Linear Algebra HW #3

Ozaner Hansha

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Problem 1

Problem: Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V then the following holds for the subspace $W_1 + W_2$:

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Solution: Since the intersection of subspaces is a subspace, there exists some finite basis $\mathcal{B}_\cap$ of $W_1 \cap W_2$:

$$\mathcal{B}_\cap = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l\}$$

Also note that since $W_1 \cap W_2$ is a subspace of both W_1 and W_2, the basis extension theorem tells us that we can extend, i.e. add vectors to, $\mathcal{B}_\cap$ to form a basis for both W_1 and W_2 respectively:

$$\mathcal{B}_1 = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l, \mathbf{v}_1, \cdots, \mathbf{v}_m\}$$
$$\mathcal{B}_2 = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l, \mathbf{w}_1, \cdots, \mathbf{w}_n\}$$

We will now show that the union of these bases, call it $\mathcal{B}$, is a basis for $W_1 + W_2$ which will in turn allows us to prove the desired identity.

(Lemma 1) First we prove that $\text{span}(\mathcal{B}) = W_1 + W_2$. Consider an arbitrary $\mathbf{x} \in W_1 + W_2$. By the definition of the sum of vector spaces, there must be vectors $\mathbf{v} \in W_1$ and $\mathbf{w} \in W_2$ such that $\mathbf{x} = \mathbf{v} + \mathbf{w}$. And since $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases of W_1 and W_2 respectively, we can express $\mathbf{v}$ and $\mathbf{w}$ as linear combinations of them:

$$\mathbf{v} = \sum_{i=1}^{l} a_i \mathbf{u}_i + \sum_{j=1}^{m} b_j \mathbf{v}_j$$
$$\mathbf{w} = \sum_{i=1}^{l} c_i \mathbf{u}_i + \sum_{k=1}^{n} d_k \mathbf{w}_k$$

And so the arbitrary vector $\mathbf{x}$ can be expressed as a linear combination of vectors in $\mathcal{B}$:

$$\mathbf{x} = \mathbf{v} + \mathbf{w} = \sum_{i=1}^{l} (a_i + c_i)\mathbf{u}_i + \sum_{j=1}^{m} b_j \mathbf{v}_j + \sum_{k=1}^{n} d_k \mathbf{w}_k$$

And so we have shown that $\mathcal{B}$ spans $W_1 + W_2$.

(Lemma 2) Now we will show that is it linearly independent. Consider coefficients a_i, b_j, c_k with at least one being nonzero such that:

$$\sum_{i=1}^{l} a_i \mathbf{u}_i + \sum_{j=1}^{m} b_j \mathbf{v}_j + \sum_{k=1}^{n} c_k \mathbf{w}_k = \mathbf{0}$$

Rearranging the equation and calling the resulting vector $\mathbf{x}$ we have:

$$\mathbf{x} = \underbrace{\sum_{j=1}^{m} b_j \mathbf{v}_j}_{\in W_1} = \underbrace{-\sum_{i=1}^{l} a_i \mathbf{u}_i - \sum_{k=1}^{n} c_k \mathbf{w}_k}_{\in W_2}$$

Note that $\mathbf{x} \neq \mathbf{0}$ since at least one of a_i, b_j, c_k is nonzero. Also note that since $\mathbf{x}$ is in both W_1 and W_2 it is certainly in $W_1 \cap W_2$. As a result, we can express it as a nonzero linear combination of vectors in $\mathcal{B}_\cap$:

$$\mathbf{x} = \sum_{i=1}^{l} d_i \mathbf{u}_i$$

But this implies that:

$$\mathbf{x} - \mathbf{x} = \underbrace{\sum_{i=1}^{l} d_i \mathbf{u}_i - \sum_{j=1}^{m} b_j \mathbf{v}_j}_{\in \text{span}(\mathcal{B}_1) = W_1} = \mathbf{0}$$

However since $\mathcal{B}_1$ is a basis, the above equation implies that $b_j = 0$ for all j. This implies that $\mathbf{x} = \mathbf{0}$ which contradicts our initial assumption that at least one of a_i, b_j, c_k is nonzero. And so we have proved linear independence.

