# Set Theory HW #7 NAME ON TOP

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November 12, 2019

## Problem 1

**Part 1:** The sum of two superCauchy sequences $s$ and $t$ is given by the following sequence:

$$s + t = \{\langle n, s_n + t_n\rangle \mid n \in \omega\}$$

For this sequence to qualify as superCauchy, there must exist some constant $C \in \mathbb{Q}^+$ such that for all naturals $n$:

$$|(s_{n+1} + t_{n+1}) - (s_n + t_n)| \leq \frac{C}{2^n}$$

We shall now produce this constant. Consider two arbitrary superCauchy sequences $s, t \in SC(\mathbb{Q})$. By definition, there exists two constants $C_s, C_t \in \mathbb{Q}^+$ such that for all natural numbers $n$:

$$|s_{n+1} - s_n| \leq \frac{C_s}{2^n} \tag{1}$$

$$|t_{n+1} - t_n| \leq \frac{C_t}{2^n} \tag{2}$$

Let us choose and fix these constants as $C_s$ and $C_t$ respectively. Now note that since they are both inequalities of positive numbers, we can add them together, implying the following for all natural numbers $n$:

$$\begin{aligned}
\frac{C_s + C_t}{2^n} &\geq |s_{n+1} - s_n| + |t_{n+1} - t_n| && \text{(sum of (1) and (2))}\\
&\geq |s_{n+1} - s_n + t_{n+1} - t_n| && \text{(triangle inequality)}\\
&\geq |s_{n+1} + t_{n+1} - s_n - t_n| && \text{(associative \& commutative prop. of } \mathbb{Q})\\
&\geq |(s_{n+1} + t_{n+1}) - (s_n + t_n)| && \text{(distributive prop. of } \mathbb{Q})
\end{aligned}$$

And so we have shown that, given two superCauchy sequences $s$ and $t$, their sum $s + t$ is also a superCauchy sequence with constant $C_s + C_t$. For the product we can multiply the inequalities together (since they are of positive numbers):

$$\begin{aligned}
|s_{n+1} - s_n||t_{n+1} - t_n| &\leq \frac{C_s}{2^n} \cdot \frac{C_t}{2^n} && \text{(product of (1) and (2))}\\
|(s_{n+1} - s_n)(t_{n+1} - t_n)| &\leq \frac{C_s C_t}{2^{n+1}} && \text{(product of absolute values)}\\
|(s_{n+1}t_{n+1} - s_n t_{n+1} - s_{n+1}t_n + t_n s_n| &\leq \frac{C_s C_t/2}{2^n}\\
|s_{n+1}t_{n+1} - s_n t_n| &\leq \frac{C_s C_t/2}{2^n}
\end{aligned}$$

And so $s \cdot t$ is superCauchy.

**Part 2:** For the sum of two null superCauchy sequences $s + t$ to qualify as null superCauchy, there must exist some constant $C \in \mathbb{Q}^+$ such that for all naturals $n$:

$$|s_n + t_n| \leq \frac{C}{2^n}$$

We shall now produce this constant. Consider two arbitrary superCauchy sequences $s, t \in SC(\mathbb{Q})$. By definition, there exists two constants $C_s, C_t \in \mathbb{Q}^+$ such that for all natural numbers $n$:

$$|s_n| \leq \frac{C_s}{2^n} \tag{1}$$

$$|t_n| \leq \frac{C_t}{2^n} \tag{2}$$

Let us choose and fix these constants as $C_s$ and $C_t$ respectively. Now note that since they are both inequalities of positive numbers, we can add them together, implying the following for all natural numbers $n$:

$$\begin{aligned} \frac{C_s + C_t}{2^n} &\geq |s_n| + |t_n| && \text{(sum of (1) and (2))} \\ &\geq |s_n + t_n| && \text{(triangle inequality)} \end{aligned}$$

And so we have shown that, given two null superCauchy sequences $s$ and $t$, their sum $s + t$ is also a null superCauchy sequence with constant $C_s + C_t$. The proof for the difference $s - t$ is much the same. With the fixed constants $C_s$ and $C_t$ we have:

$$\begin{aligned} \frac{C_s + C_t}{2^n} &\geq |s_n| + |t_n| && \text{(sum of (1) and (2))} \\ &\geq |s_n| + |-t_n| && \text{(multiply by } -1) \\ &\geq |s_n - t_n| && \text{(triangle inequality)} \end{aligned}$$

And so the difference of two null superCauchy sequences is also null with a constant of $C_s + C_t$.

