

Logical statements: Summary of definitions, notations, and terminology

Logical operations

The basic logical operations are \wedge (“and”), \vee (“or”), \neg (negation), and \Rightarrow (“implies”), and \Longleftrightarrow (“equivalent”); they are defined in the following “truth tables”:

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Longleftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Implication, converse, and contrapositive: The most important, but also the most misused and misunderstood, of these logical operations is the logical implication ($P \Rightarrow Q$), and the related operations of the **converse** ($Q \Rightarrow P$), the **contrapositive** ($\neg Q \Rightarrow \neg P$), and the negated implication ($\neg(P \Rightarrow Q)$). The following table (which follows from the above truth table for $P \Rightarrow Q$) shows the differences between these operations:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$	$\neg(P \Rightarrow Q)$
		Implication	Converse	Contrapositive	Negated implication
T	T	T	T	T	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	F

Notes:

- The best way to remember the correct logical interpretation of an implication is as follows:

An implication $P \Rightarrow Q$ is true in all cases EXCEPT for the case when P (the assumption) is true and Q (the conclusion) is false.

In particular, this shows that if P is false, then the implication $P \Rightarrow Q$ is true, regardless of the truth value of Q . **In other words, from a false statement one can derive anything!**

- The negation of an implication $P \Rightarrow Q$ is not equivalent to another implication involving P and Q .** This is easy to see from the above truth table: $\neg(P \Rightarrow Q)$ is true in exactly one of the four cases and false in three cases. By contrast, an implication with P or $\neg P$ on one side and Q or $\neg Q$ on the other side (e.g., $\neg P \Rightarrow \neg Q$, $\neg Q \Rightarrow \neg P$, $\neg Q \Rightarrow P$, etc.) is always is true in three cases, and false in one case, so it can never match the truth table of $\neg(P \Rightarrow Q)$.
- Note that the implication $P \Rightarrow Q$ and its contrapositive $\neg Q \Rightarrow \neg P$ have the same truth tables, and hence are logically equivalent.** This is the reason why “proof by contraposition” is a valid method of proof.

Quantifiers

- Existential quantifier:** \exists (“there exists”).
- Universal quantifier:** \forall (“for all”).

Rules for negation

- $\neg(P \wedge Q) \Longleftrightarrow (\neg P) \vee (\neg Q)$ (De Morgan’s Law, I)
- $\neg(P \vee Q) \Longleftrightarrow (\neg P) \wedge (\neg Q)$ (De Morgan’s Law, II)
- $\neg(\forall x \in S)(P(x)) \Longleftrightarrow (\exists x \in S)(\neg P(x))$
- $\neg(\exists x \in S)(P(x)) \Longleftrightarrow (\forall x \in S)(\neg P(x))$

English equivalences of logical operations and notations

Here are some English words and phrases that translate into logical implications, equivalences. See p. 33 and p. 29 of the text for similar tables. **Be sure to familiarize yourself with these words and phrases and with their logical interpretation.**

Phrases expressing $P \Rightarrow Q$ (“ P implies Q ”):

- “If P , then Q ”
- “ Q follows from P ”:
- “ Q is true if P is true”
- “ Q is true whenever P is true”
- “ P is true only if Q is true”
- “ P is a sufficient condition for Q ”
- “ Q is a necessary condition for P ”:

Phrases expressing $P \iff Q$ (“ P is equivalent to Q ”):

- “ P holds if and only if Q holds”
- “ P is a necessary and sufficient condition for Q ”

Phrases expressing existential quantifiers ($\exists x \dots$)

- “There exists $x \in S$ such that \dots ”
- “ \dots holds for some $x \in S$ ”
- “For some $x \in S$ we have \dots ”

Phrases expressing universal quantifiers ($\forall x \dots$)

- “For all $x \in S$, we have \dots ”
- “For any $x \in S$, \dots ”
- “Every $x \in S$ satisfies \dots ”
- “Given $x \in S$, we have \dots ”
- “If $x \in S$, then \dots ”
- “Whenever $x \in S$, then \dots ”

Further resources

This material is covered in the D’Angelo/West text on pp. 27–34; you’ll find there many examples illustrating the proper use of logical statements, and some excellent general remarks and advice. Be sure to read this section.

Final Word

Logic, like whiskey, loses its beneficial effect when taken in too large quantities.

—Lord Dunsany