Politecnico
di Torino

Complexity Analysis

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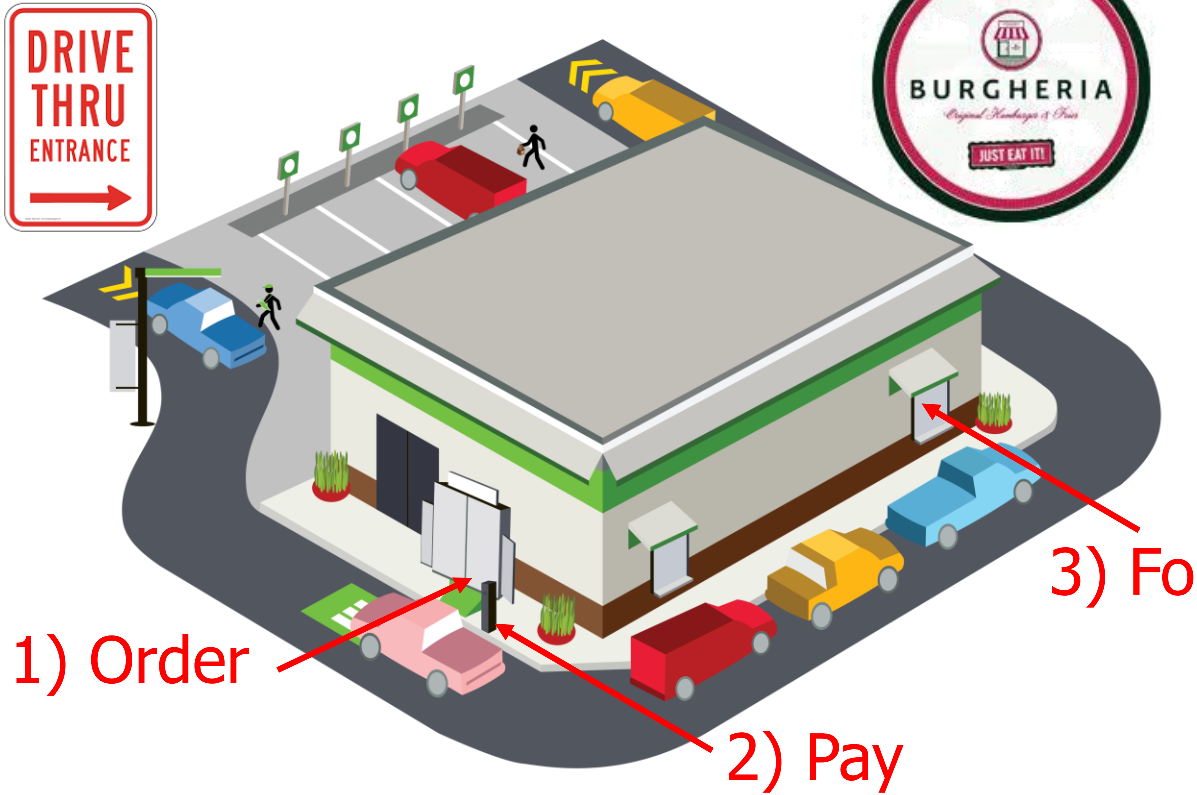
Complexity Analysis

DRIVE
THRU
ENTRANCE
1) Order
2) Pay
3) Food

Complexity Analysis



1) Order

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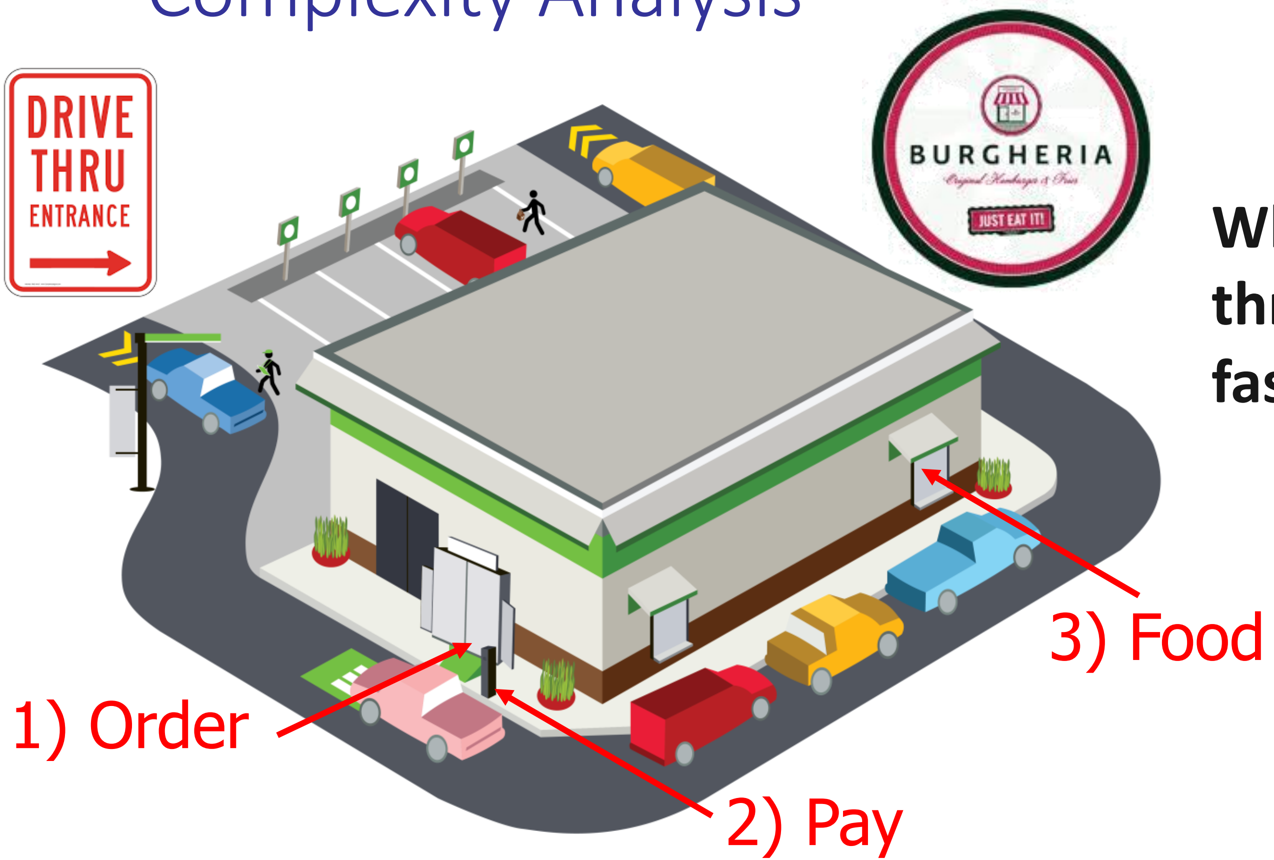
Complexity Analysis



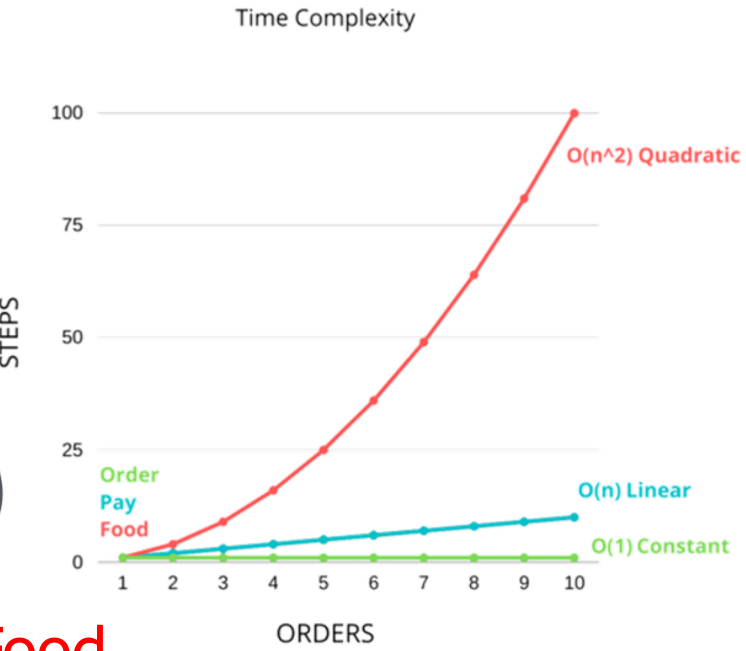
Why do some cars get through the drive-thru faster than others?

Complexity Analysis

1) Order

2) Pay

3) Food



Complexity Analysis

Definition:

- **Forecast of resources** (**memory**, **time**) needed by the algorithm for execution.
  - Empirical
  - Analytical

Complexity Analysis

Features:

- Machine-independent
- Assumption: Sequential single-processor model (traditional architecture)
- Independent of the input data of a particular instance of the problem.

Example:

- Problem **P**: sort integer data
- Instance **I**: data are 45 10 6 7 99
- Size of instance |**I**|: number of bits needed to encode **I**,

  in this case 5 x the size of the integer or simply **5**

Complexity Analysis

- It depends on the size **n** of the problem.
- Examples:
  - n **number of bits** of the operands for integer **multiplication**
  - n **size** of the file to sort
  - n **number of characters** in a string of text
  - n **number of data** to sort for a sorting algorithm
- Output:
  - **S(n): memory occupation (memory footprint)**
  - **T(n): execution time (Performance)**

Algorithm Classification

- 1: **constant**
- log n: **logarithmic**
- n: **linear**
- n log n: ***linearithmic***
- n^2: **quadratic**
- n^3: **cubic**
- 2^n: **exponential**

Worst-case Asymptotic Analysis

Goal:

- to guess an upper-bound for **T(n)** **(execution time)** for an algorithm on **n** data in the **worst possible case**

Asymptotic: n $\rightarrow \infty$:

- for small **n**, complexity is irrelevant

Worst-case Asymptotic Analysis

Why **worst-case** analysis?

- Conservative guess
- Worst case is very frequent
- Average case:
  - either it coincides with the worst case
  - or it is not definable, unless we resort to complex assumptions on data.

Importance of complexity analysis

Advantages of a lower complexity:

- it compensates hardware (in)efficiency

Example:

- **Algorithm #1**:
  - **T(n) (execution time)** = $2n^2$
  - **machine #1:** 10^8 instructions/second

- **Algorithm #2**:
  - **T(n) (execution time)** = $50n \lg_2 n$
  - **machine 2:** 10^6 instructions/second

Importance of complexity analysis

If **n** = 1M = 10^6:

- **Algorithm #1:** $2 \cdot (10^6)^2 / 10^8 = 2 \cdot 10^4 = 20000$ s = 333,33 *min*

- **Algorithm #2:** $50 \cdot 10^6 \lg_2 10^6 / 10^6 = 50 \cdot 6 \cdot \lg_2 10 = 1000$ s = 16,67 *min*

An inefficient algorithm rapidly «wastes» the increase in hardware performance!

Examples

Discrete Fourier Transform:

- decomposition of a N-sample waveform into periodic components
- applications: DVD, JPEG, astrophysics, ….
- **trivial algorithm**: quadratic (N^2)
- **FFT (Fast Fourier Transform):** N log N

Simulation of N bodies:

- simulates gravity interaction among N bodies
- **trivial algorithm:** quadratic (N^2)
- **Barnes-Hut algorithm**: N log N

Search Algorithms on Arrays

Let **v[N]** be an array of **N** distinct elements, let **k** be a key:

- **Decision problem:**

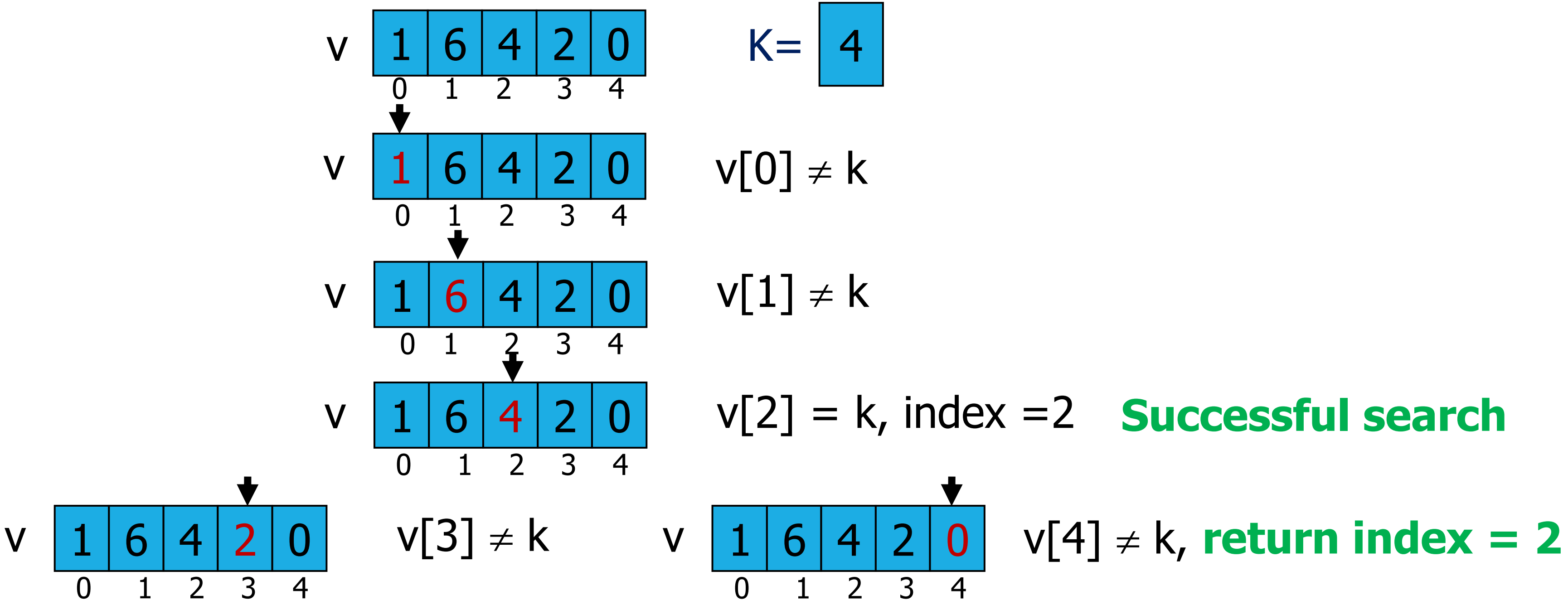
  Does key **k** appear in array **v[N]?** Yes/No

- **Search problem:**

  if **k** is in the array **v[N]**, where (***at what index***)?

Algorithm #1: Linear Search

Scan the array **v[N]** from first to possibly last element, compare at each step current element and key **k**.



v | 1 | 6 | 4 | 2 | 0
0 1 2 3 4

K= 4

v | 1 | 6 | 4 | 2 | 0
0 1 2 3 4

v[0] ≠ k

v | 1 | 6 | 4 | 2 | 0
0 1 2 3 4

v[1] ≠ k

v | 1 | 6 | 4 | 2 | 0
0 1 2 3 4

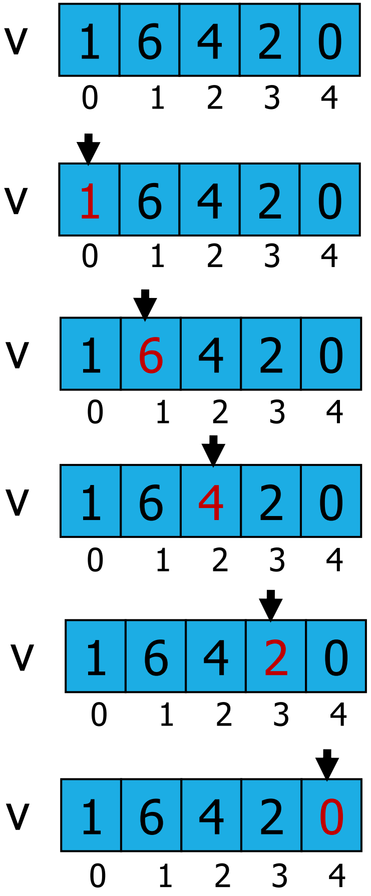
v[2] = k, index =2 **Successful search**

v | 1 | 6 | 4 | 2 | 0
0 1 2 3 4

v[3] ≠ k

v | 1 | 6 | 4 | 2 | 0
0 1 2 3 4

v[4] ≠ k, **return index = 2**



v	1	6	4	2	0
	0	1	2	3	4

k 8

v	1	6	4	2	0
	0	1	2	3	4

v[0] ≠ k

v	1	6	4	2	0
	0	1	2	3	4

v[1] ≠ k

v	1	6	4	2	0
	0	1	2	3	4

v[2] ≠ k

Unsuccessful search

v	1	6	4	2	0
	0	1	2	3	4

v[3] ≠ k

v	1	6	4	2	0
	0	1	2	3	4

v[4] ≠ k, **return index = -1**

Alternatives:

- **Solution #1: scan the array** from first to last element: always **N** operations
- **Solution #2: use a flag**: early scan stop possible, at most N operations, in the worst case **N** operations.

