Honors Calculus III Challenge Problems #1

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The problems all refer to the following vectors in $\mathbb{R}^5$:

$$\begin{aligned}
\mathbf{v}_1 &= (1, 2, 0, 2, 0) \\
\mathbf{v}_2 &= (2, 1, 1, 1, 1) \\
\mathbf{v}_3 &= (0, 1, -1, 1, -1) \\
\mathbf{v}_4 &= (-1, -1, 0, -3, 0) \\
\mathbf{v}_5 &= (1, 2, 1, 2, -1)
\end{aligned}$$

Exercise 1

Problem: Apply the Gram-Schmidt algorithm to the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ in that order to produce the orthonormal set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$.

Solution: Recall that the Gram-Schmidt algorithm generates an orthonormal set of vectors U with the same span as a finite set of vectors V by iterating through each vector $\mathbf{v}_i$ in V and subtracting its component parallel to every $\mathbf{u}_i$ *currently* in U. This guarantees the result is orthogonal to all members of U (yet possibly zero). If a non-zero vector is computed from this calculation, normalize it and add it to U. If the result *is* zero, move on to the next $\mathbf{v}_i$.

Applying this to the set V given above, we choose the first vector $\mathbf{v}_1$ and subtract its parallel component for all members of U. This is already been vacuously done as the set is currently empty. And since $\mathbf{v}_1 \neq \mathbf{0}$, we normalize it and add it to U:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 2, 0, 2, 0)}{\|(1, 2, 0, 2, 0)\|} = \boxed{\frac{1}{3}(1, 2, 0, 2, 0)}$$

Now we move onto the next vector in V, namely $\mathbf{v}_2$. First we subtract from it the component parallel to all members of U, in this case just $\mathbf{u}_1$:

$$\begin{aligned}
\mathbf{w}_2 = \mathbf{v}_2 &- \sum_{\mathbf{u}\in U} \mathrm{proj}_{\mathbf{u}}(\mathbf{v}_2) \\
&- \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) && \text{(only } \mathbf{u}_1 \text{ in } U\text{)} \\
&- \frac{(\mathbf{v}_2 \cdot \mathbf{u}_1)}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 && \text{(def. of projection)} \\
&- (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 && \text{(}\mathbf{u} \text{ is normal)}
\end{aligned}$$

We dub this intermediary vector $\mathbf{w}_2$ for convience.

Now we check if the resultant vector, $\mathbf{w}_2$, is non-zero:

$$\begin{aligned}
\mathbf{w}_2 &= \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 \\
&= (2,1,1,1,1) - ((2,1,1,1,1) \cdot \frac{1}{3}(1,2,0,2,0))\frac{1}{3}(1,2,0,2,0) \\
&= (2,1,1,1,1) - \frac{2}{3}(1,2,0,2,0) \\
&= \frac{1}{3}(4,-1,3,-1,3) \neq \mathbf{0}
\end{aligned}$$

The vector is indeed non-zero and thus we normalize it and add it to U:

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{\frac{1}{3}(4,-1,3,-1,3)}{\|\frac{1}{3}(4,-1,3,-1,3)\|} = \boxed{\frac{1}{6}(4,-1,3,-1,3)}$$

We continue these steps as we iterate through V, ending at its last vector. For $\mathbf{v}_3$ we get:

$$\begin{aligned}
\mathbf{w}_3 = \mathbf{v}_3 &- \sum_{\mathbf{u}\in U} \mathrm{proj}_{\mathbf{u}}(\mathbf{v}_3) \\
&= \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 \\
&= \mathbf{v}_3 - \frac{4}{3}\mathbf{u}_1 + \frac{4}{3}\mathbf{u}_2 \\
&= (2,1,1,1,1) - \frac{2}{9}(2,-5,2,-5,3) \\
&= \frac{1}{9}(4,-1,-3,-1,-3) \neq \mathbf{0}
\end{aligned}$$

Again, the resulting vector is non-zero, and so we normalize it and add it to U:

