Practice Problems $\boxed{\text{Solutions}}$

The problems below have been carefully selected to illustrate common situations and the techniques and tricks to deal with these. **Try to master them all; it is well worth it! What you learn in the process will be useful later in this class when we get to epsilonics in other contexts, and in many advanced math classes.**

For each problem, first try to gain an intuitive "feel" for the problem (a sketch of a typical situation may be useful) and try to understand why the statement is correct. Then try to construct a rigorous ϵ-proof.

In the following $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ denote arbitrary sequences, and L and M denote real numbers. **Unless otherwise specified, you should only use the ϵ-definition of a limit, not any theorems or properties of limits.**

1. **Warmup problems.** These are conceptually quite easy, and the results are intuitively "clear". Try to use these problems to practice proper write-ups of proofs.

(a) **Limit of constant sequence.** Prove that the limit of a constant sequence is equal to this constant; i.e., show rigorously that, if $a_n = c$ for all n, then $\lim_{n\to\infty} a_n = c$.

Solution:

Proof: *Suppose $a_n = c$ for all $n \in \mathbb{N}$. To show that $\lim_{n\to\infty} a_n = c$, we need to show that, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $(*)$ $|a_n - c| < \epsilon$. Since $a_n = c$ for all $n \in \mathbb{N}$, we have $|a_n - c| = 0$ for all $n \in \mathbb{N}$, so $(*)$ holds in fact for **all** $\epsilon > 0$ and **all** $n \in \mathbb{N}$. Thus, the above definition is satisfied with $N = 1$.*

(b) **Scaling property.** Prove that if $\lim_{n\to\infty} a_n = L$, then for any $c \in \mathbb{R}$, $\lim_{n\to\infty}(ca_n) = cL$.

Solution: ***Idea and proof strategy.*** We use a variation of the $\epsilon/2$ trick: By the definition of a limit, we have $(*)$ $|a_n - L| < \epsilon$ from some point onwards. Multiplying $(*)$ by $|c|$, gives $(**)$ $|ca_n - cL| < |c|\epsilon$, which is an estimate of the type we want except that we need ϵ instead of $|c|\epsilon$ on the right. We can achieve this if we start out in $(*)$ with $\epsilon' = \epsilon/(|c|+1)$ in place of ϵ. (Note that using $\epsilon/|c|$ would cause problems in case $c = 0$; Replacing $|c|$ in the denominator by $|c|+1$ avoids this issue.)

Below is a formal proof along these lines..

Proof: *Suppose $\lim_{n\to\infty} a_n = L$, and let c be a given real number.*
Let $\epsilon > 0$ be given.
Set $\epsilon' = \epsilon/(|c|+1)$. Since $\epsilon > 0$, we have $\epsilon' > 0$.
Since $\lim_{n\to\infty} a_n = L$, applying the definition of a limit with this ϵ', we obtain an $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$|a_n - L| < \epsilon'. \tag{1}$$

Then for all $n \geq N$ we have

$$\begin{aligned} |ca_n - cL| &= |c| \cdot |a_n - L| \quad \text{(by properties of absolute values)} \\ &< |c|\epsilon' \quad \text{(by (1))} \\ &= |c|\frac{\epsilon}{|c|+1} \quad \text{(by the definition of } \epsilon') \\ &< \epsilon. \end{aligned}$$

Hence $n \geq N$ implies $|ca_n - cL| < \epsilon$.
By the definition of a limit, this proves that $\lim_{n\to\infty}(ca_n) = cL$.

(c) **Multiplication by bounded sequence.** Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is a bounded sequence, then $\lim_{n\to\infty} a_n b_n = 0$.

Solution: ***Idea and proof strategy.*** We need to get from an inequality of the form $(*)$ $|a_n - 0| < \epsilon$ to an inequality of the form $(**)$ $|a_n b_n - 0| < \epsilon$. Since the sequence b_n is bounded, we have (with a suitable bound M) $|b_n| \leq M$ for all n, so $(*)$ implies $|a_n b_n - 0| = |a_n b_n| \leq |M| \cdot |a_n| < |M|\epsilon$. To get a bound $< \epsilon$ in the last step, we replace ϵ in $(*)$ by $\epsilon' = \epsilon/(|M|+1)$. (Note again, the trick of working with $|M|+1$ in place of $|M|$ in order to ensure that ϵ' is well-defined.)

