Set Theory HW #3

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Problem 1

Exercises 1,2,3,4,5 from pages 38-39 in the textbook.

Exercise 1: Suppose that we attempted to generalize the Kuratowski definitions of ordered pairs to ordered triples by defining:

$$\langle x, y, z\rangle^* = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

Show that this definition is unsuccessful by giving an example of objects u, v, w, x, y, z with $\langle x, y, z\rangle^* = \langle u, v, w\rangle^*$ but with either $y \neq v$ or $z \neq w$.

Solution: Consider the following 'triplet' under the proposed definition:

$$\begin{aligned}
\langle 1, 2, 1\rangle^* &= \{\{1\}, \{1, 2\}, \{1, 2, 1\}\} \\
&= \{\{1\}, \{1, 2\}, \{1, 2\}\} \\
&= \{\{1\}, \{1, 2\}, \{1, 2, 2\}\} \\
&= \langle 1, 2, 2\rangle^*
\end{aligned}$$

As we can see this triplet definition fails since the last elements of both triplets do not equal each other (i.e. $1 \neq 2$).

Exercise 2: Prove the following:

$$\begin{aligned}
&\textbf{a)}\ A \times (B \cup C) = (A \times B) \cup (A \times C) \\
&\textbf{b)}\ (A \times B = A \times C \ \&\ A \neq \varnothing) \implies B = C
\end{aligned}$$

Solution: For **a)** considering an arbitrary ordered pair $\langle x, y\rangle$ gives us following chain of logical equivalences:

$$\begin{aligned}
\langle x, y\rangle \in A \times (B \cup C) &\iff x \in A \ \&\ y \in B \cup C && \text{(def. of Cartesian product)} \\
&\iff x \in A \ \&\ (y \in B \text{ or } y \in C) && \text{(def. of union)} \\
&\iff (x \in A \ \&\ y \in B) \text{ or } (x \in A \ \&\ y \in C) && \text{(distributive prop. of } \& \text{)} \\
&\iff \langle x, y\rangle \in (A \times B) \text{ or } \langle x, y\rangle \in (A \times C) && \text{(def. of Cartesian product)} \\
&\iff \langle x, y\rangle \in (A \times B) \cup (A \times C) && \text{(def. of union)}
\end{aligned}$$

And so, by extensionality, we have that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

For **b)** consider an arbitrary set x:

$$\begin{aligned}
x \in B &\implies (\exists a \in A)\, \langle a, x\rangle \in A \times B && \text{(Assuming } A \neq \varnothing\text{)}\\
&\iff (\exists a \in A)\, \langle a, x\rangle \in A \times C && \text{(Assuming } A \times B = A \times C\text{)}\\
&\implies x \in C && \text{(def. of ordered pair)}
\end{aligned}$$

And so, by the definition of subset, $B \subseteq C$. A symmetric argument where B and C are switched gives us $C \subseteq B$ and so we have $B = C$.

Exercise 3: Prove the following:

$$A \times \bigcup B = \bigcup\{A \times X \mid X \in B\}$$

Solution: Consider an arbitrary ordered pair $\langle x, y\rangle$:

$$\begin{aligned}
\langle x, y\rangle \in A \times \bigcup B &\iff y \in \bigcup B \;\&\; x \in A && \text{(def. of ordered pair)}\\
&\iff (\exists X \in B)\, y \in X \;\&\; x \in A && \text{(def. of arbitrary union)}\\
&\iff (\exists X \in B)\, \langle x, y\rangle \in A \times X && \text{(def. of ordered pair)}\\
&\iff \langle x, y\rangle \in \bigcup\{A \times X \mid X \in B\} && \text{(def. of arbitrary union)}
\end{aligned}$$

And so by extensionality $A \times \bigcup B = \bigcup\{A \times X \mid X \in B\}$. Also note that since the LHS is a Cartesian product and the RHS is the union of Cartesian products, we are justified in denoting an arbitrary element of these sets in the form of an ordered pair $\langle x, y\rangle$ and using the axiom of extensionality.

Exercise 4: Show that there is no set to which every ordered pair belongs.

Solution: Let us assume that such a set P exists that contains all ordered pairs. Now let us consider an arbitrary x:

$$\begin{aligned}
&\langle x, x\rangle \in P && \text{(assumption)}\\
\implies &\{\{x\}\} \in P && \text{(def. of ordered pair)}\\
\implies &\{x\} \in \bigcup P && \text{(def. of arbitrary union)}\\
\implies &x \in \bigcup\bigcup P && \text{(def. of arbitrary union)}
\end{aligned}$$

And so we have shown that the set $\bigcup\bigcup P$ (which exists because the arbitrary union of any set exists) contains all sets. The existence of a set that contains all sets has already been shown to be a contradiction (Russel's paradox, etc.) and so our assumption that the set P exists was false.

Exercise 5: a) Assume A and B are given sets, and show that there exists a set C such that for any y:

$$y \in C \iff (\exists x)\, y = \{x\} \times B$$

In other words, show that the set $C = \{\{x\} \times B \mid x \in A\}$ exists. **b)** With A, B and C as above, show that $A \times B = \bigcup C$.

Solution: For **a)** consider an arbitrary set x:

$$\begin{aligned}
x \in A &\implies \Big[(\forall b \in B)\ x \in \{x\}\ \&\ b \in B \implies x \in A\ \&\ b \in B\Big] \\
&\implies \Big[(\forall b \in B)\ \langle x, b\rangle \in \{x\} \times B \implies \langle x, b\rangle \in A \times B\Big] && \text{(def. of ordered pair)} \\
&\implies \{x\} \times B \subset A \times B && \text{(def. of subset)} \\
&\implies \{x\} \times B \in \mathfrak{P}(A \times B) && \text{(def. of powerset)} \\
&\implies \{\{x\} \times B\} \subseteq \mathfrak{P}(A \times B) && \text{(def. of powerset)}
\end{aligned}$$

And so we have shown that the set $\{\{x\} \times B\}$ is a subset of $\mathfrak{P}(A \times B)$ for any $x \in A$. Since the union of subsets of a set is still a subset we have:

$$\{\{x\} \times B \mid x \in A\} \subseteq \mathfrak{P}(A \times B)$$

And since we know that 1) given sets A and B, their Cartesian product $A \times B$ exists and 2) the power set of any set exists due to the powerset axiom, the subset axiom implies that the set $C = \{\{x\} \times B \mid x \in A\}$ must exist as well.

For **b)** consider an arbitrary set c:

$$\begin{aligned}
c \in A \times B &\iff (\exists x \in A, \exists y \in B)\ c = \langle x, y\rangle && \text{(def. of Cartesian product)} \\
&\iff (\exists x \in A)\ c \in \{x\} \times B && (x \in \{x\}) \\
&\iff c \in \bigcup \{\{x\} \times B \mid x \in A\} && \text{(def. of arbitrary union)} \\
&\iff c \in \bigcup C && \text{(def. of C)}
\end{aligned}$$

And so for any set c we have $c \in A \times B \iff c \in C$ which, by extensionality, implies that $A \times B = \bigcup C$.

Problem 2

Consider the following theorem and its proof:

Theorem: If A, B are sets and $A \times B = B \times A$ then $A = B$.

Proof: If $x \in A$ and $y \in B$ then $\langle x, y\rangle \in A \times B$; since $A \times B = B \times A$, it follows that $\langle x, y\rangle \in B \times A$; so $x \in B$ and $y \in A$. This shows that if $x \in A$ then $x \in B$, so $A \subseteq B$, and that if $y \in B$ then $y \in A$, so $B \subseteq A$. Hence $A = B$. ■

As it turns out, the theorem, as stated, is false. And so the proof must be wrong.

Part i: Prove that the theorem is false, by giving a counterexample.

Solution: Consider $A = \varnothing$ and $B = \{\varnothing\}$, this gives us:

$$\begin{aligned}
A \times B &= \varnothing \times \{\varnothing\} \\
&= \varnothing \\
&= \{\varnothing\} \times \varnothing \\
&= B \times A
\end{aligned}$$

Yet we clearly have $A = \varnothing \neq \{\varnothing\} = B$. Thus, the theorem presented is false.

Part ii: Explain why the proof is wrong, that is, find the step or steps that are invalid.

Solution: We can rephrase the given proof in the following way: *Consider arbitrary sets x and y:*

$$\begin{aligned} x \in A \ \&\ y \in B &\implies \langle x, y\rangle \in A \times B && \text{(def. of ordered pair)} \\ &\implies \langle x, y\rangle \in B \times A && \text{(assume } A \times B = B \times A\text{)} \\ &\implies x \in B \ \&\ y \in A && \text{(def. of ordered pair)} \end{aligned}$$

And so we have shown that for arbitrary x and y, $x \in A$ implies that $x \in B$ and the revserse for y meaning $A \subseteq B$ and $B \subseteq A$ giving us $A = B$.

This proof, however, breaks down at the assumption that:

$$(\forall x, y) \quad x \in A \ \&\ y \in B \implies x \in B \ \&\ y \in A$$

entails the following:

$$(\forall x, y) \quad (x \in A \implies x \in B) \ \&\ (y \in B \implies y \in A)$$

Since, for instance, $x \in a \implies x \in B$ only if there exists a $y \in B$ for the ordered pair (x, y) to be created with. But this is not the case $B = \varnothing$. The same problem goes for the other direction when $A = \varnothing$.

Part iii: Fix the theorem, by adding an extra condition to the hypotheses of the theorem that makes it true.

Solution: The theorem should instead be stated as:

$$(A \times B = B \times A \ \&\ A \neq \emptyset \ \&\ B \neq \emptyset) \implies A = B$$

That is to say, the theorem given is correct if we add the assumption that neither A nor B is the empty set.

Part iv: Give a correct proof of the true theorem.

Solution: Consider an arbitrary set x:

$$\begin{aligned} & x \in A \ \&\ (\exists y \in B) && \text{(assume } B \neq \varnothing\text{)} \\ \implies & (\exists y \in B)\ \langle x, y\rangle \in A \times B && \text{(def. of ordered pair)} \\ \implies & (\exists y \in B)\ \langle x, y\rangle \in B \times A && \text{(assume } A \times B = B \times A\text{)} \\ \implies & x \in B \ \&\ (\exists y \in B)\ y \in A && \text{(def. of ordered pair)} \\ \implies & x \in B \end{aligned}$$

And so we have shown that, by assuming that $B \neq \varnothing$, that for an arbitrary set $x \in A \implies x \in B$ and thus $A \subset B$. A symmetric argument holds for the other direction under the assumption that $A \neq \varnothing$, giving us $B \subseteq A$. Along with the previous result, this implies that $A = B$.

As a side note, our theorem says nothing about the special case where $A = \varnothing = B$. In this case, we do indeed have $A \times B = \varnothing = B \times A$ and $A = B$.