Direct proof of $P \Rightarrow Q$

This is the simplest, and most natural, method of proof. *Try this first before considering other methods.*

> Assume P.
> ...
> [Logical deductions]
> ...
> Therefore Q.

Proof by contraposition of $P \Rightarrow Q$

The **contraposition** (or **contrapositive**) of an implication (1) $P \Rightarrow Q$ is the implication (2) $\neg Q \Rightarrow \neg P$ (where $\neg$ denotes negation). The two implication (1) and (2) are logically equivalent. A proof by contraposition consists of proving (2) instead of (1):

> Assume Q is false.
> ...
> [Logical deductions]
> ...
> Therefore P is false.
> Hence $\neg Q \Rightarrow \neg P$.
> By contraposition, this proves $P \Rightarrow Q$.

CAUTION: Contraposition is not the same as converse. The converse of an implication $P \Rightarrow Q$ is the same implication in reverse direction: $Q \Rightarrow P$. By contrast, the contraposition of $P \Rightarrow Q$ is the implication in reverse direction *and with both P and Q replaced by their negations*: $\neg Q \Rightarrow \neg P$.

The contraposition is logically equivalent to $P \Rightarrow Q$; hence a proof of the contrapositive statement, i.e., $\neg Q \Rightarrow \neg P$, is equivalent to a proof the original statement. By contrast, the truth of the converse $Q \Rightarrow P$ says nothing about the truth of the original statement; the converse may be true, while the original statement is not, and it may be false, while the original statement is true.

Proof by contradiction of $P \Rightarrow Q$

To prove a statement of the form $P \Rightarrow Q$ by contradiction, assume the assumption, P, is true, but the conclusion, Q, is false, and derive from this assumption a contradiction, i.e., a statement such as "$0 = 1$" or "$0 \geq 1$" that is patently false:

> Assume P is true, and that Q is false.
> ...
> [Logical deductions]
> ...
> [Contradiction]
> Hence, our assumption that P is true and Q is false cannot hold.
> Therefore, if P is true, then Q must be true.
> Hence $P \Rightarrow Q$.

Variaton: Proof of an unconditional statement P by contradiction. For **unconditional** statements (e.g., "$\sqrt{2}$ is irrational"), the structure of the proof is even simpler:

> Assume P is false.
> . . .
> [Logical deductions]
> . . .
> [Contradiction]
> Hence, our assumption that P is false cannot hold.
> Therefore P must be true.

Proof by cases

This method consists of breaking up the analysis into two or more cases that together exhaust all possibilities, and showing, by separate arguments, that for each of these cases the desired statement holds.

For example, to prove that a statement such as "$n^3 - n$ is even" holds for all integers n, it is natural to consider separately the case when n is even and the case when n is odd, and to show that the claimed result holds in each of these cases.

Choosing a Proof Technique

- **Direct proof:** In most cases, the best strategy is to try a direct proof first; this is the simplest and most natural method of proof. Only if a direct method fails, consider other methods of proof.
- **Proof by cases:** This is appropriate if the problem naturally breaks down into several cases, and if each of these cases can be handled in a relatively straightforward manner (typically, by a simple direct proof).
- **Proof by contraposition:** This is the same as a direct proof of the contrapositive statement, and is worth considering if a direct proof of the original statement does not seem to work..
- **Proof by contradiction:** A proof by contradiction is logically more complicated, and more prone to errors, but can be effective in some situations, particularly, when we want to conclude that something does *not* have a certain property (such as being rational, or being a perfect square). In this case, assuming that the property *does* hold (i.e., that the number is rational, or is a perfect square), and trying to derive a contradiction from this assumption is a natural approach that is often successful.

  That said, contradiction tends to be overused, and many students use it unnecessarily in cases when other proof methods such as a direct proof or a proof by contraposition work just as well, and are simpler and more reliable. Bottomline: *Use contradiction only as a last resort, in situations such as the one described above.*

Further resources

Proof techniques are covered in the text on pp. 35–39. (The "proof by cases" method does not appear explicitly, but it comes up implicitly in several of the examples and exercises.) The section "How to approach problems" on pp. 39–44 offers some excellent general advice on doing proofs. **Be sure to read these two sections (i.e., pp. 35–44).** (The first part of Chapter 2, on logical statements, will be covered later.)

The "Even/odd proofs" worksheet has a problems illustrating all of the above proof techniques. **Be sure to work carefully through the problems on this worksheet.**