Set Theory HW #1

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Problem 1

Exercises 1,2,3,4 from pages 6-7 in the textbook.

Exercise 1: Which of the following statements are true when $\in$ is inserted in the blank? Which are true when $\subseteq$ is inserted?

(a) $\{\varnothing\}\_\{\varnothing, \{\varnothing\}\}$

(b) $\{\varnothing\}\_\{\varnothing, \{\{\varnothing\}\}\}$

(c) $\{\{\varnothing\}\}\_\{\varnothing, \{\varnothing\}\}$

(d) $\{\{\varnothing\}\}\_\{\varnothing, \{\{\varnothing\}\}\}$

(e) $\{\{\varnothing\}\}\_\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$

Solution: Statements (a) and (d) are true when $\in$ is inserted in the blank. Statements (a), (b) and (c) are true when $\subseteq$ is inserted.

Exercise 2: Show that none of the three sets $\varnothing, \{\varnothing\}$, and $\{\{\varnothing\}\}$ are equal to any other.

Solution: We have 3 statements to disprove. Let's assume $\varnothing = \{\varnothing\}$. The axiom of extensionality tells us that:

$$\forall x(x \in \varnothing \iff x \in \{\varnothing\})$$

However, note that for the particular choice of $x = \varnothing$ we have:

$$\underbrace{\varnothing \in \varnothing}_{F} \iff \underbrace{\varnothing \in \{\varnothing\}}_{T}$$

With the LHS being false because $\varnothing$ has no elements by definition and the RHS being clear. This is a contradiction and so our initial assumption is false and $\varnothing \neq \{\varnothing\}$.

For the next case, we'll assume $\varnothing = \{\{\varnothing\}\}$. The axiom of extensionality tells us that:

$$\forall x(x \in \varnothing \iff x \in \{\{\varnothing\}\})$$

However, note that for the particular choice of $x = \{\varnothing\}$ we have:

$$\underbrace{\{\varnothing\} \in \varnothing}_{F} \iff \underbrace{\{\varnothing\} \in \{\{\varnothing\}\}}_{T}$$

The LHS being false because $\varnothing$ has no elements and the RHS being clear. This is a contradiction and so our initial assumption is false and $\varnothing \neq \{\{\varnothing\}\}$.

For the last case, we assume $\{\varnothing\} = \{\{\varnothing\}\}$. The axiom of extensionality tells us that:

$$\forall x(x \in \{\varnothing\} \iff x \in \{\{\varnothing\}\})$$

However, note that for the particular choice of $x = \{\varnothing\}$ we have:

$$\underbrace{\{\varnothing\} \in \{\varnothing\}}_{F} \iff \underbrace{\{\varnothing\} \in \{\{\varnothing\}\}}_{T}$$

The LHS being false because $\{\varnothing\}$ only contaians $\varnothing$ and not it's singleton and the RHS being clear. This is a contradiction and so our initial assumption is false and $\{\varnothing\} \neq \{\{\varnothing\}\}$.

Exercise 3: Show that if $B \subseteq C$, then $\mathcal{P}(B) \subseteq \mathcal{P}(C)$.

Solution: Assume that $B \subseteq C$. Now note that:

$$\forall x(x \in \mathcal{P}(B) \iff x \subseteq B) \qquad \text{(def. of power set)}$$

And because of our assumption that $B \subseteq C$ and the transitivity of subset, we have:

$$\begin{aligned}
\forall x(x \in \mathcal{P}(B) \iff x \subseteq B \subseteq C) &\implies \forall x(x \in \mathcal{P}(B) \implies x \subseteq C) && \text{(transitivity of subset)} \\
&\iff \forall x(x \in \mathcal{P}(B) \implies x \in \mathcal{P}(C)) && \text{(def. of power set)} \\
&\iff \mathcal{P}(B) \subseteq \mathcal{P}(C) && \text{(def. subset)}
\end{aligned}$$

Exercise 4: Assume $x, y \in B$. Show that $\{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(B))$

Solution: Since we are assuming $x, y \in B$, we have the following chain of implications:

$$\begin{aligned}
&\underbrace{\{x\} \subseteq B}_{x \in \{x\} \to x \in B} \wedge \underbrace{\{x,y\} \subseteq B}_{\substack{x \in \{x,y\} \to x \in B \\ y \in \{x,y\} \to y \in B}} && \text{(def. of subset)} \\
\implies\ & \{x\} \in \mathcal{P}(B) \wedge \{x,y\} \in \mathcal{P}(B) && \text{(def. of power set)} \\
\implies\ & \underbrace{\{\{x\}, \{x,y\}\} \subseteq \mathcal{P}(B)}_{\substack{\{x\} \in \{\{x\},\{x,y\}\} \to \{x\} \in \mathcal{P}(B) \\ \{x,y\} \in \{\{x\},\{x,y\}\} \to \{x,y\} \in \mathcal{P}(B)}} && \text{(def. of subset)} \\
\implies\ & \{\{x\}, \{x,y\}\} \in \mathcal{P}(\mathcal{P}(B)) && \text{(def. of power set)}
\end{aligned}$$

Problem 2

Exercises 5,7 from page 9 in the textbook. Where V_α below refers to the rank α of the von Neumann hierarchy.

Exercise 5: Define the rank of a set c to be the least α such that $c \subseteq V_\alpha$. Compute the rank of $\{\{\varnothing\}\}$ and $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.

Solution: The rank of $\{\{\varnothing\}\}$ is 2 as $V_2 = \mathcal{P}(V_1) = \mathcal{P}(\varnothing)$ is the first rank in which it shows up.

The highest ranked element of $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ is $\{\varnothing, \{\varnothing\}\}$, which first appears in rank 3. And because $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ contains this rank 3 element, it appears in $V_4 = \mathcal{P}(V_3)$ and so is rank 4.

Exercise 7: List all the members of V_3 and V_4.

Solution: V_3 and V_4 are given by:

$$
\begin{aligned}
V_3 = \{&\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
V_4 = \{&\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\} \\
&\{\{\{\varnothing\}\}\}, \{\varnothing, \{\{\varnothing\}\}\}, \\
&\{\{\varnothing\}, \{\{\varnothing\}\}\}, \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}, \\
&\{\{\varnothing, \{\varnothing\}\}\}, \{\varnothing, \{\varnothing, \{\varnothing\}\}\}, \\
&\{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\
&\{\{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
&\{\{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
\}&
\end{aligned}
$$

Problem 3

A set a is transitive if every member of a is a subset of a. In other words a is transitive iff:

$$\forall u(u \in a \implies u \subseteq a)$$

Part i: Prove that $\varnothing$ is transitive.

Solution: The empty set is vacuously a transitive set:

$$\forall u(u \in \varnothing \implies u \subseteq \varnothing)$$

The condition that $u \in \varnothing \implies u \subseteq \varnothing$ is always satisfied as the antecedent is false for any u, because the empty set has no elements by definition.

Part ii: Prove that the union of two transitive sets is transitive. That is to say prove that:

$$
\begin{aligned}
\forall a \forall b \big[\big(\forall u(u \in a \implies u \subseteq a) \wedge \forall u(u \in b \implies u \subseteq b) \big) \\
\implies \forall u(u \in a \cup b \implies u \subseteq a \cup b) \big]
\end{aligned}
$$

Solution: Let a and b be transitive sets. Now let $u \in a \cup b$. This implies that:

$$u \in a \vee u \in b \qquad \text{(def. of union)}$$

Also, since a and b are transitive, we have:

$$
\begin{aligned}
u \in a &\implies u \subseteq a && \text{(def. of transitive)} \\
u \in b &\implies u \subseteq b && \text{(def. of transitive)}
\end{aligned}
$$

