Linear Algebra Midterm #2

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Problem 1

Consider the following linear map T on $\mathcal{P}_2(\mathbb{R})$:

$$T(f(x)) = f''(x) + f'(x) + f(x)$$

Part a: Determine whether or not T is invertible.

Solution: Let us first compute the matrix representative of T w.r.t. to the standard basis $\beta = \{x^2, x, 1\}$. First we define our coordinatization function $\Phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ like so:

$$\Phi(ax^2 + bx + c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Then we compute the coordinatization of each of the basis vectors under T:

$$\Phi(T(x^2)) = \Phi(x^2 + 2x + 2) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\Phi(T(x)) = \Phi(x + 1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Phi(T(1)) = \Phi(1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Finally we combine them to form the matrix representation:

$$[T]_\beta = \begin{bmatrix} | & | & | \\ \Phi(T(x^2)) & \Phi(T(x)) & \Phi(T(1)) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Now we simply compute the determinant across the first row:

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = 1 \cdot (1 \cdot 1 - 0 \cdot 1) = 1$$

And because we have the following chain of equivalences:

$$\begin{aligned} \det [T]_\beta \neq 0 &\iff [T]_\beta \text{ is invertible} \\ &\iff T \text{ is invertible} \end{aligned}$$

T must be invertible as $\det [T]_\beta \neq 0$. With this we are done.

Part b: Determine whether or not T is diagonalizable.

Solution: First we compute the eigenvalues of $[T]_\beta$ by solving its characteristic equation:

$$\begin{aligned}
0 &= p(\lambda) \\
&= \det([T]_\beta - \lambda I) \\
&= \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 2 & 1 & 1-\lambda \end{bmatrix} \\
&= (1-\lambda)((1-\lambda)(1-\lambda) - 0 \cdot 1) \\
&= (1-\lambda)^3
\end{aligned}$$

Solving for λ we find that $[T]_\beta$ has a single eigenvalue, 1, with an algebraic multiplicity (AM) of 3. Now let us compute the geometric multiplicity (GM) of this eigenvalue (i.e. the dimension of its associated eigenspace):

$$\begin{aligned}
\dim \mathcal{E}_1 &= \dim \operatorname{Null}([T]_\beta - 1 \cdot I) && \text{(def. of eigenspace)} \\
&= \text{Nullity} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} && \text{(dim of nullspace = nullity)} \\
&= \text{Nullity} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 1 && \text{(row operations preserve nullspace)}
\end{aligned}$$

We find that the GM of the sole eigenvalue is actually less than its AM. This combined with the following chain of equivalences:

$$\begin{aligned}
\text{GM} < \text{AM for at least 1 eigenvalue of } [T]_\beta &\iff [T]_\beta \text{ is not diagonalizable} \\
&\iff T \text{ is not diagonalizable}
\end{aligned}$$

Implies that T is in fact not diagonalizable.

Problem 2

Problem: Find the Jordan normal form of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 1 & -2 & -2 \end{bmatrix}$$

Solution: We first compute the eigenvalues of A by solving its characteristic equation:

$$\begin{aligned} 0 &= p(\lambda) \\ &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} -\lambda & 1 & -1 & -1 \\ 0 & -\lambda & 0 & 0 \\ 0 & -1 & 2-\lambda & 2 \\ 0 & 1 & -2 & -2-\lambda \end{bmatrix} \\ &= -\lambda(-\lambda((2-\lambda)(-2-\lambda) - (2 \cdot -2)) \\ &= \lambda^4 \end{aligned}$$

As we can see, this matrix posses a single eigenvalue, 0, with an AM of 4. Now let us compute an eigenbasis for the corresponding eigenspace:

$$\begin{aligned} E_0 &= \mathrm{Null}([T]_\beta - 0 \cdot I) && \text{(def. of eigenspace)} \\ &= \mathrm{Null} \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 1 & -2 & -2 \end{bmatrix} \\ &= \mathrm{Null} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} && \text{(row operations[1])} \\ &= \mathrm{Span} \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{v_2} \right\} && (x_1 \text{ free, } x_2 = 0,\ x_3 = -x_4,\ x_4 \text{ free}) \end{aligned}$$

