Upper and lower bounds, sup and inf

In the following, S denotes a nonempty set of real numbers.

- **Upper and lower bounds:** A real number α is called an **upper bound** for S if $x \le \alpha$ for all $x \in S$. The set S is said to be **bounded above** if it has an upper bound.

  In analogous fashion, one defines a **lower bound**, and one calls a set that has a lower bound **bounded below**.

- **Sup and inf:** A real number α is called the **least upper bound** (or **supremum**, or **sup**) of S, if (i) α is an upper bound for S; and (ii) there does not exist an upper bound for S that is strictly smaller than α. The supremem, if it exists, is unique, and is denoted by $\sup S$.

  The **greatest lower bound** (or **infimum** or **inf**) is defined analogously and denoted by $\inf S$.

  ϵ-definition of sup and inf: For proofs involving sups and infs, the following equivalent definition is useful:

  $$\boxed{\alpha = \sup S \iff \text{(i) } x \le \alpha \text{ for all } x \in S, \text{ and (ii) for every } \epsilon > 0 \text{ there exists } x \in S \text{ such that } x > \alpha - \epsilon.}$$

  $$\boxed{\alpha = \inf S \iff \text{(i) } x \ge \alpha \text{ for all } x \in S, \text{ and (ii) for every } \epsilon > 0 \text{ there exists } x \in S \text{ such that } x < \alpha + \epsilon.}$$

- **Sup versus max:** The **maximum**, or **max**, of a set S is its largest element *if such an element exists*. Here are the key differences between the max and sup concepts. (Analogous remarks apply to the inf and min concepts.)

  – If a set has a maximum, then the maximum is also a sup for this set, but the converse is not true.
  – A finite set always has a maximum (which is also its sup), but an infinite set need not have a maximum.
  – The sup of a set S need not be an element of the set S itself, but the maximum of S must always be an element of S.

  Here are some examples that illustrate these differences.:

  – If $S = \{1 - 1/n : n = 1, 2, \dots\}$, then $\sup S = 1$, but $1 \notin S$ and $\max S$ does not exist.
  – If $S = \{1/n : n = 1, 2, , \dots\}$, then 1 is both the max and the sup of S.
  – If $S = \{r \in \mathbb{Q} : r^2 < 2\}$, then S does not have a max, but it has a sup, namely $\sup S = \sqrt{2}$. Note that $\sqrt{2}$ is not an element of S (since $\sqrt{2} \notin \mathbb{Q}$), but is an element of $\mathbb{R}$.

Completeness Axiom and Related Properties

The Completeness Axiom or Least Upper Bound Property, is one of the fundamental properties of the real numbers. The remaining properties are consequences of the Completeness Axiom, and you know how to deduce them from the Completeness Axiom. These properties may seem obvious (and you can use them without further justification when doing epsilon proofs), but they are closely tied to the real numbers, and there exist domains in which the properties fail (see Exercise 13.40 for an example).

Completeness Axiom: $\boxed{\text{Any nonempty subset of } \mathbb{R} \text{ that is \textbf{bounded above} has a \textbf{least upper bound}.}}$
In other words, the Completeness Axiom guarantees that, for any nonempty set of real numbers S that is bounded above, a sup exists (in contrast to the max, which may or may not exist (see the examples above). An analogous property holds for inf S: Any nonempty subset of $\mathbb{R}$ that is **bounded below** has a **greatest lower bound**.

Archimedean Property: $\boxed{\text{Given any real number } x, \text{ there exists } n \in \mathbb{N} \text{ such that } n > x.}$
In other words, this says that the set of natural numbers is not bounded (from above).

Completeness Axiom implies Archimedian Property (Proof Idea): Assume $\mathbb{N}$ is bounded above. By the Completeness Axiom $\mathbb{N}$ must have a sup, say $\alpha = \sup \mathbb{N}$. Now show that $\alpha - 1$ must also be an upper bound for $\mathbb{N}$, contradicting the definition of a sup.

Completeness Axiom implies Monotone Convergence Theorem (Proof Idea): Assume a sequence $\{a_n\}$ satisfies the assumptions of the Monotone Convergence Theorem, i.e., is bounded and monotone, say monotone increasing. Since the sequence is bounded, by the Completeness Axiom, it must have a sup, say $\alpha = \sup\{a_n : n \in \mathbb{N}\}$. Now use the ϵ-definition of a sup, to deduce that the sequence converges to α.