Relations: Definitions and Properties

Definition and Notations

Let S and T be sets.

- A **relation from** S **to** T is a subset $R \subseteq S \times T$.
- A **relation on** S is a subset $R \subseteq S \times S$.

- **Notation:** We write "$x \sim y$" if $(x, y) \in R$.
- **Note:** According to the above definition, **any subset of** $S \times S$ **is a relation on** S. Thus, for example, the empty set, $R = \emptyset$, is a valid relation on a set S, as is the entire Cartesian product, $R = S \times S$, and anything in between.
- **Examples:** Examples of mathematical relations are:
  - **Divisor relation:** $S = \mathbb{N}$, $x \sim y \Longleftrightarrow x \mid y$
  - **Congruence relation modulo** m **(where** $m \in \mathbb{N}$**):** $S = \mathbb{Z}$, $x \sim y \Longleftrightarrow x \equiv y \bmod m$
  - **Order relation:** $S = \mathbb{R}$, $x \sim y \Longleftrightarrow x \leq y$.
  - **Equality relation:** S arbitrary, $x \sim y \Longleftrightarrow x = y$. The corresponding subset of $S \times S$ is the set of "diagonal" elements: $R = \{(x, x) : x \in S\}$.

  Real-world examples include road networks, webpage links, Facebook "likes", Twitter "followers", and many others.

Properties

Relations can be classified according to the properties they satisfy. The most important properties are reflexivity, symmetry, and transitivity, defined below. **Note that a relation may or may not a given property. In fact, a relation need not have any of these properties.**

- **Reflexive:** For all $x \in S$, $x \sim x$.
- **Symmetric:** For all $x, y \in S$, $x \sim y$ implies $y \sim x$.
- **Transitive:** For all $x, y, z \in S$, $x \sim y$ and $y \sim z$ implies $x \sim z$.

Equivalence Relations

Definition. An **equivalence relation** on a set S is a relation that satisfies all of the above properties, i.e., is **reflexive**, **symmetric**, and **transitive**.

- **Examples:** Among the above examples, the congruence and equality relations are reflexive, symmetric, and transitive, and therefore are equivalence relations. The divisor and order relations are reflexive and transitive, but not symmetric, and hence are not equivalence relations.

Equivalence Classes

Definition. Given an an equivalence relation $\sim$ on S, the **equivalence class** of an element $x \in S$ is the set
$$[x] = \{y \in S : x \sim y\}$$

Partitions

Definition. Let S be a set. A collection of subsets $A_i \subseteq S$ (where i runs through some index set I) is called a **partition of** S if it satisfies the following properties:

- The sets A_i are **nonempty**: $A_i \neq \emptyset$ for all $i \in I$.
- The sets A_i are **pairwise disjoint**, i.e., $A_i \cap A_j = \emptyset$ for all $i, j \in I$ with $i \neq j$.
- The sets A_i make up all of S, i.e., $\bigcup_{i \in I} A_i = S$.

Equivalence Relations and Partitions

The following fundamental result connects the seemingly unrelated concepts of an equivalence relation on a set S (which, formally, is a subset of $S \times S$ with certain properties) and a partition on S (which is a collection of subsets of S with certain properties).

Theorem (Equivalence Relations and Partitions).

- **From equivalence relations to partitions.** Given an equivalence relation on a set S, the set of *distinct* equivalence classes for this relation forms a partition of S.
- **From partitions to equivalence relations.** Given a partition of S into sets A_i, the relation $\sim$ defined by
$$x \sim y \iff \text{“}x \text{ and } y \text{ belong to the same set } A_i\text{”}$$
is an equivalence relation on S.

Further Resources

In the text, the above definitions can be found in Chapter 7:
Relation: Definition 7.5, p. 140
Equivalence relation: Definition 7.9, p. 140.
Equivalence class: Definition 7.13, p. 141.
Connection with Partitions: Exercise 7.12, p. 152