Problem 1. Definitions and theorems. State the requested definition, theorem, or property. Be sure to use correct notation and include any necessary quantifiers in the appropriate order.

(a) Give a precise definition of the **graph** of a function $f : A \to B$, using correct set-theoretical notation.

Solution: $\{(a, b) : a \in A, b = f(a)\}$, *or, equivalently,* $\{(a, f(a)) : a \in A\}$.

(b) Without using words of negation state the definition of "f is **not increasing**" (where f is a function from $\mathbb{R}$ to $\mathbb{R}$). Write your answer in English, i.e., without using logical symbols.

Solution: "$(\exists x, y \in \mathbb{R})((x < y) \wedge (f(x) \leq f(y)))$"
"There exist real numbers $x < y$ such that $f(x) \geq f(y)$."

(c) A function f from $\mathbb{R}$ to $\mathbb{R}$ is **not bounded** if . . .

Solution: "$(\forall M \in \mathbb{R})(\exists x \in \mathbb{R})(|f(x)| > M)$."
"For all $M \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $|f(x)| > M$."

(d) Two sets A and B are said to have the **same cardinality** if ...

Solution: *"there exists a bijection from A to B."*

Problem 2. Short answers, I. For the following questions, give an answer and a brief justification.

(a) Let $f(x) = |x-1|$ if $x < 4$, and $f(x) = |x| - 1$ if $x > 2$. Determine whether f is a **function from $\mathbb{R}$ to $\mathbb{R}$**, and justify your answer (i.e., explain why, or why not, f is a function from $\mathbb{R}$ to $\mathbb{R}$).

> **Solution:** ***TRUE.*** *For f to be a function we need to check that (i) the given rules define $f(x)$ for* every *$x \in \mathbb{R}$, and (ii) the rules define* a unique *value $f(x)$ for every $x \in \mathbb{R}$. Here $f(x)$ is given by one formula, $|x-1|$, in the range $x < 4$, and by another formula, $|x| - 1$, in the range $x > 2$. Since every real number is covered by these ranges, $f(x)$ is defined (possibly ambiguously) for every $x \in \mathbb{R}$, so property (i) holds. To check whether property (ii) holds as well, we need to check whether in the overlap of these ranges, namely for $2 < x < 4$, the two formulas given agree. Now,*
>
> $$|x| - 1 = x - 1 \quad \text{for } x > 0,$$
> $$|x-1| = x - 1 \quad \text{for } x - 1 > 0, \text{ i.e., } x > 1,$$
>
> *so in the range $2 < x < 4$ we have $|x| - 1 = x - 1 = |x-1|$. Hence $f(x)$ is unambiguously defined by the given rules, and therefore is a properly defined function from $\mathbb{R}$ to $\mathbb{R}$.*

(b) Let $f(p/q) = 1/q$ if $p \in \mathbb{Z}, q \in \mathbb{N}$, and $f(x) = 0$ if x is irrational. Determine whether f is a **function from $\mathbb{R}$ to $\mathbb{R}$**, and justify your answer (i.e., explain why, or why not, f is a function from $\mathbb{R}$ to $\mathbb{R}$).

> **Solution:** ***FALSE.*** *The first of the two rules is ambiguous because of the non-unique way of writing a rational number as p/q with integers p and q. For example, the number $x = 1/2$ could be written as $1/2$, $2/4$, $3/6$, etc., corresponding to the values $1/q = 1/2, 1/4, 1/6, \ldots$. Similarly, 0 can be written as $0/1$, $0/2$, $0/3$, etc., so 0 would be mapped to multiple values under this rule: 1, $1/2$, $1/3$, etc.* ***Thus, this rule does NOT define a function.***
> ***Remarks:*** *If one requires p/q to be* in reduced form, *this ambiguity does not arise. The resulting function is well-defined, and has the remarkable property that it is continuous at all irrational points, and discontinuous at all rational points.*

(c) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ that is unbounded, but not surjective? If so, give a *specific* example of such a function; if not, explain why no such function exists.

> **Solution:** ***YES.*** *$f(x) = x^2$ is unbounded, but not surjective since it does not take on negative values.*

(d) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ that has an inverse, but is not injective? If so, give a *specific* example of such a function; if not, explain why no such function exists.

> **Solution:** ***NO.*** *No such function exists, since if f has an inverse, then f is a bijection and hence injective.*

Problem 3. Short answers, II. For the following questions, give an answer and a brief justification. For questions about cardinality and countability you can use (without proof) the following:
(i) Known results about the countability or uncountability of the following **specific** sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and the set of infinite binary sequences.
(ii) Any of the **general** results and properties about countable sets given on the cardinality handout. *If you use one of these results/properties, say so and indicate which property you are using.*

(a) Does there exist a bijection between $\mathbb{Z}$ (the set of all integers) and $\mathbb{Z}_{odd}$ (the set of odd integers)? If so, give a *specific* example of such a bijection; if not, explain why no such bijection exists.

Solution: ***YES.*** $f(n) = 2n - 1$ *is a bijection from all integers to the odd integers.*

(b) Does there exist an infinite set A such that $A \times A$ has the same cardinality as A? If so, give a *specific* example of such a set, and explain briefly why this set has the required property. If not, explain why no such set exists.

Solution: ***YES.*** *The set* $A = \mathbb{N}$ *has this property. The set* $\mathbb{N}$ *is countable, and since the cartesian product of two countable sets is countable, the set of all pairs* (a, b), *with* $a, b \in \mathbb{N}$, *i.e., the set* $\mathbb{N} \times \mathbb{N}$, *is countable as well, and hence has the same cardinality as* $\mathbb{N}$.

