

## Worksheet: Errors in Induction Proofs Solutions

By now, induction proofs should feel routine to you, to the point that you could almost do them in your sleep. However, it is important not to become complacent and careless, for example, by skipping seemingly minor details in the write-up, omitting quantifiers, or neglecting to check conditions and hypotheses. Below are some examples of false induction proofs that illustrate what can happen when some minor details are left out. In each case, the statement claimed is clearly nonsensical (e.g., that all numbers are equal), but the induction argument sounds perfectly fine, and in some cases the error is quite subtle and hard to spot. Try to find it!

### Example 1

**Claim:** For all  $n \in \mathbb{N}$ ,  $(*) \sum_{i=1}^n i = \frac{1}{2}(n + \frac{1}{2})^2$

**Proof:** We prove the claim by induction.

**Base step:** When  $n = 1$ ,  $(*)$  holds.

**Induction step:** Let  $k \in \mathbb{N}$  and suppose  $(*)$  holds for  $n = k$ . Then

$$\begin{aligned}
 \sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k+1) \\
 &= \frac{1}{2} \left( k + \frac{1}{2} \right)^2 + (k+1) \quad (\text{by ind. hypothesis}) \\
 &= \frac{1}{2} \left( k^2 + k + \frac{1}{4} + 2k + 2 \right) \quad (\text{by algebra}) \\
 &= \frac{1}{2} \left( \left( k + 1 + \frac{1}{2} \right)^2 - 3k - \frac{9}{4} + k + \frac{1}{4} + 2k + 2 \right) \quad (\text{more algebra}) \\
 &= \frac{1}{2} \left( (k+1) + \frac{1}{2} \right)^2 \quad (\text{simplifying}).
 \end{aligned}$$

Thus,  $(*)$  holds for  $n = k + 1$ , so the induction step is complete.

**Conclusion:** By the principle of induction,  $(*)$  holds for all  $n \in \mathbb{N}$ .

**Where the error lies:** Here there is no problem with the induction step, but the base case is not valid despite the claim that it is true in this case.

**MORAL:** Make sure to really check the base case. Simply stating that it is true doesn't make it true!

**Example 2****Claim:** All real numbers are equal.**Proof:** To prove the claim, we will prove by induction that, for all  $n \in \mathbb{N}$ , the following statement holds:

$$(P(n)) \quad \text{For any real numbers } a_1, a_2, \dots, a_n, \text{ we have } a_1 = a_2 = \dots = a_n.$$

**Base step:** When  $n = 1$ , the statement is trivially true, so  $P(1)$  holds.**Induction step:** Let  $k \in \mathbb{N}$  be given and suppose that  $P(k)$  is true, i.e., suppose that any  $k$  real numbers must be equal. We seek to show that  $P(k + 1)$  is true as well, i.e., that any  $k + 1$  real numbers must also be equal.Let  $a_1, a_2, \dots, a_{k+1}$  be given real numbers. Applying the induction hypothesis to the first  $k$  of these numbers,  $a_1, a_2, \dots, a_k$ , we obtain

$$(1) \quad a_1 = a_2 = \dots = a_k.$$

Similarly, applying the induction hypothesis to the last  $k$  of these numbers,  $a_2, a_3, \dots, a_k, a_{k+1}$ , we get

$$(2) \quad a_2 = a_3 = \dots = a_k = a_{k+1}.$$

Combining (1) and (2) gives

$$(3) \quad a_1 = a_2 = \dots = a_k = a_{k+1},$$

so the numbers  $a_1, a_2, \dots, a_{k+1}$  are equal. Thus, we have proved  $P(k + 1)$ , and the induction step is complete.**Conclusion:** By the principle of induction,  $P(n)$  is true for all  $n \in \mathbb{N}$ . Thus, any  $n$  real numbers must be equal.**Where the error lies:** *The base step is valid, and at first glance the induction step seems to be perfectly valid as well.**However, when  $k = 1$ , the variables  $a_1, \dots, a_k$  in (1) reduce to the single variable  $a_1$ , while the variables  $a_2, \dots, a_{k+1}$  in (2) reduce to the single variable  $a_2$ . Thus, in the case  $k = 1$  there is no overlap between these two sets of variables, so one cannot “chain together” these equalities as in (3). Hence the induction step is not valid when  $k = 1$  (though it is valid for all  $k \geq 2$ ).***MORAL:** *Make sure that the induction step is valid for ALL  $k$ -values in the given range (here all  $k \in \mathbb{N}$ ).*

**Example 3**

**Claim:** For every nonnegative integer  $n$ ,  $(*)$   $5n = 0$ .

**Proof:** We prove that  $(*)$  holds for all  $n = 0, 1, 2, \dots$ , using strong induction with the case  $n = 0$  as base case.

