Intro to Math Reasoning HW 8b

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Problem 1

Problem: Let $\leq_A$ be a partial order on A and let X be a nonempty subset of A. Prove that $m \in A$ is a minimum for X iff m is a lower bound for X and $m \in X$.

Solution: Note that the definition of a minimum on X is:

$$(\forall x \in X)\ m \in X \land m \leq_A x$$

Now note that if m was a lower bound of X it would satisfy the following:

$$(\forall x \in X)\ m \leq_A x$$

And the conjunction of this and $m \in X$ is:

$$(\forall x \in X)\ m \in X \land m \leq_A x$$

And so of course, a lower bound that is also in X is also a minimum and vice versa. They are equivalent statements. Minimums are just lower bounds in the set they minimize. It's directly implied by the definition...

Problem 2

Problem: Let $\leq_A$ be a partial order on A and let X be a nonempty finite subset of A. Prove that for all $x \in X$ there is a maximal element $m \in X$ such that $x \leq_A m$.

Solution: Recall that all finite posets are well-founded. This means, by definition, that any nonempty subset of X (including X itself) has a minimum and maximum element. The definition of a maximum element is the proposition we are trying to prove. If I use the well-foundedness of finite sets there is nothing to prove...

Problem 3

Problem: Given some set U, prove that for any nonempty set $X \in P(U)$, there is a unique set $G \in P(U)$ which is the greatest lower bound of X under the $\subseteq$ partial order.

Solution: The greatest lower bound of X under the subset relation would be the intersection of all nonempty subsets of U. This is because for a set G to be a lower bound of X, it must be a subset of all the element in X. The arbitrary intersection of n sets is precisely the maximal (that is containing the most elements possible) set that contains only the common elements of those n sets, even for infinite n.

It is clear then that the greatest lower bound is:

$$\bigcap X = \{a \mid (\forall x \in X)\ a \in x\}$$

And this set will always at least be the empty set (that is if the condition above never holds), which is contained in $P(X)$, thus still qualifying it for being a lower bound of X.

Problem 4

Problem: Prove that the following relation on $\mathbb{R}$ is an equivalence relation:

$$xRy \equiv (\exists k \in \mathbb{Z})\ x - y = 2\pi k$$

Then give the equivalence classes of 0 and $\frac{\pi}{2}$.

Solution: First we'll prove the 3 properties that make a partial order an equivalence relation.

Reflexivity

$$\begin{aligned} xRx \equiv (\exists k \in \mathbb{Z})\ x - x &= 2\pi k \\ 0 &= 2\pi k \end{aligned}$$

And since $2\pi k = 0$ when $k = 0 \in \mathbb{Z}$, the relation is satisfied and thus reflexive.

Symmetry

$$\begin{aligned} xRy &\equiv (\exists k \in \mathbb{Z})\ x - y = 2\pi k \\ &\equiv (\exists k \in \mathbb{Z})\ y - x = 2\pi(-k) && \text{(anti-commutativity of subtraction)} \\ &\equiv yRx && \text{(closure of } \mathbb{Z} \text{ under additive inverse)} \end{aligned}$$

Thus the relation is symmetric.

Transitivity

$$\begin{aligned}
xRy &\equiv (\exists k_1 \in \mathbb{Z})\ x - y = 2\pi k_1 && \text{(definition)}\\
&\to (\exists k_1 \in \mathbb{Z})\ -y = 2\pi k_1 - x && \text{(arithmetic)}\\
yRz &\equiv (\exists k_2 \in \mathbb{Z})\ y - z = 2\pi k_2 && \text{(definition)}\\
&\to (\exists k_1 \in \mathbb{Z})\ y = 2\pi k_2 + z && \text{(arithmetic)}\\
&\to (\exists k_1, k_2 \in \mathbb{Z})\ 0 = 2\pi(k_1 + k_2) - x + z && \text{(sum of lines 2 \& 4)}\\
&\equiv (\exists k_1, k_2 \in \mathbb{Z})\ z - x = 2\pi(k_1 + k_2) && \text{(arithmetic)}\\
&\equiv (\exists k \in \mathbb{Z})\ z - x = 2\pi k && \text{(closure of } \mathbb{Z} \text{ under addition)}\\
&\equiv zRx && \text{(definition)}\\
&\equiv xRz && \text{(symmetry of } R\text{)}
\end{aligned}$$

And so xRy and yRx imply xRz, thus the relation is transitive.

