

Worksheet: Epsilonics, I: Sequences and Limits

This is the first of a series of worksheets aimed at introducing you to “epsilonics”—the art and science of constructing proofs involving ϵ ’s (and δ ’s, N ’s, etc.). In this worksheet you’ll practice epsilonics in a simple, yet important context, that of limits of sequences. Sequences are relatively uncomplicated mathematical objects that are easy to understand and develop a good intuition for and that require no background beyond some basic set theory and logic. As such they provide an ideal practice ground for epsilonics. Once you have mastered ϵ -proofs in the sequence context, you should have little problems with ϵ -proofs in other contexts, e.g., limits of real functions.

The core of these worksheets is a set of carefully selected problems that represent common situations and illustrate various techniques and tricks of the trade. **Try to completely master each of these problems. Ideally you should get to a point where producing an ϵ -proof for a problem of this type becomes as routine as differentiating or integrating a function or proving a summation formula by induction.**

Achieving this kind of mastery of epsilonics may seem daunting at first, but is very much doable. Once you have gotten over the initial hump, things will get easier as you progress through the problems. Each problem you work adds to your experience, it exposes you to a new situation, you learn how to deal with it, and you may learn a trick or two in the process that may be handy in other situations. Soon you will get to a point where you have accumulated a large enough body of experience that you can handle almost any situation, and in many cases know right away what kind of approach to use.

The effort you put in at this stage to get *fully* comfortable with epsilonics is well worth it, and it will pay off in the long run. You will be better prepared for advanced math classes such as Math 424/444/447, and you will have a leg up on most of your fellow students when you take such classes. At a broader level, practicing epsilonics sharpens your logical reasoning skills, and you will be better prepared for any class that depends on these skills.

Sample proof: Proof of the sum property.

Claim: If $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = M$, then $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$.

Proof: Suppose $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = M$.

Let $\epsilon > 0$ be given.

Since $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = M$, applying the definition of a limit with $\epsilon' = \epsilon/2$, we obtain integers $N_1, N_2 \in \mathbb{N}$ such that

$$(1) \quad |a_n - L| < \frac{\epsilon}{2} \quad \text{for all } n \geq N_1, \quad \text{and} \quad |b_n - M| < \frac{\epsilon}{2} \quad \text{for all } n \geq N_2.$$

Set $N = \max(N_1, N_2)$.

Then, for any $n \geq N$ we have

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \quad (\text{by the triangle inequality}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad (\text{by (1), since } n \geq N \text{ implies } n \geq N_1 \text{ and } n \geq N_2) \\ &= \epsilon. \end{aligned}$$

Hence $n \geq N$ implies $|(a_n + b_n) - (L + M)| < \epsilon$.

By the definition of a limit, this proves $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$.

The above example illustrates several useful tricks in epsilonics:

- **The $\epsilon/2$ trick:** Apply the definition of a limit with $\epsilon' = \epsilon/2$ in place of ϵ so as to end up with an overall bound of ϵ .
- **The $N = \max(N_1, N_2)$ trick:** Choosing N to be the larger (max) of two N -values arising in the proof, say N_1 and N_2 , ensures that any $n \geq N$ satisfies both $n \geq N_1$ and $n \geq N_2$.
- **Triangle Inequality.** Often the quantity that we want to show is “small” can be written as a sum of two quantities each of which is *known* to be small. The triangle inequality (i.e., the inequality $|x + y| \leq |x| + |y|$), which holds for any real numbers x, y (along with the $\epsilon/2$ trick or some similar device) can then be used to obtain the desired estimate.

Practice Problems

The problems below have been carefully selected to illustrate common situations and the techniques and tricks to deal with these. **Try to master them all; it is well worth it! What you learn in the process will be useful later in this class when we get to epsilonics in other contexts, and in many advanced math classes.**

For each problem, first try to gain an intuitive “feel” for the problem (a sketch of a typical situation may be useful) and try to understand why the statement is correct. Then try to construct a rigorous ϵ -proof.

