Linear Algebra HW #2

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Problem 1

Problem: Recall that an $m \times n$ matrix A is called upper triangular if all entries lying below the diagonal entries are zero, that is, if $a_{ij} = 0$ whenever $i > j$. Prove that the set of upper triangular matrices U form a subspace of $\mathbb{F}^{m\times n}$.

Solution: To show that U forms a subspace, we must verify that it:

a) contains the zero element

b) is closed under addition

c) is closed under scalar multiplication

a) is easy to prove since the zero vector of $\mathbb{F}^{m\times n}$ is simply the $m \times n$ zero matrix $\mathbf{0}$. All entries of this matrix are the zero scalar $0 \in \mathbb{F}$, meaning that $\mathbf{0}_{ij} = 0$ for all i, j and in particular when $i > j$. Thus $\mathbf{0} \in U$.

b) Given two upper triangular matrices $\mathbf{a}, \mathbf{b} \in U$, we have that for all i, j s.t. $i > j$:

$$a_{ij} = b_{ij} = 0$$

As is the definition of upper triangular matrix. And so their sum, defined element wise, is also 0 for all $i > j$:

$$a_{ij} + b_{ij} = 0 + 0 = 0$$

And thus $\mathbf{a} + \mathbf{b} \in U$, i.e. U is closed under addition.

c) Given an upper triangular matrix $\mathbf{a} \in U$, and a scalar $\lambda \in \mathbb{F}$ we have that for all i, j s.t. $i > j$:

$$\lambda a_{ij} = \lambda \cdot 0 = 0$$

Which is precisely the definition of an upper triangular matrix. Thus $\lambda\mathbf{a} \in U$, i.e. U is closed under scalar multiplication.

Problem 2

Problem: Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Solution: Now recall that two vectors $\mathbf{u}, \mathbf{v}$ are linearly independent iff there exist scalars λ_1, λ_2, where at least one is non-zero, such that:

$$\lambda_1\mathbf{u} + \lambda_2\mathbf{v} = \mathbf{0}$$

(Lemma 1) First note that if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, then $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$ since all non-zero scalars have multiplicative inverses:

$$\mathbf{u} = \lambda \mathbf{v} \implies \mathbf{v} = \lambda^{-1}\mathbf{u}$$

And so it will suffice to show that at least one vector is a multiple of another.

($\Rightarrow$) for the forward direction we will assume w.l.o.g. that $\lambda_1 \neq 0$. This nets us the following:

$$\begin{aligned} \lambda_1\mathbf{u} + \lambda_2\mathbf{v} = \mathbf{0} &\implies \lambda_1\mathbf{u} = -\lambda_2\mathbf{v} \\ &\implies \mathbf{u} = -\frac{\lambda_2}{\lambda_1}\mathbf{v} \end{aligned}$$

And so $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. And by Lemma 1 we know this argument can be reversed for $\mathbf{v}$ as well.

($\Leftarrow$) for the backwards direction we will assume w.l.o.g. that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. This nets us the following:

$$\begin{aligned} \mathbf{u} = \lambda\mathbf{v} &\implies \mathbf{u} - \lambda\mathbf{v} = \mathbf{0} \\ &\implies 1 \cdot \mathbf{u} - \lambda\mathbf{v} = \mathbf{0} \end{aligned}$$

And so there is a pair of coefficients, namely $\lambda_1 = 1$ and $\lambda_2 = -\lambda$, such that $\lambda_1\mathbf{u} + \lambda_2\mathbf{v} = \mathbf{0}$.

($\Leftrightarrow$) And so these two implications, along with Lemma 1, give us the desired equivalence:

$$\lambda_1\mathbf{u} + \lambda_2\mathbf{v} = \mathbf{0} \iff (\exists\lambda_1)\, \mathbf{u} = \lambda_1\mathbf{v} \vee (\exists\lambda_2)\, \mathbf{v} = \lambda_2\mathbf{u}$$

Problem 3

Problem: Show that if S_1 and S_2 are subsets of a vector space V then:

$$\text{span}(S_1 \cup S_2) = \text{span}(S1) + \text{span}(S2)$$

Where + denotes the sum of two subspaces.

Solution: To prove that these two sets are equal we must show that each is a subset of the other.

($\subseteq$) Let $\mathbf{v} \in \text{span}(S_1 \cup S_2)$. This means that, by definition, $\mathbf{v}$ is a linear combination of the vectors in $S_1 \cup S_2$:

$$\mathbf{v} = \overbrace{\sum_i a_i\mathbf{x}_i}^{\mathbf{x}} + \overbrace{\sum_j b_j\mathbf{y}_j}^{\mathbf{y}}$$

