Linear Optimization
HW #8

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Problem 6

Problem: Let X and Y represent two binary indicator random variables. Is there a binary indicator random variable Z that represents XY? How would you interpret Z?

Solution: Consider the quantity XY. We have the following:

$$\begin{gathered}0 \leq XY \leq 1 \\ -1 \leq XY - 2 \leq 0 \\ -1 \leq XY - 1 \leq 1\end{gathered}$$

Using the definition of a binary indicator variable, we can now define Z as:

$$\begin{gathered}Z \in \{0,1\} \\ \frac{XY-1}{1} + \frac{\delta}{2} \leq Z \\ Z \leq 1 + \frac{XY-1}{1}\end{gathered}$$

And so $Z = 1 \iff XY - 1 \geq 0 \iff XY \geq 1 \iff XY = 1$. This is essentially a conjunction between X and Y.

Problem 7

Problem: Let X and Y represent two binary indicator random variables. Is there a binary indicator random variable Z that represents $X+Y$? How would you interpret Z?

Solution: Consider the quantity $X+Y$. We have the following:

$$\begin{gathered}0 \leq X+Y \leq 2 \\ -1 \leq X+Y-1 \leq 1\end{gathered}$$

Using the definition of a binary indicator variable, we can now define Z as:

$$\begin{gathered}Z \in \{0,1\} \\ \frac{X+Y-1}{1} + \frac{\delta}{2} \leq Z \\ Z \leq 1 + \frac{X+Y-1}{1}\end{gathered}$$

And so $Z = 1 \iff X+Y-1 \geq 0 \iff X+Y \geq 1$. This I essentially a disjunction between X and Y.

Problem 11

Problem: We showed how to code the inclusive and exclusive OR; show how one can code AND. For example, we want $z_{A \wedge B}$ to be 1 if A and B are non-negative, and 0 otherwise.

Solution: First we make a binary indicator variable for both A and B:

$$\begin{aligned} X_A &\in \{0,1\} \\ \frac{A}{N_A} + \frac{\delta}{2N_A} &\leq X_A \\ X_A &\leq 1 + \frac{A}{N_A} \end{aligned}$$

$$\begin{aligned} X_B &\in \{0,1\} \\ \frac{B}{N_B} + \frac{\delta}{2N_B} &\leq X_B \\ X_B &\leq 1 + \frac{B}{N_B} \end{aligned}$$

Now we let $Z = X_A X_B$ as defined in problem 6 and we are done. Z is only true if both X_A and X_B are true, and they are only true if A and B are greater than 0 respectively.

Problem 18

Problem: Frequently in problems we desire two distinct tuples, say points $(a_1, \cdots, a_k) \neq (\alpha_1, \cdots, \alpha_k)$. Find a way to incorporate such a condition within the confines of integer linear programming.

Solution: For each k we make a variable X_k such that $X_k = 1$ iff $a_k \neq \alpha_k$.

$$\begin{aligned} X_k &\in \{0,1\} \\ \frac{a_k - \alpha_k}{N_a} + \frac{\delta}{2N_a} &\geq X_k \\ X_k &\geq 1 + \frac{a_k - \alpha_k}{N_a} \end{aligned}$$

Now we just successively apply the AND operator to each of these variable:

$$\begin{aligned} Z_1 &= X_1 X_2 \\ Z_2 &= X_3 Z_1 \\ Z_3 &= X_4 Z_2 \\ &\vdots \\ Z_k &= X_k Z_{k-1} \end{aligned}$$

And so with the conditions that $Z_k = 1$ we are done.