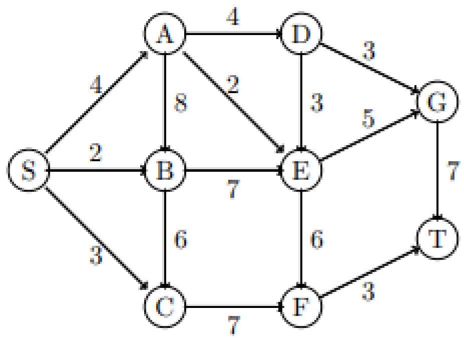
Algorithms Quiz #4-5

Ozaner Hansha

April 28, 2020

Problem 1

For the following questions consider the following directed graph:

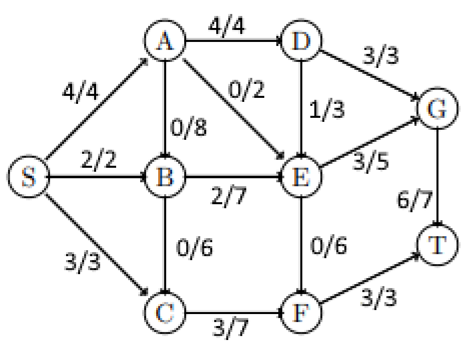


Part a: Find the maximum flow from S to T by inspection.

Solution: The maximum flow from S to T is 9. Note that there is only $4+2+3=9$ capacity across all the outgoing edges of S, which means the flow cannot be any higher than 9. This combined with the flow of value 9 given in part b implies that 9 must be the maximum.

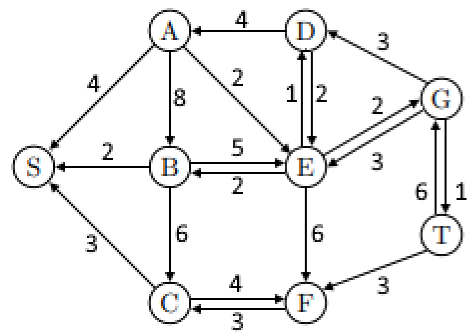
Part b: Show the amount of flow along each edge.

Solution: Below is a graph showing the flow along each edge for an optimal solution:



Part c: Give the residual graph corresponding to the optimal flow.

Solution: Below is the corresponding residual graph of the optimal solution given in part b:

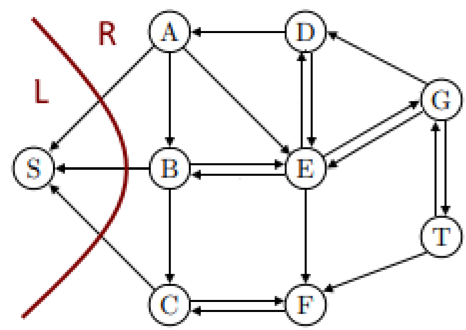


Part d: Give the minimum (S,T) cut from the residual graph in part c.

Solution: The minimum cut is given by the following partition of V:

$$L = \{S\}$$
$$R = \{A, B, C, D, E, F, G, T\}$$

Where L is the set of vertices reachable from S in the residual graph, and $R = L^{\mathsf{C}}$. We can represent this graphically:



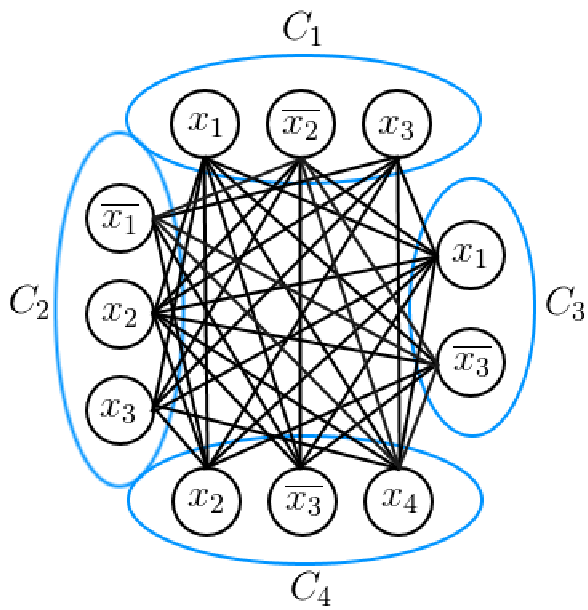
Problem 2

Problem: Convert the following CNF into a graph $G = (V, E)$ such that G has a clique of size 4 iff the CNF is satisfiable:

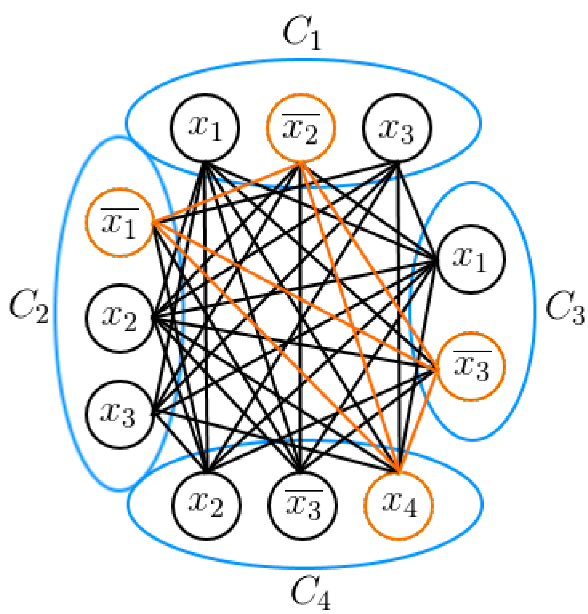
$$\underbrace{(x_1 \vee \overline{x_2} \vee x_3)}_{C_1} \wedge \underbrace{(\overline{x_1} \vee x_2 \vee x_3)}_{C_2} \wedge \underbrace{(x_1 \vee \overline{x_3})}_{C_3} \wedge \underbrace{(x_2 \vee \overline{x_3} \vee x_4)}_{C_4}$$

Does G have a clique of size 4?

Solution: In the context of the clique problem, the given CNF corresponds to the following graph:



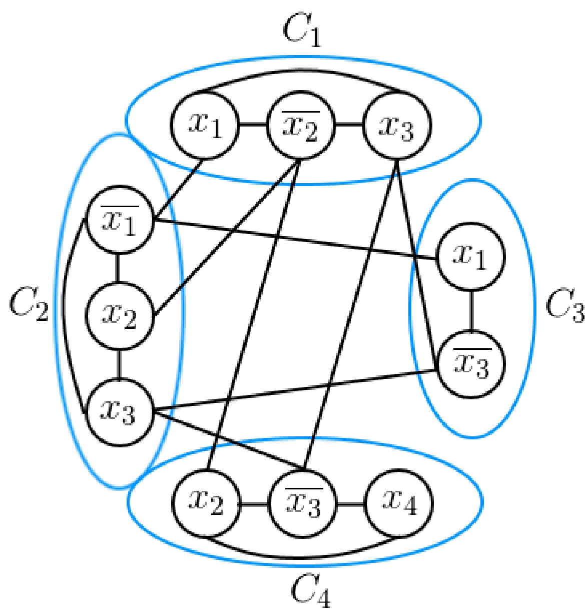
You'll note that many cliques of size 4 exist. We give one such clique below, highlighted in orange:



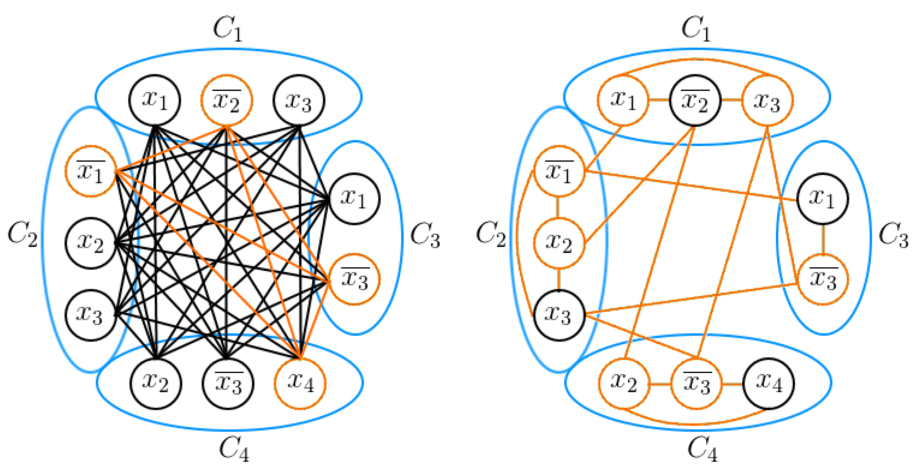
Problem 3

Problem: For the graph G from problem 2, construct the complementary graph $\overline{G} = (V, \overline{E})$. What is the largest clique in G? What is the smallest vertex-cover in $\overline{G}$?

Solution: The complement $\overline{G}$ of the graph given in problem 2 is given below:



A maximal clique in G of size 4 and a minimal vertex-cover in $\overline{G}$ of size 7 are both given below:



To prove that 4 really is the maximal clique size of G, we simply have to note that no two vertices are connected within a clause. As such, no clique can have more than 1 member per clause, totalling a maximum of 4 members. As we have provided an example of such a a clique, this is indeed the maximum clique size.

To prove that 7 really is the minimal vertex-cover size of $\overline{G}$, we simply recall the following theorem:

- if a graph G has a clique of size k, then there exists a vertex-cover of size $|V| - k$ on $\overline{G}$
- likewise, if a graph G has a vertex-cover of size k, then there exists a clique of size $|V| - k$ on $\overline{G}$

And so because G has a clique of 4, there must exist a vertex cover of at least $11 - 4 = 7$ on $\overline{G}$. Moreover, there cannot exist a smaller vertex-cover. For example, if a vertex-cover of size 6 existed on $\overline{G}$, then the theorem states there should exist a clique of size $11 - 6 = 5$ on G which, as we have already established, is impossible.