Equivalence Classes

The equivalence classes are given below:

$$\begin{aligned}
[0] &= \{x \in \mathbb{R} \mid (\exists k \in \mathbb{Z})\ 0 - x = 2\pi k\}\\
&= \{2\pi k \mid k \in \mathbb{Z}\}\\
\left[\frac{\pi}{2}\right] &= \left\{x \in \mathbb{R} \mid (\exists k \in \mathbb{Z})\ \frac{\pi}{2} - x = 2\pi k\right\}\\
&= \left\{\frac{\pi}{2} + 2\pi k \mid k \in \mathbb{Z}\right\}
\end{aligned}$$

Problem 5

Problem: Prove that the following relation on $\mathbb{R}$ is an equivalence relation:

$$\begin{aligned}
xRy &\equiv x - y \in \mathbb{Q}\\
&\equiv (\exists q \in Q)\ x - y = q
\end{aligned}$$

Then give the equivalence classes of 0, $\frac{1}{2}$, and $\frac{\pi}{2}$.

Solution: First we'll prove the 3 properties that make a partial order an equivalence relation.

Reflexivity

$$\begin{aligned}
xRx &\equiv x - x \in \mathbb{Q}\\
&\equiv 0 \in \mathbb{Q}
\end{aligned}$$

Which is clearly true. Thus R is reflexive.

Symmetry

$$\begin{aligned}
xRy &\equiv (\exists q \in Q)\ x - y = q \\
&\equiv (\exists q \in Q)\ y - x = -q && \text{(anti-commutativity of subtraction)} \\
&\equiv yRx && \text{(closure of } \mathbb{Q} \text{ under additive inverse)}
\end{aligned}$$

And of course $\mathbb{Q}$ is closed under the additive inverse because the negative of any rational number is also rational. Thus the relation is symmetric.

Transitivity

$$\begin{aligned}
xRy &\equiv (\exists q_1 \in Q)\ x - y = q_1 && \text{(definition)} \\
&\to (\exists q_1 \in Q)\ -y = q_1 - x && \text{(arithmetic)} \\
yRz &\equiv (\exists q_2 \in Q)\ y - z = q && \text{(definition)} \\
&\to (\exists q_2 \in Q)\ y = q_2 + z && \text{(arithmetic)} \\
&\to (\exists q_1, q_2 \in Q)\ 0 = q_1 + q_2 - x + z && \text{(sum of lines 2 \& 4)} \\
&\equiv (\exists q_1, q_2 \in Q)\ z - x = q_1 + q_2 && \text{(arithmetic)} \\
&\equiv (\exists q \in Q)\ z - x = q && \text{(closure of } \mathbb{Q} \text{ under addition)} \\
&\equiv zRx && \text{(definition)} \\
&\equiv xRz && \text{(symmetry of } R\text{)}
\end{aligned}$$

And of course $\mathbb{Q}$ is closed under the addition because we can always make a find a common denominator and add the numerators. Thus the relation is transitive.

Equivalence Classes

The equivalence classes are given below:

$$\begin{aligned}
[0] &= \{x \in \mathbb{R} \mid (\exists q \in \mathbb{Q})\ x - 0 = q\} = \mathbb{Q} \\
\left[\frac{1}{2}\right] &= \left\{x \in \mathbb{R} \mid (\exists q \in \mathbb{Q})\ x - \frac{1}{2} = q\right\} = \mathbb{Q} \\
\left[\frac{\pi}{2}\right] &= \left\{x \in \mathbb{R} \mid (\exists q \in \mathbb{Q})\ x - \frac{\pi}{2} = q\right\} \\
&= \left\{\frac{\pi}{2} + r \mid r \in \mathbb{R}\right\}
\end{aligned}$$