Numerical Analysis HW #5

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Problem 1

Part a

Problem: Use Euler's method with step size $h = \frac{1}{2}$ to approximate $y\left(\frac{1}{2}\right)$ and $y(1)$ where $y(x)$ is the solution to the following IVP:

$$y' = 1 - 8xy \qquad\qquad y(0) = 0$$

Solution: Recall that Euler's method is given by the following iteration for each equally spaced node x_i:

$$y_{i+1} = y_i + hf(x_i, y_i)$$

For $y\left(\frac{1}{2}\right)$ our interval is $[0, \frac{1}{2}]$. This gives us only one iteration step at $x_1 = \frac{1}{2}$:

$$y_1 = y_0 + hf(x_0, y_0) = 0 + \frac{1}{2}(1 - 8(0)(0)) = \boxed{\frac{1}{2} \approx y\left(\frac{1}{2}\right)}$$

For $y(1)$ our interval is $[0, 1]$. gives us one more iteration step for $x_2 = 1$:

$$y_2 = y_1 + hf(x_1, y_1) = \frac{1}{2} + \frac{1}{2}\left(1 - 8\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right) = \boxed{0 \approx y(1)}$$

Part b

Problem: Repeat part a, but with a Taylor series method of order two.

Solution: Recall that the Taylor Series method of degree two is given by the following iteration for each equally spaced node x_i:

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right](x_i, y_i)$$

To solve this method explicitly, we must first compute the summands of $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} 1 - 8xy = -8y \\
\frac{\partial f}{\partial y} f &= \left(\frac{\partial}{\partial y} 1 - 8xy\right)(1 - 8xy) \\
&= -8x(1 - 8xy) \\
&= 64x^2 y - 8x
\end{aligned}$$

The sum of these terms is then:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f = (-8y) + (64x^2 y - 8x) = 64x^2 y - 8x - 8y$$

Giving us a final iteration method of:

$$y_{i+1} = y_i + h(1 - 8x_i y_i) + \frac{h^2}{2}(64x_i^2 y_i - 8x_i - 8y_i)$$

For $y\left(\frac{1}{2}\right)$ our interval is $[0, \frac{1}{2}]$. This gives us only one iteration step at $x_1 = \frac{1}{2}$:

$$\begin{aligned} y_1 &= y_0 + h(1 - 8x_0 y_0) + 4h^2(8x_0^2 y_0 - x_0 - y_0) \\ &= 0 + \left(\frac{1}{2}\right)(1 - 8(0)(0)) + 4\left(\frac{1}{2}\right)^2 (8(0)^2(0) - (0) - (0)) \\ &= \boxed{\frac{3}{2} \approx y\left(\frac{1}{2}\right)} \end{aligned}$$

For $y(1)$ our interval is $[0, 1]$. This gives us one more iteration step for $x_2 = 1$:

$$\begin{aligned} y_2 &= y_1 + h(1 - 8x_1 y_1) + 4h^2(8x_1^2 y_1 - x_1 - y_1) \\ &= \frac{3}{2} + \left(\frac{1}{2}\right)\left(1 - 8\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\right) + 4\left(\frac{1}{2}\right)^2 \left(8\left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right) - \left(\frac{1}{2}\right) - \left(\frac{3}{2}\right)\right) \\ &= \boxed{0 \approx y(1)} \end{aligned}$$

Part c

Problem: Repeat part a, but with Heun's method.

Solution: Recall that Heun's method is given by:

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, \tilde{y}_{i+1})]$$

Where $\tilde{y}_{i+1}$ is an approximation of y_{i+1}:

$$\tilde{y}_{i+1} = y_i + hf(x_i, y_i)$$

For $y\left(\frac{1}{2}\right)$ we have one iteration at $x_1 = \frac{1}{2}$. First we calculate $\tilde{y}_1$:

$$\begin{aligned} \tilde{y}_1 &= y_0 + hf(x_0, y_0) \\ &= 0 + \frac{1}{2}(1 - 8(0)(0)) = \frac{1}{2} \end{aligned}$$

Now we can calculate our approximation y_1:

$$
\begin{aligned}
y_1 &= y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, \tilde{y}_1)] \\
&= 0 + \frac{1}{2}\left(\frac{1}{2}\right)\left[(1 - 8(0)(0)) + \left(1 - 8\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right)\right] \\
&= \boxed{0 \approx y\left(\frac{1}{2}\right)}
\end{aligned}
$$

And for $y(1)$ we have one more iteration at $x_2 = 1$. First we calculate $\tilde{y}_2$:

$$
\begin{aligned}
\tilde{y}_2 &= y_1 + hf(x_1, y_1) \\
&= 0 + \frac{1}{2}\left(1 - 8\left(\frac{1}{2}\right)(0)\right) = \frac{1}{2}
\end{aligned}
$$

Now we can calculate our approximation y_2:

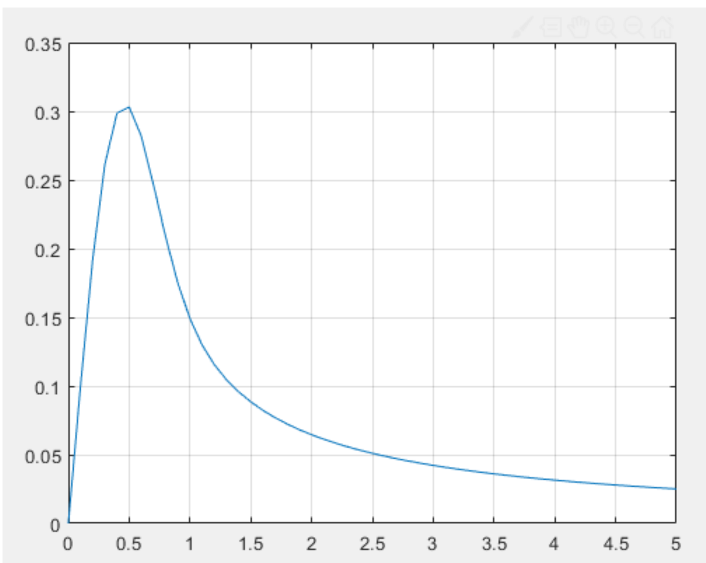
$$
\begin{aligned}
y_2 &= y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, \tilde{y}_2)] \\
&= 0 + \frac{1}{2}\left(\frac{1}{2}\right)\left[\left(1 - 8\left(\frac{1}{2}\right)(0)\right) + \left(1 - 8(1)\left(\frac{1}{2}\right)\right)\right] \\
&= \frac{1}{4}(1 - 3) = \boxed{-\frac{1}{2} \approx y(1)}
\end{aligned}
$$

Problem 2

Part a

Problem: Using the same IVP as Problem 1, approximate $y(5)$ over the interval $[0, 5]$ with a step size of $h = 0.1$ using Euler's method in MATLAB. Then plot y over the interval.

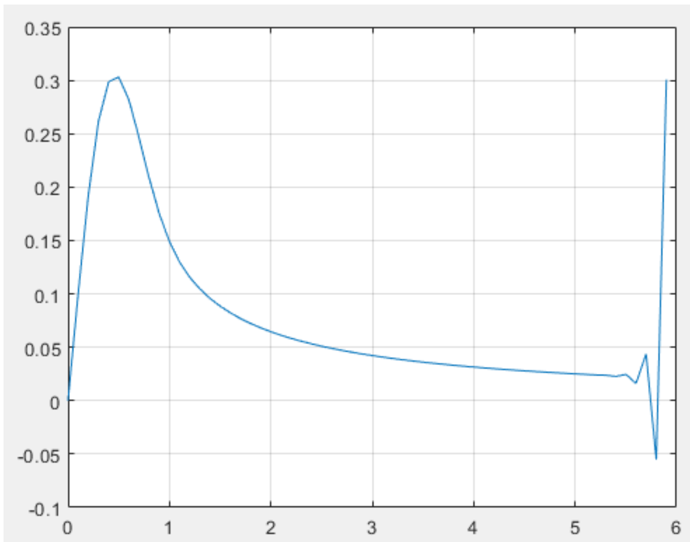
Solution: The approximation is $y(5) \approx 0.025118547520984$ and the plot is given below:



Part b

Problem: Do the same as part a, but approximate $y(5.9)$ on the interval $[0, 5.9]$. Is the numerical solution accurate?

Solution: The approximation is $y(5.9) \approx 0.300553327482678$ and the plot is given below:



The numerical solution is indeed *not* accurate. The true solution, while not elementary, is smooth and is decreasing as $x \to \infty$. This is not reflected on our numerical plot. In particular the huge spike at the end means our approximation of 0.3005 is wildly off. The true solution is approximately 0.0212634.