Linear Algebra HW #1

Ozaner Hansha

February 3, 2020

Problem 1

Problem: Are the following statements true or false? Provide justification of your answers.

a) In any vector space, $\lambda_1 v = \lambda_2 v$ implies that $\lambda_1 = \lambda_2$.

b) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.

c) The empty set is a subspace of every vector space.

d) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

Solution:

a) False. For a counterexample, consider the following statement in $\mathbb{R}^2$:

$$0 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 5 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} \not\Longrightarrow 0 = 5$$

b) False. For a counterexample, consider the $n = 2$ case and the two polynomials $f, g \in \mathbb{R}[x]$:

$$\underbrace{(x^2 + x)}_{f} + \underbrace{(-x^2 + x)}_{g} = \underbrace{2x}_{f+g}$$

As we can see, the sum $f + g$ is a 1st degree polynomial, not 2nd.

c) False. This can be seen by noting that the empty set does not contain a zero vector, or any vector for that matter, and thus does not qualify to be a subspace.

d) True. Since all vector spaces contain a zero vector, i.e. $\mathbf{0} \in V$, the condition $V \neq \{\mathbf{0}\}$ implies that V is a strict superset of the zero vector space, i.e. $\{0\} \subsetneq V$. As such we can simply choose W to be the zero vector space $\{\mathbf{0}\} = W \neq V$.

Problem 2

Problem: a) Solve the following system of linear equations using Gaussian elimination:

$$\begin{cases} 3x_1 - 7x_2 + 4x_3 = 10 \\ x_1 - 2x_2 + x_3 = 3 \\ 2x_1 - x_2 - 2x_3 = 6 \end{cases}$$

b) Does the following system of linear equations have a solution?

$$\begin{cases} x_1 + 2x_2 + 2x_3 = 2 \\ x_1 + 8x_3 + 5x_4 = -6 \\ x_1 + x_2 + 5x_3 + 5x_4 = 3 \end{cases}$$

Solution: a) We first express this system of equations as an augmented matrix, and then perform Gaussian elimination:

$$\left[\begin{array}{ccc|c} 3 & -7 & 4 & 10 \\ 1 & -2 & 1 & 3 \\ 2 & -1 & -2 & 6 \end{array}\right] \xrightarrow{r_1 - r_3} \left[\begin{array}{ccc|c} 1 & -6 & 6 & 4 \\ 1 & -2 & 1 & 3 \\ 2 & -1 & -2 & 6 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & -6 & 6 & 4 \\ 0 & 4 & -5 & -1 \\ 2 & -1 & -2 & 6 \end{array}\right]$$

$$\xrightarrow{(1/4)r_2} \left[\begin{array}{ccc|c} 1 & -6 & 6 & 4 \\ 0 & 1 & -\frac{5}{4} & -\frac{1}{4} \\ 2 & -1 & -2 & 6 \end{array}\right] \xrightarrow{r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & -6 & 6 & 4 \\ 0 & 1 & -\frac{5}{4} & -\frac{1}{4} \\ 0 & 11 & -14 & -2 \end{array}\right]$$

$$\xrightarrow{r_3 - 11r_2} \left[\begin{array}{ccc|c} 1 & -6 & 6 & 4 \\ 0 & 1 & -\frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} \end{array}\right] \xrightarrow{-4r_3} \left[\begin{array}{ccc|c} 1 & -6 & 6 & 4 \\ 0 & 1 & -\frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{r_2 + (5/4)r_3} \left[\begin{array}{ccc|c} 1 & -6 & 6 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{r_1 - 6r_3} \left[\begin{array}{ccc|c} 1 & -6 & 0 & 22 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{r_1 + 6r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right] \implies \begin{cases} x_1 = -2 \\ x_2 = -4 \\ x_3 = -3 \end{cases}$$

And so our solution is $\mathbf{x} = \begin{bmatrix} -2 \\ -4 \\ -3 \end{bmatrix}$.

b) First we express the system as an augmented matrix and put it in ref:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 1 & 0 & 8 & 5 & -6 \\ 1 & 1 & 5 & 5 & 3 \end{array}\right] \xrightarrow{r_2 - r_3} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 0 & -1 & 3 & 0 & -9 \\ 1 & 1 & 5 & 5 & 3 \end{array}\right] \xrightarrow{r_3 - r_1} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 0 & -1 & 3 & 0 & -9 \\ 0 & -1 & 3 & 5 & 1 \end{array}\right]$$

$$\xrightarrow{-r_2} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & -3 & 0 & 9 \\ 0 & -1 & 3 & 5 & 1 \end{array}\right] \xrightarrow{r_3 + r_2} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & -3 & 0 & 9 \\ 0 & 0 & 0 & 5 & 10 \end{array}\right]$$

It is at this point that we can see that the rank of the matrix without augmentation is equal to the rank of augmented matrix, that is they both have a rank of 3. This guarantees that there exists at least one solution. And since the system is of 4 variables, there is $4 - 3 = 1$ free variable and thus the solution set is infinite.