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{\frac{1}{9}(4,-1,-3,-1,-3)}{\|\frac{1}{9}(4,-1,-3,-1,-3)\|} = \boxed{\frac{1}{6}(4,-1,-3,-1,-3)}$$

For $\mathbf{v}_4$ we get:

$$\begin{aligned}
\mathbf{w}_4 &= \mathbf{v}_4 - \sum_{\mathbf{u} \in U} \text{proj}_{\mathbf{u}}(\mathbf{v}_4) \\
&= \mathbf{v}_4 - (\mathbf{v}_4 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_4 \cdot \mathbf{u}_2)\mathbf{u}_2 - (\mathbf{v}_4 \cdot \mathbf{u}_3)\mathbf{u}_3 \\
&= \mathbf{v}_4 + 3\mathbf{u}_1 - 0\mathbf{u}_2 - 0\mathbf{u}_3 \\
&= (-1, -1, 0, -3, 0) + 3 \cdot \frac{1}{3}(1, 2, 0, 2, 0) \\
&= (0, 1, 0, -1, 0) \neq \mathbf{0}
\end{aligned}$$

Once more, the resulting vector is non-zero, and so we normalize it and add it to U:

$$\mathbf{u}_4 = \frac{\mathbf{w}_4}{\|\mathbf{w}_4\|} = \frac{(0, 1, 0, -1, 0)}{\|(0, 1, 0, -1, 0)\|} = \boxed{\frac{\sqrt{2}}{2}(0, 1, 0, -1, 0)}$$

And finally for $\mathbf{v}_5$ we get:

$$\begin{aligned}
\mathbf{w}_5 &= \mathbf{v}_5 - \sum_{\mathbf{u} \in U} \text{proj}_{\mathbf{u}}(\mathbf{v}_5) \\
&= \mathbf{v}_5 - (\mathbf{v}_5 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_5 \cdot \mathbf{u}_2)\mathbf{u}_2 - (\mathbf{v}_5 \cdot \mathbf{u}_3)\mathbf{u}_3 - (\mathbf{v}_5 \cdot \mathbf{u}_4)\mathbf{u}_4 \\
&= \mathbf{v}_5 - 3\mathbf{u}_1 - 0\mathbf{u}_2 - 0\mathbf{u}_3 - 0\mathbf{u}_4 \\
&= (1, 2, 1, 2, -1) - 3 \cdot \frac{1}{3}(1, 2, 0, 2, 0) \\
&= (0, 0, 1, 0, -1) \neq \mathbf{0}
\end{aligned}$$

And for the last time, the resulting vector is non-zero, and so we normalize it and add it to U:

$$\mathbf{u}_5 = \frac{\mathbf{w}_5}{\|\mathbf{w}_5\|} = \frac{(0, 0, 1, 0, -1)}{\|(0, 0, 1, 0, -1)\|} = \boxed{\frac{\sqrt{2}}{2}(0, 0, 1, 0, -1)}$$

With that we are done. The resulting set of orthonormal vectors U, whose span is equivalent to V, is as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \frac{1}{3}(1, 2, 0, 2, 0) \\
\mathbf{u}_2 &= \frac{1}{6}(4, -1, 3, -1, 3) \\
\mathbf{u}_3 &= \frac{1}{6}(4, -1, -3, -1, -3) \\
\mathbf{u}_4 &= \frac{\sqrt{2}}{2}(0, 1, 0, -1, 0) \\
\mathbf{u}_5 &= \frac{\sqrt{2}}{2}(0, 0, 1, 0, -1)
\end{aligned}$$

Exercise 2

Part A

Problem: Where $\mathbf{x}_0 = (1, 1, 1, 0, 0)$, show that the plane P parameterized by the following:

$$\mathbf{x}_1(s,t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$$

is equivalent to the solution set of the following system of equations:

$$\begin{aligned}
\mathbf{u}_3 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \\
\mathbf{u}_4 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \\
\mathbf{u}_5 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0
\end{aligned}$$

Solution: Intuitively, we know that a 2D plane in 5D space is defined by the intersection of $5 - 2 = 3$ hyperplanes in $\mathbb{R}^5$. If we were to describe these hyperplanes in point-normal vector form, i.e:

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

Where $\mathbf{a}$ *is normal to the hyperplane and* $\mathbf{x}_0$ *is some arbitrary point on the plane.*

we would need to establish that $\mathbf{x}_0$ lies on P and that $\mathbf{u}_3, \mathbf{u}_4$, and $\mathbf{u}_5$ are orthogonal with each other and P. The former is trivial, simply consider $\mathbf{x}_1(s,t)$ for $s, t = 0$:

$$\begin{aligned}
\mathbf{x}_1(0,0) &= \mathbf{x}_0 + 0\mathbf{v}_1 + 0\mathbf{v}_2 \\
&= \mathbf{x}_0
\end{aligned}$$

Thus $\mathbf{x}_0$ is on the plane P. In regards to the other proposition, we know that $\mathbf{u}_3, \mathbf{u}_4$, and $\mathbf{u}_5$ are orthogonal to each other as they are part of the orthonormal basis we constructed in Exercise 1. Now all we have to show is that these 3 vectors are orthogonal to every vector on the plane P.

We can do this by simply dotting each vector with $\mathbf{x}_1(s,t)$. If the result of each dotting is 0 we'll know that, regardless of s or t, that any vector on P will be orthogonal to the three vectors. So we'll do just that:

$$\begin{aligned}
\mathbf{u}_3 \cdot \mathbf{x}_1(s,t) &= \mathbf{u}_3 \cdot (\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2) \\
&= (\mathbf{u}_3 \cdot \mathbf{x}_0) + (\mathbf{u}_3 \cdot s\mathbf{v}_1) + (\mathbf{u}_3 \cdot t\mathbf{v}_2) && \text{(distr. of dot prod.)} \\
&= (\mathbf{u}_3 \cdot \mathbf{x}_0) + 0 + 0 && \text{(consequence of Gram-Schmidt)} \\
&= \frac{1}{6}(4, -1, -3, -1, -3) \cdot (1, 1, 1, 0, 0) \\
&= 0
\end{aligned}$$

Notice that we made two of the terms 0 in line 3 due to a "consequence of Gram-Schmidt". To elaborate, this is because the Gram-Schmidt algorithm guarantees that the span of the set U at any point is equal to the span of the set of vectors V that have been iterated through so far. This is simply the nature of the algorithm as it subtracts all parts parallel to all the previously generated orthonormal vectors to generate new ones.

It is clear then that because the vector $\mathbf{u}_3$ was generated from $\mathbf{v}_3$, it must be orthogonal to the vectors that came before it. Those vectors are $\mathbf{v}_1$ and $\mathbf{v}_2$. If this doesn't satisfy you, we can also just manually compute the dot products and observe that they indeed equal 0.

The same argument follows for $\mathbf{u}_4$ and $\mathbf{u}_5$:

$$\begin{aligned}
\mathbf{u}_4 \cdot \mathbf{x}_1(s,t) &= \mathbf{u}_4 \cdot (\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2) \\
&= (\mathbf{u}_4 \cdot \mathbf{x}_0) + (\mathbf{u}_4 \cdot s\mathbf{v}_1) + (\mathbf{u}_4 \cdot t\mathbf{v}_2) && \text{(distr. of dot prod.)} \\
&= (\mathbf{u}_4 \cdot \mathbf{x}_0) + 0 + 0 && \text{(consequence of Gram-Schmidt)} \\
&= \frac{\sqrt{2}}{2}(0,1,0,-1,0) \cdot (1,1,1,0,0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbf{u}_5 \cdot \mathbf{x}_1(s,t) &= \mathbf{u}_5 \cdot (\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2) \\
&= (\mathbf{u}_5 \cdot \mathbf{x}_0) + (\mathbf{u}_5 \cdot s\mathbf{v}_1) + (\mathbf{u}_5 \cdot t\mathbf{v}_2) && \text{(distr. of dot prod.)} \\
&= (\mathbf{u}_5 \cdot \mathbf{x}_0) + 0 + 0 && \text{(consequence of Gram-Schmidt)} \\
&= \frac{\sqrt{2}}{2}(0,0,1,0,-1) \cdot (1,1,1,0,0) \\
&= 0
\end{aligned}$$

And so the solution set of the parameterization $\mathbf{x}_1(s,t)$ is the same as that of the system of equations given above, both giving the same plane P.

