Math Statistics
Semiweekly HW 7

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Question 1

Suppose we have a normally distributed population with unknown variance σ^2. Let S^2 be the sample variance of a sample with $n = 5$.

Part a: Find x_a such that:

$$P\left(\frac{4S^2}{\sigma^2} > x_a\right) = .05$$

Solution: Recall that $\frac{4S^2}{\sigma^2} \sim \chi^2_4$. And so, consulting our chi-square table, we have:

$$x_a \approx 9.488$$

Part b: Find x_b such that:

$$P\left(\frac{4S^2}{\sigma^2} < x_b\right) = .05$$

Solution: First note that:

$$\begin{aligned}
.05 &= P\left(\frac{4S^2}{\sigma^2} < x_b\right) \\
&= 1 - P\left(\frac{4S^2}{\sigma^2} \geq x_b\right) && \text{(complement)} \\
.95 &= P\left(\frac{4S^2}{\sigma^2} \geq x_b\right)
\end{aligned}$$

Again, we have that $\frac{4S^2}{\sigma^2} \sim \chi^2_4$. And so, consulting our chi-square table, we have:

$$x_b = .711$$

Part c: If we observe our sample variance to be $s^2 = 12$, give a 90% confidence interval of the population variance σ^2.

Solution: Note the following two events are mutually exclusive:

$$P\left(\frac{4S^2}{\sigma^2} < x_b \cap \frac{4S^2}{\sigma^2} > x_a\right) = 0$$

And so we have the following:

$$
\begin{aligned}
.05 + .05 &= P\left(\frac{4S^2}{\sigma^2} > x_a\right) + P\left(\frac{4S^2}{\sigma^2} < x_b\right) \\
.1 &= P\left(\frac{4S^2}{\sigma^2} > x_a \cup \frac{4S^2}{\sigma^2} < x_b\right) && \text{(mutually exclusive)} \\
&= 1 - P\left(x_b < \frac{4S^2}{\sigma^2} < x_a\right) && \text{(complement \& deMorgan)} \\
.9 &= P\left(x_b < \frac{4S^2}{\sigma^2} < x_a\right) \\
&= P\left(\frac{1}{x_b} > \frac{\sigma^2}{4S^2} > \frac{1}{x_a}\right) && (\sigma^2, S^2, x_a, x_b \text{ are positive}) \\
&= P\left(\frac{4S^2}{x_b} > \sigma^2 > \frac{4S^2}{x_a}\right)
\end{aligned}
$$

And so, our 90% confidence interval of σ^2 is given by:

$$
\begin{aligned}
\left[\frac{4s^2}{x_a}, \frac{4s^2}{x_b}\right] &= \left[\frac{4*12}{x_a}, \frac{4*12}{x_b}\right] && (s^2 = 12) \\
&\approx \left[\frac{48}{9.488}, \frac{48}{.711}\right] && (x_a = 9.488, x_b = .711) \\
&\approx [5.059, 67.511]
\end{aligned}
$$