Differential Equations HW #2

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Problem 1

Problem: Given the following IVP:

$$y' = t + y, \quad y(0) = 1$$

Given that the zeroth term of the Picard iteration is $y_0(t) = 1$, calculate the next two terms $y_1(t), y_2(t)$.

Solution: Recall that the Picard iteration of an IVP is given by:

$$y_{n+1} = y_0 + \int_{t_0}^{t} f(t, y_n)\, dt$$

And so the first iterate y_1 is given by:

$$\begin{aligned} y_1(t) &= 1 + \int_0^t (t+1)\, dt \\ &= 1 + \left[\frac{t^2}{2} + t\right]_0^t \\ &= \frac{t^2}{2} + t + 1 \end{aligned}$$

And the second iterate y_2 given by:

$$\begin{aligned} y_2(t) &= 1 + \int_0^t \left(t + \frac{t^2}{2} + t + 1\right) dt \\ &= 1 + \int_0^t \left(\frac{t^2}{2} + 2t + 1\right) dt \\ &= 1 + \left[\frac{t^3}{6} + t^2 + t\right]_0^t \\ &= \frac{t^3}{6} + t^2 + t + 1 \end{aligned}$$

Problem 2

Problem: Show that the following IVP does not have a solution $y(t)$ defined on any interval $(-\epsilon, \epsilon)$:

$$y' = \begin{cases} \frac{y}{t}, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases}, \qquad y(0) = 1$$

Solution: We can see that for $t \neq 0$ this is a separable equation, and so its solution set is given by:

$$\begin{aligned} \frac{dy}{dt} &= \frac{y}{t} \\ \int \frac{1}{y}\, dy &= \int \frac{1}{t}\, dt && \text{(separable equation)} \\ \ln|y| &= \ln|t| + C_1 && \text{(integration)} \\ |y| &= e^{C_1}|t| && \text{(exponentiation)} \\ y &= C_2 t && (\pm e^{C_1} = C_2 \neq 0) \end{aligned}$$

Note that by letting $C_2 = 0$, we arrive at what happens to be the equation's sole equilibrium solution: $y = 0$. Thus we can replace C_2 with a new constant C_3 that can take on any real value. This gives us the following family of solutions indexed by $C_3 \in \mathbb{R}$:

$$y_{C_3}(t) = C_3 t$$

Note, however, that there is no constant C_3 that can satisfy the initial condition:

$$(\forall C_3 \in \mathbb{R})\ y_{C_3}(0) = C_3 \cdot 0 \neq 1$$

And so the initial condition can't be satisfied by any solution of the differential equation.

Problem 3

Part a: Sketch the phase line of the following autonomous equation:

$$y' = \sin y, \qquad y \in (-3\pi, 3\pi)$$

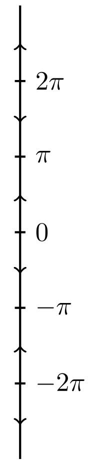
Solution: The equilibrium points of the differential equation are given by the roots of $\sin y$ which are just the integer multiples of π (within the given domain):

$$(\forall n \in \mathbb{N})\ y'(n\pi) = \sin(n\pi) = 0$$

Now we apply the first derivative test to classify these equilibria as either sources or sinks:

$$\frac{d}{dy}\sin(n\pi) = \cos(n\pi) = (-1)^n$$

This is to say that for even n the equilibrium point is a source, and for odd n it's a sink. We now have enough information to draw our phase line, complete with equilibria and decreasing/increasing intervals:



Part b: Sketch the phase line of the following autonomous equation:

$$y' = f(y) = y^3 + 2y^2 - y$$

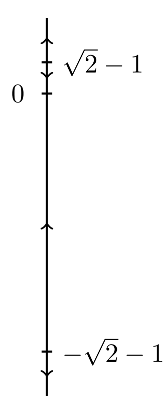
Solution: The equilibrium points are given by the roots of the differential equation:

$$\begin{aligned}
f(y) = y^3 + 2y^2 - y &= 0 \\
y(y^2 + 2y - 1) &= 0 \\
y(y + \sqrt{2} + 1)(y - \sqrt{2} + 1) &= 0 \\
y &= 0, \pm\sqrt{2} - 1
\end{aligned}$$

Now we apply the first derivative test to classify these equilibria as either sources or sinks:

$$\begin{aligned}
f'(y) &= 3y^2 + 4y - 1 \\
f'(0) &= 3 \cdot 0^2 + 4 \cdot 0 - 1 = -1 < 0 \\
f'(\sqrt{2} - 1) &= 3(\sqrt{2} - 1)^2 + 4(\sqrt{2} - 1) - 1 = 4 - 2\sqrt{2} > 0 \\
f'(-\sqrt{2} - 1) &= 3(-\sqrt{2} - 1)^2 + 4(-\sqrt{2} - 1) - 1 = 4 + 2\sqrt{2} > 0
\end{aligned}$$

