Differential Equations HW #3

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Problem 1

Problem: Find the general solution to the following system:

$$\begin{cases} \frac{dx}{dt} = 2x \\ \frac{dy}{dt} = 4y - x^2 \end{cases}$$

Solution: This is a partially decoupled system, thus we can solve for x first. Being separable, it is clear that the general solution to x is:

$$x = k_1 e^{2t}$$

Now we plug in our general solution for x into the other ODE and solve the resulting linear ODE for y.

$$\frac{dy}{dt} = 4y - k_1^2 e^{4t}$$

First we find the general solution to the homogenous equation $y' = 4y$. Like before, it is separable and so the general solution is:

$$y_h = k_2 e^{4t}$$

Via the method of undetermined coefficients, we know that a particular solution y_p to the LDE is of the form:

$$y_p = \alpha t e^{4t}$$

Plugging this into the ODE we find:

$$\begin{aligned} \frac{dy_p}{dt} &= 4y_p - x^2 \\ 4\alpha t e^{4t} + \alpha e^{4t} &= 4\alpha t e^{4t} - k_1^2 e^{4t} \\ \alpha e^{4t} &= -k_1^2 e^{4t} \\ \alpha &= -k_1^2 \end{aligned}$$

And so our general solution to y is given by:

$$y = y_h + y_p = k_2 e^{4t} - k_1^2 t e^{4t}$$

Putting it together, our general solution to the system of ODEs is:

$$\boxed{\begin{cases} x = k_1 e^{2t} \\ y = k_2 e^{4t} - k_1^2 t e^{4t} \end{cases}}$$

For arbitrary constants $k_1, k_2 \in \mathbb{R}$.

Problem 2

Problem: Rewrite the following system of ODEs in matrix form:

$$\begin{cases} \frac{dp}{dt} = 3p - 2q - 7r \\ \frac{dq}{dt} = -2p + 6r \\ \frac{dr}{dt} = 7q + 2r \end{cases}$$

Solution: Defining the following variables:

$$\mathbf{p}(t) = \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & 7 & 2 \end{bmatrix}$$

We can express the given system, supressing the argument (t), as the following matrix ODE:

$$\boxed{\frac{d\mathbf{p}}{dt} = \mathbf{A}\mathbf{p} = \begin{bmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & 7 & 2 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}}$$

Problem 3

Problem: Consider the following system of equations:

$$\begin{cases} \frac{dx}{dt} = f(x,y) = -3y(1 + x^2 + y^2) \\ \frac{dy}{dt} = g(x,y) = 2x(1 + 2x^2 + 2y^2) \end{cases}$$

a) Show that $\mathbf{y}_1(t) = (\cos 6t, \sin 6t)$ is a solution of this system.

b) Show that if $\mathbf{y}_2(t) = (x_2(t), y_2(t))$ is another solution with $\mathbf{y}_2(1) = (0.5, 0.5)$, then $x_2(t)^2 + y_2(t)^2 < 1$ for all t.

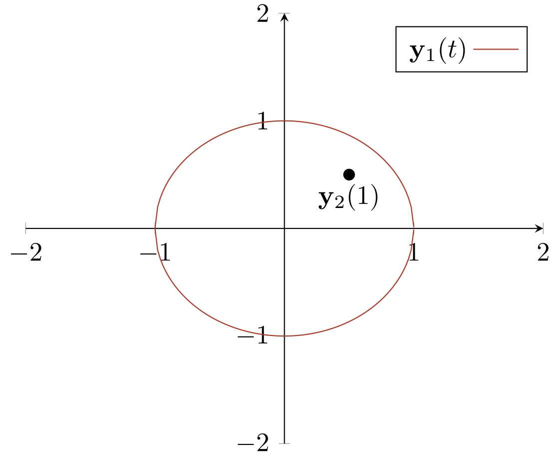
Solution: For **a)** we simply plug in the solution into the both equations of the system to verify it:

$$\begin{aligned} \frac{dx}{dt} &= -3y(1 + x^2 + y^2) \\ -6\sin 6t &= -3\sin 6t(1 + \cos^2 6t + \sin^2 6t) \\ &= -3\sin 6t(1 + 1) && \text{(trig. identity)} \\ &= -6\sin 6t \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= 2x(1 + 2x^2 + 2y^2) \\ 6\cos 6t &= 2\cos 6t(1 + 2\cos^2 6t + 2\sin^2 6t) \\ &= 2\sin 6t(1 + 2) && \text{(trig. identity)} \\ &= 6\sin 6t \end{aligned}$$

To show **b)** let us first establish the uniqueness of solutions to this system. This is guaranteed by the Picard-Lindelöf theorem as long as $\frac{d(f,g)}{d(x,y)}$ exists and is continuous over some open set. This is trivial, as both f and g are polynomials over x and y and so are continuously differentiable functions with respect to x and y.

