

Worksheet: Logical statements

Solutions

About this worksheet

The problems in this set are intended to hone your skills in working with logical statements, translating English phrases into logical notation and vice versa, negating logical statements using the rules of negation, and interpreting complex logical statements.

Before you start: Be sure to study the Logic handout, and in particular familiarize yourself with the various English phrases expressing logical statements. Also, read pp. 27–34 of the text, which has a number of examples illustrating the use of logical statements, and also offers some excellent general advice.

1. **Implications:** Express each of the following statements as a logical implication (e.g., $A \Leftarrow (\neg B)$) or equivalence (e.g., $A \Leftrightarrow B$). Also state its negation in English (in a form like “A is true, but B is false”).

- (a) If A holds, then B holds.

$A \Rightarrow B$ **Negation:** A is true, and B is false.

- (b) A is true only if B is true.

$A \Rightarrow B$ **Negation:** A is true, and B is false.

- (c) A is true whenever B is true.

$A \Leftarrow B$ $B \Rightarrow A$ **Negation:** B is true, and A is false.

- (d) A is false only if B is false.

$\neg A \Rightarrow \neg B$ **Negation:** A is false, and B is true.

- (e) A is a necessary condition for B.

$A \Leftarrow B$ $B \Rightarrow A$ **Negation:** B is true, and A is false.

- (f) A is necessary and sufficient for B.

$A \Leftrightarrow B$ **Negation:** A is false and B is true or A is true and B is false.

- (g) A holds if and only if B holds.

$A \Leftrightarrow B$ **Negation:** A is false and B is true or A is true and B is false.

Remarks: Here are some tips in correctly interpreting implications stated in English:

- **Rephrasing the given sentence (without changing its meaning) can make it easier to determine its logical structure.** For example, “P is true whenever Q is true” clearly has the same meaning in English as “Whenever Q is true, then P is true”, which in turn is the same as “If Q is true, then P is true”. In the latter form, the logical interpretation is obvious: $Q \Rightarrow P$, or equivalently $P \Leftarrow Q$.
- **Be familiar with the pairs of words that have opposite meanings:** “necessary” and “sufficient” are reverses of each other, as are “only if” and “if”. If you are not sure about the correct interpretation of one of these words in a given phrase, try replacing it by its reverse. For example, consider the statement “P is true only if Q is true”. Replacing “only if” by “if” gives “P is true if Q is true”, which is the same as “If Q is true, then P is true”, hence means $Q \Rightarrow P$, i.e., $P \Leftarrow Q$. Thus, the given statement, “P is true only if Q is true”, must be the reverse of $P \Leftarrow Q$, i.e., $P \Rightarrow Q$.

2. **Negations of English sentences.** Negate the following statements. Express the negations in English, *avoiding the use of words of negation when possible*.

- (a) All classrooms have at least one chair that is broken.

Negation: There exists a classroom in which no chair is broken.

- (b) No classroom has only chairs that are not broken.

Negation: There exists a classroom that has only chairs that are not broken.

- (c) Every student in this class has taken Math 231 or Math 241.

Negation: There exists a student in this class who has taken neither 231 nor 241.

- (d) Every student in this class has taken Math 231 and Math 241.

Negation: There exists a student in this class who has not taken both 231 and 241."

- (e) In every section of Math 347 there is a student who has taken neither Math 231 nor Math 241.

Negation: There exists a section of 347 in which every student has taken 231 or 241.

3. **Negations of mathematical statements, I.** Translate the following sentences into logical notation, negate the statement using logical rules, then translate the negated statement back into English, *avoiding the use of words of negation when possible*. (Below f is a function from \mathbb{R} to \mathbb{R} , and x_0 a given real number.)

A bit harder, but very instructive: Many of the statements define familiar properties of functions (e.g., boundedness, monotonicity, etc.), or negations of such properties. Try to uncover these definitions and express in simple language the functions that are described by the statements.

- (a) $f(x, y) \neq 0$ whenever $x \neq 0$ and $y \neq 0$.

Symbolic notation: $(\forall x, y \in \mathbb{R})[(x \neq 0 \wedge y \neq 0) \Rightarrow f(x, y) \neq 0]$

Negation: $(\exists x, y \in \mathbb{R})[x \neq 0 \wedge y \neq 0 \wedge f(x, y) = 0]$

There exist real numbers x, y such that $x \neq 0$ and $y \neq 0$ and $f(x, y) = 0$.

