Differential Equations HW #5

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Problem 1

Problem: Find the general solution to the following ODE:

$$y'' - y' - 6y = e^{4t}$$

Solution: First we find the general solution y_h to the associated homogenous equation. To do this, we first find the roots the ODE's characteristic equation:

$$s^2 - s - 6 = 0$$

These are given by:

$$s_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot (-6)}}{2} = \frac{1 \pm 5}{2}$$

And so our roots are $s_1 = -2$ and $s_2 = 3$. Thus, our general homogenous solution y_h is:

$$y_h(t) = k_1 e^{-2t} + k_2 e^{3t}$$

Now we must find a particular solution y_p of our original ODE. We use the **method of undetermined coefficients** and note that there is a particular solution of the following form, and with the following derivatives:

$$\begin{aligned} y_p(t) &= Ae^{4t} \\ y_p'(t) &= 4Ae^{4t} \\ y_p''(t) &= 16Ae^{4t} \end{aligned}$$

Where A is some constant. To find this constant, we plug this solution into the ODE and solve:

$$\begin{aligned} e^{4t} &= y_p'' - y_p' - y_p \\ &= 16Ae^{4t} - 4Ae^{4t} - 6Ae^{4t} \\ &= 6Ae^{4t} \\ 1 &= 6A \\ \frac{1}{6} &= A \end{aligned}$$

And so our particular solution y_p is:

$$y_p(t) = \frac{e^{4t}}{6}$$

And so, by the **extended linearity principle**, the general solution of the ODE y is simply the sum of the general homogenous solution y_h and the particular one y_p:

$$y(t) = y_h(t) + y_p(t) = \boxed{k_1 e^{-2t} + k_2 e^{3t} + \frac{e^{4t}}{6}}$$

Problem 2

Problem: Solve the following IVP:

$$\begin{cases} y'' + 4y' + 20y = -3\sin 2t \\ y(0) = y'(0) = 0 \end{cases}$$

Solution: First we find the general solution y_h to the associated homogenous equation. To do this, we first find the roots the ODE's characteristic equation:

$$s^2 + 4s + 20 = 0$$

These are given by:

$$s_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 20}}{2} = \frac{-4 \pm 8i}{2} = -2 \pm 4i$$

And so our roots are $s_1 = -2 + 4i$ and $s_2 = -2 - 4i$. To find the general real-valued homogenous solution y_h, we need only consider one root, say s_1:

$$\begin{aligned} y_h(t) &= k_1 \operatorname{Re}\left(e^{(-2+4i)t}\right) + k_1 \operatorname{Im}\left(e^{(-2+4i)t}\right) \\ &= k_1 \operatorname{Re}\left(e^{-2t}(\cos 4t + i \sin 4t)\right) + k_2 \operatorname{Im}\left(e^{-2t}(\cos 4t + i \sin 4t)\right) \\ &= k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t \end{aligned}$$

Now consider the associated complexified ODE:

$$y'' + 4y' + 20y = -3e^{2it}$$

Recall that for any particular solution of the complexification y_c, its imaginary part is a particular solution y_p to the original ODE. As such, we begin by noting that there exists a solution of the following form and with the following derivatives:

$$\begin{aligned} y_c(t) &= Ae^{2it} \\ y_c'(t) &= 2iAe^{2it} \\ y_c''(t) &= -4Ae^{2it} \end{aligned}$$

Where A is some constant. To find this constant, we plug this solution into the ODE and solve:

$$\begin{aligned} e^{2it} &= y_c'' + 4y_c' + 20y_c \\ &= -4Ae^{2it} + 8iAe^{2it} + 20Ae^{2it} \\ &= Ae^{2it}(16 + 8i) \\ 1 &= A(16 + 8i) \\ A &= \frac{1}{16 + 8i} = \frac{-3}{20} + \frac{3i}{40} \end{aligned}$$

And so our particular solution y_p is:

$$\begin{aligned} y_p(t) &= \operatorname{Im}(y_c(t)) \\ &= \operatorname{Im}\left(Ae^{2it}\right) \\ &= \operatorname{Im}\left(\left(\frac{-3}{20} + \frac{3i}{40}\right) e^{2it}\right) \\ &= \operatorname{Im}\left(\left(\frac{-3}{20} + \frac{3i}{40}\right)(\cos 2t + i \sin 2t)\right) \\ &= \frac{3\cos 2t}{40} - \frac{3\sin 2t}{20} \end{aligned}$$

And so, using the extended linearity principle, our general solution y the the given ODE is:

$$y(t) = y_h(t) + y_p(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t + \frac{3\cos 2t}{40} - \frac{3\sin 2t}{20}$$

Now we simply have to find k_1 and k_2 that satisfy the initial conditions. First we set $y(0) = 0$:

$$\begin{aligned} 0 &= y(0) \\ &= k_1 e^0 \cos 0 + k_2 e^0 \sin 0 + \frac{3\cos 0}{40} - \frac{3\sin 0}{20} \\ &= k_1 + 0 + \frac{3}{40} - 0 \\ -\frac{3}{40} &= k_1 \end{aligned}$$

Now we set $y'(0) = 0$:

$$\begin{aligned} y'(t) &= -2k_1 e^{-2t}(\cos 4t + 2\sin 4t) + 2k_2 e^{-2t}(2\cos 4t - \sin 4t) - \frac{3\sin 2t}{20} - \frac{3\cos 2t}{10} \\ 0 = y'(0) &= -2k_1 e^0(\cos 0 + 2\sin 0) + 2k_2 e^0(2\cos 0 - \sin 0) - \frac{3\sin 0}{20} - \frac{3\cos 0}{10} \\ &= -2k_1 + 4k_2 - \frac{3}{10} \\ &= \frac{3}{20} + 4k_2 - \frac{3}{10} \\ -4k_2 &= -\frac{3}{10} + \frac{3}{20} \\ k_2 &= \frac{3}{80} \end{aligned}$$

And so finally, the solution to the IVP y_0 is given by:

$$y_0(t) = \boxed{-\frac{3}{40}e^{-2t}\cos 4t + \frac{3}{80}e^{-2t}\sin 4t + \frac{3\cos 2t}{40} - \frac{3\sin 2t}{20}}$$

Problem 3

Problem: Find the amplitude and phase angle of the following system:

$$y'' + 2y' + 10y = \cos 3t$$

Solution: Consider the complexification of this ODE:

$$y'' + 2y' + 10y = e^{3it}$$

Now consider the following particular solution to the complexification y_c:

$$\begin{aligned} y_c(t) &= Ae^{3it} \\ y_c'(t) &= 3iAe^{3it} \\ y_c''(t) &= -9Ae^{3it} \end{aligned}$$

Where A is some constant. To find this constant, we plug this solution into the ODE and solve:

$$\begin{aligned} e^{3it} &= y_c'' + 4y_c' + 20y_c \\ &= -9Ae^{3it} + 6iAe^{3it} + 10Ae^{3it} \\ &= Ae^{3it}(1+6i) \\ 1 &= A(1+6i) \\ A &= \frac{1}{1+6i} = \frac{1-6i}{37} \end{aligned}$$

The amplitude is given by the magnitude of A:

$$|A| = \frac{1+6^2}{37^2} = \frac{37}{37^2} = \boxed{\frac{1}{37}}$$

and the phase angle is given by the phase of A:

$$\arg(A) = \tan^{-1}\left(\frac{-6/37}{1/37}\right) = \boxed{\tan^{-1}(-6) \approx 1.4056}$$

Problem 4

Problem: Solve the following IVPs and sketch their solutions:

a) $\begin{cases} y'' + 4y = \cos\frac{9t}{4} \\ y(0) = y'(0) = 0 \end{cases}$

b) $\begin{cases} y'' + 4y = \cos 2t \\ y(0) = y'(0) = 0 \end{cases}$

Solution: We start with **a)** the roots of the characteristic equation $s^2 + 4 = 0$ are simply $s = \pm 2i$. This means the homogenous solution y_h is given by:

$$\begin{aligned} y_h(t) &= k_1 \operatorname{Re}\left(e^{2it}\right) + k_1 \operatorname{Im}\left(e^{2it}\right) \\ &= k_1 \operatorname{Re}(\cos 2t + i\sin 2t) + k_2 \operatorname{Im}(\cos 2t + i\sin 2t) \\ &= k_1 \cos 2t + k_2 \sin 2t \end{aligned}$$

To find a particular solution y_c, we consider a particular solution y_c to the associated complexified equation $y'' + 4y = e^{\frac{9it}{4}}$:

$$\begin{aligned} y_c(t) &= Ae^{\frac{9it}{4}} \\ y_c'(t) &= \frac{9}{4}iAe^{\frac{9it}{4}} \\ y_c''(t) &= -\left(\frac{9}{4}\right)^2 Ae^{\frac{9it}{4}} \end{aligned}$$

To solve for A, we plug this solution into the ODE and solve:

$$\begin{aligned}
e^{\frac{9it}{4}} &= y_c'' + 4y_c \\
&= -\left(\frac{9}{4}\right)^2 Ae^{\frac{9it}{4}} + 4Ae^{\frac{9it}{4}} \\
&= Ae^{\frac{9it}{4}}\left(4 - \frac{9}{4}^2\right) \\
1 &= -\frac{17A}{16} \\
A &= -\frac{16}{17}
\end{aligned}$$

And so our particular solution y_p is:

$$\begin{aligned}
y_p(t) &= \mathrm{Re}\left(y_c(t)\right) \\
&= \mathrm{Re}\left(Ae^{\frac{9it}{4}}\right) \\
&= \mathrm{Re}\left(-\frac{16e^{\frac{9it}{4}}}{17}\right) \\
&= \mathrm{Re}\left(-\frac{16(\cos\frac{9}{4}t + i\sin\frac{9t}{4})}{17}\right) \\
&= -\frac{16\cos\frac{9t}{4}}{17}
\end{aligned}$$

So our general solution is given by:

$$y(t) = y_h(t) + y_p(t) = k_1\cos 2t + k_2\sin 2t - \frac{16\cos\frac{9t}{4}}{17}$$

Now we simply have to find k_1 and k_2 that satisfy the initial conditions. First we set $y(0) = 0$:

$$\begin{aligned}
0 &= y(0) \\
&= k_1\cos 0 + k_2\sin 0 - \frac{16\cos 0}{17} \\
&= k_1 + 0 - \frac{16}{17} \\
\frac{16}{17} &= k_1
\end{aligned}$$

Now we set $y'(0) = 0$:

$$\begin{aligned}
y'(t) &= -2k_1\sin 2t + 2k_2\cos 2t - \frac{16\cdot\frac{9}{4}\sin\frac{9t}{4}}{17} \\
0 = y'(0) &= -2k_1\sin 0 + 2k_2\cos 0 - \frac{16\cdot\frac{9}{4}\sin 0}{17} \\
&= 2k_2 \\
&= k_2
\end{aligned}$$

And so the solution to the IVP is:

$$y_0(t) = \boxed{\frac{16(\cos 2t - \cos \frac{9t}{4})}{17}}$$
$$= \frac{32}{17}\sin\left(\frac{t}{8}\right)\sin\left(\frac{17t}{8}\right) \qquad \text{(alternative form)}$$

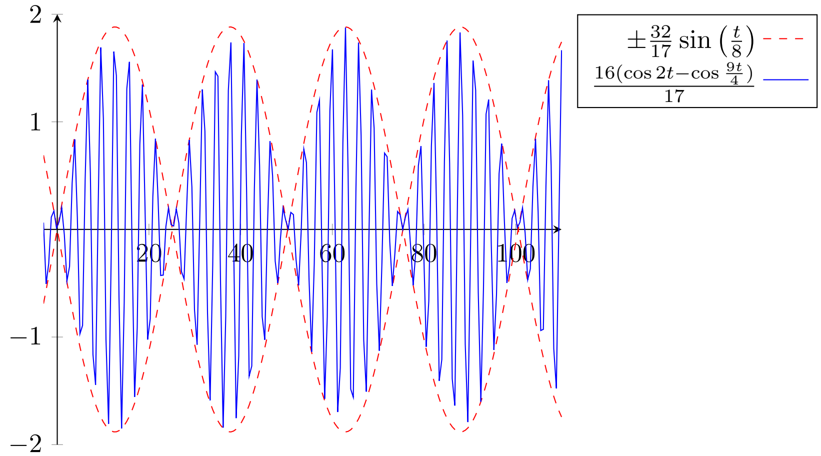
For graphing the solution, the alternative form tells us the amplitude of the red wave is given by:

$$\pm\frac{32}{17}\sin\left(\frac{t}{8}\right)$$

With the period of the red wave being:

$$\frac{4\pi}{\frac{9}{4}-2} = 16\pi \approx 50.26$$

This gives us the following graph:



For **b)** the roots of the characteristic equation $s^2+4=0$ are still $s=\pm 2i$. So homogenous solution y_h is still:

$$y_h(t) = k_1\cos 2t + k_2 \sin 2t$$

However, the particular solution of our complexified equation $y''+4y=e^{2it}$ is now of the following form:

$$\begin{aligned} y_c(t) &= Ate^{2it} \\ y_c'(t) &= Ae^{2it}(1+2it) \\ y_c''(t) &= 4Aie^{2it} - 4Ate^{2it} \\ &= 4Aie^{2it} - 4y_c \end{aligned}$$

Solving for A we find:

$$\begin{aligned} y_c'' + 4y_c &= 4Aie^{2it} \\ e^{2it} &= 4Aie^{2it} \\ 1 &= 4Ai \\ A &= \frac{1}{4i} = \frac{-i}{4} \end{aligned}$$

And so our particular solution y_p is:

$$\begin{aligned} y_p(t) &= \mathrm{Re}\left(y_c(t)\right) \\ &= \mathrm{Re}\left(Ate^{\frac{9it}{4}}\right) \\ &= \mathrm{Re}\left(-\frac{ite^{2it}}{4}\right) \\ &= \mathrm{Re}\left(-\frac{it(\cos 2t + i\sin 2t)}{4}\right) \\ &= \frac{t\sin 2t}{4} \end{aligned}$$

And so the general solution y is given by:

$$y(t) = k_1\cos 2t + k_2\sin 2t + \frac{t\sin 2t}{4}$$

Now we simply have to find k_1 and k_2 that satisfy the initial conditions. First we set $y(0) = 0$:

$$\begin{aligned} 0 &= y(0) \\ &= k_1\cos 0 + k_2\sin 0 + \frac{0}{4} \\ &= k_1 + 0 + 0 \\ 0 &= k_1 \end{aligned}$$

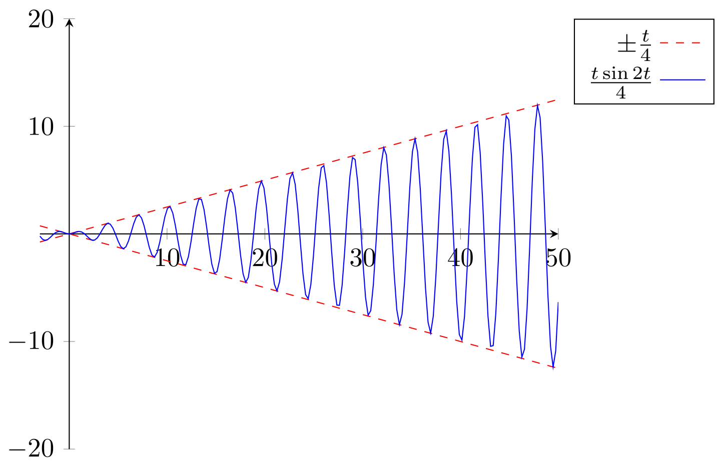
Now we set $y'(0) = 0$:

$$\begin{aligned} y'(t) &= -2k_1\sin 2t + 2k_2\cos 2t + \frac{t\cos 2t}{2} + \frac{\sin 2t}{4} \\ 0 = y'(0) &= -2k_1\sin 0 + 2k_2\cos 0 + \frac{0}{2} + \frac{\sin 0}{4} \\ &= 0 + 2k_2 + 0 + 0 \\ 0 &= k_2 \end{aligned}$$

So the solution to the IVP is simply:

$$y_0(t) = \frac{t\sin 2t}{4}$$

This gives us the following graph:



Problem 5

Problem: Find and classify all the equilibria of the following system:

$$\begin{cases} x' = x(-x - 3y + 150) = -x^2 - 3xy + 150x \\ y' = y(-2x - y + 100) = -2xy - y^2 + 100y \end{cases}$$

Solution: By inspection, we note that this system has 4 equilibria:

a) $\mathbf{p}_1 = (0, 0)$

b) $\mathbf{p}_2 = (0, 100)$

c) $\mathbf{p}_3 = (150, 0)$

d) $\mathbf{p}_4 = (30, 40)$

Where the last equilibria is found by solving $\begin{cases} -x - 3y + 150 \\ -2x - y + 100 \end{cases}$ To classify them via **linearization**, we must first compute the Jacobian of the system:

$$J\mathbf{F} = \begin{bmatrix} \frac{dx'}{dx} & \frac{dx'}{dy} \\ \frac{dy'}{dx} & \frac{dy'}{dy} \end{bmatrix} = \begin{bmatrix} -2x - 3y + 150 & -3x \\ -2y & -2x - 2y + 100 \end{bmatrix}$$

Now, applying the Jacobian to each equilibrium, we find:

a) $J\mathbf{F}(0, 0) = \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix}$ Both eigenvalues $\lambda_1 = 150$ and $\lambda_2 = 100$ are positive, thus $\mathbf{p}_1$ is a source.

b) $J\mathbf{F}(0, 100) = \begin{bmatrix} -150 & 0 \\ -200 & -100 \end{bmatrix}$ Both eigenvalues $\lambda_1 = -150$ and $\lambda_2 = -100$ are negative, thus $\mathbf{p}_2$ is a sink.

c) $J\mathbf{F}(150, 0) = \begin{bmatrix} -150 & -450 \\ 0 & -200 \end{bmatrix}$ Both eigenvalues $\lambda_1 = -150$ and $\lambda_2 = -200$ are negative, thus $\mathbf{p}_3$ is a sink.

d) $J\mathbf{F}(30, 40) = \begin{bmatrix} -30 & -90 \\ -80 & -40 \end{bmatrix}$ The eigenvalues are roots of the equation $(\lambda + 30)(\lambda + 40) - 90 \cdot 80$ which are $\lambda_1 = -120$ and $\lambda = 50$. This means $\mathbf{p}_4$ is a saddle.

Problem 6

Problem: Sketch the x-nullcline and y-nullcline of the system in problem 5. Then sketch its phase portrait.

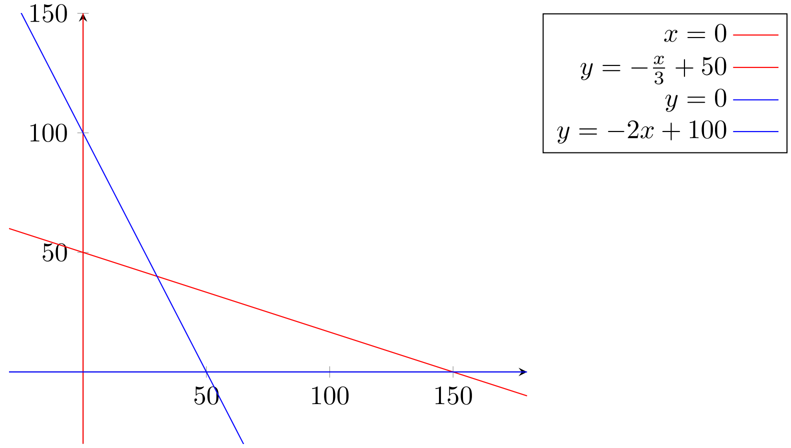
Solution: The x-nullcines are:

$$x = 0, \quad y = -\frac{x}{3} + 50$$

And the y-nullcines are:

$$y = 0, \quad y = -2x + 100$$

Graphing the x-nullclines in red, the y-nullclines in blue, and drawing in some solutions, we arrive at:



Problem 7

Problem: Find and classify all the equilibria of the following system:

$$\begin{cases} x' = x(2 - x - y) = 2x - x^2 - xy \\ y' = y(y - x^2) = y^2 - x^2 y \end{cases}$$

Solution: By inspection, we note that this system has 4 equilibria:

a) $\mathbf{p}_1 = (0, 0)$

b) $\mathbf{p}_2 = (2, 0)$

c) $\mathbf{p}_3 = (-2, 4)$

d) $\mathbf{p}_4 = (1, 1)$

Where the last two equilibria are found by solving $\begin{cases} 2 - x - y \\ y - x^2 \end{cases}$. The Jacobian of the system is given by:

$$J\mathbf{F} = \begin{bmatrix} \frac{dx'}{dx} & \frac{dx'}{dy} \\ \frac{dy'}{dx} & \frac{dy'}{dy} \end{bmatrix} = \begin{bmatrix} 2 - 2x - y & -x \\ -2xy & 2y - x^2 \end{bmatrix}$$

Now, applying the Jacobian to each equilibrium, we find:

a) $J\mathbf{F}(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ This matrix lies on the point $(2, 0)$ in the trace-determinant plane and thus $\mathbf{p}_1$ is a degenerate equilibria with lines of parallel sources around it.

b) $J\mathbf{F}(2, 0) = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix}$ Both eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$ are negative, thus $\mathbf{p}_2$ is a sink.

c) $J\mathbf{F}(-2, 4) = \begin{bmatrix} 2 & 2 \\ 16 & 4 \end{bmatrix}$ This matrix lies on the point $(6, -24)$ in the trace-determinant plane and thus $\mathbf{p}_3$ is a saddle.

d) $J\mathbf{F}(1, 1) = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$ This matrix lies on the point $(0, -3)$ in the trace-determinant plane and thus $\mathbf{p}_4$ is a saddle.

Problem 8

Problem: Sketch the x-nullcline and y-nullcline of the system in problem 7. Then sketch its phase portrait.

Solution: The x-nullcines are:

$$x = 0, \quad y = 2 - x$$

And the y-nullcines are:

$$y = 0, \quad y = x^2$$

Graphing the x-nullclines in red, the y-nullclines in blue, and drawing in some solutions, we arrive at: