Problem 1. Definitions, theorems, and examples. State the requested definition, theorem, or example. Be sure to use correct notation and include any necessary quantifiers in the appropriate order. (No proofs/explanations needed.)

(a) State the Completeness Axiom.

(b) State the Fundamental Theorem of Arithmetic.

(c) A function f from $\mathbb{R}$ to $\mathbb{R}$ is **not bounded** if ...

(d) Give a formal definition of a **relation on a set** S.

(e) Give a formal definition of a **partition of a set** S.

(f) Give a formal definition (using ϵ's and logical notation) of the **sup** of a set S of real numbers. Be sure to include any necessary quantifiers, in the correct order.

Problem 2. Logical statements, I.

Consider the following statements:

(P) *"To pass this class it is necessary to score at least* 60% *on each of the exams."*

(Q) *"Scoring at least* 60% *on each of the exams guarantees that you will pass the class."*

(a) Rewrite **statement P** in the form "**If** ... **then** ..."

(b) Write the **negation of statement P** without using words of negation (such as "not", "it is false that","without", etc.). (You can use the words "less than" as the negation of "at least" and "fail" as the negation of "pass".)

(c) Write the **contrapositive of statement Q** without using words of negation (such as "not", "it is false that","without", etc.). (You can use the words "less than" as the negation of "at least" and "fail" as the negation of "pass".)

(d) Mark each of the following statements as TRUE or FALSE. (An answer is enough; no justification needed.)

(i) P implies Q.

(ii) Q implies P.

(iii) Q is the converse of P.

(iv) Q is the contrapositive of P.

(v) Q is the negation of P.

Problem 3. Logical statements, II. Negate the following statements **without using words of negation**. Write the negation as an English sentence, use proper terminology, be sure to include any necessary quantifiers and appropriate connecting words (e.g., "such that") if necessary.

(a) For every $\epsilon > 0$ there exists a positive integer n_0 such that $|a_n - a_m| < \epsilon$ holds whenever $m, n \geq n_0$.

(b) For all $L \in \mathbb{R}$, all $\epsilon > 0$, and all $x_0 \in \mathbb{R}$ there exists a real number $x > x_0$ such that $|f(x) - L| < \epsilon$.

Bonus question: Find a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the given statement. Explain briefly why the example "works"! (Use back of page if needed.)

(c) There exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $|x| < \delta$ and $|f(x)| \geq \epsilon|x|$.

Bonus question: Describe in *simple language* (using standard calculus terminology), the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the **negation** of the given statement. Explain! (Use back of page if needed.)

Problem 4. Short answers, I. For the following questions, give an answer and a brief justification. For questions about cardinality and countability you can use (without proof) the following:
(i) Known results about the countability or uncountability of the following **specific** sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and the set of infinite binary sequences.
(ii) Any of the **general** results and properties about countable sets given on the cardinality handout. *If you use one of these results/properties, say so and indicate which property you are using.*

(a) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ that is increasing, but not injective? If so, give a *specific* example of such a function; if not, explain why no such function exists.

(b) Does there exist an infinite subset of $\mathbb{R}$ that is both countable and bounded? If so, give a *specific* example of such a set; if not, explain why no such set exists.

(c) Do there exist functions $f : A \to B$ and $g : B \to C$ such that the composition $g \circ f$ is a bijection from A to C, but neither f nor g are bijections? If so, give a *specific* example of such functions; if not, explain why no such functions exists.

Problem 5. Short answers, II. For each of the statements below, determine if it is true or false. If it is true, give a proof, e.g., by using an appropriate theorem. (With the right approach, the proofs required are quite short—a few lines at most. You can use any results proved in class, the handouts and the worksheets, but **you must state clearly which result you are using**.) If it is false, give a **specific** counterexample and explain briefly why this example "works" (e.g., in case of an example of a divergent series say why the series diverges).

(a) If for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$ for all $n \geq N$, then the sequence $\{a_n\}$ converges.

(b) If $\{a_n\}$ converges, then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $\epsilon > 0$, $|a_n - a_{n+1}| < \epsilon$.

(c) If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is monotone and bounded.

Problem 6. Short computations/proofs. The following questions can be answered with a short computation/proof using appropriate formulas or some "congruence magic". (Use back of page for work if needed.)

(a) Evaluate the sum $x^n + 2x^{n-1} + 2^2x^{n-2} + \cdots + 2^n x^0$, where n is a natural number and x a nonzero real number.

(b) Let $P(n)$ be the polynomial defined by $P(n) = n^2 + n + 41$. Find, with proof, infinitely many positive integers n such that $P(n)$ is a multiple of 47. (Hint: First find one such value n, then prove that there exist infinitely many.)

(c) Let n be an integer ≥ 2. Prove that if $2^n - 1$ is prime, then n is prime. (Use back of page for work if needed.)

Problem 7. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ and let S_f be the set of all functions g from $\mathbb{R}$ to $\mathbb{R}$ such that there exist positive constants c, a such that $|g(x)| \leq c|f(x)|$ for all $x > a$.

(a) Give a careful, step-by-step, proof of the following statement:

If g_1 and g_2 are functions in S_f, then the sum, $g_1 + g_2$, is also in S_f.

(b) Give a *specific* counterexample (i.e., an appropriate choice of the functions g_1, g_2, f) showing that the analogous property for products (i.e., the statement "if g_1 and g_2 are functions in S_f, then g_1g_2 is also in S_f") does **not** hold. Explain why your example is indeed a counterexample, i.e., does not satisfy the above statement.

Problem 8. Given a finite set A, call a subset of A **odd** if it has an odd number of elements, and **even** otherwise. **Using induction**, prove that, for any natural number n, an n-element set has exactly 2^{n-1} odd subsets.

Note: You may use the fact that the total number of subsets of an n-element set is 2^n, but not any other formulas for the number of subsets. *The proof must be done by induction, not by other methods.* Pay particular attention to the write-up. Make sure your argument is written up in the correct logical order, includes any necessary quantifiers, and appropriate justifications for each step (e.g., "by the induction hypothesis applied to ...", "since an n-element set has 2^n subsets"). It must include all details without being too wordy.

Problem 9. **Using only the ϵ-definition of a limit**, give a formal, ϵ-style proof for the following result:

If $\{a_n\}$ is a convergent sequence, then $\{a_n\}$ is bounded.

Note: The result must be derived directly from the ϵ-definition of the limit of a sequence, and *not use any of the properties and theorems on convergent sequences given in the book, the worksheets, the class handouts, and the homework.*