Now the formal proof:

Proof: *Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = 0$, applying the definition of a limit with $\epsilon' = \epsilon/(|M|+1)$, we obtain an $N \in \mathbb{N}$ such that*

$$|a_n| = |a_n - 0| < \epsilon' = \frac{\epsilon}{|M|+1} \quad \text{for all } n \geq N.$$

Then

$$|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| < \frac{\epsilon}{|M|+1} \cdot |M| < \epsilon \quad \text{for all } n \geq N.$$

Hence $n \geq N$ implies $|a_n b_n - 0| < \epsilon$.
By the definition of a limit, this proves that $\lim_{n\to\infty} a_n b_n = 0$.

(d) **Limit of shifted sequence.** Let a_n be a given sequence, and let $b_n = a_{n+1}$ be the "shifted" sequence. Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = L$.

Solution: ***Proof idea.*** We need to get from an inequality of the form $(*)$ $|a_n - L| < \epsilon$ to an inequality of the form $(**)$ $|a_{n+1} - L| < \epsilon$. Now if $(*)$ holds for all $n \geq N$, then $(**)$ holds for all $n \geq N-1$ and therefore also for all $n \geq N$. Thus the N-value that "worked" for the original sequence also "works" for the shifted sequence.

2. **Intermediate problems.** These problems illustrate a variety of techniques and tricks in working with limits.

(a) **Uniqueness of limit.** Prove that if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$. (Hint: Use contradiction.)

Solution: ***Idea and proof strategy.*** Given the form of the claim, the most natural approach is a proof by contradiction. Thus, we assume $\{a_n\}$ has two distinct limits, L and M, and seek to derive a contradiction from this assumption. Applying the limit definition with both L and M, we see that, given any $\epsilon > 0$, from some point onwards all terms a_n must fall into the intervals $(*)$ $L - \epsilon < a_n < L + \epsilon$ and $(**)$ $M - \epsilon < a_n < M + \epsilon$. We will get a contradiction if we choose ϵ small enough so that these two intervals don't overlap, for example, by choosing $\epsilon = |M - L|/3$.

Proof: *We argue by contradiction. Suppose $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_n = M$, and $L \neq M$. Then either $L < M$ or $L > M$. Without loss of generality, we may suppose $L < M$. (Otherwise we could just relabel the limits, setting $L' = M$ and $M' = L$ and working with L' and M'.)*
Applying the definition of a limit with $\epsilon = (M-L)/3$ (which is positive by our assumption $L < M$), we obtain $N_1, N_2 \in \mathbb{N}$ such that

$$|L - a_n| < \epsilon \quad \text{for all } n \geq N_1, \tag{1}$$
$$|M - a_n| < \epsilon \quad \text{for all } n \geq N_2. \tag{2}$$

Let $N = \max(N_1, N_2)$. Then a_N satisfies both (1) and (2), so in particular,

$$a_N < L + \epsilon \quad \text{(by (1))} \quad \text{and} \quad a_N > M - \epsilon \quad \text{(by (2))}.$$

This implies $M - \epsilon < L + \epsilon$, and since $\epsilon = (M-L)/3$, we obtain

$$M - \frac{1}{3}(M-L) < L + \frac{1}{3}(M-L),$$
$$\frac{2}{3}(M-L) < 0,$$

which is a contradiction to our assumption $L < M$.
Hence we must have $L = M$.

Remark: Note the "Without loss of generality" phrase used at the beginning of the argument. This is a very common phrase (often abbreviated as WLOG) used to add assumptions that are convenient in the proof, but which do not alter the nature of the problem in any material way. Here the additional assumption is $L < M$. Since the cases $L < M$ and $L > M$ are completely symmetrical (relabeling M as L' and L as M' converts one case into the other), the "WLOG" assumption is entirely justified here.

