Theory of Probability HW #8

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Problem 1

Solution: Letting X be the time the rabbit reaches the carrot and H_i be the event hole i is chosen, the expected time is given by:

$$\begin{aligned}
E[X] &= E[X \mid H_1]P(H_1) + E[X \mid H_2]P(H_2) + E[X \mid H_3]P(H_3) \\
&= \frac{2+E[X]}{3} + \frac{3+E[X]}{3} + \frac{4}{3} \\
3E[X] &= 9 + 2E[X] \\
E[X] &= 9
\end{aligned}$$

Problem 2

Solution: Letting X being the number of heads landed in 8 trials, X_1 be the number of heads landed in the first 4 trials, and X_2 the number of heads last 4 trials, we have:

$$\begin{aligned}
X &= X_1 + X_2 && \text{(convolution of binomial R.V.)} \\
X_1, X_2 &\sim B\left(4, \frac{2}{3}\right)
\end{aligned}$$

And so our desired expected value is given by:

$$\begin{aligned}
E[X] &= E[X_1 \mid X_1 = 3] + E[X_2] \\
&= E[X_1 \mid X_1 = 3] + 4 \cdot \frac{3}{4} && \text{(expectation of binomial R.V.)} \\
&= 3 + 3 = 6
\end{aligned}$$

Problem 3

Solution: First we compute the marginal distribution of Y:

$$\begin{aligned}
f_Y(y) &= \int_0^y f(x,y)\,dx \\
&= \int_0^y \frac{e^{-y}}{y}\,dx \\
&= \left[\frac{xe^{-y}}{y}\right]_0^y \\
&= \frac{ye^{-y}}{y} = e^{-y}
\end{aligned}$$

And so the conditional probability distribution is given by:

$$f_{X\,|\,Y}(x\,|\,y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{e^{-y}}{y}}{e^{-y}} = \frac{1}{y}$$

And so, finally, our desired expectation is given by:

$$\begin{aligned} E[X^5\,|\,Y=y] &= \int_{-\infty}^{\infty} x^5 f_{X\,|\,Y}(x\,|\,y)\,dx \\ &= \int_0^y \frac{x^5}{y}\,dx \\ &= \left[\frac{x^6}{6y}\right]_0^y = \frac{y^5}{6} \end{aligned}$$

Problem 4

Solution: We'd expect X and Y to be negatively correlated as, out of a pool of 3 rolls, more 1's rolled (i.e. X) mean less chances for 2's to be rolled (i.e. Y), and vice versa. First let us denote X and Y as the sum of 3 Bernoulli R.V.s rather than a single binomial R.V.:

$$\begin{aligned} X &= X_1 + X_2 + X_3 \\ Y &= Y_1 + Y_2 + Y_3 \\ X_i, Y_i &\sim \text{Bernoulli}\left(\frac{1}{6}\right) \end{aligned}$$

Compute the covariance we find:

$$\begin{aligned} Cov(X,Y) &= Cov\left(\sum_{i=1}^{3} X_i, \sum_{j=1}^{3} Y_j\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} Cov(X_i, Y_j) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} E[X_iY_j] - E[X_i]E[X_j] \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} E[X_iY_j] - \left(\frac{1}{6}\cdot\frac{1}{6}\right) && \text{(expectation of bernoulli R.V.)} \end{aligned}$$

Note that the expectation $E[X_iY_j] = E[X_i]E[Y_j]$ whenever $i \neq j$ as those represent independent trials. As such, the only case that doesn't cancel out are the 3 cases when $i = j$, giving us:

$$Cov(X,Y) = \sum_{i=1}^{3} E[X_iY_i] - \frac{1}{36}$$

However, note that if $X_i = 1$ then $Y_i = 0$ since for a 1 to be rolled, a 2 must *not* have been rolled. This applies in the other direction too. As such, we have $E[X_iY_i] = 0$, giving us a final covariance of:

$$Cov(X,Y) = \sum_{i=1}^{3} -\frac{1}{36} = -\frac{3}{36}$$

Problem 5

Solution: Part a) first we compute the marginal distribution of Y:

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\
&= \int_y^1 8xy\,dx \\
&= \left[4x^2y\right]_y^1 \\
&= 4y - 4y^3 = 4y(1-y^2)
\end{aligned}$$

Now we can compute the conditional distribution:

$$\begin{aligned}
f_{X\,|\,Y}(x\,|\,y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{8xy}{4y(1-y^2)} = \frac{2x}{1-y^2}
\end{aligned}$$

To show that these distributions are dependent, we'll compute the marginal distribution of X:

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \int_0^x 8xy\,dy \\
&= \left[4xy^2\right]_0^x \\
&= 4x^3
\end{aligned}$$

Clearly, X and Y are not independent as evidenced by the following:

$$\begin{aligned}
f_{X\,|\,Y}(x\,|\,y) &\neq f_X(x) \\
\frac{2x}{1-y^2} &\neq 4x^3
\end{aligned}$$

For part b) we **cannot** say whether the covariance is 0 or not. This is because, while the independence of two random variables implies a covariance of 0, the converse is not true. It may be the case that X and Y have a covariance of 0 despite being dependent.

Problem 6

Solution: The expectation of Y is given by:

$$\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\,dy \\
&= \int_0^1 4y^2 - 4y^4\,dy \\
&= \left[\frac{4y^3}{3} - \frac{4y^5}{5}\right]_0^1 \\
&= \frac{4}{3} - \frac{4}{5} = \frac{8}{15}
\end{aligned}$$

The expectation of X is given by:

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dy \\
&= \int_0^1 4x^4\, dy \\
&= \left[\frac{4x^5}{5}\right]_0^1 \\
&= \frac{4}{5}
\end{aligned}
$$

Finally we compute the following joint expectation:

$$
\begin{aligned}
E[XY] &= \iint\limits_{(x,y)\in\mathbb{R}^2} xyf(x,y)\, dy\, dx \\
&= \int_0^1 \int_0^x 8x^2y^2\, dy\, dx \\
&= \int_0^1 \left[\frac{8x^2y^3}{3}\right]_0^x dx \\
&= \int_0^1 \frac{8x^5}{3}\, dx \\
&= \int_0^1 \frac{8x^5}{3}\, dx \\
&= \left[\frac{4x^6}{9}\right]_0^1 = \frac{4}{9}
\end{aligned}
$$

And so, finally, our covariance is given by:

$$
\begin{aligned}
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= \frac{4}{9} - \frac{4}{5} \cdot \frac{8}{15} \\
&= \frac{4}{225}
\end{aligned}
$$