Numerical Analysis HW #4

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Problem 1

Problem: Use the composite trapezoid method with 4 equally sized subintervals to approximate the following integral:

$$\int_1^2 x \ln x \; dx$$

Also give an upper bound of the approximation's error.

Solution: Splitting up our interval of $[1, 2]$ into $n = 4$ subintervals gives us the following 5 nodes (rounded to the 5th decimal place) with which to interpolate:

$$(1, 0), (1.25, 0.27893), (1.5, 0.60820), (1.75, 0.97933), (2, 1.38629)$$

Recall that the composite trapezoid rule for a uniform distribution of 5 points is given by:

$$\frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$$

Leaving us with:

$$\int_1^2 x \ln x \; dx \approx \frac{1}{8} (0 + 2(0.27893) + 2(0.60820) + 2(0.97933) + 1.38629) = \boxed{0.63990}$$

As for the error of this approximation, recall that the maximum absolute error of the trapezoid method is given by the following:

$$\text{error}_{\max} = \frac{(\Delta x)^3 n}{12} \max_{\xi \in [1,2]} |f''(\xi)|$$

The second derivative of f is given by:

$$f(x) = x \ln x \qquad f'(x) = \ln x + 1 \qquad f''(x) = \frac{1}{x}$$

The second derivative, the inverse function, is a decreasing function over the positive reals. Since our interval $[1, 2] \subseteq \mathbb{R}^+$, the maximum of this function is at the end point $x = 1$. This gives us $f(1) = 1$. Plugging this back into our error bound we arrive at:

$$|\,\text{error}\,| \leq \frac{(\Delta x)^3 n}{12} \max_{\xi \in [1,2]} |f''(\xi)| = \frac{(1)^3 (4)}{12} (1) = \boxed{\frac{1}{3}}$$

Problem 2

Problem: Use the composite Simpson rule with 2 equally sized subintervals to approximate the same integral as problem 1 and give an upper bound of the approximation's error.

Solution: Splitting up our interval of $[1, 2]$ into $n = 2$ subintervals gives us the following 3 nodes (rounded to the 5th decimal place) with which to interpolate:

$$(1, 0)\,, \left(1.3\overline{33}, 0.383576\right), (2, 1.38629)$$

Recall that the composite trapezoid rule for a uniform distribution of 5 points is given by:

$$\frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

Leaving us with:

$$\int_1^2 x \ln x \; dx \approx \frac{1}{8}\left(0 + 2(0.27893) + 2(0.60820) + 2(0.97933) + 1.38629\right) = \boxed{0.63990}$$

As for the error of this approximation, recall that the maximum absolute error of the trapezoid method is given by the following:

$$\text{error}_{\max} = \frac{(\Delta x)^3 n}{12} \max_{\xi \in [1,2]} |f''(\xi)|$$

The second derivative of f is given by:

$$f(x) = x \ln x \qquad f'(x) = \ln x + 1 \qquad f''(x) = \frac{1}{x}$$

The second derivative, the inverse function, is a decreasing function over the positive reals. Since our interval $[1, 2] \subseteq \mathbb{R}^+$, the maximum of this function is at the end point $x = 1$. This gives us $f(1) = 1$. Plugging this back into our error bound we arrive at:

$$|\,\text{error}\,| \leq \frac{(\Delta x)^3 n}{12} \max_{\xi \in [1,2]} |f''(\xi)| = \frac{(1)^3(4)}{12}(1) = \boxed{\frac{1}{3}}$$

Problem 3

Problem: Use Gaussian Integration with 2 nodes and weights to approximate the same integral as problem 1.

Solution:

Problem 4

Problem: Use both the composite trapezoid and Simpson rule to approximate the following integrals to a tolerance of $10^{-2}, 10^{-4}, 10^{-8}$ and record the number of intervals n needed and the error for each.

$$\int_0^1 (1 - 4x(1-x))^{\frac{1}{3}} \; dx \qquad \int_0^1 xe^{-x} \; dx$$

Also use the Simpson rule on the second integral to a tolerance of 10^{-16} and see the results.

Solution: