Name:

Collaborator(s)[1]:

Math 347, Prof. Hildebrand, Summer 2019

HW 4, DUE IN CLASS MONDAY, 7/1

- **Grading:** 15 points total, broken down as follows:
  - **Presentation/effort:** **3 points**
  - **Graded problems:** **12 points**
- **Use this sheet as cover sheet and staple it to the assignment.** Write legibly and allow a full page for each problem. Group work on these problems is encouraged, provided (i) you write up the solutions yourself, using your own words, and (ii) you write the names of all student(s) in the group on the cover sheet.

Basic Induction Problems. *The following problems are of one the standard types of problems on the Induction Worksheets I and II, and they should be done in the same manner as the worksheet examples, using a properly written up induction (or strong induction) proof. See the back of the page for some hints on the write-up.*

#1 **Summation formulas (#3.28 in the text):** Find and prove by induction a formula for the sum $\sum_{i=1}^n \frac{1}{i(i+1)}$, where $n \in \mathbb{N}$.

#2 **Inequalities (#3.49(b) in the text):** Determine the exact set of natural numbers n for which the inequality $2^n \geq (n+1)^2$ holds. (Hint: For $n = 1, 2, 3, 4, 5, 6$, determine directly whether the inequality holds, then use induction to prove that it holds for all $n \geq 6$.)

#3 **Recurrences (#3.56 in the text):** Let $a_1, a_2, a_3, \ldots$ be a sequence satisfying $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$.

(a) Prove that if a_1 and a_2 are odd, then a_n is odd for all $n \in \mathbb{N}$.

(b) Prove that if $a_1 = 1$ and $a_2 = 1$, then $a_n = (1/2)(3^{n-1} - (-1)^n)$ for all $n \in \mathbb{N}$.

#4 **Representation problems (#3.44 in the text):** Determine the exact set of natural numbers that can be expressed as the sum of some nonnegative number of 3's and some nonnegative number of 10's (i.e., in the form $n = 3x + 10y$, where x, y are nonnegative integers).
(Hint: For $n = 1, 2, \ldots, 18$, determine directly if n can be expressed in the desired form, then then use strong induction to show that all integers $n \geq 18$ have such a representation. Pay particular attention to the base case(s).)

Fibonacci Problems. *The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, defined by $F_1 = 1, F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for ≥ 2, is one of the most famous mathematical sequences. It has many remarkable properties, is full of surprises, and a near endless source of amazing formulas. These formulas are often hard to* discover, *but routine to* prove, *using strong induction, as in the examples on the second induction worksheet. The exercises below are intended to practice such proofs.*

#5 **Sum of Fibonacci squares.** Using strong induction, prove that $\sum_{i=1}^n F_i^2 = F_n F_{n+1}$ for all $n \in \mathbb{N}$.

#6 **Exact formula for F_n.** Using strong induction, prove the formula $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, where $\alpha = (1+\sqrt{5})/2$ (the golden ratio) and $\beta = (1-\sqrt{5})/2$. (Hint: To simplify the algebra, prove and then use the relations $\alpha - \beta = \sqrt{5}$, $\alpha + \beta = 1$, $\alpha^2 - \beta^2 = \sqrt{5}$.)

#7 **Fibonacci rationals.** Let $a_1 = 1$ and for $n \geq 2$ define $a_n = 1 + 1/a_{n-1}$. The numbers a_n form a sequence of positive rational numbers and hence can be expressed in the form $a_n = p_n/q_n$, where p_n and q_n are positive integers. Guess and prove by induction a formula for these numbers. (Hint: The formula should be obvious after you calculate the first 6 terms a_n and express each as a rational number p_n/q_n.)

*** Turn page for comments, hints, and advice ***

[1] If you worked with another student or in a small group on this assignment, list the names of all students involved.

Hints, Advice, and Comments

General comments. All problems in this assignment are exercises in induction; you should do them by induction, and not by other methods. The probems are all of one of the standard types of induction proofs discussed in the first two Induction Worksheets.

Write-up of induction proofs. Induction proofs (especially, routine type proofs like those in this assignment) provide an ideal setting in which to practice and perfect your proof writing skills. **You should take the write-up seriously and strive for a proof that is as close to perfect as possible in every respect: the logic, the mathematics, the notation, the visual presentation (e.g., display long chains of equations), and the English (e.g., use proper spelling, grammar, and punctuation).** A properly written induction proof should include the following parts:

1. **A precise statement of the statement/proposition to be proved (i.e., the statement "$P(n)$").** It may be helpful to label the statement in some form, e.g., by an equation number (1), $(*)$, so that you can refer back to it.
2. **The base case.** Usually, this consists of checking $P(n)$ for a single initial n-value, but in some applications of strong induction (e.g., problems involving recurrence sequences and representation problems), one needs to check $P(n)$ for two or more consecutive n-values before the induction step can be applied.
3. **The induction step.** The proof of the induction step is the crux of the argument, and it must be given in full detail, with each of the steps justified (marginal notes like "by inductive hypothesis" or "by algebra" are okay). **Always clearly indicate, at the appropriate place in the induction step, where the induction hypothesis is being used.**
4. **A conclusion.** An overall conclusion, e.g.: "By the principle of induction, this proves ..."

Common errors in writing up induction proofs:

- **Undeclared/unquantified induction variable:** Starting out an induction step by saying "Suppose $P(n)$ is true for $n = k$", without saying what k is, what restrictions there have to be on k (e.g., $k \geq 2$), would make this step incomplete. There are many examples of "false" induction proofs that are due to errors of this type.

- **Not stating, explicitly and precisely, the statement/formula that one seeks to prove.** For example, using language like "suppose it is true for $n = k$", or "suppose $P(k)$", without explicitly and precisely stating what "it" or "$P(k)$" means, would be wrong.

- **Induction hypothesis not referenced.** The most crucial part of any induction proof is the place where the induction hypothesis (e.g., the case $n = k$ of $P(n)$, or the cases $n = k$ and $n = k - 1$, etc) is being used. This is the place that can make or break an induction argument, and an error there can have disastrous consequences. **Always clearly state, at the appropriate place in the induction step, where the induction hypothesis is being used.**

- **Insufficient/inappropriate base cases:** The base case(s) must be sufficient to "kick-start" the chain reaction that the induction proof represents. For example, an induction in which the $k + 1$ case depends on the cases k and $k - 1$ requires checking two consecutive n-values in the base step.

- **Base cases and induction step don't match up:** The first case of the induction step (i.e., the first k-value for which the induction step is claimed) must match up with the base case(s), so as not to leave a gap in the chain reaction. There are numerous examples of "false" induction proofs that result from carelessness in this regard.