Intro to Real Analysis
HW #5

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Problem 1

Problem: Define $f : (0,1) :\to \mathbb{R}$ by $x \mapsto \sqrt{x}\sin\left(\frac{1}{x}\right)$. Use $\epsilon - \delta$ language to find $\lim_{x\to 0} f(x)$.

Solution: Consider a fixed $\epsilon > 0$, let $\delta = \epsilon^2 > 0$. Then, for all x such that $0 < |x| < \delta$ we have:

$$\begin{aligned}
\delta &> |x| && \text{(hypothesis)}\\
\epsilon^2 &> |x| && \text{(def. of } \delta)\\
\epsilon &> \sqrt{|x|} &&\\
&> |\sqrt{x}| && (\forall x \in (0,1),\, x > 0)\\
&> |\sqrt{x}|\left|\sin\left(\frac{1}{x}\right)\right| && (\forall c \in \mathbb{R},\, 0 \le |\sin c| \le 1)\\
&> \left|\sqrt{x}\sin\left(\frac{1}{x}\right)\right| &&\\
&> |f(x)| && \text{(def. of } f(x))\\
&> |f(x) - 0| &&
\end{aligned}$$

In other words, we have shown:

$$(\forall\epsilon > 0)\,\underbrace{(\exists\delta > 0)}_{\text{namely } \epsilon^2},\, 0 < |x - 0| < \delta \implies |f(x) - 0| < \epsilon$$

Which is prescisly the definition of:

$$\lim_{x\to 0} f(x) = 0$$

Problem 2

Problem: Consider a function $f : D \to \mathbb{R}$. Suppose that $\lim_{x\to x_o} f(x) = c$, use $\epsilon - \delta$ language to show that:

$$\lim_{x\to x_0} |f(x)| = |c|$$

Solution: First let us establish the reverse triangle inequality. Consider any two reals x, y:

$$\begin{aligned}
|x + y - x| &\le |x| + |y - x| && \text{(triangle inequality)}\\
|y| &\le |x| + |y - x| &&\\
|y| - |x| &\le |y - x| &&\\
|y| - |x| \le |y - x| &\wedge |x| - |y| \le |x - y| && (x \text{ and } y \text{ are indistinguishable})\\
|x| - |y| \ge -|x - y| &\wedge |x| - |y| \le |x - y| &&\\
||x| - |y|| &\le |x - y| && \text{(reverse triangle inequality)}
\end{aligned}$$

With this in mind, note that by assuming $\lim_{x\to x_o} f(x) = c$ we have, $(\forall\epsilon > 0)(\exists\delta > 0)(\forall x \in D)$:

$$\begin{aligned}
0 < |x - x_0| < \delta &\implies |f(x) - c| < \epsilon && \text{(def. of limit)}\\
&\implies ||f(x)| - |c|| \le |f(x) - c| < \epsilon && \text{(reverse triangle inequality)}\\
&\implies ||f(x)| - |c|| < \epsilon && \text{(transitivity)}
\end{aligned}$$

Note that this is precisely the definition of:

$$\lim_{x\to x_0} |f(x)| = |c|$$

Problem 3

Problem: Define $f : (0,1) \to \mathbb{R}$ by $x \mapsto (1+x)^{1/x}$. Find $\lim_{x\to 0} f(x)$.

Solution: First recall Bernoulli's inequality for general r and y.

$$(\forall r \geq 1)(\forall x \geq -1),\ (1+x)^r \geq 1 + rx$$
$$(\forall r \in [0,1])(\forall x \leq -1),\ (1+x)^r \geq 1 + rx$$

First, note the following:

$$(1+x)^{1/x} \geq 1 + \frac{x}{x} = 2 \qquad (r = 1/x > 1 \ \&\ x = x \geq -1, \text{ Bernoulli's inequality})$$
$$\left(1 + \frac{1}{x}\right)^x \leq 1 + \frac{x}{x} = 2 \qquad (r = x \in (0,1) \ \&\ x = 1/x \geq -1, \text{ Bernoulli's inequality})$$

Leading us to the inequality:

$$\left(1 + \frac{1}{x}\right)^x \leq 2 \leq (1+x)^{1/x}$$

And now consider the following:

$$\begin{aligned}
e &= \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x && \text{(def. of } e\text{)} \\
&= \lim_{1/u\to\infty} (1+u)^{1/u} && \text{(change of variables } {}^{u=1/x}_{x=1/x}\text{)} \\
&= \lim_{|1/u|\to\infty} (1+u)^{1/u} \\
&= \lim_{u\to 0} (1+u)^{1/u} && (\lim_{u\to 0} \left|\tfrac{1}{u}\right| = \infty)
\end{aligned}$$

Problem 4

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions such that:

$$\lim_{x\to 0} f(x) = c$$
$$\lim_{x\to c} g(x) = d$$

Part a: Suppose there exists $\delta > 0$ such that for any $x \in (-\delta, 0) \cup (0, \delta)$ we have $f(x) \neq c$. Prove the following:

$$\lim_{x\to 0} g \circ f(x) = d$$

Solution: Let us call the supposition above condition a. Let us now rewrite this condition, $(\exists \delta > 0)$:

$$\begin{aligned}
x \in (-\delta, 0) \cup (0, \delta) &\implies f(x) \neq c && \text{(condition a)} \\
0 < |x| < \delta &\implies f(x) \neq c && \text{(def. of absolute value)} \\
&\implies f(x) - c \neq 0 \\
&\implies |f(x) - c| \neq 0 \\
&\implies 0 < |f(x) - c| && \text{(absolute value is nonnegative)}
\end{aligned}$$

Now let us prove the statement, $(\forall \epsilon_1 > 0)(\exists \delta_1 > 0)(\exists \delta > 0)(\forall \epsilon_2 > 0)(\exists \delta_2 > 0)$:

$$\begin{aligned}
0 < |x| < \delta_1 &\implies |f(x) - c| < \epsilon_1 && \text{(def. of } \lim_{x\to 0} f(x) = c) \\
0 < |x| < \delta &\implies 0 < |f(x) - c| < \epsilon_1 && \text{(condition a, let } \delta_1 = \delta) \\
&\implies |g(f(x)) - d| < \epsilon_2 && \text{(def. of } \lim_{x\to c} g(x) = d\text{, let } x = f(x)\text{, let } \delta_2 = \epsilon_1)
\end{aligned}$$

Simplifying, we have:

$$(\forall \epsilon_2 > 0)(\exists \delta > 0),\ 0 < |x| < \delta \implies |f(x) - d| < \epsilon_2$$

Which is precisely the definition of:

$$\lim_{x\to 0} g \circ f(x) = d$$

Part b: Without condition a, find an example where:

$$\lim_{x\to 0} g \circ f(x) \neq d$$

Solution: Let us define our functions and constants:

$$\begin{aligned}
f(x) &= 0 \\
g(x) &= \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases} \\
c &= 0 \\
d &= 0
\end{aligned}$$

Now let us verify that these definitions satisfy the conditions of the problem:

$$\begin{aligned}
\lim_{x\to 0} f(x) = \lim_{x\to 0} 0 = 0 &= c \\
\neg(\exists \delta,\ 0 < |x| < \delta &\implies f(x) \neq c) && (\forall x, f(x) = 0 = c) \\
\lim_{x\to c} g(x) = \lim_{x\to 0} \left(\begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases} \right) &= 0 = d
\end{aligned}$$

Now note the following:

$$\lim_{x\to 0} g(f(x)) = \lim_{x\to 0} g(0) = \lim_{x\to 0} 1 = 1 \neq 0 = d$$

And so we have produced our counterexample.