Linear Algebra Final Exam

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Problem 1

Solution: We first find the eigenvalues of A by solving its characteristic equation:

$$\begin{aligned}
0 &= p(\lambda) \\
&= \det(A - \lambda I) \\
&= \det \begin{bmatrix} -1-\lambda & -1 & 1 \\ 5 & 3-\lambda & -3 \\ -2 & -1 & 2-\lambda \end{bmatrix} \\
&= (-1-\lambda)\left(\lambda^2 - 5\lambda + 3\right) - (-1)(-5\lambda + 4) + 1 \cdot (-2\lambda + 1) \\
&= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 \\
&= -(\lambda - 1)^2(\lambda - 2)
\end{aligned}$$

Now we check the GM of the eigenvalue $\lambda_1 = 1$, i.e. the dimension of the eigenspace corresponding to $\lambda_1 = 1$:

$$\begin{aligned}
\dim E_1 &= \dim \ker(A - I) && \text{(def. of eigenspace)} \\
&= \text{nullity} \begin{bmatrix} -2 & -1 & 1 \\ 5 & 2 & -3 \\ -2 & -1 & 1 \end{bmatrix} && \text{(def. of nullity)} \\
&= \text{nullity} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} && (\text{rref}^{[1]}) \\
&= 1
\end{aligned}$$

This leaves us with 2 eigenvalues: $\lambda_1 = 1$ with AM 2 and GM 1, and $\lambda_2 = 2$ with AM 1 and GM 1. This corresponds to the following Jordan blocks:

$$J_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad J_2 = \begin{bmatrix} 2 \end{bmatrix}$$

Putting these together, we have the following Jordan normal form:

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Recall that every matrix is similar to its JNF, and so there exists some P such that:

$$A = P^{-1} J P$$

We can now express the trace of A^n as:

$$\begin{aligned}
\operatorname{tr}(A^n) &= \operatorname{tr}((P^{-1}JP)^n) && \text{(similar to JNF)} \\
&= \operatorname{tr}(P^{-1}J^nP) && \text{(power of similar matrices are similar)} \\
&= \operatorname{tr}(J^n) && \text{(trace is constant for similar matrices)} \\
&= \operatorname{tr}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^n\right) \\
&= \operatorname{tr}\begin{bmatrix} 1^n & \binom{n}{1}1^{n-1} & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 2^n \end{bmatrix} && \text{(power of JNF)} \\
&= 2 + 2^n
\end{aligned}$$

Problem 2

Solution: We first prove 2 lemmas before proving the main theorem:

Lemma 1: Any eigenvectors of a matrix M with distinct eigenvalues are linearly independent.

Proof: Let λ_1, λ_2 be distinct eigenvalues of M, and let $\mathbf{v}_1, \mathbf{v}_2$ be corresponding eigenvectors, i.e.:

$$M\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{Eq. 1}$$

Now consider constants c_1, c_2 such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \tag{Eq. 2}$$

Recall that $\mathbf{v}_1, \mathbf{v}_2$ are independent iff $c_1 = c_2 = 0$. Now consider the following:

$$\begin{aligned}
\mathbf{0} &= M\mathbf{0} \\
&= M(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) && \text{(from Eq. 2)} \\
&= c_1 M\mathbf{v}_1 + c_2 M\mathbf{v}_2 \\
&= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 && \text{(from Eq. 1)}
\end{aligned}$$

We take that last equality and call it (Eq. 3):

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 = \mathbf{0} \tag{Eq. 3}$$

Now, multiplying (Eq. 2) by λ_2 and subtracting from it (Eq. 3) we find:

$$\begin{aligned}
\mathbf{0} - \mathbf{0} &= \lambda_2 \cdot \text{(Eq. 2)} - \text{(Eq. 3)} \\
\mathbf{0} &= \lambda_2(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) - (c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2) \\
&= c_1(\lambda_1 - \lambda_2)\mathbf{v}_1 \\
0 &= c_1(\lambda_1 - \lambda_2) && \text{(eigenvectors are nonzero)} \\
&= c_1 && (\lambda_1 \neq \lambda_2 \implies \lambda_1 - \lambda_2 \neq 0)
\end{aligned}$$

And so we have proved that $c_1 = 0$. Even further, substituting $c_1 = 0$ into (Eq. 2), we find:

$$\begin{aligned}
\mathbf{0} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 && \text{(Eq. 2)} \\
&= c_2\mathbf{v}_2 && (c_1 = 0) \\
&= c_2 && \text{(eigenvectors are nonzero)}
\end{aligned}$$

And so $c_1 = c_2 = 0$, and thus $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. And so, since the corresponding eigenvectors of any two distinct eigenvalues of a matrix are linearly independent, we have that an $n \times n$ matrix with n distinct eigenvalues has n linearly independent eigenvalues. ■

Lemma 2: If an $n \times n$ matrix M has n linearly independent eigenvectors, then it is diagonalizable.

Proof: Assume the $n \times n$ matrix M has n linearly independent eigenvectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. Now consider a matrix P whose columns are those same eigenvectors, and a diagonal matrix D whose entries are the corresponding eigenvalues:

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \qquad D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Clearly we have:

$$
\begin{aligned}
MP = M\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{M}v_1 & \cdots & M\mathbf{v}_n\end{bmatrix} &= \begin{bmatrix}\lambda_1 v_1 & \cdots & \lambda_2\mathbf{v}_n\end{bmatrix} && (\mathbf{v}_i \text{ are eigenvectors}) \\
PD = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n\end{bmatrix} &= \begin{bmatrix}\lambda_1 v_1 & \cdots & \lambda_2\mathbf{v}_n\end{bmatrix}
\end{aligned}
$$

Since MP and PD are equal, we can say:

$$
\begin{aligned}
MP &= PD \\
M &= PDP^{-1} && (P \text{ has linearly independent columns, thus is invertible})
\end{aligned}
$$

And so M is diagonalizable. ■

Theorem: If a finite dimensional linear transformation $T : V \to V$ with $\dim V = n$ has n distinct eigenvalues, then it is diagonalizable.

Proof: Note that a finite linear transformation is diagonalizable iff (for some arbitrary basis) its matrix representative M is diagonalizable. And so we only need to prove that any $n \times n$ matrix M with n distinct eigenvalues is diagonalizable:

$$
\begin{aligned}
\text{A matrix } M \text{ has } n \text{ distinct eigenvalues} &\implies M \text{ has } n \text{ linearly independent eigenvectors} && \text{(Lemma 1)} \\
&\implies M \text{ is diagonalizable} && \text{(Lemma 2)}
\end{aligned}
$$

And so we are done. ■

Problem 3

Solution: First note that showing this result for finite dimensional linear transformations S and T is equivalent to showing it for their matrix representatives A and B, and so we do that instead.

Below we show that for any eigenvalue λ that a AB posesses, BA also posess this eigenvalue. We do this in two cases, one in which $\lambda \neq 0$ and one in which $\lambda = 0$:

Case 1: Suppose λ is a nonzero eigenvalue of AB, i.e. there exists some nonzero vector $\mathbf{v} \in V$ such that:

$$AB\mathbf{v} = \lambda\mathbf{v} \tag{Eq. 1}$$

Note from this equation that $B\mathbf{v} \neq \mathbf{0}$ because if it did, then $\lambda\mathbf{v} = \mathbf{0}$ implying that $\lambda = 0$ since $\mathbf{v}$ is an eigenvector, and thus nonzero. This contradicts our initial assumption that $\lambda \neq 0$.

Applying B to both sides of (Eq. 1) we find:

$$\begin{aligned} AB\mathbf{v} &= \lambda\mathbf{v} && \text{(Eq. 1)} \\ B(AB\mathbf{v}) &= B(\lambda\mathbf{v}) \\ BA(B\mathbf{v}) &= \lambda(B\mathbf{v}) && \text{(associativity)} \end{aligned}$$

And so, λ is an eigenvalue for BA with a corresponding eigenvector of $B\mathbf{v}$, since it is nonzero. ■

Case 2: Suppose $\lambda = 0$ is an eigenvalue of AB. This implies that AB has a nontrivial kernal, because some nonzero vectors go to $\mathbf{0}$.. We want to show that BA also has a nontrivial kernal, i.e. that it has a 0 eigenvalue. To do this we give a proof by contradiction:

Suppose BA *does* have a trivial kernal. This makes BA a linear isomorphism which implies that both A and B are linear isomorphisms. This in turn implies that AB is a linear isomorphism, since it is the product of two linear isomorphisms. This however causes a contradiction as a linear isomorphism must have a trivial kernal, contradicting our assumption that AB has a 0 eigenvalue. And so, we have that TS must have a nontrivial kernal and thus a 0 eigenvalue. ■

While we have only proved one direction, i.e. that an eigenvalue of AB is an eigenvalue of BA, note that the other direction is given by an identical proof but with A and B switched. As such, we are done. ■

Problem 4

Solution: Let E_λ be the λ-eigenspace of U. We have that for any eigenvector $\mathbf{v}$ corresponding to eigenvalue λ:

$$\begin{aligned} U(T(\mathbf{v})) &= T(U(\mathbf{v})) && \text{(commutative)} \\ &= T(\lambda \mathbf{v}) && \text{(eigenvector)} \\ &= \lambda T(\mathbf{v}) && \text{(linearity)} \end{aligned}$$

This shows that every E_λ is T-stable, i.e. T preserves the eigenspace E_λ. This stability implies that the restriction $T|_{E_\lambda}$ is a self-adjoint operator on E_λ.