Now note that the Jordan chains of v_1 and v_2 are both 2 long:

$$(A - 0I)v_1 = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 1 & -2 & -2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A - 0I)v_2 = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 1 & -2 & -2 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

As a result, we know that there is only 1 possibility for what their Jordan blocks look like: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Putting these together we have:

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[1]We will now show the row operations we skipped previously:

$$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 1 & -2 & -2 \end{bmatrix} \xrightarrow{R_4+R_3} \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow[R_1-R_3]{-R_3} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3

Problem: Give $\det(A^n)$, for $n \in \mathbb{Z}^+$, of the following matrix A:

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 5 & 3 & -3 \\ -2 & -1 & 2 \end{bmatrix}$$

Solution: To solve this, we will find the Jordan decomposition of A and raise that to the n power. We first compute the eigenvalues of A by solving its characteristic equation:

$$\begin{aligned} 0 &= p(\lambda) \\ &= \det([T]_\beta - \lambda I) \\ &= \det \begin{bmatrix} -1-\lambda & -1 & 1 \\ 5 & 3-\lambda & -3 \\ -2 & -1 & 2-\lambda \end{bmatrix} \\ &= (-1-\lambda)((3-\lambda)(2-\lambda)-3) + (5(2-\lambda)-6) + (-5+2(3-\lambda)) \\ &= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 \\ &= -(\lambda-2)(\lambda-1)^2 \end{aligned}$$

We will now find an eigenbasis for each eigenspace, starting with $\lambda = 1$:

$$\begin{aligned} E_1(A) &= \mathrm{Null}(A - I) && \text{(def. of eigenspace)} \\ &= \mathrm{Null} \begin{bmatrix} -2 & -1 & 1 \\ 5 & 2 & -3 \\ -2 & -1 & 1 \end{bmatrix} \\ &= \mathrm{Null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} && (\text{rref}^{[2]}) \\ &= \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} && (x_1 = x_3,\ x_2 = -x_3,\ x_3 \text{ free}) \end{aligned}$$

Here we can see that the eigenvalue 2 has an AM of 2 despite having a GM of 1. This corresponds to the following Jordan block: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Now let us compute an eigenbasis for $\lambda = 2$:

$$\begin{aligned} E_2(A) &= \mathrm{Null}(A - 2I) && \text{(def. of eigenspace)} \\ &= \mathrm{Null} \begin{bmatrix} -3 & -1 & 1 \\ 5 & 1 & -3 \\ -2 & -1 & 0 \end{bmatrix} \\ &= \mathrm{Null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} && (\text{rref}^{[3]}) \\ &= \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} && (x_1 = x_3,\ x_2 = -2x_3,\ x_3 \text{ free}) \end{aligned}$$

As the AM matches the GM, we have the following Jordan block: [2]. Putting both our blocks together, we then have the following JNF:

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

And so we can finally express A^n as the following:

$$
\begin{aligned}
\det A^n &= \det(SJ^nS^{-1}) && \text{(Jordan decomposition)} \\
&= \det(J^n) && (\det SS^{-1} = \det I = 1) \\
&= \det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^n\right) \\
&= \det\begin{bmatrix} 1^n & \binom{n}{1}1^{n-1} & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 2^n \end{bmatrix} && \text{(determinant of a Jordan block)} \\
&= \det\begin{bmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{bmatrix} \\
&= 2^n
\end{aligned}
$$

