Politecnico di Torino

Sorting Algorithms

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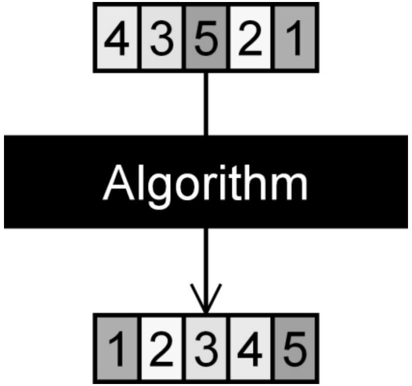
Problem definition

The sorting problem has attracted the attention of research due to the complexity of solving it efficiently despite its simple and familiar statement.

Betty Holberton (1951 - ENIAC and UNIVAC)

The most frequently used **orders** are numerical order and lexicographical order, and either ***ascending*** or ***descending***.



Problem definition

Sorting:

- **Input:** symbols <$a_1, a_2, \ldots, a_n$> belonging to a set having an order relation $\leq$
- **Output:** monolitic permutation <$a'_1, a'_2, \ldots, a'_n$> of the input for which the order relation holds $a'_1 \leq a'_2 \leq \ldots \leq a'_n$.

Order relation $\leq$

Binary relation between items of a set **A** satisfying the following properties:

- **Reflexivity** $\forall\ x \in \mathbf{A}\ x \leq x$
- **Antisymmetry** $\forall\ x, y \in \mathbf{A}\ \ x \leq y \wedge y \leq x \Rightarrow x = y$
- **Transitivity** $\forall\ x, y, z \in \mathbf{A}\ \ x \leq y \wedge y \leq z \Rightarrow x \leq z$

A is a partially ordered set (*poset*).

If relation $\leq$ holds $\forall\ x, y \in \mathbf{A}$, **A** is totally ordered set.

Order relation ≤

Examples of order relations ≤ :

- **(total) relation** ≤ on natural, relative, rational and real numbers (sets **N, Z, Q, R**)
- **(partial) relation**: e.g., divisibility on natural numbers, excluding 0

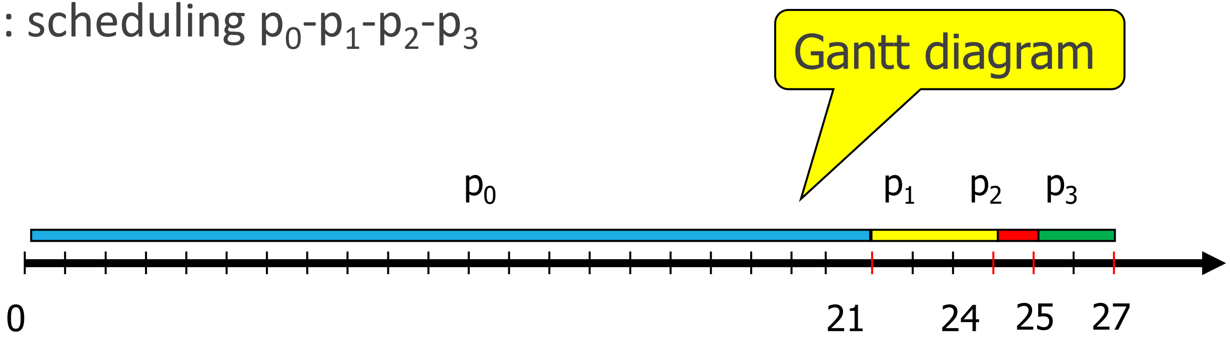
On the importance of sorting

- **30%** of CPU time is spent on **sorting** data
- **e.g., CPU scheduling**: how to select among the processes ready for execution the next one to be run on the CPU.
  - **Simple solution:** queue, thus *First-come First-Served* policy:
    the first request that arrives is the first one to be served
  - **problem:** *minimize average waiting time*: it turns into an **optimization** problem.

Example: determine the effect of scheduling in average waiting time of processes $\mathbf{p_i}$ with duration:

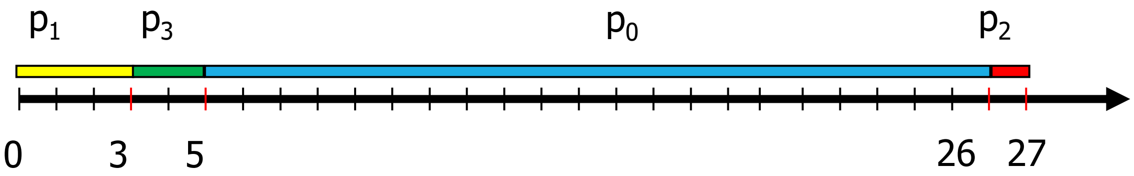
$\mathbf{p_0}$ =21, $\mathbf{p_1}$ =3, $\mathbf{p_2}$ =1, $\mathbf{p_3}$ =2

- Case1: scheduling p_0-p_1-p_2-p_3



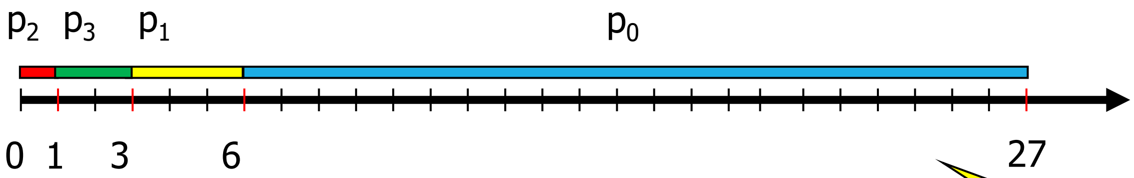
average waiting time = (0+21+24 +25)/4 =**17,5**

- Case2: scheduling p_1-p_3-p_0-p_2



average waiting time (0+3+5 +26)/4 =**8,5**

- Case3: scheduling (**sorted by increasing duration**) p_2-p_3-p_1-p_0



average waiting time (0+1+3 +6)/4 =**2,5**

shortest job first scheduling

Applications of sorting

- Sorting a **list of names**
- Organizing an **MP3** library
- Displaying **Google PageRank** results
- …

Trivial applications

Applications of sorting

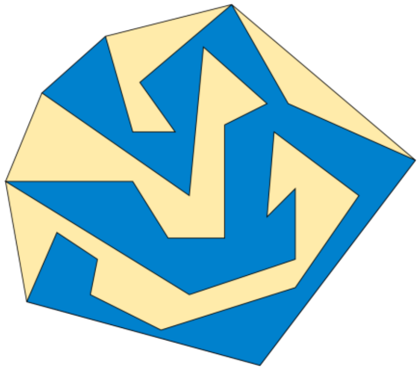
- find the **median**
- **binary search** in a database
- find **duplicates** in a mailing list
- …

