verte

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Notes on the heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + f \quad \text{in} \quad \Omega \times (0, T]$$

$$u = u_D \quad \text{on} \quad \partial\Omega \times (0, T]$$

$$u = u_0 \quad \text{at} \quad t = 0$$

Weak formulation: $\int_\Omega \frac{\partial u}{\partial t} v \, d\Omega + \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega$

$$\frac{\partial u}{\partial t} \approx \frac{u_{new} - u_{old}}{\Delta t}$$

implication of the code

Multi-stage: $\underbrace{u_{n+1}}_{u_i} = u_n + \Delta t \sum_j a_{ij} k_j$, $\boxed{k_j\Big|_{\partial\Omega} = \frac{u_D\Big|_{t=t^n+c_j\Delta t} - u_D\Big|_{t=t^n}}{\Delta t}}$

Example (LobattoIIIc): $u_1 = u_n + \Delta t(a_{11}k_1 + a_{12}k_2)$

$$u_2 = u_n + \Delta t(a_{21}k_1 + a_{22}k_2)$$

$$k_1\Big|_{\partial\Omega} = \frac{u_D(t = t^n + c_1 \cdot \Delta t) - u_D(t = t^n)}{\Delta t}$$

$$k_2\Big|_{\partial\Omega} = \frac{u_D(t = t^n + c_2 \Delta t) - u_D(t = t^n)}{\Delta t}$$

A known solution

$$u = 1 + x^2 + \alpha y^2 + \beta t$$
$$f = \beta - 2 - 2\alpha$$
$$u_0 = 1 + x^2 + \alpha y^2$$
$$u_D = 1 + x^2 + \alpha y^2 + \beta t$$

Problem declaration

$$\int \frac{u_{n+1}-u_n}{\Delta t} v \, dx + \int \nabla u_{n+1} \cdot \nabla v \, dx - \int f_{n+1} v \, dx = 0$$

$$\underbrace{\langle u_{n+1}, v\rangle + \langle \nabla u_{n+1}, \nabla v\rangle \cdot dt}_{LHS} = \underbrace{\langle u_n, v\rangle + dt \langle f_{n+1}, v\rangle}_{RHS} \rightarrow \text{Implicit}$$

$$\underbrace{\langle u_{n+1}, v\rangle}_{LHS} = \underbrace{-\langle \nabla \underline{u_n}, \nabla v\rangle dt + \langle u_n, v\rangle + dt\langle \underline{f_n}, v\rangle}_{RHS} \rightarrow \text{Explicit}$$

Lobatto IIIc from the thesis (Two-stage implicit)

Solution at each time step

$$y_{n+1} = y_n + \Delta t (b_1 k_1 + b_2 k_2)$$

with

$$k_1 = f(t_n + c_1 \Delta t, \; y_n + \Delta t(a_{11} k_1 + a_{12} k_2)$$

$$k_2 = f(t_n + c_2 \Delta t, \; y_n + \Delta t(a_{21} k_1 + a_{22} k_2)$$