Numerical Analysis HW #3

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Problem 1

Problem: Given a function f for which the following values are given:

$$\begin{aligned} f(8.1) &= 16.94410 \qquad & f(8.3) &= 17.56492 \\ f(8.6) &= 18.50515 \qquad & f(8.7) &= 18.82091 \end{aligned}$$

approximate $f(8.4)$ using both f's degree 1 and degree 2 interpolating polynomials over the interval $[8.1, 8.6]$.

Solution: Given $k+1$ data points the Lagrange basis polynomials are given by:

$$\ell_j(x) := \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{x - x_i}{x_j - x_i}$$

Linear Case And so for the 1st degree, i.e. linear, interpolating polynomial we use the following nodes:

$$\begin{aligned} (x_0, y_0) &= (8.1, 16.94410) \\ (x_1, y_1) &= (8.6, 18.50515) \end{aligned}$$

giving us the following basis polynomials:

$$\begin{aligned} \ell_0(x) &= \frac{x - 8.6}{8.1 - 8.6} = -\frac{x - 8.6}{.5} = -2(x - 8.6) \\ \ell_1(x) &= \frac{x - 8.1}{8.6 - 8.1} = \frac{x - 8.1}{.5} = 2(x - 8.1) \end{aligned}$$

The following linear combination gives us the interpolating polynomial:

$$\begin{aligned} p_1(x) &= y_0\ell_0(x) + y_1\ell_1(x) \\ &= (16.94410)(-2)(x - 8.6) + (18.50515)(2)(x - 8.1) \\ &= -33.8882(x - 8.6) + 37.0103(x - 8.1) \\ &= 3.1221x - 8.34491 \end{aligned}$$

Evaluating $p_1(8.4)$ gives us our linear approximation:

$$p_1(8.4) = 3.1221(8.4) - 8.34491 = \boxed{17.88073}$$

Quadratic Case Now we do the same for the 2nd degree, i.e. quadratic, interpolating polynomial. It's nodes are:

$$\begin{aligned}(x_0, y_0) &= (8.1, 16.94410)\\ (x_1, y_1) &= (8.3, 17.56492)\\ (x_2, y_2) &= (8.6, 18.50515)\end{aligned}$$

giving us the following Lagrange basis polynomials:

$$\begin{aligned}\ell_0(x) &= \left(\frac{x-8.3}{8.1-8.3}\right)\left(\frac{x-8.6}{8.1-8.6}\right) = 10(x^2 - 16.9x + 71.38)\\ \ell_1(x) &= \left(\frac{x-8.1}{8.3-8.1}\right)\left(\frac{x-8.6}{8.3-8.6}\right) = \frac{-50}{3}(x^2 - 16.7x + 69.66)\\ \ell_2(x) &= \left(\frac{x-8.1}{8.6-8.1}\right)\left(\frac{x-8.3}{8.6-8.3}\right) = \frac{20}{3}(x^2 - 16.4x + 67.23)\end{aligned}$$

The following linear combination of the basis' gives us our interpolation:

$$\begin{aligned}p_2(x) &= y_0\ell_0(x) + y_1\ell_1(x) + y_2\ell_2(x)\\ &= (16.94410)(10)(x^2 - 16.9x + 71.38)\\ &\quad + (17.56492)\left(\frac{-50}{3}\right)(x^2 - 16.7x + 69.66)\\ &\quad + (18.50515)\left(\frac{20}{3}\right)(x^2 - 16.4x + 67.23)\\ &= 0.0600147x^2 + 2.11987x - 4.16437\end{aligned}$$

Evaluating $p_2(8.4)$ gives us our quadratic approximation:

$$p_2(8.4) = 0.0600147(8.4)^2 + 2.11987(8.4) - 4.16437 = \boxed{17.877132}$$

Problem 2

Problem: The values above were truncated from their original values given by $f(x) = x \ln x$. Find a bound for the error in both the linear and quadratic cases and compare it to the results in problem 1.

Solution: Recall that, assuming f is $n+1$ times differentiable, the error of p_n is given by the following:

$$\text{error}(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)$$

Where $\xi \in [x_0, x_n]$. We can bound this error by simply choosing a ξ such that $f^{(n+1)}$ is maximized:

$$|\text{error}(x)| \leq \max_{x_0 \leq \xi \leq x_n}\left|f^{(n+1)}(\xi)\right|\left|\frac{1}{(n+1)!}\prod_{i=0}^{n}(x - x_i)\right|$$

Linear Case Using this, we can provide error bounds for both the linear and quadratic linear interpolations derived in problem 1. For the linear case, we note that f'' is given by:

$$f(x) = x \ln x \qquad f'(x) = \ln x + 1 \qquad f''(x) = \frac{1}{x}$$

To maximize this f'' over the interval $[x_0, x_1]$, we note that it is a decreasing function over the positive reals and since $0 < x_0 < x_1$ we can say the function reaches a maximum at x_0:

$$\max_{x_0 \leq \xi \leq x_1} |f''(\xi)| = \max_{x_0 \leq \xi \leq x_1} \left| \frac{1}{\xi} \right| = \frac{1}{x_0} = \frac{1}{8.1}$$

We can now solve for the error bound:

$$\begin{aligned}
|\operatorname{error}(x)| &\leq \max_{x_0 \leq \xi \leq x_n} \left| f^{(n+1)}(\xi) \right| \left| \frac{1}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \right| \\
&= \max_{x_0 \leq \xi \leq x_n} |f''(\xi)| \left| \frac{(x - x_0)(x - x_1)}{2} \right| \\
&= \frac{1}{8.1} \left| \frac{(x - 8.1)(x - 8.6)}{2} \right| \\
&= \frac{1}{16.2} |x^2 - 16.7x + 69.66|
\end{aligned}$$

