Intro to Math Reasoning HW 9a

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Problem 1

Problem: Prove that the distributive property holds for finitary addition.

Solution: First we'll define distributivity for an n-ary sum via the following predicate for any y and summands x_i:

$$P(n) \equiv y\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} x_i y$$

Now note that $P(1)$ is true:

$$\begin{aligned} y\left(\sum_{i=1}^{n} x_i\right) &= \sum_{i=1}^{n} x_i y && P(1) \\ y(x_1) &= yx_1 && \text{(def. of 1-ary sum)} \end{aligned}$$

Now we will prove that $P(n) \to P(n+1)$:

$$\begin{aligned} y\sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} yx_i && (P(n) \text{ given}) \\ \left(y\sum_{i=1}^{n} x_i\right) + yx_{n+1} &= \left(\sum_{i=1}^{n} yx_i\right) + yx_{n+1} \\ y\left(\left(\sum_{i=1}^{n} x_i\right) + x_{n+1}\right) &= \left(\sum_{i=1}^{n} yx_i\right) + yx_{n+1} && \text{(distrib. prop. of binary +)} \\ y\left(\sum_{i=1}^{n+1} x_i\right) &= \left(\sum_{i=1}^{n+1} yx_i\right) && \text{(def. of } (n+1)\text{-ary sum)} \end{aligned}$$

And so we have shown that $P(n+1)$ follows from $P(n)$. The PMI tells us that:

$$\begin{aligned} &P(n) \implies P(n+1) \\ &P(1) \\ \therefore\ &\forall n \in \mathbb{Z}^+,\ P(n) \end{aligned}$$

And so n-ary addition is distributive for finite positive n.

Problem 2

Problem: Prove that the associative property holds for finitary multiplication.

Solution: First we'll define distributivity for an n-ary sum via the following predicate given x_i:

$$P(n,k) \equiv (1 < k < n) \wedge \left(\prod_{i=1}^{n} x_i\right) = \left(\prod_{i=1}^{k} x_i\right)\left(\prod_{i=k+1}^{n} x_i\right)$$

Now note that $(\forall k,\ 1 < k < n)\ P(3,k)$ is true, we just need to check $k=2$:

$$\begin{aligned}
\left(\prod_{i=1}^{3} x_i\right) &= \left(\prod_{i=1}^{2} x_i\right)\left(\prod_{i=2+1}^{3} x_i\right) && P(3,2)\\
\left(\prod_{i=1}^{2} x_i\right) x_3 &= \left(\prod_{i=1}^{2} x_i\right)\left(\prod_{i=3}^{3} x_i\right) && \text{(def. of finitary multiplication)}\\
\left(\prod_{i=1}^{2} x_i\right) x_3 &= \left(\prod_{i=1}^{2} x_i\right) x_3 && \text{(def. of 1-ary multiplication)}
\end{aligned}$$

Also note that $(\forall k,\ 1 < k < n)\ P(1,k) \wedge P(2,k)$ *is vacuously true.*

Now we will prove that $(\forall k,\ 1 < k < n)\ P(n,k) \to P(n+1,k)$:

$$\begin{aligned}
\left(\prod_{i=1}^{n} x_i\right) &= \left(\prod_{i=1}^{k} x_i\right)\left(\prod_{i=k+1}^{n} x_i\right) && \text{(given)}\\
\left(\prod_{i=1}^{n} x_i\right) x_{n+1} &= \left(\prod_{i=1}^{k} x_i\right)\left(\prod_{i=k+1}^{n} x_i\right) x_{n+1}\\
\left(\prod_{i=1}^{n+1} x_i\right) &= \left(\prod_{i=1}^{k} x_i\right)\left(\prod_{i=k+1}^{n+1} x_i\right) && \text{(def. of finitary multiplication)}
\end{aligned}$$

But this only proves k bounded by the previous n not $n+1$. To resolve this we now have to prove that $(P(n,k) \wedge (1 < k+1 < n)) \to P(n,k+1)$:

$$\begin{aligned}
\left(\prod_{i=1}^{n} x_i\right) &= \left(\prod_{i=1}^{k} x_i\right)\left(\prod_{i=k+1}^{n} x_i\right) && \text{(given)} \\
&= \left(\prod_{i=1}^{k} x_i\right)\left(\prod_{i=k+2}^{n} x_i\right) x_{k+1} && \text{(def. of finitary multiplication)} \\
&= \left(\prod_{i=1}^{k} x_i\right) x_{k+1} \left(\prod_{i=k+2}^{n} x_i\right) && \text{(associativity of 3-ary mult.)} \\
&= \left(\prod_{i=1}^{k+1} x_i\right)\left(\prod_{i=k+2}^{n} x_i\right) && \text{(def. of finitary multiplication)}
\end{aligned}$$

Notice that line 2 required that $1 < k+1 < n$ in order to use that definition of finitary multiplication.

