Theory of Probability HW #2

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Part A

Problems taken from Chapters 2 and 3 of the textbook.

<u>Chapter 2</u>

Problem 31a

Problem: A 3-person basketball team consists of a guard, a forward, and a center. If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team? *(use the inclusion-exclusion principle instead of calculating directly)*

Solution: Let G denote the chosen team has 1 guard, F denote that it was onw forward, and C denote that it has one center. Then the probability we seek is $P(GFC)$. Note that we can express this in terms of the probability of the union of events:

$$\begin{aligned}
P(GFC) &= P(((GFC)^{\complement})^{\complement}) && \text{(involutory property)}\\
&= P((G^{\complement} \cup F^{\complement} \cup C^{\complement})^{\complement}) && \text{(DeMorgan's law)}\\
&= 1 - P(G^{\complement} \cup F^{\complement} \cup C^{\complement}) && \text{(complement of event)}
\end{aligned}$$

Via the inclusion-exclusion principle for $n = 3$ we have:

$$\begin{aligned}
P(G^{\complement} \cup F^{\complement} \cup C^{\complement}) &= P(G^{\complement}) + P(F^{\complement}) + P(C^{\complement})\\
&\quad - P(G^{\complement}F^{\complement}) - P(G^{\complement}C^{\complement}) - P(F^{\complement}C^{\complement})\\
&\quad + P(G^{\complement}F^{\complement}C^{\complement})
\end{aligned}$$

Now note that the probability that any particular position is not chosen from any particular team is $\frac{2}{3}$ since there are 2 valid options out of the 3 members. Since there are 3 teams the principle of counting gives us:

$$P(G^{\complement}) = P(F^{\complement}) = P(C^{\complement}) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2^3}{3^3}$$

In a similar vain, the probability that any 2 different positions are not chosen from any particular team is $\frac{1}{3}$ since there is only 1 valid option out of the 3 members. Since there are 3 teams the principle of counting gives us:

$$P(G^{\complement}F^{\complement}) = P(G^{\complement}C^{\complement}) = P(F^{\complement}C^{\complement}) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3^3}$$

And taken to the extreme, the probability that none of the three positions are chosen from any particular team is 0 since each member has a position and at least 1 member from each team must be chosen:

$$P(G^{\complement}F^{\complement}C^{\complement}) = 0 \cdot 0 \cdot 0 = 0$$

Plugging this into the inclusion-exclusion principle we have:

$$\begin{aligned} P(G^{\complement} \cup F^{\complement} \cup C^{\complement}) &= \frac{2^3}{3^3} + \frac{2^3}{3^3} + \frac{2^3}{3^3} - \frac{1}{3^3} - \frac{1}{3^3} - \frac{1}{3^3} + 0 \\ &= \frac{3 \cdot 2^3}{3^3} - \frac{3}{3^3} = \frac{7}{9} \end{aligned}$$

And so our desired probability is given by:

$$P(GFC) = 1 - P(G^{\complement} \cup F^{\complement} \cup C^{\complement}) = 1 - \frac{7}{9} = \frac{2}{9}$$

Problem 45

Problem: A woman has n keys, of which only 1 will open her door. **a)** If she tries keys at random, discarding those that do not work, what's the probability she will open the door on her kth try? **b)** What if she doesn't discard the keys?

Solution: Let E_k be the event she opens the door on the kth key. **a)** Note that the first try has a $\frac{1}{n}$ chance of being correct, the second has a $P(E_1^{\complement})\frac{1}{n-1}$ chance since the first try had to fail and since 1 key has been discarded, and so on. This pattern gives us:

$$\begin{aligned} P(E_1) &= \frac{1}{n} \\ P(E_2) &= \underbrace{\frac{n-1}{n}}_{P(E_1^{\complement})} \cdot \frac{1}{n-1} \\ P(E_3) &= \underbrace{\frac{n-1}{n}}_{P(E_1^{\complement})} \cdot \underbrace{\frac{n-2}{n-1}}_{P(E_2^{\complement})} \cdot \frac{1}{n-2} \\ &\vdots \\ P(E_k) &= \underbrace{\frac{n-1}{n}}_{P(E_1^{\complement})} \cdot \underbrace{\frac{n-2}{n-1}}_{P(E_2^{\complement})} \cdots \underbrace{\frac{n-k+1}{n-k+2}}_{P(E_{k-1}^{\complement})} \cdot \frac{1}{n-k+1} \end{aligned}$$

It is quite plain to see that this is a finite telescoping product. Canceling out the like factors, we are left with the very simple:

$$P(E_k) = \frac{1}{n}$$

b) If she doesn't discard the keys, then each try is independent of any other. And so the probability of her opening the door on the kth try is the probability that she failed the first $k-1$ tries then succeeded kth one:

$$P(E_k) = \left(\frac{n-1}{n}\right)^{k-1} \frac{1}{n}$$

Problem 55a

Problem: What's the probability that a hand of 13 cards contains both ace and king of at least one suit?