Part B

Problem: Show that the following parameterization $\tilde{\mathbf{x}}_1$ is of the same plane P as that of $\mathbf{x}_1$ from Part A:

$$\tilde{\mathbf{x}}_1(s,t) = \mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2$$

Solution: The above is true as a consequence of the Gram-Schmidt algorithm. The only vectors in the orthonormal set U after the first two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ were iterated through were $\mathbf{u}_1$ and $\mathbf{u}_2$. In other words:

$$\operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \operatorname{span}(\{\mathbf{u}_1, \mathbf{u}_2\})$$

And so whatever solution set is given by $\mathbf{x}_1$ is also given by $\tilde{\mathbf{x}}_1$ as the only terms that differ between the two have the same linear span and can even be written as a linear combination of the others:

$$(\forall s,t \in \mathbb{R})\,(\exists s',t' \in \mathbb{R})\, s\mathbf{v}_1 + t\mathbf{v}_2 = s'\mathbf{u}_1 + t'\mathbf{u}_2$$

Exercise 3

Part A

Problem: Show that the following expression is independent of s, t, u and v:

$$((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_5)^2$$

Solution: Simply distribute the dot product of $\mathbf{u}_5$ to the other terms:

$$((\mathbf{u}_5 \cdot \mathbf{x}_0) + (\mathbf{u}_5 \cdot s\mathbf{u}_1) + (\mathbf{u}_5 \cdot t\mathbf{u}_2) - (\mathbf{u}_5 \cdot u\mathbf{v}_3) - (\mathbf{u}_5 \cdot v\mathbf{v}_4))^2$$

It is clear that $(\mathbf{u}_5 \cdot s\mathbf{u}_1)$ and $(\mathbf{u}_5 \cdot t\mathbf{u}_2)$ are 0 as they belong to the same orthonormal set. It should also be clear from the explanation given in Excersice 2 that $(\mathbf{u}_5 \cdot u\mathbf{v}_3)$ and $(\mathbf{u}_5 \cdot v\mathbf{v}_4)$ are also 0 given that these vectors preceded $\mathbf{u}_5$ in the Gram-Schmidt process.

And so we have reduced the expression to one with no mention of s, t, u or v, thus it is independent:

$$(\mathbf{u}_5 \cdot \mathbf{x}_0)^2$$

Part B

Problem: What is the value of the above expression?

Solution: As we have shown in Exercise 2:

$$\mathbf{u}_5 \cdot \mathbf{x}_0 = \frac{\sqrt{2}}{2}(0, 0, 1, 0, -1) \cdot (1, 1, 1, 0, 0) = 0$$

Exercise 4

Part A

Problem: Show that the following expression depends only on v:

$$((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_4)^2$$

Solution: Again we distribute the dot product:

$$((\mathbf{u}_4 \cdot \mathbf{x}_0) + (\mathbf{u}_4 \cdot s\mathbf{u}_1) + (\mathbf{u}_4 \cdot t\mathbf{u}_2) - (\mathbf{u}_4 \cdot u\mathbf{v}_3) - (\mathbf{u}_4 \cdot v\mathbf{v}_4))^2$$

For the same reasoning given in Exercise 2 and 3, we can say:

$$(\mathbf{u}_4 \cdot \mathbf{x}_0) = (\mathbf{u}_4 \cdot s\mathbf{u}_1) = (\mathbf{u}_4 \cdot t\mathbf{u}_2) = (\mathbf{u}_4 \cdot u\mathbf{v}_3) = 0$$

