Intro to Real Analysis
HW #2

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Problem 1

Problem: Use Euclid's lemma to show that there are no rational numbers x such that $x^3 = 2021$.

Solution: Before we prove this, we must first prove 3 lemmas.

- **Lemma 1**: For $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ we have:

$$\begin{aligned} a_1 \mid b_1 \wedge a_2 \mid b_2 &\implies (\exists k_1, k_2 \in Z),\ \frac{b_1}{a_1} = k_1 \wedge \frac{b_2}{a_2} = k_2 && \text{(def. of divisibility)} \\ &\implies (\exists k_1, k_2 \in Z),\ \frac{b_1 b_2}{a_1 a_2} = k_1 k_2 \\ &\implies a_1 a_2 \mid b_1 b_2 && \text{($\mathbb{Z}$ closed under multiplication)} \end{aligned}$$

- **Lemma 2**: For all prime numbers p and integers m:

$$\begin{aligned} p \mid m^3 &\implies p \mid m^2 \vee p \mid m && \text{(Euclid's lemma)} \\ &\implies (p \mid m \vee p \mid m) \vee p \mid m && \text{(Euclid's lemma)} \\ &\implies p \mid m \\ &\implies p \mid m^2 && \text{(lemma 1, $1 \mid m$)} \\ &\implies p \mid m^3 && \text{(lemma 1, $1 \mid m$)} \end{aligned}$$

  And so by a chain of implications we have: $p \mid m \iff p \mid m^2 \iff p \mid m^3$.

- **Lemma 3**: For all $k \in \mathbb{Z}$:

$$\begin{aligned} a \mid b &\iff \exists c \in \mathbb{Z},\ \frac{b}{a} = c && \text{(def. of divisible)} \\ &\iff \exists c \in \mathbb{Z},\ \frac{kb}{ka} = c && \text{(common factor)} \\ &\implies ka \mid kb \end{aligned}$$

We can now finally prove the desired statement:

$$
\begin{aligned}
2021 &= x^3 \\
&= \left(\frac{m}{n}\right)^3 \text{ where } m, n \text{ are relatively prime} && (\text{assume } x \in \mathbb{Q}) \\
&= \frac{m^3}{n^3} \\
2021n^3 &= m^3 \\
47 \cdot 43n^2 &= m^3 && (\text{prime factorization of } 2021) \\
&\implies 47 \mid 2021n^3 && (47 \text{ is a factor of } 2021) \\
&\implies 47 \mid m^3 && (2021n^3 = m^3) \\
&\implies \boxed{47 \mid m} && (\text{lemma } 2) \\
&\implies 47^2 \mid m^2 && (\text{lemma 1, with } 47 \mid m) \\
&\implies 47^2 \mid m^3 && (\text{lemma 1, with } 1 \mid m) \\
&\implies 47^2 \mid 47 \cdot 43n^3 && (47 \cdot 43n^3 = m^3) \\
&\implies 47 \mid 43n^3 && (\text{lemma 3, } k = 47) \\
&\implies 47 \mid n^3 && (\text{Euclid's lemma, } 47 \nmid 43) \\
&\implies \boxed{47 \mid n} && (\text{lemma } 2)
\end{aligned}
$$

Taking notice of the lines highlighted in blue, we can see that in assuming that $x \in Q$ and thus equals m/n with n and m relatively prime, we have found a contradiction. Namely that 47 divides both n and m despite them being relatively prime as previously mentioned. This means our assumption that x was rational is wrong and thus we have proven by contradiction that there is no rational x such that $x^3 = 2021$.

Problem 2

Problem: Prove the following:

$$(\forall y \in \mathbb{R}, \exists x \in \mathbb{R})\ x^3 = y$$

Solution:

CASE 1: Consider the case where $y > 0$. Now, consider the following set:

$$S = \{s \in \mathbb{R} \mid s^3 < y\}$$

This set is clearly bounded by y, and is nonempty (e.g. $-1 < 0 < y$ thus $-1 \in S$). These two facts, by the L.U.B property, imply that S has a supremum, call it x:

$$\sup S = x$$

We will now prove, by contradiction, that $x^3 = y$.

- if $x^3 < y$ then, by the density of the rationals, there must exist some $\epsilon \in (0, 1)$ such that:

  $$x^3 < (x + \epsilon)^3 < y$$

  However, this implies that $\sup S = x$ is less than another element in S, namely $x + \epsilon$. This is a contradiction of the LUB property and thus our assumption was wrong and $x^3 \not< y$.

- if $x^3 > y$ then, by the density of the rationals, there must exist some $\epsilon \in (0, 1)$ such that:

  $$x^3 > (x + \epsilon)^3 > y$$

  However, this implies that $\sup S = x$ is less than another element in S, namely $x + \epsilon$. This is a contradiction of the LUB property and thus our assumption was wrong and $x^3 \not> y$.

Putting these two together we have, by the law of trichotomy, that $x^3 = y$. And so every positive real y has a cube root x given by the supremum of the set S.

CASE 2: Now consider the case where $y < 0$. We know from case 1 that there exists a real number x such that $x^3 = -y$. Now consider the following:

$$(-x)^3 = -x^3 = -(-y) = y$$

And so, by case 1, every negative real y has a cube root given by the $-x$ above.

CASE 3: The only other case to consider is $y = 0$. This is simple as $0^3 = 0 = y$. And so 0 is its own cube root.

