Sequences and Limits: Definitions and Theorems

Basic definitions

- **Sequence:** Formally, a sequence[1] is a function $f : \mathbb{N} \to \mathbb{R}$. Since specifying such a function is equivalent to specifying the values $f(1), f(2), f(3), \ldots$, we can think of a sequence as given by a (countable) list of real numbers, $a_1, a_2, a_3, \ldots$, where $a_n = f(n)$. We write the sequence as $\{a_n\}_{n=1}^{\infty}$, or $\{a_n\}$.
- **Bounded sequence:** A sequence $\{a_n\}$ is called bounded if there exists $M \in \mathbb{R}$ such that $|a_n| \le M$ for all $n \in \mathbb{N}$. (This is consistent with the definition of a bounded function if the sequence is regarded as a function from $\mathbb{N}$ to $\mathbb{R}$.)
- **Monotone sequence:** A sequence $\{a_n\}$ is called monotone if it satisfies either $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$ ("**nondecreasing**") or $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$ ("**nonincreasing**").

Limits and convergence

- **Limit of a sequence:** Given a *real number*[2] L, we say that a sequence $\{a_n\}$ **has limit** L (or **converges to** L), and write $\lim_{n\to\infty} a_n = L$ (or $a_n \to L$), if the following holds:

  $$\boxed{\text{For every } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } n \ge N \text{ implies } |a_n - L| < \epsilon.}$$

  A sequence $\{a_n\}$ is said to be **convergent** if there exists a real number L such that $\lim_{n\to\infty} a_n = L$.
- **Cauchy sequence:** A sequence $\{a_n\}$ is said to be a **Cauchy sequence** if the following holds.

  $$\boxed{\text{For every } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } m, n \ge N \text{ implies } |a_m - a_n| < \epsilon.}$$

Theorems

- **Monotone Convergence Theorem:** $\boxed{\text{Every bounded and monotone sequence is convergent.}}$
- **Cauchy Convergence Criterion:** $\boxed{\text{A sequence is convergent if and only if it is a Cauchy sequence.}}$
- **Algebraic Properties of Limits:** Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$, denote arbitrary sequences, and let L and M denote real numbers.
  1. (Scaling) If $\lim_{n\to\infty} a_n = L$ and $c \in \mathbb{R}$, then $\lim_{n\to\infty} (ca_n) = cL$.
  2. (Sums and Differences) If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $\lim_{n\to\infty} (a_n+b_n) = L+M$ and $\lim_{n\to\infty} (a_n-b_n) = L - M$.
  3. (Products) If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $\lim_{n\to\infty} (a_n b_n) = LM$.
  4. (Quotients) If $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, $b_n \ne 0$ for all $n \in \mathbb{N}$, and $M \ne 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$.

Further resources

In the text, this material is covered in Chapters 13 and 14 on pp. 259–263 and 271–276. See also the section "How to approach problems" at the end of Chapter 14 (pp. 284–286) some excellent tips and advice on epsilonics.

The basic $\epsilon - \delta$ definition of a limit, and the main theorems about limits, are also covered in Stewart's calculus text; see Sections 11.1 (limits of sequences) and 2.4 (limits of functions).

[1] Since we will only consider sequences of real numbers, we will use the word "sequence" to mean a sequence of real numbers. Sometimes it is convenient to allow the indexing of a sequence to start with $n = 0$ instead of $n = 1$; the definition extends in an obvious manner to such a situation, but make sure to use an explicit notation like $\{a_n\}_{n=0}^{\infty}$ rather than the abbreviated form $\{a_n\}$ whenever the index set is not the standard set $\mathbb{N}$.

[2] In calculus texts, the limit concept is often extended to allow ∞ as a limit, but we do not consider this case here. **Thus, convergence of a sequence $\{a_n\}$ always means convergence to a real number L.**