Simple problems if data are sorted

Applications of sorting

- **data compression** (e.g., Burrows-Wheeler Transform)
- **Computer graphics** (e.g., convex hull)
- **Computational biology** (e.g., DNA sequencing)
- **Statistics**
- …

bzip2

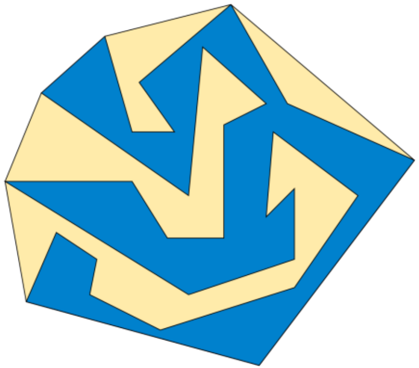


Convex hull (in blue and yellow) of a simple polygon (in blue)

Applications of sorting

- **data compression** (e.g., Burrows-Wheeler Transform)
- **Computer graphics** (e.g., convex hull)
- **Computational biology** (e.g., DNA sequencing)
- **Statistics**
- …

Non trivial applications



Convex hull (in blue and yellow) of a simple polygon (in blue)

Classification: internal/external sorting

- **Internal sorting**
  - data are in **main** memory (e.g., RAM storage)
  - **direct** access to items
- **External sorting**
  - data are on **mass** memory (e.g., CD-ROM storage)
  - **sequential** access to items

Classification: in place / stable

- **In place sorting**
  n data in array + constant number of auxiliary memory locations

- **Stable sorting**
  For Data with duplicated keys the **relative ordering is unchanged**: the output relative ordering is the same as the relative input ordering

Example of stable sorting

BEFORE	
Name	**Grade**
Dave	C
Earl	B
Fabian	B
Gill	B
Greg	A
Harry	A

AFTER	
Name	**Grade**
Greg	A
Harry	A
Earl	B
Fabian	B
Gill	B
Dave	C

Example of effect of unstable sorting

BEFORE	
Name	**Grade**
Dave	C
Earl	B
Fabian	B
Gill	B
Greg	A
Harry	A

AFTER	
Name	**Grade**
Greg	A
Harry	A
Gill	B
Fabian	B
Earl	B
Dave	C

Example

Struct with **2** keys: **name** (the first letter is the key) and **group** (the key is an integer)

Andrea	3
Lucia	3
Franco	1
Barbara	4
Roberto	2
Fabio	3
Giada	4
Chiara	3

Example

Struct with **2** keys: **name** (the first letter is the key) and **group** (the key is an integer)

Andrea	3
Lucia	3
Franco	1
Barbara	4
Roberto	2
Fabio	3
Giada	4
Chiara	3

First sorting by first letter:

Andrea	3
Barbara	4
Chiara	3
Fabio	3
Franco	1
Giada	4
Lucia	3
Roberto	2

Example

Second sorting by group:

Andrea	3
Barbara	4
Chiara	3
Fabio	3
Franco	1
Giada	4
Lucia	3
Roberto	2

NON stable algorithm

Franco	1
Roberto	2
Chiara	3
Fabio	3
Andrea	3
Lucia	3
Giada	4
Barbara	4

Stable algorithm

Franco	1
Roberto	2
Andrea	3
Chiara	3
Fabio	3
Lucia	3
Barbara	4
Giada	4

Classification: complexity

- $O(n^2)$:
  - simple, iterative, based on comparison
  - ***Insertion sort***, ***Selection sort***, ***Exchange/Bubble sort***
- $O(n^x)$ with $x \leq 2$
  - Evolution of the simple ones, iterative, based on comparison
  - ***Shell sort***, $O(n^2)$, $O(n^{3/2})$, $O(n^{4/3})$ depending on the sequence
- $O(n \log n)$:
  - more complex, recursive, based on comparison. Will be dealt with in second year Course
  - ***Merge sort***, ***Quick sort***, ***Heap sort***
- $O(n)$:
  - applicable only with restrictions on data, based on computation
- ***Counting sort***, ***Radix sort, Bin/Bucket sort***

Classification:

A mode detailed analysis is possible, distinguishing between

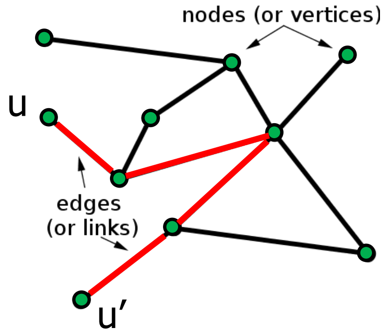
- **comparison** and
- **swap operations**.

When data are large, moving **chunks of memory** (***not just pointers***) may be expensive.

Asymptotic complexity however doesn’t change.

Graph theory: Paths

Graphs: Paths



In a graph **G** = (**V**, **E**)

a **Path** in a graph is a finite or infinite sequence of **edges** which joins a sequence of **vertices**.

Path p: $u \rightarrow_p u'$:

$\exists(v_0,v_1,v_2,\ldots,v_k) \mid u=v_0,\ u'=v_k,\ \forall i = 1,2,\ldots,k\ (v_{i-1},v_i) \in \mathbf{E}$

- k = length of the path
- u' is reachable from u $\Leftrightarrow\ \exists$ p: $u \rightarrow_p u'$.

In an **undirected** graph it is also true that u is reachable from u’ $\Leftrightarrow\ \exists$ p: $u' \rightarrow_p u$

- **simple path** p$\Leftrightarrow$ $(v_0,v_1,v_2,\ldots,v_k) \in$ p distinct

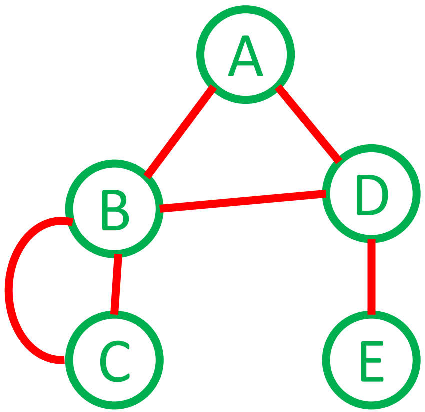
Graphs: Paths

Wall, Trains and Paths

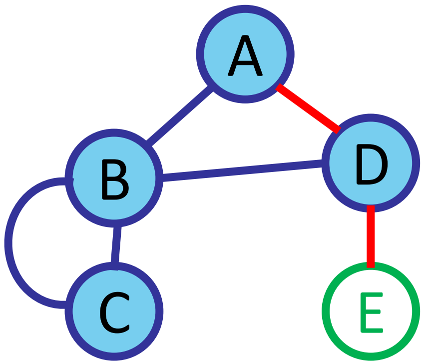
A **walk** is a finite or infinite sequence of **edges** which joins a sequence of **vertices**

A **trail** is a **walk** in which all **edges** are **distinct**

A **path** is a **trail** in which all **vertices** (and therefore also all **edges**) are **distinct**



Is it walk, trail or path?



