# Number Theory I: Worksheet

1. **Warmup: Practice with definitions.** The following quickies test your precise understanding of the definitions of divisibility and primes. In answering the questions, make sure to use the appropriate "official" definition from class (as given on the "Number Theory I" class handout), not any preconceived notions about primes and divisibilty, what your high school teacher told you, etc.

   (a) **Primes and composite numbers.** For each of the following numbers determine if it is prime, composite, or neither. For composite numbers, write out their prime factorization.
   $2, \quad -2, \quad 1, \quad 12, \quad 120$

   (b) **Divisibility properties of $1$ and $0$:**
   Which $n \in \mathbb{Z}$ satisfy (i) $1 \mid n$, (ii) $n \mid 1$, (iii) $n \mid 0$ ?

   (c) **Divisors of primes and prime powers:**
   (i) Let $p$ be a prime. List all *positive* divisors of $p$. How many positive divisors are there?
   (ii) Let $k$ be a natural number and $p$ be a prime. List all *positive* divisors of $p^k$. How many positive divisors are there?

2. **Famous numbers, I: Perfect numbers.** A number $n$ is perfect if it is equal to the sum of its *positive* divisors *excluding $n$ itself*. For example, the positive divisors of $6$ are $1, 2, 3, 6$. If we remove $6$ from the list and add up the remaining divisors, we get $1 + 2 + 3 = 6$, so $6$ is perfect.

   (a) List all positive divisors of $28$, and show that $28$ is a perfect number.

   (b) More generally, consider numbers of the form $n = 2^k \cdot (2^{k+1} - 1)$. (The above examples, $n = 6$ and $n = 28$, correspond to $k = 1$ and $k = 2$.) Show that such an $n$ is perfect whenever $2^{k+1} - 1$ is a prime number.
   **Remark:** This is a famous result of Euclid and Euler. In fact, Euler showed the only *even* perfect numbers are those of the above form. Primes of the form $2^k - 1$ form another famous class of numbers, namely *Mersenne primes.*

3. **Proof-writing practice: Divisibility properties.** For each of the following statements, give a careful proof *using only the definition of divisibility* ("$a \mid b$" means "there exists $m \in \mathbb{Z}$ such that $b = am$"). Some of these problems will appear in the homework. Others will be worked out in class.

   (In all statements, $a, b, c, d, \dots$ are assumed to be non-zero integers.)

   (a) If $a \mid b$ and $b \mid c$, then $a \mid c$.
   (b) If $d \mid a$ and $d \mid b$, then $d \mid a + b$ and $d \mid a - b$.
   (c) If $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.
   (d) (HW) If $d \mid a$ and $d \mid b$, then $d \mid ax + by$ for any $x, y \in \mathbb{Z}$.
   (e) (HW) If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

4. **Proof-writing practice: Proof of the Existence Part of FTA.** Using strong induction prove that every integer $n \geq 2$ has a prime factorization. (Pay particular attention to the write-up; be sure to include an appropriate base case, and clearly state the assumptions in the induction step.)