Iterative Linear Sorting Algorithms

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$\Theta(N)$

General Features

- Find the position of an **item by computation**, instead that **by comparison**
- The worst-case asymptotic lower bound **T(N)** = $\Omega(N\log N)$ is no more true
- Complexity is linear **T(N)** = $O(N)$
- There are restriction on use
- Algorithms:
  - **Counting sort**
  - **Radix sort**
  - **Bin/bucket sort:** requires lists, topic dealt with in second year Course

Outline

- **Counting sort**
- **Radix sort**

Counting sort

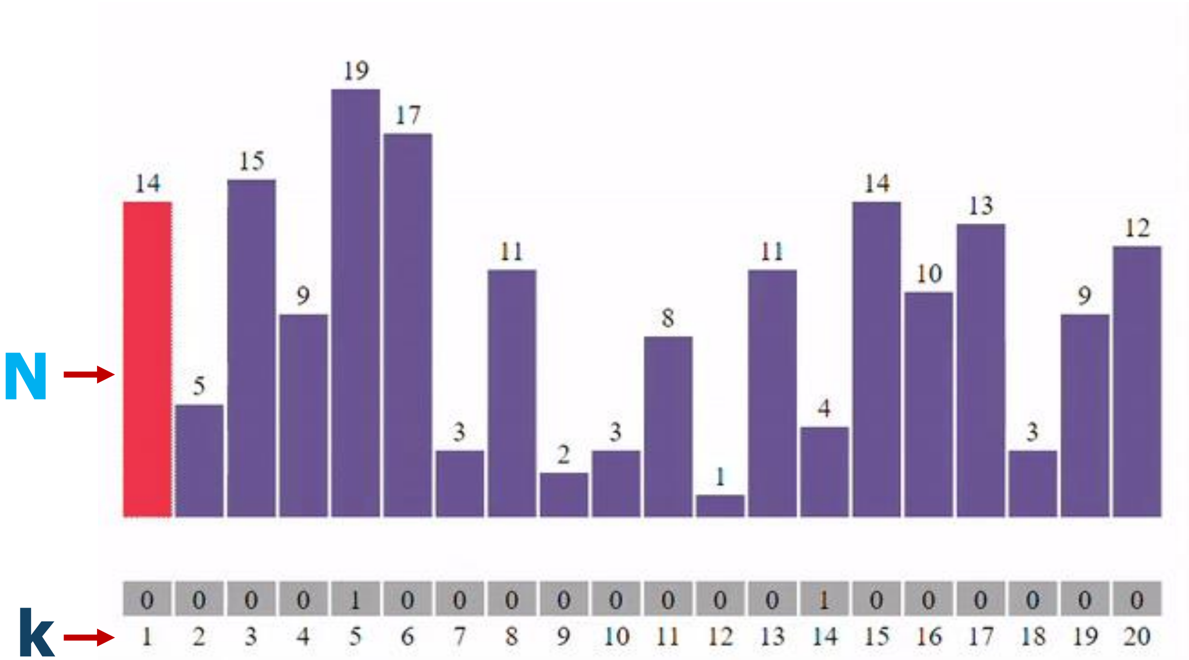
- is an algorithm that plays a pivotal role in any programming language

- **Main objective:** sort out an **object collection (**e.g., array of **N integers** in the range from **0 to k-1)** in accordance with **keys** that are present as small **integers**.
- Each finite set of **k** items (**keys**) may be matched with the **N integers** in the range (0 … **k-1)**

- Input data may be repeated or
- Input data may not contains some values in the range **0 … k-1**

Counting sort

- Example of using the counting sort to organize a sequence of integers:



Counting sort

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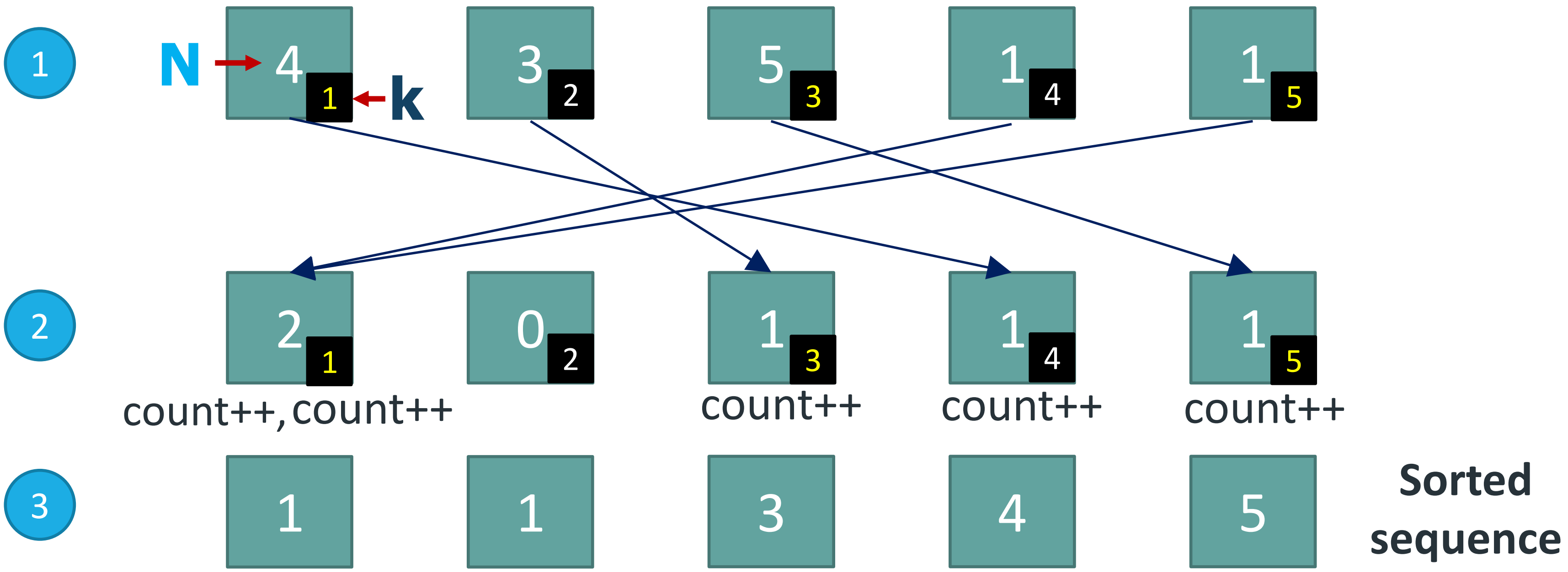
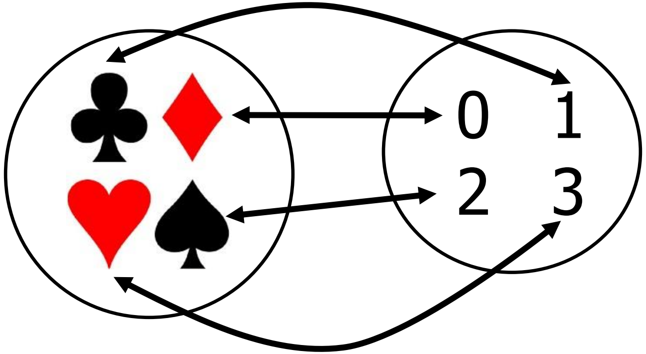


Counting sort

- **Goal:** to sort an array of **N** integers in the range **0 …k-1**



Simple analogy between integers and symbols in a set to be sort.

Approach

- Sorting **by computation** and **not by comparison**
- For each item **x** compute how many items precede it in the sorted array:
  - First compute **simple occurrences** of **x**, i.e. how many instances of **x** appear in the input
  - **Starting from simple occurrences**, compute **multiple occurrences**, i.e., how many items are ≤ **x**
- Walking the array from right to left, put item **x** in its final correct position

Data structures

Use 3 arrays:

- **Input array:** A[0..N-1] of N integers
- **Result array:** **B** [0..N-1] of N integers
- Simple/multiple occurrences array **C** of **k** integers if data belong to the range [0..**k**-1]

Example: N=8 **k**=6

A

2	5	3	0	2	3	0	3
0	1	2	3	4	5	6	7

Array to sort

C

0	1	2	3	4	5

Simple/multiple occurrences array

Data structures

Use 3 arrays:

- **Input array:** A[0..N-1] of N integers
- **Result array:** **B** [0..N-1] of N integers
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Example: N=8 **k**=6

A

2	5	3	0	2	3	0	3
0	1	2	3	4	5	6	7

Array to sort

C

0	1	2	3	4	5

Simple/multiple occurrences array

Question if: **A**

2	7	3	0	2	9	0	3
0	1	2	3	4	5	6	7

Array to sort K = ?

