Math Statistics
Weekly HW 5

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Question 1

Suppose we have a population described by a Poisson distribution with paramter λ. That is, with pdf $f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$ for a nonnegative integer x.

Part a: Compute the Fisher information about λ for a single observation X.

Solution: The Fisher information $\mathcal{I}(\lambda)$ is given by:

$$
\begin{aligned}
\mathcal{I}(\lambda) &= E\left[\left(\frac{\partial}{\partial\lambda}\ln f(X;\lambda)\right)^2\right] && \text{(def. of Fisher information)}\\
&= E\left[\left(\frac{\partial}{\partial\lambda}\ln \frac{\lambda^X e^{-\lambda}}{X!}\right)^2\right]\\
&= E\left[\left(\frac{\partial}{\partial\lambda}(X\ln\lambda - \lambda - \ln X!)\right)^2\right]\\
&= E\left[\left(\frac{X}{\lambda} - 1\right)^2\right]\\
&= \operatorname{Var}\left(\frac{X}{\lambda}\right) && \text{(def. of variance } (E[X/\mu_X] = 1))\\
&= \frac{1}{\lambda^2}\operatorname{Var}(X)\\
&= \frac{\lambda}{\lambda^2} = \boxed{\frac{1}{\lambda}}
\end{aligned}
$$

Part b: Show that the sample mean $\bar{X}$ (for sample size $n \in \mathbb{Z}^+$) is a minimum variance unbiased estimator for λ

Solution: First we will prove that our estimator is unbiased by showing that its mean equals λ:

$$
\begin{aligned}
E[\bar{X}] &= E\left[\frac{\sum_{i=1}^n X_i}{n}\right] && \text{(def. of } \bar{X})\\
&= \frac{\sum_{i=1}^n E[X_i]}{n} && \text{(linearity of expectation)}\\
&= \frac{\sum_{i=1}^n \lambda}{n} && (X_i \sim \text{Pois}(\lambda))\\
&= \frac{n\lambda}{n} = \boxed{\lambda}
\end{aligned}
$$

Now recall that if the following is satisfied then $\bar{X}$ is an MVU estimator:

$$\mathrm{Var}(\bar{X}) = \frac{1}{n\mathcal{I}(\lambda)}$$

Where each $X_i \sim \mathrm{Pois}(\lambda)$ is i.i.d. And indeed the above does hold:

$$\begin{aligned}
\mathrm{Var}(\bar{X}) &= \mathrm{Var}\left(\frac{\sum_{i=1}^n X_i}{n}\right) && \text{(def. of } \bar{X}\text{)} \\
&= \mathrm{Var}\left(\frac{\sum_{i=1}^n X_i}{n}\right) \\
&= \frac{1}{n^2}\,\mathrm{Var}\left(\sum_{i=1}^n X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^n \mathrm{Var}\left(X_i\right) && (X_i \text{ are independent}) \\
&= \frac{1}{n^2}\sum_{i=1}^n \lambda && (X_i \sim \mathrm{Pois}(\lambda)) \\
&= \frac{n\lambda}{n^2} = \frac{1}{n(1/\lambda)} = \frac{1}{n\mathcal{I}(\lambda)} && \text{(part a)}
\end{aligned}$$

And so, $\bar{X}$ is a MVU estimator for λ.

Question 2

Problem: If $\hat{\Theta}$ is an unbiased estimator for θ, is $\hat{\Theta}^2$ is an unbiased estimator for θ^2?

Solution: No. Consider a population X with mean μ and variance σ^2. Recall that the sample mean (for sample size n) is an unbiased estimator of a population's mean μ:

$$E[\bar{X}] = \mu$$

However, we find that $E[\bar{X}^2] \neq \mu^2$:

$$\begin{aligned}
\mathrm{Var}(\bar{X}) &= E[\bar{X}^2] - E[\bar{X}]^2 && \text{(true of any RV)} \\
E[\bar{X}^2] &= \mathrm{Var}(\bar{X}) + E[\bar{X}]^2 \\
&= \frac{\sigma^2}{n} + \mu^2 && \text{(mean/variance of sample mean)} \\
&\neq \mu^2
\end{aligned}$$

And so, as we have shown, it is not necessarily the case that the square of an unbiased estimator is itself an unbiased estimator of the parameter squared.

Question 3

Problem: Suppose that we have a population with known mean μ and unknown variance σ^2. Show that $\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{n}$ is an unbiased estimator for σ^2.

Solution: Note the following chain of equalities:

$$\begin{aligned}
E\left[\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{n}\right] &= \frac{1}{n}\sum_{i=1}^{n} E\left[(X_i-\mu)^2\right] && \text{(linearity of expectation)} \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{Var}(X_i) && \text{(def. of variance)} \\
&= \frac{1}{n}\sum_{i=1}^{n} \sigma^2 && \text{(def. } \sigma^2\text{)} \\
&= \frac{n\sigma^2}{n} = \sigma^2
\end{aligned}$$

And so, since the mean of this estimator is indeed the parameter σ^2, it is unbiased.

Question 4

Problem: Show that $\bar{X}^2 - \frac{S^2}{n}$ is an unbiased estimator for μ^2.

Solution: Now consider the following chain of equalities:

$$\begin{aligned}
E\left[\bar{X}^2 - \frac{S^2}{n}\right] &= E[\bar{X}^2] - \frac{E[S^2]}{n} && \text{(linearity of expectation)} \\
&= \mathrm{Var}(\bar{X}) + E[\bar{X}]^2 - \frac{E[S^2]}{n} && (E[\bar{X}^2] = \mathrm{Var}(\bar{X}) + E[\bar{X}]^2) \\
&= \frac{\sigma^2}{n} + \mu^2 - \frac{\sigma^2}{n} && (E[S^2] = \sigma^2) \\
&= \mu^2
\end{aligned}$$

And so, since the mean of this estimator is indeed the parameter μ^2, it is unbiased.

Where S^2 is the corrected sample variance $\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.