Problem 1. Logical statements, I. Negate the following statements **without using words of negation**. (You can use "composite" as negation of "prime", "even" as negation of "odd", and "good" as negation of "broken".) Write the negation as an English sentence, use proper terminology, be sure to include any necessary quantifiers and appropriate connecting words (e.g., "such that") if necessary. Below f denotes a function from $\mathbb{R}$ to $\mathbb{R}$, and n denotes a positive integer.

(a) "Every classroom in Altgeld Hall has a chair that is not broken."

(b) "If n is a Fermat number, then n is a prime."

(c) "For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x)| < \epsilon$ holds whenever $|x| < \delta$"

(d) "There exist $x_0 \in \mathbb{R}$ and $\delta > 0$ such that $|x - x_0| < \delta$ implies $f(x) < f(x_0)$."

(e) **Bonus question:** Describe, with a brief explanation/justification, in **simple language** (in about three words, *without* using logical terminology) the precise set of functions that satisfy statement (d).

Problem 2. Logical statements, II: Consider the following statement:

"To win the World Cup it is necessary to win each of the last four games."

(a) Write the **negation** of this statement without using words of negation. (You can use "lose" as a synonym for "not win".)

(b) Write the **converse** of this statement without using words of negation.

(c) Write the **contrapositive** of this statement without using words of negation.

(d) Rewrite the statement in the form "... **only if** ...".

Problem 3. Short answer problems, I: Sum/product formulas. Evaluate the given sum or product. (For all sums/products there is a simple formula involving only elementary functions and factorials. No induction proof needed, but show all work.)

(a) $\displaystyle\prod_{i=1}^{n} \left(1 + \frac{n}{i}\right)$

(b) $\displaystyle\prod_{i=1}^{n} n^{n-i}$

(c) $\displaystyle\sum_{i=1}^{n} \sum_{j=i}^{n} \frac{i}{j}$ (Note that the summation limits for j are from $j = i$ to $j = n$.)

Problem 4. Short answer problems, II: Set-theoretic notations and definitions

(a) Let $A = \{1, 2\}$, $B = \{0, 1\}$. Find $P(A)$, $P(B$, and $P(A) \cap P(B)$ (where $P(S)$ denotes the power set of S) and write this set out *explicitly (i.e., by listing all its elements), using proper set-theoretic notation.*

(b) Express the set $\mathbb{Q}$ of rational numbers in set builder notation, i.e., in the form $\mathbb{Q} = \{\dots : \dots\}$, with appropriate expressions in place of the dots.

(c) For each of the following properties (where f is a function from $\mathbb{R}$ to $\mathbb{R}$), state its definition in *symbolic form.*

(I) "f is a **decreasing** function"

(II) "f is a **nondecreasing** function"

(III) "f is a **bounded** function"

(Just state the three definitions; no further work/justifications needed.)

Problem 5. Proofs, I: Even/odd and divisibility. *Using only the basic algebraic properties of the integers and the **definitions** of even and odd numbers and divisibility*, give careful, step-by-step, proofs for each of the following statements. (You must work directly from the definitions of even/odd; you can **not** use any of the properties or results about even/odd numbers established in the worksheets or in homework problems.)
(If you need extra space for work, use the back of the page.)

(a) "If $n^2 - 1$ is even, then n is odd."

(b) "If n is odd, then $n^2 - 1$ is divisible by 8."

Problem 6. Proofs, II: Set theory. Let A, B, and C be sets. For each of the following statements below, determine whether it is true. If it is true, give a careful, step-by-step, proof; if it is false, give a counterexample. Your write-up must include all necessary steps, with appropriate justifications (e.g., "by the def. of ..."), in the correct logical order, use proper notation and terminology, and include any necessary quantifiers and connecting words (e.g., "therefore", "such that").
(If you need extra space for work, use the back of the page.)

(a) $(A \cup B) - C \subseteq (A - (B \cup C)) \cup (B - (A \cap C))$.

(b) $(A - (B \cup C)) \cup (B - (A \cap C)) \subseteq (A \cup B) - C$.