# Honors Calculus III HW #9

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## Exercise 1

**Problem:** Using the mean value theorem, prove Cauchy's Mean value theorem. That is, for two functions $f, g$ continuous on the interval $[a, b]$ and differentiable on $(a, b)$, there exists a $c$ with $a < c < b$ such that:

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$$

**Solution:** First let us consider the following function:

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

Note that this function is also continuous and differentiable on the same intervals as $f$ and $g$. Also note that the mean value theorem guarantees that for some $c$ that satsifies $a < c < b$:

$$h'(c) = \frac{h(b) - h(a)}{b - a}$$

However, notice that when evaluating $h(b) - h(a)$ we arrive at:

$$\begin{aligned} h(b) - h(a) &= f(b)g(b) - f(b)g(a) - g(b)f(b) + g(b)f(a) \\ &\quad - f(a)g(b) - f(a)g(a) - g(a)f(b) + g(a)f(a) \\ &= 0 \end{aligned}$$

And since the numerator is zero the whole of $h'(c) = 0$. Also notice that when we differentiate $h(x)$ and plug in $c$ we get the following:

$$h'(c) = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a))$$

But because $h'(c) = 0$ we can say:

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$

or if the denominators aren't zero:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

And so the $c$ we knew existed via the MVT satisfies the more general Cauchy MVT.

## Exercise 2

**Problem:** Consider a continuous function $f$ on an open interval containing $[a, b]$ where $f'$ and $f''$ exist on $(a, b)$. Show that if $x \in (a, x)$ then there exists a $c \in (a, x)$ such that:

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(c)(x - a)^2$$

**Solution:** First consider the following continuous and differentiable functions, where $a, b$ and $f$ are the same as above:

$$h(x) = f(x) - f(a) - f'(a)(x - a)$$
$$g(x) = (x - a)^2$$

Cauchy's MVT guarantees that there exists some $d$ such that:

$$\frac{h(x) - h(a)}{g(x) - g(a)} = \frac{h'(d)}{g'(d)}$$

Note that $d$ satisfies $a < d < x$. Now we if we apply Cauchy's MVT again, but this time on the functions $h'(x)$ and $g'(x)$ over the interval $[a, d]$, we get:

$$\frac{h'(x) - h'(a)}{g'(x) - g'(a)} = \frac{h''(c)}{g''(c)}$$

Since $h'(a) = g'(a) = 0$ transitivity gives us:

$$\frac{h(x) - h(a)}{g(x) - g(a)} = \frac{h''(c)}{g''(c)}$$

Also note that $a$ is clearly a root of both $h$ and $g$ leaving us with:

$$\frac{h(x)}{g(x)} = \frac{h''(c)}{g''(c)}$$

First we will list the second derivatives of $h$ and $g$:

$$h'(x) = f'(x) - f'(a)$$
$$h''(x) = f''(x)$$
$$g'(x) = 2(x - a)$$
$$g''(x) = 2$$

Plugging these in we find:

$$\frac{f(x) - f(a) - f'(a)(x - a)}{(x - a)^2} = \frac{f''(c)}{2}$$
$$f(x) - f'(a)(x - a) = \frac{f''(c)}{2}(x - a)^2$$
$$f(x) = f'(a)(x - a) + \frac{f''(c)}{2}(x - a)^2$$

And we are done.

# Exercise 3

## Part a

**Problem:** Suppose a function $f$ has continous second partials. Using Taylor's Theorem, show that there exists an $s \in (0, t)$ that satisfies the following equation on the interval $[0, t]$ for some choice of $\mathbf{x}_0$ and $\mathbf{v}$:

$$f(\mathbf{x}_0 + t\mathbf{v}) = f(\mathbf{x}_0) + t(\nabla f(\mathbf{x}_0) \cdot \mathbf{v}) + \frac{t^2}{2} Hf(\mathbf{x}_0 + s\mathbf{v})\mathbf{v} \cdot \mathbf{v}$$

**Solution:** Taylor's Theorem tells us that for some $s \in (0, a)$:

$$g_{\mathbf{v}}(t) = g_{\mathbf{v}}(0) + g'_{\mathbf{v}}(0)(t - 0) + \frac{1}{2} g''_{\mathbf{v}}(s)$$

Now recall that the first derivative of a function from $\mathbb{R}^n \to \mathbb{R}^m$ parameterized by a line (i.e. let $t$ vary in $\mathbf{x}_0 + t\mathbf{v}$) is given by it's Jacobian matrix applied to $\mathbf{v}$. Also note that the Jacobian reduces to the gradient when $m = 1$ (i.e. an $n \times 1$ Jacobian):

$$g'_{\mathbf{v}}(t) = Jf(\mathbf{x}_0 + t\mathbf{v})\mathbf{v} = \nabla f(\mathbf{x}_0 + t\mathbf{v}) \cdot \mathbf{v}$$

Also note that the second derivative of a function from $\mathbb{R}^n \to \mathbb{R}$ is given by the Hessian matrix (i.e. the Jacobian of the Jacobian). So, just applying the same process as above twice over we get:

$$g''_{\mathbf{v}}(t) = J(Jf(\mathbf{x}_0 + t\mathbf{v})\mathbf{v})\mathbf{v} = J(\nabla f(\mathbf{x}_0 + t\mathbf{v}) \cdot \mathbf{v})\mathbf{v} = Hf(\mathbf{x}_0 + t\mathbf{v})\mathbf{v} \cdot \mathbf{v}$$

