Practice problems Solutions

1. **Induction proofs, type I: Sum/product formulas:** The most common, and the easiest, application of induction is to prove formulas for sums or products of n terms. All of these proofs follow the same pattern.

   (a) $\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$
   (b) $\sum_{i=0}^n 2^i = 2^{n+1} - 1$ (sum of powers of 2)
   (c) $\sum_{i=0}^n r^i = \frac{1-r^{n+1}}{1-r}$ $(r \neq 1)$ (sum of finite geometric series)
   (d) $\sum_{i=0}^n i!i = (n+1)! - 1$.

   Solution: *All proofs follow the pattern illustrated by the sample proof (of the formula $\sum_{i=1}^n i = n(n+1)/2$). We will carry out the details for (a) and (d). The other formulas can be proved similarly. (Note that (b) is a special case of (c).)*

   Proof of (a): *We seek to show that, for all $n \in \mathbb{N}$,*

   $$(*) \qquad \sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

   Base case: *When $n = 1$, the left side of $(*)$ is $1 \cdot (1+1) = 2$, and the right side is $1 \cdot (1+1)(1+2)/3 = 2$, so both sides are equal and $(*)$ is true for $n = 1$.*

   Induction step: *Let $k \in \mathbb{N}$ be given and suppose $(*)$ is true for $n = k$. Then*

   $$\begin{aligned}\sum_{i=1}^{k+1} i(i+1) &= \sum_{i=1}^{k} i(i+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{(by induction hypothesis)} \\ &= \frac{(k+1)(k+2)(k+3)}{3}.\end{aligned}$$

   Thus, $(*)$ holds for $n = k+1$, and the proof of the induction step is complete.

   Conclusion: *By the principle of induction, it follows that $(*)$ is true for all $n \in \mathbb{N}$.*

   Proof of (d): *We seek to show that, for all $n \in \mathbb{N}$,*

   $$(*) \qquad \sum_{i=0}^n i!i = (n+1)! - 1.$$

   Base case: *When $n = 1$, the left side of $(*)$ is $0 + 1 \cdot 1! = 1$, and the right side is $(1+1)! - 1 = 1$, so both sides are equal and $(*)$ is true for $n = 1$.*

   Induction step: *Let $k \in \mathbb{N}$ be given and suppose $(*)$ is true for $n = k$. Then*

   $$\begin{aligned}\sum_{i=1}^{k+1} i \cdot i! &= \sum_{i=1}^{k} i \cdot i! + (k+1)(k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)! \quad \text{(by induction hypothesis)} \\ &= (k+1)!(k+2) - 1 \\ &= (k+2)! - 1.\end{aligned}$$

   Thus, (2) holds for $n = k+1$, and the proof of the induction step is complete.

   Conclusion: *By the principle of induction, $(*)$ is true for all $n \in \mathbb{N}$.*

2. **Induction proofs, type II: Inequalities:** A second general type of application of induction is to prove inequalities involving a natural number n. These proofs also tend to be on the routine side; in fact, the algebra required is usually very minimal, in contrast to some of the summation formulas.

   In some cases the inequalities don't "kick in" until n is large enough. By checking the first few values of n one can usually quickly determine the first n-value, say n_0, for which the inequality holds. Induction with $n = n_0$ as base case can then be used to show that the inequality holds for all $n > n_0$.

(a) $2^n > n$

(b) $2^n \geq n^2$ $(n \geq 4)$

(c) $n! > 2^n$ $(n \geq 4)$

(d) $(1-x)^n \geq 1 - nx$ $(0 < x < 1)$

(e) $(1+x)^n \geq 1 + nx$ $(x > 0)$

Solution: *We will give detailed proofs for (c), (d), (e). The other inequalities can be proved similarly.*

Proof of (c): *A direct check of the inequality for the first few values of n shows that the left-right pairs in the stated inequality are $(1,2),(2,4),(6,8),(24,16),(120,32)$. Thus, the inequality fails for $n = 1,2,3$, but holds for $n = 4,5$. This suggests that it indeed holds for all n from 4 onwards. We will prove this by induction, i.e., we will show that*

$$(*) \qquad n! > 2^n$$

holds for all $n \geq 4$.