The worst-case asymptotic complexity is the same (**N**), the second alternative improves the average case.

Solution #1

```
int LinearSearch1(int v[], int N, int k)
{
  int i = 0, index = -1;

  for (i = 0;  i < N; i++)
    if (k == v[i])
      index = i;

  return index;
}
```

Solution #2

```
int LinearSearch2(int v[], int N, int k)
{
  int i = 0;
  int found = 0;

  while (i < N && found == 0)
    if (k == v[i])
      found = 1;
    else
      i++;

  if (found == 0)
    return -1;
  else
    return i;
}
```

Algorithm #2: Binary Search

Let **v[N]** be a **sorted** array of **N** distinct elements and let **k** be a key **k**:

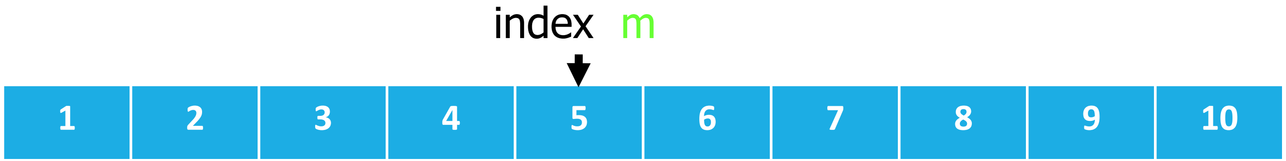
- **Decision problem:** does key **k** appear in array **v[N]**? Yes/No
- **Search problem:** if **k** is in the array, where (***at what index***)?

We work on a **subarray** identified by the contiguous elements of **v** whose indices range from a **leftmost one** (l) and a **rightmost one** (r).

Initially array and **subarray** coincide (l = 0 and r = N-1).

The **middle element of the subarray** is at index m= (l+r)/2.

Example: **k=9**



index m

1	2	3	4	5	6	7	8	9	10

Algorithm #2: Binary Search

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- **Decision problem:** does key **k** appear in array **v[N]**? Yes/No
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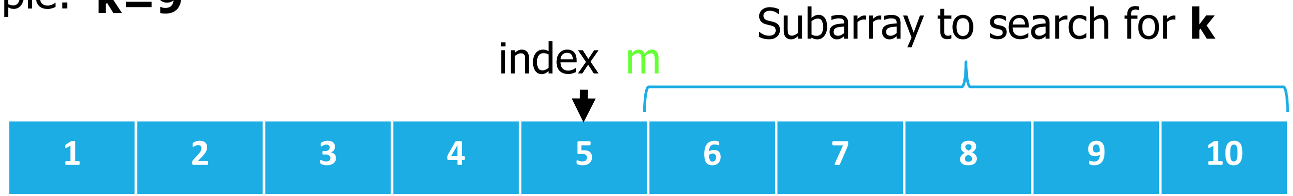


Subarray to search for **k**

index m

1	2	3	4	5	6	7	8	9	10

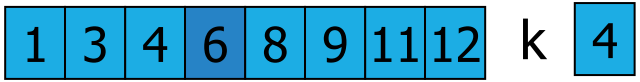
- **Loop:** at each step compare **k** to the middle element **v[m]** of the subarray
- **Loop condition:** l ≤ r && found==0: the key has not yet been found and the subarray is meaningful (l doesn’t exceed r)
- **Body of the loop:**
  - if **v[m]** == **k:**

    termination with success, found = 1
  - if **v[m]** < **k**:

    search continues in the right subarray: l=m+1, r unchanged
  - if **v[m]** > **k:**

    search continues in the left subarray: l unchanged, r = m-1

- Upon exiting the loop, test found, return -1 for failure or m for success.



1	3	4	6	8	9	11	12

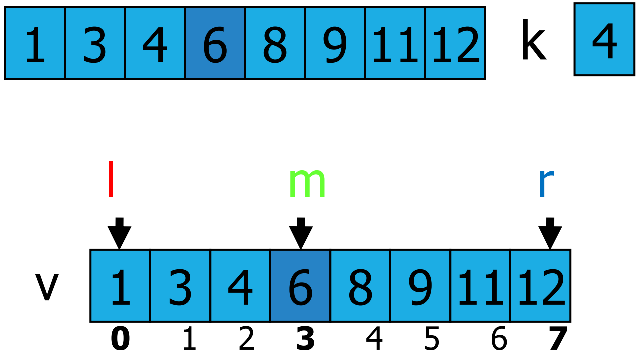
k 4

l = leftmost index, initially l = 0

r = rightmost index, initially r = N-1

m = index of middle element

v[m] = middle element



1	3	4	6	8	9	11	12

k 4

l m r

v	1	3	4	6	8	9	11	12
	0	1	2	**3**	4	5	6	**7**

l = leftmost index, initially l = 0

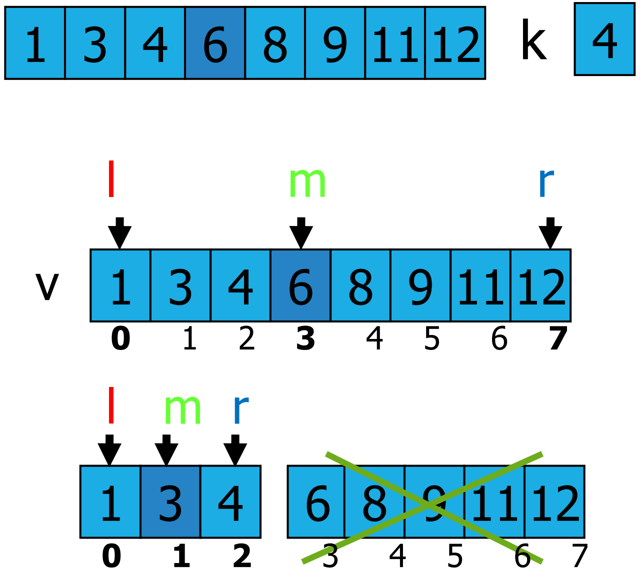
r = rightmost index, initially r = N-1

m = index of middle element

v[m] = middle element

Iteration 1:

l=**0**, r=**7**, m= (l+r)/2=**3**, v[3]>**k**, r=m-1



l = leftmost index, initially l = 0

r = rightmost index, initially r = N-1

m = index of middle element

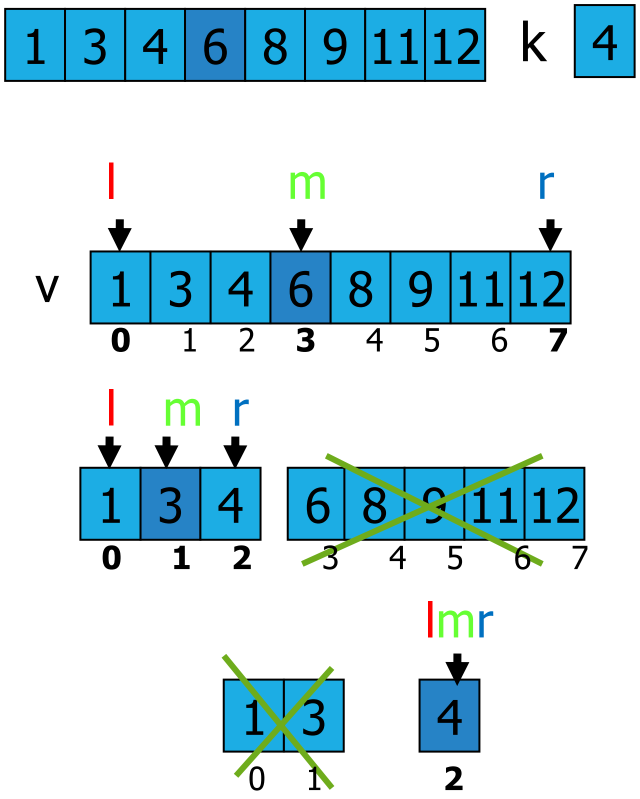
v[m] = middle element

Iteration 1:

l=**0**, r=**7**, m= (l+r)/2=**3**, v[3]>**k**, r=m-1

Iteration 2:

l=**0**, r=**2**, m= (l+r)/2=**1**, v[1]<**k**, l=m+1



l = leftmost index, initially l = 0

r = rightmost index, initially r = N-1

m = index of middle element

v[m] = middle element

Iteration 1:

l=**0**, r=**7**, m= (l+r)/2=**3**, v[3]>**k**, r=m-1

Iteration 2:

l=**0**, r=**2**, m= (l+r)/2=**1**, v[1]<**k**, l=m+1

Iteration 3:

l=**2**, r=**2**, m= (l+r)/2=**2**, v[2]=**k**

Successful search, return index =**2**

1	3	4	6	8	9	11	12

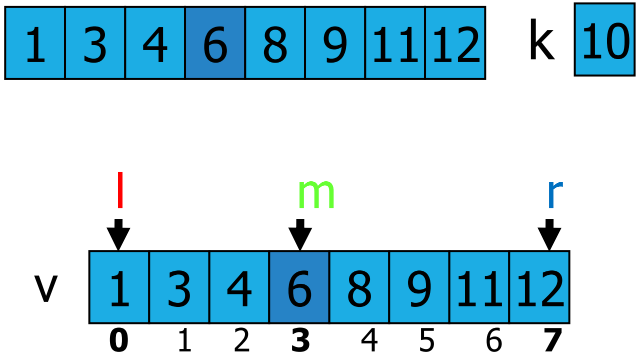
k 10

l = leftmost index, initially l = 0

r = rightmost index, initially r = N-1

m = index of middle element

v[m] = middle element



l = leftmost index, initially l = 0

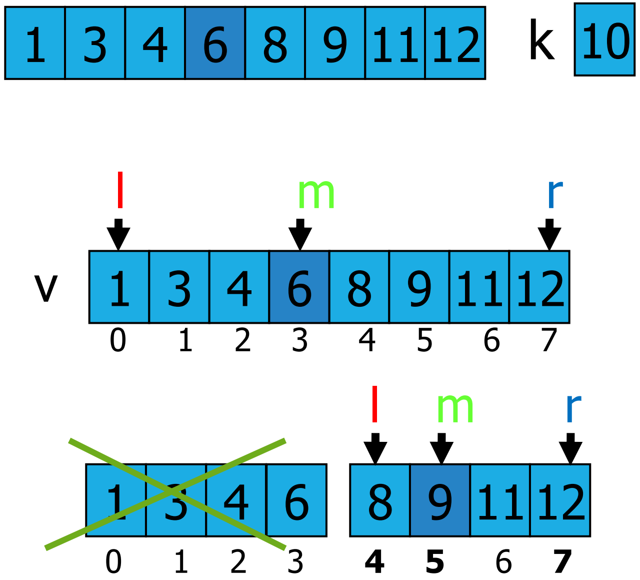
r = rightmost index, initially r = N-1

m = index of middle element

v[m] = middle element

Iteration 1:

l=0, r=**7**, m= (l+r)/2=**3**, v[3]<**k**, l=m+1



l = leftmost index, initially l = 0

r = rightmost index, initially r = N-1

m = index of middle element

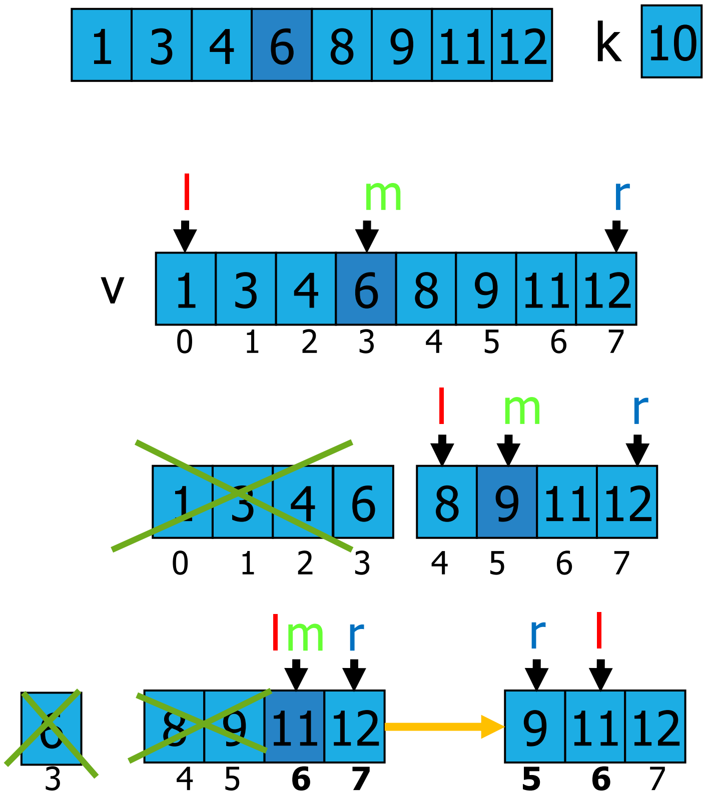
v[m] = middle element

Iteration 1:

l=0, r=7, m= (l+r)/2=3, v[3]<k, l=m+1

Iteration 2:

l=**4**, r=**7**, m= (l+r)/2=**5**, v[5]<**k**, l=m+1



l = leftmost index, initially l = 0

r = rightmost index, initially r = N-1

m = index of middle element

v[m] = middle element

Iteration 1:

l=0, r=7, m= (l+r)/2=3, v[3]<k, l=m+1

Iteration 2:

l=4, r=7, m= (l+r)/2=5, v[5]<k, l=m+1

Iteration 3:

l=**6**, r=**7**, m= (l+r)/2=**6**, v[6]>k, r=m-1

r < l, **exit from loop, search failed**

```
int BinSearch(int v[], int N, int k) {
  int m, found= 0, l=0, r=N-1;

while(l <= r && found == 0){
    m = (l+r)/2;
    if(v[m] == k)
      found = 1;
    if(v[m] < k)
      l = m+1;
    else
      r = m-1;
  }
  if (found == 0)
    return -1;
  else
    return m;
}
```

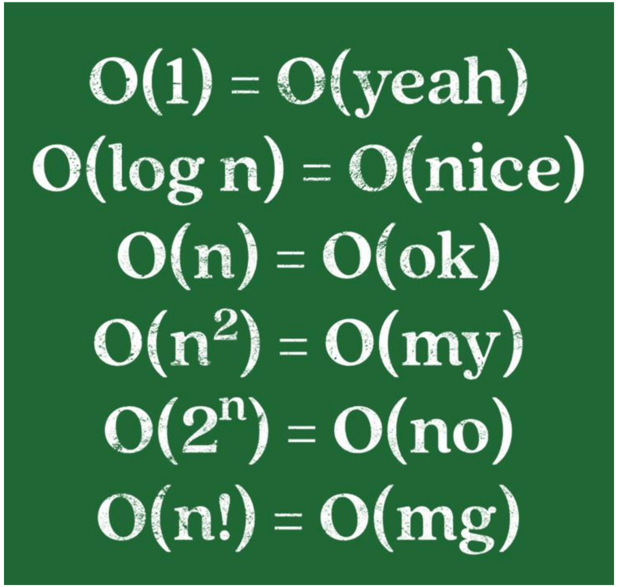
Analysis of Linear Search

- We consider **n** numbers for a search miss and in average **n/2** for a search hit
- **T(n)** **grows linearly** with **n**.