And so now if we simply plug $f$ back in for $g$, and do a little bit of simplifying, we indeed find:

$$f(\mathbf{x}_0 + t\mathbf{v}) = f(\mathbf{x}_0) + t(\nabla f(\mathbf{x}_0) \cdot \mathbf{v}) + \frac{t^2}{2} Hf(\mathbf{x}_0 + s\mathbf{v})\mathbf{v} \cdot \mathbf{v}$$

## Part b & c

**Problem:** Rewrite the above expression with the following substitutions:

$$\begin{aligned} \mathbf{x}_0 + t\mathbf{v} = \mathbf{x} &= \mathbf{x}_0 + |\mathbf{x} - \mathbf{x}_0| \left( \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \right) \\ s &= r|\mathbf{x} - \mathbf{x}_0| \end{aligned}$$

Then manipulate the resulting expression to the following:

$$\begin{aligned} &f(\mathbf{x}) - f(\mathbf{x}_0) - (\nabla f(\mathbf{x}_0) \cdot \mathbf{x} - \mathbf{x}_0) - \frac{1}{2} Hf(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \\ &\quad = \frac{1}{2}(Hf(\mathbf{x}_0 + r(\mathbf{x} - \mathbf{x}_0)) - Hf(\mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)) \end{aligned}$$

**Solution:** Plugging in these values we see:

$$
\begin{aligned}
f(\mathbf{x}) &= f(\mathbf{x}_0) + |\mathbf{x} - \mathbf{x}_0|(\nabla f(\mathbf{x}_0) \cdot \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}) + \frac{t^2}{2} Hf(\mathbf{x}_0 + s\mathbf{v})\mathbf{v} \cdot \mathbf{v} \\
&= f(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0) \cdot \mathbf{x} - \mathbf{x}_0) + \frac{|\mathbf{x} - \mathbf{x}_0|^2}{2} Hf\left(\mathbf{x}_0 + r|\mathbf{x} - \mathbf{x}_0|\left(\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}\right)\right)\left(\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}\right) \cdot \left(\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}\right) \\
&= f(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0) \cdot \mathbf{x} - \mathbf{x}_0) + \frac{1}{2} Hf(\mathbf{x}_0 + r(\mathbf{x} - \mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)
\end{aligned}
$$

Now let us manipulate just the last term:

$$
\begin{aligned}
&\frac{1}{2} Hf(\mathbf{x}_0 + r(\mathbf{x} - \mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \\
&= \frac{1}{2}(Hf(\mathbf{x}_0) + Hf(r(\mathbf{x} - \mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0)) \cdot (\mathbf{x} - \mathbf{x}_0) \\
&= \frac{1}{2}(Hf(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + Hf(r(\mathbf{x} - \mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0))) \cdot (\mathbf{x} - \mathbf{x}_0) \\
&= \frac{1}{2} Hf(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2} Hf(r(\mathbf{x} - \mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)) \\
&= \frac{1}{2} Hf(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(Hf(r(\mathbf{x} - \mathbf{x}_0)) + Hf(\mathbf{x}_0) - Hf(\mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)) \\
&= \frac{1}{2} Hf(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(Hf(\mathbf{x}_0 + r(\mathbf{x} - \mathbf{x}_0)) - Hf(\mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0))
\end{aligned}
$$

Now if we just plug this back in and subtract the first 3 terms to the left hand side we are left with:

$$
\begin{aligned}
f(\mathbf{x}) - f(\mathbf{x}_0) - (\nabla f(\mathbf{x}_0) \cdot \mathbf{x} - \mathbf{x}_0) - \frac{1}{2} Hf(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \\
= \frac{1}{2}(Hf(\mathbf{x}_0 + r(\mathbf{x} - \mathbf{x}_0)) - Hf(\mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0))
\end{aligned}
$$

## Part d

**Problem:** Use the above result and the Cauchy-Schwartz Inequality to show that:

$$
\frac{f(\mathbf{x}) - f(\mathbf{x}_0) - (\nabla f(\mathbf{x}_0) \cdot \mathbf{x} - \mathbf{x}_0) - \frac{1}{2} Hf(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|^2} \leq \frac{1}{2}|Hf(\mathbf{x}_0 + r(\mathbf{x} - \mathbf{x}_0)) - Hf(\mathbf{x}_0)|
$$

**Solution:** The Cauchy-Schwartz Inequality tells us that $|A\mathbf{v}| \leq |A||\mathbf{v}|$. So if we just apply this twice, once for the matrix multiply by $(\mathbf{x} - \mathbf{x}_0)$ and once for the dot product by $(\mathbf{x} - \mathbf{x}_0)$, which is just matrix multiplication transposed, we can divide the left hand side by this norm twice and are left with the desired statement.

## Part e

**Problem:** Prove continuous second partials imply second order differentiability:

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - (\nabla f(\mathbf{x}_0)\cdot \mathbf{x}-\mathbf{x}_0) - \frac{1}{2}Hf(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^2} = 0$$

**Solution:** Recall that we have already shown:

$$0 \le \cdots \le \frac{1}{2}|Hf(\mathbf{x}_0 + r(\mathbf{x}-\mathbf{x}_0)) - Hf(\mathbf{x}_0)|$$

Also note that the limit of the right-hand side as $\mathbf{x}\to\mathbf{x}_0$ is 0:

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{1}{2}|Hf(\mathbf{x}_0 + r(\mathbf{x}-\mathbf{x}_0)) - Hf(\mathbf{x}_0)| = \frac{1}{2}|Hf(\mathbf{x}_0) - Hf(\mathbf{x}_0)| = 0$$

And so by the squeeze theorem the desired statement is true. (this automatically implies the statement on the sheet, we just didn't need an explicit construction of $\delta$)