Worksheet: Sum/Product Notations

About this worksheet

In induction proofs you'll frequently come across formulas involving sum ($\sum$) and product ($\prod$) notations. These notations are abbreviations for repeated sums or repeated products, and it is important to learn how to work with them, and get comfortable and secure in using these notation. This worksheet is intended to practice these skills. Here are some tips:

- **Be familiar with the standard summation formulas:** The following are formulas that you should know. (All are easy to prove by induction.)

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{(Gauss' formula for sum of first } n \text{ natural numbers)}$$

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \quad (r \neq 1) \quad \text{(finite geometric series)}$$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \quad (|r| < 1) \quad \text{(infinite geomeric series)}$$

- **Write out sums or products given in $\sum$ and $\prod$ notation explicitly, using dots (...):** This can help avoid mistakes. For example, consider the sum $\sum_{i=0}^{n} i$ and the product $\prod_{i=0}^{n} i$. Writing out the sum explicitly, we get $\sum_{i=0}^{n} i = 0 + 1 + \cdots + n = \dfrac{n(n+1)}{2}$, as expected. By contrast, doing the same with the product $\prod_{i=0}^{n} i$ gives $\prod_{i=0}^{n} i = 0 \cdot 1 \cdot \cdots \cdot n = 0$, which shows that the product is in fact equal to 0, contrary to what one might have expected! The term $i = 0$ had no influence on the sum but it caused the product to be 0.

- **Double sums are analogous to double integrals:** In particular, the order of summation can be changed in the same way as the order of integration can be changed in double integrals, as long as one takes care in changing the summation limits accordingly. For example, the double sum $\sum_{i=1}^{n} \sum_{j=1}^{i}$ becomes $\sum_{j=1}^{n} \sum_{i=j}^{n}$ after interchanging the order of summation.

- **Additional resources.** This material is briefly covered in Chapter 3 (see p. 53/54). The text "Discrete Mathematics" by Rosen, which is on course reserve in the library, provides a more extensive treatment, with numerous practice problems.

***** Practice problems on back of page *****

Practice Problems

1. Write the following sums and products explicitly without using sum/product notations, and then evaluate them. (Here n denotes a positive integer and c is a positive constant. (In all cases there is a simple formula for the given sum or product.)

   (a) $\displaystyle\sum_{i=1}^{n} c$

   (b) $\displaystyle\prod_{i=1}^{n} c$

   (c) $\displaystyle\sum_{i=1}^{n}(n-i)$

   (d) $\displaystyle\prod_{i=1}^{n}(n-i)$

   (e) $\displaystyle\prod_{i=1}^{n} i^n$

   (f) $\displaystyle\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)$

   (g) $\displaystyle\prod_{i=1}^{n}\frac{n+i}{i}$

   (h) $\displaystyle\sum_{i=1}^{n}\sum_{j=i}^{n} 1$

   (i) $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)$

2. Express the following sums in summation notation and evaluate using appropriate summation formulas.

   (a) $3+7+11+\cdots+(4n-1)$

   (b) $(1/3)^2-(1/3)^4+(1/3)^6-(1/3)^8+\cdots$

   (c) $y^n+xy^{n-1}+x^2y^{n-2}+\cdots+x^{n-1}y+x^n$ $(x, y \neq 0,\ x \neq y)$.