Intro to Real Analysis
Final

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Problem 1

Part a: Note that the definition of uniform continuity for a function $f(x)$ over $(0,\infty)$ is given by:

$$\forall\epsilon > 0,\, \exists\delta > 0,\, \forall x, y \in (0,\infty), \quad |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Negating this, we have:

$$\exists\epsilon > 0,\, \forall\delta > 0,\, \exists x, y \in (0,\infty), \quad |x-y| < \delta \land |f(x) - f(y)| \geq \epsilon$$

Which is the definition of $f(x)$ *not* being uniformly continuous over $(0,\infty)$.

Part b: Consider $\epsilon = 1$, and any $\delta > 0$. Note that, by the archemdian principle, there exists an $n \in Z^+$ such that:

$$\frac{1}{2n\pi} < \delta$$

Call this number x, and call the following y:

$$y = \frac{1}{(2n+1)\pi} < \frac{1}{2n\pi} < \delta$$

Also note that both x, y are clearly in $(0,1)$. Now note that since x and y are both positive and less than δ their absolute difference is also less than delta:

$$\begin{aligned} x, y < \delta &\implies x - y < \delta && \text{(x,y¿0)} \\ &\implies |x-y| < \delta && \text{(x,y¿0)} \end{aligned}$$

Yet we also have that:

$$\begin{aligned} |f(x) - f(y)| &= \left| f\left(\frac{1}{2n\pi}\right) - f\left(\frac{1}{(2n+1)\pi}\right) \right| && \text{(def. of } x, y\text{)} \\ &= |\cos(2n\pi) - \cos((2n+1)\pi)| && \text{(def. of } f(x)\text{)} \\ &= |1 - (-1)| \\ &= 2 \geq \epsilon = 1 \end{aligned}$$

And so we have shown that:

$$\exists\epsilon > 0,\, \forall\delta > 0,\, \exists x, y \in (0,\infty), \quad |x-y| < \delta \land |f(x) - f(y)| \geq \epsilon$$

In particular with $\epsilon = 1$.

Problem 2

Part a: Consider an arbitrary real numbers x, x_0, and an arbitrary $\epsilon > 0$. By the archmedian principle, we have that there exists an n such that:

$$\frac{1}{n} < \epsilon$$

Note that the set S of numbers for which $|f(x) - 0| < \epsilon$ is given by:

$$S = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \cdots, \frac{3}{4}, \cdots, \frac{1}{n}, \cdots, \frac{n-1}{n}\}$$

Crucially, this set is finite. And so we can choose the number the smallest (non-zero) distance from our x_0:

$$d = \arg\min_{s \in S\{x_0\}} |x_0 - s|$$

And so, setting $\delta = d$ we have that:

$$0 < |x - x_0| < \delta = d \implies |f(x) - 0| < \frac{1}{n} < \epsilon$$

This is the definition of the limit of $f(x)$ at an arbitrary point x_0. So, in other words, we have shown that:

$$\forall x_0 \in (0, 1), \quad \lim_{x \to x_0} f(x) = 0$$

Part b: As we showed in part a, the function has a limit for all its values $a \in (0, 1)$:

$$\lim_{x \to a} f(x) = 0$$

For rationals ${}^{p}/_{q} \in (0, 1)$ (where p and q are co-prime), this means that $f(x)$ is discontinuous:

$$\lim_{x \to {}^{p}/_{q}} f(x) = 0 \neq f({}^{p}/_{q}) = {}^{1}/_{q}$$

Since ${}^{p}/_{q} \in \mathbb{Q} \implies f({}^{p}/_{q}) = {}^{1}/_{q}$ by the definition of f. But for irrationals $r \in (0, 1)$, this means that $f(x)$ is continuous:

$$\lim_{x \to r} f(x) = 0 = f(r)$$

Since $r \in \mathbb{R} \setminus \mathbb{Q} \implies f(r) = 0$ by the definition of f.

Part c: Since $f(x)$ is not continuous over the rationals, it is not differentiable over them either. In the case of the irrationals, $f(x)$ is not differentiable. To see this fix an irrational $x \in (0, 1)$. Suppose $f'(x)$ exists. We should have that $f'(x) = 0$ because there is a sequence of irrationals a_n such that:

$$\frac{f(a_n + h) - f(a_n)}{h} \to 0$$

Since $f(a_n) = 0$ by def. of f.

