

Even/odd proofs: Practice problems Solutions

The problems below illustrate the various proof techniques: direct proof, proof by contraposition, proof by cases, and proof by contradiction (see the separate handout on proof techniques). For each of these proof techniques there is at least one problem for which the technique is appropriate. For some problems, more than one approach works; try to find the simplest and most natural method of proof. **Unless you are using a direct proof, state the method of proof you are using.**

For the proofs you should use only the definitions and assumptions stated above; in particular, do not use any results or notations from number theory that you may know. Pay particular attention to the write-up. In all but the simplest cases, this requires doing some preliminary scratch work before writing up a formal proof. Specifically, proceed in two stages as follows:

- **Stage 1: Produce outline of argument.** Begin with scratch work to come up with a “flow chart” of the proof, showing the steps involved and their logical connections, with brief justifications for each step. For this part, you can make liberal use of logical symbols (e.g., \Rightarrow) and abbreviations (e.g., “def. of \cup ”).
- **Stage 2: Produce proper write-up:** Once you have a step-by-step outline of the argument, convert it to a proper proof, using complete sentences and English words rather than logical symbols (such as \forall , \exists , \Rightarrow). Double-check your write-up to make sure that it makes both logical and grammatical sense.

1. Sums and products of even/odd numbers. Prove the following statements:

- (a) If n and m are both odd, then $n + m$ is even.
- (b) If n is odd and m is even, then $n + m$ is odd.
- (c) If n and m are both even, then $n + m$ is even.
- (d) If n and m are both odd, then nm is odd; otherwise, nm is even.

Solution. We give a proof for the first statement, (a); the other statements, (b)–(d), can be proved in an analogous way.

(a) Proof of “ n and m odd $\Rightarrow n + m$ even”: As usual, in order to better show the structure of the proof, we write each step on a separate line. (A “pro-style” proof would consist of a single continuous paragraph, but this makes it harder to parse the argument.)

- Suppose n and m are odd integers.
- Then $n = 2k + 1$ and $m = 2l + 1$ for some $k, l \in \mathbb{Z}$, by the definition of an odd integer.
- Therefore $n + m = (2k + 1) + (2l + 1) = 2(k + l + 1)$.
- Since k and l are integers, so is $k + l + 1$.
- Hence $n + m = 2p$ with $p = k + l + 1 \in \mathbb{Z}$.
- By the definition of an even integer, this shows that $n + m$ is even. ■

2. Even/odd squares: Prove the following:

- (a) Let n be an integer. If n^2 is odd, then n is odd.
- (b) Let n be an integer. If n^2 is even, then n is even.
- (c) Let n be an integer. Then n^2 is odd if and only if n is odd.

Solution.

Proof of (a): We proof the statement in (a) by contraposition. Since the negation of “even” is “odd”, the contrapositive of the statement (a) is is:

(a)’ Let n be an integer. If n is even, then n^2 is even.

Proof of (a)’:

- Assume n is an even integer.
- Then $n = 2k$ for some $k \in \mathbb{Z}$, by the definition of an even integer.
- Squaring both sides, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
- Since k is an integer, so is $2k^2$.
- Hence $n^2 = 2p$ with $p = 2k^2 \in \mathbb{Z}$.
- Therefore n^2 is even, by the definition of an even integer. ■

(b) can be proved in the same way as (a).

Proof of (c): Here we have an “if and only if” statement, which breaks into the following two implications:

- (i) Let $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.
- (ii) Let $n \in \mathbb{Z}$. If n is odd, then n^2 is odd.

Now, (i) is the same as (a), while (ii) is the contraposition of (b), and hence equivalent to (b). Thus, (c) is a consequence of (a) and (b).

3. **Cool application I: Sums of odd perfect squares.** Can a sum of two perfect squares be another perfect square? Sure; for example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $6^2 + 8^2 = 10^2$, $7^2 + 24^2 = 25^2$. However, no matter how much you try, you won’t find any examples in which the two perfect squares on the left are both odd. Your task is to prove this, i.e.:

Prove that a sum of two odd perfect squares is never a perfect square.

(An interesting consequence of this result is that in any right triangle in which all sides have integer length at least one of the two shorter sides must be of even length.)

Proof: The argument given below makes use of the results of Problems 1 and 2. Without using these results, the proof would become a bit longer. One would need to consider separately the cases when n, m, p below have a specified parity (even or odd), and show that in each of these cases a contradiction arises.

We first restate the result to be proved in a more explicit form:

- (*) If s and t are odd perfect squares, then $s + t$ is not a perfect square.

Proof of (*): We use the method of contradiction.

- Suppose s and t are odd perfect squares, and assume that $s + t$ is also a perfect square.
- Then $s = n^2$, $t = m^2$, and $s + t = p^2$, for some $n, m, p \in \mathbb{Z}$, by the definition of a perfect square.
- Since, by assumption, $s = n^2$ and $t = m^2$ are odd, the integers n and m must be odd as well (by Problem 2).
- Hence $n = 2k + 1$ and $m = 2l + 1$ for some $k, l \in \mathbb{Z}$, by the definition of an odd integer.
- Since the sum of two odd numbers is even (by Problem 1), $s + t = p^2$ is even.
- Hence p , must be even as well (by Problem 2).
- Therefore $p = 2h$ for some $h \in \mathbb{Z}$, by the definition of an even integer.

- Substituting $n = 2k + 1$, $m = 2l + 1$, and $p = 2h$ into $n^2 + m^2 = p^2$ we get

$$\begin{aligned}(2k + 1)^2 + (2l + 1)^2 &= (2h)^2, \\ 4k^2 + 4k + 1 + 4l^2 + 4l + 1 &= 4h^2, \\ 4(k^2 + k + l^2 + l) + 2 &= 4h^2, \\ 2(k^2 + k + l^2 + l) + 1 &= 2h^2.\end{aligned}$$

- In the last equation the left side is an odd integer since it is of the form $2j + 1$, where $j = k^2 + k + l^2 + l$ is an integer. On the other hand, the right side is an even integer since it is of the form $2i$, where $i = h^2$ is an integer.
- Since an integer cannot be simultaneously even and odd, we have arrived at a contradiction.
- Therefore our assumption that $s + t$ is a perfect square is false.
- Thus, we have shown that if s and t are odd perfect squares, then $s + t$ cannot be a perfect square.

■

4. Cool application, II: Quadratic equations with no integer/rational solutions:

- (a) Prove the following:

If a, b, c are odd integers, then the equation $ax^2 + bx + c = 0$ has no integer solution.

[This is the equation considered at the beginning of Chapter 2 of the text and it is a nice illustration of the power of “parity arguments”. **Don’t look up the solution in the text; try to find it on your own.**¹ For the proof, you may use any of the properties of sums and products of even/odd integers established in Problem 1.]

- (b) Prove that the above result remains true if “integer solution” is replaced by “rational solution”.

Proof of (a): We use the method of contradiction.

- Suppose a, b, c are odd integers and there exists an integer solution x to the equation $ax^2 + bx + c = 0$.
- Then $a = 2h + 1$, $b = 2i + 1$, $c = 2j + 1$, for some $h, i, j \in \mathbb{Z}$, by the definition of an odd integer.
- We now consider separately the cases x even and x odd:
 - If x is even, then $x = 2k$ for some $k \in \mathbb{Z}$, by the definition of an even integer. Therefore,

$$\begin{aligned}ax^2 + bx + c &= (2h + 1)(2k)^2 + (2i + 1)(2k) + (2j + 1) \\ &= 2[(2h + 1)2k^2 + (2i + 1)k + j] + 1.\end{aligned}$$

Hence $ax^2 + bx + c$ is of the form $2[\dots] + 1$, where the expression $[\dots]$ is an integer. Therefore $ax^2 + bx + c$ is an odd integer.

