Intro to Real Analysis
HW #4

Ozaner Hansha

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Problem 1

Consider $(a_n)_{n=1}^{\infty}$ where $a_n \geq 0$ and:

$$\lim_{n\to\infty} a_n = A$$

Part a: Show that $(b_n)_{n=1}^{\infty}$ where $b_n = \sqrt{a_n}$ is also a convergent sequence and that:

$$\lim_{n\to\infty} b_n = \sqrt{A}$$

Solution: Before we prove this, let use first show why $A \nless 0$:

Let us suppose that indeed $A < 0$. We have $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\exists n > N)$:

$$\begin{aligned}
|a_n - A| &< \epsilon && \text{(def. of convergence)} \\
|a_n - A| &< -A && (-A > 0) \\
A < a_n - A &< -A \\
2A < a_n &< 0
\end{aligned}$$

Which is a contradiction since we know a_n is nonnegative. Thus, by the trichotomy of the reals, we have two cases to consider:

- If $A = 0$ then we have $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\exists n > N)$:

$$\begin{aligned}
|a_n - 0| &< \epsilon^2 && \text{(def. of convergence, } \epsilon^2 > 0) \\
|a_n| &< \epsilon^2 \\
\sqrt{|a_n|} &< \epsilon && \text{(both sides of inequality are positive)} \\
|\sqrt{a_n}| &< \epsilon && (a_n > 0) \\
|\sqrt{a_n} - 0| &< \epsilon \\
|b_n - 0| &< \epsilon && \text{(def. of } b_n) \\
|b_n - \sqrt{0}| &< \epsilon \\
|b_n - \sqrt{A}| &< \epsilon && (A = 0 \text{ by assumption})
\end{aligned}$$

  So in other words we have $\lim_{n\to\infty} b_n = \sqrt{A}$.

- If $A > 0$ then we have the following $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\exists n > N)$:

$$\begin{aligned}
|b_n - \sqrt{A}| &= |\sqrt{a_n} - \sqrt{A}| && \text{(def. of } b_n) \\
&= \frac{|a_n - A|}{\sqrt{a_n} + \sqrt{A}} && \text{(multiply by conjugate)} \\
&< \frac{|a_n - A|}{\sqrt{A}} && (\sqrt{a_n} > 0) \\
&< \frac{\epsilon\sqrt{A}}{\sqrt{A}} && \text{(def. of convergence, } \epsilon\sqrt{A} > 0) \\
&= \epsilon
\end{aligned}$$

  So in other words we have $\lim_{n\to\infty} b_n = \sqrt{A}$.

Part b: Prove the following:

$$(\forall c > 0)\ \lim_{n\to\infty} \sqrt[n]{c} = 1$$

Solution: We have two cases:

- $c \geq 1$. Let $c = 1 + b$. We then have:

$$\begin{aligned}
\left(1 + \frac{b}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{b}{n}\right)^k && \text{(binomial theorem)} \\
&> \binom{n}{0} + \binom{n}{1}\frac{b}{n} && \text{(first 2 terms)} \\
&= 1 + b
\end{aligned}$$

  And so we have:

$$\begin{aligned}
1 \leq c = 1 + b &< \left(1 + \frac{b}{n}\right)^n && \text{(see above)} \\
1 \leq \sqrt[n]{c} &< 1 + \frac{b}{n} && \text{(nth root, all sides positive)} \\
\lim_{n\to\infty} 1 \leq \lim_{n\to\infty} \sqrt[n]{c} &< \lim_{n\to\infty} 1 + \frac{b}{n} && \text{(squeeze theorem)} \\
1 \leq \lim_{n\to\infty} \sqrt[n]{c} &< 1 \\
\implies \lim_{n\to\infty} \sqrt[n]{c} &= 1
\end{aligned}$$

- $c < 1$. Consider an arbitrary real $\epsilon > 0$. Choose a positive integer $N > \frac{1}{c\epsilon}$. We then have for $n \geq N$:

$$\begin{aligned}
\left(\sqrt[n]{c} + \epsilon\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \sqrt[n]{c}^{\,n-k} \epsilon^k && \text{(binomial theorem)} \\
&> c + \sqrt[n]{c}^{\,n-1} n\epsilon && \text{(first two terms)} \\
&> c + cn\epsilon \\
&> c + \frac{c}{c\epsilon}\epsilon \\
&= c + 1 \\
&> c
\end{aligned}$$

  Transforming this result further, we have:

$$\begin{aligned}
1 &< \left(\sqrt[n]{c} + \epsilon\right)^n && \text{(see above)} \\
1 &< \sqrt[n]{c} + \epsilon \\
1 - \sqrt[n]{c} &< \epsilon \\
-(\sqrt[n]{c} - 1) &< \epsilon \\
|\sqrt[n]{c} - 1| &< \epsilon && (c < 1 \implies \sqrt[n]{c} < 1 \implies \sqrt[n]{c} - 1 < 0)
\end{aligned}$$

  And so we have shown that for any $\epsilon > 0$ there is a choice of N such that for all $n > N$ we have $|\sqrt[n]{c} - 1| < \epsilon$. In other words:

$$\lim_{n\to\infty} \sqrt[n]{c} = 1$$

Problem 2

Problem: Let $a, b > 0$, show that:

$$\lim_{n\to\infty} \sqrt[n]{a^n + b^n} = \max\{a, b\}$$

Solution: W.l.o.g we can assume $a \leq b$. Now consider the following:

$$\begin{aligned}
b &= \lim_{n\to\infty} b \\
&= \lim_{n\to\infty} \sqrt[n]{b^n} \\
&\leq \lim_{n\to\infty} \sqrt[n]{a^n + b^n} && (a^n + b^n \geq b^n) \\
&\leq \lim_{n\to\infty} \sqrt[n]{2b^n} && (a < b) \\
&= \lim_{n\to\infty} \sqrt[n]{2} \lim_{n\to\infty} \sqrt[n]{b^n} \\
&= \lim_{n\to\infty} \sqrt[n]{b^n} && \text{(Problem 1, Part b)} \\
&= b
\end{aligned}$$

