Theory of Probability HW #1

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Part A

Problems taken from Chapters 1 and 2 of the textbook.

Chapter 1

Problem 10

Problem: In how many ways can 8 people be seated in a row if **a)** there are no restrictions on the seating arrangement? **b)** persons A and B must sit next to each other? **c)** there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?

Solution: a) With no restrictions, there are 8! permutations of a group of 8 people and so the number of seating arrangements is given by:

$$8! = 40320$$

b) If two particular people A and B must sit next to each other, then we can consider them a single unit. And so we have 7! permutations of the AB group and the remaining 6 people. We then multiply this by 2 since we have a choice of either AB or BA:

$$2 \cdot 7! = 10080$$

c) Disregarding the identities of the particular men and women, there are only two valid arrangements of them:

$$mwmwmwmw$$
$$wmwmwmwm$$

And for each of these two arrangements, we have to choose how to arrange the 4 men and 4 women into them, giving us:

$$2 \cdot 4! \cdot 4! = 1152$$

Problem 19

Problem: Seven different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

Solution: First we choose which set of 7 children get gifts from the total group of 10. Then we choose a particular gift for each of those 7 children. This is equivalent to:

$$\underbrace{\binom{10}{7}}_{\substack{\text{Who gets}\\ \text{presents}}} \overbrace{7!}^{\substack{\text{Which presents}\\ \text{they get}}} = 604800$$

Problem 22

Problem: A person has 8 friends, of whom 5 will be invited to a party. How many choices of invitees are there if 2 of the friends are feuding and will not attend together? How many choices are there if 2 of the friends will only attend together?

Solution: The first question is given by the total number of combinations of friends minus the number of combinations that have the two friends:

$$\underbrace{\binom{8}{5}}_{\substack{\text{all combos}\\ \text{of friends}}} - \underbrace{\overbrace{\binom{2}{2}}^{\substack{\text{feuding}\\ \text{friends}}} \overbrace{\binom{6}{3}}^{\substack{\text{the}\\ \text{rest}}}}_{\text{impossible cases}} = 36$$

The second question is given by the number of combinations where the two close friends show up together plus the number of combinations where they do not:

$$\underbrace{\binom{6}{5}}_{\substack{\text{close friends}\\ \text{don't show}}} + \underbrace{\overbrace{\binom{2}{2}}^{\substack{\text{close}\\ \text{friends}}} \overbrace{\binom{6}{3}}^{\substack{\text{the}\\ \text{rest}}}}_{\substack{\text{close friends}\\ \text{show up}}} = 26$$

Chapter 2

Problem 10

Problem: 60% of the students at a school wear wear neither a ring nor a necklace. 20% wear a ring and 30% wear a necklace. What's the probability that any given student is wearing a ring *or* a necklace? What about a ring *and* a necklace?

Solution: Letting R be the event that a student wears a ring, and N the event they wear a necklace, the problem statement tells us that the event a student wears neither (i.e. $R^{\mathsf{C}}N^{\mathsf{C}}$) is 0.6. This implies that the probability a student wears a ring or necklace (i.e. $R \cup N$) is given by:

$$P((R^{\mathsf{C}}N^{\mathsf{C}})^{\mathsf{C}}) = P(R \cup N) = 1 - 0.6 = 0.4$$

We can find the probability that a student wears both (i.e. RN) by plugging in our known probabilities into the inclusion-exclusion principle and solving for $P(RN)$:

$$
\begin{aligned}
P(R \cup N) &= P(R) + P(N) - P(RN) \\
0.4 &= 0.2 + 0.3 - P(RN) \\
\implies P(RN) &= 0.1
\end{aligned}
$$

Problem 15

Problem: If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt two pairs? (This occurs when the cards have denominations a, a, b, b, c where a, b and c are all distinct.)

