Intro to Real Analysis
Midterm

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Problem 1

Problem: Consider the sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ given by:

$$a_n = \sum_{k=1}^{n} 4k$$
$$b_n = \sum_{k=1}^{n} 2k+1$$

Show that $\left(\frac{a_n}{b_n}\right)_{n=1}^\infty$ is convergent and give its limit.

Solution: Note that, if the desired limit existed, it would be an infinite series:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sum_{k=1}^n 4k}{\sum_{k=1}^n 2k+1} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{4k}{2k+1} \\
&= \sum_{k=1}^{\infty} \frac{4k}{2k+1} && \text{(def. of infinite series)}
\end{aligned}$$

However, you'll note that this series is divergent. To see this, recall that a necessary, but not sufficient, condition for a series to be convergent is for the limit of its summand to be 0. Yet this is not the case:

$$\begin{aligned}
\lim_{k\to\infty} \frac{4k}{2k+1} &= \lim_{k\to\infty} \frac{4}{2+{}^1/_k} \\
&= \lim_{k\to\infty} \frac{4}{2+0} \\
&= 2 \neq 0
\end{aligned}$$

And so, we cannot give the limit of this series as it does not exist.

Problem 2

Problem: Find a bijection from I to the following set to $[0,1]$, where I is given by:

$$I = (1,2) \cup (2,3) \cup \cdots \cup (2020, 2021)$$

Solution: We will first construct a bijective function $g_{a,b,c,d} : [a,b) \to (c,d)$ for any $a < b \land c < d$.

Consider the following sequence $(y_n)_{n=1}^\infty$ given by:

$$y_n = a + .1^n(b-a)$$

This is an injective infinite sequence of numbers in $[a,b)$. Now consider the following bijection $h_{a,b} : (a,b) \to [a,b)$:

$$h_{a,b}(x) = \begin{cases} a, & x = y_1 \\ y_{n-1}, & x = y_n,\ n > 1 \\ x, & \text{otherwise} \end{cases}$$

We can now transform this bijection to have our desired codomain, resulting in $g_{a,b,c,d} : (a,b) \to [c,d)$:

$$g_{a,b,c,d}(x) = \frac{d}{b-a+c}(h_{a,b}(x) - a + c) \qquad \text{(call this lemma 1)}$$

Now let us note one last fact:

$$[0,1] = \left[0, \frac{1}{2021}\right) \cup \left[\frac{1}{2021}, \frac{2}{2020}\right) \cup \cdots \cup \left[\frac{2020}{2021}, 1\right) \cup \{1\} \qquad \text{(call this lemma 2)}$$

And so by lemma 2, a bijection $f_1 : I \to [0,1)$ can be given as the following piecewise function:

$$f_1(x) = \begin{cases} g_{1,2,0,\frac{1}{2021}}(x), & x \in (1,2) \\ g_{2,3,\frac{1}{2021},\frac{2}{2021}}(x), & x \in (2,3) \\ \vdots & \\ g_{i,i+1\frac{i-1}{2021},\frac{i}{2021}}(x), & x \in (i,i+1) \\ \vdots & \\ g_{2020,2021,\frac{2020}{2021},1}(x), & x \in (2020,2021) \end{cases}$$

And of course, that each case of the partition of $I \setminus \{1\}$ is bijective is given by lemma 1. Now all that's left is to deal with the leftover 1 not yet in the codomain. Consider another sequence: $(z_n)_{n=1}^{\infty}$ given by:

$$z_n = 1 - .2^n$$

With this we can finally give our desired bijection $f_2 : I \to [0,1]$:

$$f_2(x) = \begin{cases} 1, & f_1(x) = z_1 \\ z_{n-1}, & f_1(x) = z_n,\ n > 1 \\ f_1(x), & \text{otherwise} \end{cases}$$

Problem 3

Problem: Compute the following limit:

$$\lim_{n\to\infty} \left(1 + \frac{1}{3n+1}\right)^{2n}$$

Solution: Consider the following:

$$
\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{3n+1}\right)^{2n} &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{2m-2}{3}} && \left(\substack{m=3n+1\\ n=\frac{2m-2}{3}}\right)\\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{2m}{3}}\left(1+\frac{1}{m}\right)^{-\frac{2}{3}}\\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{2m}{3}}\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{-\frac{2}{3}} && \text{(limit of products is product of limits)}\\
&= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right)^{\frac{2}{3}}\left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)\right)^{-\frac{2}{3}} && \left(\substack{a_m\geq 0\implies\\ \text{limit of root is root of limit}}\right)\\
&= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right)^{\frac{2}{3}}\left(1+\lim_{m\to\infty}\frac{1}{m}\right)^{-\frac{2}{3}} && \text{(limit of sum is sum of limit)}\\
&= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right)^{\frac{2}{3}}(1+0)^{-\frac{2}{3}}\\
&= e^{\frac{2}{3}} && \text{(def. of } e\text{)}
\end{aligned}
$$

