

Worksheet: Induction Proofs III: Miscellaneous examples

Practice Problems Solutions

1. **Number of subsets with an even (or odd) number of elements:** Using induction, prove that an n -element set has 2^{n-1} subsets with an even number of elements and 2^{n-1} subsets with an odd number of elements.

Solution: For brevity, we call a subset with an odd number of elements an **odd subset**, and a subset with an even number of elements an **even subset**. Let $P(n)$ denote the following statement:

$P(n)$: Any set with n elements has 2^{n-1} odd subsets and 2^{n-1} even subsets.

We will use induction to show that $P(n)$ holds for all $n \in \mathbb{N}$.

Base case: Let $n = 1$, and consider a set $A = \{a_1\}$ with one element. Then the subsets of A are $\{\}$ and $\{a_1\}$. The first of these sets is even (since the empty set has 0 elements and 0 is an even number), and the second set is odd (since it has one element). Thus a one-element set has exactly one even and one odd subset. Since, $2^{n-1} = 1$ when $n = 1$, this proves the statement $P(n)$ for $n = 1$.

Induction step: Let $k \in \mathbb{N}$ be given and suppose $P(k)$ is true, i.e., that any k -element set has 2^{k-1} even subsets and 2^{k-1} odd subsets. We seek to show that $P(k+1)$ is true as well, i.e., that any $(k+1)$ -element set has 2^k even subsets and 2^k odd subsets.

Let A be a set with $(k+1)$ elements. Choose an element a in A , and set $A' = A - \{a\}$. We classify the subsets of A into two types: (I) subsets that do not contain a , and (II) subsets that do contain a . We now count the number of even and odd subsets of each type as follows:

- **Even and odd subsets of type I:** The subsets of type I are exactly the subsets of the set A' . Since A' has k elements, the induction hypothesis can be applied to this set and we get that there are 2^{k-1} even subsets and 2^{k-1} odd subsets of type I.
- **Even and odd subsets of type II:** The subsets of type II are exactly the sets of the form $B = B' \cup \{a\}$, where B' is a subset of A' , and hence are in one-to-one correspondence with subsets B' of A' . Moreover, B is an odd subset of A if and only if the associated set B' is an even subset of A' , and B is an even subset of A if and only if the associated set B' is an odd subset of A' . By the induction hypothesis there are 2^{k-1} even subsets of A' , and 2^{k-1} odd subsets of A' . Hence there are 2^{k-1} odd subsets of type II, and 2^{k-1} even subsets of type II.

Since there are 2^{k-1} even subsets of each of the types I and II, the total number of **even** subsets of A is $2^{k-1} + 2^{k-1} = 2^k$. Similarly, the total number of odd subsets of A is $2^{k-1} + 2^{k-1} = 2^k$.

Since A was an arbitrary $(k+1)$ -element set, we have proved that any $(k+1)$ -element set has 2^k even subsets and 2^k odd subsets. Thus $P(k+1)$ is true.

Conclusion: By the principle of induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

2. **Number of regions created by n lines:** How many regions are created by n lines in the plane such that no two lines are parallel and no three lines intersect at the same point? Guess the answer from the first few cases, then use induction to prove your guess.

Solution: For brevity, we call a set of lines **generic** if it satisfies the conditions in the statement, namely that no two lines are parallel, and no three lines intersect at the same point.

Let $P(n)$ denote the following statement:

$P(n)$: The number of regions created by n generic lines in the plane is $1 + \frac{n(n+1)}{2}$.

We will use induction to show that $P(n)$ holds for all $n \in \mathbb{N}$.

Base case: A single line divides the plane into 2 regions. Since $1 + 1(1+1)/2 = 2$, this proves $P(n)$ for $n = 1$.

Induction step: Let $k \in \mathbb{N}$ be given, and suppose $P(n)$ holds for $n = k$, i.e., suppose that any k generic lines in the plane create $1 + k(k+1)/2$ regions.

Let $k + 1$ lines L_1, L_2, \dots, L_{k+1} be given that are generic in the above sense. Then the first k lines, L_1, \dots, L_k are also generic and, by the induction hypothesis, these k lines divide the plane into $1 + k(k + 1)/2$ regions.

Now consider the line L_{k+1} . By the “generic” property, this line intersects each of the lines L_1, \dots, L_k at exactly one point, and the k intersection points are all distinct and hence divide L_{k+1} into $k + 1$ segments. Each of these segments divides one of the regions created by the first k lines into two parts, and hence increases the region count by 1. Since there are $k + 1$ such segments, the added line L_{k+1} increases the region count by $k + 1$. Thus the total number of regions created by the lines L_1, \dots, L_{k+1} is

$$1 + \frac{k(k + 1)}{2} + k + 1 = \frac{k(k + 1) + 2k + 4}{2} = 1 + \frac{(k + 1)(k + 2)}{2}.$$

This is the desired formula for the number of regions created by $k + 1$ lines. Hence $P(k + 1)$ holds, and the proof of the induction step is complete.

Conclusion: By the principle of induction, $P(n)$ holds for all $n \in \mathbb{N}$.

3. **Sum of angles in a polygon:** The sum of the interior angles in a triangle is 180 degrees, or π . Using this result and induction, prove that for any $n \geq 3$, the sum of the interior angles in an n -sided polygon is $(n - 2)\pi$.

Solution: Let $P(n)$ denote the following statement:

$P(n)$: The sum of the interior angles in an arbitrary n -sided polygon is $(n - 2)\pi$.

We will use induction to show that $P(n)$ holds for all integers $n \geq 3$.

Base case: The sum of the angles in a triangle is π , which agrees with the formula $(n - 2)\pi$ when $n = 3$. Thus the statement $P(n)$ holds for $n = 3$.

Induction step: Let $k \in \mathbb{N}$ with $k \geq 3$ be given, and assume $P(n)$ holds for $n = k$, i.e., suppose that, for any k -sided polygon, the sum of the interior angles is $(k - 2)\pi$.

Let P be a $(k + 1)$ -sided polygon. Pick a vertex P_1 of P at which the interior angle is $< \pi$. (It is clear that such a vertex must exist.) Let P_0 and P_2 denote the vertices of P adjacent to P_1 , let T be the triangle $P_0P_1P_2$, and P' the polygon obtained from P by removing the triangle T , i.e., with the two sides P_0P_1 and P_1P_2 replaced by P_0P_2 .

Then P' has k sides, so by the induction hypothesis the sum of the interior angles in P' is $(k - 2)\pi$. Also, since T is a triangle, the sum of the interior angles in T is π . The sum of the interior angles in the original polygon P is equal to the sum of the interior angles of P' plus the sum of the interior angles of T , i.e., $(k - 2)\pi + \pi = ((k + 1) - 2)\pi$. This is the desired formula for the sum of the interior angles of a $(k + 1)$ -sided polygon, so we have proved $P(k + 1)$.

Conclusion: By the principle of induction, $P(n)$ holds for every integer $n \geq 3$.