And so by induction we have:

$$\begin{aligned}
&(\forall k,\ 1 < k < n)\ P(3,k) \\
&(\forall k,\ 1 < k < n)\ P(n,k) \to P(n+1,k) \\
&(\forall n \in \mathbb{Z}^+)(\forall k,\ 1 < k < 3)\ P(n,k) && \text{(special case for 1 \& 2)} \\
\therefore\ &(\forall n \in \mathbb{Z}^+)\ P(n,2) && \text{(only } k = 2 \text{ satisfies this)}
\end{aligned}$$

Using induction once more we find:

$$\begin{aligned}
&(\forall n \in \mathbb{Z}^+)\ P(n,2) \\
&(P(n,k) \wedge (1 < k+1 < n)) \to P(n,k+1) \\
\therefore\ &(\forall n,k \in \mathbb{Z}^+)\ (\forall k,\ 1 < k < n)\ P(n,k)
\end{aligned}$$

Problem 3

Problem: Prove the following for all $n \in \mathbb{Z}^+$:

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

Solution: First we define the following predicate:

$$P(n) \equiv \sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

Now note that $P(1)$ is true:

$$\begin{aligned} \sum_{i=1}^{1} i^3 &= \left(\sum_{i=1}^{1} i\right)^2 && P(1) \\ 1^3 &= (1)^2 && \text{(def. of 1-ary sum)} \end{aligned}$$

Now we will prove that $P(n) \to P(n+1)$:

$$\begin{aligned} \sum_{i=1}^{n} i^3 &= \left(\sum_{i=1}^{n} i\right)^2 && (P(n) \text{ given}) \\ \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 &= \left(\sum_{i=1}^{n} i\right)^2 + (n+1)^3 \\ \left(\sum_{i=1}^{n+1} i^3\right) &= \left(\sum_{i=1}^{n} i\right)^2 + (n+1)^3 && \text{(def. of finitary sum)} \\ &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 && (n\text{th triangular number}) \\ &= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} && \text{(algebra)} \\ &= \left(\frac{(n+1)(n+2)}{2}\right)^2 && \text{(Lemma 1)} \\ \left(\sum_{i=1}^{n+1} i^3\right) &= \left(\sum_{i=1}^{n+1} i\right)^2 && ((n+1)\text{th triangular number}) \end{aligned}$$

Notice that we justified the 5th step as 'Lemma 1'. This is because factoring is hard and so we will simply show that this is true in reverse, by multiplying out the square of the $(n+1)th$ triangular number:

$$\begin{aligned} \left(\frac{(n+1)(n+2)}{2}\right)^2 &= \left(\frac{n^2 + 3n + 2}{2}\right)^2 \\ &= \frac{(n^2 + 3n + 2)^2}{4} \\ &= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \end{aligned}$$

And so we have shown that $P(n+1)$ follows from $P(n)$. The PMI tells us that:

$$\begin{aligned} &P(n) \implies P(n+1) \\ &P(1) \\ \therefore\ &\forall n \in \mathbb{Z}^+,\ P(n) \end{aligned}$$

And so the equality holds for all positive n.

Problem 4

Problem: Given some set of k real (or complex) constants a_i, we define an LHCC recurrence relation R of order k as:

$$x(n) = \sum_{i=1}^{k} a_i x(n-i) = a_1 x(n-1) + \cdots + a_k x(n-k)$$

Show that if two sequences satisfy R, then a linear combination of the two also satisfy it. Then use induction to show that a linear combination of m solutions to R also satisfies R.

