Math Statistics
Weekly HW 4

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Question 1

Problem: Suppose we have a normally distributed population with unknown mean μ and variance σ^2, and we take two samples from it, the first of size 8 and the second of size 10. If the second sample has a sample variance of 1, give a 90% confidence interval for the sample variance of the first sample.

Solution: First note that the following functions of the first and second sample variances, S_1^2 and S_2^2 respectively, have the following distributions:

$$\frac{7S_1^2}{\sigma^2} \sim \chi_7^2 \qquad \frac{9S_2^2}{\sigma^2} \sim \chi_9^2$$

Now, recalling that the ratio of two chi-squared RVs divided by their degrees of freedom is an F-distributed RV, we have that:

$$F_{7,9} \sim \frac{\frac{7S_1^2}{\sigma^2}/7}{\frac{9S_2^2}{\sigma^2}/9} = \frac{S_1^2}{S_2^2}$$

We can now finally derive the desired confidence interval (note that $F_{7,9}(x)$ is a cdf):

$$\begin{aligned}
.90 &= P\left(F_{7,9}(.1/2) \leq \frac{S_1^2}{S_2^2} \leq F_{7,9}(1-.1/2) \mid S_2^2 = 1\right) \\
&= P\left(F_{7,9}(.05) \leq \frac{S_1^2}{S_2^2} \leq F_{7,9}(.95) \mid S_2^2 = 1\right) \\
&= P\left(F_{7,9}(.05) \leq S_1^2 \leq F_{7,9}(.95)\right) \\
&\approx P\left(.27198 \leq S_1^2 \leq 3.29275\right)
\end{aligned}$$

And so the 90% confidence interval of S_1^2 is given by:

$$[.27198, 3.29275]$$

Question 2

Problem: Suppose we take two samples from a normally distributed population with unknown mean μ and variance σ^2. If the sample sizes are 12 and 9 respectively, what is the probability that the first sample has a sample variance which is at least 3 times as large as the second sample.

Solution: First note the following:

$$F_{11,8} \sim \frac{\frac{11S_1^2}{\sigma^2}/11}{\frac{8S_2^2}{\sigma^2}/8} = \frac{S_1^2}{S_2^2}$$

At this point, calculating the desired probability is simple:

$$\begin{aligned}
P\left(\frac{S_1^2}{S_2^2} \geq 3\right) &= 1 - P\left(\frac{S_1^2}{S_2^2} < 3\right) && \text{(complement)} \\
&= 1 - F_{11,8}(3) && \left(\tfrac{S_1^2}{S_2^2} \sim F_{11,8}\right) \\
&\approx .06488
\end{aligned}$$