(b) **Preservation of inequalities.** Prove that if $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, and $a_n < b_n$ for all $n \in \mathbb{N}$, then $L \leq M$.

Solution: ***Proof idea.*** This is similar to the proof that the limit of a sequence is unique. Argue by contradiction, assume $L > M$, and apply the definition of convergence with $\epsilon = (L-M)/3$.

(c) **Squeeze Theorem.** Suppose $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Show that if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Solution: ***Idea and proof strategy.*** Since a_n and c_n both have limits L, we know that $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$ hold from some point onwards. We write these bounds as two-sided inequalities for a_n and c_n: $L - \epsilon < a_n < L + \epsilon$, and $L - \epsilon < c_n < L + \epsilon$ Since $b_n \geq a_n$, the first of these inequalities implies $b_n \geq a_n > L - \epsilon$; similarly, since $b_n \leq c_n$, the second relation implies $b_n < L + \epsilon$. Altogether we get $L - \epsilon < b_n < L + \epsilon$, i.e., $|b_n - L| < \epsilon$, from some point onwards. (Draw a picture, with the numbers ($L - \epsilon$, $L + \epsilon$, a_n, b_n, c_n marked, to visualize these relations.)

(d) **Cauchy Criterium, "easy" direction.** Prove that any convergent sequence is a Cauchy sequence. (Hint: Use the $\epsilon/2$-trick.)

Solution: ***Idea and proof strategy.*** Comparing the definitions of "convergent sequence" and "Cauchy sequence", we see that we need to convert a bound of the form $(*)$ $|a_n - L| < \epsilon$ to a bound of the form $(**)$ $|a_m - a_n| < \epsilon$. . The trick that accomplishes this is a standard one: Add and subtract L in $a_m - a_n$, then apply the triangle inequality:

$$|a_m - a_n| = |(a_m - L) + (L - a_n)| \leq |a_m - L| + |a_n - L|.$$

Combining this with another standard trick, namely working with $\epsilon' = \epsilon/2$ in place of ϵ in $(*)$, gives the desired ϵ-bound.

Here is a formal proof:

Proof: *Suppose* $\lim_{n\to\infty} a_n = L$. *Let* $\epsilon > 0$ *be given. Since* $\lim_{n\to\infty} a_n = L$, *applying the definition of a limit with* $\epsilon' = \epsilon/2$, *we obtain* $N \in \mathbb{N}$ *such that*

(1) $$|a_n - L| < \frac{\epsilon}{2} \quad \text{for all } n \geq N.$$

Let $m, n \geq N$. *Then*

$$\begin{aligned} |a_n - a_m| &= |(a_n - L) + (L - a_m)| \quad \text{(by algebra)} \\ &\leq |a_n - L| + |L - a_m| \quad \text{(by the triangle inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{(by (1) applied to } a_m \text{ and } a_n) \\ &= \epsilon. \end{aligned}$$

Hence $m, n \geq N$ *implies* $|a_n - a_m| < \epsilon$.
Thus the sequence $\{a_n\}$ *satisfies the Cauchy Criterion, and hence is a Cauchy sequence.*

(e) **Convergent implies bounded.** Prove that any convergent sequence is bounded. (Hint: Apply the definition of a limit with $\epsilon = 1$.)

Solution: ***Idea and proof strategy.*** To show that a sequence $\{a_n\}$ is bounded, we we need to show that there exists an M such that $|a_n| \leq M$ for all n. We proceed in two stages.

First, applying the definition of a limit with $\epsilon = 1$, we obtain such a bound (namely, $|L| + 1$, where L is the limit of the sequence) that holds for all terms a_n with n greater than some suitable N.

To extend this to a bound for *all* a_n, we need to deal with the terms a_n for $n \leq N$. We cannot say anything about these terms, but since there are only finitely many such terms, they have a finite upper bound (namely, the largest among the N terms $|a_1|, \dots, |a_N|$).

Combining these two bounds gives the required global bound for a_n.

