Linear Optimization
HW #5

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March 1, 2021

Problem 1

Problem: The definition of linear independence assumes the vectors are non-zero. Why is this a good condition to include?

Solution: Recall that vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent iff:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0} \implies a_1 = a_2 = \cdots = a_n = 0$$

If we had $\mathbf{v}_i = \mathbf{0}$ for some i, then we could always multiply $\mathbf{v}_i$ by some non-zero constant a_i and add it to a linear combination of vectors without changing its result. As a result, a_i is never necessarily 0 in the LHS above and so the implication cannot hold.

And since the inclusion of a zero vector automatically makes the set linearly dependent, it is reasonable to just exclude them outright from the definition.

Problem 5

Problem: Let $\mathbf{v}_1 = (1, 2, 3)$ and $\mathbf{v}_2 = (2, 1, 0)$. Describe the set of all vectors $\mathbf{v}_3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Solution: The set of vectors that are linearly *dependent* to $\{\mathbf{v}_1, \mathbf{v}_2\}$ are all the linear combinations of this set, i.e. $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

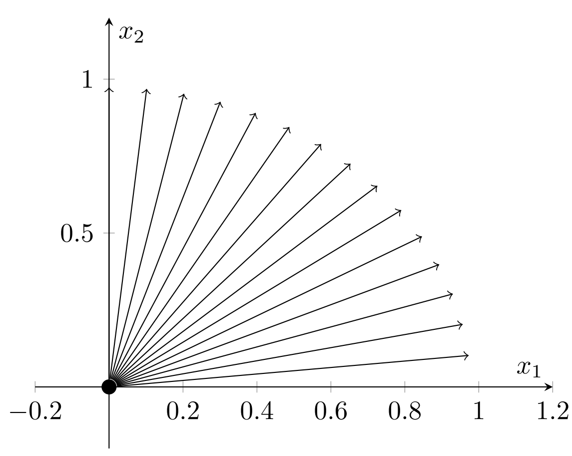
The set of vectors that are linearly *in*dependent of $\mathbf{v}_1$ and $\mathbf{v}_2$ is simply the set of vectors that are *not* dependent on them, which is to say the complement of their span. In other words, the set S of vectors $\mathbf{v}_3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent is:

$$S = \mathbb{R}^3 \setminus \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

Problem 6

Problem: Given any positive integer $n \geq 2$, construct a $2 \times n$ matrix such that any two columns are linearly independent.

Solution: Recall that two vectors are linearly independent iff they are not multiples of each other. Now note the following image of 2D vectors:



As we can see, each vector above is linearly independent of any other since none is a multiple of any other. In this example we picked 15 equally spaced angles within the interval $(0, \pi/2]$. Generalizing, for n vectors that are independent of any other, we simply have to pick n equally spaced angles from this interval:

$$\begin{bmatrix} \cos\frac{\pi}{2n} & \cos\frac{2\pi}{2n} & \cdots & \cos\frac{i\pi}{2n} & \cdots & \cos\frac{n\pi}{2n} \\ \sin\frac{\pi}{2n} & \sin\frac{2\pi}{2n} & \cdots & \sin\frac{i\pi}{2n} & \cdots & \sin\frac{n\pi}{2n} \end{bmatrix}$$

From the above reasoning it follows that this matrix is one whose columns are pairwise linearly independent.

Problem 29

Problem: Find all basic feasible solutions to:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: First let us convert this problem into an equivalent one with fewer rows:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{array}\right] &\xrightarrow{r_2 - 2r_1} \left[\begin{array}{ccc|r} 1 & 2 & 3 & 1 \\ 2 & 1 & 0 & -1 \\ 7 & 8 & 9 & 1 \end{array}\right] \\ &\xrightarrow{r_3 - 3r_1} \left[\begin{array}{ccc|r} 1 & 2 & 3 & 1 \\ 2 & 1 & 0 & -1 \\ 4 & 2 & 0 & -2 \end{array}\right] \\ &\xrightarrow{r_3 - 2r_2} \left[\begin{array}{ccc|r} 1 & 2 & 3 & 1 \\ 2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{\qquad} \left[\begin{array}{ccc|r} 1 & 2 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{array}\right] \end{aligned}$$