**Part 3:** For the product of two null superCauchy sequences $s \cdot t$ to qualify as null superCauchy, there must exist some constant $C \in \mathbb{Q}^+$ such that for all naturals $n$:

$$|s_n \cdot t_n| \leq \frac{C}{2^n}$$

We shall now produce this constant. Consider two arbitrary superCauchy sequences $s, t \in SC(\mathbb{Q})$. By definition, there exists two constants $C_s, C_t \in \mathbb{Q}^+$ such that for all natural numbers $n$:

$$|s_n| \leq \frac{C_s}{2^n} \tag{1}$$

$$|t_n| \leq \frac{C_t}{2^n} \tag{2}$$

Let us choose and fix these constants as $C_s$ and $C_t$ respectively. Now note that since they are both inequalities of positive numbers, we can multiply them together, implying the following for all natural numbers $n$:

$$\begin{aligned} |s_n||t_n| &\leq \frac{C_s}{2^n} \cdot \frac{C_t}{2^n} && \text{(product of (1) and (2))} \\ &\leq \frac{C_s C_t}{2^{n+1}} \\ |s_n t_n| &\leq \frac{C_s C_t / 2}{2^n} && \text{(product of absolute values)} \end{aligned}$$

And so we have shown that, given two null superCauchy sequences $s$ and $t$, their product $s \cdot t$ is also a null superCauchy sequence with constant $\frac{C_s C_t}{2}$.

**Part 4:** First we show that $\sim$ is reflexive, i.e. $s \sim s$ for an arbitrary superCauchy sequence $s$. Note that for this to be the case, there must be a constant $C \in \mathbb{Q}^+$ such that for all naturals $n$:

$$|s_n - s_n| \leq \frac{C}{2^n}$$

But since $|s_n - s_n| = 0$, every positive rational $C$ satisfies this (for instance 1). And so we have proved reflexivity. Now we will prove the relation is symmetric, i.e. $s \sim t \implies t \sim s$ for any arbitrary superCauchy sequences $s$ and $t$. Assuming that $s \sim t$, we have for some $C \in \mathbb{Q}^+$ and all naturals $n$:

$$\begin{aligned} |s_n - t_n| &\leq \frac{C}{2^n} \\ \implies |t_n - s - n| &\leq \frac{C}{2^n} && \text{(multiply by } -1\text{)} \end{aligned}$$

With the second line being equivalent to $t \sim s$. And so we have proved symmetry. All that's left is to prove transitivity, i.e. given 3 super cauchy sequences $s, t, u$ the following holds:

$$s \sim t \ \& \ t \sim u \implies s \sim u$$

To prove this, let us assume the antecedent. This implies that for some constants $C_1, C_2 \in \mathbb{Q}^+$ and all naturals $n$:

$$|s_n - t_n| \leq \frac{C_1}{2^n} \tag{1}$$

$$|t_n - u_n| \leq \frac{C_2}{2^n} \tag{2}$$

Let us choose and fix these constants as $C_1$ and $C_2$ respectively. Now note that since they are both inequalities of positive numbers, we can add them together, implying the following for all natural numbers $n$:

$$\begin{aligned} \frac{C_1 + C_2}{2^n} &\geq |s_n - t_n| + |t_n - u_n| && \text{(sum of (1) and (2))} \\ &\geq |s_n - t_n + t_n - u_n| && \text{(triangle inequality)} \\ &\geq |s_n - u_n| \end{aligned}$$

And so we have shown that $s \sim u$ for some constant $(C_1 + C_2)$, thus proving the relation is transitive. All three of these conditions (i.e. reflexivity, symmetry, and transitivity) taken together imply the relation is an equivalence relation.

