Linear Optimization
HW #7

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Chapter 7

Problem a: Use the two-phase simplex method to solve the following:

$$\begin{array}{ll} \text{Maximize} & x_1 + x_2 - x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 \leq 7 \\ & 2x_1 - x_2 - x_3 - 3x_4 \leq -1 \\ \text{and} & \mathbf{x} \geq 0 \end{array}$$

Solution: First let us transform the problem to standard form:

$$\begin{cases} \text{Maximize} & x_1 + x_2 - x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 \leq 7 \\ & 2x_1 - x_2 - x_3 - 3x_4 \leq -1 \\ \text{and} & \mathbf{x} \geq 0 \end{cases} \implies \underbrace{\begin{cases} \text{Maximize} & x_1 + x_2 - x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 + s_1 = 7 \\ & 2x_1 - x_2 - x_3 - 3x_4 + s_2 = -1 \\ \text{and} & \mathbf{x}, \mathbf{s} \geq 0 \end{cases}}_{\text{introduce slack variables } \mathbf{s}}$$

$$\implies \underbrace{\begin{cases} \text{Maximize} & \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix}^\top \mathbf{z} \\ \text{subject to} & \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & -3 & 0 & 1 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} \\ \text{and} & \mathbf{z} \geq 0 \end{cases}}_{\text{let } \mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{s} \end{smallmatrix}\right]}$$

Note though that the solution given by $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{b} \end{smallmatrix}\right]$ is not feasible. As such, we must proceed with the two-phase method and find another BFS. To do this, we add an artificial variable t to the constraints corresponding to the negative entries of $\mathbf{b}$, giving us an auxillary problem:

$$\underbrace{\begin{cases} \text{Maximize} & x_1 + x_2 - x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 + s_1 = 7 \\ & 2x_1 - x_2 - x_3 - 3x_4 + s_2 - t = -1 \\ \text{and} & \mathbf{x}, \mathbf{s}, t \geq 0 \end{cases}}_{\text{introduce artificial variable } t} \implies \underbrace{\begin{cases} \text{Maximize} & \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}^\top \mathbf{z} \\ \text{subject to} & \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & -1 & -1 & -3 & 0 & 1 & -1 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} \\ \text{and} & \mathbf{z} \geq 0 \end{cases}}_{\text{let } \mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{s} \\ t \end{smallmatrix}\right]}$$

A BFS to the auxillary problem is given by $\begin{bmatrix} 0 & 0 & 0 & 0 & 7 & 0 & 1 \end{bmatrix}^\top$. We can now use the tableau method to find another BFS to this problem that sets $t = 0$. To do that we want to maximize $-t$:

$$\begin{aligned} 2x_1 - x_2 - x_3 - 3x_4 + s_2 - t &= -1 \\ \implies -t &= -2x_1 + x_2 + x_3 + 3x_4 - s_2 - 1 \end{aligned}$$

Leaving us with this problem:

$$\begin{array}{ll} \text{Maximize} & \begin{bmatrix} -2 & 1 & 1 & 3 & 0 & -1 & 0 \end{bmatrix}^\top \mathbf{z} \\ \text{subject to} & \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & -1 & -1 & -3 & 0 & 1 & -1 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} \\ \text{and} & \mathbf{z} \geq 0 \end{array}$$

We can use the tableau method to solve this:

x_1	x_2	x_3	x_4	s_1	s_2	t	
1	2	1	0	1	0	0	7
-2	1	1	3	0	-1	1	1
2	-1	-1	-3	0	1	0	-1
1	1	-1	-1	0	0	0	0

$\xrightarrow[r_3/3]{r_3+r_2}$

x_1	x_2	x_3	x_4	s_1	s_2	t	
1	2	1	0	1	0	0	7
$-2/3$	$1/3$	$1/3$	1	0	$-1/3$	$1/3$	$1/3$
0	0	0	0	0	0	1	0
$-1/3$	$-4/3$	$2/3$	0	0	$1/3$	0	$-1/3$

At this point we have a positive BFS to the auxillary problem given by $\begin{bmatrix}0 & 0 & 0 & 1/3 & 7 & 0\end{bmatrix}^\top$ And since its last entry (corresponding to the artificial variable t) is 0, we also have a BFS for the original canonical problem: $\begin{bmatrix}0 & 0 & 0 & 1/3 & 7\end{bmatrix}^\top$

