Practice Problems—Solutions

1. **Recurrences:** The first few problems deal with properties of the Fibonacci sequence and related recurrence sequences. The Fibonacci sequence is defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Its first few terms are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$.

   In the following problems, use an **appropriate** form of induction (standard induction or strong induction) to establish the desired properties and formulas. (Note that some of these problems require only ordinary induction.)

   (a) **Fibonacci sums:** Prove that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ for all $n \in \mathbb{N}$.

      Solution: *We seek to show that, for all $n \in \mathbb{N}$,*

   $$(*) \qquad\qquad \sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

      Base case: *When $n = 1$, the left side of $(*)$ is $F_1 = 1$, and the right side is $F_3 - 1 = 2 - 1 = 1$, so both sides are equal and $(*)$ is true for $n = 1$.*
      Induction step: *Let $k \in \mathbb{N}$ be given and suppose $(*)$ is true for $n = k$. Then*

   $$\sum_{i=1}^{k+1} F_i = \sum_{i=1}^{k} F_i + F_{k+1}$$
   $$= F_{k+2} - 1 + F_{k+1} \quad \text{(by ind. hyp. $(*)$ with $n = k$)}$$
   $$= F_{k+3} - 1 \quad \text{(by recurrence for F_n)}$$

      Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.
      Conclusion: *By the principle of induction, it follows that $(*)$ is true for all $n \in \mathbb{N}$.*

      Remark: *Here standard induction was sufficient, since we were able to relate the $n = k + 1$ case directly to the $n = k$ case, in the same way as in the induction proofs for summation formulas like $\sum_{i=1}^{n} i = n(n+1)/2$. Hence, a single base case was sufficient.*

   (b) **Fibonacci matrix:** Let A be the 2×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Let $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ denote this matrix multiplied by itself n times. Find and prove a formula for the four entries a_n, b_n, c_n, d_n.

      Solution: *We will show that, for all $n \in \mathbb{N}$,*

   $$(*) \qquad\qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

      (For convenience, we define $F_0 = 0$; with this definition, the recurrence relation $F_n = F_{n-1} + F_{n-2}$ holds for all $n \geq 2$ and the above matrix is well defined for all $n \geq 1$.)
      Base case: *When $n = 1$, the four entries of the matrix on the right are $F_2 = 1$, $F_1 = 1$, $F_1 = 1$, and $F_0 = 0$, so $(*)$ holds in this case.*
      Induction step: *Let $k \in \mathbb{N}$ be given and suppose $(*)$ holds for $n = k$. Then*

   $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
   $$= \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{(by $(*)$ with $n = k$)}$$
   $$= \begin{pmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{pmatrix} \quad \text{(by matrix multiplication)}$$
   $$= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} \quad \text{(by recurrence for F_n).}$$

      Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.
      Conclusion: *By the principle of induction, it follows that $(*)$ is true for all $n \in \mathbb{N}$.*

   (c) **Odd/even Fibonacci numbers:** Prove that the Fibonacci numbers follow the pattern odd,odd,even: that is, show that for any positive integer m, F_{3m-2} and F_{3m-1} are odd and F_{3m} is even.

Solution: *We will use induction to show that the following statement holds for all $m \in \mathbb{N}$:*

$P(m)$: F_{3m-2} and F_{3m-1} are odd, and F_{3m} is even.

Base case: *When $m = 1$, the three Fibonacci numbers appearing in $P(m)$ are $F_1 = 1$, $F_2 = 1$, and $F_3 = 2$, and thus are of the required parity. Hence $P(1)$ is true.*
Induction step: *Let $k \in \mathbb{N}$ be given and suppose $P(m)$ is true for $m = k$. Thus, F_{3k-2} and F_{3k-1} are odd and F_{3k} is even.*
By the recurrence for F_n, we have $F_{3k+1} = F_{3k} + F_{3k-1}$. Hence F_{3k+1} is the sum of an even number F_{3k}, and an odd number, F_{3k-1}, and therefore odd.
Similarly, $F_{3k+2} = F_{3k+1} + F_{3k}$, so F_{3k+2} is the sum of an odd number, F_{3k+1}, and an even number, F_{3k}, and hence odd.
Finally, $F_{3k+3} = F_{3k+2} + F_{3k+1}$, so F_{3k+3} is the sum of two odd numbers and hence even.
Altogether, we have shown that, of the three numbers $F_{3k+1}, F_{3k+2}, F_{3k+3}$, the first two are odd and the last one is even. Thus $P(m)$ holds for $m = k + 1$, and the proof of the induction step is complete.
Conclusion: *By the principle of induction, it follows that $P(m)$ is true for all $m \in \mathbb{N}$.*