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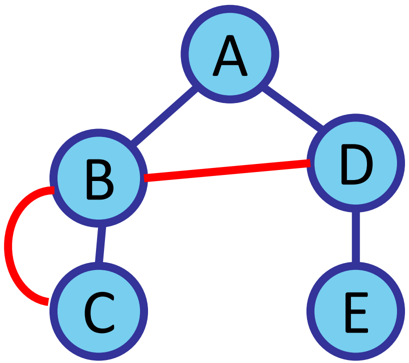
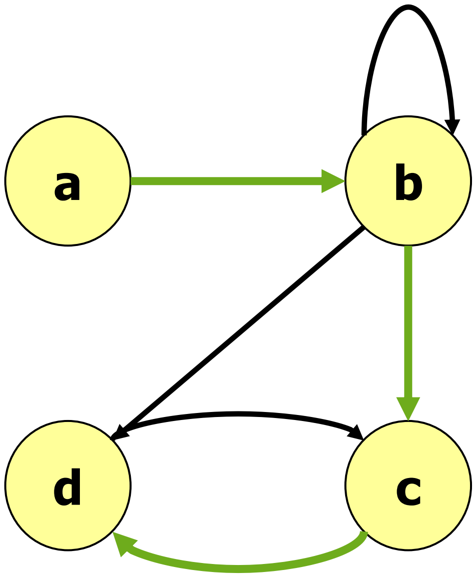


Example



G = (V, E)

p: **a** $\rightarrow_p$ **d** : (a, b), (b, c), (c, d)

k = 3

d is reachable from **a** (not necessarily vice-versa)

p is simple.

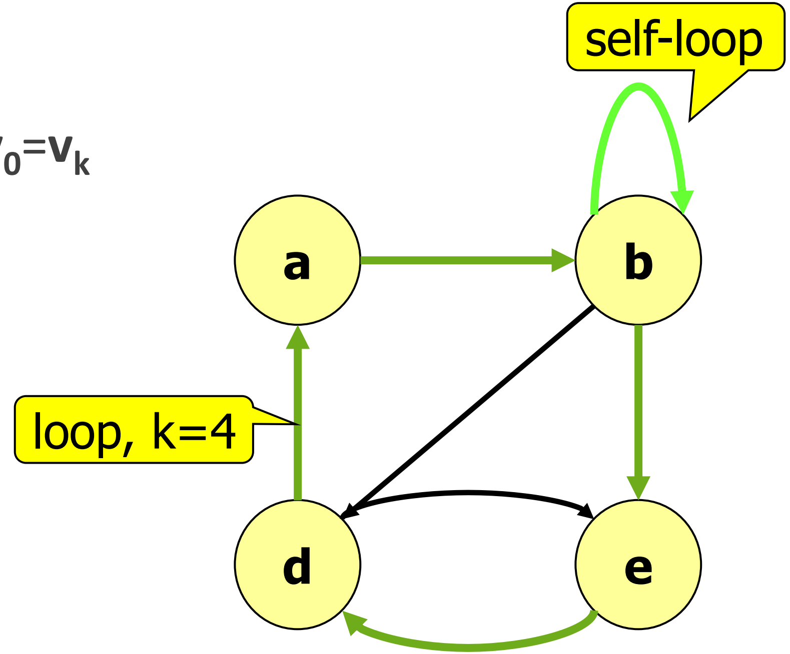
Graphs: Loops

Loop = path where $\mathbf{v_0}=\mathbf{v_k}$

Simple loop = simple path where $\mathbf{v_0}=\mathbf{v_k}$

Self-loop = unit-length loop

A graph without loops is acyclic



Undirected Graphs: Connection

Connected **undirected** graph $G = (V, E)$:

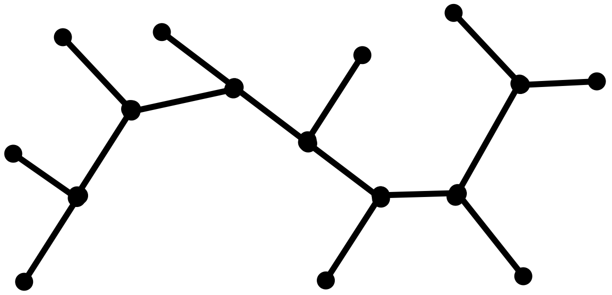
$$\forall v_i, v_j \in V \; \exists \; p \; v_i \rightarrow_p v_j$$

Connected component: maximal connected subgraph (= $\nexists$ subsets for which the property holds that include it)

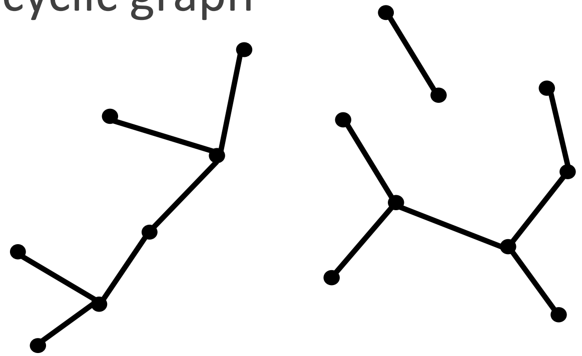
Connected undirected graph: only one connected component

Non rooted (free) trees

Non rooted tree = undirected, connected, acyclic graph



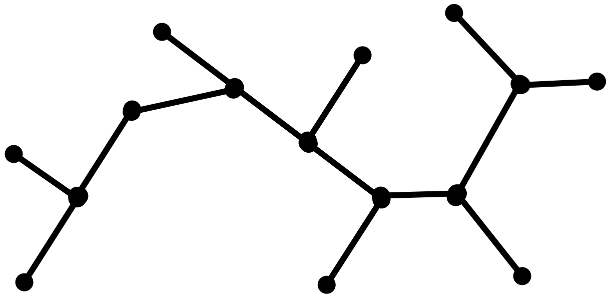
Forest = undirected acyclic graph



Properties

G = (V, E) **undirected** graph |E| edges, |V| nodes:

- G = **non rooted tree**
- Every pair of **nodes** is connected by a **single simple path**
- G **connected**, removing an **edge** disconnects the graph
- G **connected** and |E| = |V| - 1
- G **acyclic** and |E| = |V| - 1
- G **acyclic**, adding an **edge** introduces a loop.



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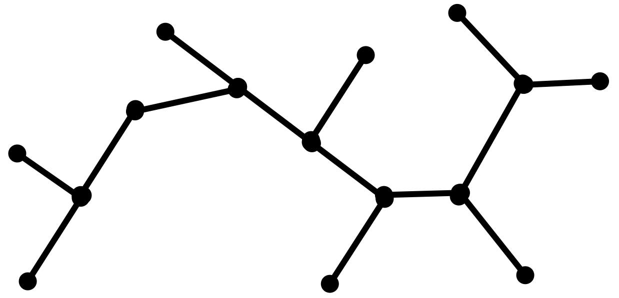
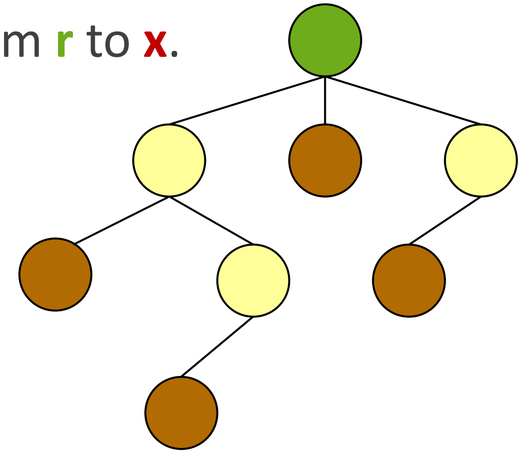


Is it a non rooted tree?

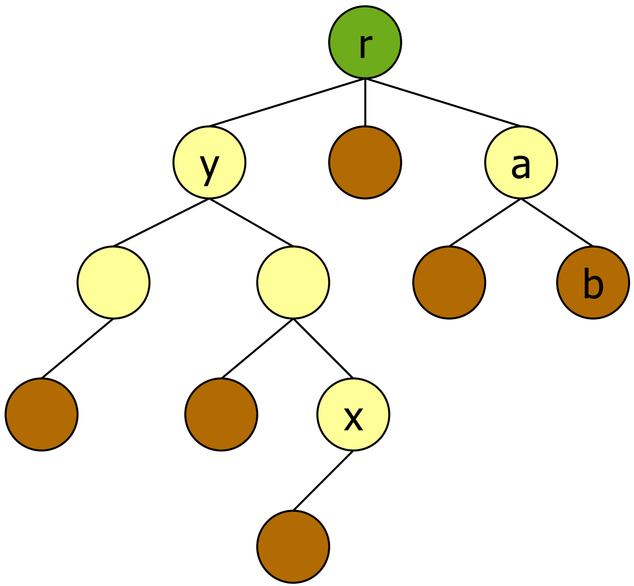
Rooted trees

∃ a node **r** called root

- **parent**/**child** relationship
  - **y** ***ancestor*** of **x** if **y** belongs to the **path** from **r** to **x**.
    «**x** ***descendant*** *of* **y**»
  - proper ***ancestor***/***descendant*** if **x** ≠ **y**
  - **parent**/**child**: adjacent nodes
- **root**: no parent
- **leaves**: no children



Example



r root

y proper ***ancestor*** of **x**

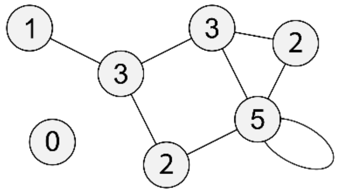
x proper ***descendant*** of **y**

a parent of **b**

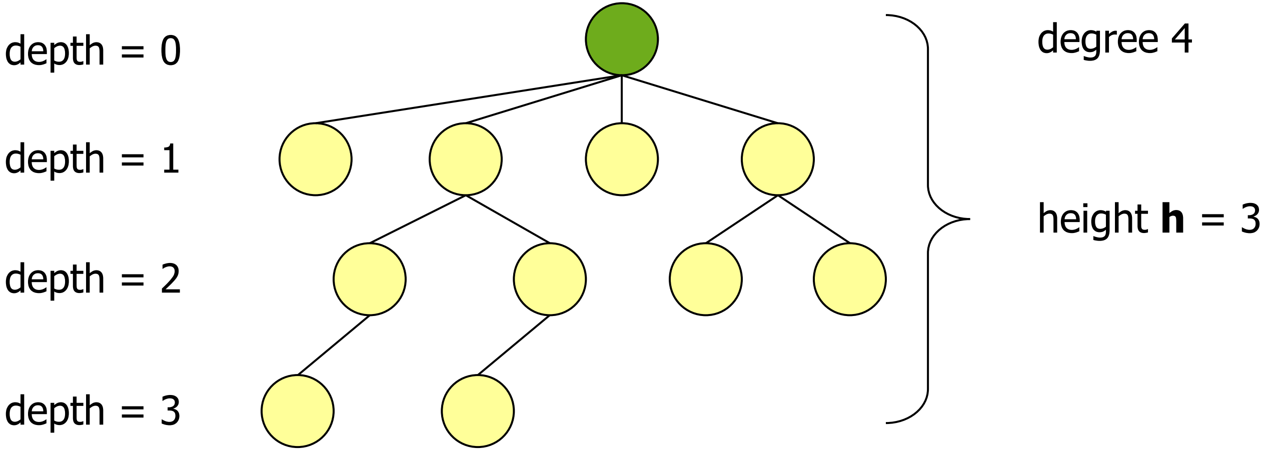
b child of **a**

Properties of a tree T

- Degree(T) = (or valency) **maximum** number of **children**
- Depth(x) = **length** of the (unique) path from **r** to **x**
- Height(T) = **maximum** depth.

