Problem 1. Definitions, theorems, and example. State the requested definition, theorem, or example. Be sure to use correct notation and include any necessary quantifiers in the appropriate order.

(a) Give a precise statement of the **negation** of the Cauchy property for sequences *without using words of negation.* That is, complete the following sentence:

*A sequence $\{a_n\}$ is **not** a Cauchy sequence if ...*

(b) Give an example of a **countable** set A of real numbers such that $\sup A$ and $\inf A$ both exist, but $\min A$ and $\max A$ do not exist.

(c) State the ϵ-definition of "$\alpha = \sup S$", where S is a non-empty set of real numbers. Be sure to include any necessary quantifiers, in the correct order.

(d) State the Archimedean Property.
Extra credit: Derive the Archimedean Property from the Completeness Axiom; i.e., assuming the Completeness Axiom, give a rigorous derivation of the Archimedean Property. (Use back of page for work.)

Problem 2. Short proofs and counterexamples, I. For each of the statements below, determine if it is true or false. If it is true, give a proof. You can use any properties and theorems on limits from the class handouts and worksheets, but **you must state clearly which result you are using**. If it is false, give a **specific** counterexample and explain briefly why this example "works" (e.g., in case of an example of a divergent series say why the series diverges).

(a) If $\{a_n\}$ is bounded and diverges, then $\{a_n\}$ is not monotone.

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(c) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} a_n$ does not exist.

Problem 3. Short proofs and counterexamples, II. For each of the statements below, determine if it is true or false. If it is true, give a proof. You can use any properties and theorems on limits from the class handouts and worksheets, but **you must state clearly which result you are using**. If it is false, give a **specific** counterexample and explain briefly why this example "works" (e.g., in case of an example of a divergent series say why the series diverges).

(a) If $\lim_{n\to\infty} a_n$ exists, then $\{a_n\}$ is a Cauchy sequence.

(b) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and and $\lim_{n\to\infty} b_n$ exists, then $\sum_{n=1}^{\infty} a_n b_n$ is also absolutely convergent.

(c) If $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = A$ and $\lim_{n\to\infty} \sum_{k=1}^{n} b_k = B$, then $\lim_{n\to\infty} \sum_{k=1}^{n} a_k b_k = AB$.

Problem 4. Using only the ϵ-definition of a limit or the Cauchy criterion, give formal, ϵ-style proofs for the following convergence/divergence results. *The proofs should not any other properties and theorems on sequences and series from the homework, class handouts, etc. (e.g., convergence tests).*

(a) Using the ϵ-definition of a limit, prove that $\lim_{n\to\infty} \sqrt{1 - 1/n} = 1$.

(b) Using the Cauchy Criterion, prove that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Problem 5. Using only the ϵ-definition of a limit, show that if $\lim_{n\to\infty} a_n = 1$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{a_n} = 1$. (The proof should be done directly from the definition of convergence, and *not use any of the properties and theorems on sequences given in the book, the worksheets, the class handouts, and the homework.*)