Intro to Real Analysis
HW #3

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Problem 1

Consider the sequence $(a_n)_{n=1}^{\infty}$ where:

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

Part a: Show that $0 < a_n < 5$ for any $n \geq 1$, hence $(a_n)_{n=1}^{\infty}$ is a bounded sequence.

Solution: First let us establish some lemmas: for $0 < n$ and $k \leq n$ we have

$$\begin{aligned} n^{\underline{k}} &= n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots (n-k+1) && \text{(def. of falling factorial)} \\ &\leq \underbrace{n \cdot n \cdot n \cdot n \cdots n}_{k \text{ times}} = n^k && \text{(lemma 1)} \end{aligned}$$

Next we have for $n > 0$:

$$\begin{aligned} \frac{1}{n!} &= \frac{1}{n(n-1)(n-2)\cdots 2 \cdot 1} && \text{(def. of factorial)} \\ &\leq \frac{1}{n(n-1)} && \text{(lemma 2)} \end{aligned}$$

And finally we have:

$$\begin{aligned} \sum_{k=2}^{n} \frac{1}{k-1} - \frac{1}{k} &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\ &= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \cdots + \frac{1}{n-1}\right) - \frac{1}{n} && \text{(telescoping sum)} \\ &= 1 - \frac{1}{n} && \text{(lemma 3)} \end{aligned}$$

Now let us prove the upper bound:

$$
\begin{aligned}
a_n &= \left(1+\frac{1}{n}\right)^n && \text{(def. of } a_n\text{)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k && \text{(binomial formula)} \\
&= 1 + 1 + \sum_{k=2}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\
&= 2 + \sum_{k=2}^{n} \frac{n^{\underline{k}}}{k!} \cdot \frac{1}{n^k} && \text{(def. of binomial coefficient)} \\
&\leq 2 + \sum_{k=2}^{n} \frac{n^k}{k!} \cdot \frac{1}{n^k} && \text{(lemma 1)} \\
&= 2 + \sum_{k=2}^{n} \frac{1}{k!} \\
&\leq 2 + \sum_{k=2}^{n} \frac{1}{k(k-1)} && \text{(lemma 2)} \\
&= 2 + 1 - \frac{1}{n} \\
&= 3 - \frac{1}{n} < 3 && (\forall n \in \mathbb{Z}^+,\ \tfrac{1}{n} \leq 1)
\end{aligned}
$$

Noting that $n \in \mathbb{Z}^+$, for the lower bound we have:

$$
\begin{aligned}
&0 < \frac{1}{n} && \text{(multiplicative inverse of a positive number is positive)} \\
&0 < 1 + \frac{1}{n} && \text{(1 plus positive number is positive)} \\
&0 < \left(1+\frac{1}{n}\right)^n = a_n && \text{(positive powers of positive numbers are positive)}
\end{aligned}
$$

Putting these two results together we find:

$$(\forall n \in \mathbb{Z}^+)\ 0 < a_n < 3 < 5$$

Part b: Show that $a_n \leq a_{n+1}$ for any $n \geq 1$, hence $(a_n)_{n=1}^{\infty}$ is an increasing sequence.

Solution: Consider the following for all $n \in \mathbb{Z}^+$:

$$
\begin{aligned}
n^2 + 2n &\leq n^2 + 2n + 1 && (n > 0) \\
n(n+2) &\leq n^2 + 2n + 1 \\
n + 1 &\leq \frac{n^2 + 2n + 1}{n} && \text{(divide both sides by positive number)} \\
\frac{1}{n+1} &\geq \frac{n}{n^2 + 2n + 1} && \text{(inverse both (positive) sides of inequality)} \\
1 - \frac{1}{n+1} &\leq 1 - \frac{n}{n^2 + 2n + 1} && \text{(negate both sides of inequality)} \\
1 - \frac{1}{n+1} &\leq \left(1 - \frac{1}{n^2 + 2n + 1}\right)^n && \text{(Bernoulli's inequality)} \\
\frac{n+1}{n+2} &\leq \left(\frac{n^2 + 2n}{n^2 + 2n + 1}\right)^n \\
\frac{n+2}{n+1} &\geq \left(\frac{n^2 + 2n + 1}{n^2 + 2n}\right)^n && \text{(inverse both (positive) sides of inequality)} \\
\frac{n+2}{n+1} &\geq \left(\frac{(n+1)^2}{n(n+2)}\right)^n \\
\frac{n+2}{n+1} &\geq \left(\frac{n+1}{n+2}\right)^n \left(\frac{n+1}{n}\right)^n \\
\frac{n+2}{n+1}\left(\frac{n+2}{n+1}\right)^n &\geq \left(\frac{n+1}{n}\right)^n && \text{(divide both sides by positive number)} \\
\left(\frac{n+2}{n+1}\right)^{n+1} &\geq \left(\frac{n+1}{n}\right)^n \\
\left(1 + \frac{1}{n+1}\right)^{n+1} &\geq \left(1 + \frac{1}{n}\right)^n \\
a_{n+1} &\geq a_n && \text{(def. of } a_n\text{)}
\end{aligned}
$$

Problem 2

Problem: Consider the sequence $(a_n)_{n=1}^{\infty}$ where $a_n = \frac{n}{3n^2+1}$. Use ϵ-N language to show that $(a_n)_{n=1}^{\infty}$ is Cauchy, then find its limit.