Alternative argument: *The above proof lumps together groups of three consecutive Fibonacci numbers and establishes the desired parity properties simultaneously for all three numbers. Alternatively, one can treat the sequences F_{3m-2}, F_{3m-1}, and F_{3m} separately using the following identity, valid for all $n \geq 4$:*

$$F_n = F_{n-1} + F_{n-2} = (F_{n-2} + F_{n-3}) + F_{n-2} = 2F_{n-2} + F_{n-3}.$$

It follows from this identity that if F_{n-3} is odd, then so is F_n, and if F_{n-3} is even, then so is F_n. Using induction with $n = 1$ as the base case then shows that the numbers $F_1 = 1, F_4, F_7, \ldots$ are all odd. With $n = 2$ as base case one gets that $F_2 = 1, F_5, F_8, \ldots$ are all odd, and taking $n = 3$ as base case shows that $F_3 = 2, F_6, F_9, \ldots$ are all even.

(d) **Inequalities for recurrence sequencs:** Let the sequence T_n ("Tribonacci sequence") be defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Prove that

$(*)$ $T_n < 2^n$

holds for all $n \in \mathbb{N}$.

Solution: *We will prove $(*)$ by strong induction.*
Base step: *For $n = 1, 2, 3$, T_n is equal to 1, whereas the right-hand side of $(*)$ is equal to $2^1 = 2$, $2^2 = 4$, and $2^3 = 8$, respectively. Thus, $(*)$ holds for $n = 1, 2, 3$.*
Induction step: *Let $k \geq 3$ be given and suppose $(*)$ is true for all $n = 1, 2, \ldots, k$. Then*

$$\begin{aligned}
T_{k+1} &= T_k + T_{k-1} + T_{k-2} \quad \text{(by recurrence for } T_n) \\
&< 2^k + 2^{k-1} + 2^{k-2} \quad \text{(by strong ind. hyp. } (*) \text{ with } n = k, k-1, \text{ and } k-2) \\
&= 2^{k+1} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \right) \\
&= 2^{k+1} \frac{7}{8} < 2^{k+1}.
\end{aligned}$$

Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.
Conclusion: *By the strong induction principle, it follows that $(*)$ is true for all $n \in \mathbb{N}$.*

2. **Representation problems.** One of the main applications of strong induction is to prove the existence of representations of integers of various types. In these applications, strong induction is usually needed in its full force, i.e., in the induction step, one needs to assume that all predecessor cases $n = 1, 2, \ldots, k$.

(a) **The postage stamp problem:** Determine which postage amounts can be created using the stamps of 3 and 7 cents. In other words, determine the exact set of positive integers n that can be written in the form $n = 3x + 7y$ with x and y nonnegative integers. (Hint: Check the first few values of n directly, then use strong induction to show that, from a certain point n_0 onwards, all numbers n have such a representation.)

Solution: *A quick direct check shows that the positive numbers $n < 15$ that have a representation $n = 3x + 7y$ with $x, y \in \mathbb{N} \cup \{0\}$) are exactly $3, 6, 7, 9, 10, 12, 13, 14$. We now use strong induction to show that from 12 onwards every integer has a representation in the above form. In other words, we will prove that the following statement holds for all $n \geq 12$:*

$(P(n))$ $\qquad\qquad\qquad$ *n has a representation $(*)$ $n = 3x + 7y$ with $x, y \in \mathbb{N} \cup \{0\}$*

