Intro to Real Analysis
HW #8

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Problem 1

Solution: Suppose, for contradiction, that f is *not* uniformly continuous on A. Then there would exist some $\epsilon_0 > 0$ and sequences whose elements $x_k, y_k \in A$ such that:

$$\lim_{k\to\infty} |x_k - y_k| = 0 \land (\forall k \in \mathbb{N})\ |f(x_k) - f(x_y)| \geq \epsilon_0$$

Now note that since A is is closed and bounded, there is a convergent subsequence x_{k_i} of x_k such that:

$$\lim_{i\to\infty} x_{k_i} = x \in A$$

Moreover, since $x_k - y_k \to 0$ as $k \to \infty$, it follows that:

$$\lim_{i\to\infty} y_{k_i} = \lim_{i\to\infty} (x_{k_i} - (x_{k_i} - y_{k_i})) = \lim_{i\to\infty} x_{k_i} - \lim_{i\to\infty} x_{k_i} - y_{k_i} = x$$

so y_{k_i} also converges to x. And since f is continuous on A, we have:

$$\lim_{i\to\infty} |f(x_{k_i}) - f(y_{k_i})| = |\lim_{i\to\infty} f(x_{k_i}) \lim_{i\to\infty} -f(y_{k_i})| = |f(x) - f(y)| = 0$$

But this contradictions what we stated previously, namely:

$$|f(x_k) - f(x_y)| \geq \epsilon_0$$

Thus, f must be uniformly continuous.

Problem 2

Solution: if $f(1/2) \neq 0$ then, since $[0,1]$ is compact, there is some neighborhood of points N around $1/2$ such that each $x \in N$ satisfies $f(x) \neq 0$. Thus $1/2$ is an accumulation point of D.

Problem 3

Solution: If $f(0) = 0$ or $f(1) = 1$ then we are done. Otherwise, define $g(x) = f(x) - x$. Certainly g is continuous as it is the difference of two continuous functions. Now note the following:

$$\begin{aligned}
0 &< f(0) && (f(0) = 0 \text{ case already considered}) \\
&< f(0) - 0 \\
&< g(0) && (\text{def. of } g) \\
1 &> f(1) && (f(1) = 1 \text{ case already considered}) \\
0 &> f(1) - 1 \\
&> g(1) && (\text{def. of } g)
\end{aligned}$$

And so we have that $g(1) < 0 < g(0)$, and so by the intermediate value theorem, there must be some $x \in [0,1]$ such that $g(x) = 0$ which is equivalent to:

$$\begin{aligned}
g(x) &= 0 \\
f(x) - x &= 0 && (\text{def. of } g) \\
f(x) &= x
\end{aligned}$$

And so there is some $x \in [0,1]$ such that $f(x) = x$.

Problem 4

Solution: Recall that a compact subset E of $\mathbb{R}$ is one that is both bounded, and closed. Since E is bounded, the supremum $\sup(E) = a$ must exist. This means there is a sequence $x_n \in E$ such that $x_n \to a$, since the supremum of a set lies either in it, or on its boundary. And since E is closed, that $a \in E$, meaning $a = \sup(E) \in E$.

Problem 5

Solution: A Cauchy sequence has a limit, and so $x_k \to x$. Next note that x is an accumulation point of E since each $x_k \in E$. Since E is closed, that accumulation point $x \in E$.