

Infinite Series: Definitions and Theorems

Definitions

- **Infinite series:** An infinite series is a formal expression of the form $\sum_{k=1}^{\infty} a_k$, i.e., a **sum** over (countably) infinitely many real numbers.¹
- **Partial sums:** The partial sums of a series $\sum_{k=1}^{\infty} a_k$, are defined as $s_n = \sum_{k=1}^n a_k$, i.e., s_n is the sum of the first n terms of the series.
- **Convergence and divergence:** Convergence of an infinite series $\sum_{k=1}^{\infty} a_k$ is defined in terms of convergence of the sequence of its partial sums:

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \text{ exists} \iff \{s_n\} \text{ converges}$$

The series $\sum_{k=1}^{\infty} a_k$ is said to be **divergent** if it does not converge in the above sense, i.e., if the sequence of partial sums $\{s_n\}$ does not converge. For a convergent series, its **sum** is defined as the limit of its partial sums:

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$$

- **Absolute convergence:** A series $\sum_{k=1}^{\infty} a_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ converges.

Theorems

The following are some standard results about series that you should know. Except for the Alternating Series Test (whose proof is a bit tricky), all of these results can easily be proved using the above definition of convergence of series and results about infinite sequences (e.g, algebraic properties, Monotone Convergence Theorem, and Cauchy Criterion). **Try to carefully work out each of these proofs. This is a great exercise in applying the definitions and theorems about sequences and series.**

- **Cauchy Criterion for infinite series:** An infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if it satisfies the **Cauchy Criterion for infinite series**:

$$\text{For every } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that for any integers } n > m \geq N, \left| \sum_{k=m+1}^n a_k \right| < \epsilon$$

- **n-th term test:** If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.
- **Algebraic properties:**
 - **Sum Property:** If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge with sums A and B respectively, then so does $\sum_{k=1}^{\infty} (a_k + b_k)$, with sum $A + B$.
 - **Scaling:** If $\sum_{k=1}^{\infty} a_k$ converges with sum A , and $c \in \mathbb{R}$, then $\sum_{k=1}^{\infty} (ca_k)$ converges, with sum cA .

¹We consider only series in which the terms a_k are real numbers.

- **Comparison test:** If $|a_k| \leq b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges as well. More generally, if there exist constants $C \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $|a_k| \leq Cb_k$ holds for all $k \geq N$, then the convergence of $\sum_{k=1}^{\infty} b_k$ implies that of $\sum_{k=1}^{\infty} a_k$.
- **Absolute convergence implies convergence:** An absolutely convergent series is convergent.
- **Alternating series test:**² If the terms a_k (i) **alternate in sign** (i.e., are negative for odd k and positive for even k , or vice versa), (ii) **are nonincreasing in absolute value** (i.e., satisfy $|a_{k+1}| \leq |a_k|$ for all $k \in \mathbb{N}$), and (iii) **have limit 0 as $k \rightarrow \infty$** , then the series $\sum_{k=1}^{\infty} a_k$ converges.

Two Classical Series

The following are important examples of series that you should be familiar with. You should know how to prove the convergence resp. divergence of these series.

- **Geometric series with ratio r :** $\sum_{k=0}^{\infty} r^k$. This series converges if $|r| < 1$, with sum $1/(1-r)$, and diverges if $|r| \geq 1$.
- **Harmonic series:** $\sum_{k=1}^{\infty} \frac{1}{k}$. This series diverges.

Series versus Sequences

By far the most common mistake in dealing with infinite series is to mix up a **series** with a **sequence**, and apply theorems and properties of sequences $\{a_n\}$ to series $\sum_{n=1}^{\infty} a_n$, or vice versa.

A sequence is simply an infinite **list** of terms, a_1, a_2, \dots , while a series is an infinite **sum** of terms, $a_1 + a_2 + a_3 + \dots$, or $\sum_{n=1}^{\infty} a_n$; Convergence of a series is defined in terms of the sequence of the corresponding **partial sums** $s_n = a_1 + a_2 + \dots + a_n$; it is **not** equivalent to the convergence of the **sequence** of its terms $\{a_n\}$. Here are some examples illustrating the difference.

- The **sequence** $\{1/n\}$ converges to 0, whereas the **series** $\sum_{n=1}^{\infty} 1/n$ is divergent.
- The **sequence** $\{1/2^n\}$ converges with limit 0, whereas the **series** $\sum_{n=0}^{\infty} 1/2^n$ converges, with the sum equal to $1/(1-1/2) = 2$ (by the geometric series formula).
- The product of two convergent **sequences** $\{a_n\}$ and $\{b_n\}$, i.e., the sequence $\{a_n b_n\}$, converges, but the analogous result does not hold for series: The convergence of the **series** $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ does **not** imply that of the series $\sum_{n=1}^{\infty} a_n b_n$. (For a counterexample consider an alternating series such as $\sum_{n=1}^{\infty} (-1)^n / \sqrt{n}$.)
- If the **series** $\sum_{n=1}^{\infty} a_n$ converges, then the **sequence** $\{a_n\}$ converges to 0 (by the n -th term test). The converse, however, is not true. (A counterexample is the harmonic series.)

²As mentioned, the proof of this result is somewhat tricky; you need to know the result, but not the proof.