[2]We will now show the row operations we skipped previously:

$$
\begin{bmatrix} -2 & -1 & 1 \\ 5 & 2 & -3 \\ -2 & -1 & 1 \end{bmatrix}
\xrightarrow[R_2+\frac{2}{5}R_1]{R_1 \leftrightarrow R_2}
\begin{bmatrix} 5 & 2 & -3 \\ 0 & -\frac{1}{5} & -\frac{1}{5} \\ -2 & -1 & 1 \end{bmatrix}
$$

$$
\xrightarrow[R_3-R_2]{R_3+\frac{2}{5}R_1}
\begin{bmatrix} 5 & 2 & -3 \\ 0 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}
\xrightarrow[R_1-2R_2]{-5R_2}
\begin{bmatrix} 5 & 0 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\xrightarrow{R_1/5}
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

[3]We will now show the row operations we skipped previously:

$$
\begin{bmatrix} -3 & -1 & 1 \\ 5 & 1 & -3 \\ -2 & -1 & 0 \end{bmatrix}
\xrightarrow[R_2+\frac{3}{5}R_1]{R_1 \leftrightarrow R_2}
\begin{bmatrix} 5 & 1 & -3 \\ 0 & -\frac{2}{5} & -\frac{4}{5} \\ -2 & -1 & 0 \end{bmatrix}
$$

$$
\xrightarrow[R_2 \leftrightarrow R_3]{R_3+\frac{2}{5}R_1}
\begin{bmatrix} 5 & 1 & -3 \\ 0 & -\frac{3}{5} & -\frac{6}{5} \\ 0 & -\frac{2}{5} & -\frac{4}{5} \end{bmatrix}
\xrightarrow[-\frac{5}{3}R_2]{R_3-\frac{2}{3}R_2}
\begin{bmatrix} 5 & 1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\xrightarrow[R_1/5]{R_1-R_2}
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}
$$

Problem 4

Problem: Let T be a linear map on vector space V whose characteristic polynomial splits, and has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. Show the following:

$$V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$$

Solution: First note that for this direct sum to even be defined it must be the case that:

$$i \neq j \implies K_{\lambda_i} \cap K_{\lambda_j} = \emptyset$$

Which is indeed the case as seen in class. As a result, the dimension of the direct sum of the vector spaces is given by:

$$\dim(K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}) = \dim K_{\lambda_1} + \dim K_{\lambda_2} + \cdots + \dim K_{\lambda_k}$$

Also note that the dimension of a generalized eigenspace K_{λ_i} is equal to the AM m_i of its respective eigenvalue, i.e.:

$$K_{\lambda_i} = m_i$$

However, since the characteristic polynomial splits, the sum of these AMs $\sum m_i$ must equal the degree of the characteristic polynomial. And since the degree of this polynomial is precisely equal to the degree of the vector space the transformation T is over, i.e. V, we have:

$$\sum_{i=1}^{k} m_i = \dim V$$

Putting this together we have:

$$\begin{aligned}
\dim \bigoplus_{i=1}^{k} K_{\lambda_i} &= \sum_{i=1}^{k} \dim K_{\lambda_i} && \text{(dimension of direct sum)} \\
&= \sum_{i=1}^{k} m_i && \text{(dim of generalized eigenspace)} \\
&= \dim V && \text{(char. polynomial splits)}
\end{aligned}$$

Having the same dimension does not alone prove that $V = \bigoplus_{i=1}^{k} K_{\lambda_i}$. It is only after noting that $\bigoplus_{i=1}^{k} K_{\lambda_i} \subseteq V$ that we can conclude:

$$V = \bigoplus_{i=1}^{k} K_{\lambda_i} = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$$

This is because any subspace (and all direct sums are subspaces) of a vector space that has equal dimension to the superspace are infact equal, i.e. for a subspace $S \subseteq V$:

$$\dim S = \dim V \implies S = V$$

If $\bigoplus_{i=1}^{k} K_{\lambda_i} \subseteq V$ isn't clear, then we can assure ourselves that it is the case by noting that any vector $v \in \bigoplus_{i=1}^{k} K_{\lambda_i}$ is itself a sum of vectors from the summands $v_1 \in K_{\lambda_1}, v_2 \in K_{\lambda_2}, \cdots$. Since each summand $K_{\lambda_i} \subseteq V$, and vector spaces are closed under addition, it must be that $v \in \bigoplus_{i=1}^{k} K_{\lambda_i}$.