We can now finally perform phase 2 of the simplex method by adjusting our last tableau and solving:

x_1	x_2	x_3	x_4	s_1	s_2	
1	2	1	0	1	0	7
$-2/3$	$1/3$	$1/3$	1	0	$-1/3$	$1/3$
$-1/3$	$-4/3$	$2/3$	0	0	$1/3$	$-1/3$

$\xrightarrow[\substack{3r_2 \\ r_1-2r_2}]{r_3+4r_2}$

x_1	x_2	x_3	x_4	s_1	s_2	
5	0	-1	-6	1	2	5
-2	1	1	3	0	-1	1
-3	0	2	4	0	-1	1

$\xrightarrow[\substack{3r_2 \\ r_1-2r_2}]{r_3+4r_2}$

x_1	x_2	x_3	x_4	s_1
1	0	$-1/5$	$-6/5$	$1/5$
0	1	$3/5$	$3/5$	$2/5$
0	0	$7/5$	$2/5$	$3/5$

With the bottom row nonnegative, we have completed the simplex algorithm. The maximum is 4, and the $\mathbf{z}$ that achieves it is given by:

$$\mathbf{z} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Chapter 8

Problem 1: Write a constraint that prevents two movies from being on the same screen at the same time.

Solution: Consider the following:

$$\begin{aligned} x_{t,m,s} = 1 \implies \sum_{\tau=t}^{t+r_m-1} \sum_{\mu=1}^{M} x_{\tau,\mu,s} = 1 &\iff x_{t,m,s} - 2 < 0 \implies \sum_{\tau=t}^{t+r_m-1} \sum_{\mu=1}^{M} x_{\tau,\mu,s} \leq 1 \\ &\iff x_{t,m,s} - 2 < 0 \implies 1 - \sum_{\tau=t}^{t+r_m-1} \sum_{\mu=1}^{M} x_{\tau,\mu,s} \geq 0 \\ &\iff \begin{cases} z \in \{0,1\} \\ Nz \geq x_{t,m,s} - 2 \\ x_{t,m,s} - 2 + (1-z)N \geq 0 \\ 1 - \sum_{\tau=t}^{t+r_m-1} \sum_{\mu=1}^{M} x_{\tau,\mu,s} \geq -zN \end{cases} \end{aligned}$$

We let our bound $N = MT$ as neither variable can exceed it.

Problem 2: Write a constraint that prevents any movie from running after the theater closes.

Solution: Consider the following:

$$\begin{aligned} x_{t,m,s} = 1 \implies t + r_m \leq T &\iff x_{t,m,s} - 2 < 0 \implies t + r_m \leq T \\ &\iff x_{t,m,s} - 2 < 0 \implies T - t - r_m \geq 0 \\ &\iff \begin{cases} z \in \{0,1\} \\ Nz \geq x_{t,m,s} - 2 \\ x_{t,m,s} - 2 + (1-z)N \geq 0 \\ T - t - r_m \geq -zN \end{cases} \end{aligned}$$

We let our bound $N = 1$ as neither variable can exceed it.

Problem 3: Write a constraint that in any 8 consecutive time blocks a movie must start somewhere in the theater.

Solution: The constraint is given by:

$$
\begin{aligned}
t \le T - 8 \implies \sum_{\tau=t}^{t+8} x_{t,m,s} \ge 1 &\iff t - T - 8 \le 0 \implies \sum_{\tau=t}^{t+8} x_{t,m,s} \ge 1 \\
&\iff t - T - 9 < 0 \implies \sum_{\tau=t}^{t+8} x_{t,m,s} - 1 \ge 0 \\
&\iff \begin{cases} z \in \{0,1\} \\ Nz \ge t - T - 9 \\ t - T - 9 + (1-z)N \ge 0 \\ \sum_{\tau=t}^{t+8} x_{t,m,s} - 1 \ge -zN \end{cases}
\end{aligned}
$$

We let our bound $N = 7$ as both variables cannot exceed it.

Problem 4: The constraint from problem 3 turns out to cause enormous trouble; prove if you do not have constraints such as this that a feasible movie schedule exists. Must a feasible schedule exist with this constraint?

Solution: Consider the following counterexample to the constraint from problem 3. If the movie theater has only one screen and if all the running times of the movies r_m exceed 8 blocks, then clearly the screen will not be able to play another movie in time after the first. In other words, a feasible schedule does *not* necessarily exist with the constraint from problem 3.

If we remove this constraint, then a feasible solution *does* exist, namely playing no movies. Indeed, playing no movies vacuously satisfies the "no movies playing simultaneously" condition as well as the "no movies playing after closing" condition from problems 1 & 2.