Differential Equations HW #4

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November 12, 2019

Problem 1

Problem: Let $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ be solutions to the following system:

$$\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{A}} \mathbf{y}$$

And let $D(t) = \det[\mathbf{y}_1(t) \quad \mathbf{y}_1(t)]$

a) Show that $D(t)$ satisfies $\frac{dD}{dt} = \operatorname{tr}(\mathbf{A})\, D$.

b) Show that if $\mathbf{y}_1(0)$ and $\mathbf{y}_2(0)$ are linearly independent, then $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are linearly independent for all t.

Solution: a) First notice the following:

$$\begin{aligned} \mathbf{y}_1' &= A\mathbf{y}_1 \\ \begin{bmatrix} y_{11}' \\ y_{21}' \end{bmatrix} &= \begin{bmatrix} ay_{11} + by_{21} \\ cy_{11} + dy_{21} \end{bmatrix} && \text{(Eq. 1)} \\ \mathbf{y}_2' &= A\mathbf{y}_2 \\ \begin{bmatrix} y_{12}' \\ y_{22}' \end{bmatrix} &= \begin{bmatrix} ay_{12} + by_{22} \\ cy_{12} + dy_{22} \end{bmatrix} && \text{(Eq. 2)} \end{aligned}$$

Now consider $D(t)$:

$$\begin{aligned} D(t) &= \det[\mathbf{y}_1(t) \quad \mathbf{y}_1(t)] \\ &= \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} = y_{11}y_{22} - y_{12}y_{21} \end{aligned}$$

We can now see that D does indeed satisfy the given ODE:

$$\begin{aligned} \frac{dD}{dt} &= y_{11}'y_{22} + y_{11}y_{22}' - y_{12}'y_{21} - y_{12}y_{21}' && \text{(product rule)} \\ &= y_{22}(ay_{11} + by_{21}) + y_{11}(cy_{12} + dy_{22}) - y_{21}(ay_{12} + by_{22}) - y_{12}(cy_{11} + dy_{21}) && \text{(Eq. 1 \& 2)} \\ &= (ay_{11}y_{22} - ay_{12}y_{21}) + (by_{21}y_{22} - by_{22}y_{21}) + (cy_{12}y_{11} - cy_{11}y_{12}) + (dy_{22}y_{11} - dy_{21}y_{12}) \\ &= a\underbrace{\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}}_{=D} + b\underbrace{\begin{vmatrix} y_{21} & y_{22} \\ y_{21} & y_{22} \end{vmatrix}}_{=0} + c\underbrace{\begin{vmatrix} y_{12} & y_{11} \\ y_{12} & y_{11} \end{vmatrix}}_{=0} + d\underbrace{\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}}_{=D} \\ &= (a+d)D = \operatorname{tr}(\mathbf{A})\, D \end{aligned}$$

b) Note that our two functions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are linearly dependent if and only if:

$$(\exists k_1, k_2)(\forall t) \quad k_1\mathbf{y}_1(t) = k_2\mathbf{y}_2(t) \tag{*}$$

However if $\mathbf{y}_1(0)$ and $\mathbf{y}_2(0)$ are not linearly dependent, i.e. linearly *in*dependent, then the following is the case:

$$(\nexists k_1, k_2) \quad k_1\mathbf{y}_1(0) = k_2\mathbf{y}_2(0)$$

And so statement (*) does not hold and thus, $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are not linearly dependent. In other words, they are linearly *in*dependent.

