Foundations of QM
HW 4

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Consider a 1D quantum system $\psi(x,t)$ consisting of a single particle with mass m moving on the interval $[0,L]$, with boundary conditions $\psi(0)=0=\psi(L)$. In the case of it having no potential, its Hamiltonian is given below:

$$H=-\frac{\hbar}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

Let $\psi_n(x,t)$ denote the solution to the (time dependent) Schrödinger equation with initial condition $\psi_n(x,0)=\phi_n(x)$. Where $\phi_n(x)$ are the solutions of the time-independent Schrödinger equation $H\phi_n=E_n\phi_n$:

$$\phi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Where $E_n=\frac{(\hbar\pi n)^2}{2mL^2}$ are the eigenvalues of H.

Question 1 & 2

Problem: Solve for $\psi_1(x,t)$ and $\psi_2(x,t)$

Solution: We will solve this for for general n:

$$\begin{aligned}
i\hbar\frac{\partial\psi_n(x,t)}{\partial t}&=H\psi_n(x,t) && \text{(Schrödinger equation)}\\
&=E_n\psi_n(x,t) && (H\psi_n(x,0)=E_n\psi_n(x,0)\text{ \& conservation of energy})\\
\frac{\partial\psi_n(x,t)}{\partial t}&=\frac{-iE_n}{\hbar}\psi_n(x,t)\\
\psi_n(x,t)&=\psi_n(x,0)\exp\left(\frac{-iE_n}{\hbar}t\right) && (\text{sol. to } y'=ay)\\
&=\phi_n(x)\exp\left(\frac{-iE_n}{\hbar}t\right) && \text{(initial condition)}\\
&=\exp\left(\frac{-iE_n}{\hbar}t\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) && (\text{def. of } \phi_n)
\end{aligned}$$

We have just solved for ψ_n, and so plugging in 1 and 2 we have:

$$\begin{aligned}
\psi_1(x,t)&=\exp\left(\frac{-iE_1}{\hbar}t\right)\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\\
&=\exp\left(\frac{-i\hbar\pi^2}{2mL^2}t\right)\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) && (E_n=\tfrac{(\hbar\pi n)^2}{2mL^2})\\
\psi_2(x,t)&=\exp\left(\frac{-iE_2}{\hbar}t\right)\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)\\
&=\exp\left(\frac{-2i\hbar\pi^2}{mL^2}t\right)\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) && (E_n=\tfrac{(\hbar\pi n)^2}{2mL^2})
\end{aligned}$$

Question 3 & 4

Problem: For $\psi_1(x,t)$, what is the probability that the position $X_{\psi_1(t)}$ of the particle will be between 0 and $\frac{L}{2}$ if measured at time $t=0$? At time $t=t$?

Solution: For a measurement at taken at time t we have:

$$\begin{aligned}
P\left(0 < X_{\psi_1(t)} < \frac{L}{2}\right) &= \int_0^{\frac{L}{2}} |\psi_1(x,t)|^2 \, \mathrm{d}x && \text{(Born rule)} \\
&= \int_0^{\frac{L}{2}} \left| \exp\left(\frac{-iE_1}{\hbar} t\right) \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \right|^2 \mathrm{d}x \\
&= \left| \exp\left(\frac{-2iE_1}{\hbar} t\right) \right|^2 \frac{2}{L} \int_0^{\frac{L}{2}} \left| \sin\left(\frac{\pi x}{L}\right) \right|^2 \mathrm{d}x \\
&= \frac{2}{L} \int_0^{\frac{L}{2}} \left| \sin\left(\frac{\pi x}{L}\right) \right|^2 \mathrm{d}x && (|e^{ix}| = 1 \text{ for real } x) \\
&= \frac{2}{L} \int_0^{\frac{L}{2}} \sin\left(\frac{\pi x}{L}\right)^2 \mathrm{d}x && (\sin(x) \text{ is real valued for real } x) \\
&= \frac{2}{L} \left[\frac{x}{2} - \frac{L \sin\left(\frac{2\pi x}{L}\right)}{4\pi} \right]_0^{\frac{L}{2}} && \text{(trig identity)} \\
&= \frac{2}{L} \left(\frac{L}{4} - 0 \right) \\
&= \frac{1}{2}
\end{aligned}$$

And since our above probability is independent of t, it is the same when $t=0$. That is to say:

$$P\left(0 < X_{\psi_1(0)} < \frac{L}{2}\right) = \frac{1}{2}$$

This should come as no surprise considering ψ_n is an eigensolution, or stationary state, of the Schrödinger equation from problems 1 & 2.

Question 5

Problem: Consider problem 1 & 2 but, instead of solving for ψ_n, solve for ψ_{1+2} which has initial condition:

$$\psi_{1+2}(x,0) = \frac{\phi_1(x) + \phi_2(x)}{\sqrt{2}}$$

Compute the probability that the position of the particle $X_{\psi_{1+2}(t)}$ at $t = \frac{2mL^2}{3\hbar\pi}$ is between 0 and $\frac{L}{2}$. (or just is it greater than 50% if its too hard)

Solution: Recall that any solution to Shrodinger's equations, including ψ_{1+2}, must be a superposition of stationary states.

$$\begin{aligned}
\psi_{1+2}(x,t) &= \sum_{i=1}^{\infty} C_n \psi_n(x,t) && \text{(superposition of stationary states)} \\
&= \sum_{i=1}^{\infty} C_n \exp\left(\frac{-iE_n}{\hbar} t\right) \phi_n(x) && \text{(problem 1 \& 2)} \\
\psi_{1+2}(x,0) &= \sum_{i=1}^{\infty} C_n \phi_n(x) \\
&= \frac{\phi_1(x)}{\sqrt{2}} + \frac{\phi_2(x)}{\sqrt{2}} && \text{(initial condition)}
\end{aligned}$$

And so we have that $C_1 = C_2 = \frac{1}{\sqrt{2}}$, and we also have that $C_i = 0$ for $i > 2$. This gives us our final solution:

$$\psi_{1+2}(x,t) = \frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{-iE_1}{\hbar}t\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{-iE_2}{\hbar}t\right)$$

We can now proceed to calculate the probability that the particle given by ψ_{1+2} will be found in the interval $[0, L/2]$ at time $t = \frac{2mL^2}{3\hbar\pi}$:

$$\begin{aligned}
P\left(0 < X_{\psi_{1+2}(t)} < \frac{L}{2}\right) &= \int_0^{\frac{L}{2}} \left|\psi_{1+2}\left(x, \frac{2mL^2}{3\hbar\pi}\right)\right|^2 \mathrm{d}x && \text{(Born rule)} \\
&= \int_0^{\frac{L}{2}} \left|\frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{-iE_1}{\hbar}\frac{2mL^2}{3\hbar\pi}\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{-iE_2}{\hbar}\frac{2mL^2}{3\hbar\pi}\right)\right|^2 \mathrm{d}x \\
&= \int_0^{\frac{L}{2}} \left|\frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{-iE_1}{\hbar}\cdot\frac{2mL^2}{3\hbar\pi}\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{-iE_2}{\hbar}\cdot\frac{2mL^2}{3\hbar\pi}\right)\right|^2 \mathrm{d}x \\
&= \int_0^{\frac{L}{2}} \left|\frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{-i\hbar^2\pi^2}{2mL^2\hbar}\cdot\frac{2mL^2}{3\hbar\pi}\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{-i\hbar^2\pi^2 4}{2mL^2\hbar}\cdot\frac{2mL^2}{3\hbar\pi}\right)\right|^2 \mathrm{d}x \\
&= \int_0^{\frac{L}{2}} \left|\frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{-i\pi}{3}\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{-4i\pi}{3}\right)\right|^2 \mathrm{d}x \\
&= \int_0^{\frac{L}{2}} \left(\frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{-i\pi}{3}\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{-4i\pi}{3}\right)\right) \\
&\qquad \left(\frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{-i\pi}{3}\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{-4i\pi}{3}\right)\right)^* \mathrm{d}x && (\|z\|^2 = z^* z) \\
&= \int_0^{\frac{L}{2}} \left(\frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{-i\pi}{3}\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{-4i\pi}{3}\right)\right) \\
&\qquad \left(\frac{\phi_1(x)}{\sqrt{2}} \exp\left(\frac{i\pi}{3}\right) + \frac{\phi_2(x)}{\sqrt{2}} \exp\left(\frac{4i\pi}{3}\right)\right) \mathrm{d}x && \text{(conjugate is distributive)} \\
&= \frac{1}{2}\int_0^{\frac{L}{2}} \phi_1(x)^2 + \phi_2(x)^2 + \phi_1(x)\phi_2(x)\exp(-i\pi) + \phi_1(x)\phi_2(x)\exp(i\pi)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{L}{2}} \phi_1(x)^2 + \phi_2(x)^2 - 2\phi_1(x)\phi_2(x)\,\mathrm{d}x && \text{(Euler's identity)} \\
&= \frac{1}{L}\int_0^{\frac{L}{2}} \sin\left(\frac{\pi x}{L}\right)^2 + \sin\left(\frac{2\pi x}{L}\right) - 2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\,\mathrm{d}x \\
&= \frac{1}{L}\left[\frac{x}{2} - \frac{L\sin\left(\frac{2\pi x}{L}\right)}{4\pi} + \frac{x}{2} - \frac{L\sin\left(\frac{4\pi x}{L}\right)}{8\pi}\right]_0^{\frac{L}{2}} - \frac{2}{L}\int_0^{\frac{L}{2}} \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\mathrm{d}x \\
&= \frac{1}{2} - \frac{2}{L}\int_0^{\frac{L}{2}} \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\mathrm{d}x \\
&= \frac{1}{2} - \frac{1}{L}\int_0^{\frac{L}{2}} \sin\left(\frac{3\pi x}{L}\right) + \sin\left(\frac{\pi x}{L}\right)\mathrm{d}x && \text{(product-to-sum formula)} \\
&= \frac{1}{2} - \frac{1}{L}\left[-\frac{L}{3\pi}\cos\left(\frac{3\pi x}{L}\right) - \frac{L}{\pi}\cos\left(\frac{\pi x}{L}\right)\right]_0^{\frac{L}{2}} \\
&= \frac{1}{2} - \frac{1}{L}\left(0 + \frac{2L}{3\pi}\right) \\
&= \frac{1}{2} - \frac{2}{3\pi}
\end{aligned}$$

And so, more succinctly, we have the following:

$$P\left(0 < X_{\psi_{1+2}(t)} < \frac{L}{2}\right) = \frac{1}{2} - \frac{2}{3\pi} < \frac{1}{2}$$

And so the probability that the particle is found within $[0, L/2]$ is less than 50%.