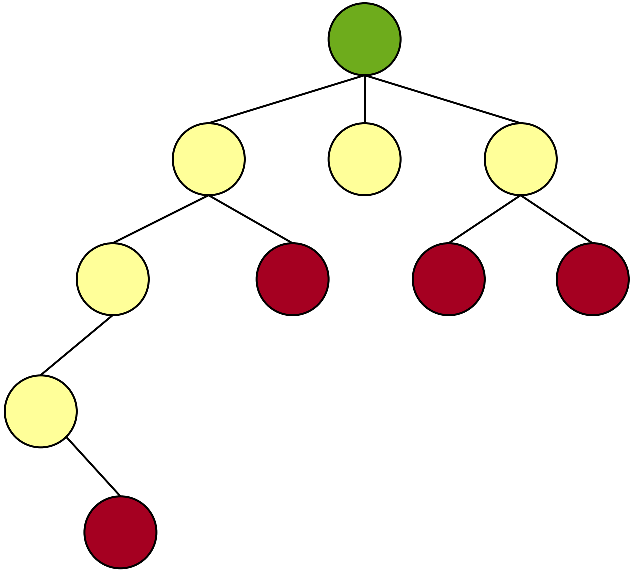


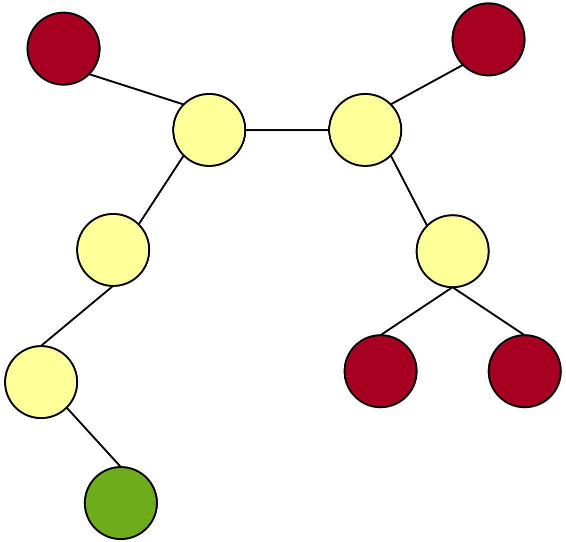
vertices labeled by degree



Quick exercise

Find the degree, depth and height of the following graphs:





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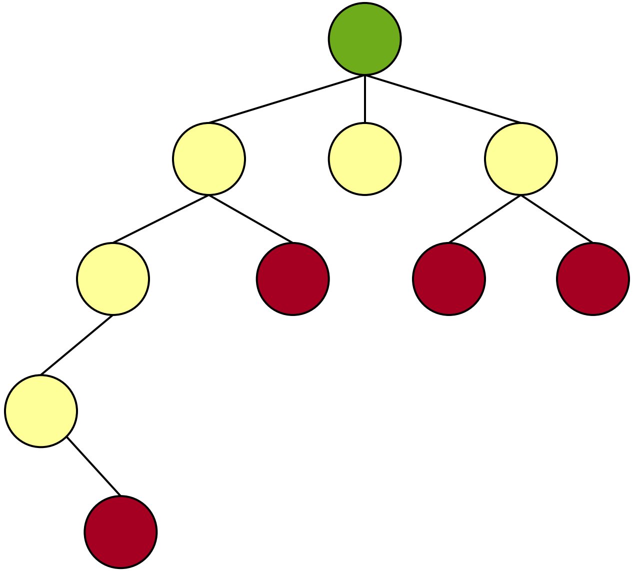

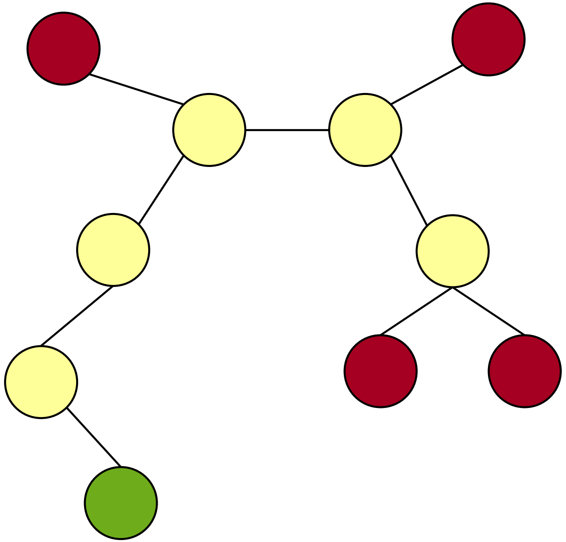

Is it a non rooted tree?

Quick exercise

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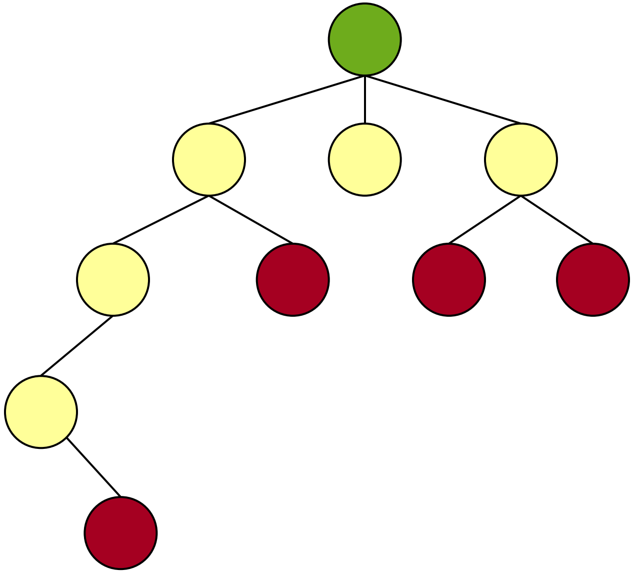
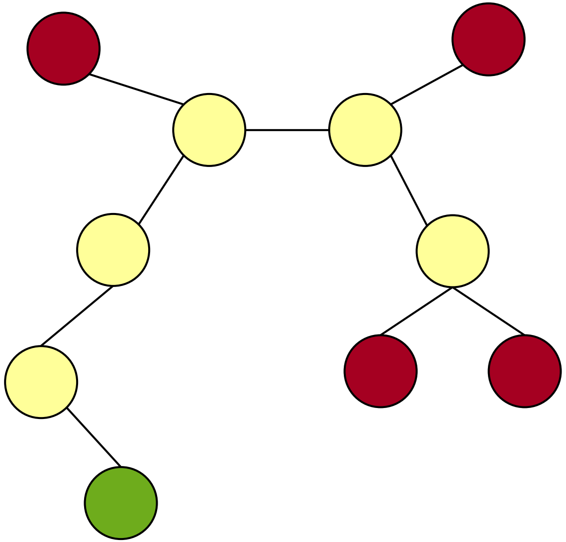


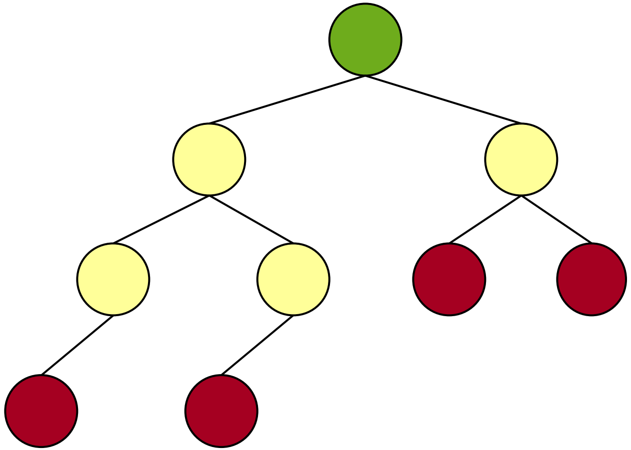


Is it a binary tree?