This leaves us with the following expression:

$$(-\mathbf{u}_4 \cdot v\mathbf{v}_4)^2$$

This expression is indeed dependent on v as when we factor it out, the dot product does not equal 0:

$$\begin{aligned}(-\mathbf{u}_4 \cdot v\mathbf{v}_4)^2 &= (-v(\mathbf{u}_4 \cdot \mathbf{v}_4))^2 \\ &= \left(-v\left(\frac{\sqrt{2}}{2}(0,1,0,-1,0)\cdot(-1,-1,0,-3,0)\right)\right)^2 \\ &= \left(-v\left(\sqrt{2}\right)\right)^2 \\ &= 2v^2\end{aligned}$$

Part B

Problem: Find a v_0 such that the following is true:

$$((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v_0\mathbf{v}_4)\cdot\mathbf{u}_4)^2 = 0$$

Solution: Using the simplified expression we found above we can simply solve for v_0:

$$2v_0^2 = 0 \to \boxed{v_0 = 0}$$

Exercise 5

Part A

Problem: Show that the following expression depends only on u and v:

$$((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4)\cdot\mathbf{u}_3)^2$$

Solution: Once again we distribute the dot product:

$$((\mathbf{u}_3\cdot\mathbf{x}_0) + (\mathbf{u}_3\cdot s\mathbf{u}_1) + (\mathbf{u}_3\cdot t\mathbf{u}_2) - (\mathbf{u}_3\cdot u\mathbf{v}_3) - (\mathbf{u}_3\cdot v\mathbf{v}_4))^2$$

For the same reasoning given in Exercise 2 and 3, we can say:

$$(\mathbf{u}_3\cdot\mathbf{x}_0) = (\mathbf{u}_3\cdot s\mathbf{u}_1) = (\mathbf{u}_3\cdot t\mathbf{u}_2) = 0$$

This leaves us with the following expression:

$$(-\mathbf{u}_3\cdot u\mathbf{v}_3 - \mathbf{u}_3\cdot v\mathbf{v}_4)^2$$

This expression would seem to be dependent on u and v but, if we evaluate $\mathbf{u}_3\cdot\mathbf{v}_4$ we see that it equals 0. As a result we are left with the following:

$$\left(-\frac{2}{3}u\right)^2 = \frac{4}{9}u^2$$

*The question erroneously (I assume) states that the expression is dependent on both u **and** v when it is not dependent on v.*

Part B

Problem: Find a u_0 such that the following is true:

$$((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u_0\mathbf{v}_3 - v_0\mathbf{v}_4) \cdot \mathbf{u}_3)^2 = 0$$

Where v_0 is the same as in Exercise 4.

Solution: Using the simplified expression we found above we can simply solve for u_0 (*due to the error we need not assume $v_0 = 0$*):

$$\frac{4}{9}u_0^2 = 0 \rightarrow \boxed{u_0 = 0}$$

Exercise 6

Part A

Problem: Show that the following expression depends only on t, u and v:

$$((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_2)^2$$

Solution: Once again we distribute the dot product:

$$((\mathbf{u}_2 \cdot \mathbf{x}_0) + (\mathbf{u}_2 \cdot s\mathbf{u}_1) + (\mathbf{u}_2 \cdot t\mathbf{u}_2) - (\mathbf{u}_2 \cdot u\mathbf{v}_3) - (\mathbf{u}_2 \cdot v\mathbf{v}_4))^2$$

Because they are part of the same orthonormal set we can say:

$$(\mathbf{u}_2 \cdot s\mathbf{u}_1) = (\mathbf{u}_2 \cdot t\mathbf{u}_2) = 0$$

Leaving us with:

$$((\mathbf{u}_2 \cdot \mathbf{x}_0) - (\mathbf{u}_2 \cdot u\mathbf{v}_3) - (\mathbf{u}_2 \cdot v\mathbf{v}_4))^2$$

This expression would seem to be dependent on u and v but, if we evaluate $\mathbf{u}_2 \cdot \mathbf{v}_4$ we see that it equals 0. As a result we are now left with the following:

$$((\mathbf{u}_2 \cdot \mathbf{x}_0) - (\mathbf{u}_2 \cdot u\mathbf{v}_3))^2 = \left(1 + \frac{4}{3}u\right)^2$$

The question erroneously (I assume again) states that the expression is dependent on t, u and v when it is not dependent on t or v.

