Practice Problems: Countable and Uncountable Sets $\boxed{\text{Solutions}}$

1. The set of all real numbers in the interval $(0, 1)$. (Hint: Use a standard calculus function to establish a bijection with $\mathbb{R}$.)

   Solution: *UNCOUNTABLE: Proof: We construct a bijection with $\mathbb{R}$ using the* arctan *function. The* arctan *function itself is a bijection between $\mathbb{R}$ and the interval $(-\pi/2, \pi/2)$. Therefore, the function $f(x) = (\arctan(x) + \pi/2)/\pi$ gives a bijection between $\mathbb{R}$ and the interval $(0, 1)$.*

2. The set of all rational numbers in the interval $(0, 1)$.

   Solution: *COUNTABLE: The rational numbers in the interval $(0, 1)$ form an infinite subset of the set of all rational numbers. Since the set of rational numbers is countable and an infinite subset of countable set is countable, the result follows.*

3. The set of all points in the plane with rational coordinates.

   Solution: *COUNTABLE. Proof: The given set is $\mathbb{Q} \times \mathbb{Q}$. Since $\mathbb{Q}$ is countable and the cartesian product of finitely many countable sets is countable, $\mathbb{Q} \times \mathbb{Q}$ is countable.*

4. The set of all "words" (defined as finite strings of letters in the alphabet).

   Solution: *COUNTABLE: Proof: The set of all words can be expressed as a union $W_1 \cup W_2 \cup W_3 \cup \cdots$, where W_k denotes the set of words of length k. Since each W_k is finite (more precisely, assuming 26 letters in the alphabet, there are 26^k words of length k), the claim follows from the fact that a countable union of finite or countable sets is finite or countable. Since the set of all words is not clearly not finite, it must be countable.*

5. The set of all computer programs in a given programming language (defined as a finite sequence of "legal" words and symbols in that language, such as "if", "for", "{", "==", etc.).

   Solution: *COUNTABLE. Proof: Use the same argument as for words, with the "alphabet" being the (finite) set of all letters in the ordinary alphabet along with symbols and digits.*

6. The set of all infinite sequences of integers.

   Solution: *UNCOUNTABLE. Proof: The set of all infinite integer sequences is a superset of the set of all infinite binary (i.e., 01) sequences. But an infinite binary sequence can be interpreted as the binary expansion of a real number in the interval $[0, 1)$, and since the set of such real numbers is uncountable, so is the set of all infinite binary sequences.*[1]

7. The set of all functions $f : \{0, 1\} \to \mathbb{N}$.

   Solution: *COUNTABLE. Proof. The functions $f : \{0, 1\} \to \mathbb{N}$ are in one-to-one correspondence with $\mathbb{N} \times \mathbb{N}$ (map f to the tuple (a_1, a_2) with $a_1 = f(1)$, $a_2 = f(2)$). Since the latter set is countable, as a Cartesian product of countable sets, the given set is countable as well.*

8. The set of all functions $f : \mathbb{N} \to \{0, 1\}$.

   Solution: *UNCOUNTABLE. Proof: Mapping a function $f : \mathbb{N} \to \{0, 1\}$ to the sequence $(a_1, a_2, \dots)$ defined by $a_i = f(i)$ for each i yields a bijection between functions of the given type and infinite binary sequences. Since the latter set is uncountable, so is the given set of all functions $f : \mathbb{N} \to \{0, 1\}$.*

[1] More precisely, to each binary sequence $\{d_1, d_2, \dots\}$ we can associate the real number $x \in [0, 1)$ whose binary expansion is given by $x = 0.d_1 d_2 \dots$. Since every real number in the interval $[0, 1)$ has a binary expansion of the form $x = 0.d_1 d_2 \dots$ and there are uncountably many real numbers in $[0, 1)$, it follows that there must be uncountably many infinite binary sequences.