Base case: *For $n = 12, 13, 14$, the representations $12 = 3 \cdot 4$, $13 = 3 \cdot 2 + 7$ and $14 = 7 \cdot 2$ show that $P(n)$ is true.*
Induction step: *Let $k \geq 14$ be given and suppose $P(k')$ is true for all k' with $k' = 12, 13, \ldots, k$, i.e., suppose that all such k' have a representation in the form $(*)$. We seek to show that $k + 1$ also has a representation of this form.*
Write $k + 1 = 3 + k'$, so that $k' = k - 2$. Note that $k' \leq k$ and also $k' \geq 12$ since we assumed $k \geq 14$. Thus, we can apply the strong induction hypothesis to k' and obtain a representation

$$k' = 3x + 7y,$$

where $x, y \in \mathbb{N} \cup \{0\}$. Adding 3 to both sides of this representation, we get

$$k + 1 = k' + 3 = 3x + 7y + 3 = 3(x + 1) + 7y,$$

which is a representation of the desired form for $k + 1$. Hence $P(k + 1)$ is true, and the proof of the induction step is complete.
Conclusion: *By the strong induction principle, it follows that $P(n)$ is true for all $n \geq 12$.*

Remark: *Note that, in the induction step, in order to be able to apply the induction hypothesis with $k' = k - 2$ we need to ensure that k' is at least 12. This in turn requires k to be at least 14 in the induction step, and the cases $k = 12, 13, 14$ to be treated as base cases.*

(b) **Binary representation:** Using strong induction prove that every positive integer n can be represented as a sum of *distinct* powers of 2, i.e., in the form $n = 2^{i_1} + \cdots + 2^{i_h}$ with integers $0 \leq i_1 < \cdots < i_h$. (Hint: To ensure distinctness, use the *largest* power of 2 as the first "building block" in the induction step.)

Solution: *We will prove by strong induction that the following statement holds for all $n \in \mathbb{N}$.*

$(P(n))$ $\qquad$ *n has a representation $(*)$ $n = 2^{i_1} + \cdots + 2^{i_h}$ with distinct integers $i_1, \ldots, i_h \in \mathbb{N} \cup \{0\}$.*

Base case: *The integer $n = 1$ has the representation $1 = 2^0$, which is of the desired form. Hence $P(n)$ holds for $n = 1$.*
Induction step: *Let $k \geq 1$ be given and suppose $P(n)$ is true for all positive integers $n \leq k$, i.e., suppose that all such n have a representation in the form $(*)$. We seek to show that $n = k + 1$ also has a representation of this form.*
Let 2^m be the largest (integer) power of 2 that satisfies $2^m \leq k + 1$.
If $2^m = k + 1$, then $k + 1$ has a representation of the desired form (namely as a sum of a single power of 2, 2^m), and we are done.
If $2^m < k + 1$, we let $k' = k + 1 - 2^m$. Since $2^m \geq 1$ and $2^m < k + 1$, k' is an integer with $1 \leq k' \leq k$. Hence we can apply the strong induction hypothesis to k' and obtain a representation of k' as a sum of distinct powers of 2.
Adding 2^m to this representation gives a representation of $k + 1 = k' + 2^m$ as a sum of powers of 2. To complete the proof of the induction step, we still need to show that the powers of 2 involved here are distinct.
Since the powers of 2 representing k' were already distinct, it suffices to show that the added power 2^m cannot occur among the powers in the representation for k'. To do this, we exploit the fact that 2^m was chosen as the largest power of 2 below $k + 1$. Thus, we have

$$2^m < k + 1 < 2^{m+1}.$$

Subtracting 2^m from both sides, we get

$$0 < k + 1 - 2^m < 2^{m+1} - 2^m = 2^m,$$

and since $k' = k + 1 - 2^m$, it follows that k' is strictly less than 2^m, and so 2^m cannot occur in the representation for k'. This is what we wanted to show.
Thus, the representation for $k + 1$ that we obtained is indeed a representation of the desired form and the proof of the induction step is complete.
Conclusion: *By the strong induction principle, it follows that $P(n)$ is true for all $n \geq 1$.*

Remarks: *The argument used above in the induction step is called a "greedy" algorithm: In constructing a binary representation for $k+1$, one starts out by using the largest possible "building block" (namely, the largest power of 2 that is $\leq k+1$), and then uses the strong induction hypothesis to make up for the left over part.*
An alternative, but less flexible, approach to the induction step is as follows: If $k+1$ is even, then $k+1 = 2k'$, where k' is an integer in the range $1 \leq k' \leq k$. By the strong induction hypothesis k' has a representation as sum of distinct powers of 2, and multiplying this representation by 2 gives a representation of the desired form for $k+1$. If $k+1$ is odd, a similar argument based on the representation $k+1 = 2k'+1$ yields the same conclusion. The latter approach, however, relies heavily on specific arithmetic properties of the powers of 2 and does not generalize to other sequences like Fibonacci numbers or factorials. By contrast, the "greedy" approach is one that can be used for many representation problems and, in fact, is the standard way to handle such problems.