Computing Simple Occurrences

- Initialize **C** to 0
- Scan input array **A**
  - **A**[i] is an occurrence of that value that belongs to the range **0… k-1**
  - **A**[i] serves as index in **C** to increment by 1 the value of that cell

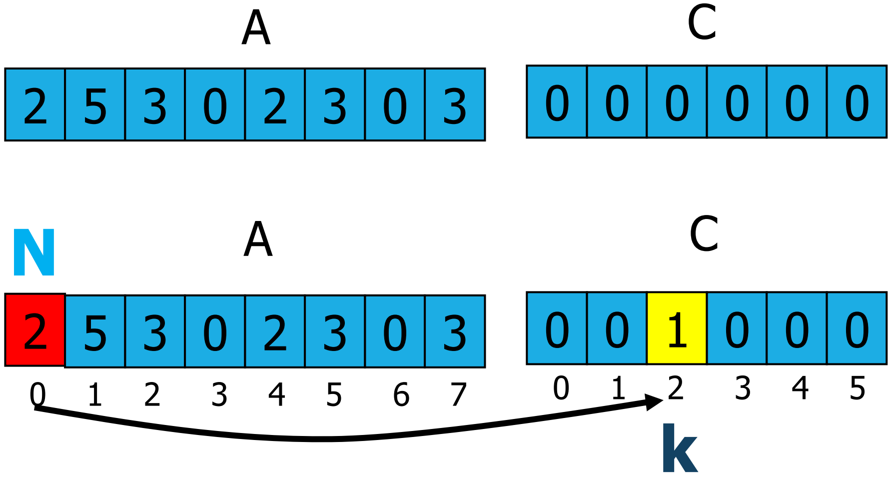
```
for (i = 1; i <= r; i++)
   C[A[i]]++;
```