Base case: *For $n = 4$, the left and right sides of $(*)$ are 24 and 16, respectively, so $(*)$ is true in this case.*

Induction step: *Let $k \geq 4$ be given and suppose $(*)$ is true for $n = k$. Then*

$$\begin{aligned}(k+1)! &= k!(k+1)\\ &> 2^k(k+1) \quad \textit{(by induction hypothesis)}\\ &\geq 2^k \cdot 2 \quad \textit{(since } k \geq 4 \textit{ and so } k+1 \geq 2))\\ &= 2^{k+1}.\end{aligned}$$

Thus, $(*)$ holds for $n = k+1$, and the proof of the induction step is complete.

Conclusion: *By the principle of induction, it follows that $(*)$ is true for all $n \geq 4$.*

Proof of (d) and (e): *We will prove that for any real number $x > -1$*

$$(*) \qquad (1+x)^n \geq 1 + nx.$$

holds for any $n \in \mathbb{N}$. This simultaneously proves both statements (d) and (e): (e) corresponds to the case $x > 0$, while (d) corresponds to the case $-1 < x < 0$ (with $x' = -x$ in place of x).

Base case: *For $n = 1$, the left and right sides of $(*)$ are both $1 + x$, so $(*)$ holds.*

Induction step: *Let $k \in \mathbb{N}$ be given and suppose $(*)$ is true for $n = k$ and any real number $x > -1$. We seek to show that $(*)$ holds for $n = k+1$ and any real number $x > -1$.*

Let $x > -1$ be given. Then

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x)\\ &\geq (1+kx)(1+x) \quad \textit{(by ind. hyp. and since } x > -1 \textit{ and thus } (1+x) > 0)\\ &= 1 + (k+1)x + kx^2 \quad \textit{(by algebra)}\\ &\geq 1 + (k+1)x \quad \textit{(since } kx^2 \geq 0).\end{aligned}$$

Hence $(*)$ holds for $n = k+1$, and the proof of the induction step is complete.

Conclusion: *By the principle of induction, it follows that $(*)$ holds for all $n \in \mathbb{N}$.*

3. **Induction proofs, type III: Extension of theorems from 2 variables to n variables:** Another very common and usually routine application of induction is to extend general results that have been proved for the case of 2 variables to the case of n variables. Below are some examples. In proving these results, use the case $n = 2$ as base case. To see how to carry out the general induction step (from the case $n = k$ to $n = k+1$), it may be helpful to first try to see how get from the base case $n = 2$ to the next case $n = 3$.

   (a) Show that if $x_1, \ldots, x_n$ are odd, then $x_1 x_2 \ldots x_n$ is odd. (Use the fact (proved earlier) that the product of 2 odd numbers is odd, as starting point, and use induction to extend this result to the product of n odd numbers.)

   Solution: *We will prove by induction on n the following statement:*

   $P(n)$: *If $x_1, \ldots, x_n$ are odd numbers, then $x_1 x_2 \ldots x_n$ is odd.*

   We will use the following fact (proved earlier):

   $(*)$ *If x and y are odd, then xy is odd.*

   Base case: *For $n = 1$, the product $x_1 \ldots x_n$ reduces to x_1, so is odd whenever x_1 is odd. Hence $P(1)$ is true.*

   Induction step.

- *Let $k \geq 1$, and suppose $P(k)$ is true, i.e., suppose that any product of k odd numbers is again odd.*
- *We seek to show that $P(k+1)$ is true, i.e., that any product of $k+1$ odd numbers is odd.*
- *Let $x_1, \dots, x_{k+1}$ be odd numbers.*
- *Applying the induction hypothesis to $x_1, \dots, x_k$, we obtain that the product $x_1 x_2 \dots x_k$ is odd.*
- *Since x_{k+1} is odd and, by $(*)$, the product of two odd numbers is again odd, it follows that $x_1 x_2 \dots x_{k+1} = (x_1 \dots x_k) x_{k+1}$ is odd.*
- *As $x_1, \dots, x_{k+1}$ were arbitrary odd numbers, we have proved $P(k+1)$, so the induction step is complete.*

Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.*

(b) Show that if a_i and b_i $(i = 1, 2, \dots, n)$ are real numbers such that $a_i \leq b_i$ for all i, then

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i.$$

(Use the fact (from Chapter 1) that $a \leq b$ and $c \leq d$ implies $a + c \leq b + d$.)