There exist real numbers $x \neq 0$ and $y \neq 0$ such that $f(x, y) = 0$.

Interpretation: The functions $f(x, y)$ defined by this statement are exactly those that are non-zero outside the coordinate axes.

Remark: This is an example involving **implied/hidden** quantifiers, which must be made explicit before negating the statement: The variables x and y are not explicitly quantified in the given statement (e.g., by an explicit statement of the form "for all $x \in \dots$ and all $y \in \dots$ "), so they can be arbitrary elements in the underlying universe, i.e., arbitrary real numbers. Thus, the given statement should be interpreted as follows: "For all $x, y \in \mathbb{R}$, $f(x, y) \neq 0$ whenever $x \neq 0$ and $y \neq 0$." In this form, the variables x and y are explicitly quantified.

- (b) For all $M \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $|f(x)| \geq M$.

Symbolic notation: $(\forall M \in \mathbb{R})(\exists x \in \mathbb{R})[|f(x)| \geq M]$.

Negation: $(\exists M \in \mathbb{R})(\forall x \in \mathbb{R})[|f(x)| < M]$.

There exists $M \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $|f(x)| < M$.

Interpretation: The negation is the definition for " f is bounded". Hence the original statement is equivalent to " f is not bounded".

- (c) For all $M \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that for all $y > x$ we have $f(y) > M$.

Symbolic notation: $(\forall M \in \mathbb{R})(\exists x \in \mathbb{R})(\forall y > x)[f(y) > M]$.

Negation: $(\exists M \in \mathbb{R})(\forall x \in \mathbb{R})(\exists y > x)[f(y) \leq M]$.

There exists $M \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ there exists $y > x$ such that $f(y) \leq M$.

Interpretation: The given statement is the definition of $\lim_{x \rightarrow \infty} f(x) = \infty$.

- (d) For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $f(y) > f(x)$.

Symbolic notation: $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[f(y) > f(x)]$.

Negation: $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[f(y) \leq f(x)]$.

There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$ we have $f(y) \leq f(x)$.

Interpretation: The given statement means that the function f does not have a maximum value; i.e., every value $f(x)$ is “beaten” by some other value $f(y)$. Its negation means that f has a maximum, namely $f(x)$, where x is the number whose existence is guaranteed by the phrase “there exists $x \in \mathbb{R}$ ”.

- (e) For every $\epsilon > 0$ there exists $x_0 \in \mathbb{R}$ such that $|f(x)| < \epsilon$ for all $x > x_0$.

Symbolic notation: $(\forall \epsilon > 0)(\exists x_0 \in \mathbb{R})(\forall x > x_0)[|f(x)| < \epsilon]$.

Negation: $(\exists \epsilon > 0)(\forall x_0 \in \mathbb{R})(\exists x > x_0)[|f(x)| \geq \epsilon]$.

There exists an $\epsilon > 0$ such that for all $x_0 \in \mathbb{R}$ there exists an $x > x_0$ such that $|f(x)| \geq \epsilon$.

Interpretation: The given statement is the definition of $\lim_{x \rightarrow \infty} f(x) = 0$.

- (f) For every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

Symbolic notation: $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon]$.

Negation: $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})[|x - x_0| < \delta \wedge |f(x) - f(x_0)| \geq \epsilon]$.

There exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an x such that $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \epsilon$.

Interpretation: The given statement is the definition for continuity of f at x_0 .

Remark: In the given statement the variable x is not explicitly quantified (i.e., no phrase such as “there exists $x \in \dots$ ” or “for all $x \in \dots$ ”), so x is to be interpreted as an arbitrary element of the underlying universe, i.e., in the sense of “ $\forall x \in \mathbb{R}$ ”.

4. **Negations of mathematical statements, II.** This problem requires the formal definitions of a bounded set or function, and increasing, decreasing, nonincreasing, nondecreasing functions. These definitions can be found in Chapter 1 of the text and are collected below. (Here S is any set of real numbers, and f denotes a function from \mathbb{R} to \mathbb{R} .)

