Worksheet: Some Common Errors in Proof-Writing Solutions

Each of the following statements has a logical, or notational, or language, error that makes it nonsensical. Some of the errors are minor issues, others are more serious logical errors. In each case try to find the problem with the given statement, then write down a correct version of the intended statement.

1. **Proof of $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$:**

   Let $A \times (B \cap C)$. Then ...

   > ***PROBLEM:*** *In a phrase of the form "Let ... ", "Suppose ... ", "Assume ... " the expression in dots (...) should be a logical statement, i.e., something that is either true or false. $A \times (B \cap C)$ is a **set**, which doesn't have a truth value.*
   > ***CORRECTION:*** *"Let $x \in A \times (B \cap C)$. Then ..."*

2. **Proof of $A - (B - C) \subseteq (A - B) \cup (A - C)$:**

   Since $x \in A - (B - C)$, then $x \in A$ and $x \in (B - C)$, by the definition of a set difference.
   Since $x \in (B - C)$, then $x \in B$ and $x \notin C$, by the definition of a set difference.
   ....

   > ***PROBLEM:*** *The main issue here is that the proof is lacking a proper start. The phrase "Since $x \in A - (B - C)$" doesn't make sense as the first sentence of a proof as x has not been defined. (Another issue is that $x \in (B - C)$ should be $x \notin (B - C)$, not $x \in (B - C)$.)*
   > ***CORRECTION:*** *Add a sentence at the beginning that says what x is:*
   > *"Let $x \in A - (B - C)$ be given. ..."*

3. Suppose $A - (B \cap C)$. Then ...

   > ***PROBLEM:*** *"Suppose" needs to be followed by a statement. "$A - (B \cap C)$" is a set, not a statement.*
   > ***CORRECTION:*** *"Suppose $x \in A - (B \cap C)$."*

4. Since $x \in B \cap C$, then $x \in B \cap x \in C$, by the definition of an intersection.

   > ***PROBLEM:*** *The intersection symbol, $\cap$, is a set-theoretic symbol that makes sense only if the expressions to the left and right of it are **sets**. Here it is presumably meant to denote a logical "and".*
   > ***CORRECTION:*** *Replace $\cap$ by "and": "then $x \in B$ and $x \in C$".*

5. Thus $x \in A$ and $x \notin B$ or $x \in A - C$.

   > ***PROBLEM:*** *The statement is of the form "P and Q or R", which can be interpreted in two ways: as "(P and Q) or R", and as "P and (Q or R)".*
   > ***CORRECTION:*** *Add parentheses or rephrase to make the statement unambiguous:*
   > *"Thus $x \in A$ and either $x \notin B$ or $X \in A - C$."*

6. Let $x \in A - B - C$. Then ...

   > ***PROBLEM:*** *Similar to the previous example: "$A - B - C$" is ambiguous, and can be interpreted as either $A - (B - C)$, or $(A - B) - C$.*
   > ***CORRECTION:*** *Add parentheses, e.g. "Let $x \in A - (B - C)$. Then ... "*

7. Therefore $x \in (A$ and $C)$.

   > ***PROBLEM:*** *The word "and", when used as a logical operator, requires logical statements to the left and right, but here it is used to connect two sets.*
   > ***CORRECTION:*** *"Therefore $x \in A$ and $x \in C$."*
   > *"Therefore $x \in (A \cap C)$."*

8. Then $x \in (B \cup C)^c \Longrightarrow B^c \cap C^c$.

PROBLEM: *The implication symbol, $\Longrightarrow$, requires a* **statement** *on the left and right; here, the right side is a* **set**. *Also, in proofs, it is better to use English words ("therefore", etc.) instead of implication symbols.*
CORRECTION: *"Then $x \in (B \cup C)^c$. Hence $x \in B^c \cap C^c$."*

9. Therefore $x \notin B \cap C \Longrightarrow x \notin B$ or $x \notin C$.

PROBLEM: *The logic now is fine, but mixing English words ("Therefore") with logical symbols ("$\Longrightarrow$") that have the same meaning is bad.*
CORRECTION: *Replace the implication symbol by an English word: "Therefore $x \notin B \cap C$. Hence $x \notin B$ or $x \notin C$."*

10. **False Proofs:** Find the error in the reasoning.

(a) **Part of an attempted proof by contradiction:**

... Therefore $2x(2x - 1) \geq 4x^2 - 1 = (2x - 1)(2x + 1)$.
Dividing by $2x - 1$ we get $2x \geq 2x + 1$.
Subtracting $2x$ on each side, we get $0 \geq 1$.
Thus we have obtained a contraction. ...

ERROR: *The error occurs when dividing by $2x - 1$. If $2x - 1 < 0$, the inequalities reverse directions.*
MORAL: Be careful when multiplying or dividing inequalities! Avoid altogether if possible, or otherwise treat separately the case when the factor by which you divide or multiply is positive, and the case when this factor is negative.

(b) **Part of another attempted proof by contradiction:**

... Therefore $2x(2x - 1) \geq 4x^2 + 2x - 1$.
Subtracting $2x - 1$ on each side, we get $2x(2x - 1) - (2x - 1) \geq 4x^2$.
Simplifying, we get $(2x - 1)^2 \geq 4x^2$.
Taking the squareroot on each side gives $2x - 1 \geq 2x$.
Subtracting $2x$ on each side, we get $-1 \geq 0$.
Thus we have obtained a contraction. ...

ERROR: *Here the error occurred when taking squareroots. The squareroot of a^2 is $|a|$, not a.*
MORAL: Be careful when taking squareroots if the numbers involved can be negative.

(c) **"Proof" that $0 = 2$ (thus creating something out of nothing):**

Obviously, $4x^2 = 4x^2$.
Rewriting the left and right sides, we get $(-2x)^2 = (2x)^2$.
Taking the squareroot, we get $-2x = 2x$.
Adding $x^2 + 1$ to each side gives $-2x + x^2 + 1 = 2x + x^2 + 1$.
By algebra, this can be written as $(x - 1)^2 = (x + 1)^2$.
Taking the squareroot, we get $x - 1 = x + 1$.
Subtracting $x - 1$ on each side, we get $x - 1 - (x - 1) = x + 1 - (x - 1)$, i.e., $0 = 2$. QED.

ERROR: *Same as before: $a^2 = b^2$ does not imply $a = b$.*