Analysis of Binary Search

- At the beginning the array to be examined contains **n** numbers
- At the **2nd iteration** the array to be examined contains about **n/2** numbers
- ….
- At the **i-th iteration** the array to be examined contains about **$n/2^i$** numbers
- **Termination** occurs when the **array** to be examined contains **1** number, thus **$n/2^i$** = 1, i = **$\log_2(n)$**
- **T(n)** grows **logarithmically** with n.

Asymptotic Notations



$O(1) = O(\text{yeah})$

$O(\log n) = O(\text{nice})$

$O(n) = O(\text{ok})$

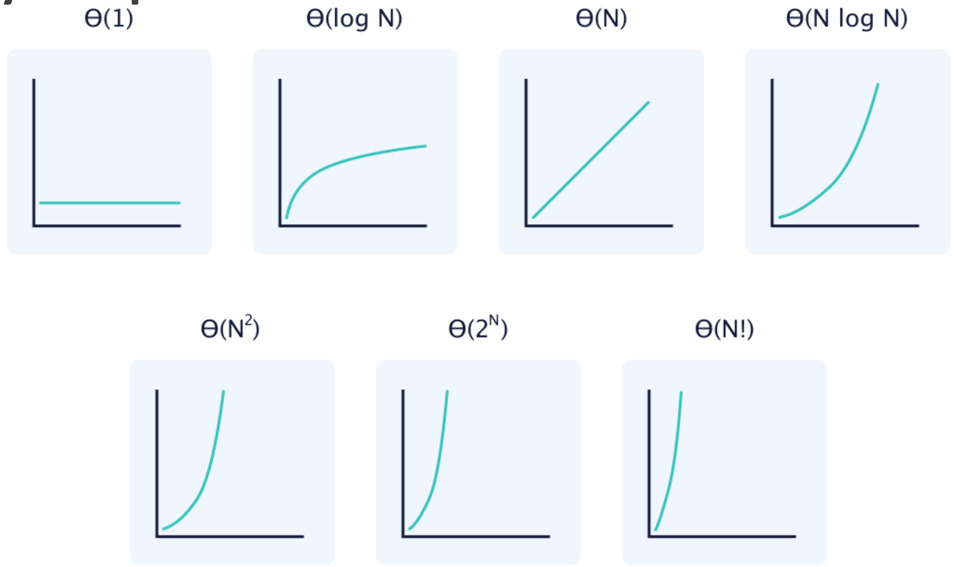
$O(n^2) = O(\text{my})$

$O(2^n) = O(\text{no})$

$O(n!) = O(\text{mg})$

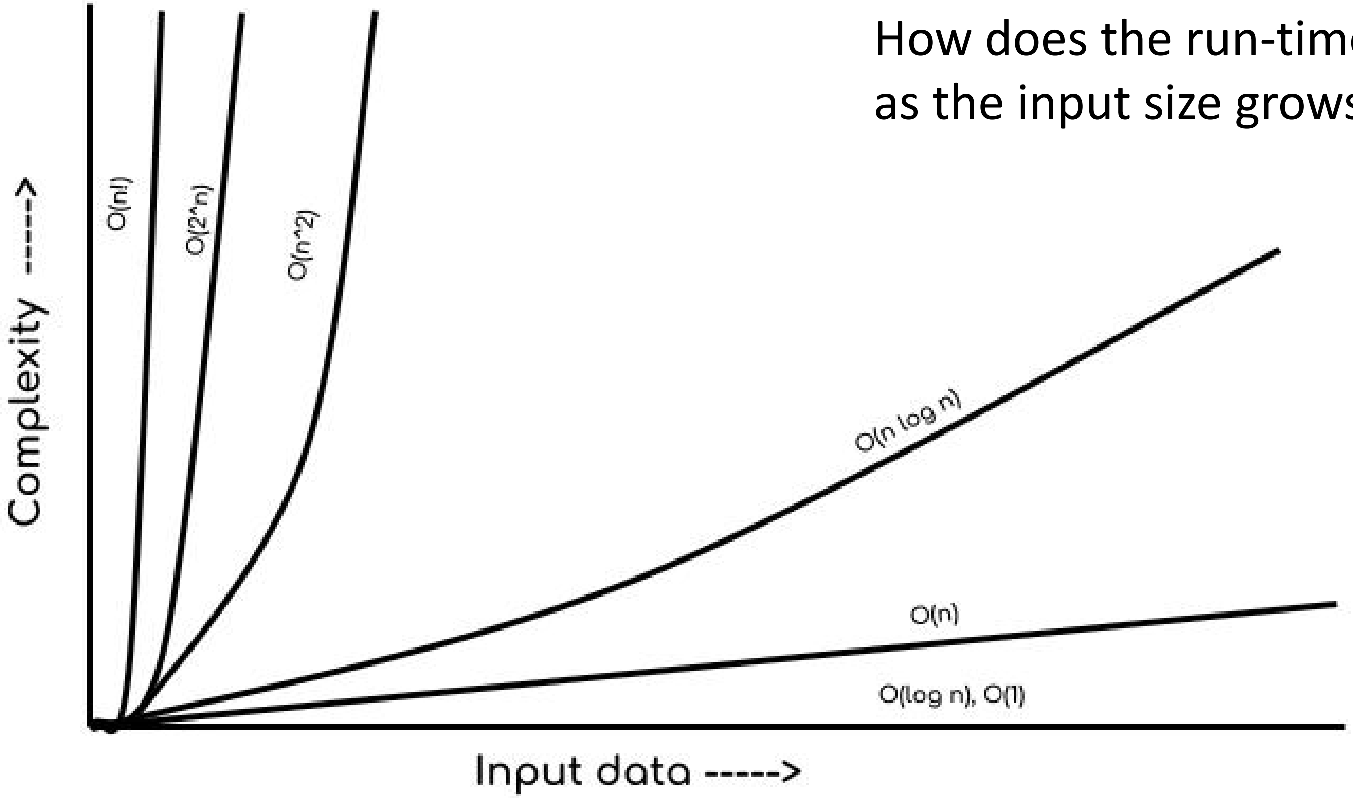
Asymptotic Notations

- Mathematical notations to describe the **running time** of an algorithm (when the input tends towards a particular value or a limiting value).
- We are talking about **efficiency** and **performance**!
- Most used:
  - **Big-Oh** notation
  - **Omega** notation
  - **Theta** notation



Common Runtimes

Asymptotic Notations



How does the run-time grow as the input size grows?

Big-Oh Asymptotic Notation

Definition:

$$\mathbf{T(n)} = \mathbf{O(g(n))} \Leftrightarrow$$
$$\exists\ \mathbf{c}>0,\ \exists\ \mathbf{n_0}>0 \text{ such that } \forall \mathbf{n} \geq \mathbf{n_0}$$
$$0 \leq \mathbf{T(n)} \leq \mathbf{cg(n)}$$

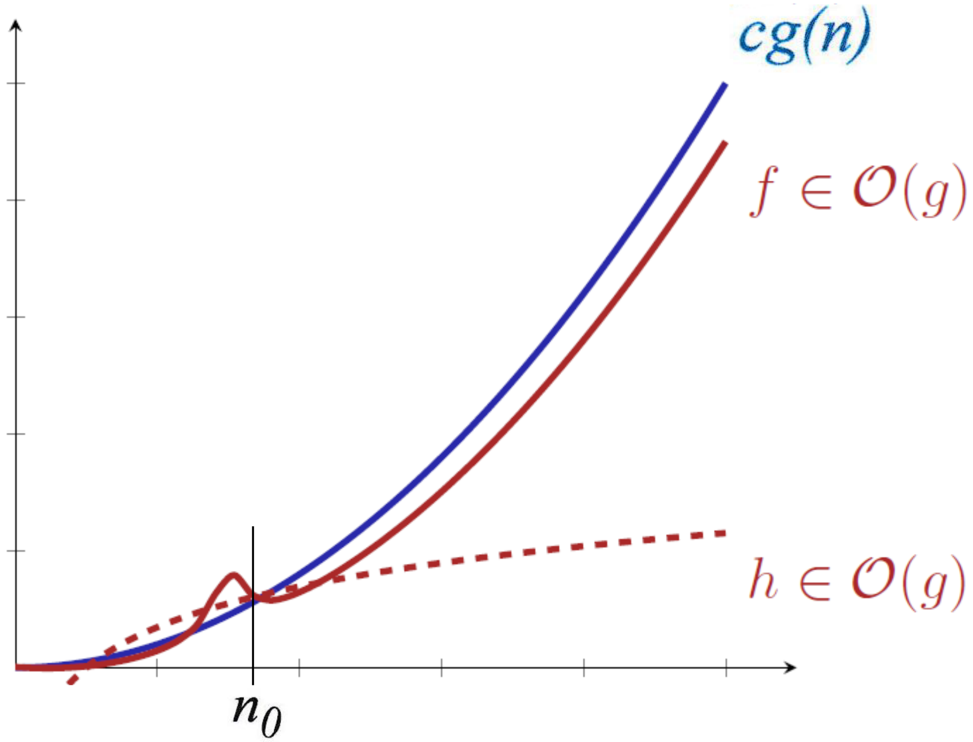
g(n) = upper bound for **T(n).** The number of steps is at most **g(n)** (constant **c** doesn't count in asymptotic analysis).

c (constant)

n (data)

Worst case scenario!

Big-Oh Asymptotic Notation

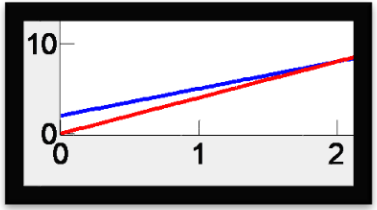


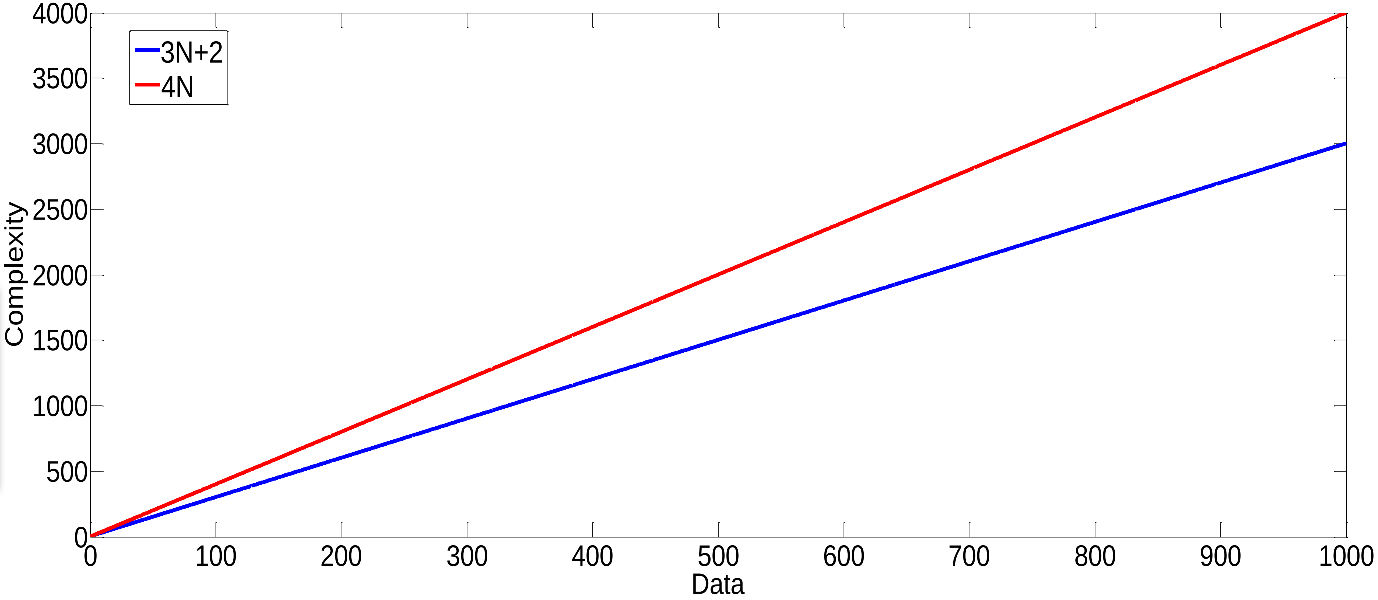
gives the **worst-case complexity** of an algorithm

Big-Oh Asymptotic Notation

Examples:

T(n) = 3n+2 = **O(n),** c=4 and n_0=2: $3n+2 \leq 4n$ $\forall n \geq 2$

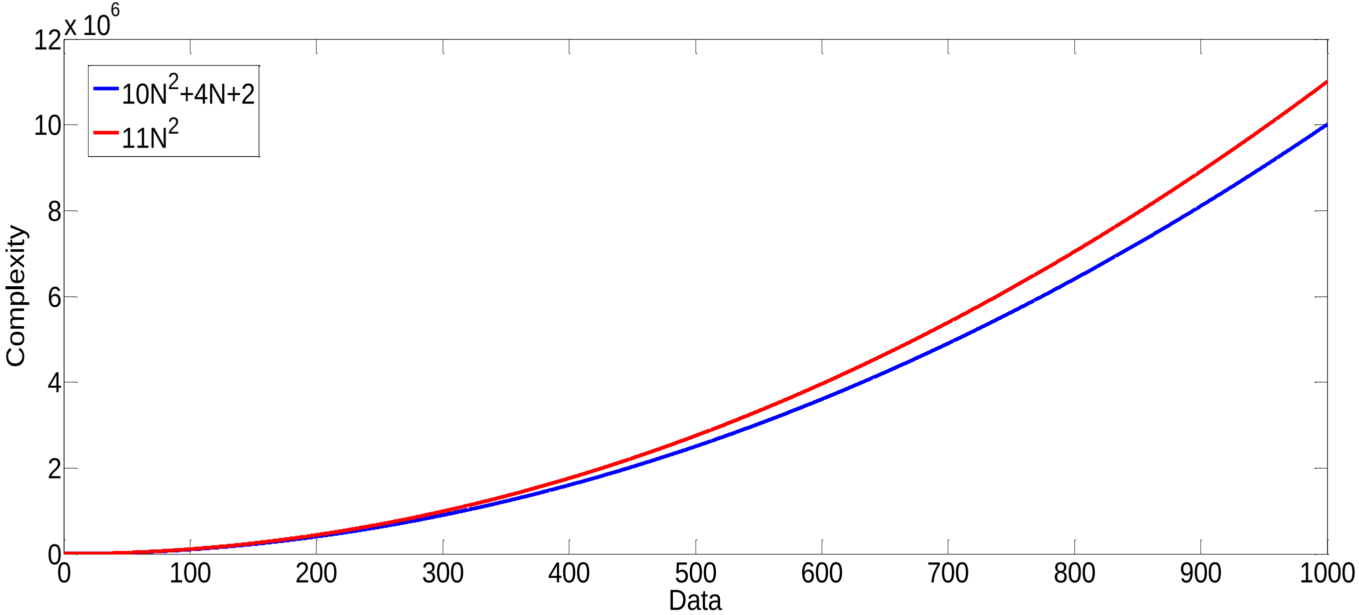




Big-Oh Asymptotic Notation

Examples:

T(n) = $10n^2+4n+2$ = **O(n^2),** c=11 and n_0=5 $10n^2+4n+2 \leq 11n^2 \;\; \forall n \geq 5$



Big-Oh Asymptotic Notation

Examples:

T(n) = $3n+2$ = **O(n^2),** $c=3$ and $n_0=2$ $3n+2 \leq 3n^2$ $\forall n \geq 2$

Theorem:

if **T(n)** = $a_m n^m + \ldots. + a_1 n + a_0$ Then **T(n)** = **O(n^m)**

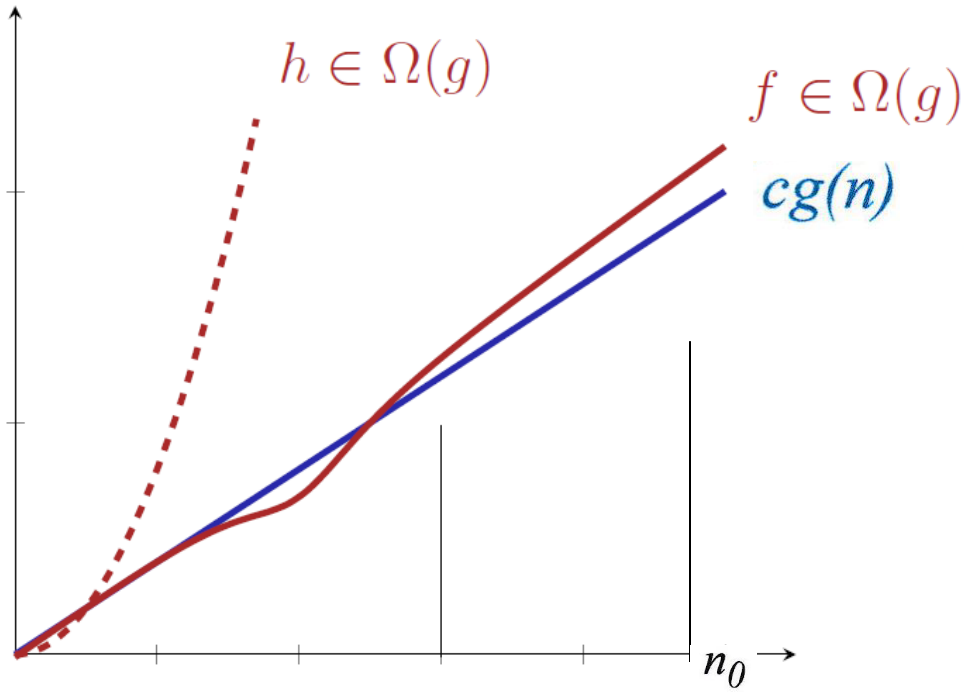
Big-Omega (Ω) Asymptotic Notation

Definition:

$$\mathbf{T(n)} = \mathbf{\Omega(g(n))} \Leftrightarrow$$
$$\exists\ \mathbf{c}>0,\ \exists\ \mathbf{n_0}>0 \text{ such that } \forall \mathbf{n} \geq \mathbf{n_0}$$
$$0 \leq c\ \mathbf{g(n)} \leq \mathbf{T(n)}$$

g(n) = lower bound for **T(n).** The number of steps is at least **g(n)** (constant c doesn’t count in asymptotic analysis).