**Part 5:** For a binary function $f$ on a set $SC(\mathbb{Q})$ to be compatible with an equivalence relation $\sim$ on that same set, the following must hold for all $s, t, s', t' \in SC(\mathbb{Q})$:

$$s \sim s' \ \& \ t \sim t' \implies f(s,t) \sim f(s', t')$$

To prove this for $+, -$ and $\cdot$, let us assume the antecedent. This implies that for some constants $C_s, C_t \in \mathbb{Q}^+$ and all naturals $n$:

$$|s_n - s'_n| \leq \frac{C_s}{2^n} \tag{1}$$

$$|t_n - t'_n| \leq \frac{C_t}{2^n} \tag{2}$$

For the $+$ case, note that since these are both inequalities of positive numbers, we can add them together, implying the following for all natural numbers $n$:

$$\begin{aligned}
\frac{C_t + C_s}{2^n} &\geq |s_n - s'_n| + |t_n - t'_n| && \text{(sum of (1) and (2))} \\
&\geq |s_n - s'_n + t_n - t'_n| && \text{(triangle inequality)} \\
&\geq |(s_n + t_n) - (s'_n + t'_n)| && \text{(assoc., comm., \& distr. prop. of } \mathbb{Q})
\end{aligned}$$

This implies that $s + t \sim s' + t'$ for some constant $(C_1 + C_2)$ and so the function $+$ is compatible. In the $-$ case we have:

$$\begin{aligned}
\frac{C_t + C_s}{2^n} &\geq |s_n - s'_n| + |t_n - t'_n| && \text{(sum of (1) and (2))} \\
&\geq |s_n - s'_n| + |-t_n + t'_n| && \text{(multiply by } -1) \\
&\geq |s_n - s'_n - t_n + t'_n| && \text{(triangle inequality)} \\
&\geq |(s_n - t_n) - (s'_n - t'_n)| && \text{(assoc., comm., \& distr. prop. of } \mathbb{Q})
\end{aligned}$$

This implies that $s - t \sim s' - t'$ for some constant $(C_1 + C_2)$ and so the function $-$ is compatible. Note that, for the $\cdot$ case, since they are both inequalities of positive numbers, we can multiply them together, implying the following for all natural numbers $n$:

$$\begin{aligned}
|s_n - s'_n||t_n - t'_n| &\leq \frac{C_s}{2^n} \cdot \frac{C_t}{2^n} && \text{(product of (1) and (2))} \\
|(s_n - s'_n)(t_n - t'_n)| &\leq \frac{C_s C_t}{2^{n+1}} && \text{(product of absolute values)} \\
|(s_n t_n - s'_n t_n - s_n t'_n + t'_n s'_n| &\leq \frac{C_s C_t/2}{2^n} \\
|s_n t_n - s'_n t'_n| &\leq \frac{C_s C_t/2}{2^n}
\end{aligned}$$

This implies that $st \sim s't'$ for some constant $(\frac{C_s C_t}{2})$ and so the function $\cdot$ is compatible.

**Part 6:** Recall problem III of HW 6, where we proved a theorem analogous to Theorem 3Q of the textbook. That is, given a binary function $f$ compatible with the relation $\sim$, there exists a unique function $\hat{f}$ such that:

$$\hat{f}([s]_\sim, [t]_\sim) = [f(s,t)_\sim]$$

Which is presumably what the problem means by "well-defined". And so the binary the operations $\hat{+}$, $\hat{-}$, and $\hat{\cdot}$ on $SC(\mathbb{Q})/\sim$ are well defined because we proved they were compatible in part 5.

# Problem 2

**Part 1:** For 1a) note that for any arbitrary $x \in \mathbb{R}$, there exists a sequence $r \in SC(\mathbb{Q})$ such that:

$$\begin{aligned}
& x - x && \text{(well-defined by prob. 1, part 6)} \\
&= [r]_\sim - [r]_\sim && \text{(i)} \\
&= [r - r]_\sim && \text{(prob. 1, part 6)} \\
&= [0_{seq}]_\sim && (- \text{ on } \mathbb{Q})
\end{aligned}$$

Now note that, for all naturals $n$, $0_{seq_n} = 0$ which implies $0_{seq_n} \leq 0$. Combining this with the fact that $0_{seq} \in x - x$, (v) tells us that $0 \leq x - x$. This can be written as $x \leq x$ thanks to (iv.a). And so we have proved reflexivity.

