Intro to Real Analysis
HW #9

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Problem 1

Part a & b: Consider the following functions:

$$A(x) = \frac{f(x) - f(x_0)}{x_0 - x}$$
$$B(x) = \frac{f(x_0 + x) - f(x_0)}{x}$$

Note that:

$$B(x - x_0) = \frac{f(x) - f(x_0)}{x - x_0} = A(x)$$

As such, we have:

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} A(x) && \text{(def. of derivative)} \\ &= \lim_{x \to x_0} B(x - x_0) && (A(x) = B(x - x_0)) \\ &= \lim_{x \to 0} B(x) && \text{(composition of continuous limits)} \end{aligned}$$

And so we have shown that the two different definitions of derivative are equivalent. Note that the composition line 3 is referring to is of the function B and the map $x \mapsto x - x_0$. This was justified because f is differentiable at x_0, and thus continuous at x_0, and so too is the map $x \mapsto x - x_0$. Note that this reasoning is two way (chain of equalities) and so we have shown both a) and b).

Problem 2

Part a: Note that the function is not continuous over any point $x_0 \neq 0$. This is because for any neighborhood $[x_0 - \delta, x_0 + \delta]$, there exists a rational number r contained within in it. The rationals are dense in $\mathbb{R}$. Since the function is not continuous for any $x_0 \neq 0$, the function cannot be differentiable for any $x_0 \neq 0$ as continuity is a prerequisite for differentiability.

Part b: Note the following:

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} \qquad (0 \in \mathbb{Q} \implies f(0) = 0^2 = 0)$$

And now we have two options, either $h \in \mathbb{Q}$ or $h \in \mathbb{R} \setminus \mathbb{Q}$. In the first case we have:

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h$$

And in the second case we have:

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

By choosing $\lambda = \epsilon$ we have:

$$\begin{aligned}
|h| < \lambda &\implies |h| < \epsilon && (\lambda = \epsilon) \\
&\implies \left|\frac{f(h)}{h}\right| \leq |h| && \text{(limit is either } h \text{ or } 0) \\
&\implies \left|\frac{f(h)}{h}\right| < \epsilon \\
&\implies \left|\frac{f(h)}{h} - 0\right| < \epsilon
\end{aligned}$$

And so the derivative of f exists at $x = 0$ and $f'(0) = 0$.

Problem 3

Solution: The first limit is equivalent to:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(5h) - f(-3h)}{h} &= \lim_{h\to 0} \frac{f(0+5h) - f(0-3h)}{h} \\
&= \lim_{h\to 0} \frac{f(0+5h) - f(0) + f(0) - f(0-3h)}{h} \\
&= \lim_{h\to 0} \left(5\frac{f(0+5h) - f(0)}{5h} + 3\frac{f(0-3h) - f(0)}{-3h}\right) \\
&= 5\lim_{h\to 0} \frac{f(0+5h) - f(0)}{5h} + 3\lim_{h\to 0} \frac{f(0-3h) - f(0)}{-3h} && \text{(limit of sum is sum of limits)} \\
&= 5f'(0) + 3f'(0) && \text{(def. of derivative, change of variables)} \\
&= 8f'(0) \\
&= 8c && (f'(0) = c)
\end{aligned}$$

The second limit can be solved in much the same way:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2h) - f(4h)}{h} &= \lim_{h\to 0} \frac{f(0+2h) - f(0+4h)}{h} \\
&= \lim_{h\to 0} \frac{f(0+2h) - f(0) + f(0) - f(0+4h)}{h} \\
&= \lim_{h\to 0} \left(2\frac{f(0+2h) - f(0)}{2h} - 4\frac{f(0+4h) - f(0)}{4h}\right) \\
&= 2\lim_{h\to 0} \frac{f(0+2h) - f(0)}{2h} - 4\lim_{h\to 0} \frac{f(0+4h) - f(0)}{4h} && \text{(limit of sum is sum of limits)} \\
&= 2f'(0) - 4f'(0) && \text{(def. of derivative, change of variables)} \\
&= -2f'(0) \\
&= -2c && (f'(0) = c)
\end{aligned}$$

Problem 4

Solution: We will prove this by contradiction. Suppose there are points x_1, x_2 such that $f(x_1) = f(x_2) = 0$. W.l.o.g let's us say that $x_1 < x_2$. Since f is continuous, we have that it is bounded on $[x_1, x_2]$ and thus achieves its maximum M over this interval at some point m. Assume $M > 0$ for now.

By the IMV, there must be some m_1 and m_2 such that $f(m_1) = f(m_2) = M/2$ and satisfies:

$$x_1, m_1 < m < m_2 < x_2$$

Now consider an n such that $f(n) = 2N$. We know that $n \in [x_1, x_2]$ since M is the maximum over that interval, so w.l.o.g say that $x_2 < n$. Applying the IVT again, there must be a solution to $f(x) = M/2$ in the new interval $[x_2, n]$. But that means we would have at least three solutions to $f(x) = M/2$. Thus, we have shown that no continuous function can achieve all values exactly twice.

Problem 5

Solution: First note that:

$$
\begin{aligned}
&(\forall x \in (a,b)), \quad f(x) \leq g(x) \leq h(x) \\
\wedge\ & f(x_0) = h(x_0) \\
\implies\ & g(x_0) = f(x_0) = h(x_0)
\end{aligned}
$$

Next note the following:

$$
\begin{aligned}
f(x) &\leq g(x) \leq h(x) \\
\frac{f(x)-f(x_0)}{x-x_0} &\leq \frac{g(x)-f(x_0)}{x-x_0} \leq \frac{h(x)-f(x_0)}{x-x_0} \\
\frac{f(x)-f(x_0)}{x-x_0} &\leq \frac{g(x)-g(x_0)}{x-x_0} \leq \frac{h(x)-h(x_0)}{x-x_0} && (g(x_0)=f(x_0)=h(x_0)) \\
\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} &\leq \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} \leq \lim_{x\to x_0}\frac{h(x)-h(x_0)}{x-x_0} && \text{(squeeze theorem)} \\
f'(x_0) &\leq \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} \leq h'(x_0) && \text{(def. of derivative)} \\
f'(x_0) &\leq \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} \leq f'(x_0) && (f'(x_0)=h'(x_0)) \\
\implies \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} &= f'(x_0) && \text{(squeeze theorem)}
\end{aligned}
$$

And so we have shown that the limit:

$$
\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}
$$

exists, and thus $g'(x_0)$ exists, and is equal to $f'(x_0) = h'(x_0)$.