Big-Omega (Ω) Asymptotic Notation



provides **the best case complexity** of an algorithm

Big-Omega (Ω) Asymptotic Notation

Examples:

T(n) = $3n+2$ = **Ω(n),** c=3 and $n_0=1$ $\quad 3n \leq 3n+2 \quad \forall n \geq 1$

T(n) = $10n^2+4n+2$ = **Ω(n²),** c=1 and $n_0=1$ $\quad n^2 \leq 10n^2+4n+2 \quad \forall n \geq 1$

T(n) = $10n^2+4n+2$ = **Ω(n),** c=30 and $n_0=3$ $\quad 30n \leq 10n^2+4n+2 \quad \forall n \geq 3$

Theorem:

if **T(n)** = $a_m n^m + \ldots. + a_1 n + a_0$ then **T(n)** = **Ω(n^m)**

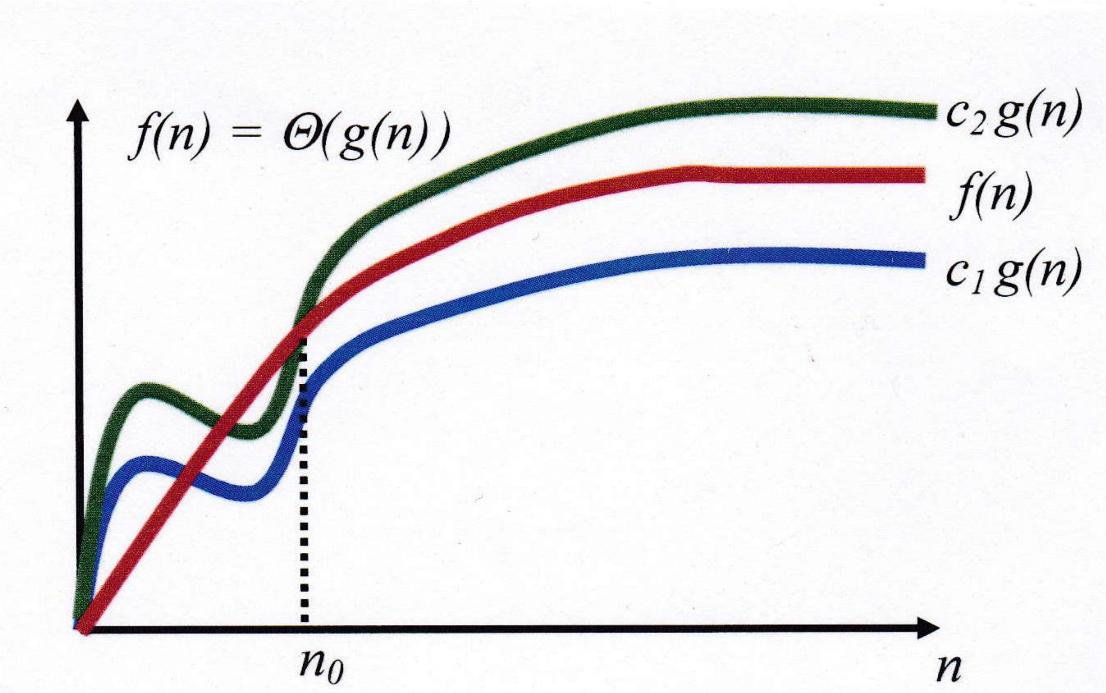
Big-Theta (Θ) Asymptotic Notation

Definition:

$$\mathbf{T(n)} = \Theta(\mathbf{g(n)}) \Leftrightarrow$$
$$\exists\ \mathbf{c_1, c_2} > 0,\ \exists\ \mathbf{n_0} > 0 \text{ such that } \forall \mathbf{n} \geq\ \mathbf{n_0}$$
$$0 \leq \mathbf{c_1\ g(n)} \leq \mathbf{T(n)} \leq \mathbf{c_2\ g(n)}$$

g(n) = tight asymptotic bound for **T(n).** The number of steps is exactly **g(n)** (constants **c_1** and **c_2** do not count in asymptotic analysis).

Big-Theta (Θ) Asymptotic Notation



f(n) can be **sandwiched** between *c_1g(n)* and *c_2g(n)*

Average-case complexity analysis of an algorithm.

Big-Theta (Θ) Asymptotic Notation

Examples:

$\mathbf{T(n)} = 3n+2 = \mathbf{\Theta(n)}$, $c_1=3$, $c_2=4$ and $n_0=2$ $\quad 3n \leq 3n+2 \leq 4n \quad \forall n \geq 1$

$\mathbf{T(n)} = 3n+2 \neq \mathbf{\Theta(n^2)}$, $\mathbf{T(n)} = 10n^2+4n+2 \neq \mathbf{\Theta(n)}$

Theorems:

- If $\mathbf{T(n)} = a_m n^m + \dots + a_1 n + a_0$, then $\mathbf{T(n)} = \mathbf{\Theta(n^m)}$
- Let **g(n)** and **T(n)** be 2 functions,

  $\mathbf{T(n)} = \mathbf{\Theta(g(n))} \Leftrightarrow \mathbf{T(n)} = \mathbf{O(g(n))}$ and $\mathbf{T(n)} = \mathbf{\Omega(g(n))}$.

Big-Theta (Θ) Asymptotic Notation

Exercise:

Given f(n), g(n), h(n), k(n):

if $f(n) = \Theta(k(n))$ and $k(n) = \Theta(g(n))$, then $g(n) = \Theta(f(n))$?

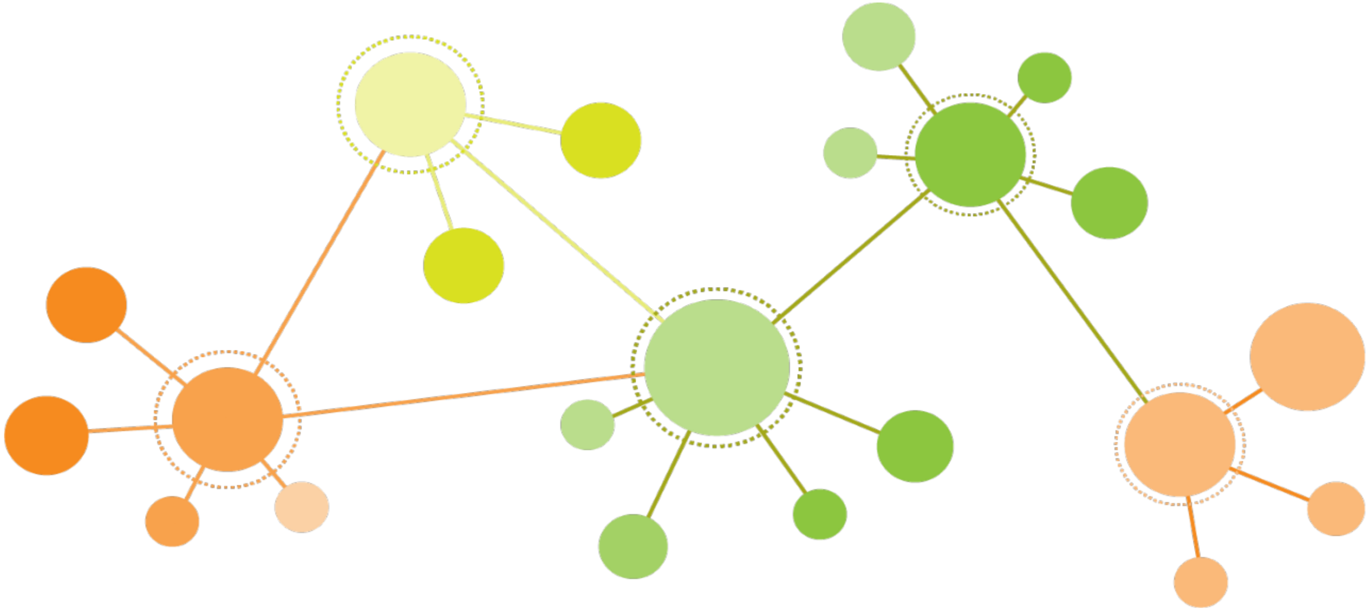
Yes / No

if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $h(n) = \Omega(f(n))$?

Yes / No

Online Connectivity

Real problem to understand the impact of the choice of the **algorithm** and of the **data structure** on **complexity**:



Online Connectivity

Real problem to understand the impact of the choice of the **algorithm** and of the **data structure** on **complexity**:

- **Undirected graph** whose **vertices** are integers and whose **edges** are pairs of integers
- **Input:** sequence of integer pairs (**p**, **q**)
- **Interpretation:** **p** is connected to **q**
- Connectivity relation:
  - **Reflexive:** ***p*** *is connected to* ***p***
  - **Symmetrical:** *if* ***p*** *is connected to* ***q****,* ***q*** *is connected to* ***p***
  - **Transitive:** *if* ***p*** *is connected to* ***q*** *and* ***q*** *is connected to* ***r****, then* ***p*** *is connected to* ***r***

Thus it is an **equivalence** relation.

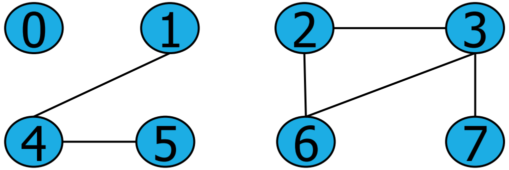
Online Connectivity

- **Output:** list of previously unknown connections (or not transitively implied by the previous ones):
  - null if **p** and **q** are already connected (directly or indirectly)
  - else (**p**, **q**)

Online Connectivity

Connected component in an **undirected graph**:
maximal subset of mutually reachable nodes

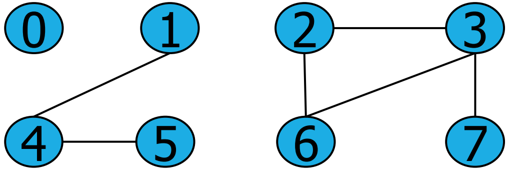
Example:

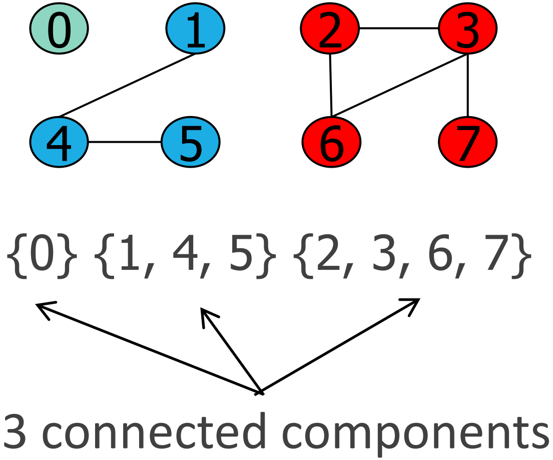


Online Connectivity

Connected component in an **undirected graph**:
maximal subset of mutually reachable nodes

Example:





{0} {1, 4, 5} {2, 3, 6, 7}

3 connected components

Applications

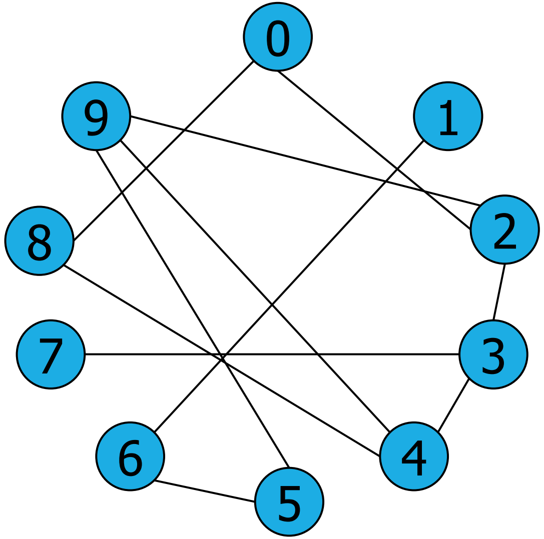
- **Pixels** in digital pictures
- **Computer networks** (computers, links)
- **Electrical networks** (components, wires)
- **Social networks** (friends)
- Mathematical **sets**
- Program **variables**.

Example

Input sequence:

3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

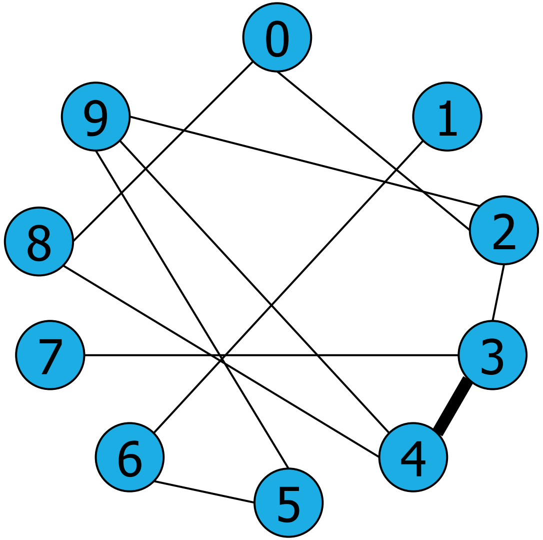
Corrisponding graph:



Example

Let's validate:

Input		Output	
3	4	3	4



Example

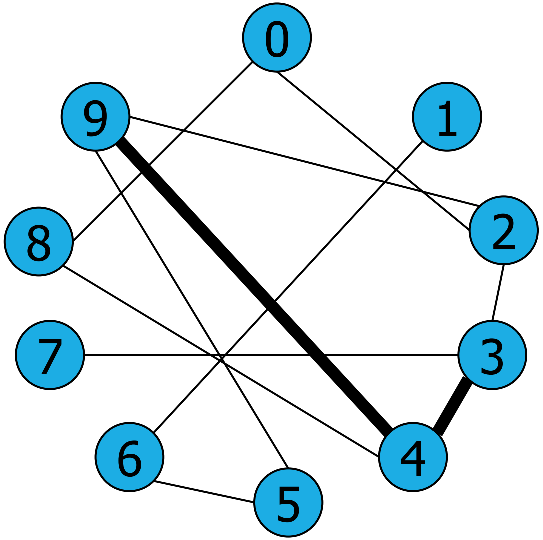
Let’s validate:

Input

9 4

Output

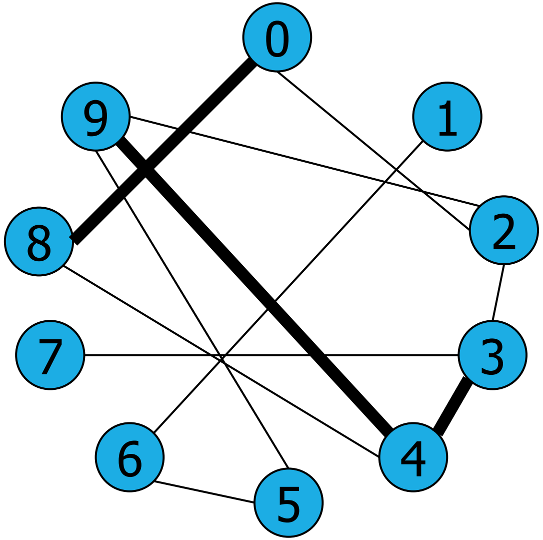
9 4



Example

Let’s validate:

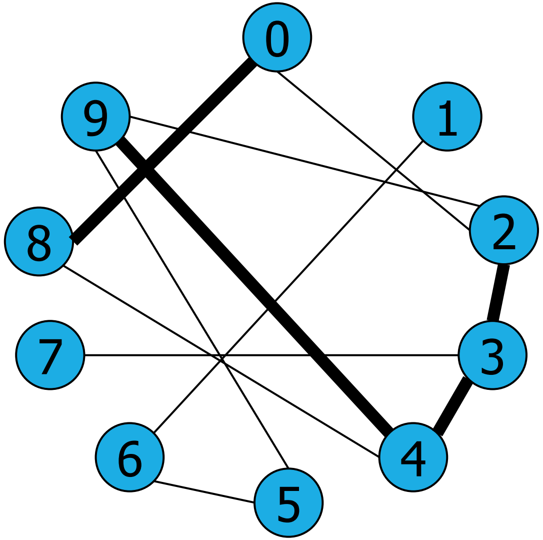
Input		Output	
8	0	8	0



Example

Let's validate:

Input		Output	
2	3	2	3



Example

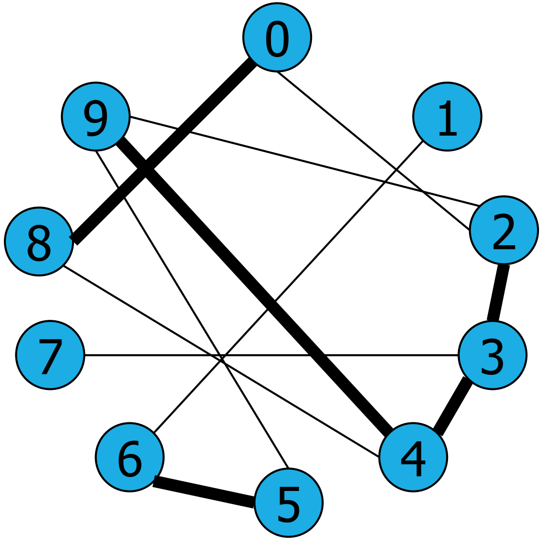
Let's validate:

Input

5 6

Output

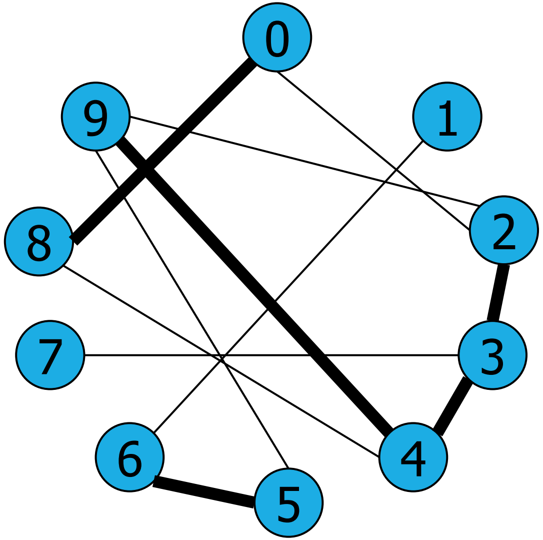
5 6



Example

Let’s validate:

Input	Output
2 9

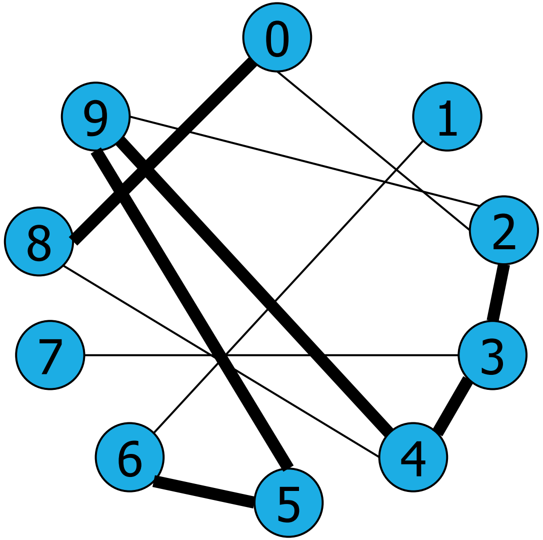


Path 2-3-4-9 already exists

Example

Let's validate:

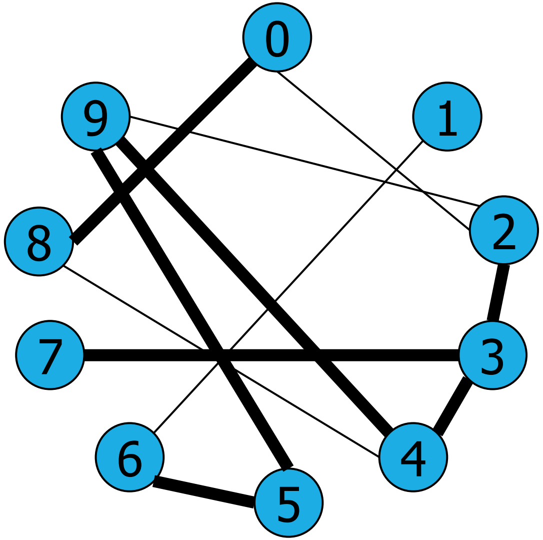
Input	Output
5 9	5 9



Example

Let's validate:

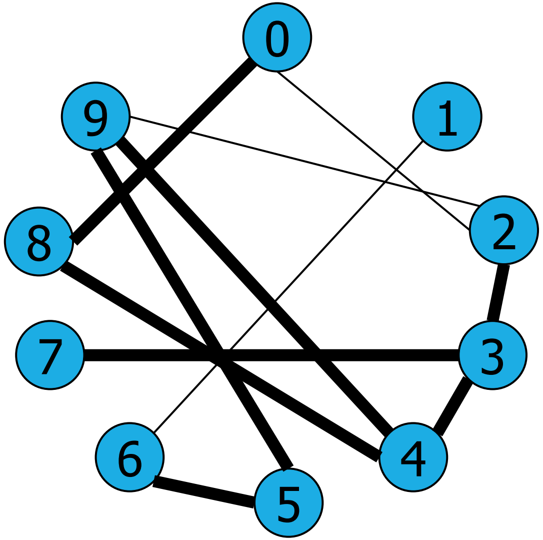
Input		Output	
7	3	7	3



Example

Let's validate:

Input		Output	
4	8	4	8



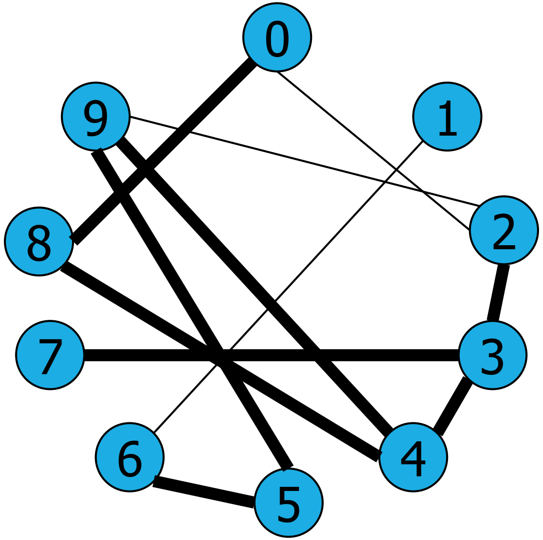
Example

Let's validate:

Input

5 6

Output



Path 5-6 already exists

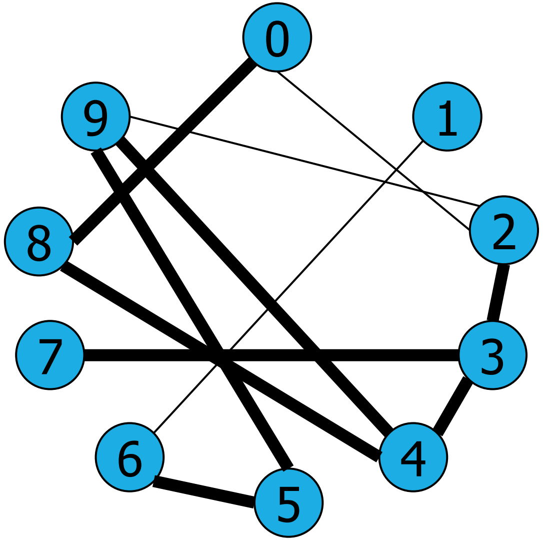
Example

Let's validate:

Input

0 2

Output



Path 0-8-4-3-2 already exists

Example

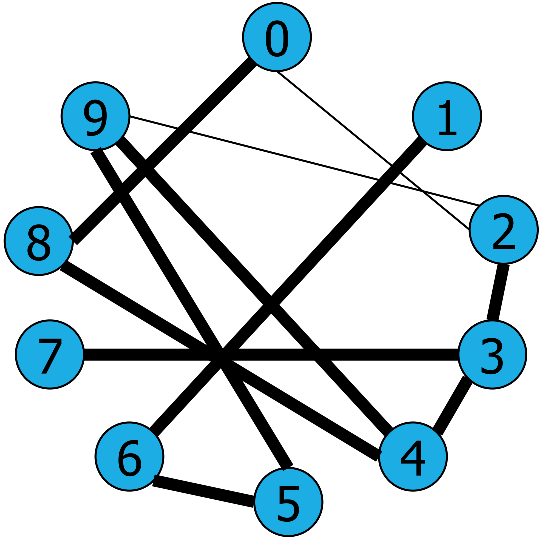
Let's validate:

Input

6 1

Output

6 1



On-line approach

Assumptions:

- We don’t have the **graph**
- We work **online** pair by pair, keeping and **updating** information necessary to find out connectivity.
- Each pair is made of **2** integers in the range from **0** to **N-1**

Sets $\mathbf{S_i}$ of connected pairs, initially as many sets as nodes, each node being connected just to itself.

Abstract operations:

- **find:** find the set an object belongs to
- **union:** merge two sets

On-line approach

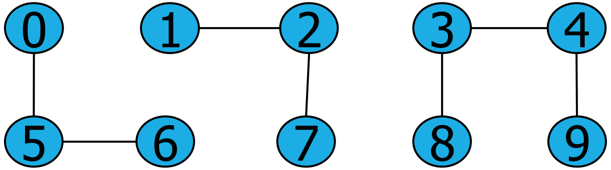
- **Algorithm:** repeat for all pairs (**p**, **q**)
  - **read** the pair (**p**, **q**)
  - **execute find on p:** find an S_p such that **p**$\in S_p$
  - **execute find on q**: find an S_q such that **q**$\in S_q$
  - if S_p and S_q coincide, consider the next pair, otherwise execute union on S_p and S_q

Quick find

Represent sets S_i of connected pairs with array id:

- initially **id[i]** = **i** (no connection)
- if **p** and **q** are connected, id[**p**] = id[**q**]

Example: the following **graph**



is represented like this:

	0	1	1	8	8	0	0	1	8	8
id	0	1	2	3	4	5	6	7	8	9

Quick find

Algorithm:

- repeat for all pairs (**p**, **q**):
  - read pair (**p, q**)
  - if pair is connected **(id[p] = id[q]),**
    - do nothing and move to the next pair,
  - else
    - scan the array**, replacing id[p]** values with **id[q]** values

Quick find

- **find:** simple reference to cell in array **id[index],** unit cost **O(1)**
- **union:** scan array to replace **id[p]** values with **id[q]** values, cost linear in array size **O(n)**

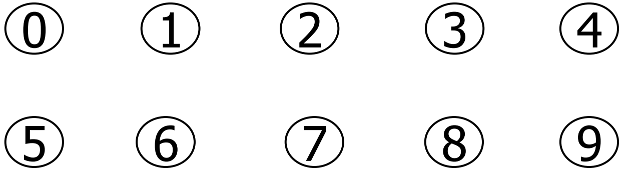
- overall **number of operations** related to

  # pairs * array size

Tree representation

- Some objects **represent** the set they belong to
- Other objects **point** to the the object that represents the set they belong to.

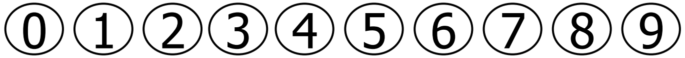
Example



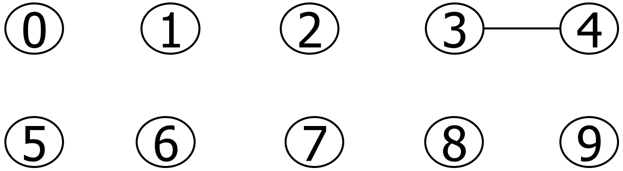
Initially

id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

S_0 = {0}, S_1 = {1}, S_2 = {2}, S_3 = {3}, S_4 = {4}
S_5 = {5}, S_6 = {6}, S_7 = {7}, S_8 = {8}, S_9 = {9}



Example



Input: **p** **q** = 3 4

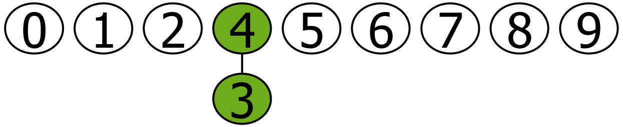
id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

id[**p**]=3 $\neq$ id[**q**]=4

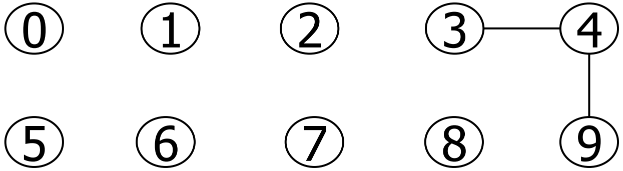
change all id[**p**] values in id[**q**]

id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

$S_0 = \{0\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_{3\text{-}4} = \{\mathbf{3,4}\}$,
$S_5 = \{5\}$, $S_6 = \{6\}$, $S_7 = \{7\}$, $S_8 = \{8\}$, $S_9 = \{9\}$



Example



Input: **p** **q** = 4 9

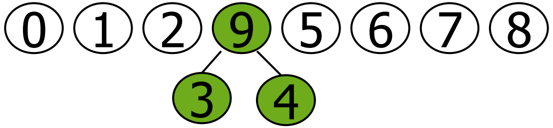
id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

id[**p**]=4 $\neq$ id[**q**]=9

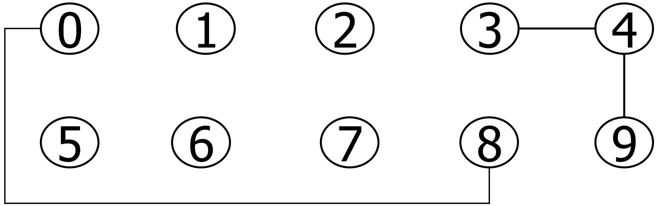
change all id[**p**] values in id[**q**]

id	0	1	2	9	9	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

S_0 = {0}, S_1 = {1}, S_2 = {2}, $S_{3\text{-}4\text{-}9}$ = {**3,4,9**},
S_5 = {5}, S_6 = {6}, S_7 = {7}, S_8 = {8}



