Linear Algebra HW #5

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Problem 1

Problem: Find k such that:

$$k \det \underbrace{\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}}_{A} = \det \underbrace{\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix}}_{A'}$$

Solution: First note the following equality:

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix}}_{A'}$$

Now, recalling that the product of determinants equals the determinant of the product, we have the following:

$$\det E \det A = \det A'$$

And so our desired constant $k = \det E$. We now calculate this determinant:

$$\begin{aligned} \det E &= 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ &= 0 - (-1) + 1 = 2 \end{aligned}$$

Problem 2

Problem: Let $A \in \mathbb{C}^{n\times n}$, then:

a) Prove that $\det \overline{A} = \overline{\det A}$.

b) Prove that if A is unitary, then $|\det A = 1|$.

Solution:

a) We can prove this holds for any matrix $A \in \mathbb{C}^{n\times n}$ for all $n \in \mathbb{N}$ using induction. The base case of $n = 1$ is trivial:

$$\det \overline{A} = \overline{a_{11}} = \overline{\det A}$$

Now we assume the inductive hypothesis, which is that $\det \overline{B} = \overline{\det B}$ for any $(n-1) \times (n-1)$ matrix B, and prove that this implies the same for $n \times n$ matrices:

$$\begin{aligned}
\det A &= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1}^{sub} && \text{(cofactor expansion over column 1)} \\
\overline{\det A} &= \overline{\sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1}^{sub}} \\
&= \sum_{i=1}^{n} \overline{(-1)^{i+1} a_{i1} \det A_{i1}^{sub}} && \text{(sum of conjugates)} \\
&= \sum_{i=1}^{n} \overline{(-1)^{i+1}}\,\overline{a_{i1}}\,\overline{\det A_{i1}^{sub}} && \text{(product of conjugates)} \\
&= \sum_{i=1}^{n} (-1)^{i+1} \overline{a_{i1}} \det \overline{A_{i1}^{sub}} && \text{(inductive hypothesis)} \\
&= \det \overline{A} && \text{(cofactor expansion over column 1)}
\end{aligned}$$

And so by induction we have that the desired identity holds for all complex valued square matrices.

Note that for an $(n+1) \times (n+1)$ matrix B, the matrix B_{ij}^{sub} refers to the $n \times n$ submatrix of B found by removing its ith row and jth column.

b) For an arbitrary unitary matrix A we have:

$$\begin{aligned}
I &= AA^* && \text{(def. of unitary matrix)} \\
&= A\overline{A^\top} && \text{(def. of conjugate transpose)} \\
\det I &= \det A \det \overline{A^\top} && \text{(determinant of product)} \\
&= \det A\, \overline{\det A^\top} && \text{(part } \mathbf{a}\text{)} \\
&= \det A\, \overline{\det A} && \text{(transpose preserves determinant)} \\
1 &= |\det A|^2 && \text{(product of conjugates is modulus)}
\end{aligned}$$

Note that the square roots of 1 are $\{1, -1\}$ but, since the modulus of a complex number is a nonnegative real, we must have that $|\det A| = 1$.

Problem 3

Problem: Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq \mathbb{F}^n$ where $\mathbb{F}$ is some field. Define $B \in \mathbb{F}^{n \times n}$ by $\mathbf{b}_{:j} = \mathbf{v}_j$, i.e.:

$$B = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{bmatrix}$$

Prove that β is a basis of $\mathbb{F}^n$ iff $\det(B) \neq 0$.

Solution: Note the following chain of equivalent conditions:

$$\begin{aligned}
\beta \text{ is a basis with } n \text{ vectors} &\iff \text{All } n \text{ of } B\text{'s columns are linearly independent} \\
&\iff B \text{ has full rank} \\
&\iff B \text{ is invertible} \\
&\iff \det B \neq 0
\end{aligned}$$

Problem 4

Problem: Consider a block matrix $M \in \mathbb{F}^{n\times n}$ of the following form:

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is an $n \times n$ matrix and C a $k \times k$ matrix, implying that M is an $(n+k) \times (n+k)$ matrix. Prove that $\det(M) = \det(A)\det(C)$.

Solution: We can prove this is true for all values of n via induction. Our base case of $n = 1$ is given by:

$$\begin{aligned}
\det M &= \sum_{i=1}^{1+k} (-1)^{i+1} m_{i1} \det M_{i1}^{sub} && \text{(cofactor expansion over column 1)} \\
&= (-1)^{1+1} m_{11} \det M_{11}^{sub} && (i > 0 \to m_{i1} = 0) \\
&= a_{11} \det C \\
&= \det A \det C && \text{(determinant of } 1 \times 1 \text{ matrix)}
\end{aligned}$$

Now we assume the inductive hypothesis, which is that identity holds for any $(n-1) \times (n-1)$ matrix A, and prove that this implies the same for $n \times n$ matrices:

$$\begin{aligned}
\det M &= \sum_{i=1}^{n+k} (-1)^{i+1} m_{i1} \det M_{i1}^{sub} && \text{(cofactor expansion over column 1)} \\
&= \sum_{i=1}^{n} (-1)^{i+1} m_{i1} \det M_{i1}^{sub} && (i > n \to m_{i1} = 0) \\
&= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det M_{i1}^{sub} && (i \leq m \to m_{i1} = a_{i1}) \\
&= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det \begin{pmatrix} A_{i1}^{sub} & B_{i0}^{sub} \\ 0 & C \end{pmatrix} && \text{(Where } B_{i0} \text{ is } B \text{ without row } i\text{)} \\
&= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1}^{sub} \det C && \text{(inductive hypothesis)} \\
&= \left(\sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1}^{sub} \right) \det C \\
&= \det A \det C && \text{(cofactor expansion over column 1)}
\end{aligned}$$

And so we have proved by induction that the desired identity holds for all values of n, regardless of k.