And so 0 is sink, and $\pm\sqrt{2} - 1$ are sources. We now have enough information to draw our phase line:



Problem 4

Part a: Sketch the bifurcation diagram of the following family of autonomous differential equations, and identify all bifurcation values:

$$y' = f_\mu(y) = 4y^2 + \mu^2 - 1$$

Solution: The bifurcation diagram is simply the graph found by setting the family equal to 0, and solving for y in terms of the parameter μ:

$$\begin{aligned} y' = 4y^2 + \mu^2 - 1 &= 0 \\ 4y^2 &= 1 - \mu^2 \\ y^2 &= \frac{1-\mu^2}{4} \\ y &= \frac{\pm\sqrt{1-\mu^2}}{2} \end{aligned}$$

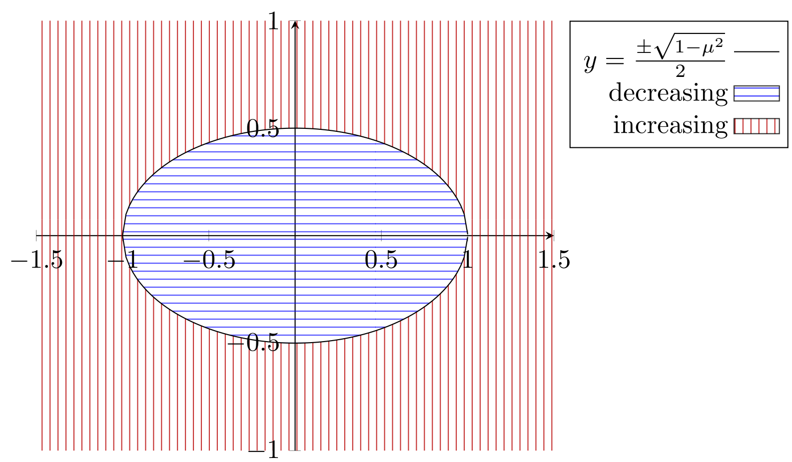
To shade in the decreasing/increasing sectors, we test $\mu = 0$ giving us the following equilibria:

$$y = \frac{\pm\sqrt{1-0^2}}{2} = \frac{\pm\sqrt{1}}{2} = \pm\frac{1}{2}$$

Performing the first derivative test on these equilibria tells us:

$$\begin{aligned} f_0'(y) &= 8y \\ f_0'\left(\frac{1}{2}\right) &= 8 \cdot \frac{1}{2} = 4 > 0 \\ f_0'\left(-\frac{1}{2}\right) &= 8 \cdot \frac{-1}{2} = -4 < 0 \end{aligned}$$

Which tells us that $\frac{1}{2}$ is a source and $-\frac{1}{2}$ is a sink. We now have enough information to graph and shade the bifurcation diagram:



We can clearly see that a change in the number of equilibria occurs when $\mu = \pm 1$, and so these are our bifurcation values.

Part b: Sketch the bifurcation diagram of the following family of autonomous differential equations, and identify all bifurcation values:

$$y' = f_\mu(y) = (y-1)(y^2 - \mu^2)$$

Solution: The bifurcation diagram is the set of all points (y, μ) such that y is an equilibrium point of $f_\mu(y) = 0$. Examining the differential equation, we see that $y = 1$ will always be an equilibrium point:

$$(\forall \mu \in \mathbb{R})\ f_\mu(1) = (1-1)(1^2 - \mu^2) = 0$$

To find the other equilibria, we simply set the other factor equal to 0 and solve for y:

$$\begin{aligned} y^2 - \mu^2 &= 0 \\ y^2 &= \mu^2 \\ y &= \pm\mu \end{aligned}$$

To shade in the decreasing/increasing sectors, we first copmute the derivative of $f_\mu(y)$:

$$f_\mu'(y) = 3y^2 - 2y - \mu^2$$

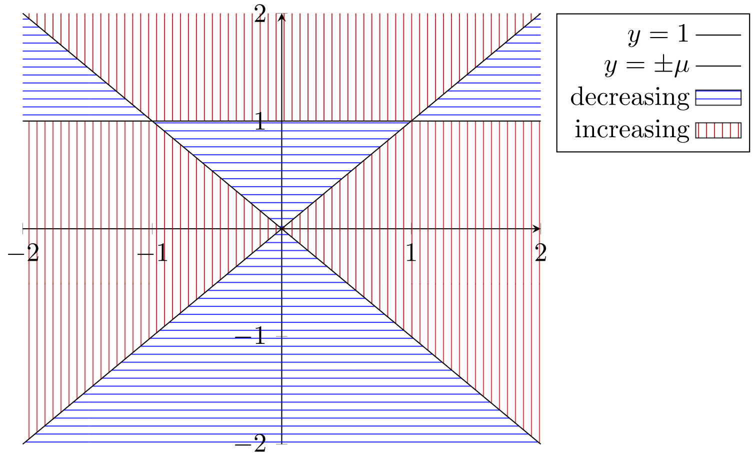
Now we test $\mu = \frac{1}{2}$ giving us the following:

$$\begin{aligned} f_{\frac{1}{2}}(y) = 0 &\implies y = 1, \pm\frac{1}{2} \\ f_{\frac{1}{2}}\left(\frac{1}{2}\right) &= \frac{3}{4} - 1 - \frac{1}{4} = -\frac{1}{2} < 0 \\ f_{\frac{1}{2}}\left(-\frac{1}{2}\right) &= \frac{3}{4} + 1 - \frac{1}{4} = \frac{3}{2} > 0 \\ f_{\frac{1}{2}}(1) &= 3 - 2 - \frac{1}{4} = \frac{3}{4} > 0 \end{aligned}$$

And so we have $\frac{1}{2}$ is a sink, and $1, -\frac{1}{2}$ are sources. For the remaining regions, we test $\mu = 2$ giving us the following:

$$\begin{aligned} f_2(y) = 0 &\implies y = 1, \pm 2 \\ f_2(2) &= 12 - 4 - 4 = 4 > 0 \\ f_2(-2) &= 12 + 4 - 4 = 12 > 0 \\ f_2(1) &= 3 - 2 - 4 = -3 < 0 \end{aligned}$$

And so we have 1 is a sink, and ± 2 are sources. We now have enough information, thanks to symmetry, to graph and shade the bifurcation diagram:



We can see that there are 3 equilibrium points for all values of μ except for $\mu = 0, \pm 1$ which all have 2 equilibria. And so these 3 values are the bifurcation values of the given autonomous family.

Problem 5

Problem: Consider the following population model with harvesting:

$$P' = kP\left(1 - \frac{P}{N}\right) - \alpha P \tag{1}$$

where $k, N > 0$ are fixed (that is, the harvesting rate is proportional to the total population). Find the critical value α_0 so that the population will become extinct if $\alpha > \alpha_0$ (you may assume that the initial population is N).

Solution: Let us first rewrite the differential equation like so:

$$P' = kP\left(1 - \alpha - \frac{P}{N}\right) \tag{2}$$

As we can see, $P = 0$ will always be an equilibrium point:

$$(\forall \alpha \in \mathbb{R})\ P'_\alpha(0) = k \cdot 0 \cdot \left((1 - \alpha - \frac{0}{N}\right) = 0$$

To find the other equilibria, we simply set the other factor equal to 0 and solve for y:

$$\begin{aligned} 1 - \alpha - \frac{P}{N} &= 0 \\ \frac{P}{N} &= 1 - \alpha \\ P &= N(1 - \alpha) \end{aligned}$$

To shade the decreasing/increasing sectors, we first test $\alpha = 0$ giving us the following equation:

$$P'_0 = kP\left(1 - \frac{P}{N}\right) = kP - \frac{kP^2}{N}$$

Its equilibrium points are $P = 0, N$. Now we just perform the first derivative test:

$$\frac{dP_0'}{dP} = f_0'(P) = k - \frac{2kP}{N}$$
$$f_0'(0) = k - 0 = k > 0$$
$$f_0'(N) = k - 2k = -k < 0$$

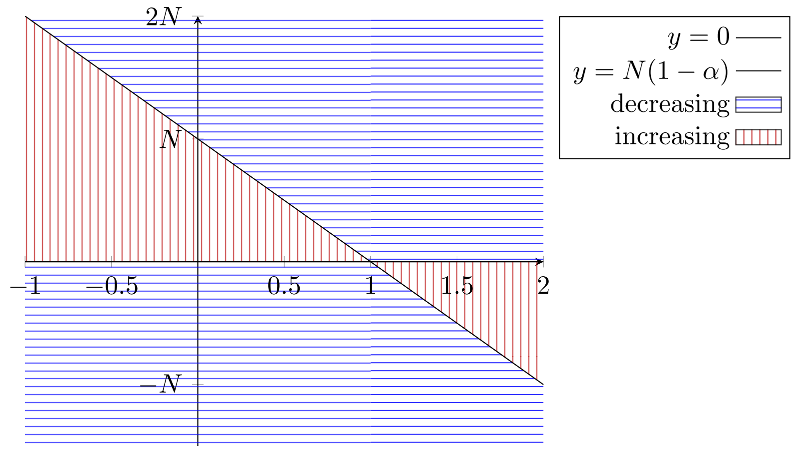
This tells us that 0 is a source, and N a sink. And now for the remaining sector we test $\alpha = 2$, giving us the following equation:

$$P_2' = kP\left(-1 - \frac{P}{N}\right) = -kP - \frac{kP^2}{N}$$

Its equilibrium points are $P = 0, -N$. Now we just perform the first derivative test:

$$\frac{dP_2'}{dP} = f_2'(P) = -k - \frac{2kP}{N}$$
$$f_2'(0) = -k - 0 = -k < 0$$
$$f_2'(-N) = k + 2k = k > 0$$

This tells us that 0 is a sink, and $-N$ a source. We now have enough information to graph the and shade the bifurcation diagram:



You'll notice that for all $\alpha < 1$, for any sufficiently small ϵ, when the population $P = N(1-\alpha)+\epsilon$, the population P tends back towards $N(1-\alpha)$ since it is a stable equilibrium.

