Intro to Real Analysis
HW #6

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Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has a limit at 0, and that satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Part a: Show that $f(x)$ has a limit at every point $x \in \mathbb{R}$.

Solution: Note the following for an arbitrary $y \in \mathbb{R}$:

$$\begin{aligned}
L &= \lim_{x\to 0} f(x) && \text{(limit exists at 0)} \\
f(y)L &= f(y)\lim_{x\to 0} f(x) \\
&= \lim_{x\to 0} f(x)f(y) && \text{(product respects limit)} \\
&= \lim_{x\to 0} f(x+y) && \text{(def. of } f\text{)} \\
&= \lim_{x\to y} f(x) && \text{(shift limit)}
\end{aligned}$$

And so we have shown that $f(x)$ has a limit for all points $y \in \mathbb{R}$, namely the product $f(y)L$ where L is the limit of $f(x)$ at 0.

Part b: Show the following:

$$\lim_{x\to 0} f(x) = 1 \vee \lim_{x\to 0} f(x) = 0$$

Solution: Note the following:

$$\begin{aligned}
L &= \lim_{x\to 0} f(x) && \text{(limit exists at 0)} \\
&= \lim_{x\to 0} f(2x) && (x \to 0 \implies 2x \to 0) \\
&= \lim_{x\to 0} f(x)f(x) && \text{(def. of } f\text{)} \\
&= \lim_{x\to 0} f(x)\lim_{x\to 0} f(x) && \text{(product of limits is limit of products)} \\
&= L^2 && \text{(limit exists at 0)}
\end{aligned}$$

We have no established that, whatever L is, it is equal to its own square L^2. Recall that exactly two real numbers satisfy this property: 0 and 1. Thus we have that:

$$\lim_{x\to 0} f(x) = L \in \{0, 1\}$$

Problem 2

Problem: Consider the following function for positive integer n:

$$f : (0, \infty) \to \mathbb{R},\ f(x) = x^{1/n}$$

Prove that this function is continuous over its domain.

Solution: First recall the reverse triangle inequality for $0 < p < 1$:

$$||a|^p - |b|^p| \leq |a - b|^p$$

Now consider an arbitrary $x_0 \in (0, \infty)$, an arbitrary $\epsilon > 0$, and let $\delta = \epsilon^n > 0$. We have:

$$\begin{aligned}
0 < |x - x_0| < \delta &\implies 0 < |x - x_0| < \epsilon^n && (\delta = \epsilon^n) \\
&\implies |x - x_0|^{\frac{1}{n}} < \epsilon && (n\text{th root is increasing}) \\
&\implies ||x|^{\frac{1}{n}} - |x_0|^{\frac{1}{n}}| < \epsilon && (\text{reverse triangle inequality}) \\
&\implies |x^{\frac{1}{n}} - x_0^{\frac{1}{n}}| < \epsilon && (\text{domain is positive})
\end{aligned}$$

This is precisely the definition of a continuous function over the domain $(0, \infty)$, and so we are done.

Problem 3

Part a: Let $(b_n)_{n=1}^{\infty}$ be a sequence of rational numbers converging to b. Show that it is a Cauchy sequence.

Solution: Note that whenever we have:

$$\left|\frac{p_n}{q_n} - \frac{p_m}{q_m}\right| < \frac{1}{N}$$

Then we must have:

$$\begin{aligned}
\left|a^{\frac{p_n}{q_n}} - a^{\frac{p_m}{q_m}}\right| &= \frac{p_m}{q_m}\left|a^{\frac{p_n}{q_n} - \frac{p_m}{q_m}} - 1\right| && (\text{all positive}) \\
&< \frac{p_m}{q_m} \cdot \max\{a^{\frac{1}{N}-1}, 1 - a^{-\frac{1}{N}}\}
\end{aligned}$$

Yet recall that:

$$\lim_{n\to\infty} \sqrt[n]{a} = 1$$
$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{a}} = 1$$

And so we can always find an N large enough that $\frac{p_m}{q_m} \cdot \max\{a^{\frac{1}{N}-1}, 1 - a^{-\frac{1}{N}}\}$ is as close to an arbitrary $\epsilon > 0$.

Part b: Let $(b_n)_{n=1}^{\infty}$ and $(b'_n)_{n=1}^{\infty}$ be two sequences of rational numbers both converging to b. Show that $(a^{b_n})_{n=1}^{\infty}$ and $(a^{b'_n})_{n=1}^{\infty}$ have the same limit.

Solution: Note that whenever we have:

$$\left|\frac{p_n}{q_n} - \frac{p'_n}{q'_n}\right| < \frac{1}{N}$$

Then we must have:

$$\begin{aligned}
\left|a^{\frac{p_n}{q_n}} - a^{\frac{p'_n}{q'_n}}\right| &= \frac{p'_n}{q'_n}\left|a^{\frac{p_n}{q_n} - \frac{p'_n}{q'_n}} - 1\right| && (\text{all positive}) \\
&< \frac{p'_n}{q'_n} \cdot \max\{a^{\frac{1}{N}-1}, 1 - a^{-\frac{1}{N}}\}
\end{aligned}$$

Yet recall that:

$$\lim_{n\to\infty} \sqrt[n]{a} = 1$$
$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{a}} = 1$$

And so we can always find an N large enough that $\frac{p'_n}{q'_n} \cdot \max\{a^{\frac{1}{N}-1}, 1 - a^{-\frac{1}{N}}\}$ is as close to an arbitrary $\epsilon > 0$.