Example



Input: **p** **q** = 8 0

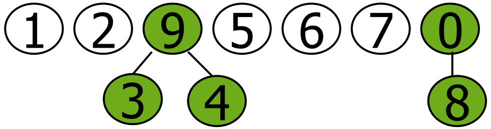
id	0	1	2	9	9	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

id[**p**]=8 $\neq$ id[**q**]=0

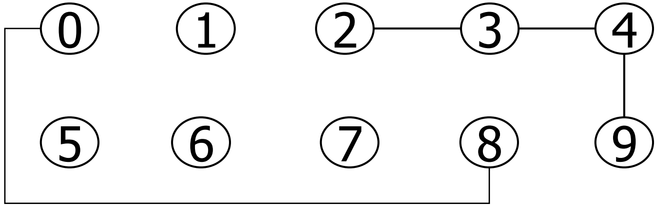
change all id[**p**] values in id[**q**]

id	0	1	2	9	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

S_{0-8} = {**0,8**}, S_1 = {1}, S_2 = {2}, S_{3-4-9} = {**3,4,9**},
S_5 = {5}, S_6 = {6}, S_7 = {7}



Example



Input: **p** **q** = 2 3

id	0	1	2	9	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

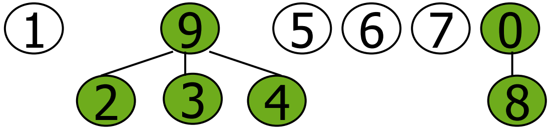
id[**p**]=2 $\neq$ id[**q**]=9

change all id[**p**] values in id[**q**]

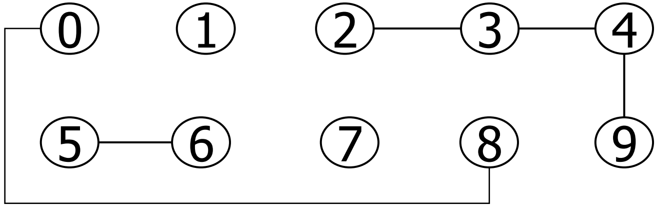
id	0	1	9	9	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8}$ = {**0,8**}, S_1 = {1}, $S_{2\text{-}3\text{-}4\text{-}9}$ = {**2,3,4,9**},

S_5 = {5}, S_6 = {6}, S_7 = {7}



Example



Input: **p** **q** = 5 6

id	0	1	9	9	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

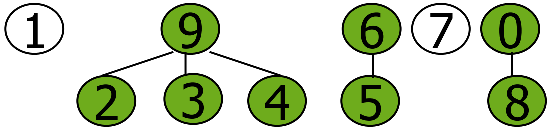
id[**p**]=5 ≠ id[**q**]=6

change all id[**p**] values in id[**q**]

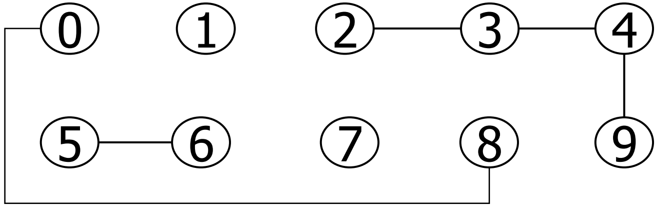
id	0	1	9	9	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8}$ = {**0,8**}, S_1 = {1}, $S_{2\text{-}3\text{-}4\text{-}9}$ = {**2,3,4,9**},

$S_{5\text{-}6}$ = {**5,6**}, S_7 = {7}



Example



Input: **p** **q** = 2 9

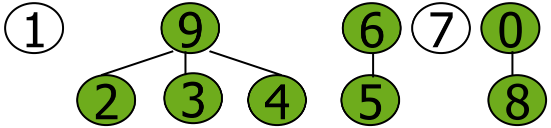
id	0	1	9	9	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[**p**]=9 = id[**q**]=9

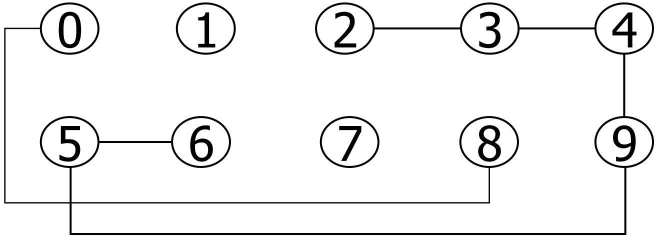
no change

id	0	1	9	9	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8} = \{\mathbf{0,8}\}$, $S_1 = \{1\}$, $S_{2\text{-}3\text{-}4\text{-}9} = \{\mathbf{2,3,4,9}\}$,
$S_{5\text{-}6} = \{\mathbf{5,6}\}$, $S_7 = \{7\}$



Example



Input: **p q** = 5 9

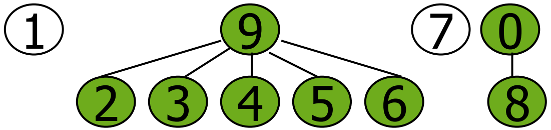
id	0	1	9	9	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[**p**]=6 ≠ id[**q**]=9

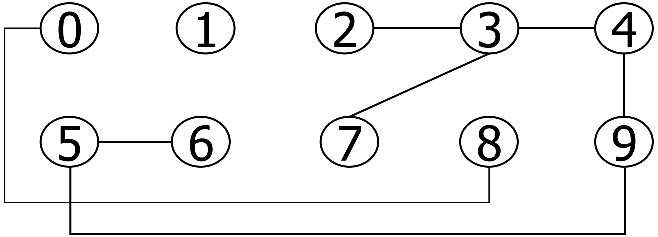
change all id[**p**] values in id[**q**]

id	0	1	9	9	9	9	9	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8} = \{\mathbf{0,8}\}$, $S_1 = \{1\}$, $S_{2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}9} = \{\mathbf{2,3,4,5,6,9}\}$,
$S_7 = \{7\}$



Example



Input: **p** **q** = 7 3

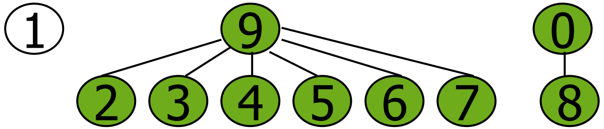
id	0	1	9	9	9	9	9	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[**p**]=7 ≠ id[**q**]=9

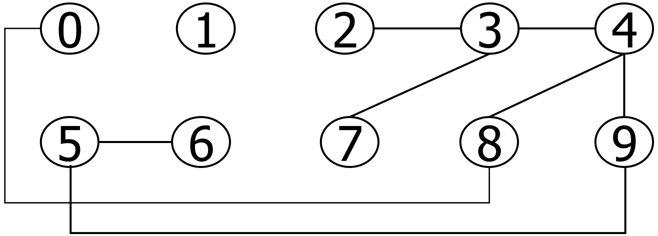
change all id[**p**] values in id[**q**]

id	0	1	9	9	9	9	9	9	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8} = \{\mathbf{0,8}\}$, $S_1 = \{1\}$, $S_{2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}9} = \{\mathbf{2,3,4,5,6,7,9}\}$



Example



Input: **p** **q** = 4 8

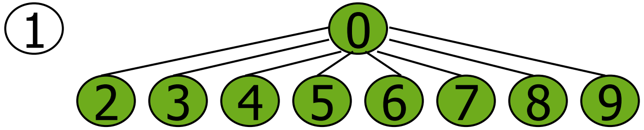
id	0	1	9	9	9	9	9	9	0	9
	0	1	2	3	4	5	6	7	8	9

id[**p**]=9 ≠ id[**q**]=0

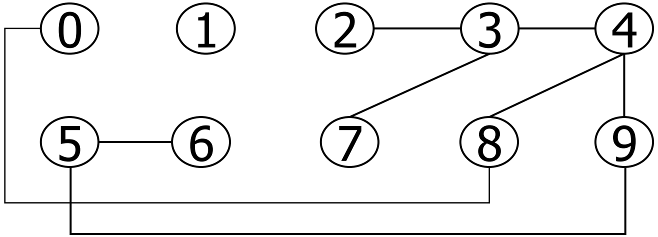
change all id[**p**] values in id[**q**]

id	0	1	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

S_1 = {1}, $S_{0\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}8\text{-}9}$ = {**0,2,3,4,5,6,7,8,9**}



Example



Input: **p** **q** = 5 6

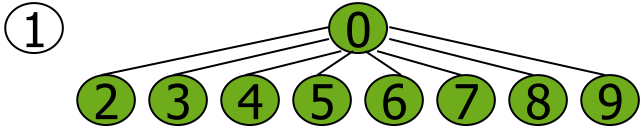
id	0	1	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

id[**p**]=0 = id[**q**]=0

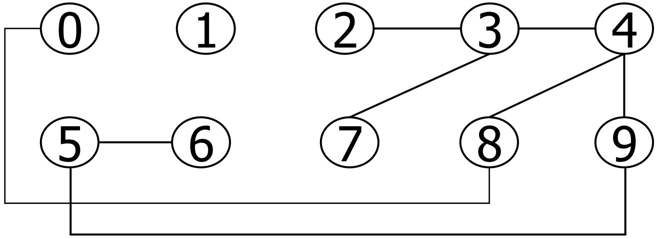
no change

id	0	1	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

S_1 = {1}, $S_{0\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}8\text{-}9}$ = {**0,2,3,4,5,6,7,8,9**}



Example



Input: **p** **q** = 0 2

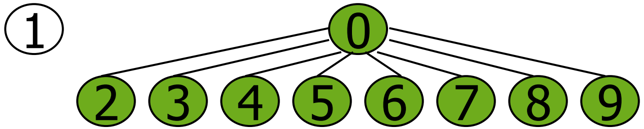
id	0	1	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

id[**p**]=0 = id[**q**]=0

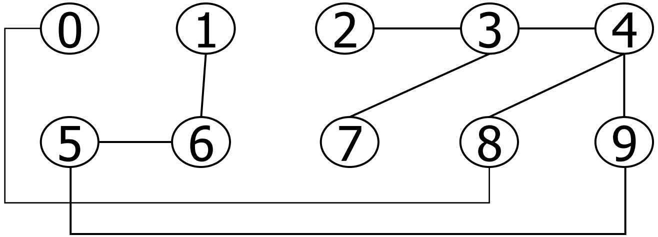
no change

id	0	1	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

S_1 = {1}, $S_{0-2-3-4-5-6-7-8-9}$ = {**0,2,3,4,5,6,7,8,9**}



Example



Input: **p q** = 6 1

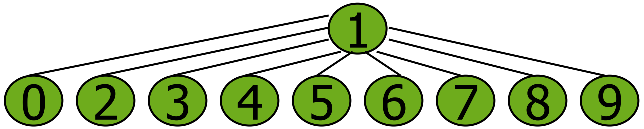
id	0	1	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

id[**p**]=0 = id[**q**]=1

change all id[**p**] values in id[**q**]

id	1	1	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}1\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}8\text{-}9}$ = {**0,1,2,3,4,5,6,7,8,9**}



```
#include <stdio.h>
#define N 10000
main() {
  int i, t, p, q, id[N];
  for (i=0; i<N; i++)
    id[i] = i;
  printf("Input pair p q:  ");
  while (scanf("%d %d", &p, &q) ==2) {
    if (id[p] == id[q])
      printf(“%d %d already connected\n", p,q);
    else {
      for (t = id[p], i = 0; i < N; i++)
        if (id[i] == t)
          id[i] = id[q];
        printf("pair %d %d not yet connected\n", p, q);
      }
      printf("Input pair p q:  ");
  }
}
```

Quick union

Represent sets S_i of connected pairs with an **array id**:

- initially all the objects point to themselves

  id[i] = i (no connection)

- each object points either to an object to which it is connected or to itself (no loops).

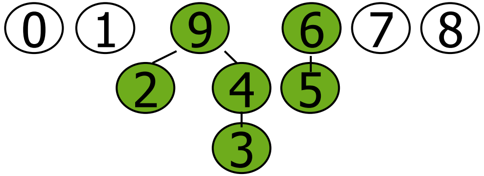
Notation **(id[i])*** stands **for id[id[id[... id[i]]]]**

If objects **i** are **j** connected

(id[i])* = **(id[j])***

Example

id	0	1	9	4	9	6	6	7	8	9
	0	1	2	3	4	5	6	7	8	9



Quick union

Algorithm:

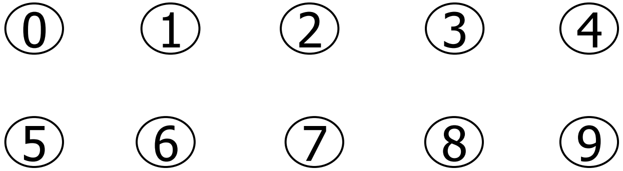
- repeat for all the pairs (**p**, **q**):
  - **read** pair (**p**, **q**)
  - **if(id[p])* = (id[q])***
    - **do nothing** (the pair is already connected) and move on to the next pair,
  - else
    - **id[(id[p])*]** = **(id[q])*** (connect the pair).

Quick union

- **find:** scan a ***"chain"*** of objects, upper bound linear cost in the number of objects, in general well below upper bound **O(n)**
- **union:** simple, as it is enough that an object points to another object, unit cost **O(1)**
- overall number of operations related to

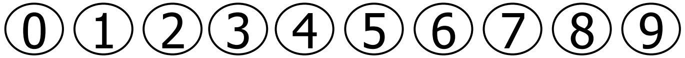
# pairs * chain length

Example

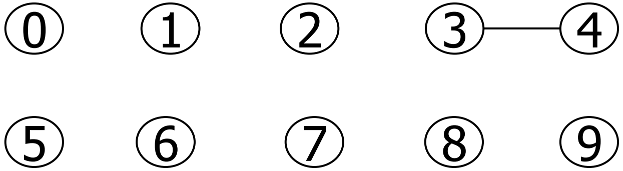


Initially

id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9



Example



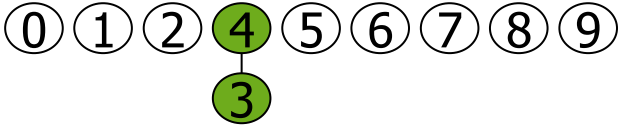
Input: **p** **q** = 3 4

id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

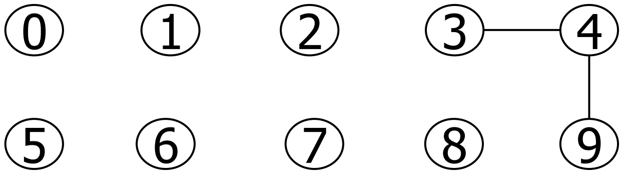
id[**p**]=3 $\neq$ id[**q**]=4

let **p** point to **q**: id[**p**]=4

id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9



Example



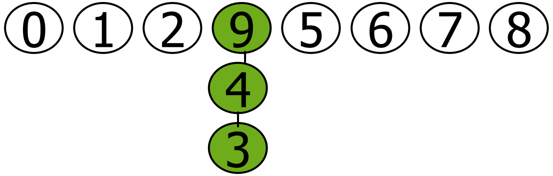
Input: **p** **q** = 4 9

id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

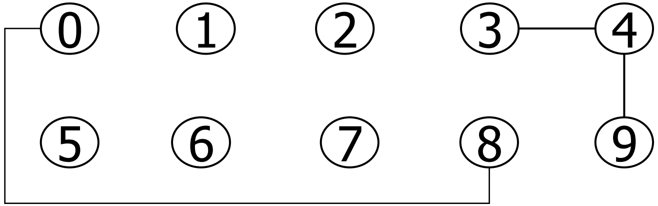
id[**p**]=4 $\neq$ id[**q**]=9

let **p** point to **q**: id[**p**]=9

id	0	1	2	4	9	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9



Example



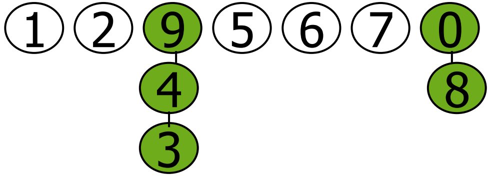
Input: **p** **q** = 8 0

id	0	1	2	4	9	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

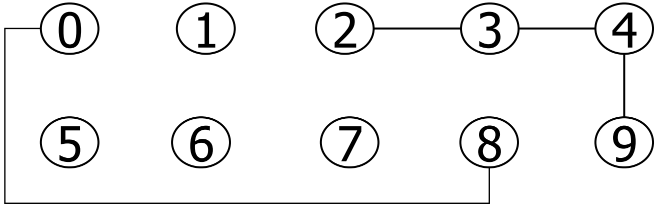
id[**p**]=8 $\neq$ id[**q**]=0

let **p** point to **q**: id[**p**]=0

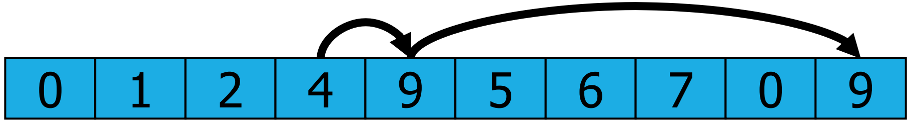
id	0	1	2	4	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9



Example



Input: **p** **q** = 2 3

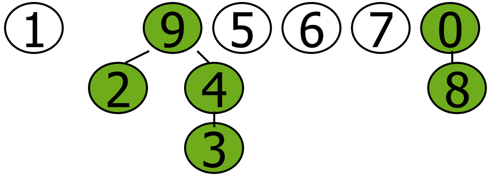


id	0	1	2	4	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

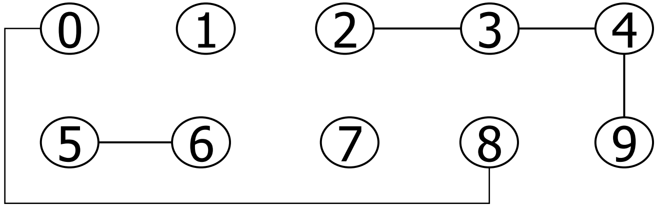
id[**p**]=2 ≠ id[id[id[**q**]]]=9

let **p** point to **q**: id[**p**]=9

id	0	1	9	4	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9



Example



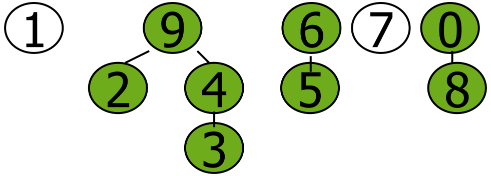
Input: **p** **q** = 5 6

id	0	1	9	4	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

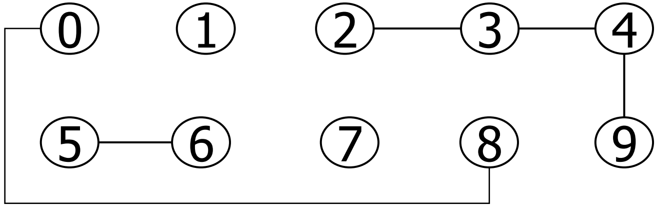
id[**p**]=5 ≠ id[**q**]=6

let **p** point to **q**: id[**p**]=6

id	0	1	9	4	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9



Example



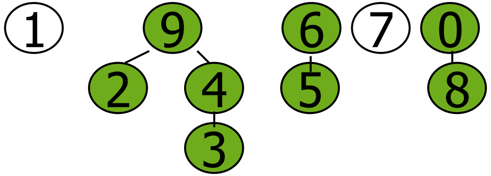
Input: **p** **q** = 2 9

id	0	1	9	4	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

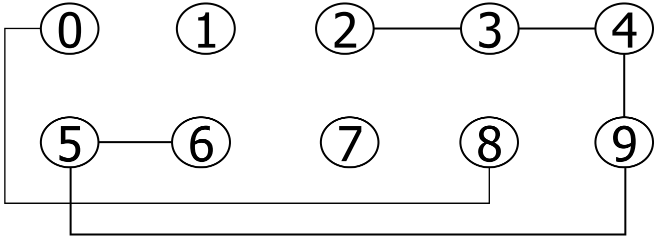
id[id[**p**]]=9 = id[**q**]=9

unchanged

id	0	1	9	4	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9



Example



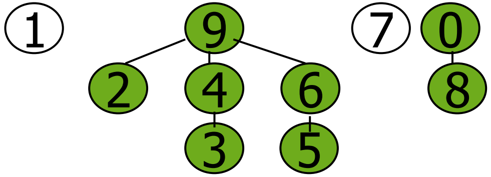
Input: **p** **q** = 5 9

id	0	1	9	4	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

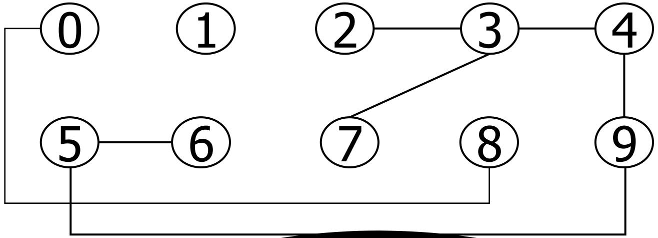
id[id[**p**]]=6 ≠ id[**q**]=9

let **p** point to **q**: id[id[**p**]]=9

id	0	1	9	4	9	6	9	7	0	9
	0	1	2	3	4	5	6	7	8	9



Example



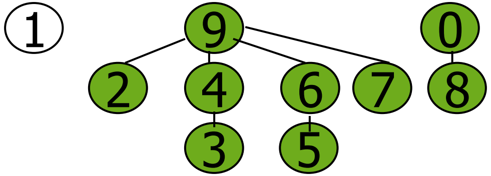
Input: **p q** = 7 3

id	0	1	9	4	9	6	9	7	0	9
	0	1	2	3	4	5	6	7	8	9

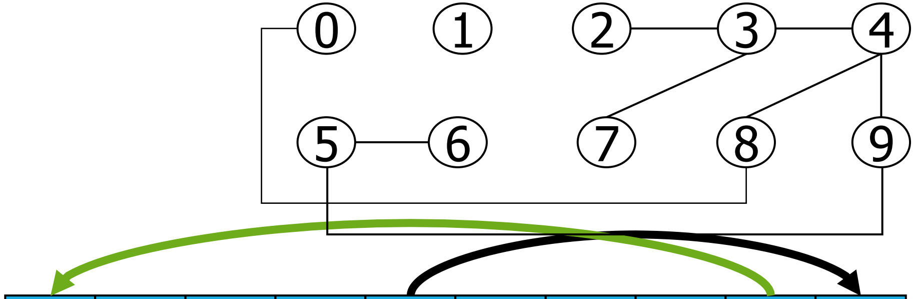
id[**p**]=7 ≠ id[id[id[**q**]]]=9

let **p** point to **q**: id[**p**]=9

id	0	1	9	4	9	6	9	9	0	9
	0	1	2	3	4	5	6	7	8	9



Example



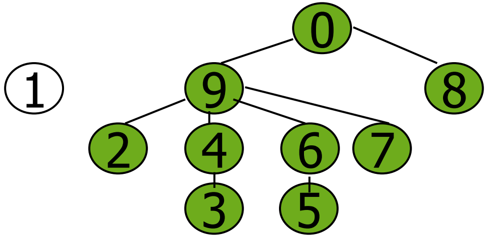
Input: **p** **q** = 4 8

id	0	1	9	4	9	6	9	9	0	9
	0	1	2	3	4	5	6	7	8	9

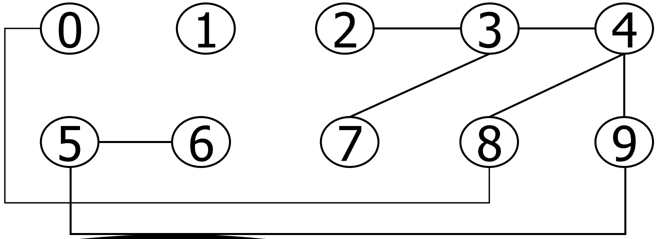
id[id[**p**]]=9 ≠ id[id[**q**]]=0

let **p** point to **q**: id[id[**p**]]=0

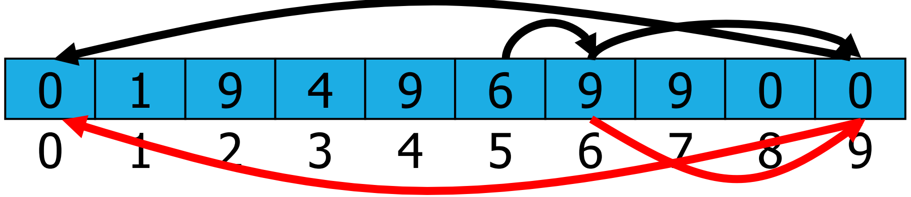
id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9



Example



Input: **p** **q** = 5 6

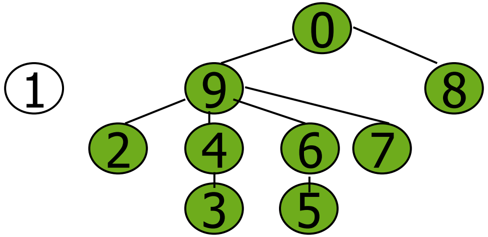


id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9

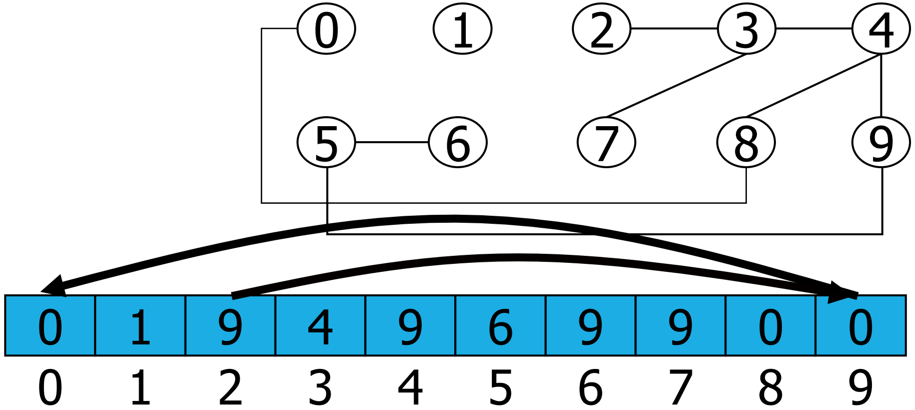
id[id[id[id[**p**]]]]=0 = id[id[**q**]]=0

unchanged

id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9



Example



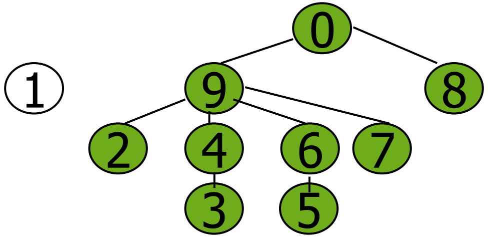
Input: **p** **q** = 0 2

id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9

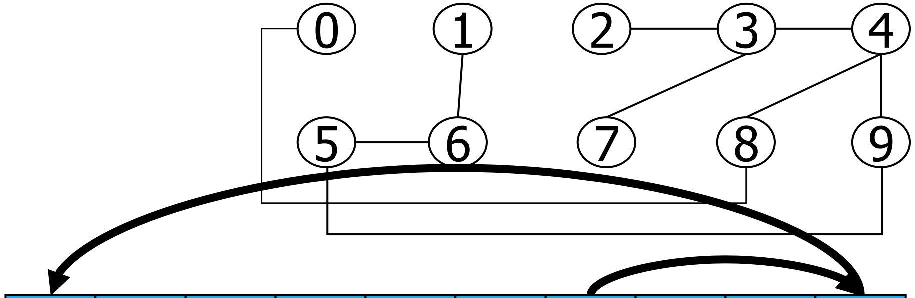
id[**p**]=0 = id[id[id[**q**]]]=0

unchanged

id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9



Example



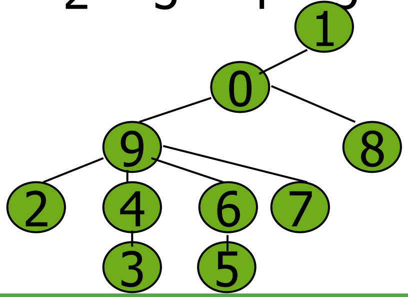
Input: **p** **q** = 6 1

id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9

id[id[id[**p**]]]=0 ≠ id[**q**]=1

let **p** point to **q**: id[id[id[**p**]]]=1

id	1	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9

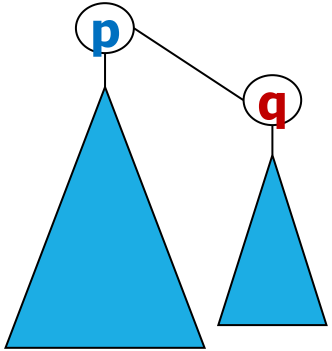


```
#include <stdio.h>
#define N 10000
main() {
  int i, j, p, q, id[N];
  for(i=0; i<N; i++)
    id[i] = i;
  printf("Input pair p q:  ");
  while (scanf("%d %d", &p, &q) ==2) {
    for (i = p; i!= id[i]; i = id[i]);
    for (j = q; j!= id[j]; j = id[j]);
    if (i == j)
      printf("pair %d %d already connected\n", p,q);
    else {
      id[i] = j;
      printf("pair %d %d not yet connected\n", p, q);
    }
    printf("Input pair p q:  ");
  }
}
```

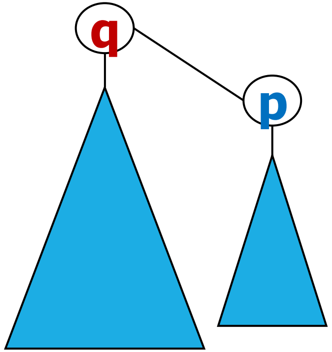
Quick union Optimization

Weighted quick union:

- To **shorten** the chain’s length, keep track of the number of elements in **each tree** (array `sz`) and **connect the smaller tree to the larger one**.
- According to which one is the larger, there might be 2 solutions:

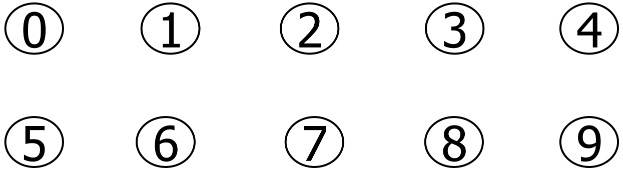


or



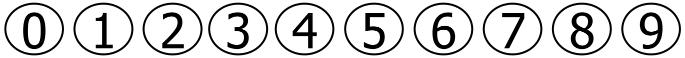
NB: it doesn’t matter whether if **p** appears at the right or at the left of **q**.

