Worksheet: Epsilonics, II: Infinite Series SOLUTIONS

Practice Problems

Below are some problems to practice proof-writing skills in the context of infinite series. All of these proofs should be done rigorously, using the "official" definitions of series convergence and results such as Cauchy's Criterion. None of these proofs is particularly difficult; try to master them all! For additional practice problems, especially of the "prove or find counterexample" variety, see the homework assignments.

> **IMPORTANT NOTE:** In proofs involving infinite series always work with **finite** sums, such as the partial sums $\sum_{k=1}^n a_k$, or the sums of finite "chunks" of the series, $\sum_{k=m+1}^n a_k$, that arise in Cauchy's criterion. **You should NEVER try to directly work with infinite sums $\sum_{k=1}^\infty a_k$ since such a sum has no meaning until you have proved that it is convergent.**
>
> For example, trying to prove the sum property by writing $\sum_{k=1}^\infty (a_k+b_k) = \sum_{k=1}^\infty a_k + \sum_{k=1}^\infty b_k$, or the Absolute Convergence Test by writing $|\sum_{k=1}^\infty a_k| \le \sum_{k=1}^\infty |a_k|$, would be totally wrong and would completely miss the point of having precise mathematical definitions of convergence and divergence of series. (With this sort of manipulation one could "prove" all sorts of nonsensical results!)

1. **Convergence/divergence of particular series:**

   (a) **Geometric series, convergent case:** Using the definition of convergence of an infinite series, prove that if $|r| < 1$, then the geometric series $\sum_{k=0}^\infty r^k$ converges with sum $1/(1-r)$. (Hint: Use the formula for the sum of a finite geometric series.)

   Proof: *Let $s_n = \sum_{k=0}^n r^k$ denote the n-th partial sum of the series. We seek to show that $\lim_{n\to\infty} s_n = 1/(1-r)$. By the formula for the sum of a **finite** geometric series, we have*

   $$s_n = \sum_{k=0}^n r^k = \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r} - \frac{r}{1-r} r^n. \tag{1}$$

   Since $|r| < 1$, we have $\lim_{n\to\infty} r^n = 0$ (for a formal proof see below). Using the properties of limits, it follows that

   $$\begin{aligned}
   \lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\frac{1}{1-r} - \frac{r}{1-r}r^n\right) && \textit{(by formula (1))}\\
   &= \lim_{n\to\infty}\frac{1}{1-r} - \frac{r}{1-r}\lim_{n\to\infty} r^n && \textit{(by algebraic properties of limits)}\\
   &= \lim_{n\to\infty}\frac{1}{1-r} - 0 && \textit{(since } (*) \lim_{n\to\infty} r^n = 0\textit{; see below)}\\
   &= \frac{1}{1-r} && \textit{(since } \lim_{n\to\infty} c = c\textit{; see Problem 1(a) on the Epsilonics Worksheet).}
   \end{aligned}$$

   Thus, the partial sums s_n converge to $1/(1-r)$. By the definition of convergence of an infinite series, this shows that the series $\sum_{k=0}^\infty r^k$ converges, with sum $1/(1-r)$.

   Proof of $(*)$: *Let $a_n = |r^n| = |r|^n$. Since $|r| < 1$, the sequence $\{a_n\}$ is bounded and monotonically decreasing. By the Monotone Convergence Theorem, it therefore has a limit, say $L = \lim_{n\to\infty} |r^n|$. To show that $L = 0$, we apply the properties of limits: $L = \lim_{n\to\infty} |r^{n+1}| = \lim_{n\to\infty} |r|\cdot|r^n| = |r|\lim_{n\to\infty}|r^n| = |r|L$, or $L(1-|r|) = 0$. Since $|r| < 1$, this implies $L = 0$, as desired.*

   (b) **Geometric series, divergent case:** Prove that if $|r| \ge 1$, then the series $\sum_{k=0}^\infty r^k$ diverges. (Hint: Use an appropriate convergence test.)

   Proof: *In the case $|r| \ge 1$ we have $|r^n| = |r|^n \ge 1$, so r^n does **not** have limit 0 as $n \to \infty$. By the n-th term test, the series $\sum_{k=0}^\infty r^k$ therefore diverges.*

   (c) **Divergence of harmonic series:** Using Cauchy's Criterion for Series, show that the harmonic series $\sum_{k=1}^\infty \frac{1}{k}$ diverges.

Proof: *To show that Cauchy's Criterion does* **not** *hold, we need to find an $\epsilon > 0$ such that, given any $N \in \mathbb{N}$, there exist integers n, m with $n > m \geq N$ such that $(*)$ $|\sum_{k=m+1}^{n} 1/k| \geq \epsilon$.*
We will show that $\epsilon = 1/2$ has the desired property: Given $N \in \mathbb{N}$, let $m = N$ and $n = 2N$. Then

$$\sum_{k=N+1}^{2N} \frac{1}{k} \geq N \cdot \frac{1}{2N} = \frac{1}{2},$$

so $(*)$ holds with $m = N$, $n = 2N$, and $\epsilon = 1/2$. Thus, the harmonic series does not satisfy the Cauchy Criterion and hence diverges.

2. **General Properties of Series.**

(a) **Proof of Cauchy Criterion for Series:** Prove the Cauchy Criterion for series, using the Cauchy Criterion for sequences.