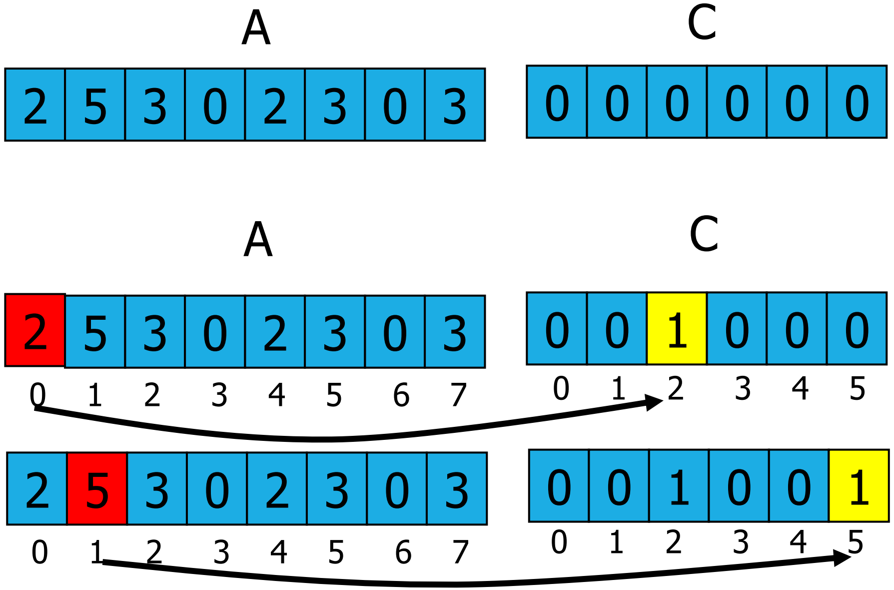
A

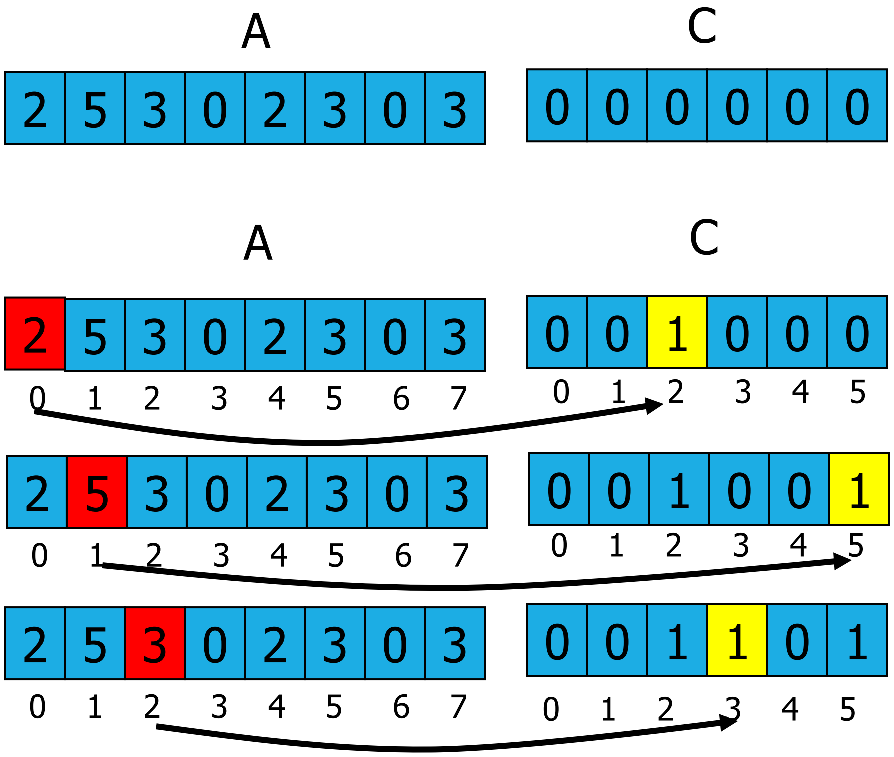
2	5	3	0	2	3	0	3

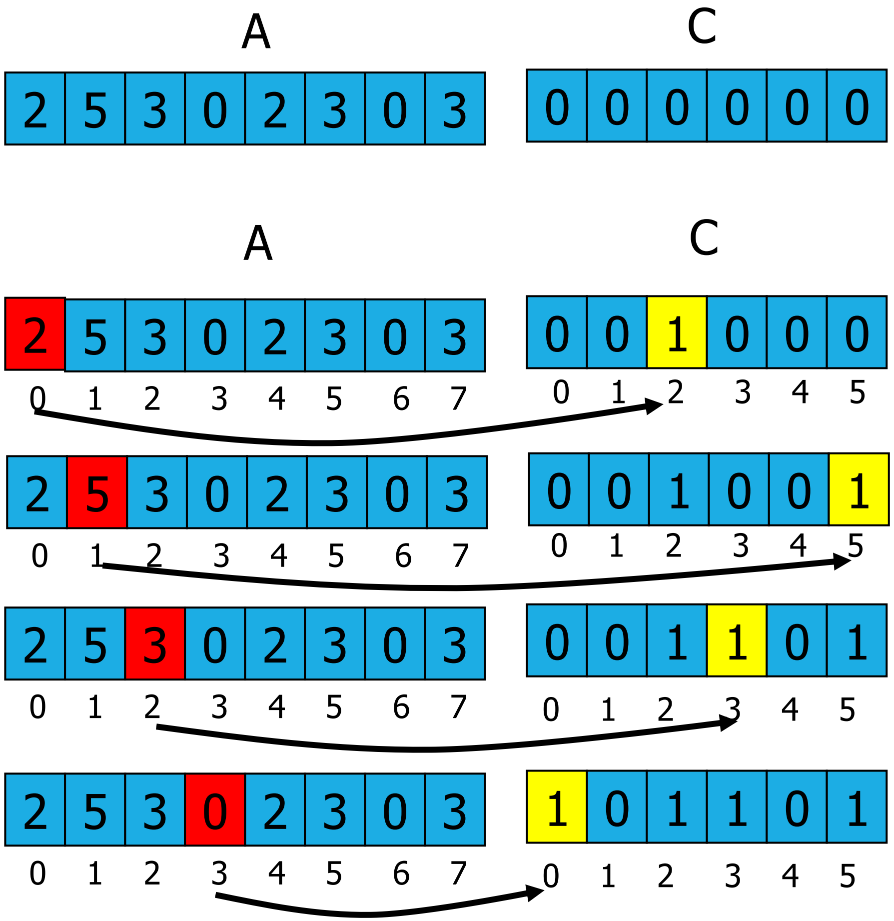
C

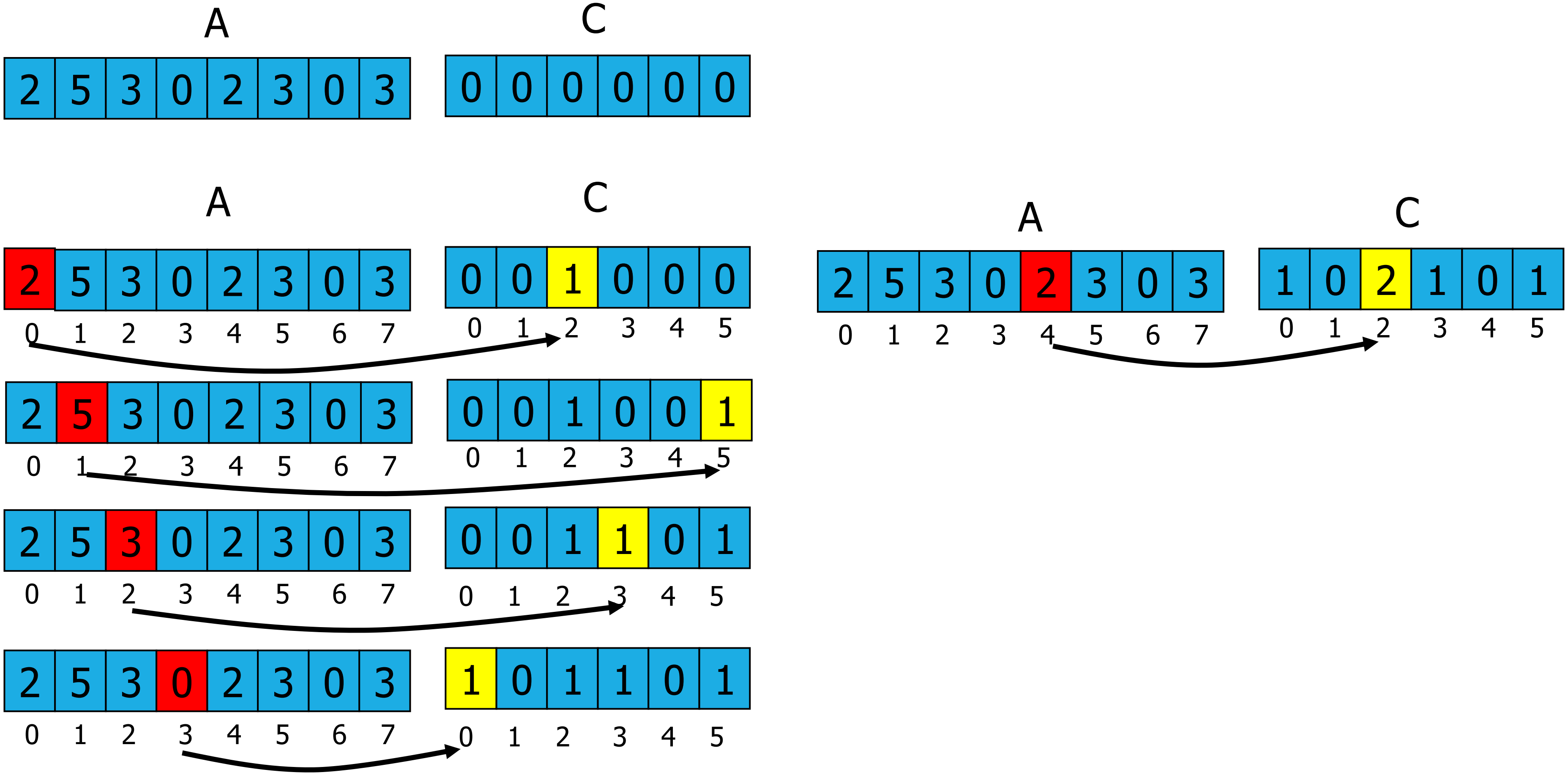
0	0	0	0	0	0


A
2 5 3 0 2 3 0 3
C
0 0 0 0 0 0
N
A
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
C
0 0 1 0 0 0
0 1 2 3 4 5
k


A
C
2 5 3 0 2 3 0 3
0 0 0 0 0 0
A
C
2 5 3 0 2 3 0 3
0 0 1 0 0 0
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 0 1 0 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5


A
C
2 5 3 0 2 3 0 3
0 0 0 0 0 0
A
C
2 5 3 0 2 3 0 3
0 0 1 0 0 0
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 0 1 0 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 0 1 1 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5


A
C
2 5 3 0 2 3 0 3
0 0 0 0 0 0
A
C
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
0 0 1 0 0 0
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
0 0 1 0 0 1
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
0 0 1 1 0 1
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
1 0 1 1 0 1
0 1 2 3 4 5


A
C
2 5 3 0 2 3 0 3
0 0 0 0 0 0
A
C
2 5 3 0 2 3 0 3
0 0 1 0 0 0
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 0 1 0 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 0 1 1 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
1 0 1 1 0 1
0 1 2 3 4 5 6 7
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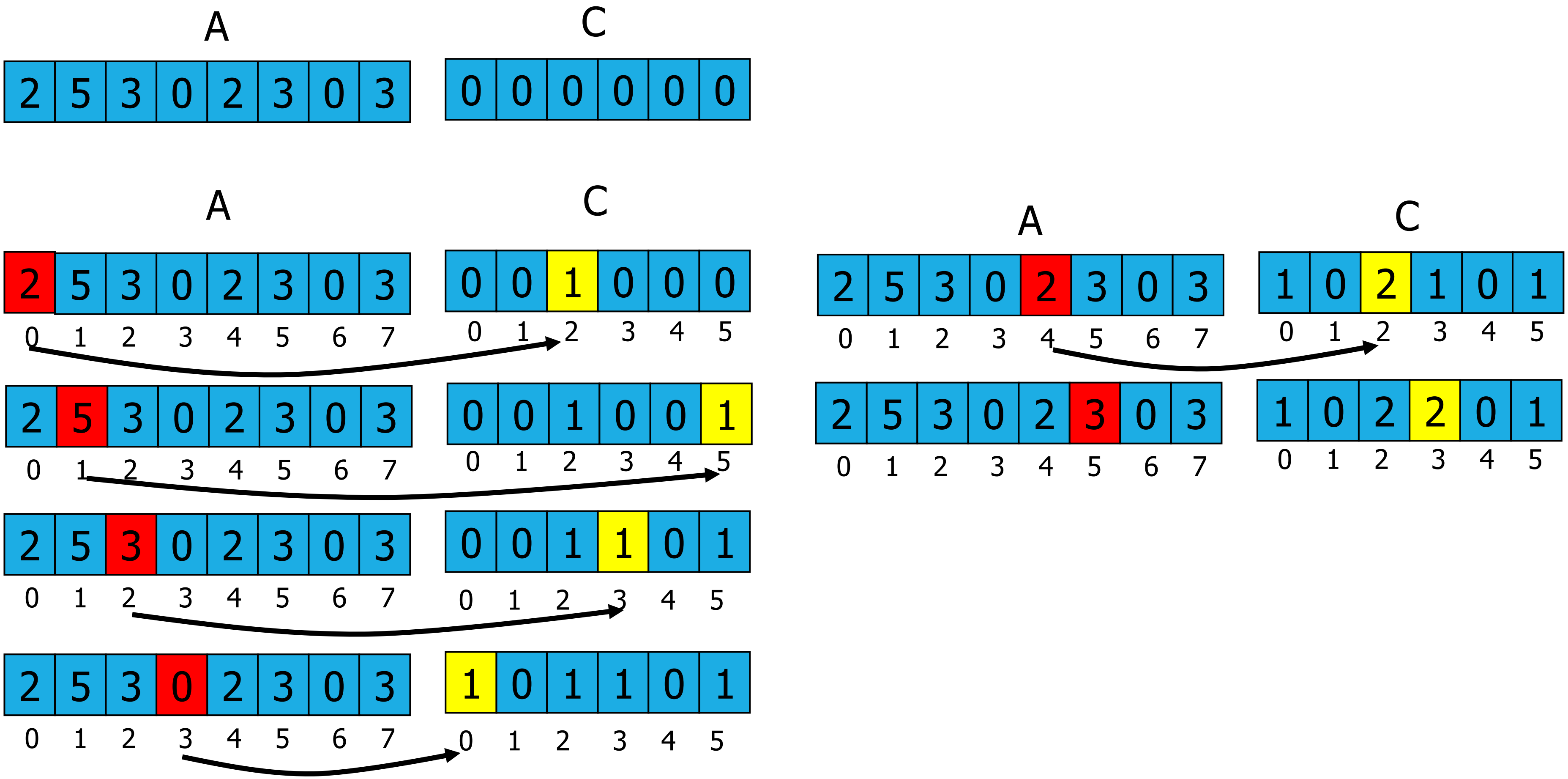
A
C
2 5 3 0 2 3 0 3
1 0 2 1 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5