(c) Does the set $\mathbb{R}$ have the same cardinality as the set $\mathbb{Q} \times \mathbb{Q}$? Explain clearly why, or why not, the two sets have the same cardinality.

Solution: ***NO.*** *Since* $\mathbb{Q}$ *is countable and the cartesian product of two countable sets is countable,* $\mathbb{Q} \times \mathbb{Q}$ *is countable. On the other hand,* $\mathbb{R}$ *is uncountable, so it cannot have the same cardinality as* $\mathbb{Q} \times \mathbb{Q}$.

Problem 4. Let the sequence a_n be defined by $a_1 = a_2 = a_3 = 1$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 4$. Using induction, prove that $a_n < 2^n$ for all $n \in \mathbb{N}$.
Pay particular attention to the write-up, be sure to include all steps, any necessary quantifiers, and provide appropriate justifications for each step (e.g., "by induction hypothesis", "by formula (1)", "by algebra", "by the AGM inequality")

Solution: *We will prove that* $(*)$ $a_n < 2^n$ *holds for all* $n \in \mathbb{N}$ *by strong induction.*

Base step: *For* $n = 1, 2, 3$, a_n *is equal to* 1, *whereas the right-hand side of* $(*)$ *is equal to* $2^1 = 2$, $2^2 = 4$, *and* $2^3 = 8$, *respectively. Thus,* $(*)$ *holds for* $n = 1, 2, 3$.

Induction step: *Let* $k \geq 3$ *be given and suppose* $(*)$ *is true for all* $n = 1, 2, \dots, k$. *Then*

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} + a_{k-2} \quad \text{(by recurrence for } a_n\text{)} \\
&< 2^k + 2^{k-1} + 2^{k-2} \quad \text{(by strong ind. hyp. } (*) \text{ with } n = k,\ k-1, \text{ and } k-2\text{)} \\
&= 2^{k+1}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \\
&= 2^{k+1}\frac{7}{8} < 2^{k+1}.
\end{aligned}$$

Thus, $(*)$ *holds for* $n = k + 1$, *and the proof of the induction step is complete.*

Conclusion: *By the strong induction principle, it follows that* $(*)$ *is true for all* $n \in \mathbb{N}$.

Problem 5. Let A, B, C be sets, $f : A \to B$, and $g : B \to C$ be functions, and let $h : A \to C$ be defined by $h(x) = g(f(x))$ for $x \in A$. For each of the following statements, determine if it is true. If the statement is true, give a careful, step-by-step, proof; be sure to use proper mathematical notation and terminology, and include any necessary quantifiers, connecting words, and justifications. If it is false, give a *specific* counterexample.

(a) If f and g are surjective, then h is surjective.

Solution: ***TRUE.***
Proof:

Suppose f and g are surjective.
We seek to show that $h = g \circ f$ is surjective.
Let $c \in C$ be given. We seek to show that there exists an $a \in A$ such that $h(a) = c$.
Since $g : B \to C$ is surjective and $c \in C$, there exists $b \in B$ such that $g(b) = c$.
Since $f : A \to B$ is surjective and $b \in B$, there exists $a \in A$ such that $f(a) = b$.
Combining these equations, we get $h(a) = g(f(a)) = g(b) = c$.
Summarizing, we have shown that, for any $c \in C$, there exists $a \in A$ such that $h(a) = c$.
Therefore, h is surjective.

(b) If h is surjective, then f is surjective.

Solution: ***FALSE.***
Counterexample: *Let $A = C = \{1\}$, $B = \{1, 2\}$, $f(1) = 1$, $g(1) = g(2) = 1$. Then $h(1) = g(f(1)) = 1$, so h maps the single element 1 in A to the single element 1 in C, and thus is a bijection from A to C, and in particular surjective. On the other hand, f is not surjective, since it does not take on the value $2 \in B$.*

Problem 6. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $f(x, y) = (x + y, x - y)$.

(a) Determine whether f is injective. If it is, give a careful, step-by-step, proof of the injectivity; if it is not, explain why.

Solution: ***TRUE.***
Proof: *Suppose* (x_1, y_1) *and* (x_2, y_2) *are elements in* $\mathbb{Z} \times \mathbb{Z}$ *such that* $(*)$ $f(x_1, y_1) = f(x_2, y_2)$. *We seek to show that* $(**)$ $(x_1, y_1) = (x_2, y_2)$.
By the definition of f, $(*)$ *implies* $(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$, *which in turn implies* $x_1 + y_1 = x_2 + y_2$ *and* $x_1 - y_1 = x_2 - y_2$. *Adding the latter two equations, we get* $2x_1 = 2x_2$, *so* $x_1 = x_2$, *and substituting this into the first of these equations gives* $y_1 = y_2$. *Thus,* $(x_1, y_1) = (x_2, y_2)$, *as desired. Therefore* f *is injective.*

(b) Determine whether f is surjective. If it is, give a careful, step-by-step, proof of the surjectivity; if it is not, explain why.

Solution: ***FALSE.***
Counterexample: *Consider the element* $(1, 0) \in \mathbb{Z} \times \mathbb{Z}$. *We will show by contradiction that* $(1, 0)$ *is not in the image of* f. *Suppose* $f(x, y) = (1, 0)$ *for some* $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. *By the definition of* f, *this implies* $1 = x + y$ *and* $0 = x - y$, *hence* $x = y$, $1 = 2x$, *and* $y = b = 1/2$. *which is a contradiction since* $(x, y) \in \mathbb{Z} \to \mathbb{Z}$. *Thus, there is no element* $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ *with* $f(x, y) = (1, 0)$, *so* f *cannot be surjective.*