In other words we have:

$$\begin{aligned}
& b \leq \lim_{n\to\infty} \sqrt[n]{a^n + b^n} \leq b && \text{(see above)} \\
\implies & \lim_{n\to\infty} \sqrt[n]{a^n + b^n} = b && \text{(squeeze theorem)} \\
\implies & \lim_{n\to\infty} \sqrt[n]{a^n + b^n} = \max\{a, b\} && (a \leq b)
\end{aligned}$$

Of course, the same argument holds when $a \geq b$ with a and b switching roles.

Problem 3

Consider a sequence $(a_n)_{n=1}^{\infty}$ whose limit is A.

Problem: Prove the following:

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = A$$

Solution: Since $a_n \to L$ we must have that $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)$:

$$|a_n - A| < \frac{\epsilon}{2} \qquad \text{(lemma 1)}$$

And so $\exists M > N$ such that $\forall n > M$:

$$\frac{|a_1 - A| + \cdots + |a_N - A|}{n} < \frac{\epsilon}{2} \qquad \text{(lemma 2)}$$

Then $(\forall n > M)$ we have:

$$\begin{aligned}
\left|\frac{a_1 + \cdots + a_n}{n} - A\right| &= \left|\frac{a_1 + \cdots a_n - nA}{n}\right| \\
&= \left|\frac{(a_1 - A) + \cdots + (a_n - A)}{n}\right| \\
&= \left|\frac{(a_1 - A) + \cdots + (a_N - A) + (a_{N+1} - A) + \cdots + (a_n - A)}{n}\right| && (n > M > N) \\
&\leq \frac{|a_1 - A| + \cdots + |a_N - A|}{n} + \frac{|a_{N+1} - A| + \cdots + |a_n - A|}{n} && \text{(triangle inequality)} \\
&< \frac{\epsilon}{2} + \frac{(n - N)\epsilon}{2n} && \text{(lemma 1 \& 2)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon && (n > N \implies 0 < \tfrac{n-N}{n} < 1)
\end{aligned}$$

And so, by the definition of convergence, we have:

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = A$$

Problem 4

Part a: Consider a sequence $(a_n)_{n=1}^{\infty}$ where $n = (1 + a_n)^2$. Show that for $n > 1$:

$$0 < a - n < \sqrt{\frac{2}{n-1}}$$

Solution: First let us solve for a_n:

$$\begin{aligned} (1 + a_n)^n &= n \\ 1 + a_n &= \sqrt[n]{n} \\ a_n &= \sqrt[n]{n} - 1 \end{aligned}$$

Recall from problem 1, part b that $c > 1 \implies \sqrt[n]{c} > 1$ for any $n \in \mathbb{N}$. And so we have:

$$\begin{aligned} a_n &= \sqrt[n]{n} - 1 && \text{(def. of } a_n\text{)} \\ &> 1 - 1 = 0 \end{aligned}$$

Now we have to prove the other side of the inequality. Consider the following:

$$\begin{aligned} n &= \left(\sqrt[n]{n}\right)^n \\ &= (1 + (\sqrt[n]{n} - 1))^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (\sqrt[n]{k})^k && \text{(binomial theorem)} \\ &\geq \binom{n}{2} (\sqrt[n]{n} - 1)^2 && \text{(second term only, } n > 1\text{)} \\ &= \frac{n(n-1)}{2} (\sqrt[n]{n} - 1)^2 \end{aligned}$$

In other words we have:

$$\begin{aligned} \frac{n(n-1)}{2} (\sqrt[n]{n} - 1)^2 &\leq n && \text{(see above)} \\ (\sqrt[n]{n} - 1)^2 &\leq \frac{2}{n-1} \\ \sqrt[n]{n} - 1 &\leq \sqrt{\frac{2}{n-1}} \\ a_n &\leq \sqrt{\frac{2}{n-1}} && \text{(def. of } a_n\text{)} \end{aligned}$$

And so, putting our two inequalities together, we have proved the desired statement:

$$(\forall n > 1)\ 0 < a_n \leq \sqrt{\frac{2}{n-1}}$$

Part b: Show that:

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Solution: Consider an arbitrary real $\epsilon > 0$. Choose a positive integer $N > 1 + \frac{2}{\epsilon^2}$. We then have:

$$\begin{aligned}
N &> 1 + \frac{2}{\epsilon^2} && (N \text{ exists by archemdiean property}) \\
N - 1 &> \frac{2}{\epsilon^2} \\
\frac{1}{N-1} &< \frac{\epsilon^2}{2} \\
\frac{2}{N-1} &< \epsilon^2 \\
\sqrt{\frac{2}{N-1}} &< \epsilon
\end{aligned}$$

And so we have for any $n \geq N$:

$$\begin{aligned}
\sqrt[n]{n} - 1 &\leq \frac{2}{n-1} && \text{(part a)} \\
|\sqrt[n]{n} - 1| &\leq \frac{2}{n-1} && (n > N > 1 \implies \sqrt[n]{n} - 1 > 0) \\
&\leq \frac{2}{N-1} \\
&< \epsilon && \text{(see above)}
\end{aligned}$$

And so we have shown that $(\forall \epsilon > 0)(\exists N > \mathbb{N})(\forall n > N)$ we have $|\sqrt[n]{n} - 1| < \epsilon$. This is the definition of:

$$\lim_{n \to \infty} \sqrt[n]{n} = 1$$