Now let us graph the initial point on the xy phase plane, along with the solution from part **a)**:



Due to uniqueness, and this being an autonomous system, no two distinct solutions can cross each other on the phase plane. As a result, whatever the solution $\mathbf{y}_2$ looks like, simply because it contains a single point in the interior of $\mathbf{y}_1$, it will never be able to cross over to its exterior.

Note that the curve $\mathbf{y}_1$ traces on the phase plane is a unit circle. This means that:

$$(\forall t \in \mathbb{R})\ \|\mathbf{y}_1(t)\| = 1$$

And since the curve $\mathbf{y}_2$ is trapped in the interior of $\mathbf{y}_1$, we have for all $t \in \mathbb{R}$:

$$\begin{aligned} \|\mathbf{y}_2(t)\| &< 1 \\ \|(x(t), y(t))\| &< 1 && \text{(def. of } \mathbf{y}_2\text{)} \\ \sqrt{x(t)^2 + y(t)^2} &< 1 && \text{(def. of } \|\cdot\|\text{)} \\ \boxed{x(t)^2 + y(t)^2 < 1} \end{aligned}$$

With the last inequality coming from the fact that $x(t)^2+y(t)^2$ is nonnegative, and that the square of any number in the interval $[0, 1)$ is less than 1.

Problem 4

Problem: In each of the following, factor the matrix $\mathbf{A}$ into a product $\mathbf{S\Lambda S^{-1}}$, with Λ a diagonal matrix:

a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

b) $\mathbf{A} = \begin{bmatrix} 5 & 6 \\ -1 & -2 \end{bmatrix}$

Solution a): First we start be finding the eigenvalues of $\mathbf{A}$, by finding the roots of its characteristic polynomial:

$$
\begin{aligned}
0 &= \det(\mathbf{A} - \lambda \mathbf{I}) \\
&= \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} \\
&= \lambda(\lambda - 1) \\
&\implies \lambda = 0, 1
\end{aligned}
$$

We now proceed to find a basis for both eigenspaces. We start with the eigenspace associated with the eigenvalue 0:

$$
\begin{aligned}
E_0(\mathbf{A}) &= \mathrm{Null}(\mathbf{A} - 0\mathbf{I}) && \text{(def. of eigenspace)} \\
&= \mathrm{Null}(\mathbf{A}) \\
&= \mathrm{Null} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \mathrm{Null} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} && \text{(ref)} \\
&= \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} && (x_2 = 0,\ x_1 \text{ free})
\end{aligned}
$$

Now we do the same for the eigenspace associated with the eigenvalue 1:

$$
\begin{aligned}
E_1(\mathbf{A}) &= \mathrm{Null}(\mathbf{A} - \mathbf{I}) && \text{(def. of eigenspace)} \\
&= \mathrm{Null} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} && (x_2 = -x_1)
\end{aligned}
$$

We can now express the desired matrix $\mathbf{S}$, whose columns are the eigenbasis of $\mathbf{A}$:

$$
\mathbf{S} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}
$$

Its inverse $\mathbf{S}^{-1}$ is given by:

$$
\mathbf{S}^{-1} = \frac{1}{|\mathbf{S}|} \begin{bmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}
$$

And finally, $\mathbf{\Lambda}$ is given by the matrix whose diagonal entries are the cooresponding eigenvalues:

$$
\mathbf{\Lambda} = \mathrm{diag} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
$$

And so we can express our original matrix $\mathbf{A}$ as the following eigendecomposition:

$$
\boxed{\mathbf{A} = \mathbf{S\Lambda S^{-1}} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}}
$$

Solution b): Again, we start be finding the roots of $\mathbf{A}$'s characteristic polynomial:

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= \begin{vmatrix} 5-\lambda & 6 \\ -1 & -2-\lambda \end{vmatrix} \\ &= (5-\lambda)(-2-\lambda) + 6 \\ &= \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) \\ &\implies \lambda = 4, -1 \end{aligned}$$

We now proceed to find a basis for both eigenspaces. We start with the eigenspace associated with the eigenvalue 4:

$$\begin{aligned} E_4(\mathbf{A}) &= \text{Null}(\mathbf{A} - 4\mathbf{I}) && \text{(def. of eigenspace)} \\ &= \text{Null} \begin{bmatrix} 1 & 6 \\ -1 & -6 \end{bmatrix} \\ &= \text{Null} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} && \text{(ref)} \\ &= \text{Span} \left\{ \begin{bmatrix} -6 \\ 1 \end{bmatrix} \right\} && (x_1 = -6x_2) \end{aligned}$$

