Honors Calculus III HW #3

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Exercise 1

Problem: Consider the curve parameterized by:

$$\mathbf{x}(t) = (1 + 2\cos t + 2\sin t, 1 - 2\cos t + \sin t, \cos t - 2\sin t)$$

Compute its $\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)$. Also compute its speed, curvature, and torsion denoted $v(t), \kappa(t)$, and $\tau(t)$ respectively.

Solution: First we find the velocity (first derivative):

$$\mathbf{v}(t) = (-2\sin t + 2\cos t, 2\sin t + \cos t, -\sin t - 2\cos t)$$

The speed is simply the magnitude of $\mathbf{v}(t)$:

$$\begin{aligned}
v(t) &= \|\mathbf{v}(t)\| \\
&= \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)} \\
&= \sqrt{(-2\sin t + 2\cos t)^2 + (2\sin t + \cos t)^2 + (-\sin t - 2\cos t)^2} \\
&= \sqrt{(4 - 8\sin t\cos t) + (1 + 4\sin t\cos t + 3\sin^2 t) + (1 + 4\sin t\cos t + 3\cos^2 t)} \\
&= \sqrt{9} = 3
\end{aligned}$$

The tangent vector is just the normalized velocity:

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{3}(-2\sin t + 2\cos t, 2\sin t + \cos t, -\sin t - 2\cos t)$$

To calculate the binormal vector we first have to calculate the acceleration (second derivative) which will be perpendicular to the velocity curve, and thus the tangent curve:

$$\mathbf{a}(t) = (-2\cos t - 2\sin t, 2\cos t - \sin t, -\cos t + 2\sin t)$$

Now we just have to take the normalized cross product of the velocity and acceleration:

$$\mathbf{B}(t) = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|} = \frac{1}{3}(1, 2, 2)$$

Because $\mathbf{B}(t)$ is a constant, $\mathbf{B}'(t) = 0$ and thus $\tau(t) = 0$. Now we compute the normal vector:

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t) = \frac{1}{3}(-2\sin t - 2\cos t, -\sin t + 2\cos t, 2\sin t - 2\cos t)$$

And finally we can compute the curvature by using quantities we have already calculated:

$$\kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{(v(t))^3} = \frac{9}{27} = \frac{1}{3}$$

Exercise 2

Problem: Consider the curve parameterized by:

$$\mathbf{x}(t) = \left(2t + t^2, 2t, t + t^2 + \frac{t^3}{3}\right)$$

Compute its $\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)$. Also compute its speed, curvature, and torsion denoted $v(t), \kappa(t)$, and $\tau(t)$ respectively.

Solution: Find the velocity:

$$\mathbf{v}(t) = (2t + 2, 2, t^2 + 2t + 1)$$

Now we find the speed:

$$\begin{aligned} v(t) &= \sqrt{(2t+2)^2 + 2^2 + (t^2 + 2t + 1)^2} \\ &= \sqrt{4(t+1)^2 + 4 + (t+1)^4} \\ &= \sqrt{(t+1)^4 + 4(t+1)^2 + 4} \\ &= \sqrt{((t+1)^2 + 2)^2} \\ &= (t+1)^2 + 2 \end{aligned}$$

Now the tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{(t+1)^2 + 2}(2t + 2, 2, t^2 + 2t + 1)$$

Now the acceleration:

$$\mathbf{a}(t) = (2, 0, 2t + 2)$$

Here's the binormal vector:

$$\mathbf{B}(t) = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|} = \frac{1}{(t+1)^2 + 2}(2t + 2, -t^2 - 2t - 1, -2)$$

Now we calculate the normal vector:

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t) = \frac{1}{(t+1)^2+2}(-t^2-2t+1, -2t-2, 2t+2)$$

The curvature:

$$\kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{(v(t))^3} = \frac{2((t+1)^2+2)}{((t+1)^2+2)^3} = \frac{2}{((t+1)^2+2)^2}$$

And finally the torsion:

$$\tau(t) = \frac{\mathbf{v}(t) \cdot (\mathbf{a}(t) \times \mathbf{a}'(t))}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|^2} = \frac{\mathbf{a}'(t) \cdot (\mathbf{v}(t) \times \mathbf{a}(t))}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|^2} = \frac{8}{4((t+1)^2+2)^2} = \frac{2}{((t+1)^2+2)^2}$$

Where $\mathbf{a}'(t) = (0, 0, 2)$.

Exercise 3

Problem: Either the curve in Exercise 1 or 2 is a planar curve. Which one is it?

Solution: A planar curve is planar iff its torsion is always 0. As we have shown, the torsion in Exercise 1 is 0 while the torsion in Exercise 2 depends on t. We can derive an equation for the plane the curve lies on from its constant binormal vector

$$\mathbf{B} = \frac{1}{3}(1, 2, 2)$$

First we'll choose some arbitrary point on the plane, say $\mathbf{x}(0) = (3, -1, 1)$. Now we can write:

$$\mathbf{B} \cdot (\mathbf{x} - \mathbf{x}(0)) = \frac{1}{3}(1, 2, 2) \cdot (x-3, y+1, z-1) = 0$$

Solving this out we arrive at:

$$\frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 1 \equiv x + 2y + 2z = 3$$

Exercise 4

Problem: Consider the parameterization given in Exercise 2. Find the function $s(t)$ which gives the arc length traveled along the curve for $t > 0$ starting at $t = 0$.

Solution: We already have the speed of the curve from Exercise 2:

$$v(t) = (t+1)^2 + 2$$

We now just integrate it from 0 to t:

$$s(t) = \int_0^t v(x)\ dx = \int_0^t \big((x+1)^2 + 2\big)\, dx = 2t + \frac{(t+1)^3}{3} - \frac{1}{3}$$

Via some algebraic manipulation we can rewrite the expression as a depressed cubic in $t+1$:

$$\frac{1}{3}(t+1)^3 + 2(t+1) - \frac{7}{3}$$

Extra Credit: Use Ferro's formula for depressed cubic roots to solve for the inverse of $s(t)$ denoted $t(s)$.

Solution: We can rewrite the equation above with the variable u:

$$s = \frac{1}{3}u^3 + 2u - \frac{7}{3}$$

Now we rearrange it in the form Ferro did:

$$x^3 + px = q$$
$$u^3 + 6u = 3s + 7$$

The unique root of a depressed cubic is the following:

$$\sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Plugging in $p = 6$ and $q = 3s + 7$ we find:

$$u = \sqrt[3]{\frac{3s+7}{2} + \sqrt{\frac{(3s+7)^2}{4} + 8}} + \sqrt[3]{\frac{3s+7}{2} - \sqrt{\frac{(3s+7)^2}{4} + 8}}$$

And so if $t + 1 = u$ then:

$$t(s) = \sqrt[3]{\frac{3s+7}{2} + \sqrt{\frac{(3s+7)^2}{4} + 8}} + \sqrt[3]{\frac{3s+7}{2} - \sqrt{\frac{(3s+7)^2}{4} + 8}} - 1$$

The arc length parameterization would then be $\mathbf{x}(t(s))$. I could write it explicitly, but it probably wouldn't be useful.

Exercise 5

Problem: Consider the curve parameterized by:

$$\mathbf{x}(t) = (1+t)^{-2}(1 + 4t + t^2, 2^{3/2}t^{1/2}, 2t + t^2 - 1)$$

Where $t > 0$. Find $v(t), \kappa(t)$ and $\tau(t)$ as well as $\mathbf{T}(1), \mathbf{N}(1)$, and $\mathbf{B}(1)$.

Solution: First we find the velocity:

$$\mathbf{v}(t) = \frac{\left(2\sqrt{t}(1-t), \sqrt{2}(1-3t), 4\sqrt{t}\right)}{(t+1)^3\sqrt{t}}$$

Now we compute the speed:

$$\begin{aligned}
v(t) &= \frac{1}{(t+1)^3\sqrt{t}} \left\| \left(2\sqrt{t}(1-t), \sqrt{2}(1-3t), 4\sqrt{t}\right) \right\| \\
&= \frac{1}{(t+1)^3\sqrt{t}} \sqrt{4t(t-1)^2 + 2(1+3t)^2 + 16t} \\
&= \frac{1}{(t+1)^3\sqrt{t}} \sqrt{4t^3 + 10t^2 + 8t + 2} \\
&= \frac{1}{(t+1)^3\sqrt{t}} \sqrt{(4t+2)(t+1)^2} \\
&= \frac{\sqrt{4t+2}}{(1+t)^2\sqrt{t}}
\end{aligned}$$

From this we can find the tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{(t+1)\sqrt{4t+2}} \left((2-2t)\sqrt{t}, \sqrt{2}(1-3t), 4\sqrt{t}\right)$$

Now we use this to find the orthogonal component of the acceleration with respect to the tangent vector:

$$\begin{aligned}
\mathbf{a}_\perp(t) = \mathbf{a}(t) - \mathrm{proj}_{\mathbf{T}(t)}(\mathbf{a}(t)) \\
- (\mathbf{a}(t) \cdot \mathbf{T}(t))\mathbf{T}(t)
\end{aligned}$$

First we need the acceleration (differentiate velocity):

$$\mathbf{a}(t) = \frac{1}{(t+1)^4 t^{3/2}} \left(4(t-2)t^{3/2}, \frac{(15t^2 - 10t - 1)}{\sqrt{2}}, -12t^2\right)$$

Now we find the dot product of $\mathbf{a}(t)$ and $\mathbf{T}(t)$:

$$\begin{aligned}
\mathbf{a}(t) \cdot \mathbf{T}(t) &= \frac{-8t^2(t-2)(t-1) + (1-3t)(15t^2 - 10 - 1) - 48t^{5/2}}{t^{3/2}(t+1)^5\sqrt{4t+2}} \\
&= \frac{-8t^4 - 21t^3 - 48t^{5/2} + 29t^2 - 7t - 1}{t^{3/2}(t+1)^5\sqrt{4t+2}}
\end{aligned}$$

Using these two pieces of information we calculate the orthogonal part to be:

$$\mathbf{a}_\perp(t) = \frac{1}{t(2t+1)(t+1)^4}(-9t^2 - 4t + 1, \sqrt{2t}(3t^2 - 6t - 5), -2(4t^2 + t - 1))$$

Now consider the following identity:

$$\|\mathbf{a}_{\perp}(t)\|^2 = (v(t))^2\kappa(t)$$

Let's find the norm squared of the orthogonal component:

$$\|\mathbf{a}_{\perp}(t)\|^2 = \frac{9t+5}{t^2(2t+1)(t+1)^5}$$

Dividing $\|\mathbf{a}_{\perp}(t)\|^2$ by the speed squared we find $\kappa(t)$ to be equal to:

$$\kappa(t) = \frac{9t+5}{t(2t+1)(t+1)^5}$$

Now we just normalize the acceleration to find the normal vector:

$$\begin{aligned}
\mathbf{N}(t) &= \frac{\mathbf{a}\perp(t)}{\|\mathbf{a}\perp(t)\|} \\
&= \kappa(t)^{-1}\frac{1}{t(2t+1)(t+1)^4}(9t^2+4t-1, \sqrt{2t}(3t^2-6t-5), 2(4t^2+t-1)) \\
&= \left(\frac{t(2t+1)(t+1)^5}{9t+5}\right)\frac{1}{t(2t+1)(t+1)^4}(9t^2+4t-1, \sqrt{2t}(3t^2-6t-5), 2(4t^2+t-1)) \\
&= \frac{t+1}{9t+5}(9t^2+4t-1, \sqrt{2t}(3t^2-6t-5), 8t^2+2t-2)
\end{aligned}$$

We can find the binormal vector by taking the cross product of the other two:

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{(9t+5)^{1/2}(t+1)^{3/2}}(6t+2, -4t^{3/2}\sqrt{2}, -3t^2-1)$$

Plugging in $t=1$ we find:

$$\begin{aligned}
\mathbf{T}(1) &= \frac{1}{\sqrt{3}}(0,-1,\sqrt{2}) \\
\mathbf{N}(1) &= \frac{1}{\sqrt{21}}(-3,-2\sqrt{2},-2) \\
\mathbf{B}(1) &= \frac{1}{\sqrt{7}}(2,-\sqrt{2},-1)
\end{aligned}$$

Now we just need to calculate the torsion $\tau(t)$. We can do this with the following formula:

$$\tau(t) = \mathbf{N}'(t)\cdot\mathbf{B}(t)$$

The derivative of the normal vector comes out to:

$$\mathbf{N}'(t) = \frac{2}{(9t+1)^2}\left(81t^3+72t^2+13t+6, \frac{135t^4-60t^3-114t^2+12t-5}{2\sqrt{2t}}, 72t^3+57t^2+10t+9\right)$$

