Intro to Math Reasoning HW 6a

Ozaner Hansha

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Problem 1

Part a

Problem: Prove that the sum of two linear functions (over real vector spaces) is linear.

Solution: Consider two linear functions f and g. We define their sum $f+g$ as:

$$(f+g)(x) = f(x) + g(x)$$

Recall we only have to prove two properties to show that $f+g$ is linear:

$$(f+g)(x+y) = (f+g)(x) + (f+g)(y) \tag{1}$$
$$(f+g)(cx) = c(f+g)(x) \tag{2}$$

Proof of property (1)

$$\begin{aligned}
(f+g)(x+y) &= f(x+y) + g(x+y) && \text{(def. of } f+g\text{)} \\
&= f(x) + f(y) + g(x) + g(y) && \text{(Linearity of } f \text{ \& } g\text{)} \\
&= f(x) + g(x) + f(y) + g(y) && \text{(Commutativity of } +\text{)} \\
&= (f+g)(x) + (f+g)(y) && \text{(def. of } f+g\text{)}
\end{aligned}$$

Proof of property (2)

$$\begin{aligned}
(f+g)(cx) &= f(cx) + g(cx) && \text{(def. of } f+g\text{)} \\
&= cf(x) + cg(x) && \text{(Linearity of } f \text{ \& } g\text{)} \\
&= c(f(x) + g(x)) && \text{(Distributivity of } + \text{ \& } \times\text{)} \\
&= c(f+g)(x) && \text{(def. of } f+g\text{)}
\end{aligned}$$

Part b

Problem: Prove that a scalar multiple of a linear function (over a real vector space) is linear.

Solution: Recall we only have to prove two properties to show that cf is linear:

$$cf(x+y) = cf(x) + cf(y) \tag{3}$$
$$cf(c_0 x) = c_0 cf(x) \tag{4}$$

Proof of property (1)

$$\begin{aligned} cf(x+y) &= c(f(x) + f(y)) && \text{(Linearity of } f\text{)} \\ &= cf(x) + cf(y) && \text{(Distributivity of + \& } \times\text{)} \end{aligned}$$

Proof of property (2)

$$\begin{aligned} cf(c_0 x) &= c(c_0 f(x)) && \text{(Linearity of } f\text{)} \\ &= c_0 cf(x) && \text{(Commutativity of } \times\text{)} \end{aligned}$$

Problem 2

Problem: In a real vector space, if the functions $f+g$ and $f-g$ are linear, prove that f and g must also be linear.

Solution: Recall that we have proved that the sum of two linear functions is linear and that any scalar multiple of a linear function is also linear. These two facts are sufficient to prove the above. Note that:

$$(f+g)(x) + (f-g)(x) = f(x) + g(x) + f(x) - g(x) = 2f(x)$$

Because $f+g$ and $f-g$ are both linear their sum, $2f(x)$ must also be linear. Now note that:

$$\frac{1}{2} \cdot 2f(x) = f(x)$$

Because $2f(x)$ is linear, any scalar multiple of it is also linear. Thus $f(x)$ is linear.

A similar argument can be made for g, just consider the following:

$$(-1) \cdot (f-g)(x) = (-1) \cdot (f(x) - g(x)) = g(x) - f(x) = (g-f)(x)$$

Because $f - g$ is linear, any scalar multiple of it is also linear. Thus $g - f$ is linear. Now we simply add this function with $f + g$ to arrive at $2g$ and multiply it by $\frac{1}{2}$ to arrive at g, both of which are linear by the same argument used above for f.

Problem 3

Problem: Prove that if $a + b$ and $a - b$ are even, then a and b are also even.

Solution: This is false. Consider $a = b = 1$:

$$\begin{aligned} a &= 1 \quad b = 1 && \text{(odd)} \\ a + b &= 2 \quad a - b = 0 && \text{(even)} \end{aligned}$$

Problem 4

Problem: Prove that $(z - a)$ is a factor of any complex polynomial $p(z)$ with a root at a.

Solution: If $p(z)$ is of degree $n > 0$ then the division theorem applies to it. And so because $(z - a)$ is of degree 1 there must exist two complex polynomials $q(z)$ and $r(z)$ such that:

$$p(z) = q(z)(z - a) + r(z)$$

Now recall that $p(a) = 0$ that is, a is a root of $p(z)$:

$$\begin{aligned} p(a) &= q(a)(a - a) + r(a) \\ &= r(a) \\ &= 0 \end{aligned}$$

The only way $r(a) = 0$ is if it is a complex polynomial with degree $n > 0$ and has a root at a or just the constant polynomial 0. However, the division theorem states that $r(z)$ is of a degree lower than $(z - a)$ which is of degree 1. This means $r(z)$ is of degree 0 and thus must be the constant 0. We are now left with:

$$p(z) = q(z)(z - a)$$

This is the definition of being a factor, and thus $(z - a)$ is a factor of $p(z)$. But if $p(z)$ is of degree 0, then it is equal to some constant. The only way for $p(a) = 0$ in this case is if $p(z) = 0$. Since this is the case, every polynomial is a factor of $p(z)$ because of the zero-product property.

Problem 5

Problem: Assuming every polynomial has at least one root a, prove that any complex polynomial $p(z)$ of degree $n \geq 1$ can be expressed in terms of n complex numbers denoted a_i and 1 non-zero complex coefficient c:

$$p(z) = c(z - a_1)(z - a_2) \cdots (z - a_n) = c \prod_{i=1}^{n} (z - a_i)$$

Solution: Call the original polynomial $p_1(z)$, call its known root a_1 and call its degree n. If n is greater than 0 than the polynomial has a factor of $(z - a_1)$ as proved above. If n is still greater than 1, call the quotient polynomial that results from the procedure $p_2(z)$. Recall that the quotient polynomial is of degree $n - 1$ because it is sans a degree 1 polynomial, namely $(z - a_1)$. The above holds for any polynomial of degree greater than 1. And so we can say $p_i(z)$ has a factor of $(z - a_i)$ as long as $i > 1$.

Because n is finite, we will eventually reach the case where the quotient polynomial is of degree 0, i.e. a constant. At that point we will have $p_{n+1} = c$. Since every $(z - a_i)$ was also a factor of every $p_j(z)$ where $j \geq i$ we can say that:

$$p(z) = c(z - a_1)(z - a_2) \cdots (z - a_n)$$

Because we have exhausted every factor until we reached an unfactorizable polynomial, $p_{n+1} = c$, we have accounted for every factor in $p(z)$ and thus it can be written as the product of those factors.