Plugging in $x = 8.4$ into the error bound, we arrive at:

$$|\operatorname{error}(8.4)| \leq \boxed{0.003704}$$

Even further, the maximum error of the linear interpolation over the entire interval can be found by finding the vertex of the error bound. The maximizing $x = \frac{-b}{2a} = 8.3994$. Plugging this in gives us:

$$\max_{x_0 \leq x \leq x_n} |\operatorname{error}(x)| \leq 0.003851$$

The actual error of our approximation was:

$$f(8.4) - p_1(8.4) = 17.87715 - 17.88073 = -0.003584$$

We can compare the absolute value of these errors like so:

$$\underbrace{0.003584}_{\text{actual}} < \underbrace{0.003704}_{\text{bound}} < \underbrace{0.003851}_{\text{max bound}}$$

Quadratic Case We can now repeat the process for the quadratic interpolation polynomial. First we note that $f'''(x)$ is given by:

$$f''(x) = \frac{1}{x} \qquad f'''(x) = \frac{-1}{x^2}$$

To maximize $|f'''(x)|$ over the interval, we note that it is an increasing function over the positive reals and since $0 < x_0 < x_2$ we can say the function reaches a maximum at x_2:

$$\max_{x_0 \leq \xi \leq x_2} |f'''(\xi)| = \max_{x_0 \leq \xi \leq x_1} \left| \frac{-1}{\xi^2} \right| = \frac{1}{x_0^2} = \frac{1}{65.61}$$

We can now solve for the error bound:

$$
\begin{aligned}
|\operatorname{error}(x)| &\le \max_{x_0 \le \xi \le x_n} \left| f^{(n+1)}(\xi) \right| \left| \frac{1}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \right| \\
&= \max_{x_0 \le \xi \le x_n} |f'''(\xi)| \left| \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} \right| \\
&= \frac{1}{65.61} \left| \frac{(x-8.1)(x-8.3)(x-8.6)}{6} \right| \\
&= \frac{1}{393.66} |x^3 - 25x^2 + 208.27x - 578.178|
\end{aligned}
$$

Plugging in $x = 8.4$ into the error bound, we arrive at:

$$|\operatorname{error}(8.4)| \le \boxed{0.000015242}$$

The actual error of our approximation was:

$$f(8.4) - p_2(8.4) = 17.87715 - 17.87713 = 0.000014$$

We can compare the absolute value of these errors like so:

$$\underbrace{0.000014}_{\text{actual}} < \underbrace{0.000015242}_{\text{bound}}$$

Problem 3

Problem: Use the divided differences method to construct an interpolating polynomial of the following 4 points:

$$(-0.1, 5.3), (0, 2), (0.2, 3.19), (0.3, 1)$$

Solution: Recall that the divided difference of n nodes is given by:

$$f[x_i, \ldots, x_{i+j}] := \frac{f[x_{i+1}, \ldots, x_{i+j}] - f[x_i, \ldots, x_{i+j-1}]}{x_{i+j} - x_i}$$

The relevant divided differences are given in the following table:

x_i	y_i	1st Order Diff.	2nd Order Diff.	3rd Order Diff.
-0.1	5.3			
		-33		
0	2		129.833	
		5.95		-556.667
0.2	3.19		-92.833	
		-21.9		
0.3	1			

The interpolating cubic polynomial is given by:

$$p(x) = [y_0] + [y_0, y_1](x - x_0) + [y_0, y_1, y_2](x - x_0)(x - x_1) + [y_0, y_1, y_2, y_3](x - x_0)(x - x_1)(x - x_2)$$

Plugging in our divided differences we arrive at:

$$
\begin{aligned}
p(x) &= 5.3 - 33(x + 0.1) + 129.833x(x + 0.1) - 556.667x(x + 0.1)(x - 0.2) \\
&= -556.667x^3 + 185.5x^2 - 8.88333x + 2
\end{aligned}
$$

Problem 4

Problem a: Use MATLAB to plot the error between the function $\frac{1}{1+x^2}$ and its nth degree polynomial interpolation over the interval $[-5, 5]$, where $n = 4, 8, 16, 32$ and the interpolation points are uniformly spaced. Record the approximate maximum error magnitude.

Solution: The apparent maximum errors are:

n	$\approx \lvert e_{\max} \rvert$
4	0.4383
8	1.045
16	14.01
32	4641

Problem b: Repeat part a but this time use the Chebyshev points to interpolate the polynomial instead:

$$x_i = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{(2i+1)\pi}{2n+2}\right)$$

Solution: The apparent maximum errors are:

n	$\approx \lvert e_{\max} \rvert$
4	0.2002
8	0.1194
16	0.03258
32	0.001304

Problem c: Based on these results, does the choice of interpolation points make a difference in the error of the interpolating polynomial? Which choice is better?

Solution: The choice of nodes clearly makes a huge impact in the error of the interpolation, especially as the number of nodes n increases. Indeed, in the first test with uniformly spaced nodes, the error seemed to grow hyper-exponentially with respect to n. Meanwhile, the error in the Chebyshev interpolation seemed to approach zero. Clearly this form of interpolation is superior to that of a uniform distribution (at least for continuous, infinitely differentiable functions).