Now we just compute the dot product of the binormal and normal prime vectors to find:

$$\tau(t) = \frac{-3\sqrt{2t}(t+1)}{9t+5}$$

Exercise 6

Part a

Problem: Consider a spherical curve $\mathbf{x}(t)$. Show that for all t the following holds:

$$\|\mathbf{x}(t) - \mathbf{c}\| \cdot \mathbf{T}(t) = 0$$

Solution: All spherical curves obey the following:

$$\|\mathbf{x}(t) - \mathbf{c}\| = r^2$$

If we differentiate both sides we get:

$$2(\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{v}(t) = 0$$

Recall that $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. Since $v(t) = 0$ then so too does $\mathbf{T}(t)$, meaning:

$$\|\mathbf{x}(t) - \mathbf{c}\| \cdot \mathbf{T}(t) = \|\mathbf{x}(t) - \mathbf{c}\| \cdot \mathbf{0} = 0$$

Part b

Problem: Show that for all t:

$$\begin{aligned}(\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{N}(t) &= \frac{-1}{\kappa(t)} \\ (\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{B}(t) &= \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\end{aligned}$$

Solution: If we differentiate both sides of the equation we found in Part a (using the product rule) we get:

$$\mathbf{v}(t) \cdot \mathbf{T}(t) + (\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{T}'(t) = 0$$

Recall the following identities

$$\mathbf{v}(t) = v(t)\mathbf{T}(t)$$

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)$$

Plugging these into the derivative we calculated we get:

$$\begin{aligned}\mathbf{v}(t) \cdot \mathbf{T}(t) + (\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{T}'(t) &= 0 \\ (\mathbf{x}(t) - \mathbf{c}) \cdot (v(t)\kappa(t)\mathbf{N}(t)) &= -v(t)\mathbf{T}(t) \cdot \mathbf{T}(t) \\ (\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{N}(t) &= \frac{-v(t)\mathbf{T}(t) \cdot \mathbf{T}(t)}{v(t)\kappa(t)} \\ (\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{N}(t) &= \frac{-1}{\kappa(t)}\end{aligned}$$

And we have proved the first part. The second part can be found by, again, differentiating the last equation:

$$\mathbf{v}(t) \cdot \mathbf{N}(t) + (\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{N}'(t) = \frac{\kappa'(t)}{\kappa^2(t)}$$

Note that $\mathbf{v}(t) \perp \mathbf{N}(t)$ meaning their dot product is 0, leaving us with:

$$(\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{N}'(t) = \frac{\kappa'(t)}{\kappa^2(t)}$$

Now we use the following fact:

$$v(t)\tau(t)\mathbf{B}(t) - v(t)\kappa(t)\mathbf{T}(t) = \mathbf{N}'(t)$$

Substituting in we arrive at:

$$(\mathbf{x}(t) - \mathbf{c}) \cdot (v(t)\tau(t)\mathbf{B}(t) - v(t)\kappa(t)\mathbf{T}(t)) = \frac{\kappa'(t)}{\kappa^2(t)}$$

As we have shown in the first part, $(\mathbf{x}(t) - \mathbf{c}) \perp \mathbf{N}(t)$ and so we are left with:

$$(\mathbf{x}(t) - \mathbf{c}) \cdot v(t)\tau(t)\mathbf{B}(t) = \frac{\kappa'(t)}{\kappa^2(t)} = (\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{B}(t) = \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}$$

And we are done.

Exercise 7

Problem: Show that for any spherical curve, the following is true:

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 = r^2 - \frac{1}{\kappa^2(t)}$$

Also explain why the radius of curvature of a spherical curve cannot be bigger than the radius of the sphere it is contained in.

Solution: Consider the orthonormal basis $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$. We can write any vector as the sum of its components in this basis:

$$\mathbf{x}(t) - \mathbf{c} = ((\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{T}(t))\mathbf{T}(t) + ((\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{N}(t))\mathbf{N}(t) + ((\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{B}(t))\mathbf{B}(t)$$

Now if we just use the equations we found in the precious Exercise:

$$\mathbf{x}(t) - \mathbf{c} = \frac{-1}{\kappa(t)}\mathbf{N}(t) + \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t)$$

So now we just compute the radius by the Pythagorean theorem (i.e the norm squared of the general vector):

$$\|\mathbf{x}(t) - \mathbf{c}\| = r^2 = \frac{1}{\kappa^2(t)} + \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2$$

This is equivalent to the first statement we set out to prove:

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 = r^2 - \frac{1}{\kappa^2(t)}$$

In regards to why the radius of curvature (i.e $\frac{1}{\kappa(t)}$) cannot be greater than r, notice that the left hand side of the above is non-negative. This means that the $r^2 - \frac{1}{\kappa^2(t)} \geq 0$ which is equivalent to:

$$r \geq \frac{1}{\kappa(t)}$$

Geometrically this is because the intersection of the osculating plane of the curve with the sphere on which the curve lies is a circle with radius γ. At any point in time t, any instantaneous change on in the curve is along this circle, which means the radius of the curvature is this circle's radius: $\frac{1}{\kappa(t)} = \gamma \leq r$.