And so we have the equivalent problem of finding the basic feasible solutions to:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Any solution must lie on the intersection of the two hyperplanes defined by the equation (i.e. satisfy the equation). Since these planes are not parallel, that intersection is a line. We have one free variable t so we'll let $x_3 = t$:

$$\begin{aligned} \begin{cases} x_1 + 2x_2 + 3t = 1 \\ 2x_1 + x_2 = -1 \end{cases} &\implies \begin{cases} x_1 = 1 - 2x_2 - 3t \\ x_2 = -1 - 2x_1 \end{cases} \\ &\implies \begin{cases} x_1 = 1 - 2(-1 - 2x_1) - 3t \\ x_2 = -1 - 2(1 - 2x_2 - 3t) \end{cases} \\ &\implies \begin{cases} x_1 = 3 + 4x_1 - 3t \\ x_2 = -3 + 4x_2 + 6t \end{cases} \\ &\implies \begin{cases} x_1 = -1 + t \\ x_2 = 1 - 2t \end{cases} \end{aligned}$$

And so our parmaterized line is given by:

$$(x_1, x_2, x_3) = (-1 + t, 1 - 2t, t)$$

And so all solutions, basic feasible or otherwise, must lie on this line. However note that no point $\mathbf{x}$ on that line satisfies $\mathbf{x} \geq 0$:

$$\begin{aligned} x_3 \geq 0 &\implies t \geq 0 && \text{(def. of } x_3\text{)} \\ &\implies -1 + t \leq -1 \\ &\implies -1 + t < 0 && (-1 < 0) \\ &\implies x_1 < 0 && \text{(def. of } x_1\text{)} \end{aligned}$$

And so, since this problem has no feasible solutions, it certainly has no *basic* feasible solutions.

Problem 33

Part a: Prove there are only finitely many basic optimal solutions.

Solution: Consider the canonical LP problem:

$$\begin{array}{ll} \text{Minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ \text{and} & \mathbf{x} \geq 0 \end{array}$$

First, assume A has size $M \times N$ with $M \leq N$. We do this w.l.o.g. because if $M > N$ then the LP would be either overdetermined (i.e. 0 solutions, which is finite), and if $M \leq N$ we can just reduce its rows to its true rank and arrive at an identical problem.

Now note that since each basic feasible solution $\mathbf{x}$ needs to be linearly independent of k columns of A where k is the number of non-zero entries in $\mathbf{x}$. Now note that the choices of these columns is finite:

$$\sum_{m=0}^{M} \binom{N}{m} \leq \sum_{m=0}^{N} \binom{N}{m} = 2^N$$

Now consider a BFS $\mathbf{x}$ with a fixed choice of independent columns. Consider the truncated matrix $A' = A_{j1}, A_{j3} \cdots, A_{jk}$ where $j1, \cdots, jk$ are the indices of the non-zero entries $\mathbf{x}$. Since A' is full rank (only has linearly independent columns) the following must hold:

$$A'\mathbf{x} = \mathbf{b} \implies \mathbf{x} = (A'^\top A')^{-1} A'^\top \mathbf{b}$$

And so for any of the finite choices of independent columns, we have a single $\mathbf{x}$ solution that satisfies the equation. For any particular choice of columns, it may be the case that $\mathbf{x} \not\geq 0$. This is not a problem however as the number of solutions is in this case is 0 rather than 1, which is still finite. And so the set of basic feasible solutions is finite.

Finally, since the set of basic optimal solutions is a subset of the set of basic feasible solutions, it too must be finite.

Part b: Prove that if there are at least two optimal solutions, then there are infinitely many optimal solutions (thus the number of optimal solutions is either 0, 1, or infinity).

Solution: Consider two distinct optimal solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ to the canonical linear optimization problem:

$$\begin{array}{ll} \text{Minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ \text{and} & \mathbf{x} \geq 0 \end{array}$$

Since they are both optimal, in this case minimal, we have:

$$\begin{aligned} \mathbf{c}^\top \mathbf{v}_1 &= M \\ \mathbf{c}^\top \mathbf{v}_2 &= M \end{aligned}$$

Where M is the minimal value they attain. Now note the following for any a, b such that $a + b = 1$:

$$\begin{aligned}
\mathbf{c}^\top(a\mathbf{v}_1 + b\mathbf{v}_2) &= a\mathbf{c}^\top\mathbf{v}_1 + b\mathbf{c}^\top\mathbf{v}_2 && \text{(objective)} \\
&= aM + bM && \text{(def. of } M\text{)} \\
&= M && (a + b = 1\text{, i.e. convex combination})
\end{aligned}$$

And so we have shown that, for any a, b such that $a + b = 1$, the linear combination of our optimal solutions given by $a\mathbf{v}_1 + b\mathbf{v}_2$ also attains the minimal value. Since there are infinitely many such pairs (a, b) we correspondingly have infinitely many minimal solutions. Of course, this applies to maximization problems just as well.