Now, by the spectral theorem, each E_λ has an orthonormal basis of eigenvectors of $T|_{E_\lambda}$. Call them $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ and call their corresponding eigenvalues $\{\mu_1, \cdots, \mu_r\}$. We now have:

$$U(\mathbf{v}_i) = \lambda \mathbf{v}_i \qquad T(\mathbf{v}_i) = \mu_i \mathbf{v}_i$$

Which is one part of our simultaneous diagonalization. we simply need to repeat this process for all eigenvalues λ of U. The set of all those eigenvectors $\mathbf{v}_i$ composes an eigenbasis which consists solely of eigenvectors of both U and T.

Now note that UT is a self-adjoint operator:

$$\begin{aligned} (UT)^* &= T^* U^* && \text{(anti-distributive)} \\ &= TU && \text{(T and U are self-adjoint)} \\ &= UT && \text{(T and U commute)} \end{aligned}$$

And so, by the spectral theorem, UT diagonalizes over V. This means that the set of simultaneous eigenvectors we found earlier spans the entirety of V. This is precisely what it means for two matrices to be simultaneously diagonalizable and so with that, we are done. ■

Problem 5

Part 1: First we compute $T(X)$ for an arbitrary matrix X:

$$\begin{aligned} T(X) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} x_{22} & 0 \\ x_{12} & 0 \end{bmatrix} + \begin{bmatrix} 0 & x_{11} \\ 0 & x_{21} \end{bmatrix} \\ &= \begin{bmatrix} x_{22} & x_{11} \\ x_{12} & x_{21} \end{bmatrix} \end{aligned}$$

Now we compute the inner product $\langle T(A), T(B) \rangle$:

$$\begin{aligned} \langle T(A), T(B) \rangle &= \operatorname{tr}(T(B)^\top T(A)) \\ &= \operatorname{tr}\left(\begin{bmatrix} b_{22} & b_{11} \\ b_{12} & b_{21} \end{bmatrix}^\top \begin{bmatrix} a_{22} & a_{11} \\ a_{12} & a_{21} \end{bmatrix} \right) \\ &= \operatorname{tr}\left(\begin{bmatrix} b_{22} & b_{12} \\ b_{11} & b_{21} \end{bmatrix} \begin{bmatrix} a_{22} & a_{11} \\ a_{12} & a_{21} \end{bmatrix} \right) \\ &= \operatorname{tr} \begin{bmatrix} b_{22}a_{22} + b_{12}a_{12} & b_{22}a_{11} + b_{12}a_{21} \\ b_{11}a_{22} + b_{21}a_{12} & b_{11}a_{11} + b_{21}a_{21} \end{bmatrix} \\ &= b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22} \end{aligned}$$

Now we compute the inner product $\langle A, B \rangle$:

$$\begin{aligned} \langle A, B \rangle &= \operatorname{tr}(B^\top A) \\ &= \operatorname{tr}\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^\top \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) \\ &= \operatorname{tr}\left(\begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) \\ &= \operatorname{tr} \begin{bmatrix} b_{11}a_{11} + b_{21}a_{21} & b_{11}a_{12} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{21} & b_{12}a_{12} + b_{22}a_{22} \end{bmatrix} \\ &= b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22} \end{aligned}$$

And so we have that $\langle T(A), T(B) \rangle = \langle A, B \rangle$ for any two matrices A and B. This is precisely what it means for T to be an orthogonal operator and so, we are done. ∎

Part 2: We want an operator T^* that, for any matrices A and B, satisfies:

$$\langle T(A), B \rangle = \langle A, T^*(B) \rangle$$

To solve for that operator, let us consider the following equality:

$$\begin{aligned} \langle T(A), B \rangle &= \langle A, T^*(B) \rangle \\ &= \langle A, M \rangle \qquad \text{(let } M := T^*(B)\text{)} \\ \operatorname{tr}(B^\top T(A)) &= \operatorname{tr}(M^\top A) \\ \operatorname{tr}\left(\begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{22} & a_{11} \\ a_{12} & a_{21} \end{bmatrix} \right) &= \operatorname{tr}\left(\begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) \\ b_{11}a_{22} + b_{21}a_{12} + b_{12}a_{11} + b_{22}a_{21} &= m_{11}a_{11} + m_{21}a_{21} + m_{12}a_{12} + m_{22}a_{22} \end{aligned}$$

Now to calculate the entries of M, i.e. what T^* mapped B to, we simply have to pick the m_{ij} such that the above equality is satisfied:

$$M = \begin{bmatrix} b_{12} & b_{21} \\ b_{22} & b_{11} \end{bmatrix}$$

And so the adjoint $T^*(X)$ for an arbitrary matrix X is given by:

$$T^*(X) = \begin{bmatrix} x_{12} & x_{21} \\ x_{22} & x_{11} \end{bmatrix}$$