Worksheet: "Even/odd" Proofs

About this worksheet

In this worksheet you will practice constructing and writing up proofs of statements involving the "parity" (even or odd) of integers and related properties, using only minimal assumptions—essentially just the definitions of "even" and "odd" and basic algebraic properties of addition and multiplication of integers.

Such "even/odd" proofs are among the simplest types of proofs you'll encounter, and they make for excellent examples to illustrate various proof techniques (direct, contrapositive, contradiction, proof by cases), and to practice proof-writing skills in a particularly simple context. Many of the statements to be proved are very intuitive, they may *seem* almost obvious, but, given the minimal assumptions we are working under, they do require a proper multi-step proof (though an extremely simple one). Others represent surprising, "cool" applications of the same type of argument.

Assumptions and definitions

- **Basic assumptions:** We assume known the sets of natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), and reals ($\mathbb{R}$), the standard algebraic operations on these sets (addition, multiplication, and division), and the basic algebraic properties of these operations (commutativity, associativity, etc.). These are our minimal assumptions under which we will work. **In particular, we do not assume any number-theoretic results, notations, or terminology (e.g., congruences or modular division).**

- **Basic definitions:** The terms "even integer", "odd integer", "perfect square", "divisible", "rational", and "irrational" are defined in the table below. **These are the definitions you should work with in the problems below.**

> *Below, n always denotes an integer, d denotes a non-zero integer, and x denotes a real number.*
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> $$n \textbf{ even} \iff \text{there exists } k \in \mathbb{Z} \text{ such that } n = 2k$$
> $$n \textbf{ odd} \iff \text{there exists } k \in \mathbb{Z} \text{ such that } n = 2k + 1$$
> $$n \textbf{ perfect square} \iff \text{there exists } k \in \mathbb{Z} \text{ such that } n = k^2$$
> $$n \textbf{ divisible by } d \iff \text{there exists } k \in \mathbb{Z} \text{ such that } n = dk$$
> $$x \textbf{ rational} \iff \text{there exist } p, q \in \mathbb{Z} \text{ with } q \neq 0 \text{ such that } x = p/q$$
> $$x \textbf{ irrational} \iff x \text{ is not rational}$$

Note that "even" is equivalent to "divisible by 2". (This follows immediately from the definitions of "even" and "divisible by d".)

- **Fact about even and odd integers:** We *assume* here the following result: *An integer is either even, or odd, but not both.* In other words, this says that **"odd" is the negation of "even"**. This may seem obvious, but it is in fact a non-trivial result that we will prove later (quite easily) using number-theoretic techniques and results (specifically, the division algorithm). **For now, we simply assume this fact.**

Even/odd proofs: Practice problems

The problems below illustrate the various proof techniques: direct proof, proof by contraposition, proof by cases, and proof by contradiction (see the separate handout on proof techniques). For each of these proof techniques there is at least one problem for which the technique is appropriate. For some problems, more than one approach works; try to find the simplest and most natural method of proof. **Unless you are using a direct proof, state the method of proof you are using.**

    For the proofs you should use only the definitions and assumptions stated above; in particular, do not use any results or notations from number theory that you may know. Pay particular attention to the write-up. In all but the simplest cases, this requires doing some preliminary scratch work before writing up a formal proof. Specifically, proceed in two stages as follows:

- **Stage 1: Produce outline of argument.** Begin with scratch work to come up with a "flow chart" of the proof, showing the steps involved and their logical connections, with brief justifications for each step. For this part, you can make liberal use of logical symbols (e.g., $\Rightarrow$) and abbreviations (e.g., "def. of $\cup$").

- **Stage 2: Produce proper write-up:** Once you have a step-by-step outline of the argument, convert it to a proper proof, using complete sentences and English words rather than logical symbols (such as $\forall$, $\exists$, $\Rightarrow$). Double-check your write-up to make sure that it makes both logical and grammatical sense.

1. **Sums and products of even/odd numbers.** Prove the following statements:

   (a) If n and m are both odd, then $n + m$ is even.

   (b) If n is odd and m is even, then $n + m$ is odd.

   (c) If n and m are both even, then $n + m$ is even.

   (d) If n and m are both odd, then nm is odd; otherwise, nm is even.

2. **Even/odd squares:** Prove the following:

   (a) Let n be an integer. If n^2 is odd, then n is odd.

   (b) Let n be an integer. If n^2 is even, then n is even.

   (c) Let n be an integer. Then n^2 is odd if and only if n is odd.

3. **Cool application I: Sums of odd perfect squares.** Can a sum of two perfect squares be another perfect square? Sure; for example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $6^2 + 8^2 = 10^2$, $7^2 + 24^2 = 25^2$. However, no matter how much you try, you won't find any examples in which the two perfect squares on the left are both odd. Your task is to prove this, i.e.:

   Prove that a sum of two odd perfect squares is never a perfect square.

   (An interesting consequence of this result is that in any right triangle in which all sides have integer length at least one of the two shorter sides must be of even length.)

4. **Cool application, II: Quadratic equations with no integer/rational solutions:**

   (a) Prove the following:

       If a, b, c are odd *integers, then the equation* $ax^2 + bx + c = 0$ *has no integer solution.*

       [This is the equation considered at the beginning of Chapter 2 of the text and it is a nice illustration of the power of "parity arguments". **Don't look up the solution in the text; try to find it on your own.**[1] For the proof, you may use any of the properties of sums and products of even/odd integers established in Problem 1.]

   (b) Prove that the above result remains true if "integer solution" is replaced by "rational solution".

5. **Cool application, III: Irrationality proofs:** Prove that $\sqrt{2}$ is irrational.

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[1]Hint: Consider the parity (even or odd) of $ax^2 + bx + c$.