Now we do the same for the eigenspace associated with the eigenvalue -1:

$$\begin{aligned} E_{-1}(\mathbf{A}) &= \text{Null}(\mathbf{A} + \mathbf{I}) && \text{(def. of eigenspace)} \\ &= \text{Null} \begin{bmatrix} 6 & 6 \\ -1 & -1 \end{bmatrix} \\ &= \text{Null} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} && \text{(ref)} \\ &= \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} && (x_1 = -x_2) \end{aligned}$$

We can now express the desired matrix $\mathbf{S}$, whose columns are the eigenbasis of $\mathbf{A}$:

$$\mathbf{S} = \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix}$$

Its inverse $\mathbf{S}^{-1}$ is given by:

$$\mathbf{S}^{-1} = \frac{1}{|\mathbf{S}|} \begin{bmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix}$$

And finally, $\mathbf{\Lambda}$ is given by the matrix whose diagonal entries are the cooresponding eigenvalues:

$$\mathbf{\Lambda} = \text{diag} \begin{bmatrix} 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

And so we can express our original matrix $\mathbf{A}$ as the following eigendecomposition:

$$\boxed{\mathbf{A} = \mathbf{S\Lambda S^{-1}} = \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{6}{5} \end{bmatrix}}$$

Problem 5

Problem: For each matrix $\mathbf{A}$ in question 4, calculate $\mathbf{A}^7$.

Solution a): As we have already decomposed $\mathbf{A}$, we can take advantage of the following property of diagonalizable matrices:

$$\begin{aligned}
\mathbf{A}^7 &= (\mathbf{S\Lambda S^{-1}})^7 && \text{(eigendecomposition)} \\
&= \mathbf{S\Lambda}^7\mathbf{S}^{-1} && \text{(diagonalizable)} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^7 \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0^7 & 0 \\ 0 & 1^7 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} && \text{(diagonal matrix)} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \mathbf{S\Lambda S^{-1}} = \mathbf{A} && \text{(eigendecomposition)} \\
&= \boxed{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}
\end{aligned}$$

Solution b): Again, we have already decomposed $\mathbf{A}$ so we can take advantage of the following property of diagonalizable matrices:

$$\begin{aligned}
\mathbf{A}^7 &= (\mathbf{S\Lambda S^{-1}})^7 && \text{(eigendecomposition)} \\
&= \mathbf{S\Lambda}^7\mathbf{S}^{-1} && \text{(diagonalizable)} \\
&= \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}^7 \begin{bmatrix} -\frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{6}{5} \end{bmatrix} \\
&= \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^7 & 0 \\ 0 & -1^7 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{6}{5} \end{bmatrix} && \text{(diagonal matrix)} \\
&= \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 16384 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{6}{5} \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 19661 & 19662 \\ -3277 & -3278 \end{bmatrix}}
\end{aligned}$$

Problem 6

Problem: For each matrix $\mathbf{A}$ in question 4, calculate $e^{t\mathbf{A}}$.

Solution a): As we have already decomposed $\mathbf{A}$, we can take advantage of the following property of exponential matrices:

$$
\begin{aligned}
e^{t\mathbf{A}} &= e^{t\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}} && \text{(eigendecomposition)}\\
&= \mathbf{S}e^{t\mathbf{\Lambda}}\mathbf{S}^{-1} && \text{(diagonalizable)}\\
&= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \exp\left(\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \right) \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^0 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} && \text{(diagonal matrix)}\\
&= \boxed{\begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}}
\end{aligned}
$$

Solution b): Again, we have already decomposed $\mathbf{A}$ so we can take advantage of the following property of exponential matrices:

$$
\begin{aligned}
e^{t\mathbf{A}} &= e^{t\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}} && \text{(eigendecomposition)}\\
&= \mathbf{S}e^{t\mathbf{\Lambda}}\mathbf{S}^{-1} && \text{(diagonalizable)}\\
&= \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} \exp\left(\begin{bmatrix} 4t & 0 \\ 0 & -t \end{bmatrix} \right) \begin{bmatrix} -\frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{6}{5} \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 6 \end{bmatrix} && \text{(diagonal matrix)}\\
&= \boxed{\frac{1}{5} \begin{bmatrix} 6e^{4t} - e^{-t} & 6e^{4t} - 6e^{-t} \\ -e^{4t} + e^{-t} & -e^{4t} + 6e^{-t} \end{bmatrix}}
\end{aligned}
$$

Problem 7

Problem: Solve the following IVP:

$$
\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}}_{\mathbf{A}} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
$$

