Name:

Collaborator(s)[1]:

Math 347, Prof. Hildebrand, Summer 2019

HW 10, DUE IN CLASS MONDAY, 7/29

- **Grading:** 15 points total, broken down as follows:
  - **Presentation/effort:** **3 points**
  - **Graded problems:** **12 points**

#1 **(14.32 in the text) The Runaway Train Puzzle.** The following is a fun (and not particularly difficult) problem that can be done in two completely different ways, one involving infinite series and another, slick approach that requires almost no calculations. Try to find both solutions, and write them up as clearly as you can. The quality of the write-up will be taken into account in the grading.

> *A train is moving towards a wall at a speed of* 100 *miles per hour. When it is two miles from the wall, a fly begins to fly repeatedly between the train and the wall at the speed of* 200 *miles per hour. Determine how far the fly travels before it gets smashed.*

#2 **Proof-writing practice: Divisibility and congruence properties.** For each of the following statements either give a proof or a counterexample. If the statement is false, a counterexample is sufficient. For true statements, give a careful step-by-step proof **using only the official definitions** of divisibility (i.e., "$a \mid b$" means "there exists $m \in \mathbb{Z}$ such that $b = ma$") and congruences (i.e., "$a \equiv b \mod m$" means "there exists $k \in \mathbb{Z}$ such that $a = b + km$"). **Your proofs should not use any properties or theorems about congruences and divisibility.**
In all statements, a, b, c, d are assumed to be integers, and m is a natural number.

(a) If $d \mid a$ and $d \mid b$, then $d \mid ax + by$ for any $x, y \in \mathbb{Z}$.

(b) If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

(c) If $a \mid b$ and $c \mid d$, then $(a + c) \mid (b + d)$.

(d) If $a \mid bc$, then $a \mid b$ or $a \mid c$.

(e) If $a \equiv b \mod m$ and $c \equiv d \mod m$, then $ac \equiv bd \mod m$.

(f) If $a \equiv b \mod m$ then $k^a \equiv k^b \bmod m$ for any $k \in \mathbb{N}$.

#3 **Congruence Magic.** The following problems should be done by hand (no calculator), using congruences, and their properties. If approached the right way, the calculations should take only a line or two, and can be done easily by hand. (You are welcome to **check** your result using a calculator, or WolframAlpha, `www.wolframalpha.com`, but you must come up with a solution based on congruences.)

(a) Find the last decimal digit of 347^{101}.

(b) Find the remainder of 347^{101} when divided by 101. (Hint: Use Fermat's Little Theorem, but be careful: The exponent, 101, is not quite the same as in the theorem ...)

(c) **(7.9 in text)** Using Fermat's Little Theorem, find a number between 0 and 12 that is congruent to 2^{100} modulo 13.

(d) **(7.6 in text)** Find the last digit in the base 8 expansion of (i) 9^{1000}, (ii) 10^{1000}, (iii) 11^{1000}.

[1] If you worked with another student or in a small group on this assignment, list the names of all students involved.