A
C
2 5 3 0 2 3 0 3
0 0 0 0 0 0
A
C
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
0 0 1 0 0 0
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
0 0 1 0 0 1
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
0 0 1 1 0 1
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
1 0 1 1 0 1
0 1 2 3 4 5
A
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2 5 3 0 2 3 0 3
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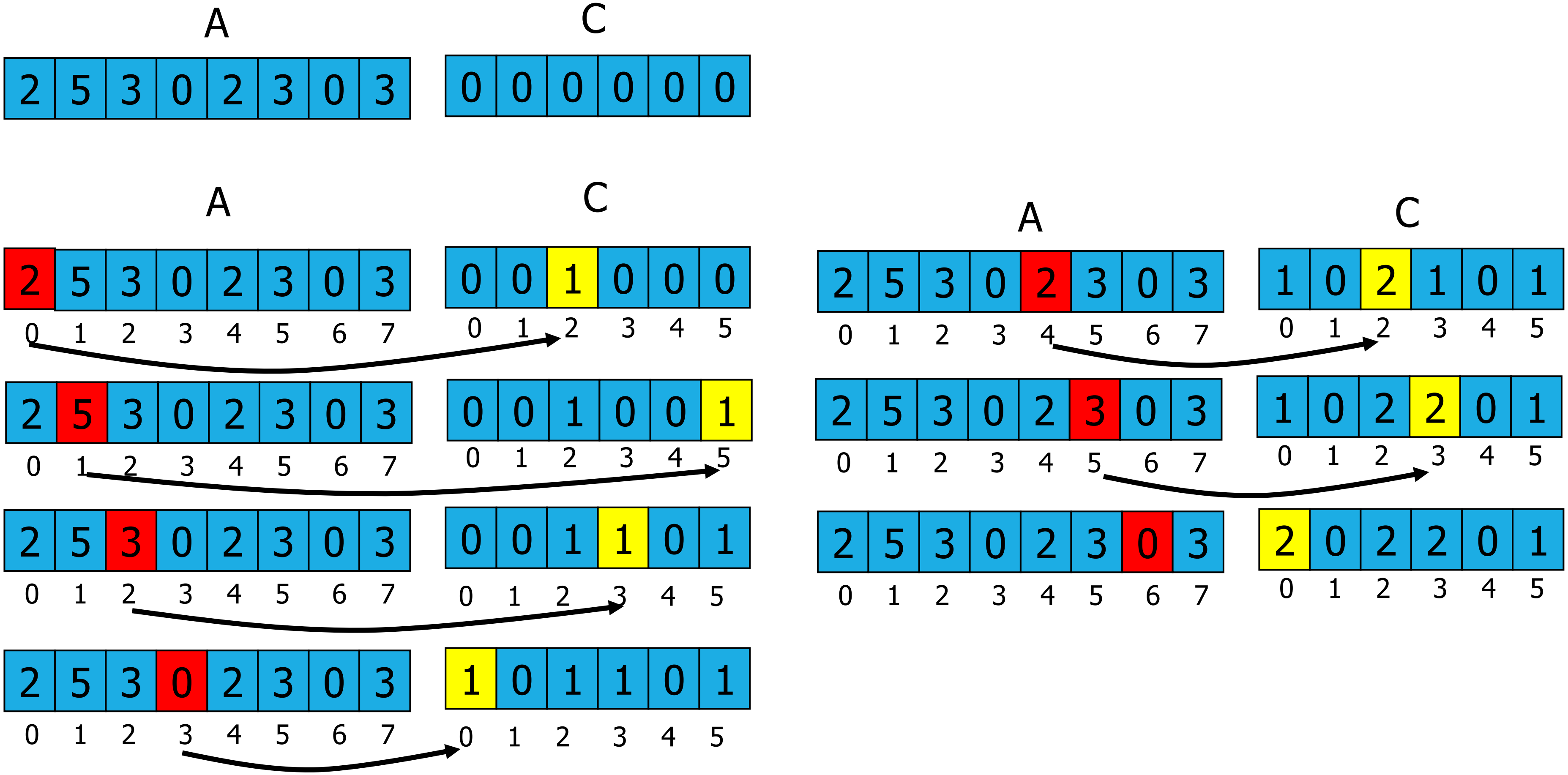
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
1 0 2 2 0 1
0 1 2 3 4 5


A
C
2 5 3 0 2 3 0 3
0 0 0 0 0 0
A
C
2 5 3 0 2 3 0 3
0 0 1 0 0 0
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
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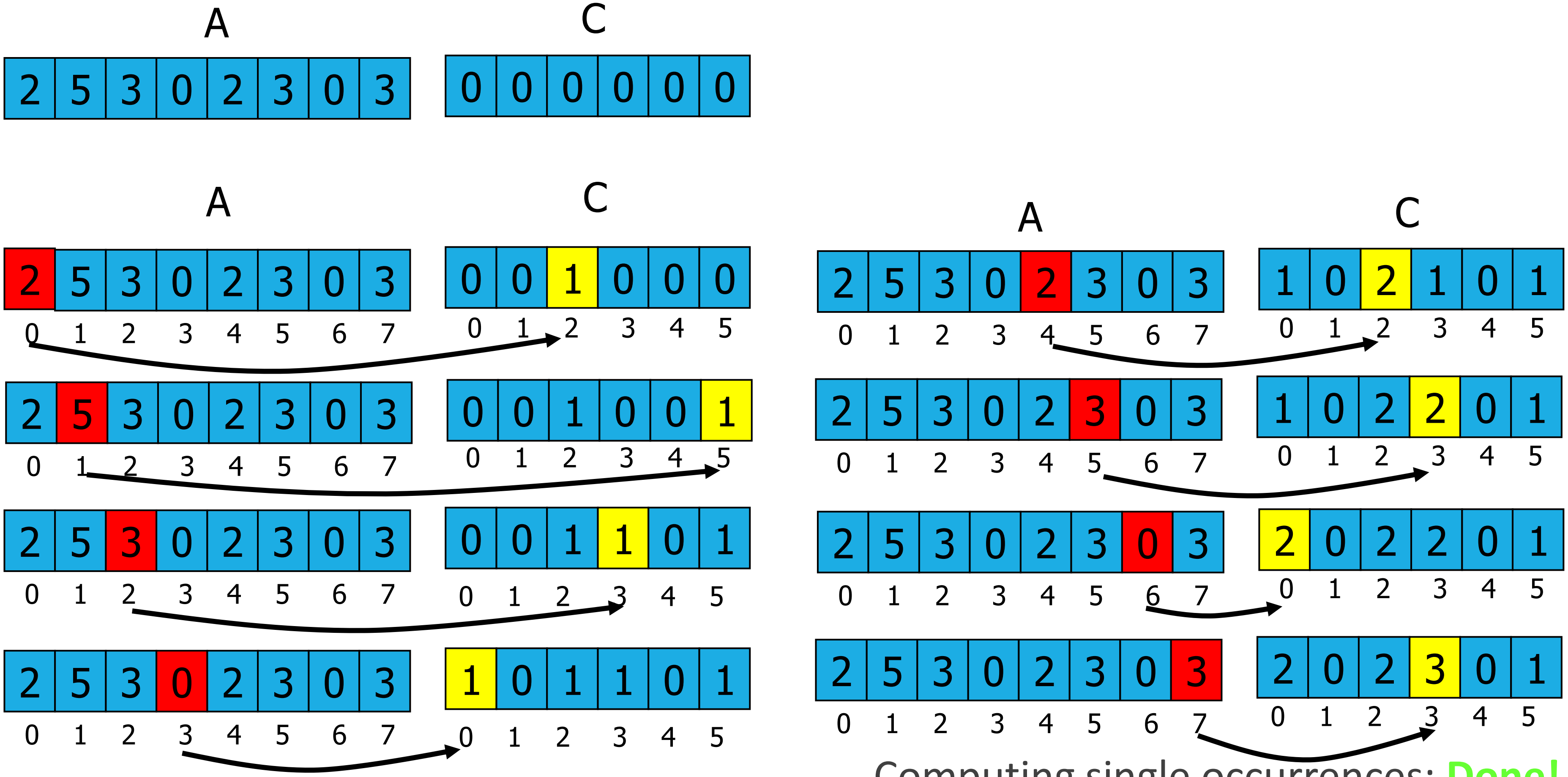
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
0 0 1 1 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
1 0 1 1 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5
A
C
2 5 3 0 2 3 0 3
1 0 2 1 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
1 0 2 2 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5
2 5 3 0 2 3 0 3
2 0 2 2 0 1
0 1 2 3 4 5 6 7
0 1 2 3 4 5



Computing single occurrences: **Done!**

Computing Multiple Occurrences

- Scan simple occurrences array **C**
  - **C**[**0**] stores the number of occurrences of **0** **and of all values that precede it** (none by definition!)
  - **C**[**i-1**] stores the occurrences of the data that precede **i** ($1 \le i < k$)
  - The occurrences of data that either precede or are equal to i are computed as **C**[**i**] = **C**[**i-1**] + **C**[**i**]

```
for (i = 1; i < k; i++)
  C[i] += C[i-1];
```

