Intro to Real Analysis
Midterm 2

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Problem 1

Part a: Consider an arbitrary $\epsilon > 0$. And a $\delta = \frac{1}{\epsilon} > 0$. We then have $\forall x \in \mathbb{R}^+$:

$$
\begin{aligned}
x > \delta \implies & \frac{1}{x} < \frac{1}{\delta} && (1/x \text{ is strictly decreasing over } \mathbb{R}^+) \\
\implies & \left|\frac{1}{x}\right| < \frac{1}{\delta} && (x > 0 \implies 1/x > 0) \\
\implies & |\sin(x)|\left|\frac{1}{x}\right| < \frac{1}{\delta} && (|\sin(x)| \leq 1) \\
\implies & \left|\frac{\sin(x)}{x}\right| < \frac{1}{\delta} \\
\implies & \left|\frac{\sin(x)}{x}\right| < \epsilon && (\text{def. of } \delta)
\end{aligned}
$$

And this is precisely the definition of:

$$\forall x \in \mathbb{R}^+, \quad \lim_{x \to \infty} \frac{\sin(x)}{x} = 0$$

Part b: Consider the sequence $A = \{1/n\}_{n=1}^{\infty}$. You'll note that:

$$\forall n \in \mathbb{Z}^+, \quad 0 < \frac{1}{n}$$

And so A is a subsequence of our interval $(0, \infty)$, since all a_n are contained within it. Now all that is left is to show that A converges to 0. Consider an arbitrary $\epsilon > 0$, and let be an integer N such that $\frac{1}{N} < \epsilon$ (this is guaranteed to us by the archemdian property). We then have:

$$
\begin{aligned}
n \geq N \implies & \frac{1}{n} \leq \frac{1}{N} && (1/x \text{ is strictly decreasing over } \mathbb{R}^+) \\
\implies & \frac{1}{n} < \epsilon && (\text{def. of } N) \\
\implies & \left|\frac{1}{n}\right| < \epsilon && (n > 0 \implies 1/n > 0) \\
\implies & \left|\frac{1}{n} - 0\right| < \epsilon
\end{aligned}
$$

And so we are done. We have shown that a subsequence of $(0, \infty)$ converges to 0, and thus it is an accumulation point of said interval.

Part c: Note the following:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(2x)}{3x(x-3)} &= \lim_{x\to 0}\frac{2\sin x\cos x}{3x(x-3)} && \text{(double angle formula)}\\
&= \lim_{x\to 0}\frac{2}{3}\cdot\frac{\sin x}{x}\cdot\frac{\cos x}{x-3}\\
&= \frac{2}{3}\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\cos x}{x-3} && \text{(product of limits is limit of products)}\\
&= \frac{2}{3}\lim_{x\to 0}\frac{\cos x}{x-3} && \text{(limit given)}\\
&= \frac{2}{3}\cdot\frac{\cos 0}{0-3} && \text{(cos is contious at 0)}\\
&= \frac{2}{3}\cdot-\frac{1}{3} = -\frac{2}{9}
\end{aligned}$$

Problem 2

Problem: First note that both pieces of this function are continuous on their own. And so, the only points in which $f(x)$ can be continuous are where they coincide:

$$\begin{aligned}
x^2+1 &= 3-x^2\\
2x^2 &= 2\\
x^2 &= 1\\
x &= \pm 1
\end{aligned}$$

And so we have that:

$$\lim x \to \pm 1 = f(\pm 1)$$

Every other point, i.e. the x in which the two functions *don't* coincide, are discontinuous. This is because there is no interval of non-zero size that does not contain a rational number. And since we are considering the points x in which the functions do not coincide, $f(x)$ cannot be continuous on such an x as the two functions approach different values. An example of this is at $x = 5$.

Problem 3

Problem: Consider $x, y \in [a, b]$ such that $x < y$. Note that:

$$\begin{aligned}
x < y &\implies [a,x] \subseteq [a,y]\\
&\implies \sup\{f(t) \mid t \in [a,x]\} \leq \sup\{f(t) \mid t \in [a,y]\}\\
&\implies g(x) \leq g(y) && \text{(def. of } g\text{)}
\end{aligned}$$

With the second implication holding because the supremum of a subset can be no larger than the supremum of its superset.

Now consider and arbitrary $x_0 \in [a, b]$. Since $f(x)$ is continuous, we have that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\forall x, x_0 \in [a, b]$:

$$\begin{aligned}
0 < |x - x_0| < \delta &\implies |f(x) - f(x_0)| < \epsilon\\
&\implies |g(x) - g(x_0)| < |f(x) - f(x_0)| < \epsilon\\
&\implies |g(x) - g(x_0)| < \epsilon
\end{aligned}$$

Which is precisely the definition of $g(x)$ being continuous over $[a, b]$.

Problem 4

Part a: Let us expand both terms in the numerator via the binomial theorem:

$$\begin{aligned}\frac{(1+mx)^n-(1+nx)^m}{x^2} &= \frac{\left(1+nmx+\binom{n}{2}m^2x^2+\cdots\right)-\left(1+mnx+\binom{m}{2}n^2x^2+\cdots\right)}{x^2}\\ &= \frac{\frac{nm(n-m)}{2}x^2+c_1x^3+c_2x^4+\cdots}{x^2}\\ &= \frac{nm(n-m)}{2}+c_1x+c_2x^2+\cdots\end{aligned}$$

Where c_k are some constants found by calculating out the binomial theorem. Now note that this is a polynomial, meaning it is continuous everywhere. As such we have:

$$\begin{aligned}\lim_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2} &= \lim_{x\to 0}\frac{nm(n-m)}{2}+c_1x+c_2x^2+\cdots && \text{(see above)}\\ &= \frac{nm(n-m)}{2}+c_1\cdot 0+c_2\cdot 0^2+\cdots && \text{(continous)}\\ &= \frac{nm(n-m)}{2}\end{aligned}$$

Part b: Note the following:

$$\begin{aligned}\lim_{x\to 1}\frac{x^m-1}{x^n-1} &= \lim_{x\to 1}\frac{(x-1)(x^{m-1}+x^{m-2}+\cdots+x+1)}{(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)}\\ &= \lim_{x\to 1}\frac{x^{m-1}+x^{m-2}+\cdots+x+1}{x^{n-1}+x^{n-2}+\cdots+x+1}\\ &= \frac{1^{m-1}+1^{m-2}+\cdots+1+1}{1^{n-1}+1^{n-2}+\cdots+1+1} && \left(\substack{\text{rational functions are continous}\\ \text{where they are defined}}\right)\\ &= \frac{m}{n}\end{aligned}$$

Part c: Note, as used in the last problem, the following identity:

$$t^N-1=(t-1)\sum_{k}^{N-1}=0t^k$$

By setting $x=t^{nm}$, and noticing that $x\to 1 \implies t^nm\to 1 \implies t\to 1$, we have:

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1} &= \lim_{t\to 1}\frac{\sqrt[m]{t^{nm}}-1}{\sqrt[n]{t^{nm}}-1} && \text{(def. of } t^nm)\\ &= \lim_{t\to 1}\frac{t^n-1}{t^m-1}\\ &= \frac{n}{m} && \text{(problem b)}\end{aligned}$$