Worksheet: Cardinality, Countable and Uncountable Sets

- **Key Tool: Bijections.**
  - **Definition:** Let A and B be sets. A bijection from A to B is a function $f : A \to B$ that is both injective and surjective.
  - **Properties of bijections:**
    - **Compositions:** The composition of bijections is a bijection.
    - **Inverse functions:** The inverse function of a bijection is a bijection.
    - **Symmetry:** The "bijection" relation is symmetric: If there is a bijection f from A to B, then there is also a bijection g from B to A, given by the inverse of f.
- **Key Definitions.**
  - **Cardinality:** Two sets A and B are said to have the **same cardinality** if there exists a bijection from A to B.
  - **Finite sets:** A set is called **finite** if it is empty or has the same cardinality as the set $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$; it is called **infinite** otherwise.
  - **Countable sets:** A set A is called **countable** (or **countably infinite**) if it has the same cardinality as $\mathbb{N}$, i.e., if there exists a bijection between A and $\mathbb{N}$. Equivalently, a set A is countable if it can be **enumerated** in a sequence, i.e., if all of its elements can be listed as an infinite sequence $a_1, a_2, \dots$.
    NOTE: The above definition of "countable" is the one given in the text, and it is reserved for **infinite** sets. **Thus finite sets are not countable according to this definition.**
  - **Uncountable sets:** A set is called **uncountable** if it is infinite and not countable.
- **Two famous results with memorable proofs.**

  The following are key results in set theory, and their proofs are among the most famous and memorable in all of mathematics.
  - **The rational numbers are countable.**
    Proof idea: "Zigzag method". Arrange the rationals in matrix and enumerate by traversing the matrix in zigzag fashion. (See text, p. 90).
  - **The real numbers are uncountable.**
    Proof idea: "Cantor's diagonalization method". Assuming the reals are countable, the decimal expansions of all real numbers in $[0, 1)$ can be put in a matrix with countably many rows and columns. Use the digits in the diagonal to construct a real number in $[0, 1)$ not accounted for, thus obtaining a contradiction. (See text, p. 266.)
- **Some general results on countable and uncountable sets.**
  - **Subsets/supersets of countable/uncountable sets:** If A is countable, then any infinite subset $B \subseteq A$ is is also countable. If A is uncountable, then any superset of A (i.e., a set B such that $A \subseteq B$) is also uncountable.
  - **Unions of countable sets:** If $A_1, A_2, \dots, A_n$ are each countable, then so is the union $A_1 \cup A_2 \cup \dots \cup A_n$. The same holds for an infinite union $A_1 \cup A_2 \cup A_3 \cup \cdots$ of countable sets if the number of sets A_i is countable.
    Proof idea: For the case of an infinite union, enumerate each A_i as $\{a_{i1}, a_{i2}, \dots\}$, so that the union $A_1 \cup A_2 \cup \cdots$ consists of all elements a_{ij}, $i \in \mathbb{N}$, $j \in \mathbb{N}$. Now arrange these elements in an infinite matrix and use a "zigzag" argument to enumerate the matrix elements.
  - **Cartesian products of countable sets:** If A and B are countable, then the cartesian product $A \times B$ is countable, too. The same holds for the cartesian product of finitely many countable sets $A_1 \times \dots A_k$.
    Proof idea: For the case of two countable sets A and B, enumerate these sets as $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$, arrange the elements (a_i, b_j) of $A \times B$ in an infinite matrix and use a "zigzag" argument to traverse this matrix and obtain an enumeration of all matrix elements. The general case can be proved by induction.

Practice Problems: Countable and Uncountable Sets

The following problems are intended to develop a "feel" for countable and uncountable sets. In each case, determine if the set is countable or uncountable and justify your answer. Here are some ways to establish countability or uncountability:

- Establish a bijection to a known countable or uncountable set, such as $\mathbb{N}$, $\mathbb{Q}$, or $\mathbb{R}$, or a set from an earlier problem. Many of the sets below have natural bijection between themselves; try to uncover these bjections!
- Establish a bijection to a *subset* of a known countable set (to prove countability) or a *superset* of a known uncountable set (to prove uncountability).
- Build up the set from sets with known cardinality, using unions and cartesian products, and use the above results on countability of unions and cartesian products.

1. The set of all real numbers in the interval $(0, 1)$. (Hint: Use a standard calculus function to establish a bijection with $\mathbb{R}$.)
2. The set of all rational numbers in the interval $(0, 1)$.
3. The set of all points in the plane with rational coordinates.
4. The set of all "words" (defined as finite strings of letters in the alphabet).
5. The set of all computer programs in a given programming language (defined as a finite sequence of "legal" words and symbols in that language, such as "if", "for", "{", "==", etc.).
6. The set of all infinite sequences of integers.
7. The set of all functions $f : \{0, 1\} \to \mathbb{N}$.
8. The set of all functions $f : \mathbb{N} \to \{0, 1\}$.