Solution: Let $E_\heartsuit$ denote the event that our hand has both an ace and king of hearts, $E_\spadesuit$ denote the event that our hand has both an ace and king of diamonds, and so on. Thus, we desire the probability of the following event $E_\heartsuit \cup E_\spadesuit \cup E_\diamondsuit \cup E_\clubsuit$. The cardinality of this event is given by the inclusion-exclusion principle:

$$|E_\heartsuit \cup E_\spadesuit \cup E_\diamondsuit \cup E_\clubsuit| = \left| \bigcup_{i \in \{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\}} E_i \right| = \sum_{k=1}^{4} \left((-1)^{k-1} \sum_{\substack{I \subseteq \{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\} \\ |I|=k}} \left| \bigcap_{i \in I} E_i \right| \right)$$

Now note that each suit we have determines 2 of the cards in our hand. For example, $|E_\heartsuit| = \binom{52-2}{13-2}$ since 2 of the cards have been chosen for us. Similarly, we have $|E_\heartsuit E_\spadesuit| = \binom{52-4}{13-4}$ since 4 of the cards (2 from hearts and 2 from spades) have been chosen for us. Generalizing this, we have for $|I| = k$:

$$\left| \bigcap_{i \in I} E_i \right| = \binom{52-2k}{13-2k}$$

Plugging this into our expression for the inclusion-exclusion principle gives us:

$$\left| \bigcup_{i \in \{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\}} E_i \right| = \sum_{k=1}^{4} \left((-1)^{k-1} \sum_{\substack{I \subseteq \{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\} \\ |I|=k}} \binom{52-2k}{13-2k} \right)$$

Now, noting that there are $\binom{4}{k}$ k-combinations of the set $\{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\}$, we have:

$$\left| \bigcup_{i \in \{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\}} E_i \right| = \sum_{k=1}^{4} \left((-1)^{k-1} \binom{4}{k} \binom{52-2k}{13-2k} \right) = 139565328072$$

We have calculated the number of hands with at least 1 pair of an ace and heart of a single suit. And because this is a discrete uniform distribution, i.e. each hand is equally likely, the probability of drawing such a hand is simply:

$$P(E_\heartsuit \cup E_\spadesuit \cup E_\diamondsuit \cup E_\clubsuit) = \frac{|E_\heartsuit \cup E_\spadesuit \cup E_\diamondsuit \cup E_\clubsuit|}{|\Omega|} = \frac{139565328072}{\binom{52}{13}} = \frac{9895443}{45023650} \approx 0.219783$$

Chapter 3

Problem 12

Problem: Suppose distinct values are written on each of 3 cards, which are then randomly named a, b, and c. Given that $a < b$ find the probability that $a < c$.

Solution: Note that we can consider the sample space Ω of this experiment the 3! passive permutations of $\{a, b, c\}$:

$$\Omega = \{(a, b, c), (a, c, b), (b, a, c), \cdots\}$$

The event $E_{a<b}$ is given by the set of all triplets in Ω in which a precedes b:

$$E_{a<b} = \{(a,b,c),(a,c,b),(c,a,b)\}$$

The event $E_{a<c}$ is given by the set of all triplets in Ω in which a precedes c:

$$E_{a<c} = \{(a,b,c),(a,c,b),(b,a,c)\}$$

The intersection of these two events $E_{a<b}E_{a<c}$ is thus given by:

$$E_{a<b}E_{a<c} = \{(a,b,c),(a,c,b)\}$$

And now, noting that this is a discrete uniform distribution, we can finally calculate the desired conditional probability of $a < c$ assuming $a < b$:

$$P(E_{a<c}|E_{a<b}) = \frac{|E_{a<b}E_{a<c}|}{|E_{a<b}|} = \frac{2}{3}$$

Problem 30

Problem: Suppose that a deck of 52 cards is shuffled and the cards are then turned over one at a time until the first ace appears. Given that the first ace is the 20th card, what is the conditional probability that the card following it is **a)** the ace of spades? **b)** the two of clubs?