So we have by the constructive dilemma, i.e.:

$$((p \implies q) \wedge (r \implies s) \wedge (p \vee r)) \implies (q \vee s)$$

the following:

$$\begin{array}{l} u \in a \implies u \subseteq a \\ u \in b \implies u \subseteq b \\ u \in a \lor u \in b \\ \hline \therefore u \subseteq a \lor u \subseteq b \end{array}$$

Now note that, by the transitivity of the subset, we have:

$$\begin{array}{l} u \subseteq a \\ a \subseteq a \cup b \\ \hline \therefore u \subseteq a \cup b \end{array}$$

And since $a \subseteq a \cup b$ is true for any a and b we arrive at the following implication:

$$u \subseteq a \implies u \subseteq a \cup b$$

And similarly we have:

$$\begin{array}{l} u \subseteq b \\ b \subseteq a \cup b \\ \hline \therefore u \subseteq a \cup b \end{array}$$

And again, since $b \subseteq a \cup b$ is true for any a and b we arrive at the following implication:

$$u \subseteq b \implies u \subseteq a \cup b$$

And so via the constructive dilemma we have:

$$\begin{array}{l} u \subseteq a \implies u \subseteq a \cup b \\ u \subseteq b \implies u \subseteq a \cup b \\ u \subseteq a \lor u \subseteq b \\ \hline \therefore u \subseteq a \cup b \end{array}$$

And so we are done. We have shown that for any transitive sets a and b, the following holds:

$$\forall u(u \in a \cup b \implies u \subseteq a \cup b)$$

which is equivalent to the statement that $a \cup b$ is a transitive set.

Part iii: Prove that if a is a transitive set then $a \cup \{a\}$ is transitive. That is, for transitive a, prove the following:

$$\forall u(u \in a \cup \{a\} \implies u \subseteq a \cup \{a\})$$

Solution: Let a be a transitive set and let u be any set. We have the following:

$$u \in a \cup \{a\} \iff u \in a \lor u \in \{a\} \qquad \text{(def. of union)}$$

As such, we can prove that in both cases $u \subseteq a \cup \{a\}$. For case 1 we will assume $u \in a$:
Because a is transitive and because $a \subseteq a \cup \{a\}$ for any a, we have:

$$\begin{aligned} u \in a &\implies u \subseteq a && (a \text{ is transitive}) \\ &\implies u \subseteq a \cup \{a\} && \text{(transitivity of subset)} \end{aligned}$$

Now we need to prove the case where $u \in \{a\}$. There is only one element to check here, $u = a$:

$$\underbrace{a \in \{a\}}_{T} \implies \underbrace{a \subseteq a \cup \{a\}}_{T}$$

This implication is certainly true and so we have the following via the consecutive dilemma:

$$\begin{array}{l} u \in a \implies u \subseteq a \cup \{a\} \\ u \in \{a\} \implies u \subseteq a \cup \{a\} \\ u \in a \vee u \in \{a\} \\ \hline \therefore u \subseteq a \cup \{a\} \end{array}$$

And so by assuming the antecedent $u \in a \cup \{a\}$ we proved the consequent and thus $a \cup \{a\}$ is a transitive set for any transitive set a.

Part iv: Prove that $\mathcal{P}(a)$ of a transitive set a is transitive. That is to say prove:

$$\forall a\big(\forall u(u \in a \implies u \subseteq a) \implies \forall u(u \in \mathcal{P}(a) \implies u \subseteq \mathcal{P}(a))\big)$$

Solution: Let $b \in \mathcal{P}(a)$, this gives us:

$$b \in \mathcal{P}(a) \implies b \subseteq a \qquad \text{(def. of power set)}$$

Now for any x we have the following chain of implications:

$$\begin{aligned} x \in b &\implies x \in a && \text{(def. of subset)} \\ &\implies x \subseteq a && (a \text{ is transitive}) \\ &\implies x \in \mathcal{P}(a) && \text{(def. of power set)} \end{aligned}$$

And so we are done. We have shown that for any set $b \in \mathcal{P}(a)$, any one of its elements x is a subset of $\mathcal{P}(a)$.