Math Statistics
Semiweekly HW 10

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Question 1

Problem: Show that if $\hat{\theta}$ is an asymptotically unbiased & asymptotically unvarianced (i.e. $\lim_{n\to\infty} \text{Var}(\hat{\theta}) = 0$) estimator for θ, then it is consistent.

Solution: First let us establish the following inequalities for any $\epsilon \in \mathbb{R}^+$, and any $n \in \mathbb{N}$:

$$0 \le P(|\hat{\theta}_n - \theta| \ge \epsilon) \le \frac{E[|\hat{\theta}_n - \theta|]}{\epsilon}$$

The LHS being the result of the nonnegativity of probabilities, and the RHS being an application of Markov's inequality. Now note that since the equality above holds for any $n \in \mathbb{N}$, we can take the limit of each term w.r.t to n:

$$\begin{aligned}
\lim_{n\to\infty} 0 \le \lim_{n\to\infty} P(|\hat{\theta}_n - \theta| \ge \epsilon) &\le \lim_{n\to\infty} \frac{E[|\hat{\theta}_n - \theta|]}{\epsilon} \\
0 \le \lim_{n\to\infty} P(|\hat{\theta}_n - \theta| \ge \epsilon) &\le \lim_{n\to\infty} \frac{E[|\hat{\theta}_n - \theta|]}{\epsilon} && \text{(limit of a constant)} \\
&= \lim_{n\to\infty} \frac{E\left[\sqrt{(\hat{\theta}_n - \theta)^2}\right]}{\epsilon} && (|x| = \sqrt{x^2}) \\
&\le \lim_{n\to\infty} \frac{\sqrt{E\left[(\hat{\theta}_n - \theta)^2\right]}}{\epsilon} && \text{(Jenson's inequality for a concave function)} \\
&= \frac{\sqrt{\lim_{n\to\infty} E\left[(\hat{\theta}_n - \theta)^2\right]}}{\epsilon} && \text{(linearity \& powers of limits)} \\
&= \frac{\sqrt{\lim_{n\to\infty} \left(\text{Bias}(\hat{\theta}_n)^2 + \text{Var}(\hat{\theta}_n)\right)}}{\epsilon} && \text{(bias-variance decomposition)} \\
&= \frac{\sqrt{\lim_{n\to\infty} \text{Bias}(\hat{\theta}_n)^2 + \lim_{n\to\infty} \text{Var}(\hat{\theta}_n)}}{\epsilon} && \text{(linearity of limits)} \\
&= \frac{\sqrt{\left(\lim_{n\to\infty} \text{Bias}(\hat{\theta}_n)\right)^2 + \lim_{n\to\infty} \text{Var}(\hat{\theta}_n)}}{\epsilon} && \text{(powers of limits)} \\
&= \frac{\sqrt{0^2 + 0}}{\epsilon} && \text{(asymptotically unbiased \& unvarianced)} \\
&= 0
\end{aligned}$$

And so, put in a cleaner form, we have:

$$0 \le \lim_{n\to\infty} P(|\hat{\theta}_n - \theta| \ge \epsilon) \le 0$$

And so, by the squeeze theorem, we have that $\hat{\theta}_n$ satisfies:

$$\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| \ge \epsilon) = 0$$

Which is precisely the definition of a (weakly) consistent estimator. ■