For 1b) we need to prove totality. Consider two arbitrary real numbers $x, y \in \mathbb{R}$. Let $s, t \in SC(\mathbb{Q})$ such that:

$$x = [s]_\sim \quad y = [t]_\sim$$

This implies the following due to problem 1, part 6:

$$x - y = [s - t]_\sim \quad y - x = [t - s]_\sim$$

Since the difference of two superCauchy sequences is superCauchy, we have for all naturals $n$:

$$\begin{aligned}
&|s_n - t_n| \leq \frac{C}{2^n} \\
\implies& s_n - t_n \leq \frac{C}{2^n} \;\&\; s_n - t_n \geq -\frac{C}{2^n} \\
\implies& s_n - t_n \leq 0 \text{ or } -(s_n - t_n) \leq 0 && \text{(0 in interval)} \\
\implies& s_n - t_n \leq 0 \text{ or } t_n - s_n \leq 0
\end{aligned}$$

And since $s_n - t_n \in [s - t]_\sim = x - y$ as well as $t_n - s_n \in [t - s]_\sim = y - x$ we have from (iv.a) that $0 \leq x - y$ or $0 \leq y - x$. And so totality is proven.

For 1c) we need to prove transitivity. Consider three arbitrary real numbers $x, y, z \in \mathbb{R}$. Now let us assume $x \leq y$ and $y \leq z$. With $s, t, u \in SC(\mathbb{Q})$ such that:

$$\begin{gathered}
x = [s]_\sim \quad y = [t]_\sim \quad z = [u]_\sim \\
0 \leq t_n - s_n \;\&\; 0 \leq u_n - t_n
\end{gathered}$$

With sequences that satisfy the second line guaranteed to exist by (iv.a). There implies the following due to problem 1, part 6:

$$y - x = [t - s]_\sim \quad z - y = [u - t]_\sim \quad z - x = [u - s]_\sim$$

We thus have the following (note we distinguish between the rational $\leq_\mathbb{Q}$ and the real $\leq$ to make the proof clear.)

$$\begin{aligned}
& x \leq y \;\&\; y \leq z \\
\implies& 0 \leq y - x \;\&\; 0 \leq z - y && \text{(v)} \\
\implies& 0 \leq [t - s]_\sim \;\&\; 0 \leq [u - t]_\sim \\
\implies& 0 \leq_\mathbb{Q} t_n - s_n \;\&\; 0 \leq u_n - t_n \\
\implies& s_n \leq_\mathbb{Q} t_n \;\&\; t_n \leq u_n \\
\implies& s_n \leq_\mathbb{Q} u_n && \text{(transitivity of } \leq_\mathbb{Q}\text{)} \\
\implies& 0 \leq_\mathbb{Q} u_n - s_n \\
\implies& 0 \leq [u - s]_\sim \\
\implies& 0 \leq z - x \\
\implies& x \leq z
\end{aligned}$$

And so we have proved the transitivity of $\leq$.

For 1d) all that is left to prove is antisymmetry. Consider two arbitrary real numbers $x, y \in \mathbb{R}$ such that $x \leq y$ & $y \leq x$, and where $s, t \in SC(\mathbb{Q})$ such that:

$$x = [s]_\sim \quad y = [t]_\sim$$

This implies the following due to problem 1, part 6:

$$x - y = [s - t]_\sim \quad y - x = [t - s]_\sim$$

We thus have the following:

$$\begin{aligned}
&x \leq y && \& \ y \leq x \\
\Longrightarrow\ &0 \leq y - x && \& \ 0 \leq x - y \\
\Longrightarrow\ &0 \leq [t - s]_\sim && \& \ 0 \leq [s - t]_\sim \\
\Longrightarrow\ &0 \leq_{\mathbb{Q}} t_n - s_n && \& \ 0 \leq_{\mathbb{Q}} s_n - t_n \\
\Longrightarrow\ &s_n \leq_{\mathbb{Q}} t_n && \& \ t_n \leq_{\mathbb{Q}} s_n \\
\Longrightarrow\ &s_n = t_n && \text{(antisymmetry of } \leq_{\mathbb{Q}}) \\
\Longrightarrow\ &[s]_\sim = [t]_\sim \\
\Longrightarrow\ &x = y
\end{aligned}$$

And thus we have proven antisymmetry. All 4 of the properties we proved in 1a,b,c, and d imply that $\leq$ is a total order over the reals.

