Intro to Math Reasoning HW 4b

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The functions f and g referenced in Problems 1-3 have the following domain and codomain: $f, g : \mathbb{R} \to \mathbb{R}$.

Problem 1

Part a

Problem: Are there two functions f, g such that they both have limits as $x \to 0$?

Solution: Yes, consider $f(x) = g(x) = x$. It clearly has a limit as $x \to 0$, namely 0.

Part b

Problem: Is there a unique pair of functions (f, g) such that they hold the same property in Part A?

Solution: No, there is more than one pair of functions that satisfy this property. We gave one above, here is another one: $f(x) = x^2$ and $g(x) = 5x + 1$. They both have limits as $x \to 0$, with $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 1$.

Part c

Problem: Is there a pair of functions (f, g) such that they do *not* satisfy the above property?

Solution: Yes, consider the following choice of f and g:

$$f(x) = g(x) = \begin{cases} x + 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

This function has a right-sided limit of 1 as $x \to 0$ but a left-sided limit of 0. Thus the limit does not exist for either f or g.

Problem 2

Problem: Is the following statement true?: if f has a limit as $x \to 0$ and g is bounded, then the product fg has a limit as $x \to 0$.

Solution: This statement is false. Consider the following choices of functions:

$$f(x) = 1$$
$$g(x) = \begin{cases} 2, & \text{if } x \geq 0 \\ -2, & \text{if } x < 0 \end{cases}$$

These functions satisfy the requirements for f and g, namely:

$$\lim_{x \to 0} f(x) = 1 \qquad \text{(limit exists)}$$
$$\forall x \ |g(x)| \leq 2 \qquad \text{(g is bounded)}$$

Also note that $\lim_{x\to 0} g(x)$ does not exist (the left hand and right hand limits are -2 and 2 respectively and thus do not line up). The reason this is important is because when we multiply the functions we get $f(x)g(x) = g(x)$. This is because we set $f(x) = 1$. As a result fg doesn't have a limit as $x \to 0$ just like g. And so the statement we set out to disprove is indeed false.

Problem 3

Problem: Prove the following statement: if $\lim_{x\to 0} f(x) = 0$ and g is bounded, then the product fg has a limit as $x \to 0$.

Solution: This is true and we can see this by writing down the definition of limit as $x \to 0$ for f:

$$\begin{aligned} \lim_{x \to 0} f(x) = 0 &\equiv (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}) \ 0 < |x - 0| < \delta \to |f(x) - 0| < \epsilon \\ &\equiv (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}) \ 0 < |x| < \delta \to |f(x)| < \epsilon \end{aligned}$$

and for the product of the functions fg:

$$\begin{aligned} \lim_{x \to 0} f(x)g(x) = 0 &\equiv (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}) \ 0 < |x - 0| < \delta \to |f(x)g(x) - 0| < \epsilon \\ &\equiv (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}) \ 0 < |x| < \delta \to |f(x)g(x)| < \epsilon \end{aligned}$$

Putting these together, this means we must prove the following:

$$\begin{gathered} (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}) \ 0 < |x| < \delta \to |f(x)| < \epsilon \\ \Longrightarrow \\ (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}) \ 0 < |x| < \delta \to |f(x)g(x)| < \epsilon \end{gathered}$$

First let's call the bound on g the constant M. As in $|g(x)| \leq M$. Now let us note that $|f(x)g(x)| \leq |f(x)||g(x)|$ via the triangle inequality. Finally let express the ϵ in the first statement as $\frac{\epsilon}{M}$. Making the antecedent:

$$(\forall \frac{\epsilon}{M} > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})\ 0 < |x| < \delta \to |f(x)| < \frac{\epsilon}{M}$$

And so for a given choice of δ we find that $|f(x)| < \frac{\epsilon}{M}$ which means that $|f(x)g(x)| \leq |f(x)||g(x)| < M \cdot \frac{\epsilon}{M} = \epsilon$

Thus, $|f(x)g(x)| < \epsilon$ making the consequent of the statement true. This chains back up the statement with each implication's consequent being true until we have proved the whole statement.

Problem 4

Problem: Prove the following:

$$(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$$

Solution: Here is a truth table:

A	B	C	$A \vee C$	$B \vee C$	$(A \vee C) \wedge (B \vee C)$	$(A \wedge B) \vee C$
F	F	F	F	F	F	F
F	F	T	T	T	T	T
F	T	F	F	T	F	F
F	T	T	T	T	T	T
T	F	F	T	F	F	F
T	F	T	T	T	T	T
T	T	F	T	T	T	T
T	T	T	T	T	T	T

Problem 5

Problem: Prove the following:

$$\neg(A \to B) \equiv (A \wedge \neg B)$$

Solution: Here is a truth table:

A	B	$\neg B$	$A \to B$	$\neg(A \to B)$	$A \wedge \neg B$
F	F	T	T	F	F
F	T	F	T	F	F
T	F	T	F	T	T
T	T	F	T	F	F