Honors Calculus III HW #7

Ozaner Hansha

November 19, 2018

Exercise 1

Consider the following matrix:

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$$

Part a

Problem: Find A^{-1}.

Solution: $\det(A) = 3 * 3 - 5 * 2 = -1$ and so:

$$A^{-1} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$$

Part b

Problem: Solve the following equations:

$$A\mathbf{x} = (3, 2) \quad A\mathbf{x} = (2, 2) \quad A\mathbf{x} = (-1, 7)$$

Solution: Using the inverse we computed above the solutions are simply:

$$\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} (3, 2)^\top = (1, 0)$$
$$\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} (2, 2)^\top = (4, -2)$$
$$\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} (-1, 7)^\top = (38, -23)$$

Exercise 2

Consider the following matrix:

$$A = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Part a

Problem: Find A^{-1}.

Solution: Consider the columns of $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, by computing the following cross products:

$$\mathbf{v}_2 \times \mathbf{v}_3 = (5, -9, 2) \mathbf{v}_1 \times \mathbf{v}_3 = (3, -5, 1)$$
$$\mathbf{v}_1 \times \mathbf{v}_2 = (-1, 2, -1)$$

We can arrange them row-wise into a matrix, then multiply it by the inverse of $\det(A) = -1$:

$$A^{-1} = \begin{bmatrix} -5 & 9 & -2 \\ -3 & 5 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Part b

Problem: Solve the following equations:

$$A\mathbf{x} = (3, 2, 1) \quad A\mathbf{x} = (2, 2, 1) \quad A\mathbf{x} = (-1, 7, 1)$$

Solution: Now we just matrix multiply by the inverse to solve for $\mathbf{x}$:

$$\begin{bmatrix} -5 & 9 & -2 \\ -3 & 5 & -1 \\ 1 & -2 & 1 \end{bmatrix} (3, 2, 1)^\top = (1, 0, 0)$$
$$\begin{bmatrix} -5 & 9 & -2 \\ -3 & 5 & -1 \\ 1 & -2 & 1 \end{bmatrix} (2, 2, 1)^\top = (6, -3, -1)$$
$$\begin{bmatrix} -5 & 9 & -2 \\ -3 & 5 & -1 \\ 1 & -2 & 1 \end{bmatrix} (-1, 7, 1)^\top = (66, -37, -14)$$

Exercise 3

Problem: Prove the following for any pair of 2×2 matrices A and B:

$$\det(AB) = \det(A)\det(B) \qquad \det(A^{-1}) = \det(A)^{-1}$$

Solution: Notice that:

$$\begin{aligned}\det(A)\det(B) &= \det\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \\ &= a_1a_2d_1d_2 - a_1b_2c_2d_1 - a_2b_1c_1d_2 + b_1b_2c_1c_2\end{aligned}$$

Now notice that:

$$\begin{aligned}\det(AB) &= \det\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}\right) \\ &= (a_1a_2 + b_1c_2)(c_1b_2 + d_1d_2) - (a_1b_2 + b_1d_2)(c_1a_2 + d_1c_2) \\ &= a_1a_2d_1d_2 - a_1b_2c_2d_1 - a_2b_1c_1d_2 + b_1b_2c_1c_2\end{aligned}$$

And so for any pair of 2×2 matrices, the first identity holds. The second one can be shown more simply. First notice that:

$$(\det(A))^{-1} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{-1} = (ad - bc)^{-1}$$

Now notice that:

$$\begin{aligned}(\det(A^{-1}) &= \det\left((ad - bc)^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) \\ &= \left((ad - bc)^{-1}\right)^2 \det\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) && (\det(kA) = k^n \det(A)) \\ &= (ad - bc)^{-2}(da - (-b)(-c)) \\ &= (ad - bc)^{-1}\end{aligned}$$

And so the second equality holds as well.

Exercise 4

Part a

Problem: Find the Jacobian of the following function, evaluate it at $(-1, 1)$ then compute its inverse:

$$\mathbf{f}(x, y) = ((x^3 - x^2)y, xy + x - y)$$

Solution: First we'll compute $J\mathbf{f}$:

$$J\mathbf{f}(x, y) = \begin{bmatrix} 3x^2y - 2xy & x^3 - x^2 \\ y + 1 & x - 1 \end{bmatrix}$$

Evaluating at $(-1, 1)$ we get:

$$J\mathbf{f}(-1,1) = \begin{bmatrix} 5 & -2 \\ 2 & -2 \end{bmatrix}$$

The inverse is found just like before:

$$(J\mathbf{f}(-1,1))^{-1} = \frac{1}{5*(-2)-(-2)*2}\begin{bmatrix} -2 & 2 \\ -2 & 5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & -5 \end{bmatrix}$$

