Intro to Math Reasoning HW 10a

Ozaner Hansha

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Problem 1

Consider the following relation R on an arbitrarily large universe of sets U:

$$ARB \equiv (\exists f : A \to B)\ \underbrace{(\forall x, y \in A)\ f(a) = f(b) \to a = b}_{f \text{ is injective}}$$

Part a

Problem: Prove that R is not symmetric.

Solution: Consider the set of all functions from $A = \{1\}$ to $B = \{1, 2\}$:

$$\text{graph}(f_1) = \{(1,1)\}$$
$$\text{graph}(f_2) = \{(1,2)\}$$

Note that both functions are injective and so ARB. Now let us consider the set of all functions from B to A:

$$\text{graph}(g) = \{(1,1)(2,1)\}$$

Notice that there exists only one such function. Also note that this function is not injective, thus $\neg BRA$. This one counterexample is sufficient to show that R is not symmetric on all universes of sets.

Part b

Problem: Prove that R is not anti-symmetric.

Solution: Consider the set of all functions from $A = \{1\}$ to $B = \{2\}$:

$$\text{graph}(f) = \{(1,2)\}$$

This function is injective, thus ARB. Now consider the set of all functions from $B = \{1\}$ to $A = \{2\}$:

$$\text{graph}(f) = \{(2,1)\}$$

This function is also injective, thus BRA. However, note that $\{1\} \neq \{2\}$ (at least I hope it doesn't). And so R does not satisfy anti-symmetry:

$$ARB \wedge BRA \not\rightarrow A = B$$

This one counterexample is sufficient to show that that R is not anti-symmetric on all universes of sets.

Problem 2

Problem: Consider the following relation based off R used in problem 1 on an arbitrary universe of sets:

$$A <_{\#} B \equiv ARB \wedge \neg \, \text{isBijective}(A, B)$$

Where the isBijective predicate simply means there exists a bijection (a function that is both injective and surjective) between A and B. Prove this relation is a strict partial order.

Solution: We have to prove three properties of this relation:

- **Anti-reflexive**

  Proving this means showing that the following is false for any set A:

$$ARA \wedge \neg \, \text{isBijective}(A, A)$$

  Note that isBijective(A, A) is true because there does exist a bijection from any A to itself, namely the identity function: $\text{id}(a) = a$. Thus the statement is always false and anti-symmetry holds.

- **Anti-symmetric**

  Proving this means showing the following $A, B \in U$:

$$\begin{aligned} ARB \wedge \neg \, \text{isBijective}(A, B) &\implies \neg(BRA \wedge \neg \, \text{isBijective}(B, A)) \\ &\implies \neg BRA \vee \text{isBijective}(B, A) \end{aligned}$$

  Note that $\neg$ isBijective(A, B) implies $\neg$ isBijective(B, A) because all bijections have bijective inverses, and so if there wasn't one for $A \to B$ then there won't be one for $B \to A$. Our only mode of attack, then, is to show that $\neg BRA$.

  This can be done by noting that the Schroder-Bernstein theorem states that if there is an injective function from $A \to B$ and one from $B \to A$, then there must exist some bijection between the two sets:

$$ARB \wedge BRA \implies \text{isBijective}(A, B)$$

  However we know that while ARB is true, the consequence isBijective(A, B) is false. This means that $\neg BRA$. And so we are done.

- **Transitive**

  This is equivalent to proving the following for any sets A, B and C:

$$(ARB \wedge \neg\,\text{isBijective}(A,B)) \wedge (BRC \wedge \neg\,\text{isBijective}(B,A)) \implies (ARC \wedge \neg\,\text{isBijective}(A,C))$$

Note that ARB guarantees the existence of at least one injective function from $A \to B$ and so we will call one such function $f : A \to B$. We will do the same for the statement BRA and call it $g : B \to C$. Now note that the composition of these two injective functions $f \circ g : A \to C$ is also a injective function (compositions of injective functions are injective). This satisfies the ARC portion of the consequent.

We can prove the second part of the consequent, namely $\neg\,\text{isBijective}(A,C)$, by contradiction. First note that if A and C are bijective then there exists injective functions from the domain to the codomain and vice versa:

$$\text{isBijective}(A,C) \implies ARC \wedge CRA$$

Now note that we are assuming that $A <_{\#} B$, which means there is an injective function from A to B:

$$A <_{\#} B \implies ARB$$

From the transitivity of injective functions we used earlier we can say:

$$CRA \wedge ARB \implies CRB$$

However note that the antecedent we are assuming includes $B <_{\#} C$ which implies BRC but also that B and C are not equinumerous:

$$B <_{\#} C \equiv BRC \wedge \neg\,\text{isBijective}(B,C)$$

This implies that $\neg CRB$ because if there was an injunctive function from C to B then Schroder-Bernstein would tell us that the sets are indeed equinumerous. Thus we are left with a contradiction:

$$CRB \wedge \neg CRB$$

And so our original assumption that there was a bijection between A and C was false.

Problem 3

Problem: Prove that if $(\exists n \in \mathbb{N})\ |A| = n$ then:

$$B \subsetneq A \implies |B| < |A|$$

Solution: Note that a subset of a set embeds that set (i.e. $B \subseteq A \to BRA$). Also note that if $|B| \neq |A|$ (i.e. no bijection) then we can use the strict partial order we defined earlier:

$$BRA \wedge |B| \neq |A| \equiv B <_{\#} A$$

Now note that for all natural numbers n (where $n = \{0, 1, \cdots, n-1\}$):

$$n <_{\#} B$$

Via the transitive property we can see that:

$$(\forall n \in \mathbb{N})\ n <_{\#} B \wedge B <_{\#} A \implies n <_{\#} A$$

Which is the same as saying A is infinite. This, however, depends on the fact that there is no bijection between A and B. To show this we just have to show that there is no injective function from A to B, (i.e. ARB). This is obvious because $|A| = n$ for some finite n and thus there is at least one element $a_0 \notin B$. Then $|A \setminus \{a_0\}| = n - 1$. And either that set is equal to B (thus equinumerous) or there is another element a_1 that is also not in B. This argument must end at some point since there are finite number of elements in A.

Problem 4

Problem: Prove that for any two sets A and B:

$$(\forall n \in \mathbb{N})\ B \subseteq A \wedge |B| > n \implies |A| > n$$

Solution: We know that all subsets of a set embed that set, thus:

$$B \subseteq A \implies BRA$$

We also are assuming that B is greater in cardinality than any finite n, meaning:

$$(\forall n \in N)\ nRB$$

This automatically entails that B is not equinumerous with any n since there is always an injection with $n+1$. We know that by the transitivity of injective functions (due to the composition of them) that:

$$(\forall n \in N)\ nRB \wedge BRA \implies nRA$$

And so now we have $|A| \geq n$ for all naturals n. But recall just like with nRB, we know that this automatically entails that $|A| \neq n$ for any finite n. Thus we are left with: $|A| > n$

Problem 5

Problem: Prove the following:

$$(\exists B \subsetneq A)\ ARB \iff (\forall n \in \mathbb{N})\ |A| > n$$

Solution: Let P be the left hand proposition and Q the right hand one. Note that $\neg Q$ means A is finite, and $\neg P$ means there is no injective function from A to B. We know then from problem 3 that $\neg Q \to \neg P$, and this is just the contraposition (tautology) of the forward direction $P \to Q$, and so we only need to prove the backwards direction.

Note that if a set A is infinite then, by the axiom of choice, it has a countable subset $\{x_1, x_2, x_3, \cdots, x_j, \cdots\} \subseteq A$. Now note that we can easily define a one-to-one map from A to $A \setminus \{x_1\}$ (i.e. a proper subset of A) like so:

$$f(x_j) = x_{j+1}$$

Thus $ARA \setminus \{x_1\}$ meaing the backwards relation is satisfied.