Solution: For this sequence to be Cauchy, it must be the case that for any $\epsilon > 0$ there is a positive integer N such that for positive integers $n, m \geq N$, we have $|a_n - a_m| < \epsilon$. Below we will characterize that N in terms of ϵ:

$$
\begin{aligned}
|a_n - a_m| &= \left|\frac{n}{3n^2+1} - \frac{m}{3m^2+1}\right| \\
&\leq \frac{n}{3n^2+1} + \frac{m}{3m^2+1} && \text{(triangle inequality)} \\
&\leq \frac{n}{n^2} + \frac{m}{m^2} && \text{(smaller denominator, larger value)} \\
&= \frac{1}{n} + \frac{1}{m} \\
&\leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} && (n, m \geq N)
\end{aligned}
$$

Now let us have $\frac{2}{N} < \epsilon$, which would satisfy the Cauchy condition. This would mean that $N < \frac{2}{\epsilon}$. And so, given any $\epsilon > 0$, we have that $\forall N > \frac{2}{\epsilon}$ where $N \in \mathbb{Z}^+$:

$$(\forall n, m \in \mathbb{Z}^+)\ n, m \geq N \implies |a_n - a_m| < \epsilon$$

Thus $(a_n)_{n=1}^{\infty}$ is Cauchy. Now we find its limit:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{3n^2+1} && \text{(def. of } a_n\text{)} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{3+\frac{1}{n^2}} \\
&= \frac{\lim_{n\to\infty} \frac{1}{n}}{3+\lim_{n\to\infty} \frac{1}{n^2}} && \text{(limit of ratio is ratio of limits)} \\
&= \frac{0}{3+0} = 0
\end{aligned}$$

Problem 3

Problem: Find the limit of the sequence $(a_n)_{n=1}^{\infty}$ where:

$$a_n = \frac{\sin(n)}{2n+1}$$

Solution: Consider the following:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\sin(n)}{2n+1} && \text{(def. of } a_n\text{)} \\
&= \lim_{n\to\infty} \frac{\frac{\sin(n)}{n}}{2+\frac{1}{n}} \\
&= \frac{\lim_{n\to\infty} \frac{\sin(n)}{n}}{2+\lim_{n\to\infty} \frac{1}{n}} && \text{(limit of ratio is ratio of limits)} \\
&= \frac{0}{2+0} = 0 && (\lim_{x\to\infty} \tfrac{\sin(x)}{x} = 0)
\end{aligned}$$

Problem 4

Problem: Find the limit of the sequence $(a_n)_{n=1}^{\infty}$ where:

$$a_n = 1 + \sqrt{n+1} - \sqrt{n}$$

Solution: Consider the following:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (1 + \sqrt{n+1} - \sqrt{n}) && \text{(def. of } a_n\text{)} \\
&= 1 + \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \\
&= 1 + \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} && \text{(multiply by conjugate)} \\
&= 1 + 0 = 1 && (\lim_{n\to\infty} \sqrt{n+1} + \sqrt{n} = \infty)
\end{aligned}$$

Problem 5

Problem: Find the limit of the sequence $(a_n)_{n=1}^{\infty}$ where:

$$a_n = \frac{n}{2^n}$$

Solution: First note that:

$$n > 4 \implies n^2 < 2^n \tag{lemma 1}$$

So we have, for $n > 4$:

$$\begin{aligned}
0 &\leq \frac{n}{2^n} && (n > 0) \\
0 &\leq \frac{n}{2^n} \leq \frac{n}{n^2} && (\text{lemma 1, } n > 4) \\
0 &\leq \frac{n}{2^n} \leq \frac{1}{n} \\
\lim_{n\to\infty} 0 &\leq \lim_{n\to\infty} \frac{n}{2^n} \leq \lim_{n\to\infty} \frac{1}{n} && (\text{squeeze theorem}) \\
0 &\leq \lim_{n\to\infty} \frac{n}{2^n} \leq 0 \\
&\implies \lim_{n\to\infty} \frac{n}{2^n} = 0
\end{aligned}$$

Problem 6

Problem: Find the limit of the sequence $(a_n)_{n=1}^{\infty}$ where:

$$a_n = \frac{2^n}{n!}$$

Solution:

$$\begin{aligned}
0 &\leq \frac{2^n}{n!} = \frac{\overbrace{2 \cdot 2 \cdot 2 \cdots 2}^{n \text{ copies}}}{1 \cdot 2 \cdot 3 \cdots n} && (n > 0) \\
0 &\leq \frac{2^n}{n!} \leq \frac{2 \cdot 2}{1 \cdot 2} \left(\frac{2}{3}\right)^{n-2} \\
\lim_{n\to\infty} 0 &\leq \lim_{n\to\infty} \frac{2^n}{n!} \leq \lim_{n\to\infty} \frac{2 \cdot 2}{1 \cdot 2} \left(\frac{2}{3}\right)^{n-2} && (\text{squeeze theorem}) \\
0 &\leq \lim_{n\to\infty} \frac{2^n}{n!} \leq 0 \\
&\implies \lim_{n\to\infty} \frac{2^n}{n!} = 0
\end{aligned}$$