Problem 1. Definitions, theorems, and examples. State the requested definition, theorem, or example. Be sure to use correct notation and include any necessary quantifiers in the appropriate order.

(a) Give a precise statement of the **negation** of the Cauchy property for sequences *without using words of negation*. That is, complete the following sentence:

*A sequence $\{a_n\}$ is **not** a Cauchy sequence if ...*

> **Solution:** *A sequence $\{a_n\}$ is **not** a Cauchy sequence if there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exist $n, m \geq N$, such that $|a_n - a_m| \geq \epsilon$.*

(b) Give an example of a **countable** set A of real numbers such that $\sup A$ and $\inf A$ both exist, but $\min A$ and $\max A$ do not exist.

> **Solution:** *One example is $A = \{(-1)^n(1 - 1/n) : n \in \mathbb{N}\}$, which has $\inf A = -1$ and $\sup A = 1$.*

(c) State the ϵ-definition of "$\alpha = \sup S$", where S is a non-empty set of real numbers. Be sure to include any necessary quantifiers, in the correct order.

> **Solution:** *$\alpha = \sup S$ means (i) $\alpha \geq x$ for all $x \in S$ and (ii) for every $\epsilon > 0$, there exists $x \in S$ such that $x > \alpha - \epsilon$.*

(d) State the Archimedean Property.
Extra credit: Derive the Archimedean Property from the Completeness Axiom; i.e., assuming the Completeness Axiom, give a rigorous derivation of the Archimedean Property. (Use back of page for work.)

> **Solution:** ***Archimedian Property:*** *Given any real number x, there exists an $n \in \mathbb{N}$ such that $n > x$.*
> ***Proof*** *We argue by contradiction. Suppose the Archimedean property does not hold. Then there exists $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $n \leq x$. Therefore the set $\mathbb{N}$ is bounded above. By the Completeness Axiom, there exists a least upper bound, say $\alpha = \sup \mathbb{N}$. Now consider $\alpha - 1$. Since α is an upper bound, we have $\alpha \geq n$ for any $n \in \mathbb{N}$. Since $n + 1 \in \mathbb{N}$ whenever $n \in \mathbb{N}$, we also have $\alpha \geq n + 1$ for any $n \in \mathbb{N}$, and hence $\alpha - 1 \geq n$ for any $n \in \mathbb{N}$. But this means that $\alpha - 1$ is also an upper bound for $\mathbb{N}$, contradicting the assumption that α is the least upper bound.*

Problem 2. Short proofs and counterexamples, I. For each of the statements below, determine if it is true or false. If it is true, give a proof. You can use any properties and theorems on sequences and series from the class handouts and worksheets, but **you must state clearly which result you are using**. If it is false, give a **specific** counterexample and explain briefly why this example "works" (e.g., in case of an example of a divergent series say why the series diverges).

(a) If $\{a_n\}$ is bounded and diverges, then $\{a_n\}$ is not monotone.

> **Solution:** *TRUE. Proof by contradiction: Suppose a sequence is bounded, divergent, and also monotone. Then, in particular, it is bounded and monotone, hence satisfies the conditions of the Monotone Convergence Theorem and therefore must be convergent. But this contradicts the assumption that the sequence diverges.*

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

> **Solution:** *FALSE. Counterexample:* $a_n = b_n = n$. $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} 1 = 1$, *but* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n$ *does not exist.*

(c) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} a_n$ does not exist.

> **Solution:** *FALSE. Counterexample:* $a_n = 1/n$. *The series* $\sum_{n=1}^{\infty} 1/n$ *is the harmonic series, which diverges, but* $\lim_{n\to\infty} 1/n = 0$

Problem 3. Short proofs and counterexamples, II. For each of the statements below, determine if it is true or false. If it is true, give a proof. You can use any properties and theorems on sequences and series from the class handouts and worksheets, but **you must state clearly which result you are using**. If it is false, give a **specific** counterexample and explain briefly why this example "works" (e.g., in case of an example of a divergent series say why the series diverges).

(a) If $\lim_{n\to\infty} a_n$ exists, then $\{a_n\}$ is a Cauchy sequence.

> **Solution:** *TRUE. By the Cauchy Criterion, any convergent sequence is a Cauchy sequence (and vice versa).*

(b) If $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = A$ and $\lim_{n\to\infty} \sum_{k=1}^{n} b_k = B$, then $\lim_{n\to\infty} \sum_{k=1}^{n} a_k b_k = AB$.

> **Solution:** *FALSE. Counterexample:* $a_k = b_k = (-1)^k/\sqrt{k}$. $\sum_{k=1}^{\infty} (-1)^k/\sqrt{k}$ *converges by the Alternating Series Test, but the "product series"* $\sum_{k=1}^{\infty} 1/k$ *is the harmonic series, which diverges.*

(c) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and and $\lim_{n\to\infty} b_n$ exists, then $\sum_{n=1}^{\infty} a_n b_n$ is also absolutely convergent.