Where $\mathbf{x}_i \in S_1$ and $\mathbf{y}_j \in S_2$. And since $\mathbf{x}$ is a linear combination of vectors in S_1 it is by definition a member of its span, and the same goes for $\mathbf{y}$ and S_2:

$$\mathbf{x} \in \text{span}(S_1) \quad \mathbf{y} \in \text{span}(S_2)$$

And so $\mathbf{v}$ can be expressed as the sum of a vector from $\text{span}(S_1)$ and $\text{span}(S_2)$ and thus is, by definition, an element of their sum:

$$\mathbf{v} = \mathbf{x} + \mathbf{y} \in \text{span}(S1) + \text{span}(S2)$$

($\supseteq$) Now we instead let $\mathbf{v} \in \text{span}(S1) + \text{span}(S2)$. We can now write $\mathbf{v}$ as the sum of two vectors $\mathbf{x} \in \text{span}(S1)$ and $\mathbf{y} \in \text{span}(S2)$:

$$\mathbf{v} = \mathbf{x} + \mathbf{y}$$

Note that a generic element $\mathbf{x}$ of S_1 is just some linear combination of its vectors and similarly for $\mathbf{y}$:

$$\mathbf{v} = \overbrace{\sum_i a_i \mathbf{x}_i}^{\mathbf{x}} + \overbrace{\sum_j b_j \mathbf{y}_j}^{\mathbf{y}}$$

And at this point it should be clear that this is a generic element of $\operatorname{span}(S_1 \cup S_2)$ as it's a linear combination of vectors from $S_1 \cup S_2$. Thus $\mathbf{v} \in \operatorname{span}(S_1 \cup S_2)$

Problem 4

Problem: Is the following set linearly independent or dependent?

$$\left\{ \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix} \right\} \subseteq \mathbb{R}^{2\times 2}$$

Solution: As we've shown in problem 2, the linear dependence of a set of two vectors can be demonstrated by showing that one is a scalar multiple of the other. In this case, that scalar is given by $-2 \in \mathbb{R}$:

$$-2 \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix}$$

And so the set is linearly dependent.

Problem 5

Problem: Is the following set a basis for $\mathcal{P}_2(\mathbb{R})$?

$$\{-x^2 + 2x + 1, x^2 - 2x + 4, -9x^2 + 18x - 1\}$$

Solution: Recalling that $\mathcal{P}_2(\mathbb{R})$ is isomorphic to $\mathbb{R}^3$, we can identify each polynomial as a vector of its coefficients. Putting these vectors in a matrix, we can perform Gaussian elimination to determine the matrix's rank and thus the dimension the polynomials span:

$$\left[\begin{array}{rrr|r} -1 & 2 & 1 & 0 \\ 1 & -2 & 4 & 0 \\ -9 & 18 & -1 & 0 \end{array}\right] \xrightarrow{r_2 + r_1} \left[\begin{array}{rrr|r} -1 & 2 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ -9 & 18 & -1 & 0 \end{array}\right]$$
$$\xrightarrow{r_3 - 9r_1} \left[\begin{array}{rrr|r} -1 & 2 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -10 & 0 \end{array}\right]$$
$$\xrightarrow{r_3 + 2r_2} \left[\begin{array}{rrr|r} -1 & 2 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Once reduced, we can see that the matrix has only 2 pivot rows and thus is of rank 2. Clearly then, this set does not form a basis of the 3-dimensional $\mathcal{P}_2(\mathbb{R})$ as it only spans 2 dimensions.

Problem 6

Problem: Find a basis, and give the dimension, of the following two subspaces of $\mathbb{F}^5$:

a) $W_1 = \{(a_1, a_2, a_3, a_4, a_5)^\top \in \mathbb{F}^5 \mid a_1 - a_3 - a_4 = 0\}$

b) $W_2 = \{(a_1, a_2, a_3, a_4, a_5)^\top \in \mathbb{F}^5 \mid a_2 = a_3 = a_4 \wedge a_1 + a_5 = 0\}$

Solution: a) For W_1 we have the following basis:

$$\left\{ \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \right\}$$

Clearly each vector satisfies $a_1 - a_3 - a_4$ and no vector is a linear combination of the others, i.e. the set is linearly independent. Now we will show that every vector $\mathbf{v} =\in W_1$ can be written as a linear combination of these 4 vectors:

$$\begin{aligned} \mathbf{v} = \begin{bmatrix}a_1\\a_2\\a_3\\a_4\\a_5\end{bmatrix} &= \begin{bmatrix}a_3 + a_4\\a_2\\a_3\\a_4\\a_5\end{bmatrix} && (a_1 - a_3 - a_4 = 0) \\ &= a_3 \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} + a_2 \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + a_4 \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix} + a_5 \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \end{aligned}$$

And so these vectors form a basis of 4 vectors, thus W_1 is 4-dimensional.

b) For W_2 we have the following basis:

$$\left\{ \begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix} \right\}$$

Clearly both vectors satisfy $a_2 = a_3 = a_4$ and $a_1 + a_5 = 0$ and neither vector is a scalar multiple of the other, i.e. the set is linearly independent. Now we will show that every vector $\mathbf{v} = W_2$ can be written as a linear combination of these 2 vectors:

$$
\begin{aligned}
\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} &= \begin{bmatrix} a_1 \\ a_2 \\ a_2 \\ a_2 \\ a_5 \end{bmatrix} && (a_2 = a_3 = a_4) \\
&= \begin{bmatrix} a_1 \\ a_2 \\ a_2 \\ a_2 \\ -a_1 \end{bmatrix} && (a_1 + a_5 = 0) \\
&= a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

And so these vectors form a basis of 2 vectors, thus W_2 is 2-dimensional.