Part B

Problem: Find a t_0 such that the following is true:

$$((\mathbf{x}_0 + s\mathbf{u}_1 + t_0\mathbf{u}_2 - u_0\mathbf{v}_3 - v_0\mathbf{v}_4) \cdot \mathbf{u}_2)^2 = 0$$

Where u_0 and v_0 are the same as in Exercise 5.

Solution: *This question assumes $u = u_0$ but that value is 0 which cannot satisfy the equation. This is probably a compounding typo (where do they end) so I guess I'll just solve for u.*

Using the simplified expression we found above we can solve for u_0:

$$\left(1 + \frac{4}{3}u\right)^2 = 0 \rightarrow \boxed{u_0 = \frac{-3}{4}}$$

Exercise 7

Part A

Problem: Show that the following expression depends only on s, u and v:

$$((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_1)^2$$

Solution: Once again we distribute the dot product:

$$((\mathbf{u}_1 \cdot \mathbf{x}_0) + (\mathbf{u}_1 \cdot s\mathbf{u}_1) + (\mathbf{u}_1 \cdot t\mathbf{u}_2) - (\mathbf{u}_1 \cdot u\mathbf{v}_3) - (\mathbf{u}_1 \cdot v\mathbf{v}_4))^2$$

Because they are part of the same orthonormal set we can say:

$$(\mathbf{u}_1 \cdot s\mathbf{u}_1) = (\mathbf{u}_1 \cdot t\mathbf{u}_2) = 0$$

Leaving us with:

$$((\mathbf{u}_1 \cdot \mathbf{x}_0) - (\mathbf{u}_1 \cdot u\mathbf{v}_3) - (\mathbf{u}_1 \cdot v\mathbf{v}_4))^2$$

Evaluating the above we find:

$$\left(1 - \frac{4}{3}u + 3v\right)^2$$

Part B

Problem: Find a s_0 such that the following is true:

$$((\mathbf{x}_0 + s_0\mathbf{u}_1 + t\mathbf{u}_2 - u_0\mathbf{v}_3 - v_0\mathbf{v}_4) \cdot \mathbf{u}_1)^2 = 0$$

Where u_0 and v_0 are the same as in Exercise 5.

Solution: *This problem can't be salvaged. The expression isn't dependent on s and cannot even be true under both assumptions. I essentially made up another problem. I know there are typos but I cannot infer what they are since many are plausible.*

Using the simplified expression we found above we can simply solve for u_0 (*due to the error I'll just assume $u_0 = 0$*):

$$\left(1 - \frac{4}{3}u_0 + 3v\right)^2 = 0 \rightarrow \boxed{v_0 = \frac{1}{3}}$$

Exercise 8

Problem: Find the distance between the plane P given in Exercise 1 and plane parameterized by the following:

$$\mathbf{x}_2(u, v) = u\mathbf{v}_3 + v\mathbf{v}_4$$

Solution: Given the preamble on the challenge sheet preceding Exercise 3, we know that the sum of all the expressions in Exercises 3-7 is the expression for the distance between the planes that we must minimize. That sum is:

$$0 + 2v^2 + \frac{4}{9}u^2 + \left(1 + \frac{4}{3}u\right)^2 + \left(1 - \frac{4}{3}u + 3v\right)^2$$

After much tedious simplification, we are left to minimize the following quadratic polynomial over two variables:

$$4u^2 - 8uv + 6v + 11v^2 + 2$$

To minimize this we find the partial derivatives with respect to u and v and set the equal to 0:

$$\frac{\partial}{\partial u}4u^2 - 8uv + 6v + 11v^2 + 2 = 8u - 8v$$
$$\frac{\partial}{\partial v}4u^2 - 8uv + 6v + 11v^2 + 2 = 22v - 8u + 6$$