Problem 4

For some real $b > 0$, consider the function:

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = x^b$$

Part a: Show that $f(x)$ is an increasing function.

Solution: We already know that x^b is an increasing function for $b \in \mathbb{Q}^+$. Call this lemma 1. Now consider a cauchy sequence $(b_n)_{n=1}^{\infty}$ that converges to $b \in \mathbb{R}$ where each $b_n \in \mathbb{Q}^+$. Note the following:

$$\begin{aligned}
x > y &\implies (\forall n \in \mathbb{Z}^+),\ x^{b_n} > y^{b_n} && \text{(lemma 1)} \\
&\implies \lim_{n\to\infty} x^{b_n} \geq \lim_{n\to\infty} y^{b_n} && \text{(limits exist \& respect inequalities)} \\
&\implies x^b \geq y^b && \text{(problem 3)}
\end{aligned}$$

And so we have shown that for any real $b > 0$, the function x^b is increasing.

Part b: Show that $f(x)$ is continuous over $\mathbb{R}^+$.

Solution: First let us note the reverse triangle inequality for $p > 1$:

$$2^{p-1}||a|^p - |b|^p| \leq |a - b|^p$$

Consider the case where $0 < b < 1$. Now consider an arbitrary $x_0 \in (0, \infty)$, an arbitrary $\epsilon > 0$, and let $\delta = \epsilon^{1/b} > 0$. We have:

$$\begin{aligned}
0 < |x - x_0| < \delta &\implies 0 < |x - x_0| < \epsilon^{1/b} && (\delta = \epsilon^{1/b}) \\
&\implies |x - x_0|^b < \epsilon && (x^b \text{ is increasing, part a)} \\
&\implies ||x|^b - |x_0|^b| < \epsilon && \text{(reverse triangle inequality } 0 < p < 1) \\
&\implies |x^b - x_0^b| < \epsilon && \text{(domain is positive)}
\end{aligned}$$

Finally, consider the case where $b > 1$. Again we consider an arbitrary $x_0 \in (0, \infty)$, an arbitrary $\epsilon > 0$, and we now let $\delta = (2^{b-1}\epsilon)^{\frac{1}{b}} > 0$. We have:

$$\begin{aligned}
0 < |x - x_0| < \delta &\implies 0 < |x - x_0| < (2^{b-1}\epsilon)^{\frac{1}{b}} && (\delta = (2^{b-1}\epsilon)^{\frac{1}{b}}) \\
&\implies |x - x_0|^b < 2^{b-1}\epsilon && (x^b \text{ is increasing, part a)} \\
&\implies 2^{b-1}||x|^b - |x_0|^b| < 2^{b-1}\epsilon && \text{(reverse triangle inequality } p > 1) \\
&\implies |x^b - x_0^b| < \epsilon && \text{(domain is positive)}
\end{aligned}$$

Since these two cases are exhaustive for $b > 0$, we have from the definition of continuity that $f(x)$ is continuous on its domain.

Problem 5

Consider the following function:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^x$$

Part a: Show that $f(x)$ is an increasing function.

Solution: Note:

$$\begin{aligned}
y > x &\implies y - x > 0 \\
&\implies e^{y-x} - 1 > 0 \\
&\implies e^x(e^{y-x} - 1) > 0 \\
&\implies e^y - e^x > 0 \\
&\implies e^y > e^x
\end{aligned}$$

Part b: Show that $\forall x, y \in \mathbb{R}$ we have:

$$f(x+y) = f(x)f(y)$$

Solution: Note:

$$\begin{aligned}
e^x e^y &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n \\
&= \lim_{n\to\infty}\left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n \\
&= \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n \\
&= e^{x+y}
\end{aligned}$$

Part c: Show that $f(x)$ is continuous over $\mathbb{R}$.

Solution: Note:

$$\begin{aligned}
& 1+x \le e^x \le \frac{1}{1-x} \\
\implies & \lim_{x\to 0} 1+x \le \lim_{x\to 0} e^x \le \lim_{x\to 0}\frac{1}{1-x} \\
\implies & 1 \le \lim_{x\to 0} e^x \le 1 \\
\implies & \lim_{x\to 0} e^x = 1
\end{aligned}$$

Which is to say $f(x)$ has a limit at 0, namely 1. Recall from problem 1 that this, plus part b, imply that $f(x)$ is continuous.

Problem 6

Consider the logarithm function:

$$f : (0,\infty) \to \mathbb{R}, \quad f(x) = \ln x$$

Part a: For any positive number x, show that there is a unique y such that $e^y = x$.

