Set Theory HW #2

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Problem 1

Exercises 5,7,10 from page 26 in the textbook.

Exercise 5: Assume that every member of A is a subset of B. Show that $\bigcup A \subseteq B$.

Solution: Consider an arbitrary set a, by the axiom of union we have:

$$a \in \bigcup A \implies \exists b \in A \, (a \in b)$$

And by the question's assumption, b is a subset of B. Putting these two together we have:

$$((a \in b) \land (b \subseteq B)) \implies a \in B \qquad \text{(def. of subset)}$$

And thus we have shown that for any $a \in \bigcup A$, the set a must also be an element of B. By the definition of subset, we have $\bigcup A \subseteq B$.

Exercise 7: Show that for any two sets A and B the following holds:

a) $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

b) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Under what conditions does equality hold?

Solution: a) Consider an arbitrary set x and note the following chain of logical equivalences:

$$\begin{aligned}
x \in \mathcal{P}(A) \cap \mathcal{P}(B) &\iff x \in \mathcal{P}(A) \land x \in \mathcal{P}(B) && \text{(def. of intersection)} \\
&\iff x \subseteq A \land x \subseteq B && \text{(def. power set)} \\
&\iff x \subseteq A \cap B && \text{(def. of intersection)} \\
&\iff x \in \mathcal{P}(A \cap B) && \text{(def. power set)}
\end{aligned}$$

And so, by extensionality, we have $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

b) Consider an arbitrary set x and note the following chain of implications:

$$\begin{aligned}
x \in \mathcal{P}(A) \cup \mathcal{P}(B) &\iff x \in \mathcal{P}(A) \lor x \in \mathcal{P}(B) && \text{(def. of union)} \\
&\iff x \subseteq A \lor x \subseteq B && \text{(def. power set)} \\
&\implies x \subseteq A \cup B && \text{(def. of union)} \\
&\iff x \in \mathcal{P}(A \cap B) && \text{(def. power set)}
\end{aligned}$$

And so, by the definition of subset, we have $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. You'll notice that on line 3 we have an implication rather than an iff. We can only make this an iff, and thus establish equality of the two sets, if we assume that $A \subseteq B$ or $B \subseteq A$.

Exercise 10: Prove that if $a \in B$, then $\mathcal{P}(a) \in \mathcal{P}(\mathcal{P}()) \bigcup B$.

Solution: Consider an arbitrary set a and note the following:

$$\begin{aligned}
a \in B &\implies \forall t(t \in a \implies t \in \bigcup B) && \text{(axiom of union)} \\
&\implies a \subseteq \bigcup B && \text{(def. of subset)} \\
&\implies (a \subseteq \bigcup B) \wedge \forall t(t \in \mathcal{P}(a) \implies t \subseteq a) && \text{(def. of powerset)} \\
&\implies \forall t(t \in \mathcal{P}(a) \implies t \subseteq \bigcup B) && \text{(transitivity of subset)} \\
&\implies \forall t(t \in \mathcal{P}(a) \implies t \in \mathcal{P}(\bigcup B)) && \text{(def. of power set)} \\
&\implies \mathcal{P}(a) \subseteq \mathcal{P}(\bigcup) B && \text{(def. of subset)} \\
&\implies \mathcal{P}(a) \in \mathcal{P}(\mathcal{P}(\bigcup B)) && \text{(def. of power set)}
\end{aligned}$$

Problem 2

Exercises 12,20,22,35 from pages 32-33 in the textbook.

Exercise 12: Verify the following identity:

$$C \setminus (A \cup B) = (C \setminus A) \cup (C \setminus B)$$

Solution: For the following set of equalities, the complement is taken with respect to the universe $A \cup B \cup C$:

$$\begin{aligned}
C \setminus (A \cap B) &= C \cap (A \cap B)^{\complement} && \text{(relative complement)} \\
&= C \cap (B^{\complement} \cup A^{\complement}) && \text{(DeMorgan's Law)} \\
&= (C \cap B^{\complement}) \cup (C \cap A^{\complement}) && \text{(distributivity of intersction)} \\
&= (C \setminus B) \cup (C \setminus A) && \text{(relative complement)}
\end{aligned}$$

Exercise 20: Let A, B and C be sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.

Solution: Consider an $x \in B$. There are two cases which exhaust all possibilities:

$$\begin{aligned}
x \in A &\implies x \in A \cap B \\
&\iff x \in A \cap C && \text{(assumption)} \\
&\implies x \in C
\end{aligned}$$

and the other case:

$$\begin{aligned} x \notin A &\implies x \in A \cup B \\ &\iff x \in A \cup C && \text{(assumption)} \\ &\implies x \in C \end{aligned}$$

This gives us $x \in B \implies x \in C$, and by replacing all occurrences of B with C we have an argument for the reverse direction. Putting these together we have $B = C$.

Exercise 22: Show that if A and B are nonempty sets, then $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$.

