Numerical Analysis HW #2

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March 7, 2019

Problem 1

Part a

Problem: Perform LUP factorization on the following matrix using Gaussian elimination with partial pivoting:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution: Below I demonstrate each step of Gaussian elimination performed on A as it becomes U. Accompanying these steps is the current L and P matrices. Our initial matrices, i.e. step 0:

$$U = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In step 1, we notice that no permutations need to be done as u_{11} has the largest magnitude of the first column. Now we find that $\frac{u_{21}}{u_{11}} = \frac{1}{3}$ and so we set ℓ_{21} equal to this and perform the following row operation: $\vec{u}_2 := \vec{u}_2 - \frac{1}{3}\vec{u}_1$, giving us:

$$U = \begin{bmatrix} 3 & -1 & 1 \\ 0 & \frac{10}{3} & -\frac{1}{3} \\ 1 & 1 & 3 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we continue with step 1, eliminating the first column of row 3. We find that $\frac{u_{31}}{u_{11}} = \frac{1}{3}$ and so we set ℓ_{31} equal to this and perform the following row operation: $\vec{u}_3 := \vec{u}_3 - \frac{1}{3}\vec{u}_1$, giving us:

$$U = \begin{bmatrix} 3 & -1 & 1 \\ 0 & \frac{10}{3} & -\frac{1}{3} \\ 0 & \frac{4}{3} & \frac{8}{3} \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \ell_{32} & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now for step 2, the final step, we again notice that no row permutations need to take place as u_{22} has the largest magnitude of the second column. We see that $\frac{u_{32}}{u_{22}} = \frac{2}{5}$ and so we set ℓ_{32} equal to this and perform the following row operation: $\vec{u}_3 := \vec{u}_3 - \frac{2}{5}\vec{u}_2$, giving us:

$$U = \begin{bmatrix} 3 & -1 & 1 \\ 0 & \frac{10}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{14}{5} \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{2}{5} & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And, as the matrix is in row echelon form, we are done. Notice that no row permutations needed to be performed and so $P = I$.

Part b

Problem: Use MATLAB's `lu` command to give the LUP factorization of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & -1 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution: Running the command `[L,U,P]=lu(1 1 3;3 -1 1;1 3 0)` returned the 3 following matrices:

$$U = \begin{bmatrix} 3.0000 & -1.0000 & 1.0000 \\ 0 & 3.3333 & -0.3333 \\ 0 & 0 & 0.4000 \end{bmatrix} \qquad L = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0.3333 & 1.0000 & 0 \\ 0.3333 & 2.8000 & 1.0000 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Problem 2

Problem: Consider the $n \times n$ Hilbert matrix:

$$(H_n)_{ij} = \frac{1}{i + j - 1}$$

Solve the linear system $H_n\vec{x} = \vec{b}$ where $\vec{b} = H_n\vec{y}$ and $y_i = \frac{1}{\sqrt{n}}$ for $n = 8, 12, 16$ using MATLAB. Record the relative error, relative residual, the condition number, and the product of the last two in a table.

Solution: Note that all matrix and vector norms $\|\cdot\|$ refer to the L^∞ norm. This is reflected in the results in table below.

n	$\frac{\|\vec{x}-\vec{z}\|}{\|\vec{x}\|}$	$\frac{\|H\vec{z}-\vec{b}\|}{\|\vec{b}\|}$	$\|H\|\|H^{-1}\|$	$\|A\|\|A^{-1}\|\frac{\|H\vec{z}-\vec{b}\|}{\|\vec{b}\|}$
8	1.9237e-07	1.1554e-16	3.3873e+10	3.9136e-06
12	0.2369	6.1967e-17	3.8420e+16	2.3807
16	60.3329	6.5680e-16	7.1626e+19	4.7044e+04

Problem 3

Part a

Problem: Give the iteration matrices for the Jacobi M_J and Gauss Seidel M_G methods of the following matrix:

$$\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$$

Solution: Recall that the iteration matrix for the Jacobi method is:

$$\begin{aligned}
M_J &= -D^{-1}(L+U) \\
&= -\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}^{-1}\left(\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}+\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\right) \\
&= -\begin{bmatrix}1 & 0\\ 0 & \frac{1}{4}\end{bmatrix}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix} \\
&= \begin{bmatrix}0 & 1\\ \frac{1}{4} & 0\end{bmatrix}
\end{aligned}$$

Recall that the iteration matrix for the Gauss Seidel method is:

$$\begin{aligned}
M_G &= -(L+D)^{-1}U \\
&= -\left(\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix} \\
&= -\begin{bmatrix}1 & 0\\ -1 & 4\end{bmatrix}^{-1}\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix} \\
&= -\frac{1}{4}\begin{bmatrix}4 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix} \\
&= \begin{bmatrix}0 & 1\\ 0 & \frac{1}{4}\end{bmatrix}
\end{aligned}$$

Part b

Problem: Determine whether the Jacobi method converges.

Solution: Recall that any iteration method converges when the spectral radius of the iteration matrix, here M_J, is less than 1:

$$\rho(M_J) < 1$$

Below we will calculate the eigenvalues of M_J to determine it's spectral radius (i.e. the max of its eigenvalues):

$$\begin{aligned}
0 &= \det(M_J - \lambda I) \\
&= \det\left(\begin{bmatrix}0 & 1\\ \frac{1}{4} & 0\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) \\
&= \det\begin{pmatrix}-\lambda & 1\\ \frac{1}{4} & -\lambda\end{pmatrix} \\
&= \lambda^2 - \frac{1}{4}
\end{aligned}$$

Factoring this quadratic polynomial, we find that $\lambda = \pm\frac{1}{2}$. The spectral radius of M_J is simply the maximum of the magnitudes of its eigenvalues:

$$\rho(M_J) = \frac{1}{2} < 1$$

And so since the spectral radius of M_J is less than 1, the method will converge.

Problem 4

Part a

Problem: Use MATLAB to implement the power method to approximate the largest eigenvalue of the following matrix:

$$A = \begin{bmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{bmatrix}$$

Perform the iteration 10 times and record the approximate eigenvalue, the successive difference of those approximations, and the successive ratio of those successive differences.

Solution

m	$\lambda_1^{(m)}$	$\lambda_1^{(m)} - \lambda_1^{(m-1)}$	$\frac{\lambda_1^{(m)} - \lambda_1^{(m-1)}}{\lambda_1^{(m-1)} - \lambda_1^{(m-2)}}$
1	13.130434782	-	-
2	14.758732361	1.628297579e+0	-
3	14.972638820	2.139064586e-1	0.1313681610
4	14.996952606	2.431378644e-2	0.1136655087
5	14.999661309	2.708702793e-3	0.1114060452
6	14.999962366	3.010569254e-4	0.1111443183
7	14.999995818	3.345188554e-5	0.1111148182
8	14.999999535	3.716889974e-6	0.1111115237
9	14.999999948	4.129879425e-7	0.1111111561
10	14.999999994	4.588755153e-8	0.1111111168

Part b

Problem: Compute the eigenvalues of A using the `eig` function in MATLAB. Sort the eigenvalues in descending order. Do the ratios in part a converge to $\frac{\lambda_2}{\lambda_1}$?

Solution: The eigenvalues of A sorted in decreasing order are:

$$15 \leq 5 \leq 5 \leq -1$$

The ratio of the second largest eigenvalue to the largest is thus:

$$\frac{\lambda_2}{\lambda_1} = \frac{5}{15} = 0.333333\overline{33}$$

This is clearly not what the ratios converge to.

Problem 5

Problem: Give the linear system satisfied by the minimizing choices of a_1, a_2, a_3:

$$E(a_1, a_2, a_3) = \int_0^1 (a_1 + a_2 x + a_3 x^2 - f(x))^2 \, dx$$

What is the relation of this system to the Hilbert matrix in problem 2?

Solution: To minimize this function we set its partial derivatives equal to 0:

$$\frac{\partial E}{\partial a_1} = 2\int_0^1 (a_1 + a_2 x + a_3 x^2 - f(x))\ dx = 0$$
$$\frac{\partial E}{\partial a_2} = 2\int_0^1 x(a_1 + a_2 x + a_3 x^2 - f(x))\ dx = 0$$
$$\frac{\partial E}{\partial a_3} = 2\int_0^1 x^2(a_1 + a_2 x + a_3 x^2 - f(x))\ dx = 0$$

Solving these out we find:

$$a_1\int_0^1 1\ dx + a_2\int_0^1 x\ dx + a_3\int_0^1 x^2\ dx = \int_0^1 f(x)\ dx$$
$$a_1\int_0^1 x\ dx + a_2\int_0^1 x^2\ dx + a_3\int_0^1 x^3\ dx = \int_0^1 xf(x)\ dx$$
$$a_1\int_0^1 x^2\ dx + a_2\int_0^1 x^3\ dx + a_3\int_0^1 x^4\ dx = \int_0^1 x^2 f(x)\ dx$$

After integration these become:

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} = \int_0^1 f(x)\ dx$$
$$\frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = \int_0^1 xf(x)\ dx$$
$$\frac{a_1}{3} + \frac{a_2}{4} + \frac{a_3}{5} = \int_0^1 x^2 f(x)\ dx$$

We can phrase this system of equations as the following linear system:

$$\underbrace{\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} \int_0^1 f(x)\ dx \\ \int_0^1 xf(x)\ dx \\ \int_0^1 x^2 f(x)\ dx \end{bmatrix}}_{\vec{b}}$$

You'll notice that H is the 3×3 Hilbert matrix. That is to say, for any nth order polynomial least square approximation of a continuous function f over the interval $[0, 1]$, the minimizing parameters $\vec{a}$ can be found by solving the following linear system:

$$H_{n+1}\vec{a} = \vec{b}$$

Where H_n is the nth order Hilbert matrix and $b_i = \int_0^1 x^{i-1} f(x)\ dx$ (indexing starts at 1).