Here is a formal proof:

Proof: *Let* $\{a_n\}$ *be a convergent sequence and let* L *be its limit. Applying the definition of a limit with* $\epsilon = 1$, *we obtain an* $N \in \mathbb{N}$ *such that*

(1) $$|a_n - L| < 1 \quad \text{for all } n \geq N.$$

Set

(2) $$M_1 = |L| + 1,$$
(3) $$M_2 = \max\{|a_1|, |a_2|, \dots, |a_N|\},$$
(4) $$M = \max(M_1, M_2).$$

We will show that, with this choice of M, *we have* $|a_n| \leq M$ *for all* $n \in \mathbb{N}$.
If $n \geq N$, *then*

$$\begin{aligned} |a_n| &= |(a_n - L) + L| \quad \text{(by algebra)} \\ &\leq |a_n - L| + |L| \quad \text{(by triangle inequality)} \\ &\leq 1 + |L| \quad \text{(by (1) since } n \geq N) \\ &= M_1 \quad \text{(by def. (2) of } M_1) \\ &\leq M \quad \text{(by def. (4) of } M). \end{aligned}$$

If $n \leq N$, *then*

$$\begin{aligned} |a_n| &\leq \max\{|a_1|, |a_2|, \dots, |a_N|\} \\ &= M_2 \quad \text{(by (def. (3) of } M_2) \\ &\leq M \quad \text{(by def. (4) of } M). \end{aligned}$$

Thus, $|a_n| \leq M$ *holds for all* $n \leq N$*.Therefore the sequence* $\{a_n\}$ *is bounded.*

Remarks: Note that we needed the definition of a limit of $\{a_n\}$ only for a single value of ϵ; we chose $\epsilon = 1$, though any other positive number would have worked. This was enough to yield the boundedness of the sequence.

Once N has been chosen, the maximum in (3) is a maximum over a finite set of real numbers and hence is well-defined. That a *finite* set of real numbers has a maximum, i.e., that one of these numbers must be greater or equal to all others, does not need to be justified, but could be proved formally by induction on the number of terms, using the fact that, for any two real numbers x and y we have either $x \le y$ (and so $y = \max(x, y)$), or $x > y$ (and so $x = \max(x, y)$). (This fact is one of the axioms defining the real numbers, so does not need a proof.)

(f) **Cauchy implies bounded.** Prove that any Cauchy sequence is bounded.

Solution: ***Idea and proof strategy.*** We use the same idea as in the previous problem, showing first that the sequence is bounded in a range $n \ge N$ with a suitable N, then extending this bound to one for the full range $n \in \mathbb{N}$. For the first part of the argument, we apply the definition of a Cauchy sequence with $\epsilon = 1$ to obtain an $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ holds for all $n, m \ge N$. Applying this with $m = N$, we get $|a_n - a_N| < 1$ for all $n \ge N$, and via the triangle inequality this gives the bound $|a_n| \le |a_N| + 1$ for all $n \ge N$. Thus, $M_1 = |a_N| + 1$ is a bound for a_n in the range $n \ge N$. (Note that in a_N the index N is a *fixed* index that has been chosen at the beginning of the proof, namely, the N that is obtained when applying the definition of a Cauchy sequence with $\epsilon = 1$; thus, a_N is also a *fixed* number, and it is okay to define M_1 in terms of a_N as was done above.)

Using the same argument as in the previous problem, one can extend this bound to one that applies to the full range $n \in \mathbb{N}$.

3. **Harder problems.** The following two problems are somewhat trickier, but very instructive.

(a) **Reciprocal property.** Prove that if $\lim_{n\to\infty} a_n = L$, $a_n \ne 0$ for all $n \in \mathbb{N}$, and $L \ne 0$, then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$. (Hint: Show first that there exists an N_1 such that $|a_n| > |L|/2$ for $n \ge N_1$.)

Solution: ***Idea and proof strategy.*** To show that $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$, we need to bound the difference $|\frac{1}{a_n} - \frac{1}{L}|$ by ϵ. We compute:

(0) $$\frac{1}{a_n} - \frac{1}{L} = \frac{L - a_n}{a_n L}.$$

The numerator, $L - a_n$, is harmless: It can be made smaller than ϵ (in absolute value) by our assumption that $\lim_{n\to\infty} a_n = L$.