Problem 2

Problem: Solve the following IVP:

$$\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}}_{\mathbf{A}} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution: To find a spanning set of solutions, we first find the eigenvalues of $\mathbf{A}$:

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= \begin{vmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-2-\lambda) - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda+3)(\lambda-2) \\ &\longrightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = -3 \end{cases} \end{aligned}$$

Now we must find an eigenvector, $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively, corresponding to each eigenvalue. To do this, we find the eigenspaces corresponding to each eigenvalue, starting with λ_1:

$$\begin{aligned} E_1(\mathbf{A}) &= \mathrm{Null}(\mathbf{A} - \lambda_1\mathbf{I}) \\ &= \mathrm{Null}\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \\ &= \mathrm{Null}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} && \text{(ref)} \\ &= \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\} && (x_2 = -2x_1) \\ &\xrightarrow{\text{let}} \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{aligned}$$

Now we do the same for λ_2:

$$\begin{aligned}
E_2(\mathbf{A}) &= \text{Null}(\mathbf{A} - \lambda_2\mathbf{I}) \\
&= \text{Null}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \\
&= \text{Null}\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} && \text{(ref)} \\
&= \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} && (x_1 = 2x_2) \\
&\xrightarrow{\text{let}} \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\end{aligned}$$

And so we have, for our desired solution $\mathbf{y}(t)$, the following:

$$\begin{aligned}
\mathbf{y}(t) &= k_1 e^{\lambda_1 t}\mathbf{v}_1 + k_2 e^{\lambda_2 t}\mathbf{v}_2 && \text{(distinct roots)} \\
\mathbf{y}(0) &= k_1 e^{0}\mathbf{v}_1 + k_2 e^{0}\mathbf{v}_2 && (\text{let } t = 0) \\
\begin{bmatrix} 1 \\ 2 \end{bmatrix} &= k_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\end{aligned}$$

We can represent this system as the following augmented matrix:

$$\begin{aligned}
\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -2 & 1 & 2 \end{array}\right] &\xrightarrow{r_2 + 2r_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 5 & 4 \end{array}\right] \\
&\xrightarrow{(1/5)r_2} \left[\begin{array}{cc|r} 1 & 2 & 1 \\ 0 & 1 & 4/5 \end{array}\right] \\
&\xrightarrow{r_1 - 2r_2} \left[\begin{array}{cc|r} 1 & 0 & -3/5 \\ 0 & 1 & 4/5 \end{array}\right]
\end{aligned}$$

And so $k_1 = \frac{-3}{5}$ and $k_2 = \frac{4}{5}$ giving us our desired solution:

$$\begin{aligned}
\mathbf{y}(t) &= k_1 e^{\lambda_1 t}\mathbf{v}_1 + k_2 e^{\lambda_2 t}\mathbf{v}_2 \\
&= \frac{-3}{5} e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{4}{5} e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= \boxed{\frac{1}{5}\begin{bmatrix} -3e^{2t} + 8e^{-3t} \\ 6e^{2t} + 4e^{-3t} \end{bmatrix}}
\end{aligned}$$

Problem 3

Problem: Find the general solution to the following system:

$$\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix}}_{\mathbf{A}} \mathbf{y}$$

Solution: To find a spanning set of solutions, we first find the eigenvalues of $\mathbf{A}$:

$$\begin{aligned}
0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\
&= \begin{vmatrix} -3-\lambda & -5 \\ 3 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)(-3-\lambda) + 15 \\
&= \lambda^2 + 2\lambda + 12 \\
&\longrightarrow \lambda = \frac{-2 \pm \sqrt{4-48}}{2} && \text{(quadratic formula)} \\
&\longrightarrow \begin{cases} \lambda_1 = -1 + i\sqrt{11} \\ \lambda_2 = -1 - i\sqrt{11} \end{cases}
\end{aligned}$$

Now we must find an eigenvector $\mathbf{v}$ that corresponds to one of the eigenvalues. To do this, we find the eigenspace corresponding to, say, λ_2:

$$\begin{aligned}
E_2(\mathbf{A}) &= \mathrm{Null}(\mathbf{A} - \lambda_2\mathbf{I}) \\
&= \mathrm{Null}\begin{bmatrix} -2+i\sqrt{11} & -5 \\ 3 & 2+i\sqrt{11} \end{bmatrix}
\end{aligned}$$

This is equivalent to the solution set of the following system of equations:

$$\begin{cases} (-2+i\sqrt{11})v_1 = 5v_2 \\ 3v_1 = (-2-i\sqrt{11})v_2 \end{cases}$$

