Problem 1. Definitions and theorems. State the requested definition, theorem, or property. Be sure to use correct notation and include any necessary quantifiers in the appropriate order.

(a) Give a precise definition of the **graph** of a function $f : A \to B$, using correct set-theoretical notation.

(b) Without using words of negation state the definition of "f is **not increasing**" (where f is a function from $\mathbb{R}$ to $\mathbb{R}$). Write your answer in English, i.e., without using logical symbols.

(c) A function f from $\mathbb{R}$ to $\mathbb{R}$ is **not bounded** if ...

(d) Two sets A and B are said to have the **same cardinality** if ...

Problem 2. Short answers, I. For the following questions, give an answer and a brief justification.

(a) Let $f(x) = |x - 1|$ if $x < 4$, and $f(x) = |x| - 1$ if $x > 2$. Determine whether f is a **function from $\mathbb{R}$ to $\mathbb{R}$**, and justify your answer (i.e., explain why, or why not, f is a function from $\mathbb{R}$ to $\mathbb{R}$).

(b) Let $f(p/q) = 1/q$ if $p \in \mathbb{Z}, q \in \mathbb{N}$, and $f(x) = 0$ if x is irrational. Determine whether f is a **function from $\mathbb{R}$ to $\mathbb{R}$**, and justify your answer (i.e., explain why, or why not, f is a function from $\mathbb{R}$ to $\mathbb{R}$).

(c) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ that is unbounded, but not surjective? If so, give a *specific* example of such a function; if not, explain why no such function exists.

(d) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ that has an inverse, but is not injective? If so, give a *specific* example of such a function; if not, explain why no such function exists.

Problem 3. Short answers, II. For the following questions, give an answer and a brief justification. For questions about cardinality and countability you can use (without proof) the following:
(i) Known results about the countability or uncountability of the following **specific** sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and the set of infinite binary sequences.
(ii) Any of the **general** results and properties about countable sets given on the cardinality handout. *If you use one of these results/properties, say so and indicate which property you are using.*

(a) Does there exist a bijection between $\mathbb{Z}$ (the set of all integers) and $\mathbb{Z}_{odd}$ (the set of odd integers)? If so, give a *specific* example of such a bijection; if not, explain why no such bijection exists.

(b) Does there exist an infinite set A such that $A \times A$ has the same cardinality as A? If so, give a *specific* example of such a set, and explain briefly why this set has the required property. If not, explain why no such set exists.

(c) Does the set $\mathbb{R}$ have the same cardinality as the set $\mathbb{Q} \times \mathbb{Q}$? Explain clearly why, or why not, the two sets have the same cardinality.

Problem 4. Let the sequence a_n be defined by $a_1 = a_2 = a_3 = 1$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 4$. Using induction, prove that $a_n < 2^n$ for all $n \in \mathbb{N}$.
Pay particular attention to the write-up, be sure to include all steps, any necessary quantifiers, and provide appropriate justifications for each step (e.g., "by induction hypothesis", "by formula (1)", "by algebra", "by the AGM inequality")

Problem 5. Let A, B, C be sets, $f : A \to B$, and $g : B \to C$ be functions, and let $h : A \to C$ be defined by $h(x) = g(f(x))$ for $x \in A$. For each of the following statements, determine if it is true. If the statement is true, give a careful, step-by-step, proof; be sure to use proper mathematical notation and terminology, and include any necessary quantifiers, connecting words, and justifications. If it is false, give a *specific* counterexample.

(a) If f and g are surjective, then h is surjective.

(b) If h is surjective, then f is surjective.

Problem 6. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $f(x, y) = (x + y, x - y)$.

(a) Determine whether f is injective. If it is, give a careful, step-by-step, proof of the injectivity; if it is not, explain why.

(b) Determine whether f is surjective. If it is, give a careful, step-by-step, proof of the surjectivity; if it is not, explain why.