Problem 3

Problem: Equip $V = \{(a_1, a_2, \cdots, a_n)^\top \mid a_i \in \mathbb{C}, i = 1, \cdots, n\}$ with the operations of coordinatewise addition and multiplication. Is V a vector space over $\mathbb{C}$? Is V a vector space over $\mathbb{R}$?

Solution: a) For closure under addition, we have the following:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \in V \tag{VS1}$$

Since $\mathbb{C}$ is a field, in particular closed under addition, any component of the sum $a_i + b_i$ is also a complex number and thus by definition the sum is a member of V.

We inherit commutativity and associativity of vector addition in much the same way:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \\ \vdots \\ b_n + a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{VS2}$$

$$\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) + c_1 \\ (a_2 + b_2) + c_2 \\ \vdots \\ (a_n + b_n) + c_3 \end{bmatrix} = \begin{bmatrix} a_1 + (b_1 + c_1) \\ a_2 + (b_2 + c_2) \\ \vdots \\ a_n + (b_n + c_3) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \left(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right) \tag{VS3}$$

Our zero vector is given by the vector with 0 as all its components:

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + 0 \\ c_2 + 0 \\ \vdots \\ c_n + 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{VS4}$$

And every vector has an additive inverse obtained by multiplying it by -1:

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} -c_1 \\ -c_2 \\ \vdots \\ -c_n \end{bmatrix} = \begin{bmatrix} c_1 - c_1 \\ c_2 - c_2 \\ \vdots \\ c_n - c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{VS5}$$

These first 5 axioms cover the additive properties of the vector space, now all that's left is to verify the multiplicative properties.

For closure under multiplication, we have the following:

$$\lambda \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{bmatrix} \in V \tag{VS6}$$

Since $\mathbb{C}$ is a field, in particular closed under multiplication, any component of the product λa_i is also a complex number and thus by definition the product is a member of V.

We inherit the distributivity and associativity of scalar multiplication in much the same way:

$$\begin{aligned}
\lambda\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\right) &= \lambda \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} \lambda(a_1 + b_1) \\ \lambda(a_2 + b_2) \\ \vdots \\ \lambda(a_n + b_n) \end{bmatrix} = \begin{bmatrix} \lambda a_1 + \lambda b_1 \\ \lambda a_2 + \lambda b_2 \\ \vdots \\ \lambda a_n + \lambda b_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{bmatrix} + \begin{bmatrix} \lambda b_1 \\ \lambda b_2 \\ \vdots \\ \lambda b_n \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad \text{(VS7)}
\end{aligned}$$

$$\begin{aligned}
(\lambda_1 + \lambda_2) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} &= \begin{bmatrix} (\lambda_1 + \lambda_2)c_1 \\ (\lambda_1 + \lambda_2)c_2 \\ \vdots \\ (\lambda_1 + \lambda_2)c_n \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 + \lambda_2 c_1 \\ \lambda_1 c_2 + \lambda_2 c_2 \\ \vdots \\ \lambda_1 c_n + \lambda_2 c_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 c_1 \\ \lambda_1 c_2 \\ \vdots \\ \lambda_1 c_n \end{bmatrix} + \begin{bmatrix} \lambda_2 c_1 \\ \lambda_2 c_2 \\ \vdots \\ \lambda_2 c_n \end{bmatrix} = \lambda_1 \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \lambda_2 \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad \text{(VS8)}
\end{aligned}$$

$$\lambda_1 \left(\lambda_2 \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right) = \lambda_1 \begin{bmatrix} \lambda_2 c_1 \\ \lambda_2 c_2 \\ \vdots \\ \lambda_2 c_n \end{bmatrix} = \begin{bmatrix} \lambda_1(\lambda_2 c_1) \\ \lambda_1(\lambda_2 c_2) \\ \vdots \\ \lambda_1(\lambda_2 c_n) \end{bmatrix} = \begin{bmatrix} (\lambda_1 \lambda_2) c_1 \\ (\lambda_1 \lambda_2) c_2 \\ \vdots \\ (\lambda_1 \lambda_2) c_n \end{bmatrix} = (\lambda_1 \lambda_2) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad \text{(VS9)}$$

Finally we verify that there is a scalar identity, namely 1:

$$1 \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 \\ 1 \cdot c_2 \\ \vdots \\ 1 \cdot c_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad \text{(VS10)}$$

b) Of course, all the same additive properties hold just as in a) for this case as they do not involve the underlying field. And since $\mathbb{R} \subseteq \mathbb{C}$, the multiplicative properties also still hold, i.e distributivity, associativity, and closure under V. The scalar identity, i.e. 1, also remains the same.

