Intro to Math Reasoning HW 4a

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Problem 1

Problem: Consider the predicate $C(x, y)$ where x and y are real numbers. Let the sets S_1 and S_2:

$$S_1 = \{x \in \mathbb{R} \mid (\forall y \in \mathbb{R})\ C(x, y)\}$$
$$S_2 = \{y \in \mathbb{R} \mid (\forall x \in \mathbb{R})\ \neg C(x, y)\}$$

Can both S_1 and S_2 be nonempty?

Solution: No, either S_1 or S_2 must be empty. We can make this clearer by renaming x and y in the definition of S_2:

$$S_1 = \{x \in \mathbb{R} \mid (\forall y \in \mathbb{R})\ C(x, y)\}$$
$$S_2 = \{x \in \mathbb{R} \mid (\forall y \in \mathbb{R})\ \neg C(y, x)\}$$

They both cannot be nonempty if $C(x, y) \equiv C(y, x)$. But even if this didn't hold, they still coudln't be nonempty. This is because whatever the predicate is, it has to hold for all y in S_1 and not hold for all y in S_2. This means we would have to be able to distinguish between x and y in the predicate but if we could then whatever held for all in one case wouldn't in the other.

Problem 2

Consider the predicate $P(A, B, C) \equiv (C \setminus A = C \setminus B) \to A = B$.

Part a

Problem: Is there an A, B and C such that $P(A, B, C)$ is true?

Solution: Yes there is. $A = \{1\}$, $B = \{1\}$, and $C = \{1, 2, 3\}$

$$C \setminus A = \{2, 3\} = C \setminus B$$
$$A = \{1\} = B$$

Both the antecedent and the consequent are true, thus the predicate is satisfied.

Part b

Problem: Is there a unique (A, B, C) such that $P(A, B, C)$ is true?

Solution: No. It suffices to show two examples of this. One was shown above, another is $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2, 3\}$

$$\begin{gathered} C \setminus A = \{1\} \neq C \setminus B = \{2\} \\ A = \{1\} \neq B = \{2\} \end{gathered}$$

The antecedent is false and the consequent is false, thus the predicate is satisfied.

Part c

Problem: Is there an A, B and C such that $P(A, B, C)$ is false?

Solution: Yes there is. $A = \{0, 1\}$, $B = \{1\}$, and $C = \{2, 3\}$

$$\begin{gathered} C \setminus A = \{2, 3\} = C \setminus B = \{2, 3\} \\ A = \{0, 1\} \neq B = \{1\} \end{gathered}$$

The antecedent is true and the consequent is false, thus the predicate is not satisfied.

Problem 3

Problem: For all sets A, B, and C is the following true:

$$A \cup B \subseteq C \implies (C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B))$$

Solution: The proposition is true. To prove it let us first note the definition of the antecedent:

$$\begin{aligned} A \cup B &= \{x \mid x \in A \vee x \in B\} \\ A \cup B \subseteq C &\equiv (x \in A \cup B) \implies x \in C \\ &\equiv (x \in A \vee x \in B) \implies x \in C \end{aligned}$$

We can rename the atomic propositions in the above compound proposition like so:

$$(a \vee b) \to c$$

Now we do the same for the first half of the consequent:

$$\begin{aligned} A \cup B &= \{x \mid x \in A \vee x \in B\} \\ C \setminus (A \cup B) &= \{x \mid x \in C \wedge \neg(x \in A \cup B)\} \\ &= \{x \mid x \in C \wedge \neg(x \in A \vee x \in B)\} \end{aligned}$$

Like above, we can rename the propositions like so:

$$c \wedge \neg(a \vee b)$$

Finally we do the same for the second half of the antecedent:

$$\begin{aligned} C \setminus A &= \{x \mid x \in C \wedge x \notin A\} \\ C \setminus B &= \{x \mid x \in C \wedge x \notin B\} \\ (C \setminus A) \cap (C \setminus B) &= \{x \mid (x \in C \setminus A) \wedge (x \in C \setminus B)\} \\ &= \{x \mid (x \in C \wedge x \notin A) \wedge (x \mid x \in C \wedge x \notin B)\} \end{aligned}$$

Again, we can rename the propositions like so:

$$\begin{aligned} (c \wedge \neg a) \wedge (c \wedge \neg b) &\equiv c \wedge (\neg a \wedge \neg b) && \text{(distributive property)} \\ &\equiv c \vee \neg(a \vee b) && \text{(De Morgan's law)} \end{aligned}$$

Now we simply have to prove the following:

$$((a \vee b) \to c) \to (c \wedge \neg(a \vee b) \equiv c \wedge \neg(a \vee b))$$

However notice that the consequent is trivially a tautology (it is literally the same expression on both sides). Since the consequent is always true, the truth of the antecedent is irrelevant and the statement as a whole is true.

Problem 4

Problem: For all sets A, B, and C is the following always true:

$$A \cup B \subseteq C \implies C \setminus (A \cup B) \subseteq C \setminus A$$

Solution: Yes this is true and we can prove it in the same way as above. Start with the antcedent:

$$\begin{aligned} A \cup B &= \{x \mid x \in A \vee x \in B\} \\ A \cup B \subseteq C &\equiv (x \in A \cup B) \implies x \in C \\ &\equiv (x \in A \vee x \in B) \implies x \in C \\ &\equiv a \vee b \to c \end{aligned}$$

Now the consequent

$$\begin{aligned}
C \setminus (A \cup B) &= \{x \mid x \in C \land \neg(x \in A \cup B)\} \\
&\equiv \{x \mid x \in C \land \neg(x \in A \lor x \in B)\} \\
&\equiv c \land \neg(a \lor b) \\
C \setminus A &= \{x \mid x \in C \land x \notin A\} \\
&\equiv c \land \neg a \\
C \setminus (A \cup B) \subseteq C \setminus A &= (x \in C \setminus (A \cup B)) \rightarrow (x \in C \setminus A) \\
&\equiv c \land \neg(a \lor b) \rightarrow c \land \neg a
\end{aligned}$$

So now we just have to prove the following:

$$(a \lor b \rightarrow c) \rightarrow (c \land \neg(a \lor b) \rightarrow c \land \neg a)$$

However consider the consequent, which is itself an implication:

$$\begin{aligned}
& c \land \neg(a \lor b) \rightarrow c \land \neg a \\
\equiv\ & c \land (\neg a \land \neg b) \rightarrow c \land \neg a && \text{(De Morgan's law)} \\
\equiv\ & (c \land \neg a) \land \neg b \rightarrow c \land \neg a && \text{(associative property)}
\end{aligned}$$

With the last statement clearly being a tautology (simplification of a conjunction). Remember that the tautology above is the consequent of the bigger statement that we set out to prove. As a result of this, the statement we set out to prove is also a tautology, since its consequent is always true.

Problem 5

Part a

Problem: Is the following true proposition always true:

$$(A \rightarrow B) \rightarrow C \equiv A \rightarrow (B \rightarrow C)$$

Solution: No. Here's a truth table:

A	B	C	$A \rightarrow B$	$B \rightarrow C$	$(A \rightarrow B) \rightarrow C$	$A \rightarrow (B \rightarrow C)$
F	F	F	T	T	F	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	F
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

As we can see the left and right hand propositions are not equivalent thus the proposition is false.

Part b

Problem: Is it the case that either $(A \to B) \to C$ or $A \to (B \to C)$ must be true?

Solution: No. Here's a truth table:

A	B	C	$(A \to B) \to C$	$A \to (B \to C)$	$(A \to B) \to C \oplus A \to (B \to C)$
F	F	F	F	T	T
F	F	T	T	T	F
F	T	F	F	T	T
F	T	T	T	T	F
T	F	F	T	F	T
T	F	T	T	T	F
T	T	F	F	F	F
T	T	T	T	T	F

The exclusive disjunction of the two statements indeed does not form a tautology. And so the statement we set out to disprove is false.