Binary Tree

Definition:

- Tree of **degree = 2**: each **node** has **0**, **1** or **2** children
- There is also a recursive definition (*topic of the 2nd-year course*)



Perfectly balanced **(full)** binary tree

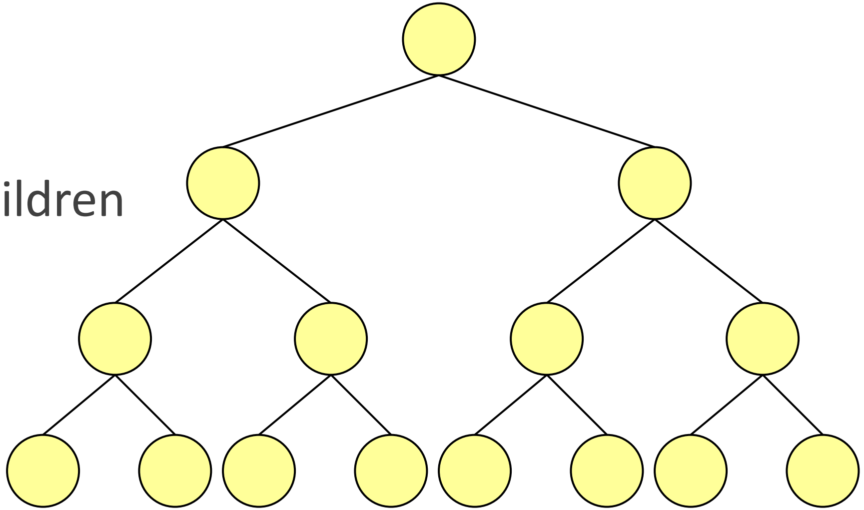
Two conditions:

- All the **leaves** have the **same depth**
- Each **node** is either a **leaf** or it has **2** children

Perfectly balanced (full) binary tree of height **h**:

- Number of **leaves**: $\mathbf{2^h}$
- Number of **nodes**: $\Sigma_{0 \le i \le h} 2^i = 2^{h+1} - 1$

Finite geometric progression with ratio =2



How many leaves?

height?

Number of nodes(vertices)?

Perfectly balanced **(full)** binary tree

Two conditions:

- All the **leaves** have the **same depth**
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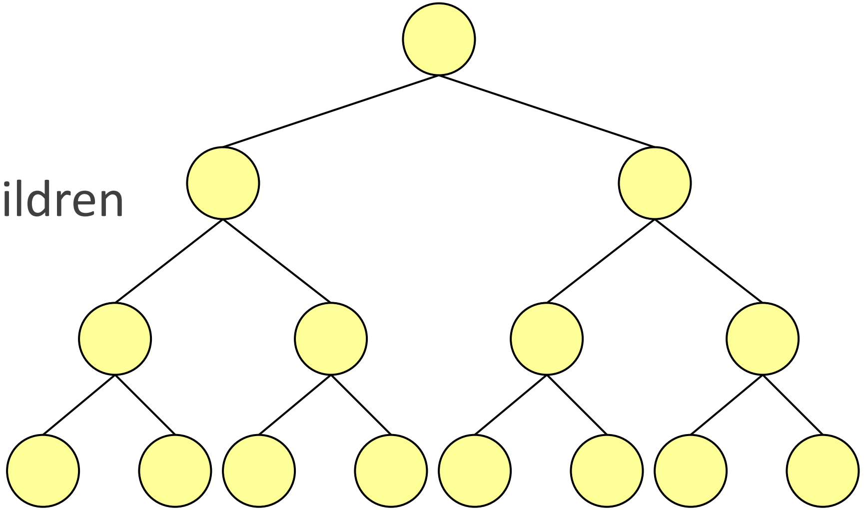


8 leaves
h = 3
15 nodes

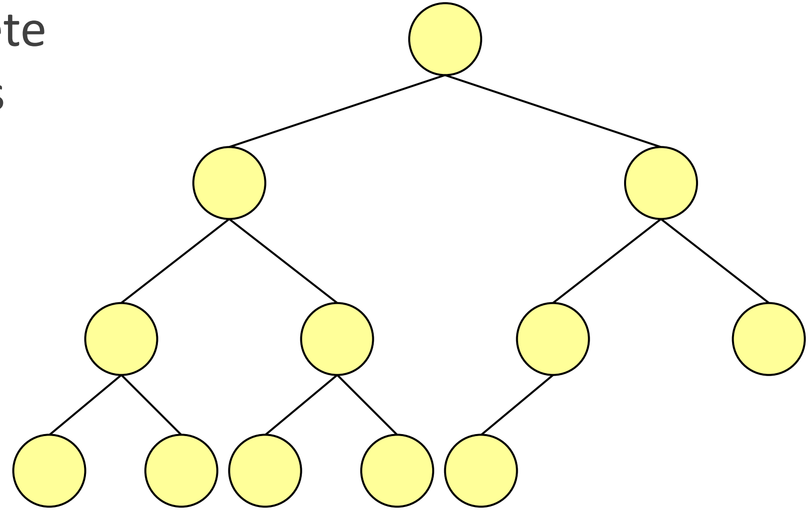
Perfectly balanced **(full)** binary tree

Is the tree a perfectly balanced binary tree?

Complete (**to the left**) binary tree

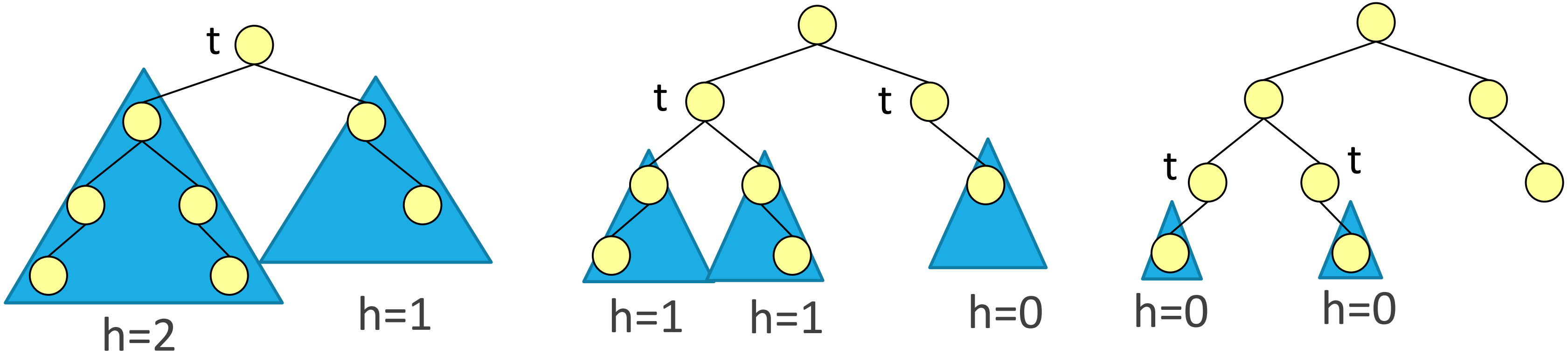
Each **level** is filled with all possible **nodes** for that level, **possibly except the last one**, filled **from left to right** (all nodes are as far left as possible)

