Intro to Real Analysis
HW #10

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Problem 1

Solution: First let us prove a lemma for $n \in \mathbb{Z}^+$:

$$\begin{aligned}
-|x^n| \leq x^n \sin(1/x) &\leq |x^n| && (-1 \leq \sin(x) \leq 1) \\
-\lim_{x\to 0} |x^n| \leq \lim_{x\to 0} x^n \sin(1/x) &\leq \lim_{x\to 0} |x^n| && \text{(squeeze theorem)} \\
0 \leq \lim_{x\to 0} x^n \sin(1/x) &\leq 0 \\
\implies \lim_{x\to 0} x^n \sin(1/x) &= 0 && \text{(lemma 1)}
\end{aligned}$$

Note that this lemma applies equally well when we replace sin with cos.

Now to prove the theorem, first note that $\frac{1}{x}$, x^3, and $\sin x$ are all differentiable over $\mathbb{R} \setminus \{0\}$. This implies that $x^3 \sin(1/x)$ is differentiable over $\mathbb{R} \setminus \{0\}$ due to the product and composition of differentiable functions being differentiable.

So we have shown that $h(x)$ is differentiable, and thus continuous, over $\mathbb{R} \setminus \{0\}$. Now, all that's left is is to deal with $x = 0$. First we will prove that $h(x)$ is differentiable at $x = 0$:

$$\begin{aligned}
\lim_{t\to 0} \frac{h(0+t) - h(0)}{t - 0} &= \lim_{t\to 0} \frac{h(t) - h(0)}{t} \\
&= \lim_{t\to 0} \frac{t^3 \sin(1/t) - 0}{t} && \text{(def. of } h\text{)} \\
&= \lim_{t\to 0} t^2 \sin(1/t) \\
&= 0 && \text{(lemma 1)}
\end{aligned}$$

And so $h'(x)$ exists at $x = 0$ and is equal to 0, meaning $h(x)$ is continuous at $x = 0$ as well. Now we will prove that $h'(x)$ is continuous:

$$\begin{aligned}
3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x} &= 3x^2 \sin\frac{1}{x} + x^3 \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) && \text{(chain-rule)} \\
&= 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}
\end{aligned}$$

Again, this function is clearly differentiable, and thus continuous, over $\mathbb{R} \setminus \{0\}$ due to the composition, addition, and product of differentiable functions being differentiable. We will now show that it is continuous at $x = 0$:

$$\begin{aligned}
\lim_{x\to 0} h'(x) &= \lim_{x\to 0} \left(3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}\right) \\
&= 3\lim_{x\to 0} x^2 \sin\frac{1}{x} - \lim_{x\to 0} x\cos\frac{1}{x} \\
&= 0 - 0 && \text{(lemma 1)} \\
&= h'(0)
\end{aligned}$$

Finally we will now show that, despite $h'(x)$ being continuous everywhere, it is *not* differentiable at $x = 0$:

$$\begin{aligned}
\lim_{t\to 0} \frac{h'(0+t) - h(0)}{t} &= \lim_{t\to 0} \frac{3t^2 \sin\frac{1}{t} - t\cos\frac{1}{x} - 0}{t} \\
&= \lim_{t\to 0} 3t\sin\frac{1}{t} - \cos\frac{1}{x} \\
&= 3\lim_{t\to 0} t\sin\frac{1}{t} - \lim_{t\to 0}\cos\frac{1}{x} \\
&= -\lim_{t\to 0}\cos\frac{1}{x}
\end{aligned}$$

Clearly, if $\lim_{t\to 0}\cos\frac{1}{x}$ doesn't exist, then the derivative doesn't either. To see that this limit DNE, consider the following sequences and their values when plugged into our function:

$$\begin{aligned}
a_n &= \frac{1}{2n\pi}, \qquad a_n \to 0 \\
b_n &= \frac{1}{(2n+1)\pi}, \qquad b_n \to 0 \\
(\forall n \in \mathbb{N}) \quad \cos\frac{1}{a_n} &= \cos 2n\pi = 1 \\
(\forall n \in \mathbb{N}) \quad \cos\frac{1}{b_n} &= \cos(2n+1)\pi = -1
\end{aligned}$$

Note that we have two sequences a_n, b_n that tend towards 0 yet $\cos {}^1\!/\!_{a_n} \to 1$ and $\cos {}^1\!/\!_{b_n} \to -1$. This is a violation of sequential continuity and thus the limit DNE.

Problem 2

Solution: As in problem 1, we know that $f(x)$ is differentiable over $\mathbb{R} \setminus \{0\}$ as it is the composition, sum, and product of differentiable functions over that same set. Now we will show that $f(x)$ is differentiable over $x = 0$ as well:

$$\begin{aligned}
\lim_{t\to 0} \frac{f(0+t) - f(0)}{t} &= \lim_{t\to 0} \frac{t + 2t^2\sin({}^1\!/\!_t) - 0}{t} \\
&= \lim_{t\to 0} 1 + 2t\sin({}^1\!/\!_t) \\
&= 1 + 2\lim_{t\to 0} t\sin({}^1\!/\!_t) \\
&= 1 && \text{(lemma 1)}
\end{aligned}$$

And so $f'(x)$ exists at $x = 0$ and is equal to 1. Even further, the a we desire is equal to 0. First note that $f(a) = f(0) = 1 > 0$. Now we will prove that $x = 0$ has no neighborhood in which $f(x)$ is increasing.

Consider an arbitrary $\epsilon > 0$. Note that, by the archmedian property, there exists an $n \in \mathbb{Z}^+$ that satisfies the following:

$$a_n = \frac{1}{2n\pi} < \epsilon$$

Note that the existence of this a_n also implies:

$$b_n = \frac{1}{(2n+1)\pi} < \epsilon$$

Note however that:

$$\begin{aligned}
f'(a_n) &= 1 + 4a_n \sin \frac{1}{a_n} - 2\cos \frac{1}{a_n} && \text{(derivative of } f(x)\text{)} \\
&= 1 + \frac{4}{2n\pi} \sin 2n\pi - 2\cos 2n\pi && \text{(def. of } a_n\text{)} \\
&= 1 + 0 - 2 \\
&= 1 \\
f'(b_n) &= 1 + 4b_n \sin \frac{1}{b_n} - 2\cos \frac{1}{b_n} && \text{(derivative of } f(x)\text{)} \\
&= 1 + \frac{4}{(2n+1)\pi} \sin((2n+1)\pi) - 2\cos((2n+1)\pi) && \text{(def. of } b_n\text{)} \\
&= 1 + 0 + 2 \\
&= 3
\end{aligned}$$

And so notice that for any $\epsilon > 0$:

$$\exists a_n, b_n \in (0 - \epsilon, 0 + \epsilon) \quad b_n < a_n \wedge f'(b_n) > f'(a_n)$$

And so clearly, there is no neighborhood around 0 for which $f'(x)$ is increasing, as we can always produce a counterexample.

Problem 3

Solution: Consider the following function and its derivative:

$$\begin{aligned}
f(x) &= (1 + x)^a \\
f'(x) &= a(1 + x)^{a-1}
\end{aligned}$$

Now note that for any $x \in \mathbb{R}$, $f(x)$ is continuous on $[0, x]$ and differentiable on $(0, x)$. And so the MVT tells us that $\exists c \in (0, x)$ such that:

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

With this fact in mind, consider the following:

$$\begin{aligned}
f'(c) &= \frac{f(x) - f(0)}{x - 0} && \text{(MVT)} \\
a(1 + c)^{a-1} &= \frac{(1 + x)^a - 1}{x} && \text{(def. of } f \text{ and } f'\text{)} \\
(1 + c)^{a-1} &= \frac{(1 + x)^a - 1}{ax}
\end{aligned}$$

Now note that because $c \in (0, x)$, i.e. c is positive, we have:

$$0 < a < 1 \implies (1 + c)^{a-1} < 1$$

Since $1 + c > 1$ and its being raised to a negative power $a - 1$. And so we have:

$$\begin{aligned}
0 < a < 1 &\implies (1 + c)^{a-1} < 1 && \text{(see above)} \\
&\implies \frac{(1 + x)^a - 1}{ax} < 1 && \text{(see above)} \\
&\implies (1 + x)^a < ax + 1
\end{aligned}$$

And we are done.

