Problem 1. Definitions, theorems, and examples. State the requested definition, theorem, or example. Be sure to use correct notation and include any necessary quantifiers in the appropriate order. (No proofs/explanations needed.)

(a) State the Completeness Axiom.

Solution: *Any nonempty set of real numbers that is bounded above has a least upper bound.*

(b) State the Fundamental Theorem of Arithmetic.

Solution: *Every integer $n \geq 2$ can be represented in the form $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, where the p_i are distinct primes, and the exponents α_i are positive integers, and this representation is unique except for the ordering of the primes p_i.*

Note: *The two key assertions of the FTA are (i) the* existence *and (ii) the* uniqueness *of a prime factorization. Both properties are essential ingredients of the FTA and important for applications. For example, irrationality proofs via the FTA rely crucially on the uniqueness of the factorization; without the uniqueness property, these proofs would break down.*

(c) A function f from $\mathbb{R}$ to $\mathbb{R}$ is **not bounded** if ...

Solution: "$(\forall M \in \mathbb{R})(\exists x \in \mathbb{R})(|f(x)| > M)$."
"For all $M \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $|f(x)| > M$."

(d) Give a formal definition of a **relation on a set** S.

Solution: *A relation is a subset of $S \times S$*

(e) Give a formal definition of a **partition of a set** S.

Solution: *A partition of S is a collection of subsets A_i of S that are (i)* ***nonempty****, (ii)* ***pairwise disjoint*** *(i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$), and (iii)* ***their union is the set*** *S (i.e., $\cup_i A_i = S$).*

(f) Give a formal definition (using ϵ's and logical notation) of the **sup** of a set S of real numbers. Be sure to include any necessary quantifiers, in the correct order.

Solution: *$\alpha = \sup S$ means (i) $\alpha \geq x$ for all $x \in S$ and (ii) for every $\epsilon > 0$, there exists $x \in S$ such that $x > \alpha - \epsilon$.*

Problem 2. Logical statements, I.

Consider the following statements:

(P) *"To pass this class it is necessary to score at least 60% on each of the exams."*
(Q) *"Scoring at least 60% on each of the exams guarantees that you will pass the class."*

(a) Rewrite **statement P** in the form "**If** ... **then** ..."

Solution: *"If you pass the class then you have scored at least 60% on each exam."*

(b) Write the **negation of statement P** without using words of negation (such as "not", "it is false that","without", etc.). (You can use the words "less than" as the negation of "at least" and "fail" as the negation of "pass".)

Solution:
"It is possible to pass this class, but score less than 60% on one of the exams."
"You can pass this class even if you scored less than 60% on one of the exams."

(c) Write the **contrapositive of statement Q** without using words of negation (such as "not", "it is false that","without", etc.). (You can use the words "less than" as the negation of "at least" and "fail" as the negation of "pass".)

Solution:
"If you fail the class, then you have scored less than 60% on one of the exams."

(d) Mark each of the following statements as TRUE or FALSE. (An answer is enough; no justification needed.)

(i) P implies Q.
(ii) Q implies P.
(iii) Q is the converse of P.
(iv) Q is the contrapositive of P.
(v) Q is the negation of P.

Solution: *Statement P is of the form $A \Rightarrow B$ (with A meaning "pass class" and B meaning "score at least 60% on each exam"), while Statement Q is of the form $B \Rightarrow A$. Thus Q is the logical converse of P, so (iii) is TRUE.*
On the other hand, (i) and (ii) are FALSE since an implication does not imply its converse, nor is it implied by its converse. (An implication may be true while its converse is false, and it may be false while its converse is true.) (iv) is also FALSE , since the contrapositive is logically different from the converse.

Problem 3. Logical statements, II. Negate the following statements **without using words of negation**. Write the negation as an English sentence, use proper terminology, be sure to include any necessary quantifiers and appropriate connecting words (e.g., "such that") if necessary.

(a) For every $\epsilon > 0$ there exists a positive integer n_0 such that $|a_n - a_m| < \epsilon$ holds whenever $m, n \geq n_0$.

Solution:
"There exists an $\epsilon > 0$ such that for all positive integers n_0 there exist integers $n \geq n_0$ and $m \geq n_0$ such that $|a_n - a_m| \geq \epsilon$.
Remark: *The given statement is is known as the "Cauchy criterion;" it is a necessary and sufficient condition for the existence of the limit $\lim_{n\to\infty} a_n$; see p. 276 in the book. We will get to this later in the course.*

(b) For all $L \in \mathbb{R}$, all $\epsilon > 0$, and all $x_0 \in \mathbb{R}$ there exists a real number $x > x_0$ such that $|f(x) - L| < \epsilon$.

Bonus question: Find a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the given statement. Explain briefly why the example "works"! (Use back of page if needed.)

Solution:
Negation: *"There exist real numbers L, x_0 and $\epsilon > 0$ such that, for all $x > x_0$, $|f(x) - L| \geq \epsilon$.*
Interpretation: *To get the proper meaning of this statement, first peel off the two outer layers "for all $L \in \mathbb{R}$" and "for all $\epsilon > 0$". The resulting statement, "for all $x_0 \in \mathbb{R}$ there exists a real number $x > x_0$ such that $|f(x) - L| < \epsilon$" means that there exist arbitrarily large x-values at which the function value, $f(x)$, is within ϵ of L.*
Wrapping the next layer, "for all $\epsilon > 0$", around this statement, gives "there exist arbitrarily large x-values at which $f(x)$ gets arbitrarily close to L."
Finally, the outside layer, "for all $L \in \mathbb{R}$", means that this should hold for every L. Thus, the functions described by the statement are those that get arbitrarily close to any given value L for arbitrarily large values of x. An example of a function with this property is $f(x) = x \sin x$ (or any function that oscillates with amplitudes going to infinity).

(c) There exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $|x| < \delta$ and $|f(x)| \geq \epsilon|x|$.

Bonus question: Describe in *simple language* (using standard calculus terminology), the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the **negation** of the given statement. Explain! (Use back of page if needed.)

Solution:
Negation: *"For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < \epsilon|x|$."*
Bonus question: *First note that the statement of the negation implies $f(0) = 0$. Keeping this in mind, the statement is precisely the definition of the limit $\lim_{x \to 0} f(x)/x = (f(x) - f(0)/x = 0$. The latter in turn is the definition of $f'(0) = 0$. Thus, the functions satisfying the negated statement are exactly those that satisfy $\boxed{f(0) = 0 \text{ and } f'(0) = 0}$.*

Problem 4. Short answers, I. For the following questions, give an answer and a brief justification. For questions about cardinality and countability you can use (without proof) the following:
(i) Known results about the countability or uncountability of the following **specific** sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and the set of infinite binary sequences.
(ii) Any of the **general** results and properties about countable sets given on the cardinality handout. *If you use one of these results/properties, say so and indicate which property you are using.*

(a) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ that is increasing, but not injective? If so, give a *specific* example of such a function; if not, explain why no such function exists.

Solution: ***NO.*** *No such function exists: If $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_2 < x_1$, thus, by the "increasing" property, $f(x_1) < f(x_2)$ or $f(x_2) < f(x_1)$, so in either case $f(x_1) \neq f(x_2)$.*

(b) Does there exist an infinite subset of $\mathbb{R}$ that is both countable and bounded? If so, give a *specific* example of such a set; if not, explain why no such set exists.

Solution: ***YES.*** *The set $\{1/n : n \in \mathbb{N}\}$ is both countable and bounded (with bound 1). Another example is the set of rational numbers in $[0, 1]$.*

(c) Do there exist functions $f : A \to B$ and $g : B \to C$ such that the composition $g \circ f$ is a bijection from A to C, but neither f nor g are bijections? If so, give a *specific* example of such functions; if not, explain why no such functions exists.

Solution: ***YES.*** *For an example, let $A = C = \{1\}, B = \{1, 2\}$, $f(1) = 1$, $g(1) = g(2) = 1$. Then $g \circ f$ is a bijection from $\{1\}$ to $\{1\}$, but f is not surjective and g is not injective.*

Problem 5. Short answers, II. For each of the statements below, determine if it is true or false. If it is true, give a proof, e.g., by using an appropriate theorem. (With the right approach, the proofs required are quite short—a few lines at most. You can use any results proved in class, the handouts and the worksheets, but **you must state clearly which result you are using**.) If it is false, give a **specific** counterexample and explain briefly why this example "works" (e.g., in case of an example of a divergent series say why the series diverges).

(a) If for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$ for all $n \geq N$, then the sequence $\{a_n\}$ converges.

Solution: *FALSE. The given statement is equivalent to: "If $\lim_{n \to \infty} (a_{n+1} - a_n) = 0$, then $\{a_n\}$ converges". A counterexample is $a_n = \sqrt{n}$. We have $\sqrt{n+1} - \sqrt{n} = 1/(\sqrt{n+1} + \sqrt{n})$, so $\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) = 0$, but the sequence $\{\sqrt{n}\}$ does not converge.*

(b) If $\{a_n\}$ converges, then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $\epsilon > 0$, $|a_n - a_{n+1}| < \epsilon$.

