Set-theoretic Proofs

Proofs of set-theoretic relations and equalities among sets are among the simplest types of proofs and therefore present an excellent opportunity to familiarize yourself with the logical structure of a mathematical proof and to practice proofwriting in a particularly simple context.

Tools and prerequisites

You should be familiar with the basic set-theoretic operations and relations and know their *precise* definitions. The following table summarizes the key definitions and shows how to correctly "unwind" expressions like "$x \in A \cup B$", "$x \in A - B$", etc.

$$x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B \tag{1}$$
$$x \notin A \cup B \Leftrightarrow x \notin A \text{ and } x \notin B \tag{2}$$
$$x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B \tag{3}$$
$$x \notin A \cap B \Leftrightarrow x \notin A \text{ or } x \notin B \tag{4}$$
$$x \in A - B \Leftrightarrow x \in A \text{ and } x \notin B \tag{5}$$
$$x \notin A - B \Leftrightarrow x \notin A \text{ or } x \in B \tag{6}$$
$$x \in A \times B \Leftrightarrow \ x = (a, b) \text{ for some } a \in A \text{ and } b \in B \tag{7}$$
$$A \subseteq B \Leftrightarrow \text{If } x \in A, \text{ then } x \in B. \tag{8}$$
$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A. \tag{9}$$

Proving a subset relation "$S \subseteq T$"

Using the definition (8) of a subset relation yields a proof of the following structure:

Let $x \in S$.
...
[Logical deductions]
...
Therefore $x \in T$.
This proves that $S \subseteq T$.

Proving a set equality "$S = T$"

By definition (9), equality between two sets S and T is equivalent to the subset relations (i) $S \subseteq T$ and (ii) $T \subseteq S$ both being true. Thus, the proof of $S = T$, breaks down into two parts, (i) the proof of $S \subseteq T$, and (ii) the proof of $T \subseteq S$, each of which follows the above template.

Example 1: Proof of $A - (A - B) \subseteq B$ (where A and B are arbitrary sets)

We apply the above template with $A - (A - B)$ as the set S, and B as the set T.

- Let $x \in A - (A - B)$.
- Then $x \in A$ and $x \notin A - B$, by the definition of a set difference (see (5)).
- By the definition of a set difference (in the negated form (6)), "$x \notin A - B$" is equivalent to "$x \notin A$ or $x \in B$".
- Therefore we have $x \in A$ and ($x \notin A$ or $x \in B$).
- Since $x \in A$, the first of the two alternatives in "$x \notin A$ or $x \in B$" is impossible, so the second alternative must hold, i.e., $x \in B$.
- Thus we have $x \in A$ and $x \in B$.
- Hence $x \in B$.
- This proves that $A - (A - B) \subseteq B$.

Remark: The reverse inclusion, $B \subseteq A - (A - B)$, does not hold in general. Therefore the sets $A - (A - B)$ and B are, in general, not equal. To prove this, we exhibit a **counterexample**: Let $A = \{1, 2\}$, $B = \{2, 3\}$. Then $A - B = \{1\}$, $A - (A - B) = \{2\}$, while $B = \{2, 3\}$. Thus, $A - (A - B)$ is a *proper* subset of B, but not equal to B.

To *find* a counterexample, a Venn diagram can be helpful. In the above case, a Venn diagram suggests that if B overlaps with A, but is not entirely contained in A, the reverse inclusion does not hold, thus leading to the above construction. Keep in mind that, while Venn diagrams can be useful in visualizing set relations and pointing to possible counterexamples, **a Venn diagram does not constitute a proof**. Once you have found a likely candidate for a counterexample (such as the sets $A = \{1, 2\}$ and $B = \{2, 3\}$ above), you still need to prove that this is indeed a counterexample to the relation $B = A - (A - B)$ by explicitly evaluating the sets on the left and on the right and showing that they are not equal.

Example 2: Proof of $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (where A, B, C are arbitrary sets)

We apply the above template with $S = A \cup (B \cap C)$ and $T = (A \cup B) \cap (A \cup C)$.

- Let $x \in A \cup (B \cap C)$.
- Then $x \in A$ or $x \in B \cap C$, by the definition of a union (see (1)).
- Therefore $x \in A$ or ($x \in B$ and $x \in C$), by the definition of an intersection (see (3)).
  - In the first case (i.e., the case "$x \in A$"), we have $x \in A \cup B$ and $x \in A \cup C$, by the definition of a union. By the definition of an intersection it follows that $x \in (A \cup B) \cap (A \cup C)$.
  - In the second case (i.e., the case "$x \in B$ and $x \in C$"), we have $x \in A \cup B$ and $x \in A \cup C$, by the definition of a union. As before, by the definition of an intersection it follows that $x \in (A \cup B) \cap (A \cup C)$.
- Thus in either case we have $x \in (A \cup B) \cap (A \cup C)$.
- This proves that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Remark: In this example, the reverse inclusion relation does hold, and it can be proved in much the same way as above. Therefore, we have the set equality $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, for any sets A, B, C. (This shows that the union and intersection of sets satisfy a distributive law.)