Solution: The number of combinations of valid hands is given by:

$$
\underbrace{\binom{13}{2}}_{\substack{\text{Pick 2 of 13} \\ \text{denominations}}} \overbrace{\binom{4}{2}}^{\substack{\text{Pick 2 suits} \\ \text{for pair } a}} \underbrace{\binom{4}{2}}_{\substack{\text{Pick 2 suits} \\ \text{for pair } b}} \overbrace{\binom{11}{1}\binom{4}{1}}^{\substack{\text{1 card of remaining} \\ \text{denominations}}} = 123552
$$

As this is a uniform distribution, putting the number of valid hands over the total number of hands gives us the probability:

$$
P(E) = \frac{123552}{\binom{52}{5}} = \frac{198}{4165}
$$

Problem 43

Problem: If n people including A and B, are randomly arranged in a line, what is the probability that A and B are next to each other? What if the the people were randomly arranged in a circle?

Solution: By considering AB as a single object, we are left with $(n-1)!$ permutations of AB and the rest of the $n-2$ people. However, since AB can also be internally arranged as BA we have the following number of valid arrangements:

$$
\overbrace{2}^{AB \leftrightarrow BA} \underbrace{(n-1)!}_{\substack{\text{arrangements of } AB/BA \\ \text{and } n-2 \text{ people}}}
$$

Since each arrangement is equally likely, putting this over the total number of arrangements $n!$ we have the following probability:

$$
P(E_n) = \frac{2(n-1)!}{n!} = \frac{2}{n}
$$

When we permutate n things in a circle, shifting the all elements of the circle by any number from 1 to n results in an equivalent permutation. As such the number of circular arrangements of n elements is $\frac{n!}{n} = (n-1)!$. This gives us, via the same reasoning as above, the following probability:

$$
P(E_{n\circ}) = P(E_{n-1}) = \frac{2(n-2)!}{(n-1)!} = \frac{2}{n-1}
$$

Problem 53

Problem: If 8 people, consisting of 4 couples, are randomly arranged in a row, what's the probability that no person is next to their partner?

Solution: Let's label the couples from 1 to 4 and let E_i be the event that the ith couple sit together. Let's first compute the cardinality of the event that at least 1 couple sits next to each other via the inclusion-exclusion principle:

$$\left|\bigcup_{i=1}^{4} E_i\right| = \sum_{k=1}^{4}\left(\sum_{\substack{I\subseteq[1..4]\\|I|=k}}\left|\bigcap_{i\in I} E_i\right|\right)$$

For any particular couple i, we can consider them a single group and, because there are now 7 groups and the couple has 2 internal rearrangements, the number of outcomes where they sit together is given by:

$$|E_i| = 2\cdot 7!$$

A similar argument holds for any two different couples i and j:

$$|E_iE_j| = 2\cdot 2\cdot 6! = 2^2\cdot 6!$$

And in general for any set I of k couples we have:

$$\left|\bigcap_{i\in I} E_i\right| = 2^k(8-k)!$$

Plugging this into our original equation we have:

$$\left|\bigcup_{i=1}^{4} E_i\right| = \sum_{k=1}^{4}\left(\sum_{\substack{I\subseteq[1..4]\\|I|=k}} 2^k(8-k)!\right)$$

And since there are $\binom{4}{k}$ subsets of couples of size k, we have:

$$\left|\bigcup_{i=1}^{4} E_i\right| = \sum_{k=1}^{4}\binom{4}{k}2^k(8-k)! = 26496$$

Now note that since there are 8! total arrangements and each one is equally likely (i.e. this is a discrete uniform distribution) we have the following probability:

$$P\left(\bigcup_{i=1}^{4} E_i\right) = \frac{\left|\bigcup_{i=1}^{4} E_i\right|}{|\Omega|} = \frac{26496}{8!} = \frac{23}{35}$$

This is the probability that at least 1 couple sits next to each other. The probability that none do is simply its complement. Thus, our desired probability is given by:

$$P\left(\left(\bigcup_{i=1}^{4} E_i\right)^{\complement}\right) = 1 - P\left(\bigcup_{i=1}^{4} E_i\right) = 1 - \frac{23}{35} = \frac{12}{35}$$

