Intro to Math Reasoning HW 3b

Ozaner Hansha

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1 Problem 1

Problem: Let $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^3 \geq 1\}$. Let $(B_y : y \in \mathbb{R})$ be the indexed family of subsets of $\mathbb{R}$ with $B_y = \{x \in \mathbb{R} \mid (x,y) \in S\}$. For each $y \in \mathbb{R}$, express B_y as an interval or a union of intervals in $\mathbb{R}$.

Solution: We can solve for the interval B_y represents:

$$\begin{aligned}
B_y &= \{x \mid x^2 + y^2 \geq 1\} \\
&= \{x \mid x^2 \geq 1 - y^2\} \\
&= \{x \mid x \leq -\sqrt{1-y^2} \land x \geq \sqrt{1-y^2}\} \\
&= (-\infty, -\sqrt{1-y^2}\,] \cup [\,\sqrt{1-y^2}, \infty)
\end{aligned}$$

However, the above fails when $y > 1$ as the radicand will be negative resulting in a complex number (which has no standard order). As such we can make a conditional definition of B_y:

$$B_y = \begin{cases} \emptyset, & \text{for } y > 1 \\ (-\infty, -\sqrt{1-y^2}\,] \cup [\,\sqrt{1-y^2}, \infty), & \text{for } y \leq 1 \end{cases}$$

If we are adamant about the interval representation, we can represent even the null set case as a degenerate interval: $B_{y>1} = (0,0) = \emptyset$.

2 Problem 2

2.1 Part a

Problem: Give two distinct real polynomials of a real variable.

Solution: $x^2 + 3x + 4$ and $x^{60} - \pi$.

2.2 Part b

Problem: What is the minimum information needed to specify a polynomial?

Solution: A given real polynomial of a single real variable is fully described by an element of $\bigcup_{n \in \mathbb{Z}_{>0}} \mathbb{R}^n$ (i.e an d-tuple of real numbers where d is any positive integer). $d-1$ represents the degree of the polynomial.

2.3 Part c

Problem: Use the information in part a to define a polynomial.

Solution: Given an element of $\bigcup_{n \in \mathbb{Z}_{>0}} \mathbb{R}^n$ denoted p with its ith entry denoted p_i (indexing starts at 0) and its length denoted d, we can define a polynomial as:

$$\sum_{i=0}^{d-1} p_i x^i$$

3 Problem 3

3.1 Part a

Problem: Identify the free and bounded variables in the following predicate: "For every positive integer n the set $\{m \in \mathbb{Z} \mid m^2 - r \text{ is divisible by } n\}$ is nonempty."

Solution: n and m are bound variable while r is unbound.

3.2 Part b

Problem: Identify the free and bounded variables in the following predicate: "x is not a member of S and for all real numbers $\epsilon > 0$, there exists a member y of S such that $|x - y| < \epsilon$."

Solution: ϵ and y are bound variables while x and S are unbound.

3.3 Part c

Problem: Identify the free and bounded variables in the following predicate: "For every function f from $\mathbb{R}$ to $\mathbb{R}$, There is a function g and a function h such that for every real number x, $f(x) = g(x) + h(x)$ and $g(-x) = g(x)$ and $h(-x) = -h(x)$."

Solution: f, g, h and x are bound variables.

4 Problem 4

4.1 Part a

Problem: Give a set A that contains 3 sets such that any 2 distinct sets in A intersect in exactly one element, and no element belongs to more than 2 sets.

Solution:

$$\{\{1, 2, 3\}\{1, 4, 5\}\{3, 4, 6\}\}$$

4.2 Part b

Problem: Generalize the previous example: For each positive integer $k \geq 3$, give an example of a collection of k sets such that any 2 distinct members of the collection intersect in exactly one element and no element belongs to more than 2 sets.

Solution: To create a collection of sets that satisfy the above, the collection must contain k sets all of size k. The first set will simply be:

$$\{1, 2, 3, 4, \cdots, k\}$$

The second set will be the first element of the 1st set followed by the integers that come after k until the set is of size k:

$$\{1, k+1, k+2, \cdots, n_1\}$$

Where n_1 represents the last number we reach once the set is of size k.

The third set will be the first element of the 1st set that hasn't been used by another set (in this case the second element: 2) followed by the first element of the 2nd set that hasn't been used by another set (in this case the second element: $k+1$) followed by the integers that come after n_1 until the set is of size k:

$$\{2, k+1, n_1+1, n_1+2, \cdots, n_2\}$$

The fourth set will be the first element of the 1st set that hasn't been used by another set (in this case the second element: 3) followed by the first element of the 2nd set that hasn't been used by another set (in this case the second element: $k+2$) followed by the first element of the 3rd set that hasn't been used by another set (in this case the second element: n_1+1) followed by the integers that come after n_2 until the set is of size k:

$$\{3, k+2, n_1+1, n_2+1, n_2+2, \cdots, n_3\}$$

And so on. Once we reach the kth set, we will have constructed a collection of k sets that satisfy the given conditions.

5 Problem 5

Problem: Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$.

Solution: Here's the truth table:

p	q	r	$p \wedge q$	$p \vee r$	$q \vee r$	$(p \wedge q) \vee r$	$(p \vee r) \wedge (q \vee r)$
F	F	F	F	F	F	F	F
F	F	T	F	T	T	T	T
F	T	F	F	F	T	F	F
F	T	T	F	T	T	T	T
T	F	F	F	T	F	F	F
T	F	T	F	T	T	T	T
T	T	F	T	T	T	T	T
T	T	T	T	T	T	T	T