(Proof) Together, Lemma 1 and Lemma 2 prove that $\mathcal{B}$ is a basis of $W_1 + W_2$. To summarize then, we have established the following bases:

$$\begin{aligned}
\mathcal{B}_\cap &= \{\mathbf{u}_1, \cdots, \mathbf{u}_l\} && \text{(basis of } W_1 \cap W_2\text{)} \\
\mathcal{B}_1 &= \{\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{v}_1, \cdots, \mathbf{v}_m\} && \text{(basis of } W_1\text{)} \\
\mathcal{B}_2 &= \{\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{w}_1, \cdots, \mathbf{w}_n\} && \text{(basis of } W_2\text{)} \\
\mathcal{B} &= \{\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{v}_1, \cdots, \mathbf{v}_m, \mathbf{w}_1, \cdots, \mathbf{w}_n\} && \text{(basis of } W_1 + W_2\text{)}
\end{aligned}$$

From here it is simple to show the desired identity:

$$\begin{aligned}
\dim(W_1 + W_2) &= l + m + n \\
&= (l + m) + (l + n) - l \\
&= \dim(W_1) + \dim(W_2) - \dim(W_1 \cup W_2)
\end{aligned}$$

Problem 2

Problem: Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. What is $T(8, 11)$?

Solution: To prove this we will first construct the desired operator T and then demonstrate that it is both satisfies the desired conditions and is linear.

First note that $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]$ are linearly independent and thus span all of $\mathbb{R}^2$. As such, we can express an arbitrary vector $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ as a linear combination of them:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 + 3c_2 \end{bmatrix}$$

This gives us the following system of equations in c_1 and c_2:

$$\begin{cases} c_1 + 2c_2 = x \\ c_1 + 3c_2 = y \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & x \\ 1 & 3 & y \end{array}\right]$$

$$\xrightarrow{r_2 - r_1} \left[\begin{array}{cc|r} 1 & 2 & x \\ 0 & 1 & y - x \end{array}\right]$$

$$\xrightarrow{r_1 - 2r_2} \left[\begin{array}{cc|r} 1 & 0 & 3x - 2y \\ 0 & 1 & y - x \end{array}\right]$$

$$\longrightarrow \begin{cases} c_1 \quad = 3x - 2y \\ \quad c_2 = y - x \end{cases}$$

Now we rewrite our arbitrary vector in terms of x and y as well as apply the linear operator T:

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= (3x - 2y) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (y - x) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
T \begin{pmatrix} x \\ y \end{pmatrix} &= T \left((3x - 2y) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (y - x) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \\
&= (3x - 2y) T \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (y - x) T \begin{pmatrix} 2 \\ 3 \end{pmatrix} && \text{(linearity)} \\
&= (3x - 2y) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (y - x) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} && \text{(given)}
\end{aligned}$$

Now that we have an operator $T : \mathbb{R}^2 \to \mathbb{R}^3$, we will demonstrate it that it takes $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} 1 \\ 0 \\ 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} 1 \\ -1 \\ 4 \end{smallmatrix}\right]$:

$$\begin{aligned}
T \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= (3 - 2) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (1 - 1) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \\
T \begin{pmatrix} 2 \\ 3 \end{pmatrix} &= (6 - 6) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (3 - 2) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}
\end{aligned}$$

All that's left is to show that T is a linear operator:

$$\begin{aligned}
T \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} &= (3\lambda x - 2\lambda y) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (\lambda y - \lambda x) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \\
&= \lambda(3x - 2y) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \lambda(y - x) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \\
&= \lambda \left((3x - 2y) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (y - x) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right) \\
&= \lambda T \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
T\begin{pmatrix} x+x' \\ y+y' \end{pmatrix} &= (3(x+x') - 2(y+y')) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + ((y+y') - (x+x')) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \\
&= ((3x-2y) + (3x'-2y')) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + ((y-x) + (y'-x')) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \\
&= (3x-2y) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (3x'-2y') \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (y-x) \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + (y'-x') \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \\
&= T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} x' \\ y' \end{pmatrix}
\end{aligned}
$$

And so we have given a linear operator T that satisfies the question.