- If x is odd, then $x = 2k + 1$ for some $k \in \mathbb{Z}$, by the definition of an odd integer. Therefore,

$$\begin{aligned}ax^2 + bx + c &= (2h + 1)(2k + 1)^2 + (2i + 1)(2k + 1) + (2j + 1) \\ &= (2h + 1)(4k^2 + 4k + 1) + 2k(2i + 1) + 2i + 1 + 2j + 1 \\ &= 4(2h + 1)(k^2 + k) + 2h + 1 + 4ki + 2k + 2i + 1 + 2j + 1 \\ &= 2[2(2h + 1)(k^2 + k) + h + 2ki + k + i + 1] + 1.\end{aligned}$$

Hence $ax^2 + bx + c$ is of the form $2[\dots] + 1$, where the expression $[\dots]$ is an integer. Therefore $ax^2 + bx + c$ is an odd integer.

- Thus, in either case, we obtain that $ax^2 + bx + c$ is an odd integer and hence cannot be equal to 0.
- Therefore, our assumption that there is an integer solution to the equation $ax^2 + bx + c = 0$ is false, so this equation has no integer solutions. ■

¹Hint: Consider the parity (even or odd) of $ax^2 + bx + c$.

Outline of proof of (b): This is harder, but the basic idea is the same as for (a). We assume that a, b, c are odd integers and that there exists a rational solution x to the equation $ax^2 + bx + c = 0$, and we try to derive a contradiction from this. Writing x as $x = p/q$ for some integers p and q , we get the equation $ap^2 + bpq + cq^2 = 0$, which is of a similar type as the equation considered in (a), and can be handled by considering separately the cases (i) p even and q odd, (ii) p odd and q even, (iii) p odd and q odd. (By the FACT (*) stated below, we don’t need to consider the case when p and q are both even.)

5. Cool application, III: Irrationality proofs: Prove that $\sqrt{2}$ is irrational.

We will prove this result using the following **FACT**:²

(*) Any rational number x can be written in the form $x = p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $q \neq 0$, and **at least one of p and q is odd**.

We will also use the following result proved in Problems 2:

(**) Let n be an integer. Then n^2 is even if and only if n is even.

Proof of irrationality of $\sqrt{2}$:

- We argue by contradiction and suppose that the statement is false, i.e., that $\sqrt{2}$ is **not** irrational.
- Then $\sqrt{2}$ is rational. By the definition of rational numbers, this means that $\sqrt{2}$ can be written as $\sqrt{2} = p/q$, where p and q are integers with $q \neq 0$. By (*) we can further assume that at least one of p and q is odd.
- Squaring each side of $\sqrt{2} = p/q$ and clearing denominators, we get

$$2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2},$$

$$2q^2 = p^2.$$

- Thus p^2 is 2 times an integer, and hence must be even, by the definition of an even integer.
- By (**) it follows that p must be even as well.
- Hence $p = 2k$, for some $k \in \mathbb{Z}$, by the definition of an even integer.
- Substituting this into (1) gives

$$2q^2 = (2k)^2,$$

$$q^2 = 2k^2.$$

- Thus q^2 is 2 times an integer, and hence must be even, by the definition of an even integer.
- Therefore, **both p and q must be even**. This is a contradiction to (*).
- Hence our assumption that $\sqrt{2}$ is rational was false.
- Therefore $\sqrt{2}$ must be irrational. ■.

²The part in boldface, “at least one of p and q is odd”, is the nontrivial part here. It is intuitively obvious: if you cancel as many factors 2 as possible in p/q , then in the resulting fraction, say p^*/q^* , there are no 2’s left to cancel, so at least one of p^* or q^* is not divisible by 2, and thus must be odd. A fully rigorous proof requires the “Well-Ordering Principle”, which will be covered later.