Now note that for each prime q, we can pick a k_q to be a multiple of ${}^{1}/_{q}$ satisfying $|x - k_q| \leq {}^{1}/_{q}$. We would then have that:

$$\frac{|f(x) - f(k_q)|}{|x - k_q|} \geq 1$$

So $|f'(x)| \geq 1$. This is a contradiction and so our assumption that $f'(x)$ existed for irrationals x is false.

Problem 3

Part a: The limit is equivalent to:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(2h)-f(-2h)}{h} &= \lim_{h\to 0}\frac{f(0+2h)-f(0-2h)}{h} \\
&= \lim_{h\to 0}\frac{f(0+2h)-f(0)+f(0)-f(0-2h)}{h} \\
&= \lim_{h\to 0}\left(2\frac{f(0+2h)-f(0)}{2h}+2\frac{f(0-2h)-f(0)}{-2h}\right) \\
&= 2\lim_{h\to 0}\frac{f(0+2h)-f(0)}{2h}+2\lim_{h\to 0}\frac{f(0-2h)-f(0)}{-2h} && \text{(limit of sum is sum of limits)} \\
&= 2f'(0)+2f'(0) && \text{(def. of derivative, change of variables)} \\
&= 4f'(0) \\
&= 4c && (f'(0)=c)
\end{aligned}$$

Part b: Consider the following function $f:\mathbb{R}\to\mathbb{R}$:

$$f(x)=\begin{cases}50, & x=0\\ x, & \text{otherwise}\end{cases}$$

Clearly $f(x)$ is discontinuous at $x=0$ and thus non-differentiable at $x=0$ as well, satisfying our constraint. Now observe that, despite this, the limit still exists:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(2h)-f(-2h)}{h} &= \lim_{h\to 0}\frac{2h-(-2h)}{h} && (h\neq 0) \\
&= \lim_{h\to 0}\frac{4h}{h} \\
&= \lim_{h\to 0}4 \\
&= 4
\end{aligned}$$

Problem 4

Solution: First let us compute the following limit:

$$\begin{aligned}
\lim_{x\to\infty}xf(x) &= \lim_{x\to\infty}\frac{xe^x f(x)}{e^x} \\
&= \lim_{x\to\infty}\frac{xe^x f'(x)+(x+1)e^x f(x)}{e^x} && \text{(L'Hopitals Rule)} \\
&= \lim_{x\to\infty}(xf'(x)+(x+1)f(x)) \\
&= \lim_{x\to\infty}(xf'(x)+f(x)+xf(x)) \\
&= \lim_{x\to\infty}(xf'(x)+f(x))+\lim_{x\to\infty}xf(x) \\
&= 3+\lim_{x\to\infty}xf(x) \\
&= 3+3+\lim_{x\to\infty}xf(x) \\
&= 3+3+\cdots+\lim_{x\to\infty}xf(x)
\end{aligned}$$

Clearly, this limit does not exist, as assuming its existence produces a contradiction for any assumed finite limit L (i.e. $L=3+L$). In fact we have shown that its limit is infinite:

$$\lim_{x\to\infty}xf(x)=\infty$$

Now note the desired limit:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{xe^x f(x)}{xe^x} \\
&= \lim_{x\to\infty} \frac{xe^x f'(x) + (x+1)e^x f(x)}{(x+1)e^x} && \text{(L'Hopitals Rule)} \\
&= \lim_{x\to\infty} \frac{xf'(x) + (x+1)f(x)}{x+1} \\
&= \lim_{x\to\infty} \frac{xf'(x) + f(x) + xf(x)}{x+1} \\
&= \lim_{x\to\infty} (xf'(x) + f(x)) \lim_{x\to\infty} \frac{1}{x+1} + \lim_{x\to\infty} \frac{xf(x)}{x+1} \\
&= 3 \cdot 0 + \lim_{x\to\infty} \frac{xf(x)}{x+1} \\
&= \lim_{x\to\infty} \frac{f(x) + xf'(x)}{2} && \text{(L'Hopitals Rule)} \\
&= \frac{1}{2} \lim_{x\to\infty} f(x) + xf'(x) \\
&= \frac{3}{2}
\end{aligned}$$

Note that the L'Hopital's rule was the following:

$$\lim_{x\to\infty} \frac{g(x)}{h(x)} = \lim_{x\to\infty} \frac{g'(x)}{h'(x)}$$

Which only holds when both:

$$\begin{aligned}
\lim_{x\to\infty} g(x) = \infty \\
\lim_{x\to\infty} h(x) = \infty
\end{aligned}$$