The denominator $a_n L$ in (0), however, could cause problems as a_n may be close to 0. Thus, we need to ensure that a_n cannot get too small. Here is one way to do this: Since $\{a_n\}$ converges to L and, by assumption, $L \ne 0$, from some point onwards, a_n has to be within $|L|/2$ of L, and hence must satisfy $|a_n| \ge |L|/2$ (draw a picture to see this!). With this lower bound the denominator in (0) becomes $\ge |L|^2/2$ in absolute value, so the expression on the right of (0) becomes $< \epsilon(2/|L|^2)$. To get rid of the constant $(2/|L|^2)$, we use the now familiar "ϵ/c" trick and apply the definition of $\lim_{n\to\infty} a_n = L$ with a scaled version of ϵ, namely $\epsilon' = \epsilon|L|^2/2$.

Here is a formal proof:

Proof: *Suppose that $\lim_{n\to\infty} a_n = L$, $L \ne 0$, and $a_n \ne 0$ for all $n \in \mathbb{N}$.*

Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = L$, applying the definition of a limit with $\epsilon' = \epsilon|L|^2/2$ (which is > 0 since $L \ne 0$), we obtain an $N_1 \in \mathbb{N}$ such that

(1) $$|a_n - L| < \frac{\epsilon|L|^2}{2} \quad \text{for all } n \ge N_1.$$

Also, applying the definition of a limit with $\epsilon = |L|/2$, we obtain an $N_2 \in \mathbb{N}$ such that

(2) $$|a_n - L| < \frac{|L|}{2} \quad \text{for all } n \ge N_2,$$

If $L > 0$, then (2) implies $a_n > L - |L/2| = |L|/2 > 0$, and if $L < 0$, (2) implies $a_n < L + |L/2| = -|L| + |L|/2 = -|L|/2$, so in either case we have

(3) $$|a_n| > \frac{|L|}{2} \quad \text{for all } n \ge N_2.$$

Now, let $N = \max(N_1, N_2)$*. Then, for* $n \ge N$*, we have*

$$\begin{aligned}\left|\frac{1}{a_n} - \frac{1}{L}\right| &= \frac{|L - a_n|}{|La_n|} \\ &< \frac{\epsilon|L|^2}{2|La_n|} \quad \textit{(by (1))} \\ &< \frac{\epsilon|L|^2}{2|L|(|L|/2)} \quad \textit{(by (3))}. \\ &= \epsilon\end{aligned}$$

Hence $n \ge N$ *implies* $|(1/a_n) - (1/L)| < \epsilon$*.*
Therefore, $\lim_{n\to\infty} 1/a_n = 1/L$*.*

Remarks: The second application of the convergence (with $|L|/2$ playing the role of ϵ) in the above argument was necessary to get a lower bound on the denominator a_nL in (0).

(b) **Product property.** Prove that if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $\lim_{n\to\infty}(a_nb_n) = LM$. (You can use the result above that a convergent sequence is bounded.) (Hint: Use a bit of algebraic magic: $ab - cd = a(b-d) + d(a-c)$.)

Solution: ***Idea and proof strategy.*** We try to proceed as in the proof of the sum property, but there are two additional difficulties we need to overcome.

First, we need to relate the difference $a_nb_n - LM$ (which we seek to estimate) to the differences $a_n - L$ and $b_n - M$ (which we know how to estimate, by our assumptions that $a_n \to L$ and $b_n \to M$). This can be done by a variation of the familiar trick of adding and subtracting the same term: Namely

$$a_nb_n - LM = (a_n - L + L)b_n - LM = (a_n - L)b_n + L(b_n - M).$$

(This a trick well worth remembering!) Applying the triangle inequality as usual, we get a bound involving the differences $|a_n - L|$ and $|b_n - M|$, but now another difficulty appears:

These differences come attached with factors, $|b_n|$ and $|L|$, that we also need to estimate. The second of these factors, $|L|$, is no problem, since it is a constant. The first one requires a bit more thought. One way to deal with this factor would be to apply the result that any convergent sequence is bounded. An alternative approach that does not depend on this result is to use the add/subtract trick and the triangle inequality in the familiar way: $|a_n| = |a_n - L + L| \le |a_n - L| + |L|$. The latter expression can be made $\le 1 + |L|$ by applying the definition of $\lim_{n\to\infty} a_n = L$ with $\epsilon = 1$.

