc. In the text, these definitions are covered as part of Chapter 1 (pp. 10–13) and Chapter 4 (pp. 80–13).

You will need to know these formal definitions *exactly* in order to do proofs, but you should also develop an intuitive grasp of these concepts in terms of arrow diagrams (see p. 84 of the text), and you should try to see how the formal definitions relate to properties in arrow diagrams, and why the definitions are the way they are. **Ideally, you should try to reach a level of comfort with this material where the formal definitions feel completely natural to you, you don't have to make an effort to memorize them, and you could easily write down the formal definition of a concept based on your intuitive understanding of this concept.**

- **Function, domain and target:** Given two nonempty sets A and B, **a function f from A to B**, denoted by $f : A \to B$, is an assignment of a *unique* element $f(a)$ in B to *each* element a in A. The set A is called the **domain** of f, the set B is called the **target** (or **codomain**) of f, and the set $\{f(a) : a \in A\}$ (a subset of B) is the called **image** of A under f.
  Note: The sets A and B are an integral part of a definition of a function, and they must be specified, along with the rule that describes the assignment of values $f(a)$.

- **Graph of a function:** The **graph** of a function $f : A \to B$ is the set (a subset of the Cartesian product $A \times B$)
$$\{(a, b) : a \in A, b = f(a)\}.$$

- **Image:** Given any subset $S \subseteq A$, the **image of S under f** is defined as
$$f(S) = \{f(a) : a \in S\} = \{b \in B : b = f(a) \text{ for some } a \in S\ \},$$
  The **image of the function** f is the set $f(A)$.

- **Composition of functions:** Given functions $f : A \to B$ and $g : B \to C$, the composition of g and f is the function $g \circ f : A \to C$ defined by
$$(g \circ f)(a) = g(f(a)) \quad (a \in A).$$

- **Inverse function:** The **inverse** of a function $f : A \to B$, is a function $g : B \to A$ satisfying
$$g(f(a)) = a \text{ for all } a \in A, \text{ and } f(g(b)) = b \text{ for all } b \in B.$$
  Remark: Not every function has an inverse. The functions that have an inverse are precisely the bijective functions.

- **Injective, surjective, and bijective:** Let $f : A \to B$ be an arbitrary function.
$$\begin{aligned} f \textbf{ is injective ("one-to-one")} &\iff (\forall x, y \in A)[f(x) = f(y) \implies x = y] \\ &\iff (\forall x, y \in A)[x \neq y \implies f(x) \neq f(y)] \\ f \textbf{ is surjective ("onto")} &\iff (\forall b \in B)(\exists a \in A)[f(a) = b] \\ f \textbf{ is bijective} &\iff f \text{ is injective and surjective} \end{aligned}$$

- **Decreasing, increasing, etc.:** Let f be a function[1] **from $\mathbb{R}$ to $\mathbb{R}$.**
$$\begin{aligned} f \textbf{ is increasing} &\iff (\forall x, y \in \mathbb{R})[x < y \implies f(x) < f(y)] \\ f \textbf{ is decreasing} &\iff (\forall x, y \in \mathbb{R})[x < y \implies f(x) > f(y)] \\ f \textbf{ is nonincreasing} &\iff (\forall x, y \in \mathbb{R})[x < y \implies f(x) \geq f(y)] \\ f \textbf{ is nondecreasing} &\iff (\forall x, y \in \mathbb{R})[x < y \implies f(x) \leq f(y)] \\ f \textbf{ is bounded} &\iff (\exists M \in \mathbb{R})(\forall x \in \mathbb{R})[|f(x)| \leq M] \end{aligned}$$

[1]These concepts make only sense for real-valued functions defined on real numbers (or subsets of real numbers), since they involve inequalities ($x < y$, $x \leq y$, $|f(x)| \leq M$, etc.), which are only defined for real numbers.)