Via substitution we find:

$$\begin{aligned}
(-2-i\sqrt{11})v_2 &= \frac{5\cdot 3}{-2+i\sqrt{11}} \\
v_2 &= \frac{15}{(-2-i\sqrt{11})(-2+i\sqrt{11})} = 1
\end{aligned}$$

Plugging this into the second equation we find:

$$v_1 = \frac{-1}{3}(2+i\sqrt{11})$$

And so our eigenspace is given by:

$$E_2(\mathbf{A}) = \mathrm{Span}\left\{ \begin{bmatrix} \frac{-1}{3}(2+i\sqrt{11}) \\ 1 \end{bmatrix} \right\}$$

And so, for ease of calculation, we'll let the following be our vector $\mathbf{v}$:

$$\mathbf{v} = \begin{bmatrix} 2+i\sqrt{11} \\ -3 \end{bmatrix}$$

Now consider the following particular solution to the system:

$$\begin{aligned}
\mathbf{y}_p(t) &= e^{(-1-i\sqrt{11})t}\mathbf{v} && \text{(straight line solution)}\\
&= e^{-t}e^{-it\sqrt{11}}\mathbf{v}\\
&= e^{-t}(\cos -t\sqrt{11} + i\sin -t\sqrt{11})\mathbf{v} && \text{(Euler's formula)}\\
&= e^{-t}\cos(t\sqrt{11}) - e^{-t}i\sin(t\sqrt{11})\begin{bmatrix} 2+i\sqrt{11} \\ -3 \end{bmatrix}\\
&= \begin{bmatrix} 2e^{-t}\cos(t\sqrt{11}) - 2ie^{-t}\sin(t\sqrt{11}) + ie^{-t}\sqrt{11}\cos(t\sqrt{11}) + e^{-t}\sqrt{11}\sin(t\sqrt{11}) \\ -3e^{-t}\cos(t\sqrt{11}) + 3e^{-1}i\sin(t\sqrt{11}) \end{bmatrix}\\
&= \underbrace{\begin{bmatrix} 2e^{-t}\cos(t\sqrt{11}) + e^{-t}\sqrt{11}\sin(t\sqrt{11}) \\ -3e^{-t}\cos(t\sqrt{11}) \end{bmatrix}}_{\mathbf{y}_{\mathcal{R}}(t)} + i\underbrace{\begin{bmatrix} -2e^{-t}\sin(t\sqrt{11}) + e^{-t}\sqrt{11}\cos(t\sqrt{11}) \\ 3e^{-t}\sin(t\sqrt{11}) \end{bmatrix}}_{\mathbf{y}_{\mathcal{I}}(t)}
\end{aligned}$$

And so the general real solution to the system is given by:

$$\begin{aligned}
\mathbf{y}(t) &= k_1\mathbf{y}_{\mathcal{R}}(t) + k_2\mathbf{y}_{\mathcal{I}}(t)\\
&= \boxed{k_1e^{-t}\begin{bmatrix} 2\cos(t\sqrt{11}) + \sqrt{11}\sin(t\sqrt{11}) \\ -3\cos(t\sqrt{11}) \end{bmatrix} + k_2e^{-t}\begin{bmatrix} -2\sin(t\sqrt{11}) + \sqrt{11}\cos(t\sqrt{11}) \\ 3\sin(t\sqrt{11}) \end{bmatrix}}
\end{aligned}$$

Problem 4

Problem: Solve the following IVP:

$$\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix}}_{\mathbf{A}}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution: To find the general solution, we first find the eigenvalues of $\mathbf{A}$:

$$\begin{aligned}
0 &= \det(\mathbf{A} - \lambda\mathbf{I})\\
&= \begin{vmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{vmatrix}\\
&= (-2-\lambda)(-4-\lambda) + 1\\
&= \lambda^2 + 6\lambda + 9\\
&= (\lambda-3)^2 \longrightarrow \lambda = -3
\end{aligned}$$

