Set Theory HW #2

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Problem 1

Exercises 5,7,10 from page 26 in the textbook.

Exercise 5: Assume that every member of A is a subset of B. Show that $\bigcup A \subseteq B$.

Solution: Consider an arbitrary set a, by the axiom of union we have:

$$a \in \bigcup A \implies \exists b \in A\, (a \in b)$$

And by the question's assumption, b is a subset of B. Putting these two together we have:

$$((a \in b)\ \&\ (b \subseteq B)) \implies a \in B \qquad \text{(def. of subset)}$$

And thus we have shown that for any $a \in \bigcup A$, the set a must also be an element of B. By the definition of subset, we have $\bigcup A \subseteq B$.

Exercise 7: Show that for any two sets A and B the following holds:

a) $\mathfrak{P}A \cap \mathfrak{P}B = \mathfrak{P}(A \cap B)$

b) $\mathfrak{P}A \cup \mathfrak{P}B \subseteq \mathfrak{P}(A \cup B)$. Under what conditions does equality hold?

Solution: a) Consider an arbitrary set x and note the following chain of logical equivalences:

$$\begin{aligned}
x \in \mathfrak{P}A \cup \mathfrak{P}B &\iff x \in \mathfrak{P}A\ \&\ x \in \mathfrak{P}B && \text{(def. of intersection)}\\
&\iff x \subseteq A\ \&\ x \subseteq B && \text{(def. power set)}\\
&\iff x \subseteq A \cap B && \text{(def. of intersection)}\\
&\iff x \in \mathfrak{P}(A \cap B) && \text{(def. power set)}
\end{aligned}$$

And so, by extensionality, we have $\mathfrak{P}A \cap \mathfrak{P}B = \mathfrak{P}(A \cap B)$.
b) Consider an arbitrary set x and note the following chain of implications:

$$\begin{aligned}
x \in \mathfrak{P}A \cup \mathfrak{P}B &\iff x \in \mathfrak{P}A \text{ or } x \in \mathfrak{P}B && \text{(def. of union)}\\
&\iff x \subseteq A \text{ or } x \subseteq B && \text{(def. power set)}\\
&\implies x \subseteq A \cup B && \text{(def. of union)}\\
&\iff x \in \mathfrak{P}(A \cap B) && \text{(def. power set)}
\end{aligned}$$

And so, by the definition of subset, we have $\mathfrak{P}A \cup \mathfrak{P}B \subseteq \mathfrak{P}(A \cup B)$. You'll notice that on line 3 we have an implication rather than an iff. We can only make this an iff, and thus establish equality of the two sets, if we assume that $A \subseteq B$ or $B \subseteq A$.

Exercise 10: Prove that if $a \in B$, then $\mathfrak{P}a \in \mathfrak{P}\mathfrak{P}\bigcup B$.

Solution: Consider an arbitrary set a and note the following:

$$\begin{aligned}
a \in B &\implies \forall t(t \in a \implies t \in \bigcup B) && \text{(axiom of union)}\\
&\implies a \subseteq \bigcup B && \text{(def. of subset)}\\
&\implies (a \subseteq \bigcup B) \ \&\ \forall t(t \in \mathfrak{P}a \implies t \subseteq a) && \text{(def. of powerset)}\\
&\implies \forall t(t \in \mathfrak{P}a \implies t \subseteq \bigcup B) && \text{(transitivity of subset)}\\
&\implies \forall t(t \in \mathfrak{P}a \implies t \in \mathfrak{P}\bigcup B) && \text{(def. of power set)}\\
&\implies \mathfrak{P}a \subseteq \mathfrak{P}\bigcup B && \text{(def. of subset)}\\
&\implies \mathfrak{P}a \in \mathfrak{P}\mathfrak{P}\bigcup B && \text{(def. of power set)}
\end{aligned}$$

Problem 2

Exercises 12,20,22,35 from pages 32-33 in the textbook.

Exercise 12: Verify the following identity:

$$C - (A \cup B) = (C - A) \cup (C - B)$$

Solution: For the following set of equalities, the complement is taken with respect to the universe $A \cup B \cup C$:

$$\begin{aligned}
C - (A \cap B) &= C \cap (A \cap B)^{\complement} && \text{(relative complement)}\\
&= C \cap (B^{\complement} \cup A^{\complement}) && \text{(DeMorgan's Law)}\\
&= (C \cap B^{\complement}) \cup (C \cap A^{\complement}) && \text{(distributivity of intersction)}\\
&= (C - B) \cup (C - A) && \text{(relative complement)}
\end{aligned}$$

Exercise 20: Let A, B and C be sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.

Solution: Consider an $x \in B$. There are two cases which exhaust all possibilities:

$$\begin{aligned}
x \in A &\implies x \in A \cap B\\
&\iff x \in A \cap C && \text{(assumption)}\\
&\implies x \in C
\end{aligned}$$

and the other case:

$$\begin{aligned} x \notin A &\implies x \in A \cup B \\ &\iff x \in A \cup C && \text{(assumption)} \\ &\implies x \in C \end{aligned}$$

This gives us $x \in B \implies x \in C$, and by replacing all occurrences of B with C we have an argument for the reverse direction. Putting these together we have $B = C$.

Exercise 22: Show that if A and B are nonempty sets, then $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$.