C	2	0	2	3	0	1
	0	1	2	3	4	5

C

2	0	2	3	0	1
0	1	2	3	4	5

2	0	2	3	0	1
0	1	2	3	4	5

⇨

2	2	2	3	0	1
0	1	2	3	4	5

There are **2** occurrences of data ≤**1**

C

0	1	2	3	4	5
2	0	2	3	0	1

0	1	2	3	4	5
2	0	2	3	0	1

→

0	1	2	3	4	5
2	2	2	3	0	1

There are 2 occurrences of data ≤1

0	1	2	3	4	5
2	2	2	3	0	1

→

0	1	2	3	4	5
2	2	4	3	0	1

There are **4** occurrences of data ≤**2**

C

2	0	2	3	0	1
0	1	2	3	4	5

2	0	2	3	0	1
0	1	2	3	4	5

⇨

2	2	2	3	0	1
0	1	2	3	4	5

There are 2 occurrences of data ≤1

2	2	2	3	0	1
0	1	2	3	4	5

⇨

2	2	4	3	0	1
0	1	2	3	4	5

There are 4 occurrences of data ≤2

2	2	4	3	0	1
0	1	2	3	4	5

⇨

2	2	4	7	0	1
0	1	2	3	4	5

There are **7** occurrences of data ≤**3**

C

0	1	2	3	4	5
2	0	2	3	0	1

0	1	2	3	4	5
2	0	2	3	0	1

⇨

0	1	2	3	4	5
2	2	2	3	0	1

There are 2 occurrences of data ≤1

0	1	2	3	4	5
2	2	2	3	0	1

⇨

0	1	2	3	4	5
2	2	4	3	0	1

There are 4 occurrences of data ≤2

0	1	2	3	4	5
2	2	4	3	0	1

⇨

0	1	2	3	4	5
2	2	4	7	0	1

There are 7 occurrences of data ≤3

0	1	2	3	4	5
2	2	4	7	0	1

⇨

0	1	2	3	4	5
2	2	4	7	7	1

There are **7** occurrences of data ≤**4**

C

0	1	2	3	4	5
2	0	2	3	0	1

0	1	2	3	4	5
2	0	2	3	0	1

0	1	2	3	4	5
2	2	2	3	0	1

There are 2 occurrences of data ≤1

0	1	2	3	4	5
2	2	2	3	0	1

0	1	2	3	4	5
2	2	4	3	0	1

There are 4 occurrences of data ≤2

0	1	2	3	4	5
2	2	4	3	0	1

0	1	2	3	4	5
2	2	4	7	0	1

There are 7 occurrences of data ≤3

0	1	2	3	4	5
2	2	4	7	0	1

0	1	2	3	4	5
2	2	4	7	7	1

There are 7 occurrences of data ≤4

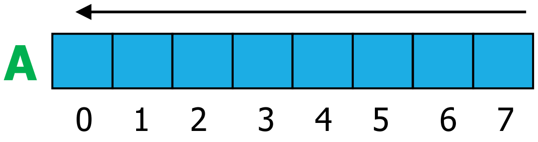
0	1	2	3	4	5
2	2	4	7	7	1

0	1	2	3	4	5
2	2	4	7	7	8

There are **8** occurrences of data ≤**5**

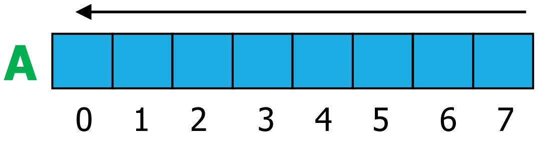
Computing multiple occurrences: **Done!**

Computing Final Positions



- Scan input array **A** from right to left
  - **C**[**A**[**i**]] stores the number of multiple occurrences of **A**[**i**] and of all the items that precede it
  - The final position in array **B** of **A**[**i**] is at index **C**[**A**[**i**]]-1.

Computing Final Positions



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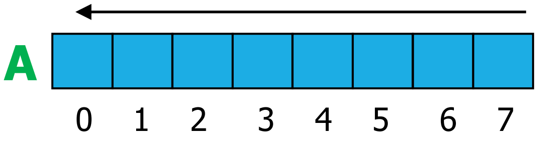
Why -1?

Computing Final Positions



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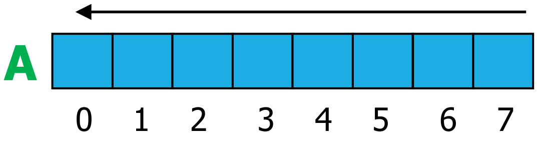
  - The final position in array **B** of **A**[**i**] is at index **C**[**A**[**i**]]-1.

Why -1?

Don't forget: in the C language array indices start from 0 **!**

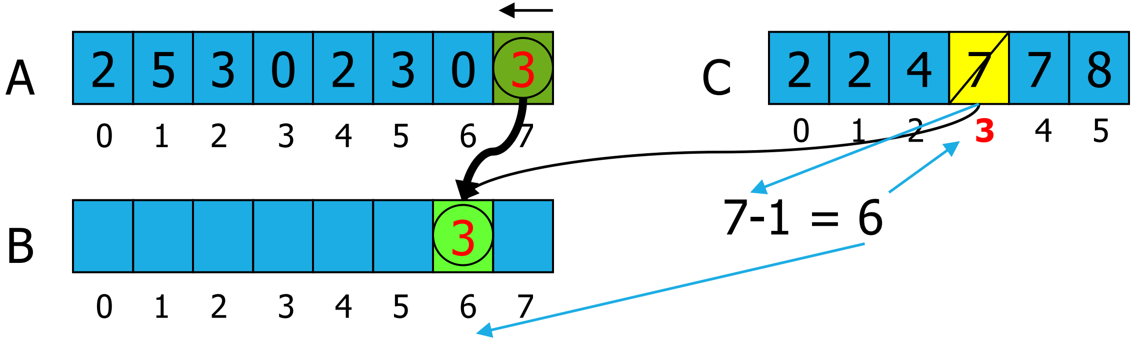
Computing Final Positions



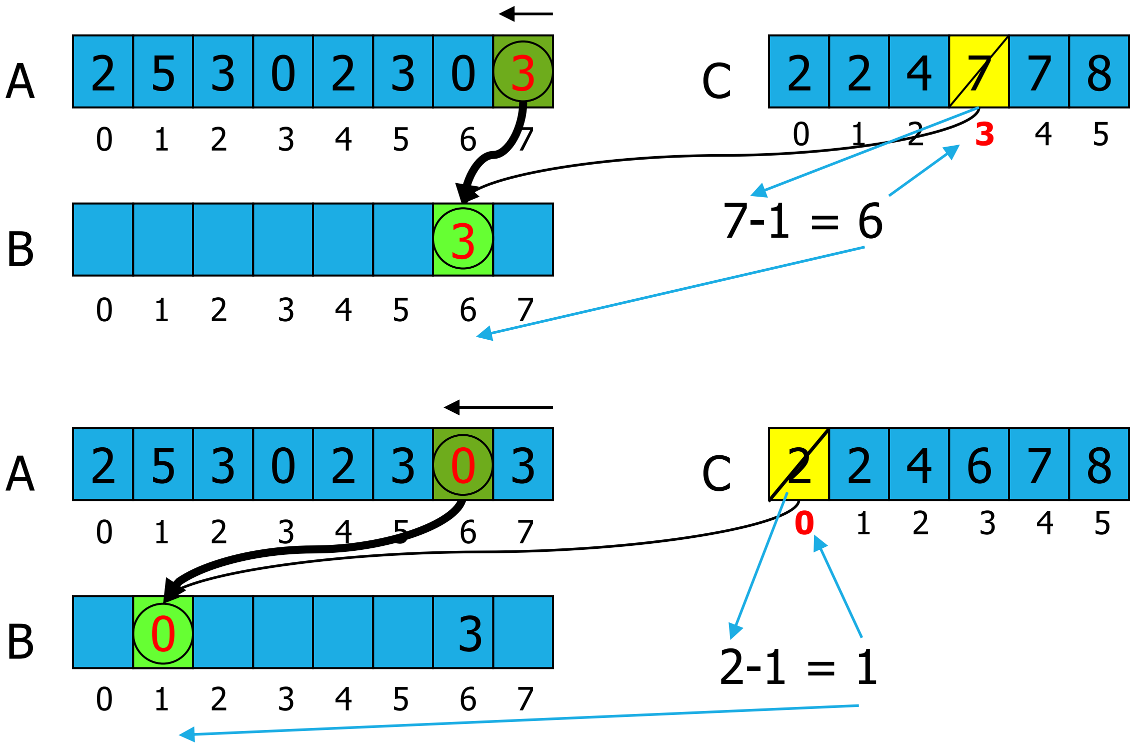
- Scan input array **A** from right to left
  - **C**[**A**[**i**]] stores the number of multiple occurrences of **A**[**i**] and of all the items that precede it
  - The final position in array **B** of **A**[**i**] is at index **C**[**A**[**i**]]-1.
  - Once **A**[**i**] is stored in its final position, update the multiple occurrences array **C** at index **A**[**i**] decrementing it by 1

```
for (i = r; i >= l; i--) {
   B[C[A[i]]-1] = A[i];
   C[A[i]]--;
}
```

Example

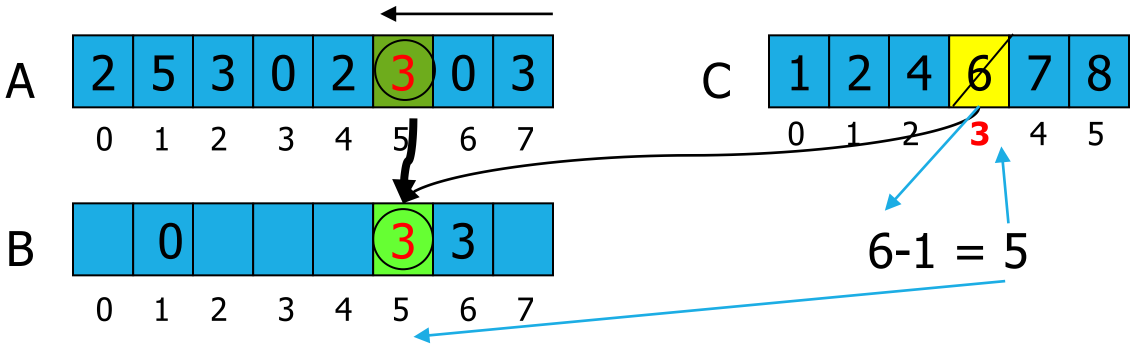


Example

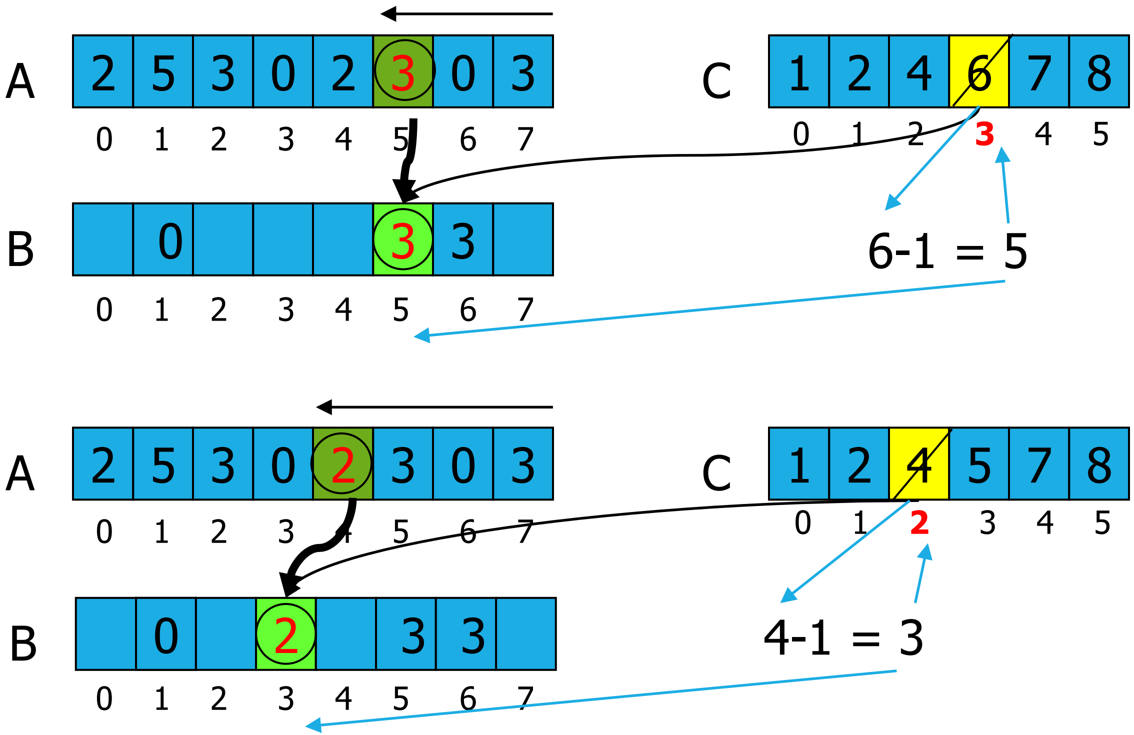


A
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
B
3
0 1 2 3 4 5 6 7
C
2 2 4 7 7 8
0 1 2 3 4 5
7-1 = 6
A
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
B
0 3
0 1 2 3 4 5 6 7
C
2 2 4 6 7 8
0 1 2 3 4 5
2-1 = 1

Example

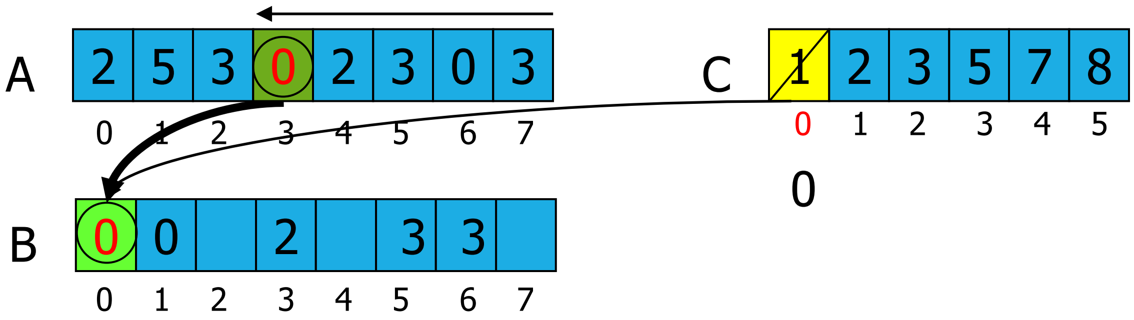


Example

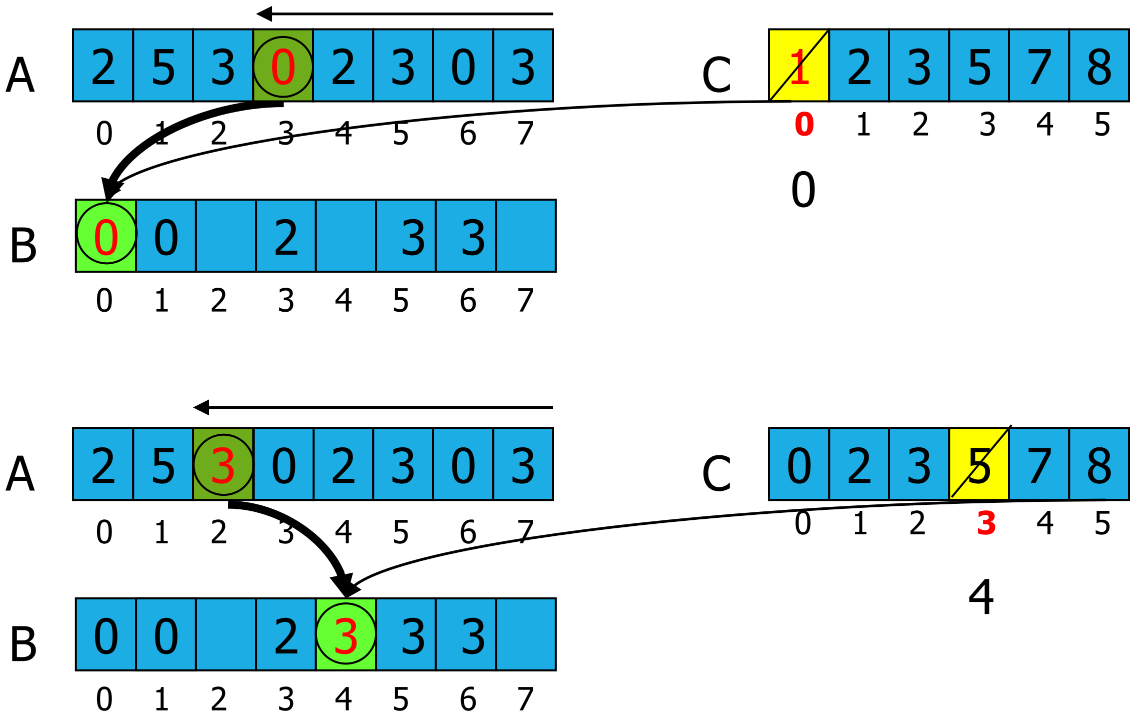


A
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
B
0 3 3
0 1 2 3 4 5 6 7
C
1 2 4 6 7 8
0 1 2 3 4 5
6-1 = 5
A
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
B
0 2 3 3
0 1 2 3 4 5 6 7
C
1 2 4 5 7 8
0 1 2 3 4 5
4-1 = 3