And so $\mathbf{A}$ has a single repeated eigenvalue of -3. Recall that this implies that the general solution is of the following form:

$$\mathbf{y}(t) = e^{\lambda t}\mathbf{v}_0 + te^{\lambda t}\mathbf{v}_1$$

Where $\mathbf{v}_0$ is an arbitrary vector and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_0$. However, noting that plugging in 0 gives us $\mathbf{y}(0) = \mathbf{v}_0$, we can calculate $\mathbf{v}_1$ to be:

$$
\begin{aligned}
\mathbf{v}_1 &= (\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_0 \\
&= \left(\begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

And so, our particular solution is given by:

$$
\begin{aligned}
\mathbf{y}(t) &= e^{\lambda t}\mathbf{v}_0 + te^{\lambda t}\mathbf{v}_1 \\
&= e^{-3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + te^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} e^{-3t} + te^{-3t} \\ te^{-3t} \end{bmatrix}}
\end{aligned}
$$

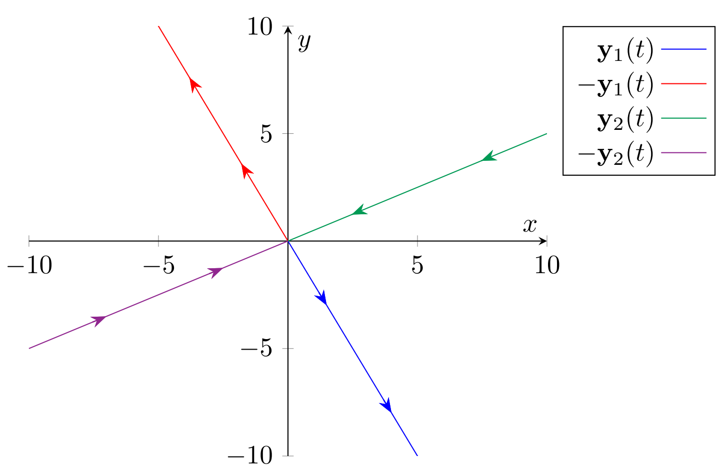
Problem 5

Problem: Sketch phase portraits of the systems in problems 2, 3, and 4.

Solution: For **problem 2** our straight-line solutions are:

$$
\begin{aligned}
\mathbf{y}_1(t) &= e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \lambda_1 = 2 > 0 \\
\mathbf{y}_2(t) &= e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_2 = -3 < 0
\end{aligned}
$$

Graphing these and their negatives, we see that the origin forms a saddle equilibrium:



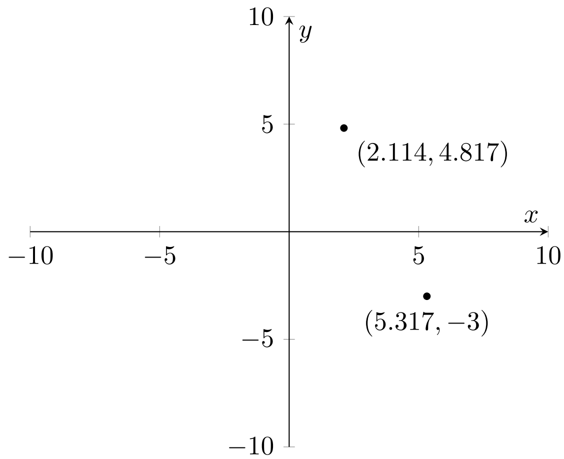
The system in **problem 3** has only complex eigenvalues and thus, it has no straight line solutions. Such a system has a spiral equilibrium at the origin. To see the spiral's direction we note that:

$$
\mathcal{R}(\lambda) = R(-1 \pm i\sqrt{11}) = -1 < 0
$$

Meaning the solutions all spiral towards to origin. Next we plug in two test points to help guide our curve. With constants $k_1 = k_2 = 1$ we have at $t = 0, \frac{\pi}{2\sqrt{11}}$:

$$\begin{aligned}
\mathbf{y}(0) &= e^0 \begin{bmatrix} 2\cos(0) + \sqrt{11}\sin(0) \\ -3\cos(0) \end{bmatrix} + e^0 \begin{bmatrix} -2\sin(0) + \sqrt{11}\cos(0) \\ 3\sin(0) \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} \sqrt{11} \\ 0 \end{bmatrix} \approx \begin{bmatrix} 5.317 \\ -3 \end{bmatrix} \\
\mathbf{y}\left(\frac{\pi}{2\sqrt{11}}\right) &= e^{\frac{\pi}{2\sqrt{11}}} \begin{bmatrix} 2\cos\left(\frac{\pi}{2}\right) + \sqrt{11}\sin\left(\frac{\pi}{2}\right) \\ -3\cos\left(\frac{\pi}{2}\right) \end{bmatrix} + e^{\frac{\pi}{2\sqrt{11}}} \begin{bmatrix} -2\sin\left(\frac{\pi}{2}\right) + \sqrt{11}\cos\left(\frac{\pi}{2}\right) \\ 3\sin\left(\frac{\pi}{2}\right) \end{bmatrix} \\
&= e^{\frac{\pi}{2\sqrt{11}}} \begin{bmatrix} \sqrt{11} \\ 0 \end{bmatrix} + e^{\frac{\pi}{2\sqrt{11}}} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \approx \begin{bmatrix} 2.114 \\ 4.817 \end{bmatrix}
\end{aligned}$$

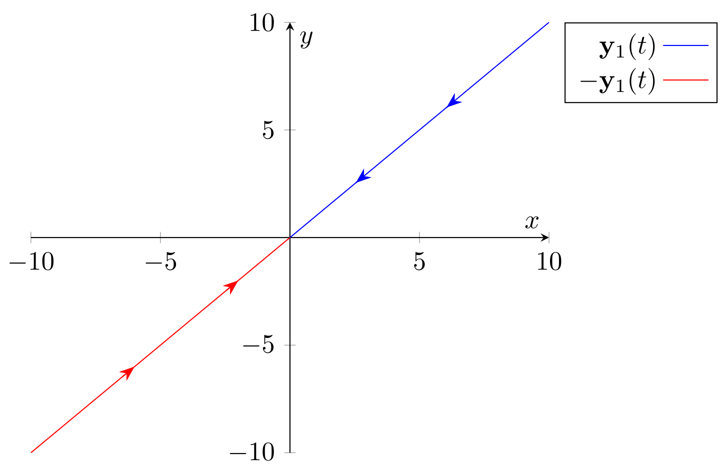
Now we can graph the phase portrait of the system's spiral sink:



For **problem 4** we have only one straight-line solution: for when $\mathbf{v}_0 = [1, 1]^\top$ (i.e. an eigenvector):

$$\mathbf{y}_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda = -3 < 0$$

Graphing this and and its negative, we see that the origin forms an unstable node equilibrium:



Problem 6

Problem: Let $\mathbf{B}$ be a matrix with a repeated zero eigenvalue. Show that $\mathbf{B}^2 = \mathbf{0}$. Then show that if a matrix $\mathbf{A}$ has a repeated eigenvalue λ_0, then $(\mathbf{A} - \lambda_0\mathbf{I})^2 = \mathbf{0}$.

