Intro to Math Reasoning HW 7b

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October 31, 2018

Problem 1

Problem: Prove that for any relation R on A that for all $a \in A$:

$$R \text{ is reflexive } \iff \text{graph}(I_A) \subseteq \text{graph}(R)$$

where I_A is the identity relation on A.

Solution: This is obvious because, by definition, for a relation R to be reflexive its graph must contain the set of all pairs of the form (a, a) where $a \in A$:

$$(\forall a \in A)\ aRa \equiv (a, a) \in \text{graph}(R)$$

and the graph of the identity relation on A, by definition, is simply the set of all (a, a) where $a \in A$:

$$\text{graph}(I_A) \equiv \{(a, a) \mid a \in A\}$$

And so any pair $(a, b) \in \text{graph}(I_A)$ must also be in R:

$$aI_Ab \to aRb$$

And so the left is a subset of the right.

Problem 2

Problem: Prove that for any relation R on A that for all $a \in A$:

$$R \text{ is anti-reflexive } \iff \text{graph}(R) \setminus \text{graph}(I_A) = \emptyset$$

Solution: By definition, for a relation R to be anti-reflexive its graph cannot contain any pair of the form (a, a) where $a \in A$:

$$(\forall a \in A)\ \neg(aRa) \equiv (a, a) \notin \text{graph}(R)$$

and the graph of the identity relation on A, by definition, is simply the set of all (a, a) where $a \in A$:

$$\text{graph}(I_A) \equiv \{(a, a) \mid a \in A\}$$

And so any pair $(a, b) \in \text{graph}(I_A)$ cannot also be in R:

$$aI_Ab \to \neg(aRb)$$

This means that no element of the graph of I_A is in the graph of the R and so they are disjoint.

Problem 3

Problem: Prove that for any relation R on A that for all $a \in A$:

$$R \text{ is symmetric} \iff (R = \text{Я})$$

where Я is the reverse of R.

Solution: This should be quite clear as all symmetric relations satisfy the following:

$$(\forall a, b \in A)\ aRb \to bRa$$

And since a and b were arbitrary indistinguishable variables from A, the stronger statement:

$$(\forall a, b \in A)\ aRb \equiv bRa$$

holds as well. Along with the definition of the reverse of R:

$$a\text{Я}b \equiv bRa$$

it's clear that for all a and b:

$$aRb \equiv bRa \equiv a\text{Я}b$$

Thus the relations are actually equivalent, given symmetry.

Problem 4

Problem: Prove that for any relation R on A that for all $a, b \in A$:

$$R \text{ is anti-symmetric} \iff (aRb \wedge \neg(a\text{Я}b) \to a = b)$$

Solution: All anti-symmetric relations satisfy the following:

$$(\forall a, b \in A)\ aRb \wedge \neg(bRa) \to a = b$$

And as shown in Problem 3, $aRb \equiv bRa \equiv a\text{Я}b$, and so the two statements above are actually equivalent.

Problem 5

Part a

Problem: Prove that the following relation C on the set of finite subsets of $\mathbb{Z}$ is a partial order:

$$xCy \equiv |x| \leq |y|$$

Solution: This is not a partial order because, due to anti-symmetry, no two *distinct* elements x and y can satisfy the following:

$$xCy \wedge yCx$$

Yet, consider the sets $\{1, 2\}$ and $\{-1, -2\}$. These sets are both clearly finite subsets of $\mathbb{Z}$ as well as not equivalent. Yet the definition of C states that:

$$\{1, 2\}C\{-1, -2\} \equiv |\{1, 2\}| \leq |\{-1, -2\}| \equiv 2 \leq 2$$
$$\{-1, -2\}C\{1, 2\} \equiv |\{-1, -2\}| \leq |\{1, 2\}| \equiv 2 \leq 2$$

It cannot be the case that both of those statements are true *and* C is a partial order. Thus, because $2 \leq 2$, C is not a partial order.

Part b

Problem: Prove that the following relation E on the set of finite subsets of $\mathbb{Z}$ is a partial order:

$$xEy \equiv |x| = |y|$$

Solution: This is not a partial order because, due to anti-symmetry, no two *distinct* elements x and y can satisfy the following:

$$xCy \wedge yCx$$

Yet, consider the sets $\{1, 2\}$ and $\{-1, -2\}$. These sets are both clearly finite subsets of $\mathbb{Z}$ as well as not equivalent. Yet the definition of C states that:

$$\{1, 2\}C\{-1, -2\} \equiv |\{1, 2\}| = |\{-1, -2\}| \equiv 2 = 2$$
$$\{-1, -2\}C\{1, 2\} \equiv |\{-1, -2\}| = |\{1, 2\}| \equiv 2 = 2$$

It cannot be the case that both of those statements are true *and* C is a partial order. Thus, because 2=2, C is not a partial order.