# Numerical Analysis HW #4

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## Problem 1

**Problem:** Use the composite trapezoid method with 4 equally sized subintervals to approximate the following integral:

$$\int_1^2 x \ln x \; dx$$

Also give an upper bound of the approximation's error.

**Solution:** Splitting up our interval of $[1,2]$ into $n = 4$ subintervals (with $\Delta x = \frac{1}{4}$) gives us the following 5 nodes (rounded to the 5th decimal place) with which to interpolate:

$$(1,0)\,,(1.25,0.27893)\,,(1.5,0.60820)\,,(1.75,0.97933)\,,(2,1.38629)$$

Recall that the composite trapezoid rule for a uniform distribution of 5 points is given by:

$$\frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

Leaving us with:

$$\int_1^2 x \ln x \; dx \approx \frac{1}{8}\left(0 + 2(0.27893) + 2(0.60820) + 2(0.97933) + 1.38629\right) = \boxed{0.63990}$$

As for the error of this approximation, recall that the maximum absolute error of the trapezoid method is given by the following:

$$\frac{(\Delta x)^3 n}{12} \max_{\xi \in [1,2]} |f''(\xi)|$$

The second derivative of $f$ is given by:

$$f(x) = x \ln x \qquad f'(x) = \ln x + 1 \qquad f''(x) = \frac{1}{x}$$

The second derivative, the inverse function, is a decreasing function over the positive reals. Since our interval $[1,2] \subseteq \mathbb{R}^+$, the maximum of this function is at the end point $x = 1$. This gives us $f(1) = 1$. Plugging this back into our error bound we arrive at:

$$|\,\text{error}\,| \leq \frac{(\Delta x)^3 n}{12} \max_{\xi \in [1,2]} |f''(\xi)| = \frac{\left(\frac{1}{4}\right)^3 (4)}{12}(1) = \boxed{\frac{1}{192}}$$

# Problem 2

**Problem:** Use the composite Simpson rule with 2 equally sized subintervals to approximate the same integral as problem 1 and give an upper bound of the approximation's error.

**Solution:** Splitting up our interval of $[1, 2]$ into $n = 2$ subintervals (with $\Delta x = \frac{1}{2}$) gives us the following 3 nodes (rounded to the 5th decimal place) with which to interpolate:

$$(1, 0), (1.5, 0.60820), (2, 1.38629)$$

Recall that the composite trapezoid rule for a uniform distribution of 3 points is given by:

$$\frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

Leaving us with:

$$\int_1^2 x \ln x \; dx \approx \frac{1}{6}(0 + 4(0.60820) + 1.38629) = \boxed{0.63651}$$

As for the error of this approximation, recall that the maximum absolute error of the trapezoid method is given by the following:

$$\frac{(\Delta x)^5 n}{180} \max_{\xi \in [1,2]} |f^{(4)}(\xi)|$$

The fourth derivative of $f$ is given by:

$$f''(x) = \frac{1}{x} \qquad f'''(x) = \frac{-1}{x^2} \qquad f^{(4)}(x) = \frac{2}{x^3}$$

The fourth derivative of $f$ is a decreasing function over the positive reals. Since our interval $[1, 2] \subseteq \mathbb{R}^+$, the maximum of this function is at the end point $x = 1$. This gives us $f(1) = 2$. Plugging this back into our error bound we arrive at:

$$|\,\text{error}\,| \leq \frac{(\Delta x)^5 n}{180} \max_{\xi \in [1,2]} |f^{(4)}(\xi)| = \frac{\left(\frac{1}{2}\right)^5 (2)}{180}(2) = \boxed{\frac{1}{1440}}$$

# Problem 3

**Problem:** Use the Gaussian–Legendre quadrature rule with $n = 2$ to approximate the same integral as problem 1.

**Solution:** Before we can apply the rule, we must first perform a change of interval from $[1, 2]$ to the bi-unit interval $[-1, 1]$. The formula for doing so is given by:

$$\begin{aligned}
\int_a^b f(x) \; dx &= \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) \; dx \\
\int_1^2 f(x) \; dx &= \frac{2-1}{2} \int_{-1}^1 f\left(\frac{2-1}{2}x + \frac{1+2}{2}\right) \; dx \\
&= \frac{1}{2} \int_{-1}^1 f\left(\frac{1}{2}x + \frac{3}{2}\right) \; dx \\
&= \frac{1}{2} \int_{-1}^1 g(x) \; dx \qquad \text{(where } g(x) = f\left(\tfrac{1}{2}x + \tfrac{3}{2}\right)\text{)}
\end{aligned}$$

Now recall that Gaussian–Legendre quadrature on a function $g$ over $[-1,1]$ with $n=2$ is given by the following:

$$\int_{-1}^{1} g(x)\ dx \approx g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

Plugging this into our change of interval, our approximation is:

$$\begin{aligned}
\int_{1}^{2} f(x)\ dx &= \frac{1}{2}\int_{-1}^{1} g\left(x\right)\ dx \\
&\approx \frac{1}{2}\left(g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)\right) \\
&= \frac{1}{2}\left(f\left(-\frac{1}{2\sqrt{3}}+\frac{3}{2}\right) + f\left(\frac{1}{2\sqrt{3}}+\frac{3}{2}\right)\right) \\
&= \boxed{0.636149}
\end{aligned}$$

# Problem 4

**Problem:** Use both the composite trapezoid and Simpson rule to approximate the following integrals to a tolerance of $10^{-2}, 10^{-4}, 10^{-8}$ and record the number of intervals $n$ needed and the error between the last two iterations for each.

$$\underbrace{\int_0^1 (1-4x(1-x))^{\frac{1}{3}}\ dx}_{I1} \qquad \underbrace{\int_0^1 xe^{-x}\ dx}_{I2}$$

Also use the Simpson rule on the second integral to a tolerance of $10^{-16}$ and see the results.

**Solution:** Below are the results for the composite trapezoid rule:

| tol | intervals I1 | error I1 | intervals I2 | error I2 |
|---|---|---|---|---|
| $10^{-2}$ | 16 | 0.007944902207349 | 8 | 0.003894761455888 |
| $10^{-4}$ | 256 | 9.364872715644790e-05 | 64 | 6.103234470516972e-05 |
| $10^{-8}$ | 65536 | 9.903734277116882e-09 | 8192 | 3.725290242950763e-09 |

and the results for the composite Simpson's rule:

| tol | intervals I1 | error I1 | intervals I2 | error I2 |
|---|---|---|---|---|
| $10^{-2}$ | 4 | 0.009130429264851 | 4 | 4.551028438593008e-05 |
| $10^{-4}$ | 64 | 9.066353416042894e-05 | 4 | 4.551028438593008e-05 |
| $10^{-8}$ | 16384 | 8.784285410179393e-09 | 64 | 7.028795323549275e-10 |
| $10^{-16}$ | - | - | 4096 | 5.551115123125783e-17 |