Name:

Collaborator(s)[1]:

Math 347, Prof. Hildebrand, Summer 2019

HW 5, DUE IN CLASS FRIDAY, 7/5

- **Grading:** 15 points total, broken down as follows:
  - **Presentation/effort:** **3 points**
  - **Graded problems:** **12 points**
- **Use this sheet as cover sheet and staple it to the assignment.** Write legibly and allow a full page for each problem. Group work on these problems is encouraged, provided (i) you write up the solutions yourself, using your own words, and (ii) you write the names of all student(s) in the group on the cover sheet.

Miscellaneous Induction Problems. *To conclude the induction chapter, here is one last set of induction problems that showcase the remarkable variety of problems to which induction can be applied. Enjoy!*

#1 **A mystery product.** Find, and prove by induction, a simple formula (involving at most two terms) for the product $P_n = \prod_{i=0}^{n}(1 + 2^{2^i}) = 3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537 \cdots (1 + 2^{2^n})$. ***See hint on back of page.***

#2 **A mystery sequence.** Let $a_1 = 2$ and $a_n = 1 + a_1 \cdots a_{n-1}$ for $n \geq 2$. The first few few terms of this sequence are $2, 3, 7, 43, 1807, 3263443, \ldots$. Find, and prove by induction, a simple formula (involving at most two terms) for $S_n = \sum_{i=1}^{n} 1/a_i$, the sum of the first n reciprocals of this sequence. ***See hint on back of page.***

#3 **Recurrences (3.57 in the text).** Let $a_1 = 1$, $a_2 = 1$, and $a_n = (1/2)(a_{n-1} + 2/a_{n-2})$ for all $n \geq 3$. Prove that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$. ***See hint on back of page.***

#4 **Fibonacci representations.** Prove that every positive integer n has a representation as a sum of **distinct** Fibonacci numbers. (For example, $15 = 8 + 5 + 2$ is such a representation for $n = 15$.) ***See hint on back of page.***

#5 **Number of two-element subsets.** Using induction, prove that, for any integer $n \geq 2$, the number of 2-element subsets of an n-element set is exactly $n(n-1)/2$.

Functions. *The following problems deal with the formal concept of a function (see p. 10/11 in the text), and the definition of a bounded function from* $\mathbb{R}$ *to* $\mathbb{R}$.

#6 **Definition of functions (1.45 in the text).** Determine whether the rules below define functions from $\mathbb{R}$ to $\mathbb{R}$; in each case explain why (or why not) the given rule defines a function from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = |x - 1|$ if $x < 4$ and $f(x) = |x| - 1$ if $x > 2$.

(c) $f(x) = ((x + 3)^2 - 9)/x$ if $x \neq 0$ and $f(x) = 6$ if $x = 0$.

(e) $f(x) = \sqrt{x^2}$ if $x \geq 2$, $f(x) = x$ if $0 \leq x \leq 4$ and $f(x) = -x$ for $x < 0$.

#7 **(1.49 in the text).** Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Determine if the following statements are true. Give proofs for true statements and counterexamples for false statements. (Note: Any counterexamples must be functions from $\mathbb{R}$ to $\mathbb{R}$; for example, the function $f(x) = 1/x$ cannot be used as a counterexample since it is not defined on all of $\mathbb{R}$.)

(a) If f and g are bounded, then $f + g$ is bounded.

(b) If f and g are bounded, then fg is bounded.

(c) If $f + g$ is bounded, then f and g are bounded.

(d) If fg is bounded, then f and g are bounded.

*** Turn page for comments, hints, and advice ***

[1] If you worked with another student or in a small group on this assignment, list the names of all students involved.

Hints, Advice, and Comments

Common errors in writing up induction proofs:

- **Undeclared/unquantified induction variable:** Starting out an induction step by saying "Suppose $P(n)$ is true for $n = k$", without saying what k is, what restrictions there have to be on k (e.g., $k \geq 2$), would make this step incomplete. There are many examples of "false" induction proofs that are due to errors of this type.
- **Not stating, explicitly and precisely, the statement/formula that one seeks to prove.** For example, using language like "suppose it is true for $n = k$", or "suppose $P(k)$", without explicitly and precisely stating what "it" or "$P(k)$" means, would be wrong.
- **Induction hypothesis not referenced.** The most crucial part of any induction proof is the place where the induction hypothesis (e.g., the case $n = k$ of $P(n)$, or the cases $n = k$ and $n = k - 1$, etc) is being used. This is the place that can make or break an induction argument, and an error there can have disastrous consequences. **Always clearly state, at the appropriate place in the induction step, where the induction hypothesis is being used.**
- **Insufficient/inappropriate base cases:** The base case(s) must be sufficient to "kick-start" the chain reaction that the induction proof represents. For example, an induction in which the $k + 1$ case depends on the cases k and $k - 1$ requires checking two consecutive n-values in the base step.
- **Base cases and induction step don't match up:** The first case of the induction step (i.e., the first k-value for which the induction step is claimed) must match up with the base case(s), so as not to leave a gap in the chain reaction. There are numerous examples of "false" induction proofs that result from carelessness in this regard.

Comments on specific problems.

- **Problems 1 and 2.** In these problems you have to first come up with a reasonable guess for the correct formula, then use induction prove that this formula holds for all $n \in \mathbb{N}$ To arrive at the correct guess, compute the first few terms until you see a pattern emerge. In the case of Problem 1 the pattern should be very obvious after four or five terms. For Problem 2, the pattern is also easy to spot if you write S_n as a rational number and look at the numerators and denominators in this number.
- **Problem 3.** Note that proving the two-sided inequality $1 \leq a_n \leq 2$ requires showing that both (1) $a_n \geq 1$ and (2) $a_n \leq 2$ hold. Thus, the statement $P(n)$ to be proved consists of two inequalities, (1) and (2). In the induction step you will need to use each of these inequalities (i.e., (1) and (2)) at some point. **Make sure to clearly indicate which of the inequalities you are using, and for which value of the index n you are using it.** You will need to use properties of inequalities between real numbers, such as the fact that, $a < b$ implies $1/a > 1/b$ whenever a and b are positive real numbers.

  As an interesting side note, a proof of (1) or (2) alone by induction will not succeed. (Try it and see what goes wrong!) One must simultaneously prove both inequalities in order for the induction step to succeed.
- **Problem 4.** Follow the argument used in class (see also Induction Worksheet II) for binary representations, i.e., representations as sums of distinct powers of 2. Here we want Fibonacci numbers instead of powers of 2 as the "building blocks" in our representations, but the basic approach is the same in both cases. As in the case of binary representations, the key here is to ensure that the Fibonacci numbers involved in the representation must be distinct, and you must be clearly explain/justify why they are distinct.