Problem 3

Problem: Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear. Prove that if $\dim(V) < \dim(W)$ then T cannot be surjective.

Solution: Assume that T is surjective. This gives us the following:

$$
\begin{aligned}
R(T) &= W && \text{(def. of surjective)} \\
\dim(R(T)) &= \dim(W) \\
\operatorname{rank}(T) &= \dim(W) && \text{(def. of rank)}
\end{aligned}
$$

Equipped with this knowledge we can now consider the following:

$$
\begin{aligned}
\dim(W) &> \dim(V) \\
&= \operatorname{nullity}(T) + \operatorname{rank}(T) && \text{(dimension theorem)} \\
&= \operatorname{nullity}(T) + \dim(W) && \text{(above reasoning)} \\
0 &> \operatorname{nullity}(T)
\end{aligned}
$$

This however is a contradiction as the dimension of an operator's nullspace, or any vector space for that matter, cannot be any lower than 0.

Problem 4

Problem: Let V be a vector space and W_1 and W_2 be subspaces. The sum $W_1 + W_2$ is called a direct sum if $W_1 \cap W_2 = \{0\}$, and denoted $W_1 \oplus W_2$. Suppose that $V = W_1 \oplus W_2$. A map $T : V \to V$ is called the projection of W_1 along W_2 if, for all $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \in W_1 \oplus W_2$, $T(\mathbf{x}) = \mathbf{x}_1$.

a) Show that T is linear.

b) Show that $W_1 = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{v}\}$.

c) Prove that $W_1 = R(T)$ and $W_2 = \mathcal{N}(T)$.

Solution:

a) Note that any vector $\mathbf{x} \in V$ can be uniquely written as the sum $\mathbf{x}_1 + \mathbf{x}_2 \in W_1 \oplus W_2$. And so we can prove linearity like so:

$$T(\lambda\mathbf{x}) = T(\lambda\mathbf{x}_1 + \lambda\mathbf{x}_2) = \lambda\mathbf{x}_1 = \lambda T(\mathbf{x})$$
$$T(\mathbf{x} + \mathbf{x}') = T(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_1' + \mathbf{x}_2') = T((\mathbf{x}_1' + \mathbf{x}_1') + (\mathbf{x}_2 + \mathbf{x}_2')) = \mathbf{x}_1 + \mathbf{x}_1' = T(\mathbf{x}) + T(\mathbf{x}')$$

b) Consider an arbitrary vector $\mathbf{v} \in V$. If $\mathbf{v} \in W_1$ then it could be uniquely written as the sum $\mathbf{v} = \mathbf{v} + \mathbf{0}$ and thus:

$$T(\mathbf{v}) = T(\mathbf{v} + \mathbf{0}) = \mathbf{v}$$

Satisfying the set criterion. Now consider the case where $\mathbf{v} \notin W_1$. This would imply that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ for some $\mathbf{v}_2 \neq \mathbf{0}$:

$$T(\mathbf{v}) = T(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1 \neq \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$$

And so we have shown that the only vectors in V that are in the set $\{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{v}\}$ are those in W_1. Thus they are equivalent.

c) It should be clear that $R(T) \subseteq W_1$, since T outputs the W_1 component of any vector it is given. Now consider an $\mathbf{v} \in W_1$. Since $T(\mathbf{v}) = \mathbf{v}$, as shown in **b)**, we have that $W_1 \subseteq R(T)$. These together imply that $R(T) = W_1$.

For the other equality, consider a $\mathbf{v} \in W_2$. This implies that $T(\mathbf{v}) = T(\mathbf{0} + \mathbf{v}) = \mathbf{0}$ and so $W_2 \subseteq \mathcal{N}(T)$. For the other direction, consider $\mathbf{v} \in \mathcal{N}(T)$. This implies that $T(\mathbf{v}) = \mathbf{0}$. This must mean that:

$$T(\mathbf{v}) = T(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1 = \mathbf{0}$$

And since $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ we have that $\mathbf{v} = \mathbf{v}_2 \in W_2$, giving us $\mathcal{N}(T) \subseteq W_2$. Putting both directions together we have $\mathcal{N}(T) = W_2$.