In the following $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ denote arbitrary sequences, and L and M denote real numbers. **Unless otherwise specified, you should only use the ϵ -definition of a limit, not any theorems or properties of limits.**

1. **Warmup problems.** These are conceptually quite easy, and the results are intuitively “clear”. Try to use these problems to practice proper write-ups of proofs.
 - (a) **Limit of constant sequence.** Prove that the limit of a constant sequence is equal to this constant; i.e., show rigorously that, if $a_n = c$ for all n , then $\lim_{n \rightarrow \infty} a_n = c$.
 - (b) **Scaling property.** Prove that if $\lim_{n \rightarrow \infty} a_n = L$, then for any $c \in \mathbb{R}$, $\lim_{n \rightarrow \infty} (ca_n) = cL$.
 - (c) **Multiplication by bounded sequence.** Prove that if $\lim_{n \rightarrow \infty} a_n = 0$ and $\{b_n\}$ is a bounded sequence, then $\lim_{n \rightarrow \infty} a_n b_n = 0$.
 - (d) **Limit of shifted sequence.** Let a_n be a given sequence, and let $b_n = a_{n+1}$ be the “shifted” sequence. Prove that if $\lim_{n \rightarrow \infty} a_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.
2. **Intermediate problems.** These problems illustrate a variety of techniques and tricks in working with limits.
 - (a) **Uniqueness of limit.** Prove that if $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} a_n = M$, then $L = M$. (Hint: Use contradiction.)
 - (b) **Preservation of inequalities.** Prove that if $\lim_{n \rightarrow \infty} a_n = L$, $\lim_{n \rightarrow \infty} b_n = M$, and $a_n < b_n$ for all $n \in \mathbb{N}$, then $L \leq M$.
 - (c) **Squeeze Theorem.** Suppose $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Show that if $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.
 - (d) **Cauchy Criterion, “easy” direction.** Prove that any convergent sequence is a Cauchy sequence. (Hint: Use the $\epsilon/2$ -trick.)
 - (e) **Convergent implies bounded.** Prove that any convergent sequence is bounded. (Hint: Apply the definition of a limit with $\epsilon = 1$.)
 - (f) **Cauchy implies bounded.** Prove that any Cauchy sequence is bounded.
3. **Harder problems.** The following two problems are somewhat trickier, but very instructive.
 - (a) **Reciprocal property.** Prove that if $\lim_{n \rightarrow \infty} a_n = L$, $a_n \neq 0$ for all $n \in \mathbb{N}$, and $L \neq 0$, then $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{L}$. (Hint: Show first that there exists an N_1 such that $|a_n| > |L|/2$ for $n \geq N_1$.)
 - (b) **Product property.** Prove that if $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = M$, then $\lim_{n \rightarrow \infty} (a_n b_n) = LM$. (You can use the result above that a convergent sequence is bounded.) (Hint: Use a bit of algebraic magic: $ab - cd = a(b - d) + d(a - c)$.)
4. **Examples and counterexamples.** These problems are intended to develop some intuition about the behavior of sequences. All are quite easy.
 - (a) **Multiplication by bounded sequence.** Give an example of a convergent sequence a_n and a bounded sequence b_n such that the product sequence $a_n b_n$, does *not* converge. (The conclusion is true if the sequence a_n converges to 0, but not in general.)
 - (b) **Preservation of inequalities.** Give an example of sequences a_n and b_n satisfying $a_n < b_n$ for all $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$. (We know that $a_n < b_n$ implies $(*) \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$ by an earlier problem. The example thus shows that the inequality sign in $(*)$ cannot be replaced by a strict inequality.)
 - (c) **Reciprocal property.** Give an example of a *convergent* sequence a_n satisfying $a_n \neq 0$ for all n for which the reciprocal sequence $1/a_n$ is *not* convergent.
 - (d) **Sequences with convergent subsequences.** Given an example of a sequence a_n such that the subsequence over odd-indexed and even-indexed terms, $b_n = a_{2n-1}$ and $c_n = a_{2n}$, both converge, but the sequence a_n itself does not converge.