Proof: *By definition, convergence of an infinite series is equivalent to convergence of the sequence of its partial sums s_n; by the Cauchy Criterion for sequences, the sequence $\{s_n\}$ converges if and only if*

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)[|s_n - s_m| < \epsilon]. \tag{1}$$

Next, observe that we may in (1) assume that $n \geq m$ since interchanging m and n does not affect the quantity $|s_n - s_m|$; in addition, we may assume $n \neq m$ since when $n = m$, $|s_n - s_n| = 0$, so the inequality $|s_n - s_n| < \epsilon$ is satisfied for any $\epsilon > 0$. Thus, (1) is equivalent to

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n > m \geq N)[|s_n - s_m| < \epsilon]. \tag{2}$$

Now, by the definition of s_n as the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$, we have (under the condition $n > m$)

$$s_n - s_m = \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k = \sum_{k=m+1}^{n} a_k. \tag{3}$$

Substituting this into (2), we can rewrite (2) as

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n > m \geq N)\left[\left|\sum_{k=m+1}^{n} a_k\right| < \epsilon\right]. \tag{4}$$

The latter is the Cauchy Criterion for Infinite Series. To summarize, we have the following chain of equivalences:

$$\begin{aligned}
\sum_{k=1}^{\infty} a_k \text{ converges} &\iff \{s_n\} \text{ converges} \quad \textit{(definiion of convergence of series)} \\
&\iff \{s_n\} \text{ is a Cauchy sequence} \quad \textit{(by Cauchy criterion for } \textbf{sequences}\textit{)} \\
&\iff \{s_n\} \text{ satisfies (2)} \quad \textit{(shown above)} \\
&\iff \sum_{k=1}^{\infty} a_k \text{ satisfies (4)} \quad \textit{(shown above)} \\
&\iff \sum_{k=1}^{\infty} a_k \text{ satisfies Cauchy Criterion for infinite series}
\end{aligned}$$

(b) **Proof of Absolute Convergence Test:** Prove the Absolute Convergence Test using the Comparison Test.

Proof: Method 1: Apply the Comparison Test: *If $\sum_{k=1}^{\infty} |a_k|$ converges, then applying the Comparison Test with $b_k = |a_k|$ (the condition $|a_k| \leq b_k$ is trivially satisfied in this case), we obtain that the series $\sum_{k=1}^{\infty} a_k$ converges as well.*
Method 2: Direct proof, using the Cauchy Criterion. *Alternatively, one can directly show, exactly as in the proof of the Comparison Test, that if $\sum_{k=1}^{\infty} |a_k|$ satisfies the Cauchy Criterion, then so does the series $\sum_{k=1}^{\infty} a_k$.*

(c) **Series of nonnegative terms:** Prove that if a series has (i) only nonnegative terms and (ii) bounded partial sums, then it converges. (Hint: What does the "nonnegative terms" condition mean in terms of the partial sums s_n?)

Proof: *Suppose $\sum_{k=1}^{\infty} a_k$ satisfies the given conditions, i.e., (i) $a_k \geq 0$ for all $k \in \mathbb{N}$ and (ii) the sequence $\{s_n\}$ of partial sums is bounded.*
Note that

$$s_{n+1} - s_n = \sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k = a_{n+1},$$

and since, by assumption (i), $a_{n+1} \geq 0$ for all $n \in \mathbb{N}$, it follows $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$. Hence the sequence $\{s_n\}$ is monotone. But by (ii) this sequence is also bounded. Hence the sequence of partial sums $\{s_n\}$ satisfies the conditions of the Monotone Convergence Theorem and therefore converges. By the definition of a convergent series, this means that the series $\sum_{k=1}^{\infty} a_k$ converges.

(d) **Proof of the n-th term test:** The n-th term test says that if $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$. Give a careful proof of the this test, by two methods: (I) using the Cauchy Criterion for Series; (II) using the definition of convergence of infinite series and algebraic properties of limits of sequences.

Proof: ***Method I:*** *Assume $\sum_{k=1}^{\infty} a_k$ converges. Let $\epsilon > 0$ be given. By Cauchy's Criterion there exists an $N_1 \in \mathbb{N}$ such that*

$$(1) \qquad \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon \quad \textit{for all } n, m \in \mathbb{N} \textit{ with } n > m \geq N_1.$$

In particular, applying this with $n = m + 1$, (1) yields

$$(2) \qquad |a_n| < \epsilon \quad \textit{for all } n \in \mathbb{N} \textit{ with } n > N_1.$$

Now let $N = N_1 + 1$. Then $n > N_1$ is equivalent to $n \geq N$, so (2) says that $n \geq N$ implies $|a_n - 0| < \epsilon$. By the definition of a limit, this proves $\lim_{n\to 0} a_n = 0$.

Method II: *We will use the sum/difference property for **sequences** (if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then $\lim_{n\to\infty}(a_n + b_n) = A + B$ and $\lim_{n\to\infty}(a_n - b_n) = A - B$) and the "shifted sequence" property (if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n+1} = L$, and vice versa) from the Epsilonics I worksheet.*
*Suppose the **series** $\sum_{k=1}^{\infty} a_k$ converges. By definition, this means that the **sequence** $\{s_n\}$ of partial sums converges, i.e., there exists a real number L such that $\lim_{n\to\infty} s_n = L$.*
Then, by the above properties

$$\lim_{n\to\infty}(s_{n+1} - s_n) = \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n = L - L = 0.$$

But $s_{n+1} - s_n = a_{n+1}$, so we get $\lim_{n\to\infty} a_{n+1} = 0$, which, by the "shifted sequence" property, implies $\lim_{n\to\infty} a_n = 0$.