> **Solution:** *TRUE. Since any convergent sequence is bounded, there exists* $M \in \mathbb{R}$ *such that* $|b_n| \le M$ *for all* $n \in \mathbb{N}$. *Then* $|a_n b_n| \le M|a_n|$ *for all* $n \in \mathbb{N}$, *and since* $\sum_{n=1}^{\infty} |a_n|$ *converges, by the comparison test so does* $\sum_{n=1}^{\infty} |a_n b_n|$. *Thus* $\sum_{n=1}^{\infty} a_n b_n$ *is absolutely convergent.*

Problem 4. Using only the ϵ-definition of a limit or the Cauchy criterion, give formal, ϵ-style proofs for the following convergence/divergence results. *The proofs should not any other properties and theorems on sequences and series from the homework, class handouts, etc. (e.g., convergence tests).*

(a) Using the ϵ-definition of a limit, prove that $\lim_{n\to\infty} \sqrt{1-1/n} = 1$.

Solution: *Let $a_n = \sqrt{1-1/n}$. To show that $\lim_{n\to\infty} a_n = 1$ we need to show that given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $|a_n - 1| < \epsilon$.*
Let $\epsilon > 0$ be given, and let

$$N = \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1. \tag{1}$$

(Here $\lfloor x \rfloor$ denotes the "floor" function, i.e., the largest integer $\leq x$; it satisfies $\lfloor x \rfloor \leq x$ and $\lfloor x \rfloor + 1 > x$.)
Note that (1) implies $N \in \mathbb{N}$ and

$$N > \frac{1}{\epsilon}. \tag{2}$$

Then, for $n \geq N$ we have

$$\begin{aligned}
|a_n - 1| &= \left|\sqrt{1-1/n} - 1\right| \\
&= 1 - \sqrt{1-1/n} \quad \text{(since } \sqrt{1-1/n} < 1\text{)} \\
&\leq 1 - \left(1 - \frac{1}{n}\right) = \frac{1}{n} \quad \text{(since } \sqrt{x} > x \text{ for } 0 < x < 1\text{)} \\
&\leq \frac{1}{N} \quad \text{(since } n \geq N\text{)} \\
&< \epsilon \quad \text{(by (2)}.
\end{aligned}$$

Hence $n \geq N$ implies $|a_n - 1| < \epsilon$.
By the definition of a limit, this proves that $\lim_{n\to\infty} a_n = 1$.

(b) Using the Cauchy Criterion, prove that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Solution: *Let $\epsilon = 1/2$. Then, for any $N \in \mathbb{N}$, we have*

$$\sum_{k=N+1}^{2N} \frac{1}{k} \geq N \cdot \frac{1}{2N} = \frac{1}{2}.$$

Thus, the harmonic series does not satisfy the Cauchy Criterion (with the choices $\epsilon = 1/2$ and $m = N, n = 2N$), and hence diverges.

Problem 5. Using only the ϵ-definition of a limit, show that if $\lim_{n\to\infty} a_n = 1$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{a_n} = 1$. (The proof should be done directly from the definition of convergence, and *not use any of the properties and theorems on sequences given in the book, the worksheets, the class handouts, and the homework.*)

Solution: *Suppose* $\lim_{n\to\infty} a_n = 1$.

Let $\epsilon > 0$ *be given.*

Applying the definition of a limit with $\epsilon' = \epsilon/2$, *we obtain an* $N_1 \in \mathbb{N}$ *such that*

$$|a_n - 1| < \frac{\epsilon}{2} \quad \textit{for } n \geq N_1. \tag{1}$$

Applying the definition of a limit a second time with $\epsilon' = 1/2$, *we obtain an* $N_2 \in \mathbb{N}$ *such that*

$$|a_n - 1| < \frac{1}{2} \quad \textit{for } n \geq N_2, \tag{2}$$

and therefore

$$a_n > 1 - \frac{1}{2} = \frac{1}{2} \quad \textit{for } n \geq N_2. \tag{3}$$

Let $N = \max(N_1, N_2)$. *Then, for* $n \geq N$, *we have*

$$\begin{aligned} \left| \frac{1}{a_n} - 1 \right| &= \left| \frac{1 - a_n}{a_n} \right| \\ &< \frac{\epsilon/2}{|a_n|} \quad \textit{(by (1))} \\ &< \frac{\epsilon/2}{1/2} \quad \textit{(by (3))} \\ &= \epsilon. \end{aligned}$$

Hence we shown that $n \geq N$ *implies* $|1/a_n - 1| < \epsilon$.

By the definition of a limit, this proves that $\lim_{n\to\infty} 1/a_n = 1$.