Worksheet: Epsilonics, II: Infinite Series

Sample Proof: Sum Property for Infinite Series

Many properties and results about **series** can be deduced from corresponding properties/results for **sequences**. The following is an example of a proof of this type.

Claim: If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge with sums A and B respectively, then so does $\sum_{k=1}^{\infty}(a_k + b_k)$, with sum $A + B$.

Proof. Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge to A and B, respectively.

By the definition of convergence of **series**, this means that the corresponding partial sums

$$s_n = \sum_{k=1}^{n} a_k, \quad t_n = \sum_{k=1}^{n} b_k$$

converge with limits

$$\lim_{n \to \infty} s_n = A, \quad \lim_{n \to \infty} t_n = B.$$

By the sum property of limits of **sequences**, it follows that

$$\lim_{n \to \infty} (s_n + t_n) = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n = A + B.$$

Since

$$s_n + t_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(a_k + b_k)$$

this proves that the series $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, with sum $A + B$, proving our claim.

Sample Proof: Comparison Test

One of the most useful *direct* methods of proof of convergence (and divergence) of series is the Cauchy Criterion for Series. This example illustrates such an application. Here Cauchy's Criterion is applied twice, first to the comparison series that is assumed to be convergent and then to the series whose convergence one seeks to prove.

Claim: If $|a_k| \le b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Proof. Suppose the series $\sum_{k=1}^{\infty} b_k$ converges, and that $|a_k| \le b_k$ for all $k \in \mathbb{N}$.

Let $\epsilon > 0$ be given.

Since $\sum_{k=1}^{\infty} b_k$ converges, this series satisfies Cauchy's Criterion, so there exists an $N \in \mathbb{N}$ such that, for all $n, m \in \mathbb{N}$ with $n > m \ge N$,

$$(1) \qquad \left| \sum_{k=m+1}^{n} b_k \right| < \epsilon.$$

Let $n > m \ge N$. Then

$$\left| \sum_{k=m+1}^{n} a_k \right| \le \sum_{k=m+1}^{n} |a_k| \quad \text{(by the triangle inequality, } |x_1 + \cdots + x_k| \le |x_1| + \cdots + |x_k|\text{)}$$

$$\le \sum_{k=m+1}^{n} b_k \quad \text{(by the assumption } |a_k| \le b_k \text{ for all } k \in \mathbb{N}\text{)}$$

$$< \epsilon \quad \text{(by (1))}$$

Hence, $n > m \ge N$ implies $\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon$.

Thus the series $\sum_{k=1}^{\infty} a_k$ satisfies Cauchy's Criterion and therefore is convergent.

Practice Problems

Below are some problems to practice proof-writing skills in the context of infinite series. All of these proofs should be done rigorously, using the "official" definitions of series convergence and results such as Cauchy's Criterion. None of these proofs is particularly difficult; try to master them all! For additional practice problems, especially of the "prove or find counterexample" variety, see the homework assignments.

> **IMPORTANT NOTE:** In proofs involving infinite series always work with **finite sums**, such as the partial sums $\sum_{k=1}^{n} a_k$, or the sums of finite "chunks" of the series, $\sum_{k=m+1}^{n} a_k$, that arise in Cauchy's criterion. **You should NEVER try to directly work with infinite sums $\sum_{k=1}^{\infty} a_k$ since such a sum has no meaning until you have proved that it is convergent.**
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> For example, trying to prove the sum property by writing $\sum_{k=1}^{\infty}(a_k+b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$, or the Absolute Convergence Test by writing $|\sum_{k=1}^{\infty} a_k| \le \sum_{k=1}^{\infty} |a_k|$, would be totally wrong and would completely miss the point of having precise mathematical definitions of convergence and divergence of series. (With this sort of manipulation one could "prove" all sorts of nonsensical results!)

1. **Convergence/divergence of particular series:**

   (a) **Geometric series, convergent case:** Using the definition of convergence of an infinite series, prove that if $|r| < 1$, then the geometric series $\sum_{k=0}^{\infty} r^k$ converges with sum $1/(1-r)$. (Hint: Use the formula for the sum of a finite geometric series.)

   (b) **Geometric series, divergent case:** Prove that if $|r| \ge 1$, then the series $\sum_{k=0}^{\infty} r^k$ diverges. (Hint: Use an appropriate convergence test.)

   (c) **Divergence of harmonic series:** Using Cauchy's Criterion for Series, show that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

2. **General Properties of Series.**

   (a) **Proof of Cauchy Criterion for Series:** Prove the Cauchy Criterion for series, using the Cauchy Criterion for sequences.

   (b) **Proof of Absolute Convergence Test:** Prove the Absolute Convergence Test using the Comparison Test.

   (c) **Series of nonnegative terms:** Prove that if a series has (i) only nonnegative terms and (ii) bounded partial sums, then it converges. (Hint: What does the "nonnegative terms" condition mean in terms of the partial sums s_n?)

   (d) **Proof of the n-th term test:** The n-th term test says that if $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n \to \infty} a_n = 0$. Give a careful proof of the this test, by two methods: (I) using the Cauchy Criterion for Series; (II) using the definition of convergence of infinite series and algebraic properties of limits of sequences.