Worksheet: Induction Proofs III: Miscellaneous examples

Sample induction proof: A non-formula example. Below is a sample proof of the statement that any n-element set (i.e., any set with n elements) has 2^n subsets. This illustrates a case where the result we seek to prove is not a formula, but a statement that must be expressed verbally, and where the induction step requires some verbal explanation, and not just a chain of equalities. Additional practice problems follow below.

We will prove by induction that, for all $n \in \mathbb{N}$, the following holds:

$(P(n))$ Any n-element set has 2^n subsets.

Base case: Since any 1-element set has 2 subsets, namely the empty set and the set itself, and $2^1 = 2$, the statement $P(n)$ is true for $n = 1$.
Induction step:

- Let $k \in \mathbb{N}$ be given and suppose $P(k)$ is true, i.e., that any k-element set has 2^k subsets. We seek to show that $P(k+1)$ is true as well, i.e., that any $(k+1)$-element set has 2^{k+1} subsets.
- Let A be a set with $k+1$ elements.
- Let a be an element of A, and let $A' = A - \{a\}$ (so that A' is a set with k elements).
- We classify the subsets of A into two types: (I) subsets that do *not* contain a, and (II) subsets that do contain a.
- The subsets of type (I) are exactly the subsets of the set A'. Since A' has k elements, the induction hypothesis can be applied to this set and we conclude that there are 2^k subsets of type (I).
- The subsets of type (II) are exactly the sets of the form $B = B' \cup \{a\}$, where B' is a subset of A'. By the induction hypothesis there are 2^k such sets B', and hence 2^k subsets of type (II).
- Since there are 2^k subsets of each of the two types, the total number of subsets of A is $2^k + 2^k = 2^{k+1}$.
- Since A was an arbitrary $(k+1)$-element set, we have proved that any $(k+1)$-element set has 2^{k+1} subsets. Thus $P(k+1)$ is true, completing the induction step.

Conclusion: By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Practice Problems

1. **Number of subsets with an even (or odd) number of elements:** Using induction, prove that an n-element set has 2^{n-1} subsets with an even number of elements and 2^{n-1} subsets with an odd number of elements.

2. **Number of regions created by n lines:** How many regions are created by n lines in the plane such that no two lines are parallel and no three lines intersect at the same point? Guess the answer from the first few cases, then use induction to prove your guess.

3. **Sum of angles in a polygon:** The sum of the interior angles in a triangle is 180 degrees, or π. Using this result and induction, prove that for any $n \geq 3$, the sum of the interior angles in an n-sided polygon is $(n-2)\pi$.