Linear Optimization
HW #12

Ozaner Hansha

April 27, 2021

Problem 11.5.2

Problem: Explicitly give an example of a Traveling Salesman Problem (other than ones in this chapter!) where there is no solution.

Solution: Consider a TSP with three cities $\mathcal{C} = \{1, 2, 3\}$ and where no city is accessible from any other:

$$S_1 = S_2 = S_2 = \emptyset$$

Clearly then, the TSP has no solution as given any starting point j, it is impossible to reach any other city, much less all of them, in a finite amount of time.

Problem 11.5.4

Problem: Assume that every city is accessible from every other city. Show there is a valid route that satisfies the constraints.

Solution: Given a TSP with n cities where for each j we have that:

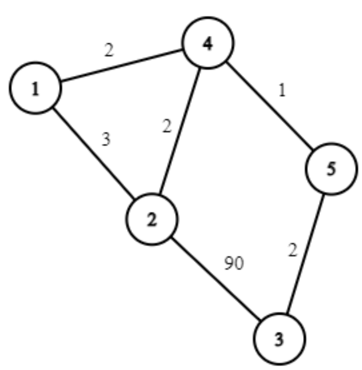
$$S_j = [1..n]$$

There is always a solution. Namely, starting at city j, at time t the following step is made: travel to city $j + t \mod n$. After $n - 1$ time steps, every city will have been visited.

Problem 11.5.6

Problem: Choose a collection of cities, a starting city, and distances between them so that the greedy algorithm does not give a route of minimal distance.

Solution: Consider a TSP with 5 cities and the given connections:



Starting at city 1, the greedy algorithm would give the following path:

$$\underbrace{1 \to 4 \to 5 \to 3 \to 2}_{2+1+2+90=95}$$

We can show this is not a route of minimal distance by demonstrating a shorter route:

$$\underbrace{1 \to 2 \to 4 \to 5 \to 3}_{3+2+1+2=8}$$

Problem 11.5.8

Problem: Without doing out the algebra, find upper and lower bounds for the sum approximating the Insertion Algorithm's run-time:

$$\sum_{k=1}^{n} k(n-(k-1))$$

How close can you get to the true value?

Solution: Note the following:

$$\sum_{k=1}^{n} k(n-(k-1)) = \sum_{k=1}^{n} kn - \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

And so an upper bound on this run time is given by:

$$\begin{aligned}\sum_{k=1}^{n} kn + \sum_{k=1}^{n} k &= (n+1)\sum_{k=1}^{n} k \\ &= (n+1)\frac{n(n+1)}{2} \\ &= \frac{n(n+1)^2}{2}\end{aligned}$$

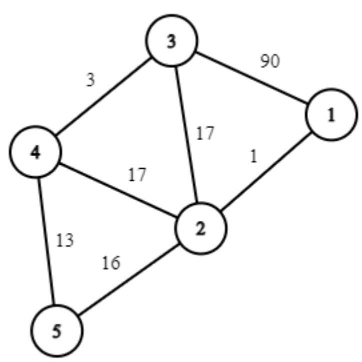
A lower bound is of course 0. Note sure if you wanted something more clever, while not doing algebra...

There is a closed form for the true value since we can express the sum of squares from 1 to n as a closed expression just as we did the sum of integers. So we can get arbitrarily close to this approximation formula.

Problem 11.5.11

Problem: Choose a collection of cities, a starting city, and distances between them so that the insertion algorithm does not give a route of minimal distance.

Solution: Consider a TSP with 5 cities and the given connections:



Starting at city 1, the insertion path algorithm (that is, adding new cities in the most locally optimal way, in order) gives:

$$\underbrace{1 \to 2 \to 3 \to 4 \to 5}_{1+17+3+13=34}$$

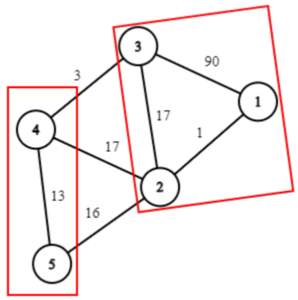
We can show that this s not optimal by demonstrating a shorter path:

$$\underbrace{1 \to 2 \to 5 \to 4 \to 3}_{1+16+13+3=33}$$

Problem 11.5.13

Problem: Choose a collection of cities, a starting city, and distances between them so that the sub-problem method does not give a route of minimal distance.

Solution: For an example with group size greater than 1, consider the following:



Here the optimal path for the subgroups are:

$$\underbrace{1 \to 2 \to 3}_{1+17=18} \qquad \underbrace{4 \to 5}_{13}$$

And, assuming our strategy is to connect them with the smallest edge possible (i.e. connect them using $(3,4)$), the total path length is:

$$\underbrace{\underbrace{1 \to 2 \to 3}_{1+17=18} \underbrace{\to}_{3} \underbrace{4 \to 5}_{13}}_{18+3+13=34}$$

Notice that this is the same example used in problem 11, which we have already shown to have a shorter path given by:

$$\underbrace{1 \to 2 \to 5 \to 4 \to 3}_{1+16+13+3=33}$$

And so, the sub-divide method need not provide an optimal route.

Problem 13.7.1

Problem: Find all fixed points for the function $f : [0,1] \to [0,1]$ given by:

$$f(x) = x(1-x)$$

Solution: This is the same as finding the solutions to:

$$\begin{aligned} f(x) &= x \\ x(1-x) &= x \\ 1-x &= 1 && (x = 0 \text{ is a solution}) \\ 0 &= x \end{aligned}$$

And so the only fixed point of $f(x)$ is at $x = 0$.

Problem 13.7.2

Problem: Find all fixed points for the function $f : [0,1]^2 \to [0,1]^2$ given by:

$$f(x, y) = \left(xy, \frac{x^2 + y^2}{2} \right)$$

Solution: First we will find the conditions for which the x component is equal to x:

$$\begin{aligned} xy &= x \\ y &= 1 && (x = 0 \text{ is a solution}) \end{aligned}$$

Now, keeping $x = 0$ we will find when the y component is y:

$$\begin{aligned} f(0, y) &= y \\ \frac{y^2}{2} &= y \\ y^2 &= 2y \\ y &= 2 && (y = 0 \text{ is a solution}) \end{aligned}$$

And so our first set of fixed points are: $(0, 2), (0, 0)$. Next we will do the same with $y = 1$:

$$\begin{aligned} f(x, 1) &= 1 \\ \frac{x^2 + 1}{2} &= 1 \\ x^2 + 1 &= 2 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

And so our second set of fixed points are: $(1, 1), (-1, 1)$. In total, this function has 4 real fixed points given by:

$$(0, 2), (0, 0), (1, 1), (-1, 1)$$