Linear Optimization
HW #1

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Problem a

Problem 4: Solve for $\mathbf{x}$:

$$\begin{bmatrix}1 & 3 & 3\\ 2 & 3 & 5\\ 2 & 1 & 3\end{bmatrix}\mathbf{x}=\begin{bmatrix}4\\0\\1\end{bmatrix}$$

Solution: First we formulate our problem in terms of an augmented matrix, then we perform Guassian elimination:

$$\left[\begin{array}{ccc|c}1 & 3 & 3 & 4\\ 2 & 3 & 5 & 0\\ 2 & 1 & 3 & 1\end{array}\right] \xrightarrow{r_2-2r_1} \left[\begin{array}{ccc|c}1 & 3 & 3 & 4\\ 0 & -3 & -1 & -8\\ 2 & 1 & 2 & 1\end{array}\right] \xrightarrow{r_3-2r_1} \left[\begin{array}{ccc|c}1 & 3 & 3 & 4\\ 0 & -3 & -1 & -8\\ 0 & -5 & -3 & -7\end{array}\right]$$

$$\xrightarrow{(-1/3)r_2} \left[\begin{array}{ccc|c}1 & 3 & 3 & 4\\ 0 & 1 & 1/3 & 8/3\\ 0 & -5 & -3 & -7\end{array}\right] \xrightarrow{r_1-3r_2} \left[\begin{array}{ccc|c}1 & 0 & 2 & -4\\ 0 & 1 & 1/3 & 8/3\\ 0 & -5 & -3 & -7\end{array}\right]$$

$$\xrightarrow{r_3+5r_2} \left[\begin{array}{ccc|c}1 & 0 & 2 & -4\\ 0 & 1 & 1/3 & 8/3\\ 0 & 0 & -4/3 & 19/3\end{array}\right] \xrightarrow{(-3/4)r_3} \left[\begin{array}{ccc|c}1 & 0 & 2 & -4\\ 0 & 1 & 1/3 & 8/3\\ 0 & 0 & 1 & -19/4\end{array}\right]$$

$$\xrightarrow{r_1-2r_3} \left[\begin{array}{ccc|c}1 & 0 & 0 & 11/2\\ 0 & 1 & 1/3 & 8/3\\ 0 & 0 & 1 & -19/4\end{array}\right] \xrightarrow{r_2-(1/3)r_3} \left[\begin{array}{ccc|c}1 & 0 & 0 & 11/2\\ 0 & 1 & 0 & 17/4\\ 0 & 0 & 1 & -19/4\end{array}\right]$$

$$\implies \begin{cases} x_1 & = 11/2 \\ \quad x_2 & = 17/4 \\ \qquad x_3 & = -19/4 \end{cases}$$

And so our solution is $\mathbf{x}=\begin{bmatrix}11/2\\17/4\\-19/4\end{bmatrix}$.

Problem 16: Find the solutions to:

$$\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\mathbf{x}=\begin{bmatrix}1\\2\end{bmatrix}$$

Give the the solutions that maximize and minimize (if they exist) the cost function $C(x_1,x_2)=x_1^2+x_2^2$.

Solution: First note that, by performing the row operation r_2-2r_1, the system above reduces to:

$$\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}\mathbf{x}=\begin{bmatrix}1\\0\end{bmatrix} \implies x_1+2x_2=1$$

And so the solution set to the problem is given by:

$$\{x_1,x_2\in\mathbb{R} \mid x_1+2x_2=1\}$$

Now let us restrict the cost function to the solution set found above:

$$\begin{aligned}
C(x_1, x_2) &= x_1^2 + x_2^2 \\
&= (1 - 2x_2)^2 + x_2^2 && (x_1 + 2x_2 = 1 \implies x_1 = 1 - 2x_2) \\
&= (1 - 4x_2 + 4x_2^2) + x_2^2 \\
&= \underbrace{5x_2^2 - 4x_2 + 1}_{a=5, b=-4, c=1}
\end{aligned}$$

As the cost function is a quadratic polynomial, it has a single extremum at its vertex. That extremum is a global minimum if $a > 0$ (i.e. the parabola opens upwards) and a global maximum if $a < 0$ (i.e. it opens downwards). In our case it is a minimum:

$$\begin{aligned}
\arg\min_{x_2} 5x_2^2 - 4x_2 + 1 &= \frac{-b}{2a} && \text{(vertex of quadratic w/ } a > 0) \\
&= \frac{4}{10}
\end{aligned}$$

