Intro to Math Reasoning HW 8a

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Problem 1

Problem: Prove that for any integer $j \geq 1$:

$$j < 10^{j+1}$$

Solution: First note that when $j = 1$ we have:

$$1 < 10^{1+1} = 1 < 100$$

Which is true. Now we just need to show the inductive step:

$$\begin{aligned} j &< 10^{j+1} \\ 10j &< 10^{j+1}10 \\ 10j &< 10^{j+2} \end{aligned}$$

Now we just need to show that $j + 1 < 10j$ for all $j \geq 1$. Now we will prove the statement $1 < 9j$ for any positive j. For the $j = 1$ case we have $1 < 9$ and for the inductive step:

$$\begin{aligned} 1 &< 9j && \text{(given)} \\ 9j &< 9j + 9 \\ 9j &< 9(j + 1) \\ 1 < 9j &< 9(j + 1) \\ 1 &< 9(j + 1) \end{aligned}$$

Now we can use this to prove the statement we set out to show:

$$\begin{aligned} 1 &< 9j && \text{(for any } j \geq 1) \\ 1 + j &< 9j + j \\ j + 1 &< 10j \end{aligned}$$

Now we can write them together as:

$$j + 1 < 10j < 10^{j+2} \implies j + 1 < 10^{j+2}$$

And so by induction the statement is true.

Problem 3

Problem: Prove that if $m \leq 0$ then $10^m - 1$ is divisible by 9.

Solution: First we show the base case of $m = 1$:

$$10^1 - 1 = 9$$

And 9 is clearly divisible by itself. Now we show the inductive step:

$$\begin{aligned} 10^m - 1 &= 9q \\ 10(10^m - 1) &= 10(9q) \\ 10^m 10 - 10 &= 90q \\ 10^{m+1} - 10 &= 90q \\ 10^{m+1} - 10 + 9 &= 90q + 9 \\ 10^{m+1} - 1 &= 9(10q + 1) \end{aligned}$$

Where $10q + 1 \in \mathbb{Z}$ since q is also an integer. And so we are done.

Problem 4

Problem: Prove that am integer greater than or equal to 1 is divisible by 9 iff the sum of its digits are divisible by 9.

Solution: (I'm starting the indexing at 0 and restating above result in terms of modulo) Note that the decimal expansion of a number n is:

$$n = 10^n d_n + 10^{n-1} d_{n-1} \cdots + 10 d_1 + d_0$$

And since $10 \equiv 1 \pmod 9$ we have $10^m \equiv 1 \pmod 9$. This means in modulo arithmetic we can say:

$$n \equiv 1 d_n + 1 d_{n-1} \cdots + 1 d_1 + d_0 \pmod 9$$

Which is just the sum of its digits. And so if n is divisible by 9 then the right hand side (the sum of the digits) must also be divisible by 9. This is because they are in the same equivalence class under mod 9.

Problem 5

Problem: Prove that $j \geq 1$ and $b \geq 2$ imply that $j < b^{j+1}$

Solution: We can prove this using a base case and 2 inductive steps. The base case is simply $1 < 2^2$ which is clearly true. Now we need to show that $j+1 < b^{j+2}$ follows from $j < b^{j+1}$:

$$\begin{aligned}
j &< b^{j+1} \\
j &< b^{j+2} && \text{(multiply by positive integer, always bigger)} \\
j+1 &< b^{j+2} && \text{(as long as } j \leq b\text{)}
\end{aligned}$$

Now we need to prove $j < (b+1)^{j+1}$ which should be clear since $b+1 > b$ and all the other variables are positive integers and so $j < b^{j+1} < (b+1)^{j+1}$.

And so using induction and the base case the statement is true for all $b \leq 2$ and $j \leq 1$