Problem 1. Logical statements, I. Negate the following statements **without using words of negation**. (You can use "composite" as negation of "prime", "even" as negation of "odd", and "good" as negation of "broken".) Write the negation as an English sentence, use proper terminology, be sure to include any necessary quantifiers and appropriate connecting words (e.g., "such that") if necessary. Below f denotes a function from $\mathbb{R}$ to $\mathbb{R}$, and n denotes a positive integer.

(a) "Every classroom in Altgeld Hall has a chair that is not broken."

Solution: *"There exists a classroom in Altgeld Hall in which all chairs are broken."*

(b) "If n is a Fermat number, then n is a prime."

Solution: *"There exists an integer n that is a Fermat number and composite."*

Remark: The quantifier "there exists an integer n" here is essential; without it, the statement wouldn't make sense.

(c) "For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x)| < \epsilon$ holds whenever $|x| < \delta$"

Solution: *"There exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x \in \mathbb{R}$ such that $|x| < \delta$ and $|f(x)| \geq \epsilon$."*

Remark: Note again the quantifier "there exists $x \in \mathbb{R}$", which must be explicitly stated in order for the statement to make sense.

(d) "There exist $x_0 \in \mathbb{R}$ and $\delta > 0$ such that $|x - x_0| < \delta$ implies $f(x) < f(x_0)$."

Solution: *"For every $x_0 \in \mathbb{R}$ and every $\delta > 0$ there exists a number x in the interval $(x - \delta, x + \delta)$ such that $f(x_0) > f(x)$."*

(e) **Bonus question:** Describe, with a brief explanation/justification, in **simple language** (in about three words, *without* using logical terminology) the precise set of functions that satisfy statement (d).

Solution: *The functions satisfying the statement are exactly those that* ***have a local maximum****. To see this, analyze the statement from the inside out as follows:*

- *At the innermost level, "$|x - x_0| < \delta$ implies $f(x) < f(x_0)$" says that on the interval $(x_0 - \delta, x_0 + \delta)$, $f(x)$ has a maximal value at x_0.*
- *At the next level, "there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $f(x) < f(x_0)$" says that $f(x)$ has a* ***local*** *maximum at x_0.*
- *The full statement, "there exits $\delta > 0$ such that $|x - x_0| < \delta$ implies $f(x) < f(x_0)$" says that* ***there exists an x_0 such that $f(x)$ has a local maximum at x_0****. But this is the same as saying that $f(x)$* ***has a local maximum****.*

Problem 2. Logical statements, II: Consider the following statement:

"To win the World Cup it is necessary to win each of the last four games."

(a) Write the **negation** of this statement without using words of negation. (You can use "lose" as a synonym for "not win".)

Solution: *"A team can lose at least one of the last four games, and still win the World Cup."*

Remark: The given statement has the logical form $P \implies Q$, where P stands for "win World Cup" and Q stands for "win each of the last four games". The correct negation of $P \implies Q$ is "$P \wedge \neg Q$." The negation of an implication is NEVER another implication (e.g., statements like $P \Rightarrow \neg Q$ are all incorrect).

(b) Write the **converse** of this statement without using words of negation.

Solution: *"If a team win each of the last four games, it will win the World Cup."*

(c) Write the **contrapositive** of this statement without using words of negation.

Solution: *"If a team loses at least one of the last four games, it will lose the World Cup."*

(d) Rewrite the statement in the form "... **only if** ...".

Solution: *"A team can win the World Cup only if it wins each of the last four games."*

Problem 3. Short answer problems, I: Sum/product formulas. Evaluate the given sum or product. (For all sums/products there is a simple formula involving only elementary functions and factorials. No induction proof needed, but show all work.)

(a) $\displaystyle\prod_{i=1}^{n}\left(1+\frac{n}{i}\right)$

Solution: $\dfrac{n+1}{1}\cdot\dfrac{n+2}{2}\cdots\dfrac{2n}{n}=\dfrac{(2n)!}{n!^2}$ *(or $\binom{2n}{n}$).*

(b) $\displaystyle\prod_{i=1}^{n} n^{n-i}$

Solution: $n^{(n-1)+(n-2)+\cdots+1}=n^{n(n-1)/2}$ *(See Problem 6 in HW 3)*

(c) $\displaystyle\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{i}{j}$ (Note that the summation limits for j are from $j=i$ to $j=n$.)

Solution: *(See Problem 9 in HW 3)*

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{i}{j}=\sum_{j=1}^{n}\sum_{i=1}^{j}\frac{i}{j}=\sum_{j=1}^{n}\frac{j(j+1)/2}{j}=\frac{1}{2}\sum_{j=1}^{n}(j+1)=\frac{1}{2}\left(\frac{(n+2)(n+1)}{2}-1\right)=\frac{n(n+3)}{4}$$

Problem 4. Short answer problems, II: Set-theoretic notations and definitions

(a) Let $A = \{1,2\}$, $B = \{0,1\}$. Find $P(A)$, $P(B$, and $P(A) \cap P(B)$ (where $P(S)$ denotes the power set of S) and write this set out *explicitly (i.e., by listing all its elements), using proper set-theoretic notation.*

> **Solution:** $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, $P(B) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$, $P(A) \cap P(B) = \{\emptyset, \{1\}\}$

(b) Express the set $\mathbb{Q}$ of rational numbers in set builder notation, i.e., in the form $\mathbb{Q} = \{\dots : \dots\}$, with appropriate expressions in place of the dots.

> **Solution:** $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$

(c) For each of the following properties (where f is a function from $\mathbb{R}$ to $\mathbb{R}$), state its definition in *symbolic form.*

(I) "f is a **decreasing** function"

(II) "f is a **nondecreasing** function"

(III) "f is a **bounded** function"

(Just state the three definitions; no further work/justifications needed.)

> **Solution:**
>
> (I) *"f is a **decreasing** function"* $\iff (\forall x, y \in \mathbb{R})[x < y \implies f(x) > f(y)]$
>
> (II) *"f is a **nondecreasing** function"* $\iff (\forall x, y \in \mathbb{R})[x < y \implies f(x) \leq f(y)]$
>
> (III) *"f is a **bounded** function"* $(\exists M \in \mathbb{R})(\forall x \in \mathbb{R}))[|f(x)| \leq M]$
>
> **Remark: The order of quantifiers in (III) is crucial. As shown in class, if the order of $\exists M$ and $\forall x$ is reversed, then any function satisfies the statement.**

Problem 5. Proofs, I: Even/odd and divisibility. *Using only the basic algebraic properties of the integers and the **definitions** of even and odd numbers and divisibility*, give careful, step-by-step, proofs for each of the following statements. (You must work directly from the definitions of even/odd; you can **not** use any of the properties or results about even/odd numbers established in the worksheets or in homework problems.)
(If you need extra space for work, use the back of the page.)

(a) "If $n^2 - 1$ is even, then n is odd."

(b) "If n is odd, then $n^2 - 1$ is divisible by 8."

Solution: ***Proof of (a).*** *We use the method of contraposition, i.e., we prove the contrapositive statement: "If n is even, then $n^2 - 1$ odd".*

- *Suppose n is even.*
- *Then $n = 2k$ for some $k \in \mathbb{Z}$, by the definition of an even integer.*
- *Hence,*

$$n^2 - 1 = (2k)^2 - 1 = 4k^2 - 1 = 2(2k^2 - 1) + 1.$$

- *Since k is an integer, so is $2k^2 - 1$.*
- *Hence $n^2 - 1$ is of the form $2p + 1$, where $p = 2k^2$ is an integer.*
- *Therefore $n^2 - 1$ is odd, by the definition of an odd integer.*
- *Thus we have shown that "n even" implies "$n^2 - 1$ odd".*
- *By contraposition, it follows that "$n^2 - 1$ even" implies "n odd".*

Proof of (b). *(See Problem 2 of HW 1.)*

- *Suppose n is odd.*
- *Then $n = 2k + 1$ for some $k \in \mathbb{Z}$, by the definition of an odd integer.*
- *Hence,*

$$(1) \qquad n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k).$$

- *We now conssider two cases, (i) k even, and (ii) k odd.*
  - ***Case (i): k even:***
    - *In this case, $k = 2l$ for some $l \in \mathbb{Z}$, by the definition of an even integer.*
    - *Substituting this into (1) we get*

$$n^2 - 1 = 4((2l)^2 + (2l)) = 8(2l^2 + l).$$

    - *Since l is an integer, so is $2l^2 + l$.*
    - *Hence n is of the form $n = 8p$, where $p \in \mathbb{Z}$.*
    - *By definition, this means that n is divisible by 8*
  - ***Case (ii): k odd:***
    - *In this case $k = 2l + 1$ for some $l \in \mathbb{Z}$, by the definition of an odd integer.*
    - *Substituting this into (1) we get*

$$n^2 - 1 = 4((2l + 1)^2 + (2l + 1)) = 4(4l^2 + 6l + 2) = 8(2l^2 + 3l + 1).$$

    - *Since l is an integer, so is $2l^2 + 3l + 1$.*
    - *Hence n is of the form $n = 8p$, where $p \in \mathbb{Z}$.*
    - *By definition, this means that n is divisible by 8*
- *Thus we have shown that, if n is odd, then $n^2 - 1$ is divisible by 8.*