Solution: Recall that we have already shown that e^x is both continuous and strictly increasing. The strictly increasing implies that it is injective, while being continuous over $\mathbb{R}$ implies it is surjective over $\mathbb{R}$. An injective, surjective function has a bijection by the Cantor-Bernstein theorem. A bijective function must have an inverse (i.e. $\ln x$) such that $f^{-1}(f(x)) = x$ In other words, for any x, there is always a unique y such that $e^y = x$.

Part b: Prove that ln is increasing.
Solution: Note that $\ln x$ is the inverse of e^x. Also note that e^x is strictly increasing. As a result $\ln x$ must be strictly monotone. Whether it is increasing or decreasing can simply be tested:

$$\begin{aligned}
e^0 = 1 &\implies \ln 1 = 0 \\
e^1 = e &\implies \ln e = 1
\end{aligned}$$

Since $2 < e < 3$, i.e. $1 < e$, the ln function must be increasing.

Part c: Prove that ln is continuous.

Solution: Consider an arbitrary $\epsilon > 0$ and let $\delta = x_0(e^\epsilon - 1)$:

$$\begin{aligned}
|x - x_0| &< \delta \\
&< x_0(e^\epsilon - 1) && \text{(def. of } \delta\text{)} \\
x - x_0 &< x_0(e^\epsilon - 1) && \text{(positive domain)} \\
x &< x_0(e^\epsilon - 1) + x_0 \\
x &< x_0(e^\epsilon - 1 + 1) \\
x &< x_0 e^\epsilon \\
\ln x &< \ln x_0 e^\epsilon && \text{(ln is increasing, part b)} \\
\ln x &< \ln x_0 \ln e^\epsilon && (e^{x+y} = e^x e^y) \\
\ln x - \ln x_0 &< \ln e^\epsilon \\
\ln x - \ln x_0 &< \epsilon \\
|\ln x - \ln x_0| &< \epsilon && \text{(positive domain)}
\end{aligned}$$

And with that we are done.

Problem 7

Part a: Compute the following limits for positive x:

$$\lim_{x \to 0}((1+x)^{1/3} - x^{1/3})$$
$$\lim_{x \to \infty}((1+x)^{1/3} - x^{1/3})$$

Solution: Since $\frac{1}{3}$ is a positive rational less than 1, it follows from part b below that:

$$\lim_{x \to 0}((1+x)^{1/3} - x^{1/3}) = 1$$
$$\lim_{x \to \infty}((1+x)^{1/3} - x^{1/3}) = 0$$

Part b: Compute the following limits for a positive rational $\frac{p}{q} < 1$, and positive x:

$$\lim_{x \to 0}((1+x)^{p/q} - x^{p/q})$$
$$\lim_{x \to \infty}((1+x)^{p/q} - x^{p/q})$$

Solution: First we compute the first limit:

$$\begin{aligned}
\lim_{x \to 0}((1+x)^{p/q} - x^{p/q}) &= \lim_{x \to 0}(1+x)^{p/q} - \lim_{x \to 0} x^{p/q} && (x^c \text{ is continuous, problem 4}) \\
&= (\lim_{x \to 0}(1+x))^{p/q} - (\lim_{x \to 0} x)^{p/q} && \text{(limit of power is power of limit)} \\
&= 1^{p/q} - 0^{p/q} \\
&= 1
\end{aligned}$$

Now we compute the second limit via the squeeze theorem. First we find an upper bound:

$$\begin{aligned}
(1+x)^{p/q} - x^{p/q} &= ||1+x|^{\frac{p}{q}} - |x|^{\frac{p}{q}}| && (x \in \mathbb{R}^+) \\
&\leq |1 + x - x|^{p/q} && \text{(reverse triangle inequality } 0 < p < 1\text{)} \\
&= 1
\end{aligned}$$

Next we find a lower bound:

$$
\begin{aligned}
(1+x)^{p/q} - x^{p/q} &\geq \frac{1}{(1+x)^{p/q} - x^{p/q}} \\
&= \frac{1}{||1+x|^{\frac{p}{q}} - |x|^{\frac{p}{q}}|} && (x \in \mathbb{R}^+) \\
&\geq \frac{1}{|1+x-x|^{p/q}} && (\text{reverse triangle inequality } 0 < p < 1) \\
&= 1
\end{aligned}
$$

And so, applying the squeeze theorem we have:

$$
\begin{aligned}
& 1 \leq (1+x)^{p/q} - x^{p/q} \leq 1 \\
\implies & \lim_{x\to\infty} 1 \leq \lim_{x\to\infty} ((1+x)^{p/q} - x^{p/q}) \leq \lim_{x\to\infty} 1 && (\text{squeeze theorem}) \\
\implies & 1 \leq \lim_{x\to\infty} ((1+x)^{p/q} - x^{p/q}) \leq 1 \\
\implies & \lim_{x\to\infty} ((1+x)^{p/q} - x^{p/q}) = 1
\end{aligned}
$$