Problem 5

Problem: Find the matrix $A \in M_2(\mathbb{R})$ such that:

$$A^3 = \begin{bmatrix} -34 & -105 \\ 14 & 43 \end{bmatrix}$$

Solution: To solve this we must first find the eigendecomposition of A^3. First we compute the eigenvalues of A^3 by solving its characteristic equation:

$$\begin{aligned} 0 &= p(\lambda) \\ &= \det([T]_\beta - \lambda I) \\ &= \det \begin{bmatrix} -34 - \lambda & -105 \\ 14 & 43 - \lambda \end{bmatrix} \\ &= (-34 - \lambda)(43 - \lambda) + 105 \cdot 14 \\ &= \lambda^2 - 9\lambda + 8 \\ &= (\lambda - 1)(\lambda - 8) \end{aligned}$$

Now let us calculate an eigenbasis for each of the corresponding eigenspaces:

$$\begin{aligned} E_1 &= \text{Null}([T]_\beta - I) && \text{(def. of eigenspace)} \\ &= \text{Null} \begin{bmatrix} -35 & -105 \\ 14 & 42 \end{bmatrix} \\ &= \text{Null} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} && (\text{rref}^{[4]}) \\ &= \text{Span} \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\} && (x_1 = -3x_2,\ x_2 \text{ free}) \end{aligned}$$

$$\begin{aligned} E_8 &= \text{Null}([T]_\beta - I) && \text{(def. of eigenspace)} \\ &= \text{Null} \begin{bmatrix} -42 & -105 \\ 14 & 35 \end{bmatrix} \\ &= \text{Null} \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix} && (\text{ref}^{[5]}) \\ &= \text{Span} \left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix} \right\} && (2x_1 = -5x_2,\ x_2 \text{ free}) \end{aligned}$$

We can now express A as follows:

$$\begin{aligned}
A &= (A^3)^{1/3} \\
&= \left(\begin{bmatrix} -3 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} -3 & -5 \\ 1 & 2 \end{bmatrix}^{-1} \right)^{1/3} && \text{(eigendecomposition)} \\
&= \left(\begin{bmatrix} -3 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix} \right)^{1/3} && \text{(inverse of a } 2 \times 2 \text{ matrix)} \\
&= \begin{bmatrix} -3 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}^{1/3} \begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix} && \text{(}n\text{th power of an eigendecomposition)} \\
&= \begin{bmatrix} -3 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1^{1/3} & 0 \\ 0 & 8^{1/3} \end{bmatrix} \begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix} && \text{(}n\text{th power of a diagonal matrix)} \\
&= \begin{bmatrix} -3 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -4 & -15 \\ 2 & 7 \end{bmatrix}
\end{aligned}$$

And we can verify that this is indeed the desired matrix A:

$$\begin{aligned}
AAA &= \begin{bmatrix} -4 & -15 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -4 & -15 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -4 & -15 \\ 2 & 7 \end{bmatrix} \\
&= \begin{bmatrix} -14 & -45 \\ 6 & 19 \end{bmatrix} \begin{bmatrix} -4 & -15 \\ 2 & 7 \end{bmatrix} \\
&= \begin{bmatrix} -34 & -105 \\ 14 & 43 \end{bmatrix} = A^3
\end{aligned}$$

[4]We will now show the row operations we skipped previously:

$$\begin{bmatrix} -35 & -105 \\ 14 & 42 \end{bmatrix} \xrightarrow[-\frac{1}{35}R_1]{R_2+\frac{2}{5}R_1} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

[5]We will now show the row operations we skipped previously:

$$\begin{bmatrix} -35 & -105 \\ 14 & 42 \end{bmatrix} \xrightarrow[-\frac{1}{21}R_1]{R_2+\frac{1}{3}R_1} \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix}$$