Solution: First we perform eigendecomposition on $\mathbf{A}$, and to do this we first find $\mathbf{A}$'s eigenvalues:

$$
\begin{aligned}
0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\
&= \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)(4-\lambda) + 2 \\
&= \lambda^2 - 5\lambda + 6 \\
&= (\lambda - 2)(\lambda - 3) \\
&\implies \lambda = 2, 3
\end{aligned}
$$

Now we find bases of both corresponding eigenspaces:

$$\begin{aligned}
E_2(\mathbf{A}) &= \text{Null}(\mathbf{A} - 2\mathbf{I}) && \text{(def. of eigenspace)} \\
&= \text{Null}\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \\
&= \text{Null}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} && \text{(ref)} \\
&= \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} && (x_1 = x_2)
\end{aligned}$$

$$\begin{aligned}
E_3(\mathbf{A}) &= \text{Null}(\mathbf{A} - 3\mathbf{I}) && \text{(def. of eigenspace)} \\
&= \text{Null}\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \\
&= \text{Null}\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} && \text{(ref)} \\
&= \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} && (x_1 = 2x_2)
\end{aligned}$$

Letting $S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, we now calculate $\mathbf{S}^{-1}$:

$$\begin{aligned}
\mathbf{S}^{-1} &= \frac{1}{|\mathbf{S}|}\begin{bmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{bmatrix} \\
&= -\frac{1}{1-2}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}
\end{aligned}$$

And so we can express our original matrix $\mathbf{A}$ as the following eigendecomposition:

$$\mathbf{A} = \mathbf{S\Lambda S^{-1}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

We can now easily compute the matrix exponential $e^{t\mathbf{A}}$:

$$\begin{aligned}
e^{t\mathbf{A}} &= e^{t\mathbf{S\Lambda S}^{-1}} && \text{(eigendecomposition)} \\
&= \mathbf{S}e^{t\mathbf{\Lambda}}\mathbf{S}^{-1} && \text{(diagonalizable)} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\exp\left(\begin{bmatrix} 2t & 0 \\ 0 & 3t \end{bmatrix}\right)\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} && \text{(diagonal matrix)} \\
&= \begin{bmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} - e^{2t} & -e^{3t} + 2e^{2t} \end{bmatrix}
\end{aligned}$$

Finally, we can express the desired solution to the given IVP as the following matrix vector product:

$$\mathbf{y}(t) = e^{t\mathbf{A}}\mathbf{y}(0) = \begin{bmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} - e^{2t} & -e^{3t} + 2e^{2t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 3e^{2t} - 2e^{3t} \\ 3e^{2t} - e^{3t} \end{bmatrix}}$$

Problem 8

Problem: Let $\mathbf{A}$ be a 2×2 matrix. Assume that the following vector functions:

$$\mathbf{y}_1(t) = \begin{bmatrix} e^t \\ -2e^t \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} 3e^{-2t} \\ e^{-2t} \end{bmatrix}$$

are solutions to the system $\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}$. Solve the following IVP:

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Solution: Recall that the solution set to a homogenous system of linear ODEs forms a vector space. Also note that the two given solutions $y_1(t)$ and $y_2(t)$ span the entirety of the solution set. We can verify this by noting that the Wronskian $W(y_1, y_2)(t) \neq 0$. This means that the desired solution $y(t)$ is simply a linear combination of $y_1(t)$ and $y_2(t)$:

$$\begin{aligned} k_1 y_1(t) + k_2 y_2(t) &= y(t) \\ k_1 y_1(0) + k_2 y_2(0) &= y(0) \\ k_1 \begin{bmatrix} e^0 \\ -2e^0 \end{bmatrix} + k_2 \begin{bmatrix} 3e^0 \\ e^0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ k_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} \end{aligned}$$

We can express this system of linear equations as the following augmented matrix:

$$\begin{aligned} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ -2 & 1 & 5 \end{array}\right] &\xrightarrow{r_2 + 2r_1} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 7 & 7 \end{array}\right] \\ &\xrightarrow{(1/7)r_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1 \end{array}\right] \\ &\xrightarrow{r_1 - 3r_2} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right] \end{aligned}$$

This leaves us with the constants $k_1 = -2$ and $k_2 = 1$. And so, our desired solution $y(t)$ is given by:

$$\begin{aligned} y(t) &= -2y_1(t) + y_2(t) \\ &= -2 \begin{bmatrix} e^t \\ -2e^t \end{bmatrix} + \begin{bmatrix} 3e^{-2t} \\ e^{-2t} \end{bmatrix} \\ &= \boxed{\begin{bmatrix} -2e^t + 3e^{-2t} \\ 4e^t + e^{-2t} \end{bmatrix}} \end{aligned}$$