Linear Optimization
HW #4

Ozaner Hansha

February 22, 2021

Problem a

Problem 15: Find the dual problem of the following:

$$\begin{aligned} &\text{Minimize} && 20x_1 + 2x_2 \\ &\text{subject to} && 30x_1 + 5x_2 \geq 60 \\ & && 15x_1 + 10x_2 \geq 70 \\ &\text{and} && \mathbf{x} \geq 0 \end{aligned}$$

Solution: Putting this problem into matrix form we find:

$$\begin{aligned} &\text{Minimize} && \underbrace{\begin{bmatrix} 20 \\ 2 \end{bmatrix}}_{\mathbf{c}} \cdot \mathbf{x} \\ &\text{subject to} && \underbrace{\begin{bmatrix} 30 & 5 \\ 15 & 10 \end{bmatrix}}_{A} \mathbf{x} \geq \underbrace{\begin{bmatrix} 60 \\ 70 \end{bmatrix}}_{\mathbf{b}} \\ &\text{and} && \mathbf{x} \geq 0 \end{aligned}$$

The dual problem is a maximization problem rather than minimization problem, and the constants $\mathbf{c}_D, A_D, \mathbf{b}_D$ that define it are given by:

$$\begin{aligned} \mathbf{c}_D &= \mathbf{b} \\ \mathbf{b}_D &= \mathbf{c} \\ A_D &= A^\top \end{aligned}$$

And so the dual problem is given by:

$$\begin{aligned} &\text{Maximize} && \underbrace{\begin{bmatrix} 60 \\ 70 \end{bmatrix}}_{\mathbf{c}_D = \mathbf{b}} \cdot \mathbf{x} \\ &\text{subject to} && \underbrace{\begin{bmatrix} 30 & 15 \\ 5 & 10 \end{bmatrix}}_{A_D = A^\top} \mathbf{x} \leq \underbrace{\begin{bmatrix} 20 \\ 2 \end{bmatrix}}_{\mathbf{b}_D = \mathbf{c}} \\ &\text{and} && \mathbf{x} \geq 0 \end{aligned}$$

Problem 16: Find the dual problem of problem 15.

Solution: You'll note that the process for finding the dual problem we used above is involutory. This means that that the dual of the dual of the primal problem is just the primal problem. To see this note the following:

$$\begin{aligned} \mathbf{c}_{D^2} &= \mathbf{b}_D = \mathbf{c} \\ \mathbf{b}_{D^2} &= \mathbf{c}_D = \mathbf{b} \\ A_{D^2} &= A_D^\top = A \end{aligned}$$

And the maximization is again switched to minimization. Thus, the dual of problem 15 is just:

$$
\begin{aligned}
&\text{Minimize} && \underbrace{\begin{bmatrix} 20 \\ 2 \end{bmatrix}}_{\mathbf{c}_{D^2}=\mathbf{c}} \cdot \mathbf{x} \\
&\text{subject to} && \underbrace{\begin{bmatrix} 30 & 5 \\ 15 & 10 \end{bmatrix}}_{A_{D^2}=A} \mathbf{x} \geq \underbrace{\begin{bmatrix} 60 \\ 70 \end{bmatrix}}_{b_{D^2}=\mathbf{b}} \\
&\text{and} && \mathbf{x} \geq 0
\end{aligned}
$$

Problem b

Problem: Find the dual problem of the following:

$$
\begin{aligned}
&\text{Maximize} && 3x_1 + x_2 + 4x_3 \\
&\text{subject to} && 3x_1 + 3x_2 + x_3 \leq 18 \\
& && 2x_1 + 2x_2 + 4x_3 = 12 \\
&\text{and} && \mathbf{x} \geq 0
\end{aligned}
$$