Given the number of **nodes n**, the complete (o the left) binary tree always exists and is unique



Height-balanced binary tree

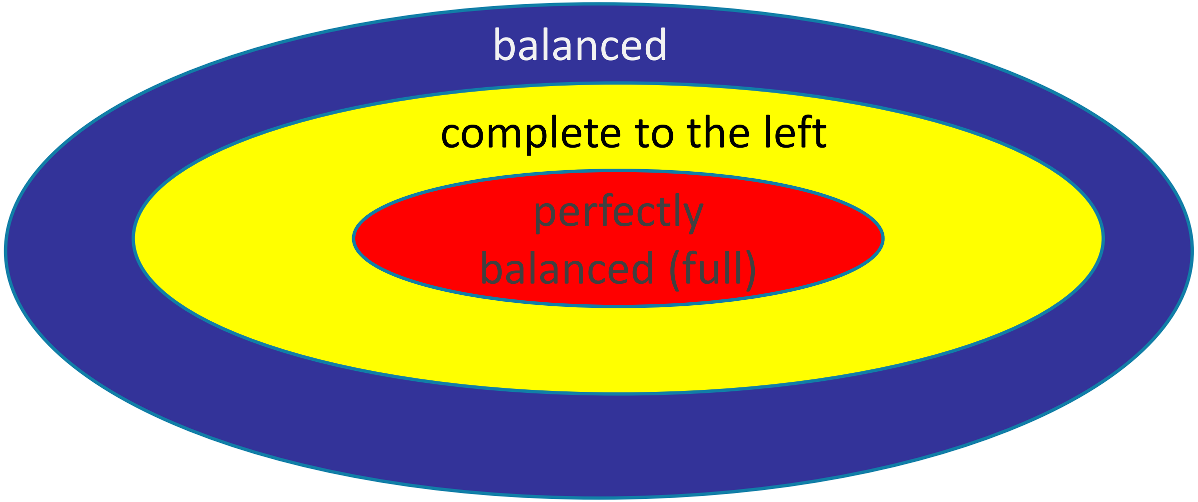
A binary tree is **height-balanced** (or **balanced**) if and only if, for every subtree t rooted, in one of its nodes, the heights (**h**) of the left and right subtrees differ for **at most 1**.



Perfectly balanced (full) binary trees are a **proper subset** of the complete (**to the left**) binary trees.

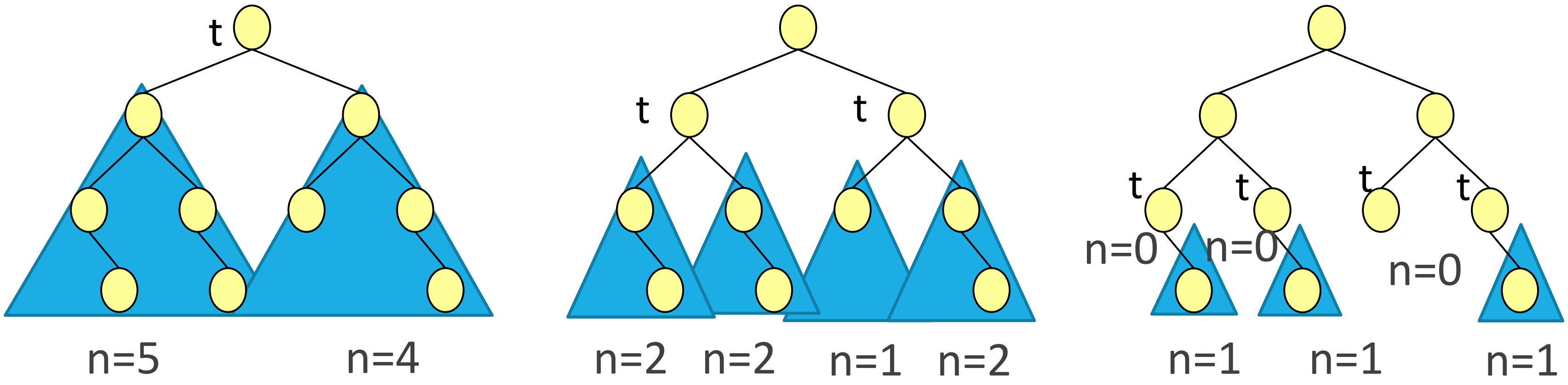
In turn, complete (**to the left**) binary trees are a **proper subset** of **balanced** trees.

Height-balanced = balanced



Node-balanced binary tree

A binary tree is **node-balanced** if and only if, for each subtree **t** rooted at one of its **nodes**, the numbers of **nodes** in the left and right subtrees **differ at most** by 1.



Lower Bound(Ω)

Goal: to find a ***lower bound*** on worst-case asymptotic complexity for ALL sorting algorithms based on comparison.

Demonstration is algorithm-INDEPENDENT.

Basic operation: comparison between **2** items $\mathbf{a_i : a_j}$

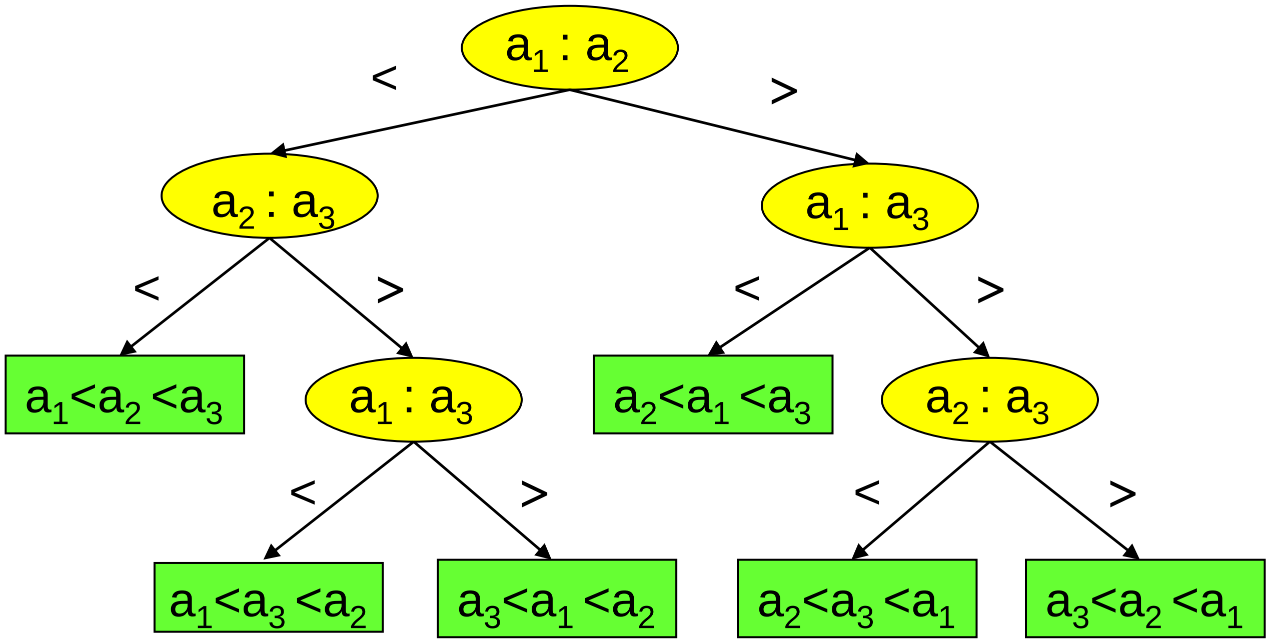
Outcome: decision ($a_i > a_j$ or $a_i \leq a_j$), shown on a binary tree called decision tree.