Solution: Let A denote the event that the first ace drawn is the 20th card, B denote that the 21st card is the ace of spades, and C denote that the 21st card is the two of clubs. The cardinality of A is given by:

$$|A| = \overbrace{\underbrace{48}_{\text{c1}} \cdot \underbrace{47}_{\text{c2}} \cdot \underbrace{46}_{\text{c3}} \cdots \underbrace{30}_{\text{c19}}}^{\text{non-ace cards}} \cdot \overbrace{\underbrace{4}_{\text{c20}}}^{\text{ace cards}} \cdot \overbrace{\underbrace{32!}_{\text{c21-c52}}}^{\text{all other cards}} = 48^{\underline{19}} \cdot 4 \cdot 32!$$

Via similar reasoning, the cardinality of BA is given by:

$$|BA| = \overbrace{\underbrace{48}_{\text{c1}} \cdot \underbrace{47}_{\text{c2}} \cdot \underbrace{46}_{\text{c3}} \cdots \underbrace{30}_{\text{c19}}}^{\text{non-ace cards}} \cdot \overbrace{\underbrace{3}_{\text{c21}}}^{\text{aces of } \diamondsuit,\clubsuit,\heartsuit} \cdot \overbrace{\underbrace{1}_{\text{c21}}}^{\text{ace of } \spadesuit} \cdot \overbrace{\underbrace{31!}_{\text{c22-c52}}}^{\text{all other cards}} = 48^{\underline{19}} \cdot 3 \cdot 1 \cdot 31!$$

Yet again via similar reasoning, the cardinality of CA is given by:

$$|CA| = \overbrace{\underbrace{47}_{\text{c1}} \cdot \underbrace{46}_{\text{c2}} \cdot \underbrace{45}_{\text{c3}} \cdots \underbrace{29}_{\text{c19}}}^{\text{non-ace/two of } \clubsuit \text{ cards}} \cdot \overbrace{\underbrace{4}_{\text{c20}}}^{\text{ace cards}} \cdot \overbrace{\underbrace{1}_{\text{c21}}}^{\text{two of } \clubsuit} \cdot \overbrace{\underbrace{31!}_{\text{c22-c52}}}^{\text{all other cards}} = 47^{\underline{19}} \cdot 4 \cdot 31!$$

The probabilities we seek are **a)** $P(B|A)$ and **b)** $P(C|A)$ and since this is a discrete uniform distribution, we have:

$$P(B|A) = \frac{|BA|}{|A|} = \frac{48^{\underline{19}} \cdot 3 \cdot 1 \cdot 31!}{48^{\underline{19}} \cdot 4 \cdot 32!} = \frac{3}{4 \cdot 32} = \frac{3}{128}$$

$$P(C|A) = \frac{|CA|}{|A|} = \frac{47^{\underline{19}} \cdot 4 \cdot 31!}{48^{\underline{19}} \cdot 4 \cdot 32!} = \frac{29}{48 \cdot 32} = \frac{29}{1536}$$

Problem 35

Problem: On rainy days Joe is late to work with probability 0.3, and on nonrainy days he is late with probability 0.1. It will rain tomorrow with probability 0.7. **a)** Find the probability that Joe is early tomorrow. **b)** Given that Joe is early, what is the conditional probability that it rained?

Solution: Let L denote the event that joe is late, and R denote the event that it is raining. The problem statement and the complement rule gives us the following probabilities:

$$\begin{aligned} P(R) &= 0.7 \quad P(R^{\mathsf{C}}) = 0.3 \\ P(L|R) &= 0.3 \quad P(L|R^{\mathsf{C}}) = 0.1 \end{aligned}$$

Now note that via the law of total probability, $P(L)$ is given by:

$$\begin{aligned} P(L) &= P(L|R)P(R) + P(L|R^{\mathsf{C}})P(R^{\mathsf{C}}) \\ &= 0.3 \cdot 0.7 + 0.1 \cdot 0.3 = 0.24 \end{aligned}$$

And so for **a)** the probability that Joe is early (i.e. not late) is given by:

$$P(L^{\mathsf{C}}) = 1 - P(L) = 1 - 0.24 = 0.76$$

For **b)** the desired probability is given by $P(R|L^{\mathsf{C}})$. We can solve for this probability via Bayes' theorem:

$$\begin{aligned} P(R|L^{\mathsf{C}}) &= \frac{P(L^{\mathsf{C}}|R)P(R)}{P(L^{\mathsf{C}})} && \text{(Bayes' theorem)} \\ &= \frac{(1 - P(L|R))P(R)}{P(L^{\mathsf{C}})} && \text{(complement of conditional)} \\ &= \frac{(1 - 0.3)0.7}{0.76} = \frac{49}{76} \approx 0.644734 \end{aligned}$$

Problem 47a

Problem: There is a 30% chance that A can fix her busted computer. If A cannot, then there is a 40% chance that her friend B can fix it. Find the probability it will be fixed by either A or B.

