Linear Optimization
HW #11

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Problem b

Problem: Solve the following problem using branch and bound:

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 5x_2 \\ \text{subject to} \quad & 2x_1 + 4x_2 \leq 25 \\ & x_1 \leq 8 \\ & x_2 \leq 5 \\ \text{and} \quad & \mathbf{x} \in \mathbb{N}^2 \end{aligned}$$

Solution: To find the root node, we must first solve the LP relaxation of this problem. First we will put it into canonical form:

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 5x_2 \\ \text{subject to} \quad & 2x_1 + 4x_2 \leq 25 \\ & x_1 \leq 8 \\ & x_2 \leq 5 \\ \text{and} \quad & \mathbf{x} \geq 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \text{Maximize} \quad & -3x_1 - 5x_2 + z = 0 \\ \text{subject to} \quad & 2x_1 + 4x_2 + s_1 = 25 \\ & x_1 + s_2 = 8 \\ & x_2 + s_3 = 5 \\ \text{and} \quad & \mathbf{x}, \mathbf{s} \geq 0 \end{aligned}$$

Now we can apply the tabluex method:

	x_1	x_2	s_1	s_2	s_3	z	c
s_1	2	4	1	0	0	0	25
s_2	1	0	0	1	0	0	8
s_3	0	1	0	0	1	0	5
z	-3	-5	0	0	0	1	0

$\xrightarrow[r_1-4r_3]{r_4+5r_3}$

	x_1	x_2	s_1	s_2	s_3	z	c
s_1	2	0	1	0	-4	0	5
s_2	1	0	0	1	0	0	8
x_2	0	1	0	0	1	0	5
z	-3	0	0	0	5	1	25

$\xrightarrow[\substack{r_2-r_1 \\ r_4+3r_1}]{r_1/2}$

	x_1	x_2	s_1	s_2	s_3	z	c
x_1	1	0	$1/2$	0	-2	0	$5/2$
s_2	0	0	$-1/2$	1	2	0	$11/2$
x_2	0	1	0	0	1	0	5
z	0	0	$3/2$	0	-1	1	$65/2$

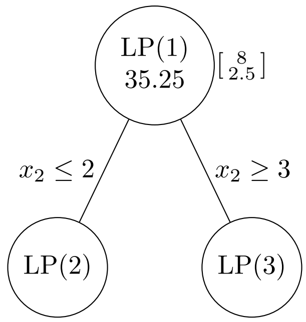
$\xrightarrow[\substack{r_2/2 \\ r_3-r_2 \\ r_4+r_2}]{r_1+r_2}$

	x_1	x_2	s_1	s_2	s_3	z	c
x_1	1	0	0	1	0	0	8
s_3	0	0	$-1/4$	$1/2$	1	0	$11/4$
x_2	0	1	$1/4$	$-1/2$	0	0	$9/4$
z	0	0	$5/4$	$1/2$	0	1	$141/4$

With this, we are done. Our tabluex method has resulted in a maximum of $141/4 = 35.25$ at:

$$\mathbf{x} = \begin{bmatrix} 8 \\ 9/4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2.25 \end{bmatrix}$$

With this solution in hand, we can begin our tree:



We will first solve solve LP(2):

$$\begin{array}{ll} \text{Maximize} & z = 3x_1 + 5x_2 \\ \text{subject to} & 2x_1 + 4x_2 \leq 25 \\ & x_1 \leq 8 \\ & x_2 \leq 2 \\ \text{and} & \mathbf{x} \geq 0 \end{array} \implies \begin{array}{ll} \text{Maximize} & -3x_1 - 5x_2 + z = 0 \\ \text{subject to} & 2x_1 + 4x_2 + s_1 = 25 \\ & x_1 + s_2 = 8 \\ & x_2 + s_3 = 2 \\ \text{and} & \mathbf{x}, \mathbf{s} \geq 0 \end{array}$$

	x_1	x_2	s_1	s_2	s_3	z	c
s_1	2	4	1	0	0	0	25
s_2	1	0	0	1	0	0	8
s_3	0	1	0	0	1	0	2
z	-3	-5	0	0	0	1	0

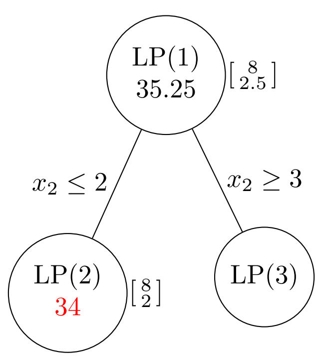
$\xrightarrow[r_1-4r_3]{r_4+5r_3}$

	x_1	x_2	s_1	s_2	s_3	z	c
s_1	2	0	1	0	-4	0	17
s_2	1	0	0	1	0	0	8
x_2	0	1	0	0	1	0	2
z	-3	0	0	0	5	1	10

$\xrightarrow[r_4+3r_2]{r_1-2r_2}$

	x_1	x_2	s_1	s_2	s_3	z	c
s_1	0	0	1	-2	-4	0	1
x_1	1	0	0	1	0	0	8
x_2	0	1	0	0	1	0	2
z	0	0	0	3	5	1	34

LP(2) has a maximum of 34 at $\mathbf{x} = \left[\begin{smallmatrix}8\\2\end{smallmatrix}\right]$. Note that this is a feasible solution to IP(1) (i.e. it is an integer solution). As such, we can set this to be a lower bound on our solutions:



Now let us solve LP(3):

$$\begin{array}{ll} \text{Maximize} & z = 3x_1 + 5x_2 \\ \text{subject to} & 2x_1 + 4x_2 \leq 25 \\ & x_1 \leq 8 \\ & x_2 \geq 3 \\ \text{and} & \mathbf{x} \geq 0 \end{array} \implies \begin{array}{ll} \text{Maximize} & -3x_1 - 5x_2 + z = 0 \\ \text{subject to} & 2x_1 + 4x_2 + s_1 = 25 \\ & x_1 + s_2 = 8 \\ & x_2 - s_3 + s_4 = 3 \\ \text{and} & \mathbf{x}, \mathbf{s} \geq 0 \end{array}$$

	x_1	x_2	s_1	s_2	s_3	s_4	z	c
s_1	2	4	1	0	0	0	0	25
s_2	1	0	0	1	0	0	0	8
s_4	0	1	0	0	-1	1	0	2
z	-3	-5	0	0	0	0	1	0

$\xrightarrow[r_1-4r_3]{r_4+5r_3}$

	x_1	x_2	s_1	s_2	s_3	s_4	z	c
s_1	2	0	1	0	4	0	0	25
s_2	1	0	0	1	0	0	0	8
x_2	0	1	0	0	-1	1	0	2
z	-3	0	0	0	-5	5	1	15

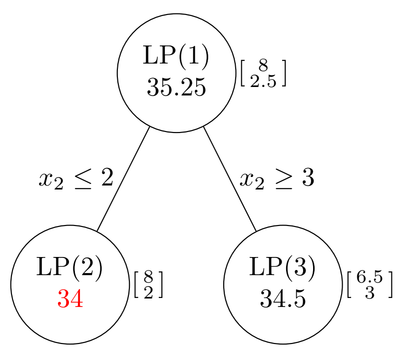
$\xrightarrow{r_1/4,\ r_3+r-1,\ r_4+5r_1}$

	x_1	x_2	s_1	s_2	s_3	z	c
s_3	$1/2$	0	$1/4$	0	1	0	$13/4$
s_2	1	0	0	1	0	0	8
x_2	$1/2$	1	$1/4$	0	0	0	$25/4$
z	$-1/2$	0	$5/4$	0	0	1	$125/4$

$\xrightarrow{r_3-r_2,\ r_4+r_1,\ 2r_1,\ r_2-r_1}$

	x_1	x_2	s_1	s_2	s_3	z	c
x_1	1	0	$1/2$	0	2	0	$13/2$
s_2	0	0	$-1/2$	1	-2	0	$3/2$
x_2	0	1	0	0	-1	0	3
z	0	0	$3/2$	0	1	1	$69/2$

LP(3) has a maximum of 34.5 at $\mathbf{x} = \left[\begin{smallmatrix} 13/2 \\ 3 \end{smallmatrix}\right]$. updating our tree we now have:



Note that, at this point, the maximum M of IP(1) must satisfy:

$$34 \le M \le 34.5$$

But also note that $M \in \mathbb{Z}$ as M is a linear combination of integers. As a result, M must equal 34 as there is no other integer that satisfies the above inequality. Thus, the solution to IP(1) is given by $\mathbf{x} = \left[\begin{smallmatrix} 8 \\ 2 \end{smallmatrix}\right]$, reaching a maximum of 34.

Problem c

Problem 17: Prove $x^n = O(e^x)$ for any $n > 0$.

Solution: Consider an arbitrary $n > 0$. Let:

$$x_0 = 1, \qquad C = \frac{1}{(e^{1/n} - 1)^n}$$

Now note the following:

$$
\begin{aligned}
|x^n| &= x^n && (x, n > 0) \\
&= \frac{(e^{1/n}-1)^n}{(e^{1/n}-1)^n} x^n \\
&= \frac{1}{(e^{1/n}-1)^n}((e^{1/n}-1)x)^n \\
&= C((e^{1/n}-1)x)^n && (\text{def. of } C) \\
&< C(1 + (e^{1/n}-1)x)^n \\
&\leq C((1 + e^{1/n}-1)^x)^n && (\text{Bernoulli's inequality}) \\
&= C(e^{1/n})^{xn} \\
&= Ce^x
\end{aligned}
$$

And with this, we are done.

Problem 18: Prove $\log^a x = O(x^r)$ as $x \to \infty$ for any $a, r > 0$. What is the relation between these two functions as $x \to 0$?

Solution: Note that this problem asks a question regarding asymptotic notation as $x \to 0$. For this, we'll need a more general definition of Big-O. That is, $\forall a \in \mathbb{R}$:

$$f(x) = O(g(x)) \quad \text{as } x \to a$$

is equivalent to:

$$\limsup_{x \to a} \frac{|f(x)|}{g(x)} < \infty$$

Now we can solve the first question. Note that we assume $\log = \ln$, ultimately our choice of base $b > 0$ doesn't matter:

$$
\begin{aligned}
\limsup_{x \to \infty} \frac{|\log^a x|}{x^r} &= \limsup_{x \to \infty} \frac{\log^a x}{x^r} && (x, a > 0) \\
&= \limsup_{x \to \infty} \frac{1}{\prod_{i=0}^{r-1} \log^i x} \cdot \frac{1}{rx^{r-1}} && (\text{L'Hopital's rule}) \\
&= 0 < \infty
\end{aligned}
$$

Note that for any function f, we have $f^0(x) = x$.

And so we have proven that $\log^a x = O(x^r)$. For the second question, note the following:

$$\limsup_{x \to 0} \left| \frac{\log^a x}{x^r} \right| = \infty$$

And so, depending on our definitions, it would seem that:

$$\log^a x = \Omega(x^r) \quad \text{as } x \to 0$$

Note that a caveat here is that, for the real numbers, log only approaches $-\infty$, and thus $|\log|$ approaches ∞, on the right hand side.