Part B

Problem a

For the next three parts, suppose that we roll a pair of fair six-sided dice. Before we answer the questions, let's establish that the sample space Ω of this experiment is given by:

$$\Omega = [1..6] \times [1..6]$$

Also note that, because this experiment has a discrete uniform distribution, the probability of an event E occurring is given by:

$$P(E) = \frac{|E|}{|\Omega|}$$

In this case $|\Omega|$ is, by the basic principle of counting, $6 \cdot 6 = 36$.

Part I: What is the probability that the sum of the upturned faces equals 6?

Solution: By simple enumeration, we can see that the outcomes that satisfy the event that the dice sum to 6, which we'll denote E_6, is given by:

$$E_6 = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

Now all we have to do is note that $|E_6| = 5$ and thus the probability of it occurring is:

$$P(E_6) = \frac{|E_6|}{|\Omega|} = \frac{5}{36}$$

Part II: What is the probability that the sum of the upturned faces equals 7?

Solution: Again, by simple enumeration, we find that the event that the dice sum to 7, denoted E_7, is given by:

$$E_7 = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

Again, we note that $|E_7| = 6$ and thus the probability of it occurring is:

$$P(E_7) = \frac{|E_7|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$

Part III: What is the probability that the sum of the upturned faces neither equals 6 nor 7?

Solution: Note that E_6 and E_7 are disjoint since two numbers can't simultaneously sum two to different numbers. As such we have the following via the disjoint addition axiom:

$$P(E_6 \cup E_7) = \frac{5}{36} + \frac{6}{36} = \frac{11}{36}$$

And since the event that neither sum is 6 or 7 is simply the complement of this event, we have:

$$P((E_6 \cup E_7)^{\mathtt{C}}) = 1 - \frac{11}{36} = \frac{25}{36}$$

Problem b

Problem: Suppose that you roll a pair of fair six-sided dice j times. Let E_j denote the event that 6 appears as a sum on the jth roll, but neither 6 nor 7 appear earlier. What is $P(Ej)$?

Solution: From part III we have that on any given roll the probability that neither a 6 or 7 is:

$$P((E_6 \cup E_7)^{\complement}) = \frac{25}{36}$$

And since each roll is independent of each other, the probability that this happens $j-1$ times in a row is given by $\left(\frac{25}{36}\right)^{j-1}$ and since $P(E_6) = \frac{5}{36}$ we have:

$$P(E_j) = \left(\frac{25}{36}\right)^{j-1} \frac{5}{36}$$

Problem c

Problem: Using the same definition of E_j from problem b, interpret the event $E = \bigcup_{j=1}^{\infty} E_j$ and compute $P(E)$.

Solution: E represents the event that a 6 is rolled after some finite amount of rolls, could be 0, where a 6 or 7 had never been rolled prior.

To calculate the probability of E we first note that:

$$(\forall i, j \in \mathbb{N})\, i \neq j \implies (E_i E_j = \emptyset)$$

That is to say $(E_j)_{j=1}^{\infty}$ is a sequence of disjoint events. As such, we can employ the axiom of disjoint addition in the following way:

$$\begin{aligned}
P(E) = P\left(\bigcup_{j=1}^{\infty} E_j\right) &= \sum_{j=1}^{\infty} P(E_j) && \text{(axiom of disjoint addition)} \\
&= \sum_{j=1}^{\infty} \left(\frac{25}{36}\right)^{j-1} \frac{5}{36} && \text{(def. of } P(E_j)) \\
&= \sum_{j=0}^{\infty} \left(\frac{25}{36}\right)^{j} \frac{5}{36} && \text{(change of index)} \\
&= \frac{\left(\frac{5}{36}\right)}{1 - \frac{25}{36}} = \frac{5}{11} && \text{(geometric series)}
\end{aligned}$$

And we are done.