Problem 4

Problem: Let $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R} \right\}$. Define addition coordinatewise, and scalar multiplication for each $\lambda \in \mathbb{R}$ by:

$$\lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if } \lambda = 0, \\ \begin{bmatrix} \lambda a_1 \\ \frac{a_2}{\lambda} \end{bmatrix} & \text{if } \lambda \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution: No, V fails to satisfy the distributivity of vectors over scalars, i.e. the following axiom:

$$(\lambda_1 + \lambda_2)\mathbf{v} = \lambda_1\mathbf{v} + \lambda_2\mathbf{v} \tag{VS8}$$

To see this, consider the following counterexample:

$$\begin{aligned}(2+3)\begin{bmatrix}18\\30\end{bmatrix} &\stackrel{?}{=} 2\begin{bmatrix}18\\30\end{bmatrix} + 3\begin{bmatrix}18\\30\end{bmatrix}\\ \begin{bmatrix}5\cdot 18\\ \frac{30}{5}\end{bmatrix} &\stackrel{?}{=} \begin{bmatrix}2\cdot 18\\ \frac{30}{2}\end{bmatrix} + \begin{bmatrix}3\cdot 18\\ \frac{30}{3}\end{bmatrix}\\ \begin{bmatrix}90\\6\end{bmatrix} &\neq \begin{bmatrix}90\\25\end{bmatrix}\end{aligned}$$

Problem 5

Problem: Is the set $W = \{f(x) \in \mathbb{F}[x] \mid f(x) = 0 \vee f(x) \text{ has degree } n\}$ a subspace of $\mathbb{F}[x]$ if $n \geq 1$? Justify your answer.

Solution: No. Consider the case where $\mathbb{F} = \mathbb{R}$ and $n = 2$. The following example shows that W is *not* closed under vector addition:

$$(x^2 + 2x) + (-x^2 + 2x) = 4x \notin W$$

You'll notice that both summands are 2nd degree polynomials and thus are members of W. Yet their sum, the polynomial $4x$, is of degree 1 and thus by definition is not in W. Thus W fails additive closure.

Problem 6

Problem: Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $w_1 + w_2$, where $w_i \in W_i$.

Solution: ($\Rightarrow$) To prove the forward implication we assume that $V = W_1 \oplus W_2$, or equivalently that $V = W_1 + W_2$ and that $W_1 \cap W_2 = \{\mathbf{0}\}$. Firstly, note that every vector $\mathbf{v} \in V$ can be represented as the sum of two vectors $\mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_i \in W_i$, because $V = W_1 + W_2$. Now let us show that the choice of these two vectors is unique. Let

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1' + \mathbf{w}_2'$$

With $w_i, w_i' \in W_i$. Note that $\mathbf{w}_1 - w_1' \in W_1$ since W_1 is a vector space and closed under addition and has inverses, likewise for W_2. However, from the equation above, we note that $w_1 - w_1' = w_2' - w_2 \in W_2$. But since $W_1 \cap W_2 = \{\mathbf{0}\}$, it must be the case that:

$$\begin{aligned} w_1 - w_1' = 0 &\implies w_1 = w_1' \\ w_2' - w_2 = 0 &\implies w_2' = w_2 \end{aligned}$$

And so we have shown $V = W_1 \oplus W_2 \implies (\forall \mathbf{v} \in V, \exists! \mathbf{w}_i \in W_i)\ \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{v}$.

($\Leftarrow$) Now to prove the converse, we will assume that every vector $\mathbf{v} \in V$ has a unique representation $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{v}$ with $\mathbf{w}_i \in W_i$. We can see that this immediately results in $V = W_1 + W_2$ since every vector in V is a sum of two vectors from these subspaces. To show that $W_1 \cap W_2 = \{\mathbf{0}\}$, let us assume there exists an $\mathbf{x} \in W_1 \cap W_2$ such that $\mathbf{x} \neq \mathbf{0}$. Note that since W_1 and W_2 are subspaces, their intersection must include the zero vector $\mathbf{0}$.

Note that we would then have that $\mathbf{x} = \mathbf{x} + \mathbf{0}$ with $\mathbf{x} \in W_1$ and $\mathbf{0} \in W_2$ but that we'd also have $\mathbf{x} = \mathbf{0} + \mathbf{x}$ with $\mathbf{0} \in W_1$ and $\mathbf{x} \in W_2$, since both $\mathbf{0}, \mathbf{x} \in W_i$. This clearly violates the uniqueness we assumed in proving the converse and thus means that there is no nonzero element in the intersection of W_1 and W_2.

Put together, these two facts imply that $V = W_1 \oplus W_2$. Thus showing that:

$$(\forall \mathbf{v} \in V, \exists! \mathbf{w}_i \in W_i)\ \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{v} \implies V = W_1 \oplus W_2$$

($\Leftrightarrow$) Since we have shown both conditions are sufficient for the other, we have established that they are equivalent.

$$V = W_1 \oplus W_2 \iff (\forall \mathbf{v} \in V, \exists! \mathbf{w}_i \in W_i)\ \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{v}$$