Problem 6. Proofs, II: Set theory. Let A, B, and C be sets. For each of the following statements below, determine whether it is true. If it is true, give a careful, step-by-step, proof; if it is false, give a counterexample. Your write-up must include all necessary steps, with appropriate justifications (e.g., "by the def. of ..."), in the correct logical order, use proper notation and terminology, and include any necessary quantifiers and connecting words (e.g., "therefore", "such that").
(If you need extra space for work, use the back of the page.)

(a) $(A \cup B) - C \subseteq (A - (B \cup C)) \cup (B - (A \cap C))$.

(b) $(A - (B \cup C)) \cup (B - (A \cap C)) \subseteq (A \cup B) - C$.

Solution: ***(a) Proof of*** $(A \cup B) - C \subseteq [A - (B \cup C)] \cup (B - (A \cap C)]$**:**

- *Let* $x \in (A \cup B) - C$.
- *Then* $x \in (A \cup B)$ *and* $x \notin C$*, by the definition of a set difference.*
- *Therefore* ($x \in A$ *or* $x \in B$) *and* $x \notin C$*, by the definition of a union.*
- *Thus, in either case we have* $x \notin C$*, while the condition "*$x \in A$ *or* $x \in B$*" breaks down into three cases: (i) "*$x \in A$ *and* $x \notin B$*", (ii) "*$x \in B$ *and* $x \notin A$*", and (iii)* $x \in A$ *and* $x \in B$*. We analyze these three cases separately:*
  - ***Case (i):*** $x \in A$**,** $x \notin B$**,** $x \notin C$**:** *In this case we have* $x \in A$ *and* $x \notin B$ *and* $x \notin C$*. The latter two conditions imply that* $x \notin B \cup C$*, by the definition of a union. Since* $x \in A$*, it follows that* $x \in A - (B \cup C)$*, by the definition of a set difference.*
  - ***Case (ii):*** $x \in B$**,** $x \notin A$**,** $x \notin C$**:** *In this case we have* $x \in B$ *and* $x \notin C$*. The latter condition implies that* $x \notin A \cap C$*, by the definition of an intersection. Since* $x \in B$*, it follows that* $x \in B - (A \cap C)$*, by the definition of a set difference.*
  - ***Case (iii):*** $x \in B$**,** $x \in A$**,** $x \notin C$**:** *In this case we also have* $x \in B$ *and* $x \notin C$*, so as before* $x \in B - (A \cap C)$*, by the definition of a set difference.*
- *Thus, we have either* $x \in A - (B \cup C)$ *or* $x \in B - (A \cap C)$.
- *Hence* $x \in (A - (B \cup C)) \cup (B - (A \cap C))$*, by the definition of a union.*
- *This proves that* $(A \cup B) - C \subseteq [A - (B \cup C)] \cup (B - (A \cap C)]$.

(b) Disproof of $(*)$ $(A - (B \cup C)) \cup (B - (A \cap C)) \subseteq (A \cup B) - C$**:**
We show that $(*)$ *is false by a* ***counterexample****:*
Let $A = \{0\}$, $B = \{1\}$, $C = \{1\}$. *Then*

$$(A - (B \cup C)) \cup (B - (A \cap C)) = (\{0\} - \{1\}) \cup (\{1\} - \emptyset) = \{0, 1\},$$
$$(A \cup B) - C = \{0, 1\} - \{1\} = \{0\}.$$

Since $\{0, 1\} \not\subseteq \{0\}$, $(*)$ *does not hold in this case.*

> **Remark:** A counterexample to a claim is a **specific** example for which the claim is false. In our case, this means that we have to find **specific** sets A, B, C for which the relation $(*)$ does not hold. If one tries to prove this relation in the usual way, one runs into problems. **However, the mere fact that a particular argument does not succeed in proving a result, is not proof that the claim is false.** (After all, there may be other, perhaps more clever, methods that do lead to the claimed result.)