Solution: Let us first put the primal problem in a different form:

$$
\begin{cases}
\text{Maximize} & 3x_1 + x_2 + 4x_3 \\
\text{subject to} & 3x_1 + 3x_2 + x_3 \leq 18 \\
 & 2x_1 + 2x_2 + 4x_3 = 12 \\
\text{and} & \mathbf{x} \geq 0
\end{cases}
\implies
\begin{cases}
\text{Maximize} & 3x_1 + x_2 + 4x_3 \\
\text{subject to} & 3x_1 + 3x_2 + x_3 \leq 18 \\
 & 2x_1 + 2x_2 + 4x_3 \leq 12 \\
 & 2x_1 + 2x_2 + 4x_3 \geq 12 \\
\text{and} & \mathbf{x} \geq 0
\end{cases}
$$

$$
\implies
\begin{cases}
\text{Maximize} & 3x_1 + x_2 + 4x_3 \\
\text{subject to} & 3x_1 + 3x_2 + x_3 \leq 18 \\
 & 2x_1 + 2x_2 + 4x_3 \leq 12 \\
 & -2x_1 - 2x_2 - 4x_3 \leq -12 \\
\text{and} & \mathbf{x} \geq 0
\end{cases}
$$

$$
\implies
\begin{cases}
\text{Maximize} & \underbrace{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}}_{\mathbf{c}} \cdot \mathbf{x} \\
\text{subject to} & \underbrace{\begin{bmatrix} 3 & 3 & 1 \\ 2 & 2 & 4 \\ -2 & -2 & -4 \end{bmatrix}}_{A} \mathbf{x} \leq \underbrace{\begin{bmatrix} 18 \\ 12 \\ -12 \end{bmatrix}}_{\mathbf{b}} \\
\text{and} & \mathbf{x} \geq 0
\end{cases}
$$

And since the constants describing the dual are given by:

$$
\begin{aligned}
\mathbf{c}_D &= \mathbf{b} \\
\mathbf{b}_D &= \mathbf{c} \\
A_D &= A^\top
\end{aligned}
$$

We have the dual:

$$
\begin{aligned}
&\text{Minimize} && \underbrace{\begin{bmatrix} 18 \\ 12 \\ -12 \end{bmatrix}}_{\mathbf{c}_D = \mathbf{b}} \cdot \mathbf{x} \\
&\text{subject to} && \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 3 & 2 & -2 \\ 1 & 4 & -4 \end{bmatrix}}_{A_D = A^\top} \mathbf{x} \geq \underbrace{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}}_{\mathbf{b}_D = \mathbf{c}} \\
&\text{and} && \mathbf{x} \geq 0
\end{aligned}
$$

Problem c

Problem: Check that $\mathbf{x} = \begin{bmatrix} 5/26 \\ 5/2 \\ 27/26 \end{bmatrix}$ is an optimal solution to the following linear optimization problem:

$$
\begin{aligned}
&\text{Maximize} && \underbrace{\begin{bmatrix} 9 \\ 14 \\ 7 \end{bmatrix}}_{\mathbf{c}} \cdot \mathbf{x} \\
&\text{subject to} && \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 1 \\ 0 & 2 & 0 \end{bmatrix}}_{A} \mathbf{x} \leq \underbrace{\begin{bmatrix} 6 \\ 12 \\ 5 \end{bmatrix}}_{\mathbf{b}}
\end{aligned}
$$

Solution: Note that the only restriction on $\mathbf{x}$ is that it satisfies the given inequality. Also note that $A, \mathbf{b}$ and $\mathbf{c}$ all have only nonnegative entries. As such, since this is a maximization problem, if a solution $\mathbf{x}_E$ exists that satisfies the following:

$$
\begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 1 \\ 0 & 2 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 12 \\ 5 \end{bmatrix}
$$

Then for any other solution $\mathbf{x}$ we'd have:

$$
\mathbf{c} \cdot \mathbf{x} \leq \mathbf{c} \cdot \mathbf{x}_E
$$

In other words, if $\mathbf{x}_E$ exists then it it is optimal. And indeed such a $\mathbf{x}_E$ does exist as A is invertible:

$$
\begin{aligned}
\mathbf{x}_E &= A^{-1}\mathbf{b} \\
&= \begin{bmatrix} -1/13 & 3/13 & -11/26 \\ 0 & 0 & 1/2 \\ 5/13 & -2/13 & 3/26 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 5/26 \\ 5/2 \\ 27/26 \end{bmatrix}
\end{aligned}
$$

And so the given solution is indeed optimal.