Solution: Note that by the axiom of union we have:

$$\begin{aligned} x \in \bigcap(A \cup B) &\implies (\forall y \in A \cup B)\, x \in y && \text{(def. of arbitrary intersection)} \\ &\implies (\forall y \in A)\, x \in y && (A \subseteq A \cup B) \\ &\iff x \in \bigcap A && \text{(def. of arbitrary intersection)} \end{aligned}$$

Similarly we have:

$$\begin{aligned} x \in \bigcap(A \cup B) &\implies (\forall y \in A \cup B)\, x \in y && \text{(def. of arbitrary intersection)} \\ &\implies (\forall y \in B)\, x \in y && (B \subseteq A \cup B) \\ &\iff x \in \bigcap B && \text{(def. of arbitrary intersection)} \end{aligned}$$

Putting these together we have:

$$\begin{aligned} x \in \bigcap(A \cup B) &\implies x \in \bigcap A \;\&\; x \in \bigcap B \\ &\implies x \in \bigcap A \cap \bigcap B && \text{(def. of intersection)} \end{aligned}$$

This proves one direction. The other direction can be proved by first recalling that:

$$\begin{aligned} x \in \bigcap A \cap \bigcap B &\implies x \in \bigcap A \;\&\; x \in \bigcap B \\ &\implies (\forall y \in A)\, x \in y \;\&\; (\forall y \in B)\, x \in y \\ &\implies (\forall y \in A)\, x \in y \text{ or } (\forall y \in B)\, x \in y \end{aligned}$$

This allows us to state the following:

$$\begin{aligned} & (\forall t \in A \cup B)\, t \in A \text{ or } t \in B && \text{(def. of union)} \\ \implies & (\forall t \in A \cup B)\, x \in t && \text{(above section)} \\ \implies & x \in \bigcap(A \cup B) && \text{(def. of arbitrary intersection)} \end{aligned}$$

And with both sides of the implication proved, the equality holds true.

Exercise 35: Assume that $\mathfrak{P}A = \mathfrak{P}B$. Prove that $A = B$.

Solution: Consider an arbitrary set x and note the following chain of implications:

$$
\begin{aligned}
x \in A &\implies \{x\} \subseteq A && \text{(def. of subset)} \\
&\implies \{x\} \in \mathfrak{P}A && \text{(def. of powerset)} \\
&\iff \{x\} \in \mathfrak{P}B && \text{(assumption)} \\
&\implies \{x\} \subseteq B && \text{(def. of powerset)} \\
&\implies x \in B && \text{(def. of subset)}
\end{aligned}
$$

This gives us $x \in A \implies x \in B$, and by replacing all occurrences of A with B we have an argument for the reverse direction. Putting these together we have $A = B$.

Problem 3

Exercises 32,33,36 from pages 33-34 in the textbook.

Exercise 32: Let S be the set $\{\{a\}, \{a, b\}\}$. Evaluate and simplify:

a) $\bigcup\bigcup S$

b) $\bigcap\bigcap S$

c) $\bigcap\bigcup S \cup (\bigcup\bigcup S - \bigcup\bigcap S)$

Solution: For a) we have:

$$
\begin{aligned}
\bigcup\bigcup S &= \bigcup\bigcup\{\{a\}, \{a, b\}\} \\
&= \bigcup\{a, b\} \\
&= a \cup b
\end{aligned}
$$

For b) we have:

$$
\begin{aligned}
\bigcap\bigcap S &= \bigcap\bigcap\{\{a\}, \{a, b\}\} \\
&= \bigcap\{a\} \\
&= a
\end{aligned}
$$

For c) we have:

$$
\begin{aligned}
\bigcap\bigcup S \cup \left(\bigcup\bigcup S - \bigcup\bigcap S\right) &= \bigcap\{a, b\} \cup \left(\bigcup\{a, b\} - \bigcup\{a\}\right) \\
&= (a \cup b) \cup ((a \cup b) - a) \\
&= (a \cup b) \cup (b - a) \\
&= b
\end{aligned}
$$

Exercise 33: With S as in the preceding exercise, evaluate $\bigcup(\bigcup S - \bigcap S)$ when $a \neq b$ and when $a = b$.

Solution: Evaluating the expression we arrive at:

$$\bigcup\left(\bigcup S - \bigcap S\right) = \bigcup(\{a, b\} - \{a\})$$

For the case that $a \neq b$ we have:

$$\bigcup(\{a, b\} - \{a\}) = \bigcup\{b\} = b$$

For the case that $a = b$ we have:

$$\bigcup(\{a, b\} - \{a\}) = \bigcup(\{a\} - \{a\}) = \bigcup\emptyset = \emptyset$$

Exercise 36: Verify that for all sets A and B the following are correct:

a) $A - (A \cap B) = A - B$

b) $A - (A - B) = A \cap B$

Solution: For both a) an b) the complement is taken with respect to the universe $A \cup B$. For a) we have:

$$\begin{aligned}
A - (A \cap B) &= A \cap (A \cap B)^{\mathsf{C}} && \text{(relative complement)}\\
&= A \cap (A^{\mathsf{C}} \cup B^{\mathsf{C}}) && \text{(DeMorgan's Law)}\\
&= (A \cap A^{\mathsf{C}}) \cup (A \cap B^{\mathsf{C}}) && \text{(distributivity of intersection)}\\
&= (A - A) \cup (A - B) && \text{(relative complement)}\\
&= \emptyset \cup (A - B)\\
&= A - B
\end{aligned}$$

For b) we have:

$$\begin{aligned}
A - (A \cap B) &= A \cap (A - B)^{\mathsf{C}} && \text{(relative complement)}\\
&= A \cap (A \cap B^{\mathsf{C}})^{\mathsf{C}} && \text{(relative complement)}\\
&= A \cap (A^{\mathsf{C}} \cup B) && \text{(DeMorgan's Law)}\\
&= (A \cap A^{\mathsf{C}}) \cup (A \cap B) && \text{(distributivity of intersection)}\\
&= \emptyset \cup (A \cap B)\\
&= A \cap B
\end{aligned}$$