**Part 2:** Consider a nonempty set $S$ with a rational upper bound $B_1$, if bound is not rational take the next highest rational. Since this set is nonempty, there must be some rational number that is *not* an upper bound of $S$. Choose such a number and call it $A_1$. Now we define the following iteration:

If $\frac{1}{2} \cdot (A_n + B_n)$ is an upper bound of $S$, let $A_{n+1} = A_n$ and $B_{n+1} = \frac{1}{2} \cdot (A_n + B_n)$. Otherwise, there must be some number $s \in S$ such that $\frac{1}{2} \cdot (A_n + B_n) < s$. Let $A_{n+1}$ equal such an $s$ and let $B_{n+1} = B_n$.

As a result of this particular construction, our sequences have the following properties: $A_n$ is increasing, $B_n$ is decreasing, and for every $i, j \in \mathbb{N}$ we have $A_i \leq B_j$. Visually we can express this as:

$$A_1 \leq A_2 \leq \cdots \leq B_2 \leq B_1$$

But also note that multiplying by $\frac{1}{2}$ at each iteration give us the following for all naturals $n$:

$$|A_n| \leq \frac{C_a}{2^n}$$
$$|B_n| \leq \frac{C_b}{2^n}$$

And so both $A$ and $B$ are null superCauchy sequences and, as we've proved before, so is their difference $A - B$. As a result, $A \sim B$ meaning there is only one unique real bound of the set $S$ and no other is less than it. This bound being given by $[A]_\sim = [B]_\sim = r$.

# Problem 3

**Solution:** First we note that $r \neq \varnothing$ because it contains $0_{\mathbb{R}}$, i.e. the st of all rationals less than 0.

Second we note that $r \neq \mathbb{Q}$. For example, $2 \notin r$ because $2 \cdot 2 \not< 2$ and thus does not satisfy the conditions for being a member of $r$.

Third we note that for any element $a \in r$, all rationals $b$ that satisfy $b < a$ are also in $r$. To show this, we note that all negative rationals are in $r$ (because $0_{\mathbb{R}} \subset r$) and so we need only consider the $b$ such that $0 < b < a$. Note:

$$\begin{aligned}
0 < b < a &\implies b^2 < a^2 = a \cdot a < 2 \\
&\implies b^2 = b \cdot b < 2 && \text{(transitive prop.)}
\end{aligned}$$

And so $b$ is in $r$.

Finally, to prove that $r$ is a real number we show that it has no greatest element. To do this simply consider an arbitrary rational $a$ such that $a^2 < 2$. We can construct a new rational $b$ such that $a < b$ yet $b^2 < 2$. Such a rational is given by:

$$b = \frac{2a+2}{a+2}$$

Now we just need to show that $r \cdot r = 2_{\mathbb{R}}$. The definition of nonnegative real multiplication tells us:

$$r \cdot r = \{ab \mid a, b \in r \text{ \& } a, b \geq 0\} \cup 0_{\mathbb{R}}$$

We know that $r \cdot r \leq 2$ since the only numbers $q$ in $r$ are those such that $q \cdot q$, and w.l.o.g. if $q_1 > q_2$ and both are in $r$, then $q_1 q_2 < 2$ since $q_1 q_2 < q_2 q_2 < 2$. And since all elements (bar the negative ones) of $r \cdot r$ take this form, it too is $\leq 2_{\mathbb{R}}$.

And since $r \cdot r \geq 2_{\mathbb{R}}$, we have by antisymmetry that $r \cdot r = 2_{\mathbb{R}}$

## Problem 4

**Solution:** Consider an arbitrary natural $n$ and an arbitrary positive real $r$. Consider the following set:

$$s = \{t \in \mathbb{R} \mid t > 0 \text{ \& } t^n \leq r\}$$

Note that $s \neq \varnothing$. Consider $t = \frac{r}{r+1}$. It satisfies:

$$t < 1 \text{ \& } t < r$$

And since $t < 1$ we must have that $t^{n-1} < 1$ so:

$$t^n < t < r$$

And thus $t$ is a member of $s$. Now we show that $s$ has an upper bound. Consider $r + 1$, this satisfies both:

$$1 < r + 1 \text{ \& } r < r + 1$$

And so since $r + 1 > 1$ we have:

$$(r+1)^n \geq r + 1 > r$$

And so $r + 1$ is an upper bound. Now recall the LUB property which states that all subsets of the reals that are bounded above have a least upper bound. Call this bound $y$. Since this bound must exist and since $y^n = r$, we are done and there must be an $n$th root $y$ of every real number $r$.