Solution: *We will prove by induction on n the following statement:*

$P(n)$: *For all real numbers a_i and b_i $(i = 1, \dots, n)$ such that $a_i \leq b_i$ for all i we have*

$$(*) \qquad \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i.$$

(Note that the quantifier "for all real numbers a_i and b_i" must be part of the induction statement we seek to prove.)

Base case: *For $n = 1$, the left and right sides are a_1 and b_1, respectively, and the inequality $(*)$ therefore follows from our hypothesis that $a_i \leq b_i$ for all $i = 1, \dots, n$. Hence $P(1)$ is true.*

Induction step:

- *Let $k \geq 1$, and suppose $P(k)$ is true, i.e., suppose that* for $n = k$ and any choice of real numbers $a_1, \dots, a_k$ and $b_1, \dots, b_k$ satisfying $a_i \leq b_i$ for each i, the inequality $(*)$ holds.
- *We seek to show that $P(k+1)$ is true, i.e., that* for $n = k+1$ any choice of real numbers $a_1, \dots, a_{k+1}$ and $b_1, \dots, b_{k+1}$ satisfying $a_i \leq b_i$ for each i, the inequality $(*)$ holds.
- *Let $a_1, \dots, a_{k+1}$ and $b_1, \dots, b_{k+1}$ be given real numbers such that $a_i \leq b_i$ for each i.*
- *Then*

$$\begin{aligned}
\sum_{i=1}^{k+1} a_i &= \sum_{i=1}^{k} a_i + a_{k+1} \\
&\leq \sum_{i=1}^{k} b_i + a_{k+1} \quad \text{(by induction hypothesis applied to } a_1, \dots a_k) \\
&\leq \sum_{i=1}^{k} b_i + b_{k+1} \quad \text{(by assumption } a_{k+1} \leq b_{k+1}) \\
&= \sum_{i=1}^{k+1} b_i.
\end{aligned}$$

- *Thus, $(*)$ holds for $n = k+1$ and the given numbers $a_1, \dots, a_{k+1}$ and $b_1, \dots, b_{k+1}$.*
- *Since the $a_1, \dots, a_{k+1}$ and $b_1, \dots, b_{k+1}$ were arbitrary real numbers satisfying $a_i \leq b_i$ for each i, we have obtained statement $P(k+1)$, and the proof of the induction step is complete.*

Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.*

(c) Show that if $x_1, \dots, x_n$ are real numbers, then

$$\left| \sin \left(\sum_{i=1}^{n} x_i \right) \right| \leq \sum_{i=1}^{n} |\sin x_i| .$$

(Use the trig identity for $\sin(\alpha + \beta)$.)

Solution: *We seek to prove by induction on n the following statement:*

$P(n)$: *For all real numbers $x_1, \dots, x_n$ we have*

$$(*) \qquad \left| \sin \left(\sum_{i=1}^{n} x_i \right) \right| \le \sum_{i=1}^{n} |\sin x_i| .$$

The key to the argument is the trig identity

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha,$$

which is valid for any real α and β. Since $|\cos x| \le 1$, this identity implies, via the triangle inequality,

$$(**) \qquad \begin{aligned} |\sin(\alpha+\beta)| &\le |\sin\alpha\cos\beta| + |\sin\beta\cos\alpha| \\ &\le |\sin\alpha| + |\sin\beta|. \end{aligned}$$

The inequality $(**)$ is the case $n = 2$ of the statement $(*)$ we seek to prove, and will be needed in the induction proof. (One could also use it as the base case of an induction proof that starts with $n = 2$, but it is easier to start the induction with $n = 1$, where the base case is trivial.)