Here is the formal proof using the latter approach:

Proof: *Suppose* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$*.*
Let $\epsilon > 0$ *be given, and define* ϵ' *by*

(1) $$\epsilon' = \min\left\{\frac{\epsilon}{|L| + |M| + 1}, 1\right\}.$$

Note that this definition ensures that $0 < \epsilon' \le 1$ *and that* $\epsilon'(|L| + |M| + 1) \le \epsilon$*.*
Since $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$*, there exist* $N_1, N_2 \in \mathbb{N}$ *such that*

(2) $$|a_n - L| < \epsilon' \quad \textit{for all } n \ge N_1,$$
(3) $$|b_n - M| < \epsilon' \quad \textit{for all } n \ge N_2.$$

From (3) we get, for all $n \ge N_2$*,*

$$\begin{aligned}|b_n| &= |b_n - M + M| \\ &\le |b_n - M| + |M| \quad \textit{(by triangle inequality)} \\ &< \epsilon' + |M| \quad \textit{(by (3))} \\ &\le 1 + |M| \quad \textit{(by (1))}. \qquad (4)\end{aligned}$$

Let $N = \max(N_1, N_2)$*. Then, for* $n \ge N$ *we have*

$$\begin{aligned}|a_nb_n - LM| &= |(a_n - L)b_n + L(b_n - M)| \quad \textit{(by algebra)} \\ &\le |(a_n - L)b_n| + |L(b_n - M)| \quad \textit{(by triangle inequality)} \\ &\le |b_n|\epsilon' + |L|\epsilon' \quad \textit{(by (2) and (3))} \\ &< (|M| + 1 + |L|)\epsilon' \quad \textit{(by (4))} \\ &\le \epsilon \quad \textit{(by (1))}.\end{aligned}$$

Hence $n \geq N$ implies $|a_n b_n - LM| < \epsilon$.
This proves $\lim_{n\to\infty}(a_n b_n) = LM$.

4. **Examples and counterexamples.** These problems are intended to develop some intuition about the behavior of sequences. All are quite easy.

   (a) **Multiplication by bounded sequence.** Give an example of a convergent sequence a_n and a bounded sequence b_n such that the product sequence $a_n b_n$, does *not* converge. (The conclusion is true if the sequence a_n covnerges to 0, but not in general.)

   Solution: Example: $a_n = 1$, $b_n = (-1)^n$. Then a_n converges with limit 1, b_n is bounded, but the product sequence, $a_n b_n = (-1)^n$, does not converge.

   (b) **Preservation of inequalities.** Give an example of sequences a_n and b_n satisfying $a_n < b_n$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. (We know that $a_n < b_n$ implies $(*)$ $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$ by an earlier problem. The example thus shows that the inequality sign in $(*)$ cannot be replaced by a strict inequality.)

   Solution: Example: $a_n = 1 - 1/n$, $b_n = 1 + 1/n$. These sequences satisfy $a_n < b_n$ for all n, but they both have the same limit 1.

   (c) **Reciprocal property.** Give an example of a *convergent* sequence a_n satisfying $a_n \neq 0$ for all n for which the reciprocal sequence $1/a_n$ is *not* convergent.

   Solution: Example: $a_n = 1/n$. This sequence converges with limit 0, but the reciprocal sequence, $1/a_n = n$, does not converge.

   (d) **Sequences with convergent subsequences.** Given an example of a sequence a_n such that the subsequence over odd-indexed and even-indexed terms, $b_n = a_{2n-1}$ and $c_n = a_{2n}$, both converge, but the sequence a_n itself does not converge.

   Solution: Example: $a_n = (-1)^n$. This sequence does not converge, but the subsequences $a_{2n-1} = -1$ and $a_{2n} = 1$ are constant sequences and thus converge.