Intro to Math Reasoning HW 5b

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Problem 1

Problem: Prove that for all indexed collections of sets $(A_\alpha)_{\alpha\in J}$ and any set B:

$$\left(\bigcup_{\alpha\in J} A_\alpha\right) \cap B = \left(\bigcup_{\alpha\in J} A_\alpha \cap B\right)$$

Solution: The definitions of the set on the left hand side is the following:

$$\begin{aligned}
\left(\bigcup_{a\in J} A_\alpha\right) &= \{x \mid (\exists\alpha \in J)\ x \in A_\alpha\} \\
\left(\bigcup_{a\in J} A_\alpha\right) \cap B &= \{x \mid x \in (\bigcup_{a\in J} A_\alpha) \land x \in B\} \\
&= \{x \mid (\exists\alpha \in J)\ x \in A_\alpha \land x \in B\}
\end{aligned}$$

The other set is defined to be:

$$\begin{aligned}
\left(\bigcup_{\alpha\in J} A_\alpha \cap B\right) &= \{x \mid x \in (\bigcup_{a\in J} A_\alpha) \land x \in B\} \\
&= \{x \mid (\exists\alpha \in J)\ x \in A_\alpha \land x \in B\}
\end{aligned}$$

And thus the two sets are equivalent.

Problem 2

Problem: Prove that for any universe U and sets in it A, B the following holds:

$$(A \setminus B)^{\complement} = A^{\complement} \cup B$$

Solution: We'll write down the definition of the left hand set (the universe is assumed to be U):

$$\begin{aligned} A \setminus B &= \{x \mid x \in A \land x \notin B\} \\ (A \setminus B)^{\complement} &= \{x \mid x \notin (A \setminus B)\} \\ &= \{x \mid \neg(x \in A \land x \notin B)\} \\ &= \{x \mid x \notin A \lor x \in B\} \end{aligned}$$

Now the definition of the right hand set (again, the universe is assumed to be U):

$$\begin{aligned} A^{\complement} &= \{x \mid x \notin A\} \\ A^{\complement} \cup B &= \{x \mid x \in A^{\complement} \lor x \in B\} \\ &= \{x \mid x \notin A \lor x \in B\} \end{aligned}$$

The two sets are equal and we are done.

Problem 3

Problem: Prove the following for all sets A, B, C, D:

$$A \subseteq C \land B \subseteq D \implies A \setminus B \subseteq C \setminus D$$

Solution: Here are the definitions for the left hand side:

$$\begin{aligned} A \subseteq C &\equiv x \in A \to x \in C \\ B \subseteq D &\equiv x \in B \to x \in D \\ A \subseteq C \land B \subseteq D &\equiv (x \in A \to x \in C) \land (x \in B \to x \in D) \end{aligned}$$

Now the right hand side:

$$\begin{aligned} A \setminus B &\equiv \{x \mid x \in A \land x \notin B\} \\ C \setminus D &\equiv \{x \mid x \in C \land x \notin D\} \\ A \setminus B \subseteq C \setminus D &\equiv (x \in A \setminus B) \to (x \in C \setminus D) \\ &\equiv (x \in A \land x \notin B) \to (x \in C \land x \notin D) \end{aligned}$$

We'll just use a truth table to verify this:

a	b	c	d	$a \to c$	$b \to d$	$a \wedge \neg b$	$c \wedge \neg d$	$\underbrace{(a \to c) \wedge (b \to d)}_{P}$	$\underbrace{(a \wedge \neg b) \to (c \wedge \neg d)}_{Q}$	$P \to Q$
F	F	F	F	T	T	F	F	T	T	T
F	F	F	T	T	T	F	F	T	T	T
F	F	T	F	T	T	F	T	T	T	T
F	F	T	T	T	T	F	F	T	T	T
F	T	F	F	T	F	F	F	F		
F	T	F	T	T	T	F	F	T	T	T
F	T	T	F	T	F	F	T	F		
F	T	T	T	T	T	F	F	T	T	T
T	F	F	F	F	T	T	F	F		
T	F	F	T	F	T	T	F	F		
T	F	T	F	T	T	T	T	T	T	T
T	F	T	T	T	T	T	F	T	T	T
T	T	F	F	F	F	F	F	F		
T	T	F	T	F	T	F	F	F		
T	T	T	F	T	F	F	T	F		
T	T	T	T	T	T	F	F	T	T	T

And we have proved it. Notice that we only care to check the truth table when the antecedent P is true. The other cases are irrelevant to our cause.

Problem 4

Problem: Prove that for any sets A, B, C:

$$(A \triangle B) \triangle C = A \triangle (B \triangle C)$$

Solution: Let's expand out the left hand side:

$$\begin{aligned} x \in (A \triangle B) \triangle C &\equiv (x \in A \triangle B \wedge x \notin C) \vee (x \notin A \triangle B \wedge x \in C) \\ &\equiv (x \in A \wedge x \notin B \wedge x \notin C) \vee (x \in A \wedge x \in B \wedge x \in C) \\ &\vee (x \notin A \wedge x \in B \wedge x \notin C) \vee (x \notin A \wedge x \notin B \wedge x \in C) \end{aligned}$$

Now let's do the same for right hand side:

$$\begin{aligned} x \in A \triangle (B \triangle C) &\equiv (x \in A \wedge x \notin B \triangle C) \vee (x \notin A \wedge x \in B \triangle C) \\ &\equiv (x \in A \wedge x \notin B \wedge x \notin C) \vee (x \in A \wedge x \in B \wedge x \in C) \\ &\vee (x \notin A \wedge x \in B \wedge x \notin C) \vee (x \notin A \wedge x \notin B \wedge x \in C) \end{aligned}$$

Notice that when expanded out both statements say the same thing: for an element to be the set it must be in either only 1 or all 3 of the sets A, B, C. Thus they are equal and the symmetric difference is associative.

Problem 5

Part a

Problem: Prove or give a counterexample:

$$\mathcal{P}(A) \setminus \mathcal{P}(B) \subseteq \mathcal{P}(A \setminus B)$$

Solution: This isn't true, here's an example:

$$\begin{aligned}
A &= \{a, b\} \\
B &= \{b\} \\
A \setminus B &= \{a\} \\
\mathcal{P}(A) &= \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \\
\mathcal{P}(B) &= \{\emptyset, \{b\}\} \\
\mathcal{P}(A \setminus B) &= \{\emptyset, \{a\}\} \\
\mathcal{P}(A) \setminus \mathcal{P}(B) &= \{\{a\}, \{a, b\}\}
\end{aligned}$$

As we can see, $\{a, b\} \notin \mathcal{P}(A \setminus B)$ but $\{a, b\} \in \mathcal{P}(A) \setminus \mathcal{P}(B)$.

Part b

Problem: Prove or give a counterexample:

$$\mathcal{P}(A \setminus B) \subseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$$

Solution: This isn't true, here's an example:

$$\begin{aligned}
A &= \{a, b\} \\
B &= \{b\} \\
A \setminus B &= \{a\} \\
\mathcal{P}(A) &= \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \\
\mathcal{P}(B) &= \{\emptyset, \{b\}\} \\
\mathcal{P}(A \setminus B) &= \{\emptyset, \{a\}\} \\
\mathcal{P}(A) \setminus \mathcal{P}(B) &= \{\{a\}, \{a, b\}\}
\end{aligned}$$

As we can see, $\emptyset \in \mathcal{P}(A \setminus B)$ but $\emptyset \notin \mathcal{P}(A) \setminus \mathcal{P}(B)$.