Base case: *For $n = 1$, the left and right sides of $(*)$ are both equal to $|\sin x_1|$, so $(*)$ holds trivially in this case. Hence $P(1)$ is true.*
Induction step:

- *Let $k \ge 1$, and suppose $P(k)$ is true, i.e., suppose that $(*)$ holds for $n = k$ and any choice of real numbers $x_1, \dots, x_k$.*
- *We seek to show that $P(k+1)$ is true, i.e., that for any choice of real numbers $x_1, \dots, x_{k+1}$ the inequality $(*)$ holds.*
- *Let $x_1, \dots, x_{k+1}$ be given real numbers.*
- *Then*

$$\begin{aligned} \left| \sin \left(\sum_{i=1}^{k+1} x_i \right) \right| &= \left| \sin \left(\left(\sum_{i=1}^{k} x_i \right) + x_{k+1} \right) \right| \\ &\le \left| \sin \left(\sum_{i=1}^{k} x_i \right) \right| + |\sin x_{k+1}| \quad \textit{(by (**) with } \alpha = \sum_{i=1}^{k} x_i \textit{ and } \beta = x_{k+1}\textit{)} \\ &\le \sum_{i=1}^{k} |\sin x_i| + |\sin x_{k+1}| \quad \textit{(by induction hypothesis applied to } x_1,\dots,x_k\textit{)} \\ &= \sum_{i=1}^{k+1} |\sin x_i| . \end{aligned}$$

- *Thus, $(*)$ holds for $n = k+1$ and the given numbers $x_1, \dots, x_{k+1}$.*
- *Since the $x_1, \dots, x_{k+1}$ were arbitrary real numbers, we have obtained statement $P(k+1)$, and proof of the induction step is complete.*

Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.*

(d) Show that if $A_1, \dots, A_n$ are sets, then

$$(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c.$$

(This is a generalization of De Morgan's Law to unions of n sets. Use De Morgan's Law for two sets ($(A \cup B)^c = A^c \cap B^c$) and induction to prove this result.)

Solution: *We seek to prove by induction on n the following statement:*

$P(n)$: *For all sets $A_1, \dots, A_n$ we have*

$$(*) \qquad (A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c.$$

The key to the argument is two set version of De Morgan's Law:

$$(**) \qquad (A \cup B)^c = A^c \cap B^c,$$

which holds for any sets A and B.
Base case: *For $n = 1$, the left and right sides of $(*)$ are both equal to A_1^c, so $(*)$ holds trivially in this case. Hence $P(1)$ is true.*
Induction step:

- *Let* $k \geq 1$*, and suppose* $P(k)$ *is true, i.e., suppose that* $(*)$ *holds for* $n = k$ *and any sets* $A_1, \dots, A_k$.
- *We seek to show that* $P(k+1)$ *is true, i.e., that for any sets* $A_1, \dots, A_{k+1}$, $(*)$ *holds.*
- *Let* $A_1, \dots, A_{k+1}$ *be given sets.*
- *Then*

$$\begin{aligned}
(A_1 \cup \cdots \cup A_{k+1})^c &= ((A_1 \cup \cdots \cup A_k) \cup A_{k+1})^c \\
&= (A_1 \cup \cdots \cup A_k)^c \cap A_{k+1}^c \quad \textit{(by } (**) \textit{ with } A = (A_1 \cup \cdots \cup A_k) \textit{ and } B = A_{k+1}) \\
&= (A_1^c \cap \cdots \cap A_k^c) \cap A_{k+1}^c \quad \textit{(by induction hypothesis applied to } A_1, \dots, A_k) \\
&= A_1^c \cap \cdots \cap A_k^c \cap A_{k+1}^c.
\end{aligned}$$

- *Thus,* $(*)$ *holds for* $n = k+1$ *and the given sets* $A_1, \dots, A_{k+1}$.
- *Since the* $A_1, \dots, A_{k+1}$ *were arbitrary sets, we have obtained statement* $P(k+1)$*, and the proof of the induction step is complete.*

Conclusion: *By the principle of induction, it follows that* $P(n)$ *is true for all* $n \in \mathbb{N}$.