Solution

2 **Sequence Case**

$$\begin{aligned}
y_1(n) &= a_1 y_1(n-1) + \cdots + a_k y_1(n-k) && \text{(given)} \\
&= \sum_{i=1}^{k} a_i y_1(i) \\
y_2(n) &= a_1 y_2(n-1) + \cdots + a_k y_2(n-k) && \text{(given)} \\
&= \sum_{i=1}^{k} a_i y_2(n)
\end{aligned}$$

Now we can simply multiply the equations by any real (or complex) constants b_1 and b_2 then sum the resulting equations to show the 2 sequence case does indeed satisfy R:

$$\begin{aligned}
b_1 y_1(n) &= b_1 \sum_{i=1}^{k} a_i y_1(n-i) \\
b_2 y_2(n) &= b_2 \sum_{i=1}^{k} a_i y_2(n-i) \\
b_1 y_1(n) + b_2 y_2(n) &= b_1 \sum_{i=1}^{k} a_i y_1(n-i) + b_2 \sum_{i=1}^{k} a_i y_2(n-i) \\
&= \sum_{i=1}^{k} a_i (b_1 y_1(n-i)) + \sum_{i=1}^{k} a_i (b_2 y_2(n-i)) \\
& && \text{(finitary distrib. prop.)} \\
&= \sum_{i=1}^{k} a_i (b_1 y_1(n-i) + b_2 y_2(n-i)) && \text{(finitary distrib. prop.)} \\
&= a_1(b_1 y_1(n-1) + b_2 y_2(n-1)) + \cdots + a_k(b_1 y_1(n-k) + b_2 y_2(n-k))
\end{aligned}$$

m **Sequence Case**

Let $y_j(i)$ be any list of m solutions to R and let b_j be any list of m real (or complex) constants. We'll use this to define a predicate:

$$
\begin{aligned}
P(m) \equiv \sum_{j=1}^{m} b_j y_j(n) &= \sum_{i=1}^{k}\sum_{j=1}^{m} a_i b_j y_j(n-i) \\
&= \sum_{j=1}^{m}\sum_{i=1}^{k} a_i b_j y_j(n-i)
\end{aligned}
$$

We proved $P(2)$ above for the arbitrary solutions and constants y_1, y_2, b_1, b_2. Now we just need to prove that $P(m) \rightarrow P(m+1)$ given some new solution $y_{m+1}(i)$ and constant b_{m+1}:

$$
\begin{aligned}
\sum_{j=1}^{m} b_j y_j(n) &= \sum_{j=1}^{m}\sum_{i=1}^{k} a_i b_j y_j(n-i) && \text{(given)} \\
\sum_{j=1}^{m} b_j y_j(n) + b_{m+1}y_{m+1}(n) &= \left(\sum_{j=1}^{m}\sum_{i=1}^{k} a_i b_j y_j(n-i)\right) + b_{m+1}y_{m+1}(n) \\
\sum_{j=1}^{m+1} b_j y_j(n) &= \left(\sum_{j=1}^{m}\sum_{i=1}^{k} a_i b_j y_j(n-i)\right) + b_{m+1}y_{m+1}(n) \\
& && \text{(def. finitary addition)} \\
&= \left(\sum_{j=1}^{m}\sum_{i=1}^{k} b_j y_j(n-i)\right) + \sum_{i=1}^{k} a_i b_{m+1} y_{m+1}(n-i) \\
& && \text{(def. of solution)} \\
&= \sum_{j=1}^{m+1}\sum_{i=1}^{k} b_j y_j(n-i) && \text{(def. finitary addition)}
\end{aligned}
$$

And so we have shown that $P(m+1)$ follows from $P(m)$. The PMI tells us that:

$$
\begin{aligned}
& P(2) \\
& P(m) \implies P(m+1) \\
\therefore\ & \forall m \leq 2,\ P(m)
\end{aligned}
$$

Problem 5

Problem: Derive the explicit formula for the Fibonacci sequence:

$$f_1 = f_2 = 1;\ f_n = f_{n-1} + f_{n-2}$$

Then evaluate it at $n = 3, 4, 5$.

Solution: First we solve for the roots of the characteristic polynomial:

$$x^2 - x - 1 = 0$$
$$r = \frac{1 \pm \sqrt{5}}{2}$$

Denoting the roots r_1 and r_2, we know that there exists constants b_1 and b_2 such that:

$$f_n = b_1 r_1^n + b_2 r_2^n$$

And since $f_1 = f_2 = 1$ we can solve for the constants:

$$f_1 = b_1 r_1 + b_2 r_2 = 1$$
$$f_2 = b_1 r_1^2 + b_2 r_2^2 = 1$$
$$b_2 = \frac{1 - b_1 r_1}{r_2}$$
$$b_1 r_1^2 + r_2(1 - b_1 r_1) = 1$$

Plugging in the roots and solving for b_1 we find:

$$b_1 = \frac{1}{\sqrt{5}}$$

Plugging this back into the first equation, we can now solve for b_2

$$b_2 = -\frac{1}{\sqrt{5}}$$

This leaves us with the following explicit formula for the Fibonacci sequence:

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 + \sqrt{5}}{2} \right)^n \right)$$