Example

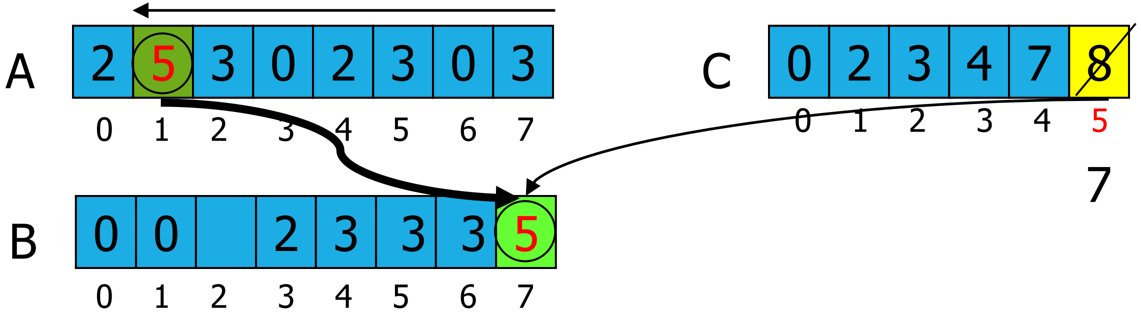


Example

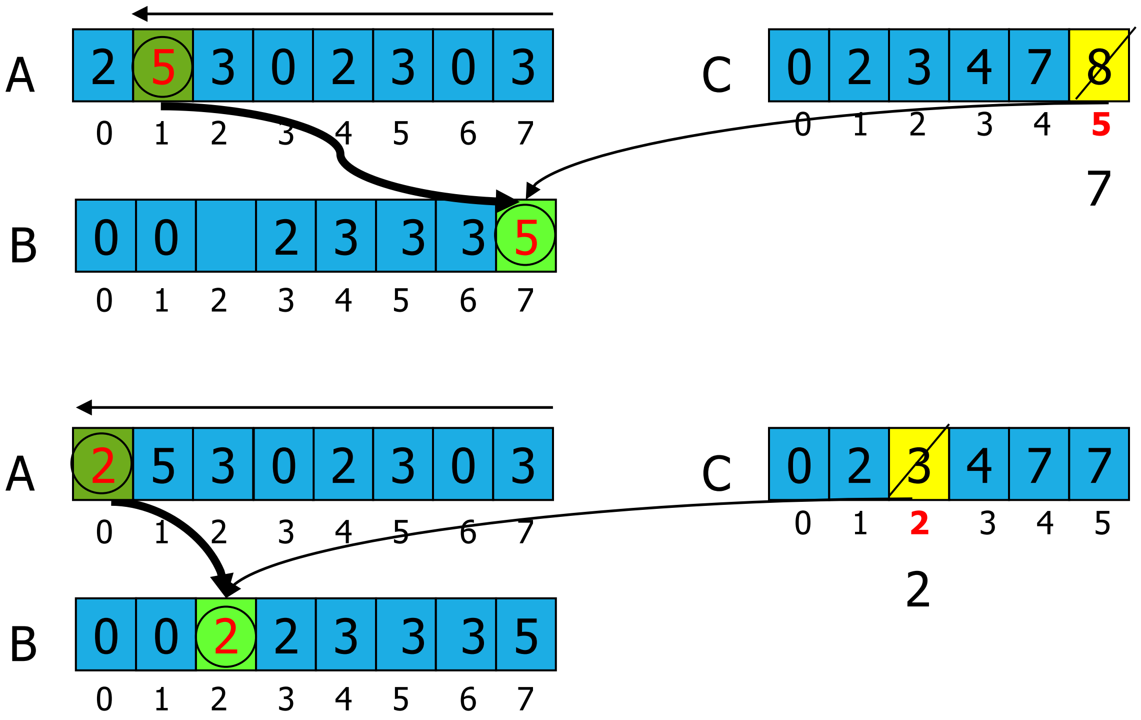


A
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
B
0 0 2 3 3
0 1 2 3 4 5 6 7
C
1 2 3 5 7 8
0 1 2 3 4 5
0
A
2 5 3 0 2 3 0 3
0 1 2 3 4 5 6 7
B
0 0 2 3 3 3
0 1 2 3 4 5 6 7
C
0 2 3 5 7 8
0 1 2 3 4 5
4

Example



Example


A
2
5
3
0
2
3
0
3
0 1 2 3 4 5 6 7
C
0
2
3
4
7
8
0 1 2 3 4 5
7
B
0
0
2
3
3
3
5
0 1 2 3 4 5 6 7
A
2
5
3
0
2
3
0
3
0 1 2 3 4 5 6 7
C
0
2
3
4
7
7
0 1 2 3 4 5
2
B
0
0
2
2
3
3
3
5
0 1 2 3 4 5 6 7

```c
void CountingSort(int A[],int B[],int C[],int N,int k){
  int i, l=0, r=N-1;
  for (i = 0; i < k; i++)
    C[i] = 0;
  for (i = l; i <= r; i++)
    C[A[i]]++;
  for (i = 1; i < k; i++)
    C[i] += C[i-1];
  for (i = r; i >= l; i--) {
    B[C[A[i]]-1] = A[i];
    C[A[i]]--;
  }
  for (i = l; i <= r; i++)
    A[i] = B[i];
}
```

Arrays allocated in `main` and passed as parameters

Initialization of C

Simple occurrences

Multiple occurrences

Correct item positioning

Copy of result

Example of code implementation in C

Counting sort Features

- **Non in-place algorithm: requires additional space ‘memory’ to operate**
  - Arrays **B** and **C** are required, in addition to **A**

- **Stable**: stability is guaranteed **by scanning array from right to left** when finding the correct positions of the items: if there are duplicate keys, the last one is the first that is stored and its position is as rightmost as possible. No other duplicate key could ever «jump over», since the corresponding cell in the multiple occurrences array **C** is decremented

- **Scanning from left to right doesn’t guarantee stability.** However it results in a sorted array.

Complexity Analysis of Counting sort

- **Loop to initialize** **C**: $\Theta(k)$
- **Loop to compute simple occurrences**: $\Theta(N)$
- **Loop to compute multiple occurrences**: $\Theta(k)$
- **Loop to position item** in **B**: $\Theta(N)$
- **Loop to copy** **B** in **A**: $\Theta(N)$

$T(N) = \Theta(N+k)$.

If $k = \Theta(N)$, $T(N) = \Theta(N)$.

Applicability: **k** and **N** must be "reasonably" comparable in size. If k=**10^6**, N=**3** and **A**= 999999, 1, 1000, it makes no sense to allocate an array of size k=**10^6** to sort 3 items!

What if the elements on an array are in the range from 1 to n^2?

$$T(n^2) = \Theta(n^2+k).$$

Even worst than comparison-based sorting algorithms!!

Is there any solution?

Radix sort

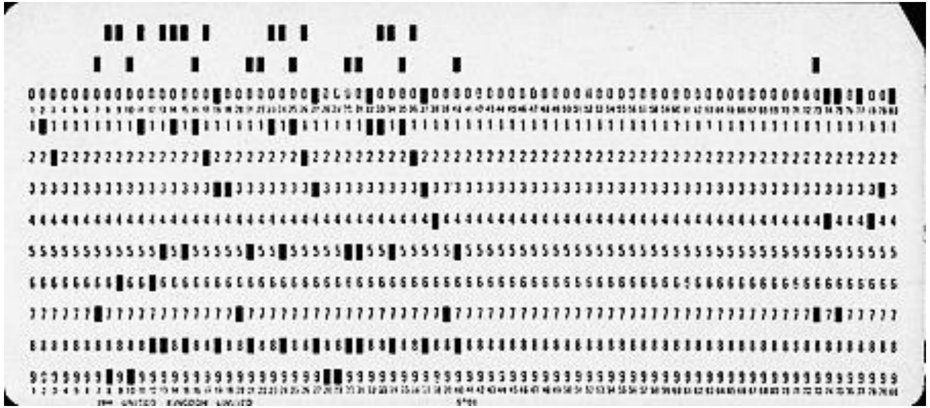
Origins:

- **1890:** first US «modern» census: **large amounts of complex data**
- **Herman Hollerith** introduces:
  - Punched cards to **store** information in **binary** form
  - «Tabulating machines» to **mechanically sort data**

Herman Hollerith

Punched cards

- Stiff **paper sheet** organized in **rows** and **columns**
- **Holes** to indicate for a certain row/column the **presence**/**absence** of an information
- **Features:**
  - Information items in **binary** form
  - Information items on **several** fields



The «tabulating machine»

Electromechanical device able to

- **«Read»** punched cards
- **Count information** items depending on the **presence**/**absence** of a hole in a certain column

Hollerith's Tabulating Machine Company becomes **International Business Machines (IBM)** in 1924.

Computing-Tabulating-Recording Company

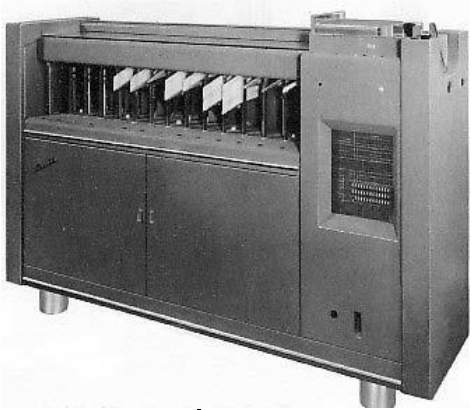
Punched Card Sorting ('60)

- **Starting** from the **rightmost column**, a machine distributed cards into bins depending on the information stored in the column
- Cards in bins were picked up keeping **the order** (stability)
- Distribution in bins continued on the **next column**
- Termination: **leftmost column** processed.

Card puncher

Punched cards

Card reader



Card sorter

Radix sort

Main idea:

- Until now only «monolithic» item considered, like **1234**, **VCDF**, etc.
- In **Radix sort** items consist of **fields**, whose values belong to a set of cardinality **n**

Example:

3-digit decimal numbers

d=3, **n**=10, **values**=0,1,2…9

329

457

657

839

Example:

Car plates: 3 fields: 2 letters, 3 digits, 2 letters

d=3,

letters **n**=22, **values**=A,…,Z **(no** I, O, Q or U)

digits **n**=10, **values**=0,1,2…9

FA 457 AA

GC 657 SD

Field-by-field «Intuitive» Sorting

- Sort according to **leftmost** column, then according to the **next column** to the right, until **rightmost** column is processed
- Intuitive for numbers, as they are represented according to a positional notation

329	329	329	720
457	355	720	355
657	436	436	436
839	457	355	457
436	657	457	657
720	720	657	329
355	859	859	859

Result is not sorted! To get a correct result, recursion is required. Recursion is a topic of the second year Course

Field-by-field «Counter-Intuitive» Sorting

- Sort according to **rightmost** column, then according to the next column to the left, until **leftmost** column is processed
- Counterintuitive for numbers, as it doesn't consider their positional notation

329	720	720	329
457	355	329	355
657	436	436	436
839	457	355	457
436	657	457	657
720	329	657	720
355	859	859	859

Result sorted!

- Constraint of the sorting algorithm used for each column: **it must be** STABLE!
- **Counting sort** is an excellent choice:
  - **It is stable**
  - **It is applicable:** the size **k** of array **C** is fixed and depends on the radix of the numbering system (hence the name **Radix sort**) of the digits that appear in each column. We may sort:
    - Base-10 numbers: **k** = 10
    - Strings of letters A…Z: **k** = 26
    - Strings of ASCII characters: **k** = 128

Sorting integers (in base 10)

- Given **n integers** stored in array **A** and consisting of (non necessarily identical) **number of digits**
- Find the maximum number of **digits d**, **left padding** with **0s** shorter numbers

170		0170
45		0045
2375		2375
90	With padding	0090
802		0802
24		0024
2		0002
66		0066

- Apply **d steps** of **Counting** sort starting from the **rightmost** column (weight 10^0) until the **leftmost** column (weight 10^{d-1}) is processed

```c
void radixSort(int A[], int B[], int C[], int D[], int n) {
  int largest, d=1, i;
  largest = getMax(A, n);

  while (largest/10 > 0){
    d++;
    largest /= 10;
  }
  for (i = 0; i < d; i++)
    CountingSort(A, B, C, D, n , i);
}
```

Identify maximum of A

Compute number of digits d

Iterate d times
Counting sort

Example of implementation in C

Example

Step 0
Column of units

A	D	A after Counting sort on column of units
17**0**	0	170
4**5**	5	90
237**5**	5	802
9**0**	0	2
80**2**	2	24
2**4**	4	45
2	2	2375
6**6**	6	66

Step 1
Column of tens

A after
Counting sort
on column
of tens

A	D	
170	7	802
90	9	2
802	0	24
2	0	45
24	2	66
45	4	170
2375	7	2375
66	6	90

A
802
2
24
45
66
170
2**3**75
90

Step 2
Column of hundreds

D
8
0
0
0
0
1
3
0

A after
Counting sort
on column
of hundreds

2
24
45
66
90
170
2375
802

Step 3
Column of thousands

A after Counting sort on column of thousands

A	D	A after Counting sort on column of thousands
2	0	2
24	0	24
45	0	45
66	0	66
90	0	90
170	0	170
2375	2	802
802	0	2375

Identifying digits

Positional representation of numbers in base **b**:

- **Digits** in the range from **0** to **b-1**
  - in **base** 2 — **b**=2, — **digits** 0, 1
  - in **base** 10 — **b**=10, — **digits** 0,1,2,3,4,5,6,7,8,9
  - in **base** 8 — **b**=8, — **digits** 0,1,2,3,4,5,6,7
  - In **base** 16 — **b**=16, — **digits** 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

$$\mathbf{x}_{(\text{in } \mathbf{base\ b}\text{, on } \mathbf{d\ digits})} = a_{d-1}\mathbf{b}^{d-1} + a_{d-2}\mathbf{b}^{d-2} + a_{d-3}\mathbf{b}^{d-3} + \ldots\ldots\ a_1\mathbf{b}^1 + a_0\mathbf{b}^0$$

Example

$$12345_{(\text{in } \mathbf{base}\ 10\text{, on } 5\ \mathbf{digits})} = 1\cdot10^4 + 2\cdot10^3 + 3\cdot10^2 + 4\cdot10^1 + 5\cdot10^0$$

$()\%b$: remainder of the integer division of () by b

- **units:** $(x / b^0)\% \, b$
- **tens:** $(x / b^1)\% \, b$
- **hundreds:** $(x / b^2)\% \, b$
- ....

Example

x = $12345_{(\text{in } \mathbf{base}\ 10)}$

- units: $(x / b^0)\% \, b$ $(12345/1) \% 10 = 5$
- tens: $(x / b^1)\% \, b$ $(12345/10) \% 10 = 1234 \% 10 = 4$
- hundreds: $(x / b^2)\% \, b$ $(12345/100) \% 10 = 123 \% 10 = 3$
- thousands: $(x / b^3)\% \, b$ $(12345/1000) \% 10 = 12 \% 10 = 2$
- tens of thousands: $(x / b^4)\% \, b$ $(12345/10000) \% 10 = 1 \% 10 = 1$

The auxiliary array **D** of **n integers** stores at each step the corresponding column and is is used by **Counting sort** to sort **A**

```
...
int i, ..., weight=1;
for (i=0; i < step; i++)
  weight *= 10;

for (i = 1; i <= r; i++)
  D[i] =(A[i]/weight) % 10;
...
```

```
void CountingSort(int A[],int B[],int C[],int D[], int N, int step){
  int i, l=0, r=N-1, weight=1;

  for (i=0; i < step; i++) weight *= 10;

  for (i = 0; i < 10; i++) C[i] = 0;

  for (i = l; i <= r; i++) D[i] =(A[i]/weight)%10;

  for (i = l; i <= r; i++) C[D[i]]++;

  for (i = 1; i < 10; i++) C[i] += C[i-1];

  for (i = r; i >= l; i--) {
    B[C[D[i]]-1] = A[i];
    C[D[i]]--;
  }
  for (i = l; i <= r; i++) A[i] = B[i];
}
```

compute 10^{step}

identify column

simple occorrences

multiple occurrences

sorting step

copy

Radix sort Features

- Not in-place: arrays **B**, **C** and **D** used. **D** could be avoided recomputing its current item whenever necessary
- **stable:** stability is guaranteed by the use at each step of a stable algorithm like **Counting sort**.

Complexity Analysis of Radix sort

- Worst-case asymptotic complexity of **Counting sort** is $T(N) = \Theta(N+k)$, where items to sort are integers in the range (**0… k-1**)
- Run **Counting sort** **d** times
- Complexity is $T(N) = \Theta(d(N+k))$.
- For numbers in base 10, **k** is fixed and is 10, thus

$$T(N) = \Theta(dN)$$

- If the number of digits **d** is fixed

$$T(N) = \Theta(N).$$