Solution: Note that by the axiom of union we have:

$$\begin{aligned} x \in \bigcap(A \cup B) &\implies (\forall y \in A \cup B)\, x \in y && \text{(def. of arbitrary intersection)} \\ &\implies (\forall y \in A)\, x \in y && (A \subseteq A \cup B) \\ &\iff x \in \bigcap A && \text{(def. of arbitrary intersection)} \end{aligned}$$

Similarly we have:

$$\begin{aligned} x \in \bigcap(A \cup B) &\implies (\forall y \in A \cup B)\, x \in y && \text{(def. of arbitrary intersection)} \\ &\implies (\forall y \in B)\, x \in y && (B \subseteq A \cup B) \\ &\iff x \in \bigcap B && \text{(def. of arbitrary intersection)} \end{aligned}$$

Putting these together we have:

$$\begin{aligned} x \in \bigcap(A \cup B) &\implies x \in \bigcap A \wedge x \in \bigcap B \\ &\implies x \in \bigcap A \cap \bigcap B && \text{(def. of intersection)} \end{aligned}$$

This proves one direction. The other direction can be proved by first recalling that:

$$\begin{aligned} x \in \bigcap A \cap \bigcap B &\implies x \in \bigcap A \wedge x \in \bigcap B \\ &\implies (\forall y \in A)\, x \in y \wedge (\forall y \in B)\, x \in y \\ &\implies (\forall y \in A)\, x \in y \vee (\forall y \in B)\, x \in y \end{aligned}$$

This allows us to state the following:

$$\begin{aligned} &(\forall t \in A \cup B)\, t \in A \vee t \in B && \text{(def. of union)} \\ \implies &(\forall t \in A \cup B)\, x \in t && \text{(above section)} \\ \implies &x \in \bigcap(A \cup B) && \text{(def. of arbitrary intersection)} \end{aligned}$$

And with both sides of the implication proved, the equality holds true.

Exercise 35: Assume that $\mathcal{P}(A) = \mathcal{P}(B)$. Prove that $A = B$.

Solution: Consider an arbitrary set x and note the following chain of implications:

$$
\begin{aligned}
x \in A &\iff \{x\} \subseteq A && \text{(def. of subset)}\\
&\iff \{x\} \in \mathcal{P}(A) && \text{(def. of powerset)}\\
&\iff \{x\} \in \mathcal{P}(B) && \text{(assumption)}\\
&\iff \{x\} \subseteq B && \text{(def. of powerset)}\\
&\iff x \in B && \text{(def. of subset)}
\end{aligned}
$$

And so by extensionality we have $A = B$.

Problem 3

Exercises 32,33,36 from pages 33-34 in the textbook.

Exercise 32: Let S be the set $\{\{a\}, \{a, b\}\}$. Evaluate and simplify:

a) $\bigcup\bigcup S$

b) $\bigcap\bigcap S$

c) $\bigcap\bigcup S \cup (\bigcup\bigcup S \setminus \bigcup\bigcap S)$

Solution: For a) we have:

$$
\begin{aligned}
\bigcup\bigcup S &= \bigcup\bigcup\{\{a\}, \{a, b\}\}\\
&= \bigcup\{a, b\}\\
&= a \cup b
\end{aligned}
$$

For b) we have:

$$
\begin{aligned}
\bigcap\bigcap S &= \bigcap\bigcap\{\{a\}, \{a, b\}\}\\
&= \bigcap\{a\}\\
&= a
\end{aligned}
$$

For c) we have:

$$
\begin{aligned}
\bigcap\bigcup S \cup \left(\bigcup\bigcup S \setminus \bigcup\bigcap S\right) &= \bigcap\{a, b\} \cup \left(\bigcup\{a, b\} \setminus \bigcup\{a\}\right)\\
&= (a \cap b) \cup ((a \cup b) \setminus a)\\
&= (a \cap b) \cup (b \setminus a)\\
&= b
\end{aligned}
$$

Exercise 33: With S as in the preceding exercise, evaluate $\bigcup(\bigcup S \setminus \bigcap S)$ when $a \neq b$ and when $a = b$.

Solution: Evaluating the expression we arrive at:

$$\bigcup\left(\bigcup S \setminus \bigcap S\right) = \bigcup(\{a,b\} \setminus \{a\})$$

For the case that $a \neq b$ we have:

$$\bigcup(\{a,b\} \setminus \{a\}) = \bigcup\{b\} = b$$

For the case that $a = b$ we have:

$$\bigcup(\{a,b\} \setminus \{a\}) = \bigcup(\{a\} \setminus \{a\}) = \bigcup\varnothing = \varnothing$$

Exercise 36: Verify that for all sets A and B the following are correct:

a) $A \setminus (A \cap B) = A \setminus B$

b) $A \setminus (A \setminus B) = A \cap B$

Solution: For both a) an b) the complement is taken with respect to the universe $A \cup B$. For a) we have:

$$\begin{aligned}
A \setminus (A \cap B) &= A \cap (A \cap B)^{\complement} && \text{(relative complement)}\\
&= A \cap (A^{\complement} \cup B^{\complement}) && \text{(DeMorgan's Law)}\\
&= (A \cap A^{\complement}) \cup (A \cap B^{\complement}) && \text{(distributivity of intersection)}\\
&= (A \setminus A) \cup (A \setminus B) && \text{(relative complement)}\\
&= \varnothing \cup (A \setminus B)\\
&= A \setminus B
\end{aligned}$$

For b) we have:

$$\begin{aligned}
A \setminus (A \setminus B) &= A \cap (A \setminus B)^{\complement} && \text{(relative complement)}\\
&= A \cap (A \cap B^{\complement})^{\complement} && \text{(relative complement)}\\
&= A \cap (A^{\complement} \cup B) && \text{(DeMorgan's Law)}\\
&= (A \cap A^{\complement}) \cup (A \cap B) && \text{(distributivity of intersection)}\\
&= \varnothing \cup (A \cap B)\\
&= A \cap B
\end{aligned}$$