Example

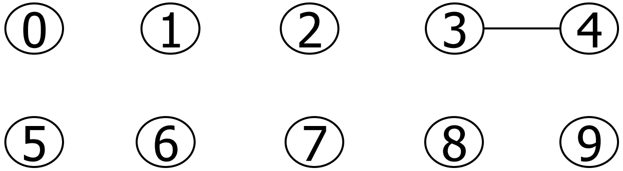


Initially

id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9



Example



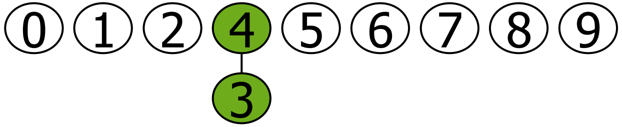
Input: **p** **q** = 3 4

id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

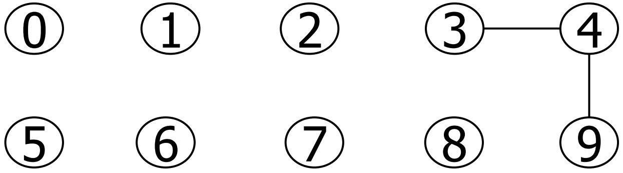
id[**p**]=3 ≠ id[**q**]=4

let **p** point to **q**: id[**p**]=4

id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9



Example



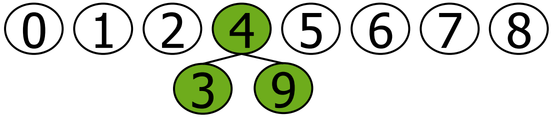
Input: **p** **q** = 4 9

id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

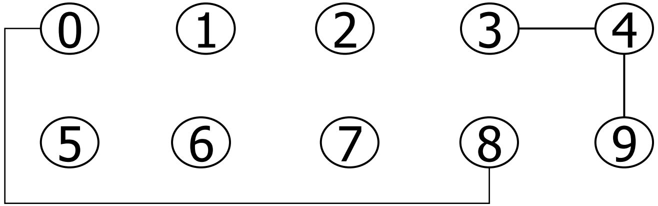
id[**p**]=4 $\neq$ id[**q**]=9

let the smaller tree **q** pointo to the larger tree **p**: id[**q**]=4

id	0	1	2	4	4	5	6	7	8	4
	0	1	2	3	4	5	6	7	8	9



Example



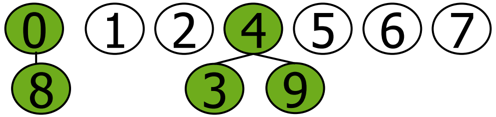
Input: **p** **q** = 8 0

id	0	1	2	4	4	5	6	7	8	4
	0	1	2	3	4	5	6	7	8	9

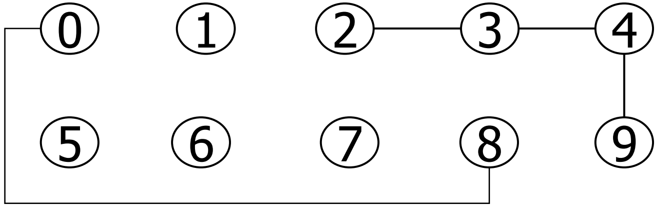
id[**p**]=8 ≠ id[**q**]=0

let **p** point to **q**: id[**p**]=0

id	0	1	2	4	4	5	6	7	0	4
	0	1	2	3	4	5	6	7	8	9



Example



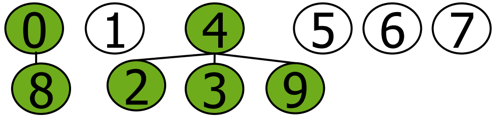
Input: **p** **q** = 2 3

id	0	1	2	4	4	5	6	7	0	4
	0	1	2	3	4	5	6	7	8	9

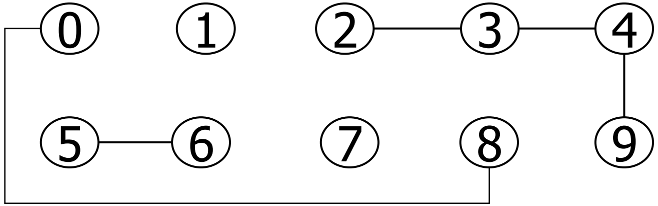
id[**p**]=2 ≠ id[id[**q**]]=4

let the smaller tree **q** pointo to the larger tree **p** : id[**p**]=4

id	0	1	4	4	4	5	6	7	0	4
	0	1	2	3	4	5	6	7	8	9



Example



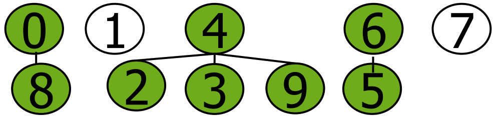
Input: **p** **q** = 5 6

id	0	1	4	4	4	5	6	7	0	4
	0	1	2	3	4	5	6	7	8	9

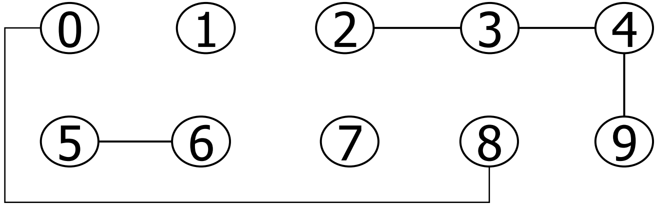
id[**p**]=5 $\neq$ id[**q**]=6

let **p** point to **q**: id[**p**]=6

id	0	1	4	4	4	6	6	7	0	4
	0	1	2	3	4	5	6	7	8	9



Example



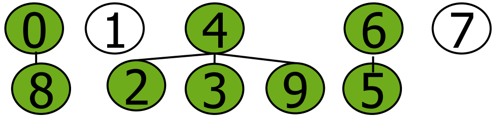
Input: **p** **q** = 2 9

id	0	1	4	4	4	6	6	7	0	4
	0	1	2	3	4	5	6	7	8	9

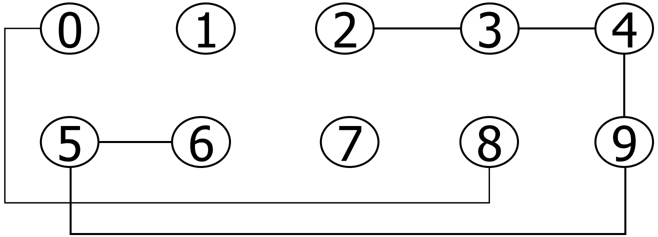
id[id[**p**]]=4 = id[**q**]=4

unchanged

id	0	1	4	4	4	6	6	7	0	4
	0	1	2	3	4	5	6	7	8	9



Example



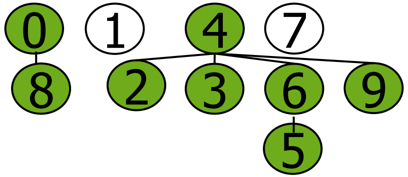
Input: **p** **q** = 5 9

id	0	1	4	4	4	6	6	7	0	4
	0	1	2	3	4	5	6	7	8	9

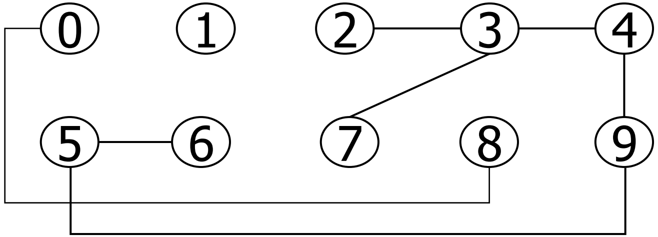
id[id[**p**]]=6 ≠ id[id[**q**]]=4

let the smaller tree **q** pointo to the larger tree **p**: id[id[**p**]]=4

id	0	1	4	4	4	6	4	7	0	4
	0	1	2	3	4	5	6	7	8	9



Example



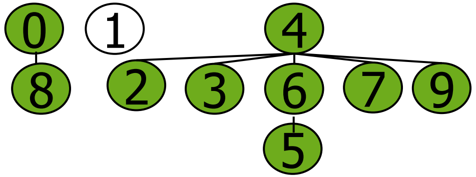
Input: **p** **q** = 7 3

id	0	1	4	4	4	6	4	7	0	4
	0	1	2	3	4	5	6	7	8	9

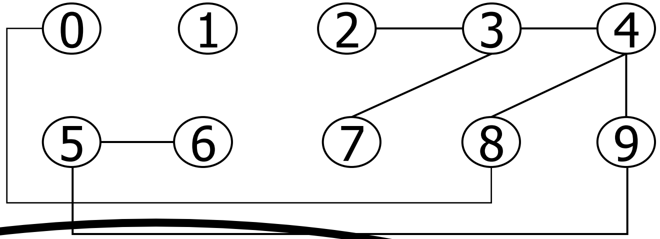
id[**p**]=7 ≠ id[id[**q**]]=4

let the smaller tree **q** pointo to the larger tree **p**: id[**p**]=4

id	0	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9



Example



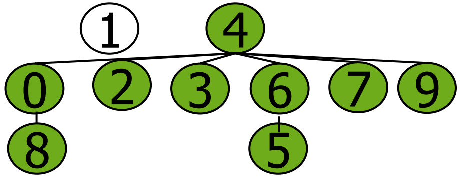
Input: **p** **q** = 4 8

id	0	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9

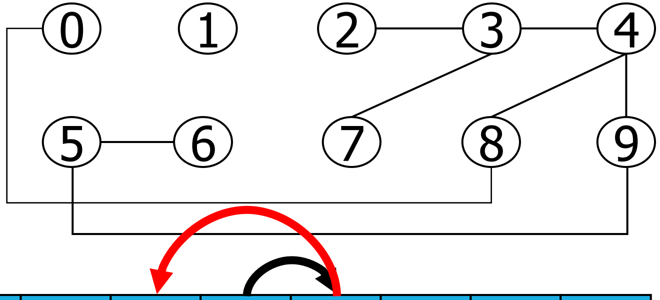
id[**p**]=4 $\neq$ id[id[**q**]]=0

let the smaller tree **q** pointo to the larger tree **p**: id[id[q]]=4

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9



Example



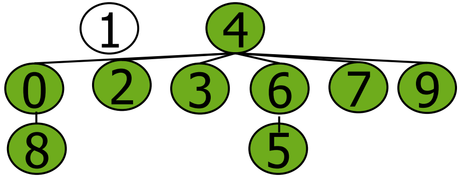
Input: **p** **q** = 5 6

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9

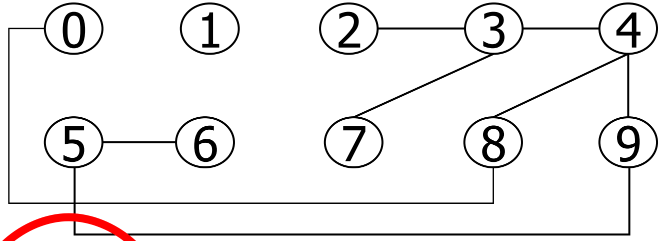
id[id[id[**p**]]]=4 = id[id[**q**]]=4

unchanged

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9



Example



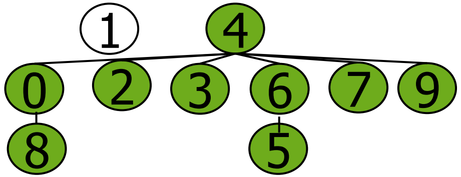
Input: **p** **q** = 0 2

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9

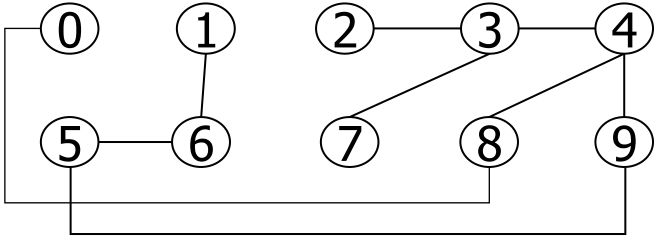
id[id[**p**]]=4 = id[id[**q**]]=4

unchanged

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9



Example



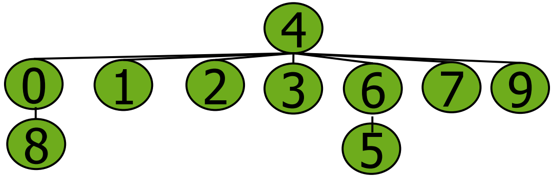
Input: **p** **q** = 6 1

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9

id[id[**p**]]=4 ≠ id[**q**]=1

let the smaller tree **q** pointo to the larger tree **p**: id[**q**]=4

id	4	4	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9



```
...
  int i, j, p, q, id[N], sz[N];
  for(i=0; i<N; i++) { id[i] = i; sz[i] =1; }
  printf("Input pair p q:  ");
  while (scanf("%d %d", &p, &q) ==2) {
    for (i = p; i!= id[i]; i = id[i]);
    for (j = q; j!= id[j]; j = id[j]);
    if (i == j)
      printf("pair %d %d already connected\n", p,q);
    else {
      printf("pair %d %d not yet connected\n", p, q);
      if (sz[i] <= sz[j]) {
        id[i] = j; sz[j] += sz[i]; }
       else { id[j] = i; sz[i] += sz[j];}
    }
    printf("Input pair p q:  ");
  }
...
```

Quick union Optimization

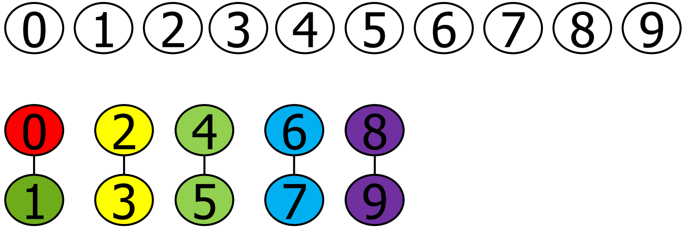
- **find:** scanning a *“chain”* of objects, cost at most logarithmic in the number of objects **O(logn)**
- **union:** simple, because it is enough that an object points to another object, unit cost **O(1)**
- globally the number of operations is bounded by

numb. of pairs * “chain” length

but the chain’s length grows logarithmically!

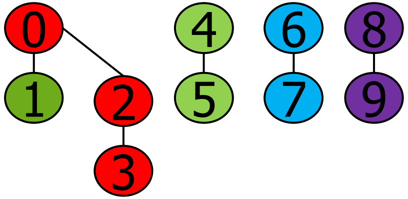
Why logatithmic?

Worst-case: given **n** elements, each union connects **2** trees of the same size

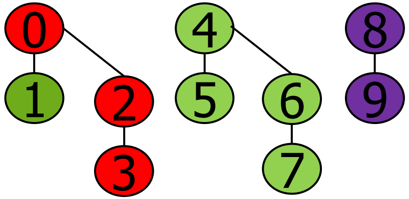


p **q** = 0 1
p **q** = 2 3
p **q** = 4 5
p **q** = 6 7
p **q** = 8 9

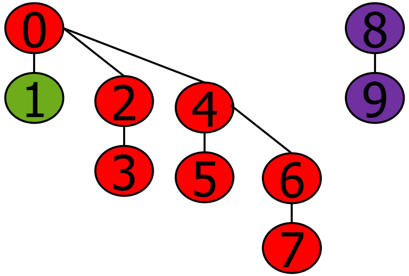
Why logatithmic?



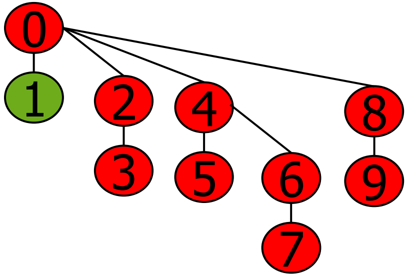
p q = 1 3



p q = 5 7



p q = 3 7



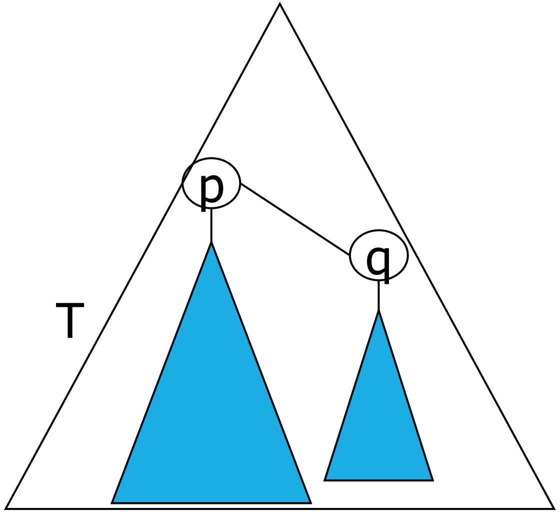
p q = 7 9

Why logatithmic?

Each **tree** containing **2^h nodes** has height **h.**
With a **union operation**, in the worst case, we merge **2 trees** with the same number of **nodes 2^h**. The result is a tree **with 2^{h+1} nodes**, thus its height is h+1.
Height grows linearly with the number of union operations.

How many union operations are required?

If $\mathbf{T_1} \geq \mathbf{T_2}$, each time we merge a smaller tree into a larger one, we create a tree whose size **T** is at least **twice** the size of $\mathbf{T_2}$.



If, at each step, the number of elements in the tree doubles at least and if there are **N** elements, after **i** steps there will be at least $\mathbf{2^i}$ elements in the tree.

As the inequality $\mathbf{2^i \leq N}$ holds, the number of union operations required is $\mathbf{i \leq \log_2 N}$.