Exercise 8

Problem: Show that a spherical curve satisfies the following:

$$\mathbf{c} = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) + \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t)$$

Also show that the conclusion from Exercise 7 implies $\mathbf{y}'(t) = 0$

Solution: The first equation is clear from splitting up $\mathbf{c}$ as $\mathbf{x}(t) - (\mathbf{x}(t) - \mathbf{c})$ and plugging in the results form the previous exercise. If we differentiate it then use the Frenet-Seret formulas we find:

$$v(t)\mathbf{T}(t) - \frac{\kappa'(t)}{\kappa^2(t)}\mathbf{N}(t) + \frac{1}{\kappa(t)}\mathbf{N}'(t) - \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t)\right)' - \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}'(t) = \mathbf{0}$$

We can split this up into three orthonormal components and add in some extraneous variables:

$$\begin{aligned}
\mathbf{0} = &\left(v(t) - \frac{1}{\kappa(t)}v(t)\kappa(t)\right)\mathbf{T}(t) \\
&+ \left(-\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)} + \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}v(t)\tau(t)\right)\mathbf{N}(t) \\
&+ \left(\frac{1}{\kappa(t)}v(t)\tau(t) - \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t)\right)'\right)\mathbf{B}(t)
\end{aligned}$$

The coefficients of these vectors must all be 0 since they are all orthonormal. In particular we find from the last coefficient that:

$$\frac{\tau(t)}{\kappa(t)} = \frac{1}{v(t)}\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)'$$

We can write the following as a consequence of Exercise 7:

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right) = \pm\sqrt{r^2 - \frac{1}{\kappa^2(t)}}$$

Differentiating both sides we get:

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)' = \pm\frac{1}{\sqrt{r^2 - \frac{1}{\kappa^2(t)}}}\frac{\kappa'(t)}{\kappa^3(t)}$$

Plugging in for the root in the denominator we find that (regardless of sign):

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)' = \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^{-1}\frac{\kappa'(t)}{\kappa^3(t)} = \frac{v(t)\tau(t)}{\kappa(t)}$$

And so whenever the equation from Exercise 7 is satisfied for $\mathbf{x}(t)$ and some curve $\mathbf{y}(t)$ is defined as shown in the problem sheet, $\mathbf{y}'(t) = 0$ because $\mathbf{y}(t)$ will be constant.

Exercise 9

Problem: Show that a curve is spherical iff the following is constant:

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)}$$

Solution: We already know that for a curve to be spherical the following must be true:

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 = r^2 - \frac{1}{\kappa^2(t)}$$

As so it must be the case that $\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)}$ is a constant. Now we have to prove it in the other direction.

Suppose that $\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)}$ is constant. Let's call that value r^2. We have already seen that the curve $\mathbf{y}(t)$ defined in the problem sheet is constant. Call that constant c. By the Pythagorean theorem we find that:

$$\|\mathbf{x}(t) - \mathbf{c}\| = \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 = r^2 - \frac{1}{\kappa^2(t)} = r^2$$

And so the curve is spherical. We have proved the condition is both necessary and sufficient.

Exercise 10

Problem: Show that the curve given in Exercise 5 is spherical and find its radius and center.

Solution: First we calculate the following:

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)}$$

As shown in the previous exercise, if this value is constant then the curve is spherical and the value is its radius squared. Using exercise 5 we can see:

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 = \frac{(3t^2+1)^2}{(t+1)^3(9t+5)}$$

$$\frac{1}{\kappa^2(t)} = \frac{4(2t+1)^3}{(t+1)^3(9t+5)}$$

Adding these together we find that they simplify to the constant value 1. So it is a sphere with radius 1, now we just need to find the center. Again, given the results we found previously, we can say:

$$\mathbf{c} = \mathbf{x}(1) + \frac{1}{\kappa(1)}\mathbf{N}(1) - \frac{\kappa'(t)}{v(1)\tau(1)\kappa^2(1)}\mathbf{B}(1)$$

This evaluates to $\mathbf{c} = (1, 0, 0)$