**Base step:** When  $n = 0$ ,  $5n = 5 \cdot 0 = 0$ , so  $(*)$  holds in this case.

**Induction step:** Let  $k \in \mathbb{N}$  be given and suppose  $(*)$  is true for all integers  $n$  in the range  $0 \leq n \leq k$ , i.e., that for all integers in this range  $5n = 0$ . We will show that  $(*)$  then holds for  $n = k + 1$  as well, i.e., that  $5(k + 1) = 0$ .

Write  $k + 1$  as a sum  $k + 1 = i + j$ , where  $i, j$  are integers satisfying  $0 \leq i, j \leq k$ . Since  $0 \leq i, j \leq k$ , we can apply the induction hypothesis to  $i$  and  $j$  to get  $5i = 0$  and  $5j = 0$ . Then

$$5(k + 1) = 5(i + j) = 5i + 5j = 0 + 0 = 0.$$

Hence  $5(k + 1) = 0$ , so  $(*)$  holds for  $n = k + 1$ .

**Conclusion:** By the principle of strong induction,  $(*)$  holds for all nonnegative integers  $n$ .

**Where the error lies:** The base step is clearly valid, as is the induction step for any  $k \in \mathbb{N}$ . However, since our base case is  $n = 0$ , we need to start the induction step at  $k = 0$ , not at  $k = 1$ . But in this case, the “splitting trick”  $k + 1 = i + j$  no longer works: For  $k = 0$  we have  $k + 1 = 1$ , and the number 1 cannot be written as a sum  $1 = i + j$  with  $0 \leq i, j \leq 0$ . Thus the induction step does not hold for  $k = 0$ . (When  $k \geq 1$ , there is no problem since choosing  $i = 1$  and  $j = k$  we can satisfy both  $i + j = k + 1$  and the constraints  $0 \leq i, j \leq k$ .)

**MORAL:** Make sure that the induction step and base case match up and that there is no gap, or missing link, between the base case and the first case of the induction step.

**Example 4**

**Claim:** For every natural number  $n$ ,  $(*)$   $2^n = 2$ .

**Proof:** We prove that  $(*)$  holds for all  $n \in \mathbb{N}$ , using strong induction.

**Base step:** When  $n = 1$ ,  $2^1 = 2$ , so  $(*)$  holds in this case.

**Induction step:** Suppose  $(*)$  is true for all natural numbers  $n \leq k$ , i.e., assume that for all natural numbers  $n \leq k$  we have  $2^n = 2$ . We will show that  $(*)$  then holds for  $n = k + 1$  as well, i.e., that  $2^{k+1} = 2$ .

We have

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2 \quad (\text{by algebra}) \\ &= 2^k \cdot \frac{2^k}{2^{k-1}} \quad (\text{by algebra}) \\ &= 2 \cdot \frac{2}{2} \quad (\text{by induction hypothesis applied to each term}) \\ &= 2 \quad (\text{simplifying}). \end{aligned}$$

Hence  $2^{k+1} = 2$ , so  $(*)$  holds for  $n = k + 1$ .

**Conclusion:** By the principle of strong induction,  $(*)$  holds for all  $n \in \mathbb{N}$ .

**Where the error lies:** *The first clue that something may be wrong is the fact that the variable  $k$  in the induction step is not quantified: Nowhere in the proof does it say what the range for  $k$  is.*

*Since here the base case is  $n = 1$ , the first case of the induction step must be  $k = 1$ . However, we need the induction hypothesis for both  $n = k$  and  $n = k - 1$ , and in the case  $k = 1$  the latter value is  $n = 1 - 1 = 0$ , which is out of range of the induction hypothesis.*

**MORAL:** *An induction step should always start out with a statement such as “Let  $k \in \mathbb{N}$  be given” that clearly states what  $k$  is, and what values it can take on. Then make sure that the step is valid for ALL of these values (in particular the first one).*