Intro to Real Analysis
HW #7

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Problem 1

Solution: Consider an arbitrary $x_0 \in [a,b]$. Now consider an arbitrary $\epsilon > 0$. Since $h(x) \in \{f(x), g(x)\}$, we have 2 cases:

1. $h(x_0) = f(x_0)$:

$$\begin{aligned} \exists \delta_f, \quad 0 < |x - x_0| < \delta_f &\implies |f(x) - f(x_0)| < \epsilon && (f(x) \text{ is continuous}) \\ &\implies |h(x) - h(x_0)| < \epsilon && (h(x_0) = f(x_0)) \end{aligned}$$

2. $h(x_0) = g(x_0)$:

$$\begin{aligned} \exists \delta_f, \quad 0 < |x - x_0| < \delta_f &\implies |g(x) - g(x_0)| < \epsilon && (g(x) \text{ is continuous}) \\ &\implies |h(x) - h(x_0)| < \epsilon && (h(x_0) = g(x_0)) \end{aligned}$$

In both cases, $h(x)$ has been shown to be continuous at $x = x_0$. You'll notice that in the case that $h(x_0) = f(x_0) = g(x_0)$, both cases apply and we can simply pick one.

Since we have shown that $h(x)$ is continuous on any $x_0 \in [a,b]$, we have shown that it is continuous over that whole interval.

Problem 2

Solution: First let us prove that $\sqrt[3]{4}$ is irrational by contradiction. Let us assume that it is indeed rational, and thus there are coprime integers $a, b \in \mathbb{Z}^+$ such that $\frac{a}{b} = \sqrt[3]{4}$. This implies:

$$\begin{aligned} \frac{a}{b} = 4^{1/3} &\implies a = 4^{1/3} b \\ &\implies a^3 = 4b^3 \\ &\implies 4|a^3 && (a \text{ is an integer}) \\ &\implies 2|a^3 && (2|4) \\ &\implies 2|a && (2 \text{ is prime}) \\ &\implies a = 2k && (a \text{ is even}) \end{aligned}$$

Plugging this in to our original expression, this implies that:

$$\begin{aligned} 4b^3 = 8k^3 &\implies b^3 = 4k^3 && (a = 2k) \\ &\implies 4|b^3 \\ &\implies 2|b^3 && (2|4) \\ &\implies 2|b && (2 \text{ is prime}) \\ &\implies b = 2k && (b \text{ is even}) \end{aligned}$$

Yet this is a contradiction as now a and b are clearly not coprime, since they are both even.

Problem 3

Solution: Yes. Let us first establish the following lemma:

$$
\begin{aligned}
e^x &\geq x+1 \\
e^{x-1} &\geq x \\
x-1 &\geq \ln x && \text{(lemma 1)}
\end{aligned}
$$

Now on to the proof. Define $f(0) = 0$ Now consider an arbitrary $\epsilon > 0$, and $\delta = \epsilon + 1$. We then have for $x_0 = 0$:

$$
\begin{aligned}
|f(x)| &= |x \ln x| \\
&= x \ln x && (x > 0) \\
&\leq x(x-1) && \text{(lemma 1)} \\
&< x - 1 \\
&< \delta - 1 && (\text{assume } 0 < |x| < \delta) \\
&< \epsilon && (\delta = \epsilon + 1)
\end{aligned}
$$

And since we have only considered $x > 0$ as $x \ln x$ isn't defined elsewhere, we have:

$$\lim_{x \to 0^+} f(0) = \lim_{x \to 0^+} x \ln x = 0$$

And we are done.

Problem 4

Solution: First let us show that a continuous function $f : [a, b] \to \mathbb{R}$ must also be uniformly continuous. We do this by contradiction. Note that if G was not uniformly continuous, then there exists an $\epsilon > 0$ such that for each $\delta > 0$, there exists x, c such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon$. By the archimedean property this implies that there is some $n \in \mathbb{N}$ such that this holds for some $\delta = \frac{1}{n}$.

This gives us two sequences:

$$
\begin{aligned}
(x_n) &\subseteq [a, b] \\
(c_n) &\subseteq [a, b]
\end{aligned}
$$

such that $|x_n - c_n| < \frac{1}{n}$ and $|f(x_n) - f(c_n)| \geq \epsilon$. But the Bolzano-Weierstrass theorem tells us that (c_n) must itself have a convergent subsequence (c_{n_k}). Say this subsequence converges to c. Since $|x_{n_k} - c_{n_k}| < \frac{1}{n_k}$, we must have that the subsequence (x_{n_k}) converges to c as well. But since f is continuous we have:

$$
\begin{aligned}
\lim(f(x_{n_k}) - f(c)) = 0 \\
\lim(f(c_{n_k}) - f(c)) = 0
\end{aligned}
$$

This tell us:

$$
\begin{aligned}
|f(c_{n_k}) - f(x_{n_k})| &= |f(c_{n_k}) - f(c) + f(c) - f(x_{n_k})| \\
&\leq |f(c_{n_k}) - f(c)| + |f(c) - f(x_{n_k})| && \text{(triangle inequality)}
\end{aligned}
$$

And so we have $f(c_{n_k}) - f(x_{n_k}) = 0$. But this contradicts our assumption that $|f(c_n) - f(x_n)| \geq \epsilon$.

With this established, consider a restriction of our function $f_0 : [0, h] \to \mathbb{R}$. Since f is continuous everywhere, it must be that f_0 is continuous over $[0, h]$ and thus is bounded over said interval. This means that for any $\epsilon > 0$ there is some $\delta > 0$ that satisfies the continuity definition of any $x, x_0 \in [0, h]$.

Note that the same holds for $f_1 : [h, 2h] \to \mathbb{R}$ and in general $f_k : [kh, (k+1)h] \to \mathbb{R}$. Not only that but since these restrictions are identical but just shifted (since f is periodic), for any $\epsilon > 0$ the same $\delta > 0$ can be used for each of these restrictions. And since we have:

$$f(x) = \begin{cases} f_0(x), & x \in [0, h] \\ & \vdots \\ f_k(x), & x \in [kh, (k+1)h] \\ & \vdots \end{cases}$$

Then we have that our choice of δ given ϵ is independent of x, x_0 over the whole domain of $\mathbb{R}$ for the complete function f. In other words, f is uniformly continuous.

Problem 5

Solution: Because f, g are uniformly continuous we have:

$$\forall \epsilon > 0, \exists \delta_1, \forall x, x_0 \in (a, b), \quad |x - x_0| < \delta_1 \implies |f(x) - f(x_0)| < \epsilon < \frac{\epsilon}{2} \tag{1}$$

$$\forall \epsilon > 0, \exists \delta_2, \forall x, x_0 \in (a, b), \quad |x - x_0| < \delta_1 \implies |g(x) - g(x_0)| < \epsilon < \frac{\epsilon}{2} \tag{2}$$

$$\tag{3}$$

And also note that by the triangle inequality:

$$|(f+g)(x) - (f+g)(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)|$$

And so letting $\delta_0 = \min\{\delta_1, \delta_2\}$ we have:

$$\forall \epsilon > 0, \exists \delta_0, \forall x, x_0 \in (a, b), \quad |x - x_0|, \delta_0 \implies |(f+g)(x) - (f+g)(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

And so uniform continuity is preserved by addition.

For multiplication, this is not true for an open interval (a, b).