Proof: Since all real numbers y must either be 0, greater than 0, or less than 0, and since we proved that y has a cube root in all those cases, we can now be sure that all real numbers have a cube root.

Problem 3

Problem: Find a bijection from $(0, 1)$ to $[-2021, 2021]$.

Solution: First let us define a sequence $(y_n)_{n\in\mathbb{N}}$:

$$y_n = .1^{n+1}$$

Now let us define a bijective function $f_1 : (0, 1) \to [0, 1]$:

$$f_1(x) = \begin{cases} 0, & x = y_0 \\ 1, & x = y_1 \\ y_{n+2}, & x = y_n,\ n > 1 \\ x, & \text{otherwise} \end{cases}$$

Finally we can define our bijection $f_2 : (0, 1) \to [-2021, 2021]$:

$$f_2(x) = 2021(2f_1(x) - 1)$$

Problem 4

Problem: Find two bounded sets A and B such that $A \cap B = \emptyset$ and $\sup A = \sup B$.

Solution: Consider the following sets:

$$\begin{aligned} A &= \{0, 1\} \\ B &= \{1 - .1^n \mid n \in \mathbb{Z}^+\} \end{aligned}$$

First note that $A \cap B = \emptyset$ as there is no $n \in \mathbb{Z}^+$ such that $1 - .1^n$ equals 0 or 1. For 0 this is obvious. For 1, while $1 - .1^0 = 1$, we still have that $0 \notin \mathbb{Z}+$ and so $1 \notin B$.

Now note that the supremum of A is 1 as it has only two elements and $0 < 1$. For B note that since it is bounded and has no single largest element, we can find its supremum with the following limit:

$$\begin{aligned} \sup B &= \lim_{n\to\infty} 1 - .1^n \\ &= 1 - \lim_{n\to\infty} .1^n \\ &= 1 - 0 = 1 \end{aligned}$$

And so we have that $\sup A = \sup B = 1$, despite $A \cap B = \emptyset$.

Problem 5

Consider a bounded set $A \subseteq R$.

Problem a: For $B = \{x + 1 \mid x \in A\}$, show that $\sup B = (\sup A) + 1$.

Solution: Consider the following:

$$\begin{aligned}
&\sup A = \sup A \\
(\forall r_1 \in \mathbb{R}, \forall a \in A)\ &\sup A \geq a \wedge \neg(\sup A > r_1 \geq a) && \text{(def. of supremum)} \\
(\forall r_1 \in \mathbb{R}, \forall a \in A)\ &\sup A + 1 \geq a + 1 \wedge \neg(\sup A + 1 > r_1 + 1 \geq a + 1) && (x + 1 \text{ is an increasing function}) \\
(\forall r_2 \in \mathbb{R}, \forall a \in A)\ &\sup A + 1 \geq a + 1 \wedge \neg(\sup A + 1 > r_2 \geq a + 1) && ((\forall r_1 \in \mathbb{R}, \exists r_2 \in \mathbb{R})\ r_2 = r_1 + 1) \\
(\forall r_2 \in \mathbb{R}, \forall b \in B)\ &\sup A + 1 \geq b \wedge \neg(\sup A + 1 > r_2 \geq b) && (a \in A \iff a + 1 \in B \text{ (i.e. def. of } B)) \\
&\sup B = \sup A + 1 && \text{(def. of supremum)}
\end{aligned}$$

And so we are done.

Problem b: For $C = \{x^3 + 1 \mid x \in A\}$, show that $\sup C = (\sup A)^3 + 1$.

Solution:

$$\begin{aligned}
&\sup A = \sup A \\
(\forall r_1 \in \mathbb{R}, \forall a \in A)\ &\sup A \geq a \wedge \neg(\sup A > r_1 \geq a) && \text{(def. of supremum)} \\
(\forall r_1 \in \mathbb{R}, \forall a \in A)\ &(\sup A)^3 + 1 \geq a^3 + 1 \wedge \neg((\sup A)^3 + 1 > r_1^3 + 1 \geq a^3 + 1) \\
& && (x^3 + 1 \text{ is an increasing function}) \\
(\forall r_2 \in \mathbb{R}, \forall a \in A)\ &(\sup A)^3 + 1 \geq a^3 + 1 \wedge \neg((\sup A)^3 + 1 > r_2 \geq a^3 + 1) \\
& && ((\forall r_1 \in \mathbb{R}, \exists r_2 \in \mathbb{R})\ r_2 = r_1^3 + 1 \text{ (see problem 2)}) \\
(\forall r_2 \in \mathbb{R}, \forall a \in A)\ &(\sup A)^3 + 1 \geq b \wedge \neg((\sup A)^3 + 1 > r_2 \geq b) && (a \in A \iff a^3 + 1 \in B \text{ (i.e. def. of } B)) \\
&\sup C = (\sup A)^3 + 1 && \text{(def. of supremum)}
\end{aligned}$$

And so we are done.

Problem c: For $D = \{x^2 + 1 \mid x \in A\}$, is it the case that $\sup D = (\sup A)^2 + 1$?

Solution: No, consider the following counterexample:

$$\begin{aligned}
A &= \{-2, 1\} \\
D &= \{x^2 + 1 \mid x \in A\} \\
&= \{5, 2\}
\end{aligned}$$

We have:

$$\begin{aligned}
\sup A &= 1 \\
(\sup A)^2 + 1 &= 2 \\
&\neq 5 \\
&= \sup D
\end{aligned}$$

And so we have that $\sup D \neq (\sup A)^2 + 1$.