However, when $\alpha > 1$, this is no longer the case. $N(1-\alpha)$ becomes unstable and 0 stable. And since populations can't be negative in the real world, this means that the population will tend to zero for $\alpha > 1$. Thus the critical $a_0 = 1$.

Problem 6

Part a: Find the general solution to the following differential equation:

$$y' = -2y + \sin 2t$$

Solution: Rewriting the ODE in standard form we have:

$$y' + 2y = \sin 2t$$

Now we note that an LDE of this form always has a particular solution y_p of the following form:

$$y_p = \alpha \sin 2t + \beta \cos 2t$$

The derivative of which is given by:

$$y_p' = 2\alpha \cos 2t - 2\beta \sin 2t$$

Plugging these two into the LDE we find:

$$\begin{aligned} y_p' + 2y_p &= \sin 2t \\ 2\alpha \cos 2t - 2\beta \sin 2t + 2\alpha \sin 2t + 2\beta \cos 2t &= \sin 2t \\ (2\alpha - 2\beta) \sin 2t + (2\alpha + 2\beta) \cos 2t &= \sin 2t \end{aligned}$$

This gives us the following system of equations:

$$\begin{cases} 2\alpha - 2\beta = 1 \\ 2\alpha + 2\beta = 0 \end{cases} \implies (\alpha, \beta) = \left(\frac{1}{4}, -\frac{1}{4}\right)$$

And so we now have the following particular solution to the LDE:

$$y_p = \frac{\sin 2t - \cos 2t}{4}$$

The general solution to its associated homogenous equation $y' = -2y$ is given by the following integral:

$$y_h = e^{\int 2\, dt} = Ce^{-2t}$$

Recall that the sum of a particular solution y_p and the general solution to the associated homogenous equation y_h, gives the general solution to the LDE indexed by C:

$$\begin{aligned} y &= y_p + y_h \\ &= \frac{\sin 2t - \cos 2t}{4} + Ce^{-2t} \end{aligned}$$

Part b: Find the general solution to the following differential equation:

$$y' = -y + e^t + e^{-t}$$

Solution: Rewriting the ODE in standard form we have:

$$y' + y = e^t + e^{-t}$$

It's integrating factor $u(t)$ is given by the following integral:

$$u(t) = e^{\int dt} = Ce^t$$

Using this, we can now express the general solution to the linear differential equation:

$$
\begin{aligned}
y &= \frac{\int u(t)(e^t + e^{-t})\,dt}{u(x)} = \frac{C\int e^t(e^t + e^{-t})\,dt}{Ce^t} \\
&= \frac{\int e^t(e^t + e^{-t})\,dt}{e^t} = \frac{\int (e^{2t} + 1)\,dt}{e^t} \\
&= \frac{\int e^{2t}\,dt + \int dt}{e^t} = \frac{\frac{e^{2t}}{2} + t + C_1}{e^t} \\
&= \frac{e^{2t} + 2t + C_2}{2e^t} = \frac{e^t}{2} + te^{-t} + C_3 e^{-t} \\
&= \frac{e^t}{2} + e^{-t}(t + C_3)
\end{aligned}
$$

Problem 7

Problem: Find the general solution of the following differential equation:

$$y' = -\frac{y}{1+t} + t^2$$

Solution: Rewriting the ODE in standard form we have:

$$y' + \frac{y}{1+t} = t^2$$

It's integrating factor $u(t)$ is given by the following integral:

$$u(t) = e^{\int \frac{1}{1+t}\,dt} = Ce^{\ln(t+1)} = C(t+1)$$

Using this, we can now express the general solution to the linear differential equation:

$$
\begin{aligned}
y &= \frac{\int u(t)t^2)\,dt}{u(x)} \\
&= \frac{C\int t^2(t+1))\,dt}{C(t+1)} \\
&= \frac{\int t^3 + t^2\,dt}{t+1} \\
&= \frac{\frac{t^4}{4} + \frac{t^3}{3} + C_1}{t+1} \\
&= \frac{t^4}{4(t+1)} + \frac{t^3}{3(t+1)} + \frac{C_1}{t+1}
\end{aligned}
$$