Problem 4

Consider the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ given by:

$$
\begin{aligned}
a_n &= \left(1+\frac{1}{n}\right)^n\\
b_n &= 1+\sum_{k=1}^{n}\frac{1}{k!} = \sum_{k=0}^{n}\frac{1}{k!} && (0! = 1)
\end{aligned}
$$

Part a: Show that $a_n \leq b_n$ for all $n \geq 1$.

Solution: We wish to prove $(\forall n \in N)\, P(n)$ where:

$$P(n) \equiv a_n \leq b_n$$

First we will show $P(1)$:

$$
\begin{aligned}
P(1) &\iff \left(1+\frac{1}{1}\right)^1 \leq 1+\sum_{k=1}^{1}\frac{1}{k!} && \text{(def. of } P(1)\text{)}\\
&\iff (1+1)^1 \leq 1+\frac{1}{1!}\\
&\iff 2 \leq 2\\
&\iff T
\end{aligned}
$$

Now we will show $P(n) \implies P(n+1)$

$$\begin{aligned}
(n+1)^{n+1} &\geq (n+1)! \\
\frac{1}{(n+1)^{n+1}} &\leq \frac{1}{(n+1)!} \\
\frac{1}{(n+1)^{n+1}} + a_n &\leq \frac{1}{(n+1)!} + b_n && \text{(assume } P(n)) \\
\frac{1}{(n+1)^{n+1}} + \left(1+\frac{1}{n}\right)^n &\leq \frac{1}{(n+1)!} + \sum_{k=0}^{n} \frac{1}{k!} && \text{(def. of } a_n \text{ \& } b_n) \\
\frac{1}{(n+1)^{n+1}} + \left(1+\frac{1}{n+1}\right)^n &\leq \frac{1}{(n+1)!} + \sum_{k=0}^{n} \frac{1}{k!} && \left(\tfrac{1}{n+1} < \tfrac{1}{n}\right) \\
\frac{1}{(n+1)^{n+1}} + \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(n+1)^k} &\leq \frac{1}{(n+1)!} + \sum_{k=0}^{n} \frac{1}{k!} && \text{(binomial theorem)} \\
\sum_{k=0}^{n+1} \binom{n}{k} \frac{1}{(n+1)^k} &\leq \sum_{k=0}^{n+1} \frac{1}{k!} \\
\left(1+\frac{1}{n+1}\right)^{n+1} &\leq \sum_{k=0}^{n} \frac{1}{k!} && \text{(binomial theorem)} \\
a_{n+1} &\leq b_{n+1} && \text{(def. of } a_{n+1} \text{ \& } b_{n+1})
\end{aligned}$$

And so we have shown that both $P(1)$ and $P(n) \implies P(n+1)$. Thus, by the PMI, we have that:

$$(\forall n \geq 1)\ \underbrace{a_n \leq b_n}_{P(n)}$$

Part b: Find the limit of $(b_n)_{n=1}^{\infty}$.

Solution: First let us establish the following identity:

$$\begin{aligned}
\sum_{k=0}^{\infty} \binom{n}{k} \frac{1}{n^k} &= \binom{n}{0}\frac{1}{n^0} + \binom{n}{1}\frac{1}{n^1} + \binom{n}{2}\frac{1}{n^2} + \binom{n}{3}\frac{1}{n^3} \cdots \\
&= 1 + 1 + \frac{1}{2!}\left(\frac{n-1}{n}\right) + \frac{1}{3!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) + \cdots \\
&= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{j=1}^{k-1} \left(1-\frac{j}{n}\right)
\end{aligned}$$

Now consider the following:

$$
\begin{aligned}
e &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n && \text{(def. of } e\text{)}\\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} && \text{(binomial theorem)}\\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\binom{n}{k}\frac{1}{n^k}\\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{1}{k!}\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right) && \text{(identity from above)}\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\prod_{j=1}^{k-1}\lim_{n\to\infty}\left(1-\frac{j}{n}\right) && \text{(limit of sum/product is sum/product of limit)}\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\prod_{j=1}^{k-1}(1-0)\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!} && \text{(def. of infinite series)}\\
&= \lim_{n\to\infty} b_n && \text{(def. of } b_n\text{)}
\end{aligned}
$$

And so we are done. We have shown that the desired limit is equivalent to e.