Math Statistics
Weekly HW 2

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September 17, 2020

Question 1

Problem: If a random sample of size 100 is taken from a normally distributed population X with $\sigma^2 = 40$, what is the probability that the sample mean will differ from the population mean by more than 1?

Solution: Note that since each sample is normally distributed, so to must the sample mean. As such we have the following:

$$\begin{aligned}
P(|\bar{X} - \mu_X| > 1) &= P(|\bar{X} - \mu_{\bar{X}}| > 1) && (\mu_{\bar{X}} = \mu_X) \\
&= P\left(\left|\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}\right| > \frac{1}{\sigma_{\bar{X}}}\right) \\
&= P\left(\left|\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}\right| > \frac{1}{\sigma_X/\sqrt{n}}\right) && (\sigma_{\bar{X}} = \sigma_X/\sqrt{n}) \\
&= P\left(|Z| > \frac{1}{\sigma_X/\sqrt{n}}\right) && \text{(standard normal RV)} \\
&= P\left(|Z| > \frac{5}{\sqrt{10}}\right) \\
&= P\left(Z > \frac{5}{\sqrt{10}}\right) + P\left(Z < -\frac{5}{\sqrt{10}}\right) \\
&= 1 - P\left(Z \leq \frac{5}{\sqrt{10}}\right) + P\left(Z < -\frac{5}{\sqrt{10}}\right) && \text{(complement)} \\
&= 1 - \Phi\left(\frac{5}{\sqrt{10}}\right) + \Phi\left(-\frac{5}{\sqrt{10}}\right) && \text{(cdf of standard normal RV)} \\
&= 2\Phi\left(-\frac{5}{\sqrt{10}}\right) && (1 - \Phi(z) = \Phi(-z)) \\
&\approx 2(0.0569231) = \boxed{0.1138461} && \text{(standard table)}
\end{aligned}$$

Question 2

Problem: Suppose a random sample of size 100 is taken from a normally distributed population with $\sigma^2 = 40$. Find a number a such that the probability is 9/10 that the sample mean will differ from the population mean by no more than a. In other words, find a so that $P(|\bar{X} - \mu| < a) = 9/10$.

Solution: Using a similar argument to question 1, we have:

$$\begin{aligned}
\frac{9}{10} &= P(|\bar{X} - \mu_X| < a) \\
&= P\left(\left|\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}\right| < \frac{a}{\sigma_X/\sqrt{n}}\right) && \text{(see Q1)} \\
&= P\left(|Z| < \frac{5a}{\sqrt{10}}\right) && \text{(standard normal RV)} \\
&= P\left(-\frac{5a}{\sqrt{10}} < Z < \frac{5a}{\sqrt{10}}\right) \\
&= \Phi\left(\frac{5a}{\sqrt{10}}\right) - \Phi\left(-\frac{5a}{\sqrt{10}}\right) && \text{(cdf of standard normal RV)} \\
&= \Phi\left(\frac{5a}{\sqrt{10}}\right) - \left(1 - \Phi\left(\frac{5a}{\sqrt{10}}\right)\right) && (1 - \Phi(z) = \Phi(-z)) \\
\frac{19}{20} &= \Phi\left(\frac{5a}{\sqrt{10}}\right) \\
\Phi^{-1}\left(\frac{19}{20}\right) &= \frac{5a}{\sqrt{10}}
\end{aligned}$$

And so we have that:

$$a = \frac{\sqrt{10}}{5}\Phi^{-1}\left(\frac{19}{20}\right) \approx \boxed{1.0403}$$