Now we set them equal to 0 and solve the system of equations:

$$8u - 8v = 0$$
$$22v - 8u + 6 = 0$$

After solving the system we arrive at $u = \frac{-3}{7}$ and $v = \frac{-3}{7}$. Plugging this into our original quadratic we see that the distance between the planes is $\frac{5}{7}$. We can find the points that minimize this distance by simply plugging u and v into $\mathbf{x}_2$ and... *this is where I realized that the distance between two points on the planes is (given the botched questions) dependent only on the second plane. This means that any point on P would suffice which makes no sense.*

Exercise 9

Part A

Problem: Apply the Gram-Schmidt algorithm to the vectors $\{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$ in that order to produce the orthonormal set $\{\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \tilde{\mathbf{u}}_3, \tilde{\mathbf{u}}_4, \tilde{\mathbf{u}}_5\}$.

Solution: The process was already shown in Exercise 1. I'll show the bare minimum of calculations here.

For $\tilde{\mathbf{u}}_1$:

$$\tilde{\mathbf{u}}_1 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{2}(0, 1, -1, 1, -1)$$

For $\tilde{\mathbf{u}}_2$:

$$\mathbf{w}_2 = \mathbf{v}_4 - (\mathbf{v}_4 \cdot \tilde{\mathbf{u}}_1)\tilde{\mathbf{u}}_1 = (-1, 0, -1, -2, -1)$$

$$\tilde{\mathbf{u}}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{\sqrt{7}}{7}(-1, 0, -1, -2, -1)$$

For $\tilde{\mathbf{u}}_3$:

$$\mathbf{w}_3 = \mathbf{v}_1 - (\mathbf{v}_1 \cdot \tilde{\mathbf{u}}_1)\tilde{\mathbf{u}}_1 - (\mathbf{v}_1 \cdot \tilde{\mathbf{u}}_2)\tilde{\mathbf{u}}_2 = \frac{1}{7}(2, 1, 2, -3, 2)$$

$$\tilde{\mathbf{u}}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{\sqrt{70}}{70}(2, 7, 2, -3, 2)$$

For $\tilde{\mathbf{u}}_4$:

$$\mathbf{w}_4 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \tilde{\mathbf{u}}_1)\tilde{\mathbf{u}}_1 - (\mathbf{v}_2 \cdot \tilde{\mathbf{u}}_2)\tilde{\mathbf{u}}_2 - (\mathbf{v}_2 \cdot \tilde{\mathbf{u}}_3)\tilde{\mathbf{u}}_3 = \frac{1}{5}(4, -1, -1, -1, -1)$$

$$\tilde{\mathbf{u}}_4 = \frac{\mathbf{w}_4}{\|\mathbf{w}_4\|} = \frac{\sqrt{5}}{10}(4, -1, -1, -1, -1)$$

For $\tilde{\mathbf{u}}_5$:

$$\mathbf{w}_5 = \mathbf{v}_5 - (\mathbf{v}_5 \cdot \tilde{\mathbf{u}}_1)\tilde{\mathbf{u}}_1 - (\mathbf{v}_5 \cdot \tilde{\mathbf{u}}_2)\tilde{\mathbf{u}}_2 - (\mathbf{v}_5 \cdot \tilde{\mathbf{u}}_3)\tilde{\mathbf{u}}_3 - (\mathbf{v}_5 \cdot \tilde{\mathbf{u}}_4)\tilde{\mathbf{u}}_4 = (0, 0, 1, 0, -1)$$

$$\tilde{\mathbf{u}}_5 = \frac{\mathbf{w}_5}{\|\mathbf{w}_4\|} = \frac{\sqrt{2}}{2}(0, 0, 1, 0, -1)$$

Part B

Problem: Find a system of equations for $\mathrm{span}(\{\mathbf{v}_3, \mathbf{v}_4\})$.

Solution: Using the orthonormal base we created in Part A, we can simply use all the vectors that are orthogonal to both $\mathbf{v}_3$ and $\mathbf{v}_4$ just like we did in Exercise 2. (i.e. need 3 orthogonal vectors in 5-space):

$$\begin{aligned}
\tilde{\mathbf{u}}_3 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \\
\tilde{\mathbf{u}}_4 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \\
\tilde{\mathbf{u}}_5 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0
\end{aligned}$$