Solution: To prove both statements, we will first prove the Cayley-Hamilton theorem for 2×2 matrices. Consider a generic 2×2 matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Now consider its characteristic polynomial $p_{\mathbf{M}}(\lambda)$:

$$\begin{aligned} p_{\mathbf{M}}(\lambda) &= \det(\mathbf{M} - \lambda\mathbf{I}) \\ &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \end{aligned}$$

Now let us adapt this function of a scalar λ into a function of a matrix $\mathbf{X}$ by multiplying the constants with the identity matrix:

$$p_{\mathbf{M}}(\mathbf{X}) = \mathbf{X}^2 - (a + d)\mathbf{X} + (ad - bc)\mathbf{I}$$

The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation. The 2×2 case is proven below:

$$\begin{aligned} p_{\mathbf{M}}(\mathbf{M}) &= \mathbf{M}^2 - (a + d)\mathbf{M} + (ad - bc)\mathbf{I} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - (a + d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad - bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \end{aligned}$$

Armed with this result, we can prove the desired statements. Consider a 2×2 matrix $\mathbf{B}$ with a single repeated eigenvalue of $\lambda_0 = 0$. The characteristic polynomial of this matrix is given by:

$$p_{\mathbf{B}}(\mathbf{X}) = (\mathbf{X} - \lambda_0\mathbf{I})(\mathbf{X} - \lambda_0\mathbf{I}) = \mathbf{X}^2$$

Plugging our matrix $\mathbf{B}$ into the polynomial, the Cayley-Hamilton theorem tells us:

$$p_{\mathbf{B}}(\mathbf{B}) = \boxed{\mathbf{B}^2 = \mathbf{0}}$$

Now consider a 2×2 matrix $\mathbf{A}$ with a repeated eigenvalue λ_0. Its characteristic polynomial is given by:

$$p_{\mathbf{A}}(\mathbf{X}) = (\mathbf{X} - \lambda_0\mathbf{I})(\mathbf{X} - \lambda_0\mathbf{I}) = (\mathbf{X} - \lambda_0\mathbf{I})^2$$

And once again, plugging our matrix $\mathbf{A}$ into the polynomial, the Cayley-Hamilton theorem tells us:

$$p_{\mathbf{A}}(\mathbf{A}) = \boxed{(\mathbf{A} - \lambda_0 \mathbf{I})^2 = \mathbf{0}}$$

Problem 7

Problem: Given the following family of systems with parameter a:

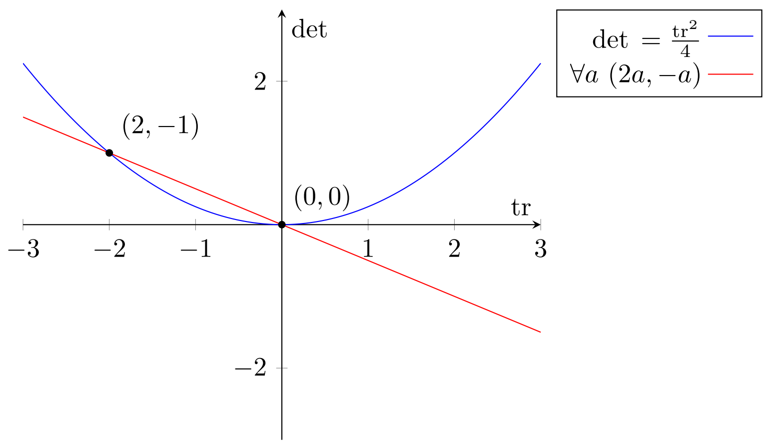
$$\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} a & a^2 + a \\ 1 & a \end{bmatrix}}_{\mathbf{X}} \mathbf{y}$$

sketch the corresponding curve in the trace-determinant plane. Identify the values of a where the type of the system changes, i.e. the bifurcation values of the family.

Solution: First we compute the trace and determinant of the given matrix:

$$\operatorname{tr}(\mathbf{X}_a) = 2a$$
$$\det(\mathbf{X}_a) = a^2 - a^2 - a = -a$$

Now we can graph this and the repeated root parabola on the trace-determinant plane:



A bifurcation occurs when the solution curve intersects the parabola or either of the axes. The two points this happens at translate to the following bifurcation values:

$$(0, 0) = a_0(2, -1) \to a_0 = 0$$
$$(-2, 1) = a_1(2, -1) \to a_1 = -1$$

And so the bifurcation values of the system are $a_0 = 0$ and $a_1 = -1$.