And using the equation we solved for earlier we know that the corresponding value of x_1 is given by:

$$x_1 = 1 - 2x_2 = 1 - 2 \cdot \frac{4}{10} = \frac{1}{5}$$

And so the minimizing solution to the system of equations under cost function C is given by $(1/5, 4/10)$. On the other hand there is no maximum, as the cost function is a quadratic with $a > 0$ (i.e. the parabola opens upwards). As such the cost can grow arbitrarily high as we increase x_2.

Problem 22: Give a differentiable function $f : \mathbb{R} \to \mathbb{R}$ that doesn't have an extremum at one of its critical points.

Solution: The function $f(x) = x^3$ has a single critical point at $x = 0$:

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}x} x^3 \\
&= 3x^2 \\
0 &= x
\end{aligned}$$

To see that this critical point is neither a min or max but instead an inflection point, consider the second derivative of x^3:

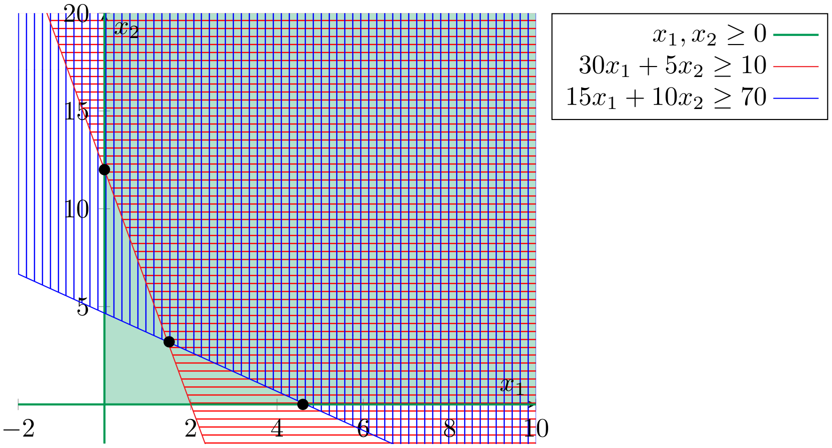
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} x^3 = \frac{\mathrm{d}}{\mathrm{d}x} 3x^2 = 6x$$

You'll note that the second derivative is negative until it reaches the critical point $x = 0$ and then turns positive after. In other words, x^3 is concave downwards to the left of the critical point and concave upwards to the right, and is thus an inflection point and not a min/max.

Problem 39: Consider the following linear constraints:

$$\begin{aligned} 30x_1 + 5x_2 &\geq 60 \\ 15x_1 + 10x_2 &\geq 70 \\ x_1, x_2 &\geq 0 \end{aligned}$$

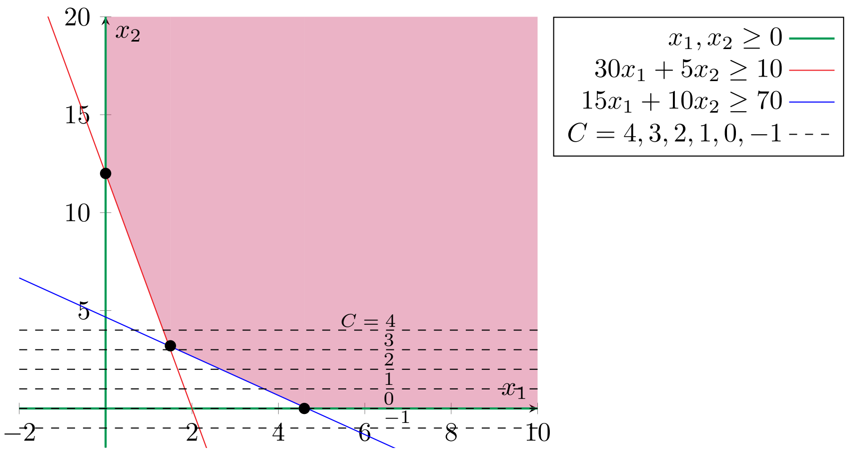
Graphed, these constraints give us the following:



Where the red lines, blue lines, and solid green background all overlap is the space of feasible solutions (i.e. solutions that satisfy all constraints). The 3 black dots are the vertices of the feasible space, the solution to any linear optimization problem under these constraints will be one of them.