Solution: *FALSE. Here ϵ is chosen after n, so it can depend on a_n and a_{n+1}. For a counterexample, consider $a_n = 1/2^n$. Then, with $\epsilon = 1/2^{n+1}$, the condition $|a_n - a_{h+1}| < \epsilon$ is false. More generally, any convergent sequence that satisfies $a_n \neq a_{n+1}$ for all n greater than some N is a counterexample.*

(c) If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is monotone and bounded.

Solution: *FALSE. Counterexample: $a_n = (-1)^n/n$ converges to 0, hence is a Cauchy sequence, but is not monotone.*

Problem 6. Short computations/proofs. The following questions can be answered with a short computation/proof using appropriate formulas or some "congruence magic". (Use back of page for work if needed.)

(a) Evaluate the sum $x^n + 2x^{n-1} + 2^2x^{n-2} + \cdots + 2^nx^0$, where n is a natural number and x a nonzero real number.

Solution: *By the geometric series formula, the given sum is*

$$x^n \sum_{k=0}^{n} (2/x)^k = x^n \frac{(2/x)^{n+1} - 1}{(2/x) - 1} = \boxed{\frac{2^{n+1} - x^{n+1}}{2 - x}}$$

provided $x \neq 2$. If $x = 2$, all terms in the given sum are equal to 2^n, so the sum becomes $\boxed{(n+1)2^n}$.

(b) Let $P(n)$ be the polynomial defined by $P(n) = n^2 + n + 41$. Find, with proof, infinitely many positive integers n such that $P(n)$ is a multiple of 47. (Hint: First find one such value n, then prove that there exist infinitely many.)

Solution: ***Proof:*** *We consider congruences mod 47. Since $P(2) = 2^2 + 2 + 41 = 47$, we have $P(2) \equiv 0 \bmod 47$. Now, by a general property of congruences, congruences are "preserved" by polynomial functions with integer coefficients, so we have*

$$n \equiv 2 \bmod 47 \Longleftrightarrow P(n) \equiv P(2) \equiv 0 \mod 47.$$

But this means that $P(n)$ is divisible by 47 whenever n is of the form $n = 2 + 47k$ for some integer k. The first few values in this sequence are $2, 49, 96, 143, \ldots$.

(c) Let n be an integer ≥ 2. Prove that if $2^n - 1$ is prime, then n is prime. (Use back of page for work if needed.)

Solution: *We show the contrapositive statement: If n is composite, then so is $2^n - 1$. So suppose $n \in \mathbb{N}$ is composite, and let $n = ab$ with $a, b \in \mathbb{N}$, $1 < a, b < n$ be a nontrivial factorization of n. Then*

$$2^a \equiv 1 \bmod 2^a - 1,$$
$$2^n = (2^a)^b \equiv 1^b = 1 \bmod 2^a - 1,$$

so $2^a - 1 \mid 2^n - 1$. Since $1 < a < n$, we have $1 < 2^a - 1 < 2^n - 1$, so $2^a - 1$ is a nontrivial divisor of $2^n - 1$. Thus, $2^n - 1$ is composite as claimed.

Problem 7. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ and let S_f be the set of all functions g from $\mathbb{R}$ to $\mathbb{R}$ such that there exist positive constants c, a such that $|g(x)| \le c|f(x)|$ for all $x > a$.

(a) Give a careful, step-by-step, proof of the following statement:

If g_1 and g_2 are functions in S_f, then the sum, $g_1 + g_2$, is also in S_f.

(b) Give a *specific* counterexample (i.e., an appropriate choice of the functions g_1, g_2, f) showing that the analogous property for products (i.e., the statement "if g_1 and g_2 are functions in S_f, then $g_1 g_2$ is also in S_f") does **not** hold. Explain why your example is indeed a counterexample, i.e., does not satisfy the above statement.

Solution: *(a) Suppose $g_1, g_2 \in S_f$. By the definition of the set S_f, this means that there exist positive constants a_1, a_2, c_1, c_2 such that*

$$|g_1(x)| \le c_1|f(x)| \quad \text{for all } x > a_1,$$
$$|g_2(x)| \le c_2|f(x)| \quad \text{for all } x > a_2.$$

Adding the above inequalities, we get

$$|g_1(x)| + |g_2(x)| \le (c_1 + c_2)|f(x)| \quad \text{for all } x > \max(a_1, a_2),$$

where $\max(a_1, a_2)$ denotes the larger of the two numbers a_1 and a_2. Since, by the triangle inequality,

$$|g(x)| = |g_1(x) + g_2(x)| \le |g_1(x)| + |g_2(x)|,$$

it follows that

$$|g(x)| \le c|f(x)| \quad \text{for all } x > a.$$

where $c = c_1 + c_2$ and $a = \max(a_1, a_2)$ are positive constants. Hence, g is in the set S_f, as we had to show.

