Numerical Analysis HW #1

Ozaner Hansha

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Problem 1

Consider the following function:

$$f(x) = \frac{1 - \cos x}{x^2}$$

Part a

Problem: Prove the following:

$$\lim_{x \to 0} f(x) = \frac{1}{2}$$

Solution: First let us plug the Taylor expansion of $\cos x$ into $f(x)$:

$$\begin{aligned} f(x) &= \frac{1 - \cos x}{x^2} \\ &= \frac{1 - (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots)}{x^2} \\ &= \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots}{x^2} \\ &= \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots \end{aligned}$$

The limit as x approaches 0 of this series becomes clear:

$$\begin{aligned} \lim_{x \to 0} f(x) &= \frac{1 - \cos x}{x^2} \\ &= \lim_{x \to 0} \left(\sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n)!} (-1)^{n+1} \right) \\ &= \lim_{x \to 0} \left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots \right) \\ &= \frac{1}{2!} - 0 + 0 - \cdots = \frac{1}{2} \end{aligned}$$

Part b

Problem: Find $\alpha \in \mathbb{R}$ such that:

$$(\exists c \in \mathbb{R}) \lim_{x \to 0} \frac{|f(x) - 1/2|}{x^\alpha} = c \neq 0$$

Solution: Recall that in part a, $f(x)$ was shown to be equal to:

$$\begin{aligned} f(x) = \frac{1 - \cos x}{x^2} &= \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n)!}(-1)^{n+1} \\ &= \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots \end{aligned}$$

And so:

$$\begin{aligned} \frac{|f(x) - 1/2|}{x^\alpha} &= \frac{\left|\left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots\right) - 1/2\right|}{x^\alpha} \\ &= \frac{\left|-\frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} \cdots\right|}{x^\alpha} \end{aligned}$$

At this point it should be obvious the lowest degree term (which has the largest value as $x \to 0$) is 2. This leaves us with $\alpha = 2$:

$$\lim_{x \to 0} \frac{|f(x) - 1/2|}{x^2} = \lim_{x \to 0} \frac{\left|-\frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^4}{6!} \cdots\right|}{x^2} = \lim_{x \to 0} \left|-\frac{1}{4!} + \frac{x^2}{6!} - \frac{x^4}{8!} \cdots\right| = \frac{1}{4!}$$

Part c

Problem: Write a Matlab program to calculate $f(x)$ and its error $|f(x) - 1/2|$ for the values $10^{-1}, 10^{-2}, \cdots, 10^{-8}$.

Solution:

x	$f(x)$	$\lvert f(x) - 1/2 \rvert$
10^{-1}	4.9958e-01	4.1653e-04
10^{-2}	5.0000e-01	4.1667e-06
10^{-3}	5.0000e-01	4.1674e-08
10^{-4}	5.0000e-01	3.0387e-09
10^{-5}	5.0000e-01	4.1370e-08
10^{-6}	5.0004e-01	4.4450e-05
10^{-7}	4.9960e-01	3.9964e-04
10^{-8}	0	5.0000e-01

Part d

Problem: Why doesn't the error, and thus the result, converge in part c as it was proven to in part b?

Solution:

Problem 2

Consider the following function:

$$f(x) = x^3 + x - 4$$

Part a

Problem: Show that $f(x)$ has at least one root in the interval $[1, 4]$.

Solution: Recall the intermediate value theorem. That is, for any continous function $f : [a, b] \to \mathbb{R}$, the following must be true:

$$(\forall y \in [\min(f(a), f(b)), \max(f(a), f(b))]) \ (\exists c \in (a, b)) \ y = f(c)$$

Since a root of f is simply a value c such that $f(c) = 0$, we can simply evalaute the f at the bounds of the interval I to find:

$$f(1) = -2 \qquad f(4) = 64$$

And because:

$$-2 < 0 < 64$$

the intermediate value theorem holds and thus this function must have at least 1 root.

Part b

Problem: Show that $(\forall x \in \mathbb{R}) \ f'(x) > 0$. Use this to show that $f(x)$ has only 1 real root.

Solution: First note that the derivative of f is:

$$f'(x) = 3x^2 + 1$$

It is should be clear that this is strictly positive, but to spell it out:

$$\begin{aligned} x^2 &\geq 0 && \text{(even-valued exponentiation)} \\ 3x^2 &\geq 0 && \text{(ordered field multiplication)} \\ 3x^2 + 1 &\geq 1 && \text{(ordered field addition)} \end{aligned}$$

Now for the second part of the question, we have shown that f has at least 1 root. Now let us assume that it has at least 2 distinct roots r_1 and r_2. If this was the case then the mean value theorem tells us:

$$(\exists c \in \mathbb{R}) \ f'(c) = \frac{f(r_1) - f(r_2)}{r_1 - r_2} = 0$$

But this contradicts our previously establiushed result of $(\forall x \in \mathbb{R}) \ f'(x) > 0$. Thus f has only 1 real root.

Part c

Problem: Write Matlab programs to approximate the root of f using both Newton and the bisection method to within a tolerance of 10^{-3} (for the input in the bisection method and for the iterations in Newton's method). For Newton's method use an initial guess of $x_0 = 1$ and for the bisection method use the same interval $[1, 4]$. Give each iteration and the total number of iterations for both programs.

Solution:

Iteration	Bisection	Newton
1	2.500000000000000	1.500000000000000
2	1.750000000000000	1.387096774193548
3	1.375000000000000	1.378838947597994
4	1.562500000000000	1.378796701230898
5	1.468750000000000	
6	1.421875000000000	
7	1.398437500000000	
8	1.386718750000000	
9	1.380859375000000	
10	1.377929687500000	
11	1.379394531250000	
12	1.378662109375000	

Problem 3

Problem: Note the following recurrence relation:

$$x_{n+1} = 2 - (1 + c)x_n + cx_n^3$$

This sequence will converge to the value $s = 1$ for certain c, assuming x_0 is sufficiently close to s. Find those c values. Also find which of them lead the sequence to converge quadratically.

Solution:

Problem 4

Problem:

Solution:

Problem 5

Problem:

Solution: