Honors Calculus III Challenge Problems #3

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Exercise 1

Solution: The problem sheet introduces Q like so:

$$\mathbf{v}_i = \sum_{j=1}^{r} R_i^j \mathbf{u}_j = \sum_{j=1}^{r} R_i^j (\sum_{k=1}^{m} Q_j^k \mathbf{f}_k)$$

This means that for every j up to r:

$$\mathbf{u}_j = \sum_{k=1}^{m} Q_j^k \mathbf{f}_k$$

And so the r columns of Q are just m-vectors that are equal to the vectors produced by the Gram-Schmidt process which are, by definition, orthonormal.

Exercise 2

Solution: We showed above that the matrix Q is orthogonal (i.e. all it's columns are orthonormal vectors), and so equivalently:

$$Q^\top Q = I$$

Where I is the m dimensional identity matrix. This should be clear as matrix multiplication is a series of dot products of the rows of Q and the columns of $Q^\top$ (or vice versa). Transposing the matrix means this dot product will be between 2 identical *orthogonal* vectors when $i = j$ (equaling 1) but between two non-identical orthogonal vectors when $i \neq j$ (i.e. 0). Thus the matrix will have only 1's on the main diagonal and nowhere else (i.e. the identity matrix).

Recall that the dot product of two vectors can also be written in terms of matrix multiplication (by considering $\mathbf{a}$ as a column matrix):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\top \mathbf{b}$$

We can now prove our initial proposition:

$$\begin{aligned}
Q\mathbf{x} \cdot Q\mathbf{y} &= (Q\mathbf{x})^\top (Q\mathbf{y}) && \text{(dot product is matrix mult.)} \\
&= (\mathbf{x}^\top Q^\top)(Q\mathbf{y}) && \text{(anti-distributivity of transpose)} \\
&= \mathbf{x}^\top (Q^\top Q)\mathbf{y} && \text{(associativity of matrix mult.)} \\
&= \mathbf{x}^\top I\mathbf{y} && \text{(def. of orthogonal matrix)} \\
&= \mathbf{x}^\top \mathbf{y} && \text{(def. identity matrix)} \\
&= \mathbf{x} \cdot \mathbf{y} && \text{(dot product is matrix mult.)}
\end{aligned}$$

And so indeed, all orthogonal matrices respect the dot product.

Exercise 3

Solution: Remember that $\ker(Q) = \{0\}$ is equivalent to all $\mathbf{a}$ such that:

$$a_1\mathbf{Q}_1 + a_2\mathbf{Q}_2 + \cdots + a_r\mathbf{Q}_r = 0$$

Where $\mathbf{Q}_j$ denotes the jth column of Q.

However, recall from Exercise 1 that $\mathbf{Q}_j = \mathbf{u}_j$ and so for there to be no non-trivial solution to the homogenous equation the following must be true:

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_r\mathbf{u}_r = 0$$

Since the vectors $\mathbf{u}_j$ were created via GS orthonormalization, they form an orthonormal basis of $\mathbb{R}$ and so are linearly independent. This necessarily implies that there are no non-trivial solutions thus, $\ker(Q) = \{\mathbf{0}\}$.

Exercise 4

Solution: Recall that $A = QR$ which means $A\mathbf{x} = QR\mathbf{x}$. Note that all linear transformations (whose kernel just contains $\mathbf{0}$) have a unique mapping from $\mathbf{0}$ to itself. Because of this, for Q to map $R\mathbf{x}$ to $\mathbf{0}$ means that $R\mathbf{x} = \mathbf{0}$. And so if $A\mathbf{x} = \mathbf{0}$ then it must be the case that $R\mathbf{x} = \mathbf{0}$ due to Q mapping it as such.

Proving it the other way around is as simple as noting that all linear transformations map $\mathbf{0}$ to itself (uniquely or not) and so if $R\mathbf{x} = 0$ then so does Q and thus A since $A = QR$

And so the kernals of A and R are equivalent.

Exercise 5

Solution: First we need to apply the GS process to the columns $\mathbf{a}_i$ of A, giving us:

$$\mathbf{u}_1 = \frac{1}{3}(1,2,2) \quad \mathbf{u}_2 = \frac{1}{3}(2,-2,1) \quad \mathbf{u}_1 = \frac{1}{3}(2,1,-2)$$

Putting these into a matrix we get:

$$Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

Now we just need to express the columns of A in terms of this new basis:

$$\begin{aligned}
\mathbf{r}_1 &= (\mathbf{v}_1 \cdot \mathbf{u}_1)\mathbf{e}_1 + (\mathbf{v}_1 \cdot \mathbf{u}_2)\mathbf{e}_2 + (\mathbf{v}_1 \cdot \mathbf{u}_3)\mathbf{e}_3 \\
\mathbf{r}_2 &= (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{e}_1 + (\mathbf{v}_2 \cdot \mathbf{u}_2)\mathbf{e}_2 + (\mathbf{v}_2 \cdot \mathbf{u}_3)\mathbf{e}_3 \\
\mathbf{r}_3 &= (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{e}_1 + (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v}_3 \cdot \mathbf{u}_3)\mathbf{e}_3
\end{aligned}$$

Evaluating these vectors this and organizing them into a matrix, we are left with:

$$R = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Since the columns of R are linearly independent, $\ker(R) = \{\mathbf{0}\}$ which in turn implies $\ker(A) = \{\mathbf{0}\}$.

Exercise 6

Solution: If for every $\mathbf{u}_i$ there was an $\mathbf{x}_i$ that R maps from, we could simply note the following:

$$\begin{aligned}
R(a_1\mathbf{x}_1 + \cdots + a_r\mathbf{x}_r) &= \mathbf{b} \\
a_1R\mathbf{x}_1 + \cdots + a_rR\mathbf{x}_r &= \mathbf{b} && \text{(Linearity of } R\text{)} \\
a_1\mathbf{u}_1 + \cdots + a_r\mathbf{u}_r &= \mathbf{b} && \text{(def. of } \mathbf{x}_i\text{)}
\end{aligned}$$

And since all vectors in $\mathbb{R}^r$ are expressible as a linear combination of the orthonormal basis $\mathbf{u}_i$, $\mathbf{b}$ must be as well.

Exercise 7

Solution: Now remember the span of R is just the set of all linear combinations of its column vectors. But recall that the ith column of R was some scalar multiple of the canonical basis vector $\mathbf{e}_i$ (and some possibly 0 multiple of all the previous $\mathbf{e}_{<i}$). Since there were r columns, the span of R must be $\mathbb{R}^r$. This means there must be some solution to $R\mathbf{x}_i = \mathbf{u}_i$ because $\mathbf{u}_i \in \mathbb{R}^r$.

Exercise 8

Solution: Just let $\mathbf{y} := R\mathbf{x}$. And so if there exists an $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ then:

$$A\mathbf{x} = QR\mathbf{x} = Q\mathbf{y} = \mathbf{b}$$

Exercise 9

Solution: Recall that the $\ker(Q) = \{0\}$, meaning that there are no non-trivial solutions to the homogenous equation and thus every non-homogenous equation (i.e. anything of the form $Q\mathbf{x} = \mathbf{y}$) has a unique solution. And so of course, we can solve for it by doing the following:

$$\begin{aligned} Q\mathbf{y} &= \mathbf{b} \\ Q^\top Q\mathbf{y} &= Q^\top \mathbf{b} \\ \mathbf{y} &= Q^\top \mathbf{b} \end{aligned}$$

And so $\mathbf{y}$ is uniquely determined by the vector $\mathbf{b}$, and the transpose of Q which is just the vectors $\mathbf{u}_i$ arranged by rows.

Exercise 10

Solution: Recall that $\mathbf{y} = Q^\top \mathbf{b}$, this means that:

$$\begin{aligned} \mathbf{y}_i &= (Q^\top)^i \cdot \mathbf{b} \\ &= Q_i \cdot \mathbf{b} \\ &= \mathbf{u}_i \cdot \mathbf{b} \end{aligned}$$

Where $(Q^\top)^i$ is the ith row of $Q^\top$ and Q_i is the ith column of Q.

Exercise 11

Solution: Remember that in Exercise 7, we established that the ith column of R is equal to some scalar times $\mathbf{e}_i$ (plus some possibly 0 multiples of the previous canonical bases). And since R has r columns, that rth column too must be nonzero multiple of $\mathbf{e}_r$ (and potentially plus some orthogonal components). And so the last row of R must at least have some nonzero scalar at the rth (final) column.

The case where all the $\mathbf{v}_i = \mathbf{0}$ is the case where the matrix A that we decomposed didn't exist at all and so our reasoning is vacuously true.

Exercise 12

Solution: Recall that R spans all of $\mathbb{R}^r$. If the last row was all zeros then there would be no column vector in R that had a component in the $\mathbf{e}_r$ direction. This directly contradicts the fact that R spans all of $\mathbb{R}^r$ and thus the last row can't all be zeros.

Exercise 13

Solution: We can turn the matrix into a system of equations:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 0 \\ 3x_3 + 6x_4 - 15x_5 &= 9 \\ -2x_4 - 6x_5 &= 4 \end{aligned}$$

Solving this out via backsubtituing the last row into the upper ones, we get the following parameterization:

$$\mathbf{x}(t, s) = t(-2, 1, 0, 0, 0) + s(-26, 0, 11, , -3, 1) + (-13, 0, 7, -2, 0)$$

Exercise 14

Solution: First we perform the GS process on the columns of A:

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{5}(4, 2, 2, -1) \\ \mathbf{u}_2 &= \frac{1}{5}(1, -2, 2, 4) \end{aligned}$$

This gives us the matrix:

$$Q = \frac{1}{5}\begin{bmatrix} 4 & 1 & 0 \\ 2 & -2 & 0 \\ 2 & 2 & 0 \\ -1 & 4 & 0 \end{bmatrix}$$

Now we just need to express the columns of A in terms of this new basis:

$$\begin{aligned} \mathbf{r}_1 &= (\mathbf{a}_1 \cdot \mathbf{u}_1)\mathbf{e}_1 \\ \mathbf{r}_2 &= (\mathbf{a}_2 \cdot \mathbf{u}_1)\mathbf{e}_1 + (\mathbf{a}_2 \cdot \mathbf{u}_2)\mathbf{e}_2 \\ \mathbf{r}_3 &= (\mathbf{a}_3 \cdot \mathbf{u}_1)\mathbf{e}_1 + (\mathbf{a}_3 \cdot \mathbf{u}_2)\mathbf{e}_2 + (\mathbf{a}_3 \cdot \mathbf{u}_3)\mathbf{e}_3 \\ &= (\mathbf{a}_3 \cdot \mathbf{u}_1)\mathbf{e}_1 + (\mathbf{a}_3 \cdot \mathbf{u}_2)\mathbf{e}_2 \end{aligned}$$

Note that $\mathbf{u}_3$ *is* $\mathbf{0}$.

Evaluating these vectors and organizing them into a matrix, we are left with:

$$R = \begin{bmatrix} 5 & 5 & 5 \\ 0 & 10 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

We can finally solve for the following:

$$R\mathbf{x} = Q^\top \mathbf{b}$$

First we compute $Q^\top \mathbf{b}$:

$$Q^\top \mathbf{b} = (15, -5, 0)$$

Now we can write down the system of equations:

$$\begin{aligned} 5x_1 + 5x_2 + 5x_3 &= 15 \\ 10x_2 - 5x_3 &= -5 \\ 0 &= 0 \end{aligned}$$

We can now parameterize this via back substitution to find:

$$\mathbf{x}(t) = \frac{1}{5}\left(t, \frac{10-t}{3}, \frac{20-2t}{3}\right)$$