Solution: Let A denote the probability that she fixes the computer, and B denote that the friend fixes the computer. Thus the problem statement gives us:

$$P(A) = 0.3 \quad P(B|A^{\mathsf{C}}) = 0.4$$

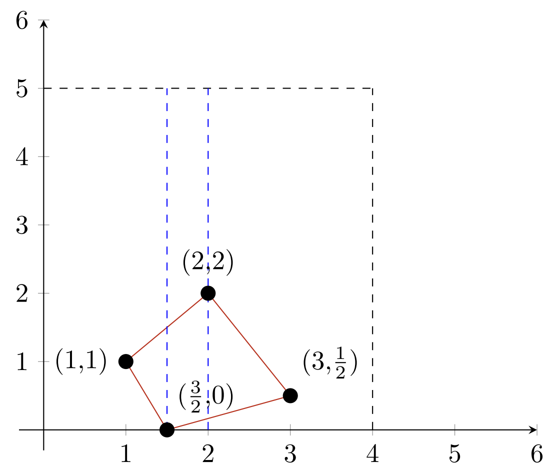
Now, note that by Bayes' theorem we have the following chain of equalities:

$$\begin{aligned}
P(B|A^{\complement}) &= \frac{P(A^{\complement}|B)P(B)}{P(A^{\complement})} && \text{(Bayes' theorem)}\\
0.4 &= \frac{P(A^{\complement}|B)P(B)}{1-0.3}\\
0.4 \cdot 0.7 &= P(A^{\complement}|B)P(B)\\
0.28 &= (1-P(A|B))P(B) && \text{(complement of conditional)}\\
0.28 &= P(B) - P(A|B)P(B)\\
0.28 &= P(B) - P(AB) && \text{(def. of conditional probability)}
\end{aligned}$$

Now, via the inclusion-exclusion principle for $n = 2$, we can solve for our desired probability, that either A or B fix the computer, $P(A \cup B)$:

$$P(A \cup B) = P(A) + P(B) - P(AB) = 0.3 + 0.28 = 0.58$$

Part B



Problem a

Problem: Express the area in the red bounded region E as a sum of integrals.

Solution: As we can see, the red bounded region can be expressed as the 3 regions split up by the blue dashed lines. The functions of each side, starting with the top left line and going clockwise, are given by:

$$\begin{aligned}
y &= x\\
y &= \frac{-3x}{2} + 5\\
y &= \frac{x}{3} - \frac{1}{2}\\
y &= -2x + 3
\end{aligned}$$

And so we can express our integral sum like so:

$$\int_{1}^{\frac{3}{2}}\int_{-2x+3}^{x} dy\,dx+\int_{\frac{3}{2}}^{2}\int_{\frac{x}{3}-\frac{1}{2}}^{x} dy\,dx+\int_{2}^{3}\int_{\frac{x}{3}-\frac{1}{2}}^{\frac{-3x}{2}+5} dy\,dx$$

Problem b

Problem: Compute the area of E.

Solution: Computing the integral sum, we have:

$$\begin{aligned}
&\int_{1}^{\frac{3}{2}}\int_{-2x+3}^{x} dy\,dx+\int_{\frac{3}{2}}^{2}\int_{\frac{x}{3}-\frac{1}{2}}^{x} dy\,dx+\int_{2}^{3}\int_{\frac{x}{3}-\frac{1}{2}}^{\frac{-3x}{2}+5} dy\,dx\\
&=\int_{1}^{\frac{3}{2}} [y]_{-2x+3}^{x}\,dx+\int_{\frac{3}{2}}^{2} [y]_{\frac{x}{3}-\frac{1}{2}}^{x}\,dx+\int_{2}^{3} [y]_{\frac{x}{3}-\frac{1}{2}}^{\frac{-3x}{2}+5}\,dx\\
&=\int_{1}^{\frac{3}{2}} 3x-3\,dx+\int_{\frac{3}{2}}^{2} \frac{2x}{3}+\frac{1}{2}\,dx+\int_{2}^{3} \frac{-11x}{6}+\frac{11}{2}\,dx\\
&=\left[\frac{3x^2}{2}-3x\right]_1^{\frac{3}{2}}+\left[\frac{2x^2}{6}+\frac{x}{2}\right]_2^{\frac{3}{2}}+\left[\frac{-11x^2}{12}+\frac{11x}{2}\right]_2^{3}\\
&=\frac{3}{8}+\frac{5}{6}+\frac{11}{12}=\frac{17}{8}
\end{aligned}$$

Problem c

Problem: Suppose the rectangle R (demarcated by the black dashed lines) represents the backyard of a house where a drone is trying to drop a (very small) parcel, and E represents a swimming pool in the backyard. If the drone drops the parcel in R at random, what is the probability that the parcel will not fall in the swimming pool?

Solution: Assuming the parcel is point-like and that the area R represents a uniform distribution, we can imagine that the probability the parcel lands in the pool $P(L)$ is given by:

$$P(L)=\frac{\text{Area of } E}{\text{Area of } R}=\frac{17}{8\cdot 20}=\frac{17}{160}$$

And so the probability it doesn't fall in is simply the complement:

$$P(L^{\complement})=1-P(L)=1-\frac{17}{160}=\frac{143}{160}$$