Math Statistics
Semiweekly HW 2

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**note that we characterize gamma distributions in terms of their shape α and their rate β.*

Question 1

Suppose X_1 and X_2 are i.i.d, with support $(0, \infty)$ and the following pdf:

$$f(x) = 3e^{-3x}$$

Part a: Are X_1 and X_2 gamma RVs?

Solution: Yes. To see this, note that all exponential RVs are gamma RVs:

$$\begin{aligned}
&\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} && \text{(pdf of gamma RV)} \\
&= \frac{\lambda^1}{\Gamma(1)} x^{1-1} e^{-\lambda x} && \text{(let } \alpha = 1 \text{ and } \beta = \lambda\text{)} \\
&= \lambda e^{-\lambda x} && \text{(pdf of exponential RV)}
\end{aligned}$$

Since X_1 and X_2 are clearly exponential RVs with paramater $\lambda = 3$, they must also be gamma RVs with parameters $\alpha = 1$ and $\beta = 3$.

Part b: Find the pdf of $X_1 + X_2$.

Solution: Since they are gamma RVs, we must have $X_1 \sim \text{Gamma}(a_1, \beta)$ and $X_2 \sim \text{Gamma}(a_2, \beta)$, for some a_1, a_2. Now consider the mgf of their sum $X_1 + X_2$:

$$\begin{aligned}
\mathcal{M}_{X_1+X_2}(t) &= E[e^{t(X_1+X_2)}] && \text{(def. of mgf)} \\
&= E[e^{tX_1} e^{tX_2}] && \text{(power rule)} \\
&= E[e^{tX_1}] E[e^{tX_2}] && \text{(independence of } X_1 \text{ and } X_2\text{)} \\
&= \left(\frac{\beta}{\beta - t}\right)^{a_1} \left(\frac{\beta}{\beta - t}\right)^{a_2} && \text{(mgf of a gamma RV)} \\
&= \left(\frac{\beta}{\beta - t}\right)^{a_1 + a_2} && \text{(power rule)}
\end{aligned}$$

As we can see, the mgf of $X_1 + X_2$ is that of a gamma RV. In particular, we know that $\alpha_1 + \alpha_2 = 2$ and that $\beta = 3$. And so we have that $X_1 + X_2 \sim \text{Gamma}(2, 3)$. The corresponding pdf for such a RV is given by:

$$\begin{aligned}
f_{X_1+X_2}(x) &= \frac{3^2}{\Gamma(2)} x^{2-1} e^{-3x} && \text{(pdf of gamma RV)} \\
&\boxed{= 9xe^{-3x}}
\end{aligned}$$