Find a linear cost function $C(x_1, x_2)$ such that, under these constraints, the minimizing point is the right most vertex.

Solution: Choosing $C(x_1, x_2) = x_2$ as our cost function, the following is a graph of its level sets for $C = 4, 3, 2, 1, 0, -1$:



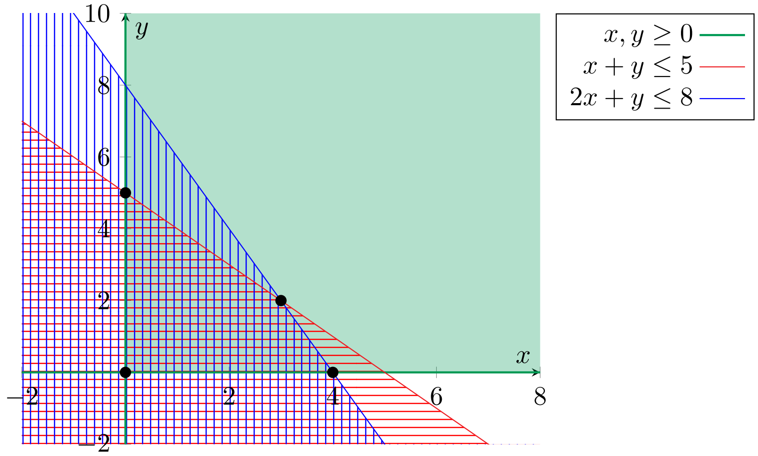
As we can see, of the feasible solutions (shaded in pink), the cost function reaches a minimum at $C = 0$. This level set is presicely where the right most vertex is located (along with an infinite set of other optimal solutions on the line $x_1 = 0$). In other words, the linear cost function $C(x_1, x_2) = x_2$ has the rightmost vertex as a minimal solution.

Problem b

Problem: Given the following linear constraints, graph the set of feasible solutions:

$$
\begin{aligned}
x + y &\leq 5 \\
2x + y &\leq 8 \\
x, y &\geq 0
\end{aligned}
$$

Solution: Graphing the constraints we have:



Where the space of feasible solutions is the are where the red lines, blue lines, and solid green backgrounds overlap. The vertices are given by the 4 black dots.

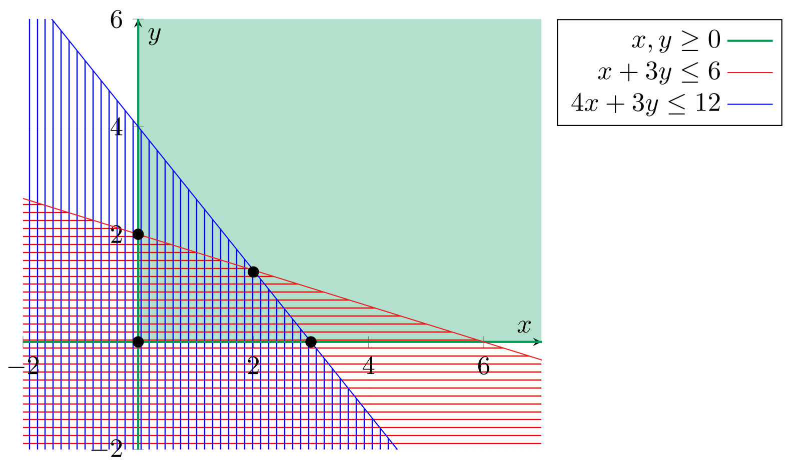
Problem c

Problem: Consider the following linear optimization problem:

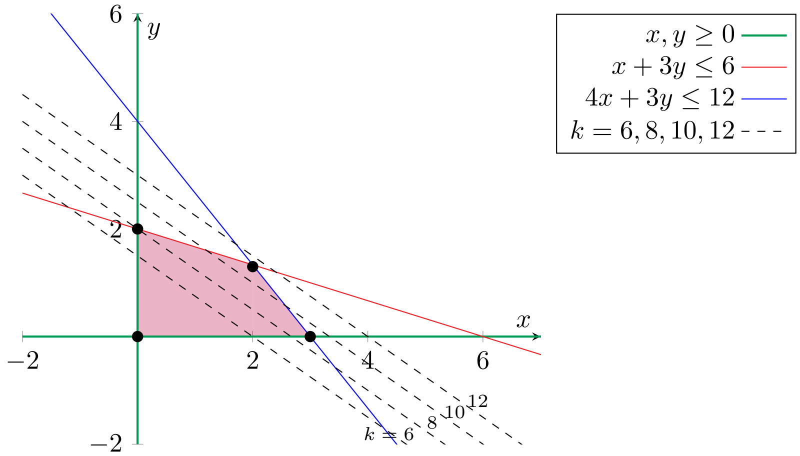
$$
\begin{aligned}
&\text{Maximize} && 3x + 4y \\
&\text{subject to} && x + 3y \leq 6 \\
& && 4x + 3y \leq 12 \\
&\text{and} && x, y \geq 0
\end{aligned}
$$

Graph the set of feasible solutions, plot the level curves of the cost function for $k = 6, 8, 10, 12$, find the maximum value and the point(s) at which this maximum is achieved.

Solution: Below is a graph of the constraints and resulting vertices and feasible region:



Below we shade only the feasible region (in pink) as well as graph the desired level sets of the cost function:



As we can see from the level sets, the cost function increases in the $(+x, +y)$ direction, while it decreases in the $(-x, -y)$ direction.

Clearly then, since the feasible solution the farthest in the $(+x, +y)$ direction is the top left vertex, it is the maximal solution. This point is the intersection of the blue and red line. To solve for x we have:

$$\begin{aligned} 4x + 3y &= 12 && \text{(blue line)} \\ -(x + 3y &= 6) && \text{(red line)} \\ \hline 3x &= 6 \\ \implies x &= 2 \end{aligned}$$

With x in hand, we can solve for y quite simply by plugging it into either line:

$$\begin{aligned} 6 &= x + 3y && \text{(red line)} \\ &= 2 + 3y && \text{(plug in } x = 2\text{)} \\ \frac{4}{3} &= y \end{aligned}$$

And so the minimal point is at $(2, {}^4\!/_3)$. We can find the maximal value by simply plugging this point into the cost function:

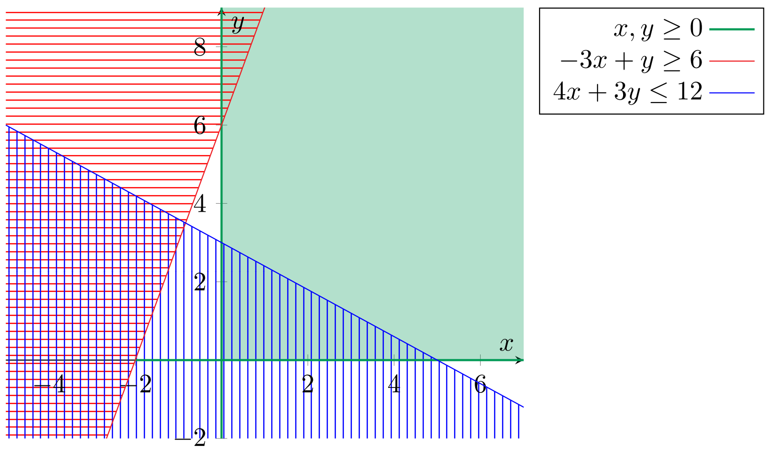
$$C(2, {}^4\!/_3) = 3 \cdot 2 + 4 \cdot \frac{4}{3} = \frac{34}{3}$$

Problem d

Problem: Solve the following linear optimization problem:

$$\begin{array}{ll} \text{Maximize} & 3x + y \\ \text{subject to} & -3x + y \geq 6 \\ & 3x + 5y \leq 15 \\ \text{and} & x, y \geq 0 \end{array}$$

Solution: Below is a graph of the constraints and their resulting vertices and feasible region:



You'll notice that there is no region in which all three constraints overlap, which is to say, there is no feasible region. As such, this linear optimization problem has no solutions.