Example

Task: Find the complexity of sorting **3-item** array **A** with distinct items $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$

Solution: Build a decision tree where each **node** is labelled with the current comparison **(a_i : a_j)** and the 2 **edges** with the outcome (**>** or **<**). Keep on comparing till a solution is found (**leaf**).

Example


$a_1 : a_2$
<
>
$a_2 : a_3$
$a_1 : a_3$
<
>
<
>
$a_1<a_2<a_3$
$a_1 : a_3$
$a_2<a_1<a_3$
$a_2 : a_3$
<
>
<
>
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$a_3<a_1<a_2$
$a_2<a_3<a_1$
$a_3<a_2<a_1$

Example

Complexity? It is related to the **number of comparisons.**

- What is the minimum number of comparisons in the worst case? **3**

The decision tree has height **h=3**. The minimum number of comparisons executed in the worst case equals height **h**.

⇓

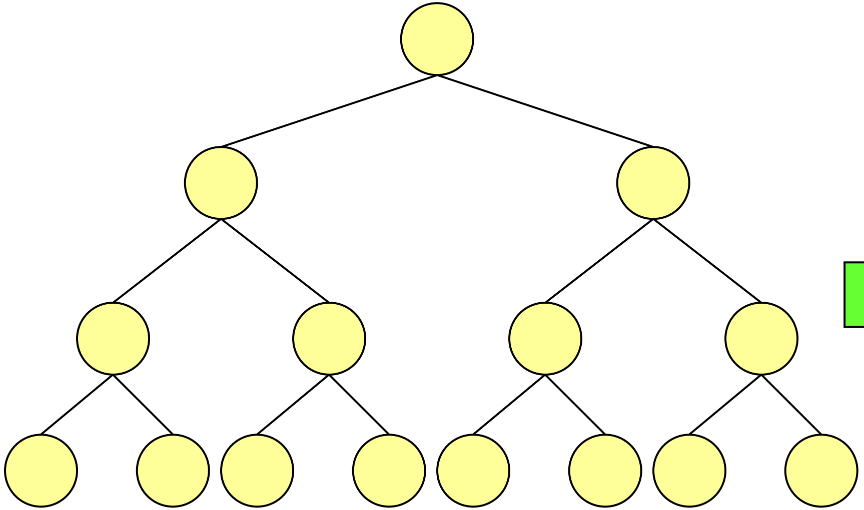
Minimum worst-case complexity is O(h). **Complexity** is a function of height **h**, not of the number of items **n**.

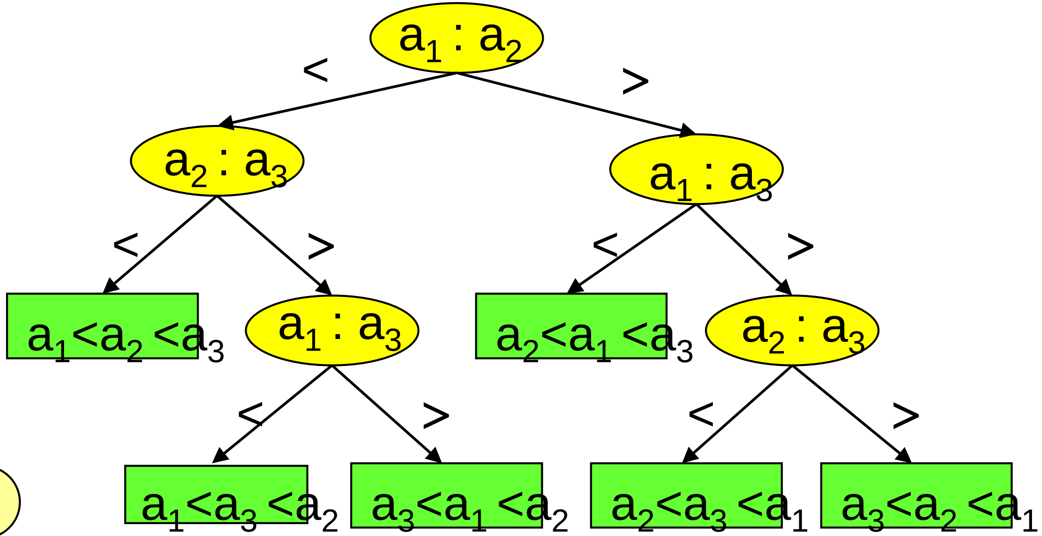
⇓

What is the **height** of a decision tree able to store decisions for an **n-item** array?

A perfectly balanced (full) binary tree of height **h** has:

- 2^h **leaves**
- $\Sigma_{0 \leq i \leq h} 2^i = 2^{h+1} - 1$ **nodes**





For **n** distinct data: the number of possible orderings is the number of permutations **n!**

Orderings are stored in tree **leaves**, so there must be at least as many **leaves** as the **number** of orderings

$$2^{\mathbf{h}} \geq n!$$

Resorting to ***Stirling's approximation*** $n! > (n/e)^n$

$$2^{\mathbf{h}} \geq n! > (n/e)^n$$

Taking the log of both sides

$$\mathbf{h} > \lg(n/e)^n = n \lg n - n \lg e = \Omega(n \lg n)$$

Comparison-based sorting algorithms whose worst-case asymptotic complexity is better than linearithmic do not exist!

Comparison-based sorting algorithms whose complexity is $\Omega(n \lg n)$ are **OPTIMAL**