(b) Take $f(x) = x$, $g_1(x) = g_2(x) = x$. Then g_1 and g_2 satisfy (trivially) $|g_i(x)| \le 1 \cdot |f(x)|$ for $x > 0$, so these two functions are in the set S_f with $c = 1$ and $a = 1$ as possible choices for c and a. However, for the product function $g(x) = g_1(x)g_2(x) = x^2$, the ratio $|g(x)|/|f(x)| = x$ tends to infinity as $x \to \infty$, so an inequality of the form $|g(x)| \le c|f(x)|$ cannot hold for all x greater than some a. Thus, g is not in S_f.

Problem 8. Given a finite set A, call a subset of A **odd** if it has an odd number of elements, and **even** otherwise. **Using induction**, prove that, for any natural number n, an n-element set has exactly 2^{n-1} odd subsets.

Note: You may use the fact that the total number of subsets of an n-element set is 2^n, but not any other formulas for the number of subsets. *The proof must be done by induction, not by other methods.* Pay particular attention to the write-up. Make sure your argument is written up in the correct logical order, includes any necessary quantifiers, and appropriate justifications for each step (e.g., "by the induction hypothesis applied to ...", "since an n-element set has 2^n subsets"). It must include all details without being too wordy.

Solution: *Let $P(n)$ denote the statement that any set with n elements has 2^{n-1} odd subsets. We use induction to show that $P(n)$ holds for all $n \in \mathbb{N}$.*

Base case: *A 1-element set $A = \{a_1\}$ has exactly one odd subset, $\{a_1\}$, so $P(1)$ is true.*

Induction step: *Let $k \in \mathbb{N}$ with $k \in \mathbb{N}$ be given and suppose $P(k)$ is true, i.e., that any k-element set has 2^{k-1} odd subsets. We seek to show that $P(k+1)$ is true as well, i.e., that any $(k+1)$-element set has 2^k odd subsets.*

Let A be a set with $(k+1)$ elements.

Let a be an element of A, and let $A' = A - \{a\}$.

We classify the odd subsets of A into two types: (I) subsets that do not *contain a, and (II) subsets that do contain a.*

The subsets of type (I) are exactly the subsets of the set A'. Since A' has k elements, the induction hypothesis can be applied to this set and we get that there are 2^{k-1} odd subsets of type (I).

The subsets of type (II) are exactly the sets of the form $B = B' \cup \{a\}$, where B' is an even *subset of A'. The induction hypothesis guarantees that there are 2^{k-1}* odd *subsets of A', and since A' has a total of 2^k subsets, it follows there must be $2^k - 2^{k-1}$* even *subsets B' of A'. Hence, A has 2^{k-1} odd subsets of type (II).*

Since there are 2^{k-1} odd subsets of each of the types (I) and (II), the total number of odd subsets of A is $2^{k-1} + 2^{k-1} = 2^k$. Thus, $P(k+1)$ is true, completing the induction step.

Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.*

Problem 9. Using only the ϵ-definition of a limit, give a formal, ϵ-style proof for the following result:

If $\{a_n\}$ is a convergent sequence, then $\{a_n\}$ is bounded.

Note: The result must be derived directly from the ϵ-definition of the limit of a sequence, and *not use any of the properties and theorems on convergent sequences given in the book, the worksheets, the class handouts, and the homework.*

Solution: *Suppose $\{a_n\}$ is a convergent sequence, and let L be its limit. By definition, this means that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. We apply this with $\epsilon = 1$ and let N_1 be the corresponding N-value. Then $|a_n - a_N| < 1$ for all $n \geq N$, and hence*

$$(1) \qquad |a_n| = |(a_n - a_N) + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N| \quad \text{for all } n \geq N.$$

Now set

$$M = \max\{|a_1|, |a_2|, \ldots, |a_N| + 1\}.$$

If $n \leq N$, then $|a_n| \leq M$, by the definition of a maximum. If $n > N$, then $|a_n| \leq |a_N| + 1 \leq M$ by (1) and the definition of M Thus, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded, with bound M.