Math Statistics
Weekly HW 3

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For the following questions, consider a population with an unknown mean and variance. Suppose that we take the following size 5 sample: $2, 3, 2, 2.5, 4$.

Question 1

Problem: What is the sample mean $\bar{X}$?

Solution: The sample mean is given by:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{(def. of } \bar{X}\text{)}$$

$$\bar{x} = \frac{2+3+2+2.5+4}{5} = \boxed{2.7} \qquad \text{(realization of } \bar{X}\text{)}$$

Question 2

Problem: What is the sample variance S^2?

Solution: The sample variance is given by:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \qquad \text{(def. of } S^2\text{)}$$

$$s^2 = \frac{(.7^2 + .3^2 + .7^2 + .2^2 + 1.3^2)}{5-1} = \boxed{.7} \qquad \text{(realization of } S^2\text{)}$$

Question 3

Problem: Based on the sample, using the chi-squared distribution, give a 95% confidence interval for the standard deviation of the population.

Solution: Recall that, for a confidence level $1-\alpha$, the confidence interval for the variance is given by:

$$P\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}\right) = 1-\alpha$$

In our case we have a confidence level of $1-\alpha = .95 \implies \alpha = .05$ and a sample size of $n = 5$. And so our interval is given by:

$$\begin{aligned}
.95 &= P\left(\frac{(5-1).7}{\chi^2_{.05/2,5-1}} \leq \sigma^2 \leq \frac{(5-1).7}{\chi^2_{1-.05/2,5-1}}\right) \\
&= P\left(\frac{2.8}{\chi^2_{.025,4}} \leq \sigma^2 \leq \frac{2.8}{\chi^2_{.975,4}}\right) \\
&= P\left(\frac{2.8}{11.1433} \leq \sigma^2 \leq \frac{2.8}{0.4844}\right) \\
&= P(0.25127 \leq \sigma^2 \leq 5.78035) \\
&= \boxed{P(0.50127 \leq \sigma \leq 2.40419)}
\end{aligned}$$

Question 4

Problem: Based on the sample, using the t-distribution, give a 95% confidence interval for the mean of the population.

Solution: Recall that, for a confidence level $1-\alpha$, the endpoint of the confidence interval for the mean are given by:

$$\bar{X} \pm t_{1-\alpha/2,n-1}\sqrt{\frac{s^2}{n}}$$

In our case we have a confidence level of $1-\alpha = .95 \implies \alpha = .05$ and a sample size of $n = 5$. And so our end points are given by:

$$\begin{aligned}
\bar{x} \pm t_{1-.05/2,5-1}\sqrt{\frac{.7}{5}} &= (2.7) \pm t_{.975,4}(.3742) \\
&= 2.7 \pm 2.776(.3742) \\
&= 2.7 \pm 1.03878
\end{aligned}$$

This results in the following confidence interval:

$$\boxed{P(1.6612 \leq \mu \leq 3.7388)}$$