Problem 4

Part a: Consider the following for an arbitrary $x \in \mathbb{R}$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{f(x)f(h)-f(x)f(0)}{h} && \text{(def. of } f\text{)} \\
&= f(x) \lim_{h\to 0} \frac{f(h)-f(0)}{h} && \text{(linearity of limit)} \\
&= f(x)f'(0) && \text{(def. of } f'(0) \text{ \& assume } f'(0) \text{ exists)}
\end{aligned}$$

And so we have shown that $f'(x)$ exists for any $x \in \mathbb{R}$ and that it is equivalent to $f(x)f'(0)$.

Part b: First note the following:

$$\begin{aligned}
f(0) &= f(0+0) \\
&= f(0)^2 && \text{(def. of } f\text{)} \\
&\in \{0, 1\}
\end{aligned}$$

If $f(0) = 0$ then we'd have the following for any real number x:

$$f(x) = f(x+0) = f(x)f(0) = 0$$

And so f would be identically 0. Barring this trivial case then, we have that $f(0) = 1$. Now note the following for any $n \in \mathbb{Z}^+$:

$$f(n) = f(\underbrace{1+1+\cdots+1}_{n \text{ times}}) = \underbrace{f(1)f(1)\cdots f(1)}_{n \text{ times}} = f(1)^n$$

Now note the following:

$$\begin{aligned}
f(-1) &= f(1-2) \\
&= f(1)f(-2) \\
&= f(1)f(-1)f(-1) \\
1 &= f(1)f(-1) \\
f(1)^{-1} &= f(-1)
\end{aligned}$$

And so, we have that for $n \in \mathbb{Z}$

$$f(n) = f(\underbrace{-1-1-\cdots-1}_{n \text{ times}}) = f(-1)^{-n} = (f(1)^{-1})^{-n} = f(1)^n$$

Now note the following for $q, n \in \mathbb{Z}$:

$$\begin{aligned}
f(n) &= f(\underbrace{1/q + 1/q + \cdots + 1/q}_{qn \text{ times}}) \\
&= f(1/q)^{qn} \\
f(1)^n &= f(1/q)^{qn} && \text{(proven above for all } n \in Z\text{)} \\
f(1) &= f(1/q)^q \\
f(1)^{1/q} &= f(1/q)
\end{aligned}$$

And so we can now prove this identity for rationals $p/q \in \mathbb{Q}$:

$$f(p/q) = f(\underbrace{1/q + 1/q + \cdots + 1/q}_{p \text{ times}}) = f(1/q)^p = (f(1)^{1/q})^p = f(1)^{p/q}$$

Now, finally, recall that the rationals are dense in the reals and that f is continuous (as a result of being differentiable). This means that for any $x \in \mathbb{R}$ we can find a sequence $r_n \in \mathbb{Q}$ that will converge to x while satisfying:

$$\forall n \in \mathbb{N}, \quad f(r_n) = f(1)^{r_n}$$

Thus, we have that $f(x) = f(1)^x$ for all real numbers x. Now note that $f(x)$ is strictly positive:

$$f(x) = f(x/2 + x/2) = f(x/2)^2 > 0 \qquad (f(x) \neq 0)$$

As such, $f(1) > 0$ and so $\exists k, \quad \ln f(1) = k$. And so we can say:

$$\begin{aligned} f(x) &= f(1)^x && \text{(we proved this for all reals)} \\ &= (\exp \ln f(1))^x && (\exp \ln f(x) = f(x)) \\ &= (\exp k)^x \\ &= e^{kx} \end{aligned}$$

To wrap this up, let us note that:

$$\begin{aligned} f'(x) &= ke^{kx} \\ f'(0) &= k \\ c &= k && \text{(def. of } c\text{)} \end{aligned}$$

And with that we can finally conclude that f, barring the trivial 0 case, must be given by:

$$f(x) = e^{cx}$$

Problem 5

Part a: Consider $f(x) = \ln x$, which is indeed defined on $(0, \infty)$. Let us first verify that this is the correct choice of $f(x)$:

$$\begin{aligned} \lim_{x\to\infty} xf'(x) &= \lim_{x\to\infty} x\frac{1}{x} && \left(\tfrac{\mathrm{d}}{\mathrm{d}x} \ln x = \tfrac{1}{x}\right) \\ &= \lim_{x\to\infty} 1 \\ &= 1 \end{aligned}$$

And so our desired limit is:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \ln x = \infty \qquad \text{(ln can grow arbitrarily large as } x \to \infty\text{)}$$

Or more specifically, we know that for every $c > 0$, there is an $x > 0$ such that $\ln x > c$.