Part b

Problem: Find the Jacobian of the following function, evaluate it at $(-2, -3)$ then compute its inverse:

$$\mathbf{g}(x,y) = \left(u^2 - v^2 + uv, \frac{u^3}{3} - v^2\right)$$

Solution: First we'll compute $J\mathbf{g}$:

$$J\mathbf{g}(u,v) = \begin{bmatrix} 2u+v & u-2v \\ u^2 & -2v \end{bmatrix}$$

Evaluating at $(-2, -3)$ we get:

$$J\mathbf{f}(-2,-3) = \begin{bmatrix} -7 & 7 \\ 4 & 6 \end{bmatrix}$$

The inverse is found just like before:

$$(J\mathbf{f}(-1,1))^{-1} = \frac{1}{(-7)*6-7*4}\begin{bmatrix} 6 & -7 \\ -4 & -7 \end{bmatrix} = \frac{1}{70}\begin{bmatrix} -6 & 7 \\ 4 & 7 \end{bmatrix}$$

Exercise 5

Consider the functions $\mathbf{f}$ and $\mathbf{g}$ defined above. Notice that their codomain and domain match up meaning we can define the following composition:

$$\mathbf{h}(x,y) = \mathbf{g}(\mathbf{f}(x,y))$$

Part a

Problem: Give an explicit formula for $\mathbf{h}(x,y)$ then find the Jacobian of $\mathbf{h}$ at $(-1, 1)$.

Solution: Finding the explicit formula is a matter of substitution:

$$h_1(x,y) = y^2(x^6 - 2x^5 + 2x^4 - 2x^3 + 2x - 1) + y(x^4 - x^3 - 2x^2) - x^2$$
$$h_2(x,y) = \frac{y^3}{3}(x^9 - 3x^8 + 3x^7 - x^6) - y^2(x^2 - 2x + 1) + 2y(x^2 - x) - x^2$$

Computing the Jacobian and plugging in $(-1,1)$ we find:

$$J\mathbf{h}(-1,1) = \begin{bmatrix} -27 & 6 \\ 32 & -20 \end{bmatrix}$$

Part b

Problem: Verify that the chain rule holds for these functions at the point $\mathbf{x}_0 = (1,2)$. That is to say:

$$J(\mathbf{g} \circ \mathbf{f})(\mathbf{x}_0) = [J(\mathbf{g}(\mathbf{f}(\mathbf{x}_0))][J(\mathbf{f}(\mathbf{x}_0)]$$

Solution: Now notice that $\mathbf{f}(-1,1) = (-2,-3)$ meaning:

$$[J(\mathbf{g}(\mathbf{f}(\mathbf{x}_0))][J(\mathbf{f}(\mathbf{x}_0)] = [J(\mathbf{g}(\mathbf{f}(-2,3))][J(\mathbf{f}(-1,1)]$$

Multiplying these two matrices we find:

$$[J(\mathbf{g}(\mathbf{f}(-2,3))][J(\mathbf{f}(-1,1)] = \begin{bmatrix} -8 & 1 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} -5 & -2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -27 & 6 \\ 32 & -20 \end{bmatrix}$$

And so, in accordance with our result from Part a, we have verified the chain rule.

Exercise 6

Part a

Problem: Show that the following function $\mathbf{m} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is differentiable and calculate the Jacobian.

$$\mathbf{m}(x_1, \cdots, x_n, x_{n+1}) = x_{n+1}(x_1, \cdots, x_n)$$

Solution: Notice that the partial derivatives of $\mathbf{m}$ take the form:

$$\frac{\partial m_j}{\partial x_i} = \delta_{ij}$$

Notice this leaves out the last case, which we can deal with for the whole vector:

$$\frac{\partial \mathbf{m}}{\partial x_{n+1}} = (x_1, \cdots, x_n)$$

And since all these partials are defined and continuous, the function is certainly differentiable. The Jacobian then is simply the organization of the above partial derivatives into an $(n+1) \times n$ matrix:

$$J\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & x_1 \\ 0 & 1 & 0 & \cdots & 0 & x_2 \\ 0 & 0 & 1 & \cdots & 0 & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x_n \end{bmatrix}$$

Or column-wise $J\mathbf{m} = (\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{x})$.

Part b

Problem: Consider two differentiable functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^l \to \mathbb{R}^k$. Now consider the function:

$$\mathbf{h}(\mathbf{x}, \mathbf{y}) = (\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{y}))$$

where $\mathbf{h} : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^k$. Assuming it exists, calculate the Jacobian. Then verify that this linear map satisfies the definition of the derivative for $\mathbf{h}$.

Solution:

Part c

Problem: whatever

Solution: