Name:

Collaborator(s)[1]:

Math 347, Prof. Hildebrand, Summer 2019

HW 7, DUE IN CLASS MONDAY, 7/15

- **Grading:** 15 points total, broken down as follows:
  - **Presentation/effort:** $\boxed{\textbf{3 points}}$
  - **Graded problems:** $\boxed{\textbf{12 points}}$

Puzzle problems. *The first few problems are puzzles that some clear and logical thinking, but otherwise are largely independent of the topics covered in class. A key goal of these problems is to practice proof-writing in situations where you cannot follow a standard template, so focus on the write-up/explanation of your answer. Try to be as clear and as convincing as possible!*

$\boxed{\#1}$ **(4.13 in the text). The** 1089 **puzzle.** (See back of page for complete statement.)

$\boxed{\#2}$ **(3.62 in the text). The December 31 puzzle.** (See back of page for complete statement.)

$\boxed{\#3}$ **A magic matrix.** Consider the $n \times n$ matrix obtained by filling the rows of this matrix with the numbers $1, 2, \ldots, n^2$, so that the first row consists of the numbers $1, 2, \ldots, n$, the second row of the numbers $n+1, n+2, \ldots, 2n$, and so on. Now choose n entries in this matrix such that each row and each column contains exactly one of these entries, and add these n entries. For example, in the case $n = 4$, here are two possible such choices of these 4 entries:

$$\begin{bmatrix} 1 & 2 & \boxed{3} & 4 \\ 5 & 6 & 7 & \boxed{8} \\ \boxed{9} & 10 & 11 & 12 \\ 13 & \boxed{14} & 15 & 16 \end{bmatrix}, \quad \begin{bmatrix} 1 & \boxed{2} & 3 & 4 \\ \boxed{5} & 6 & 7 & 8 \\ 9 & 10 & 11 & \boxed{12} \\ 13 & 14 & \boxed{15} & 16 \end{bmatrix},$$

The sum of the boxed entries is $3 + 8 + 9 + 14 = 34$ for the matrix on the left, and $2 + 5 + 12 + 15 = 34$ for the matrix on the right. Trying out more examples one always obtains 34 as sum of the chosen entries, suggesting that 34 is a "magic number" for this matrix. Your task is to prove that this property holds in general, for matrices of any dimension: ***Prove that, no matter how these n entries are chosen, for a given n-value their sum is always the same, and equal to a certain magic number M_n.***

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Proof practice with the ϵ-definition of a limit. *For the following problems, give formal proofs* ***using the definition of a limit****. These proofs should be done directly from the "$\epsilon - N$" definition of the limit of a sequence, and* ***not*** *use any of the properties, lemmas, propositions, etc. of limits established in the book or in class.*

$\boxed{\#4}$ **(13.25 in the text).** Using the ϵ-definition of a limit, prove that $\lim_{n\to\infty} \sqrt{1 + \frac{1}{n}} = 1$.

$\boxed{\#5}$ **(13.26 in the text).** Using the ϵ-definition of a limit, prove that if $\lim_{n\to\infty} a_n = 1$ and $a_n > -1$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} 1/(1 + a_n) = 1/2$.

$\boxed{\#6}$ Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that the limits $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$ both exist. Using the ϵ-definition of a limit, prove the following:

(a) (13.11(a) in the text). If $L < M$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n < b_n$.

(b) If $a_n < b_n$ for all $n \in \mathbb{N}$, then $L \leq M$.

(c) Show, by a counterexample, that "$a_n < b_n$ for all $n \in \mathbb{N}$" does **not** imply $L < M$.

*** Turn page for comments, hints, and advice ***

[1] If you worked with another student or in a small group on this assignment, list the names of all students involved.

HW 7 Hints, Advice, and Comments

Comments on Puzzle Problems

- **Problem 1. The 1089 puzzle (4.13).** Here is the statement of this problem.

  Let x be an arbitrary 3-digit number whose first and last digit are different. Reverse the digits of x to get another 3-digit number, x'. Let $y = |x - x'|$, and let $z = y + y'$, where y' is obtained from y by reversing its digits. Here are some examples:

  If $x = 347$, then $x' = 743$, $y = 743 - 347 = 396$, $z = 396 + 693 = 1089$.
  If $x = 173$, then $y = 371 - 173 = 198$, $z = 198 + 891 = 1089$.
  If $x = 213$, then $y = 312 - 213 = 099$, $z = 099 + 990 = 1089$.

  Surprisingly, the final answer is 1089 in all three cases.
  Prove that this is true in general: No matter which number x you start out with (assuming it is a 3-digit number with different first and last digits), show that the final number, z, obtained in this way is always 1089.

  Hint: If a, b, c are the decimal digits of x, then $x = 100a + 10b + c$.

- **Problem 2. The December 31 puzzle (3.62).** Here is the statement of this problem:

  Two players alternately name dates. On each move, a player must pick a new date in which either the month or the day of the month is increased, but not both. The starting position is Jan. 1, and the player who first names Dec. 31 wins.

  For example, a possible sequence of moves for Players A and B is: (Player A) Jan. 5, (Player B) March 5, (Player A) March 15, (Player B) Nov. 15, (Player A) Nov. 30, (Player B) Dec. 30, (Player A) Dec. 31. Player A wins since he is the first to reach Dec. 31.

  In fact, once Player A reaches Nov. 30, he is guaranteed to win, since Player B is forced to pick Dec. 30, and then Player A wins by choosing Dec. 31. Thus, Nov. 30 is a "winning date" for Player A.
  Determine, with proof, all "winning dates" for this game.

  Hint: Encode the winning dates as pairs (m, d), where $m \in \{1, 2, \ldots, 12\}$ is the month and $d \in \{1, 2, \ldots, 31\}$ is the day, and find and prove a general formula for the pairs (m, d) that are winning dates.

Advice on Limit Problems

The ϵ-definition of a limit is a logical statement involving multiple quantifiers ("for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ ..."), and the structure of a proof can sometimes be quite complex. Here is some general advice on writing up such proofs:

- **Match the structure of a proof to the logical structure of the statement you seek to prove.** For example, the ϵ-definition of a limit is a statement of the form "for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ ...", so a proof of such a statement would normally start out with "Let $\epsilon > 0$ be given", then proceed (perhaps after some intermediate steps) by choosing/specifying an N, and showing that this N "works".
- **Split the proof into individual steps, and write each step in a separate line or as a bullet item in a list, rather than writing the entire proof as a single contiguous paragraph.** This makes it a lot easier to see the logical connection between steps, and to spot any logical errors. Make up a list of the necessary steps, in the proper order, during the scratch phase of your work.
- **After you have written up the proof, read through it carefully, and check that it makes logical sense.** Do this again a day later, with fresh eyes; you may notice issues you overlooked the first time!
- **Make sure that all variables are declared and quantified before they are used.** For example, starting out a proof with "By the definition of a limit, there exists N such that $|a_n - 1| < \epsilon$ for all $n \geq N$" would be wrong since ϵ hasn't been declared yet.
- **Use the examples in the worksheets and homework solutions as models for your own write-up.** Most proofs are similar to one of a handful of standard examples (e.g., the proof of the sum property, or proof of the uniqueness of limits), and can be modeled after one of these examples.