Uniquess of representation: *The above argument proves only the existence of a representation, not its uniqueness. One way to prove the uniqueness is by contradiction: Assume there are positive integers with multiple representations, let n be the smallest of these exceptional integers, and derive a contradiction from this assumption.*
Another way to prove uniquess is to incorporate the uniqueness claim into the statement $P(n)$ to be proved. The strengthened statement requires an additional argument in the induction step showing that uniqueness holds for $k+1$, provided it holds for all $k' \leq k$. This is not difficult; the key observation is that any representation of $k+1$ must necessarily involve the power 2^m defined above.

(c) **Factorial representation.** Show that any integer $n \geq 1$ has a represention in the form $n = d_1 1! + d_2 2! + \cdots + d_r r!$ with "digits" d_i in the range $d_i \in \{0, 1, \ldots, i\}$. (Hint: Use again the "greedy" trick (pick the largest factorial that "fits" as your first building block), and use the fact (established in an earlier problem) that $\sum_{i=1}^{k} i! i = (k+1)! - 1$.)

> **Solution:** *We will prove by strong induction that the following statement holds for all $n \in \mathbb{N}$.*
>
> $(P(n))$ *n has a representation $(*)$ $\displaystyle\sum_{i=1}^{r} d_i i!$ with $d_i \in \{0, 1, \ldots, i\}$.*
>
> **Base case:** *The integer $n = 1$ has the representation $1 = 1 \cdot 1!$, which is of the desired form. Hence $P(n)$ holds for $n = 1$.*
> **Induction step:** *Let $k \geq 1$ be given and suppose $P(n)$ is true for all positive integers $n \leq k$, i.e., suppose that all such n have a representation in the form $(*)$. We seek to show that $n = k+1$ also has a representation of this form.*
> *Let r be the largest integer such that $r! \leq k+1$; i.e., r is the unique integer for which*
>
> (1) $r! \leq k+1 < (r+1)!.$
>
> *If $r! = k+1$, then $k+1$ has a representation of the desired form, and we are done.*
> *If $r! < k+1$, we let $k' = k+1 - r!$. Since $1 \leq r! < k+1$, k' is an integer in the range $1 \leq k' \leq k$. Hence we can apply the strong induction hypothesis to k' and obtain a representation of k' as a finite sum of terms $d_i i!$, with "digits" d_i in the range $0 \leq d_i \leq i$.*
> *Adding $r!$ to this representation gives a representation of $k+1 = k' + r!$ as a sum of factorials. To complete the induction step, we need to make sure that this new representation still satisfies the constraints $0 \leq d_i \leq i$ on the digits.*
> *If $r!$ does not occur in the representation of k', this is clearly the case.*
> *If $r!$ does occur in the representation of k' with an associated "digit" d_r satisfying $d_r \leq r - 1$, then adding $r!$ to this representation gives a representation with d_r replaced by $d_r + 1$ and all other digits unchanged, and since $d_r \leq r - 1$, the new digit $d_r + 1$ satisfies the required constraint, $d_r + 1 \leq r$.*
> *It remains to consider the case when k' has a representation involving $r!$ in which the associated digit is maximal, i.e., $d_r = r$. But then*
>
> $$k + 1 = k' + r! \geq d_r r! + r! = r \cdot r! + r! = (r+1)!,$$
>
> *so $(r+1)! \leq k+1$, contradicting (1). Therefore this case is impossible.*
> *Hence, in each case we have obtained a representation of $k+1$ of the desired form and the proof of the induction step is complete.*
> **Conclusion:** *By the strong induction principle, it follows that $P(n)$ is true for all $n \geq 1$.*