- S is *bounded* if there exists M such that $|x| \leq M$ for all $x \in S$.
- f is *bounded* if there exists M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.
- f is *increasing* (or *strictly increasing*) if $f(x) < f(y)$ whenever $x < y$.
- f is *nondecreasing* (or *weakly increasing*) if $f(x) \leq f(y)$ whenever $x < y$.
- f is *decreasing* (or *strictly decreasing*) if $f(x) > f(y)$ whenever $x < y$.
- f is *nonincreasing* (or *weakly decreasing*) if $f(x) \geq f(y)$ whenever $x < y$.

- (a) Express the statement “ f is *not bounded*” without using words of negation.

“($\forall M \in \mathbb{R}$)($\exists x \in \mathbb{R}$)($|f(x)| > M$).”

“For all $M \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $|f(x)| > M$. ”

- (b) Express the statement “ f is *not increasing*” (i.e., the negation of the “increasing” property) without using words of negation.

“($\exists x, y \in \mathbb{R}$)($x < y \wedge (f(x) \geq f(y))$)”

“There exist real numbers x, y such that $x < y$ and $f(x) \geq f(y)$. ”

- (c) Compare the definitions of “nonincreasing” and “not increasing” (the latter being the negation of “increasing”). Does one imply the other? Are there functions that satisfy one property, but not the other?

“Not increasing” means that **there exist** real numbers x, y such that $x < y$ and $f(x) \geq f(y)$. “Nonincreasing” means that **for all** real numbers x, y we have $x < y$ and

$f(x) \geq f(y)$. Thus, “nonincreasing” is a stronger property than “not increasing” (“for all” is stronger than “there exists”), so “nonincreasing” implies “not increasing”. The converse (i.e., the implication “not increasing” \implies “increasing”) is in general **not true**. For example, the function $f(x) = \sin x$ is “not increasing”, but it is **not** “nonincreasing”.

5. **Practice with epsilon-delta definitions.** Here is the epsilon-delta definition of

“ $\lim_{x \rightarrow 0} f(x) = 0$ ”:

(*) “For every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x)| < \epsilon$ whenever $|x| < \delta$.”

The following statements are small perturbations of this definitions, some of which are equivalent to the original definition, while others are “botched” versions of this definition that have a drastically different meaning.

Which versions are equivalent to the above limit definition, and which are not?

Harder, but very instructive: For those definitions that are not equivalent to $\lim_{x \rightarrow 0} f(x) = 0$, try to determine, in as simple a language as possible, what they really define. Find examples (if they exist) of functions that satisfy the definition, and of functions that don’t satisfy it. (Cf. Exercises 2.25–2.27 in the text for similar problems. In some cases this can be quite some quite tricky!)

(a) For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < \epsilon$.

Analysis: This is the same as (*), except that “ $|f(x)| < \epsilon$ whenever $|x| < \delta$ ” is replaced by “ $|x| < \delta$ implies $|f(x)| < \epsilon$.” Since the latter statements are equivalent, the given statement is equivalent to that in (*).

(b) For every $\delta > 0$ there exists $\epsilon > 0$ such that for all $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < \epsilon$.

Analysis: This differs from (*) in that the types of quantifiers (\forall or \exists) for δ and ϵ have been switched. This, however, changes the meaning completely, and the statement is no longer equivalent to that in (*). (In fact, the statement given here is satisfied, for example, for any function f that is bounded, since one can choose ϵ to be the bound M .)

(c) There exists $\delta > 0$ such that for every $\epsilon > 0$ and for all $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < \epsilon$.

Analysis: This differs from (*) in that the order of the quantifiers $\forall \epsilon > 0$ and $\exists \delta > 0$ has been switched. This changes the meaning completely, and the statement is not equivalent to (*). (For example, the function $f(x) = x$ satisfies (*), but does not satisfy the given statement.)

(d) For every $\epsilon > 0$ and for all $x \in \mathbb{R}$ there exists $\delta > 0$ such that $|x| < \delta$ implies $|f(x)| < \epsilon$.

Analysis: Here the order of the quantifiers $\forall x \in \mathbb{R}$ and $\exists \delta > 0$ has been switched, allowing the δ to be chosen after x has been chosen. This alters the situation completely. For example, any function f with $f(0) = 0$ satisfies this statement: Simply choose $\delta = |x|/2$ when $x \neq 0$.

6. **Additional resources.**

This material is covered at the beginning of Chapter 2, on pp. 27–34 of the text; be sure to read this section, study the examples and the general remarks and comments given there. Additional practice problems can be found in Homework 2; particularly instructive are Problems 2.10, 2.23, and 2.24 from HW 2.