Math Statistics
Weekly HW 1

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Question 1

Problem: Show that if events A and B are independent, then so are $A^\complement$ and $B^\complement$.

Solution: Note the following:

$$\begin{aligned}
P(A^\complement B^\complement) &= P((A \cup B)^\complement) && \text{(DeMorgan's Law)} \\
&= 1 - P(A \cup B) && \text{(prob. of complement)} \\
&= 1 - P(A) - P(B) + P(AB) && \text{(inclusion-exclusion principle)} \\
&= 1 - P(A) - P(B) + P(A)P(B) && \text{(independence of } A \text{ and } B\text{)} \\
&= (1 - P(A))(1 - P(B)) && \text{(factorization)} \\
&= P(A^\complement)P(B^\complement) && \text{(prob. of complement)}
\end{aligned}$$

And so, assuming A and B are independent, we have that $P(A^\complement B^\complement) = P(A^\complement)P(B^\complement)$ and thus, by definition, $A^\complement$ is independent of $B^\complement$.

Question 2

Problem: Show that if events A and B are independent, then so are $A^\complement$ and B.

Solution: Note the following:

$$\begin{aligned}
P(A^\complement B) &= P(A^\complement \mid B)P(B) && \text{(chain rule)} \\
&= (1 - P(A \mid B))P(B) && \text{(prob. of complement)} \\
&= P(B) - P(A \mid B)P(B) && \text{(multiplicative distributivity)} \\
&= P(B) - P(AB) && \text{(chain rule)} \\
&= P(B) - P(A)P(B) && \text{(independence of } A \text{ and } B\text{)} \\
&= (1 - P(A))P(B) && \text{(multiplicative distributivity)} \\
&= P(A^\complement)P(B) && \text{(prob. of complement)}
\end{aligned}$$

And so, assuming A and B are independent, we have that $P(A^\complement B) = P(A^\complement)P(B)$ and thus, by definition, we have that $A^\complement$ is independent of B.

Question 3

Problem: Consider a RV X with the following cdf:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{4} & \text{if } 1 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$

Find $P(X = 2), P(2 \le X \le 5)$, and $P(X = 3)$.

Solution: The desired probabilites are given by:

$$\begin{aligned}
P(X=2) &= F_X(2) - \lim_{x\to 2^+} F_X(x) \\
&= \frac{1}{4} - \frac{1}{4} = \boxed{0} \\
P(2 \le X \le 5) &= F_X(5) - F_X(2) \\
&= 1 - \frac{1}{4} = \boxed{\frac{3}{4}} \\
P(X=3) &= F_X(3) - \lim_{x\to 3^+} F_X(x) \\
&= 1 - \frac{1}{4} = \boxed{\frac{3}{4}}
\end{aligned}$$

Question 4

Problem: Consider a RV X with support $[0,5]$ and the following pdf:

$$f_X(x) = ke^{-3x}$$

Solve for k, and give $P(X \ge 3)$.

Solution: Recalling that the measure of a pdf over its support should be 1, we can solve for k like so:

$$\begin{aligned}
1 &= \int_0^5 f_X(x)\,dx \\
&= \int_0^5 ke^{-3x}\,dx \\
&= -k\left[\frac{e^{-3x}}{3}\right]_0^5 \\
&= -k\frac{e^{-15}-1}{3} \\
k &= \boxed{\frac{3}{1-e^{-15}}}
\end{aligned}$$

Now that we have k, we can solve for $P(X \ge 3)$:

$$\begin{aligned}
P(X \ge 3) &= \int_3^5 f_X(x)\,dx \\
&= \frac{3}{1-e^{-15}}\int_3^5 e^{-3x}\,dx \\
&= \frac{3}{1-e^{-15}}\left[\frac{e^{-3x}}{3}\right]_3^5 \\
&= \frac{3}{1-e^{-15}}\left(\frac{e^{-9}-e^{-15}}{3}\right) \\
&= \boxed{\frac{e^{-9}-e^{-15}}{1-e^{-15}}}
\end{aligned}$$

Question 5

Problem: Let X denote the number of heads obtained by flipping a fair coin once. Find $E[X]$.

Solution: Clearly, $X \sim$ Bernoulli(0.5). And so it's expected value is:

$$E[X] = 0(0.5) + 1(0.5) = \boxed{0.5}$$

Question 6

Problem: Suppose a fair coin is tossed n times. Let X_i denote the number of heads on the ith toss. Find $E\left[\sum_{i=1}^{n} X_i\right]$.

Solution: Just as in question 5, we have that each $X_i \sim$ Bernoulli(0.5), giving us:

$$\begin{aligned}
E\left[\sum_{i=1}^{n} X_i\right] &= \sum_{i=1}^{n} E[X_i] && \text{(linarity of expectation)} \\
&= \sum_{i=1}^{n} 0.5 && \text{(mean of Bernoulli distribution)} \\
&= \boxed{0.5n}
\end{aligned}$$

Question 7

Problem: Consider a RV X with support $[1, 5]$ and the following pdf:

$$f_X(x) = \frac{1}{x \ln 5}$$

Find $E[X], E[X^2]$, and $E[X^3]$.

Solution: The desired expectations are given below:

$$\begin{aligned}
E[X] &= \int_1^5 \frac{x}{x \ln 5}\, dx \\
&= \frac{1}{\ln 5} \int_1^5 1\, dx \\
&= \boxed{\frac{4}{\ln 5}}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \int_1^5 \frac{x^2}{x \ln 5}\, dx \\
&= \frac{1}{\ln 5} \int_1^5 x\, dx \\
&= \frac{1}{\ln 5} \left[\frac{x^2}{2}\right]_1^5 \\
&= \frac{25 - 1}{2 \ln 5} = \boxed{\frac{12}{\ln 5}}
\end{aligned}$$

$$
\begin{aligned}
E[X^3] &= \int_1^5 \frac{x^3}{x \ln 5}\, dx \\
&= \frac{1}{\ln 5} \int_1^5 x^2\, dx \\
&= \frac{1}{\ln 5} \left[\frac{x^3}{3} \right]_1^5 \\
&= \frac{125 - 1}{3 \ln 5} = \boxed{\frac{124}{3 \ln 5}}
\end{aligned}
$$

Question 8

Problem: Consider a RV X with support $[1, 5]$ and the following pdf:

$$
f_X(x) = \frac{1}{4}
$$

Find $E[X]$ and $\mathrm{Var}(X)$.

Solution: Clearly $X \sim \mathcal{U}(1, 5)$. It's expectation is given by:

$$
\begin{aligned}
E[X] &= \int_1^5 \frac{x}{4}\, dx \\
&= \frac{1}{4} \left[\frac{x^2}{2} \right]_1^5 \\
&= \frac{1}{4} \left(\frac{25 - 1}{2} \right) = \boxed{3}
\end{aligned}
$$

And the variance is given by:

$$
\begin{aligned}
\mathrm{Var}(X) &= E[(X - E[X])^2] \\
&= E[(X - 1)^2] \\
&= \int_1^5 \frac{(x-1)^2}{4} \\
&= \frac{1}{4} \int_1^5 x^2 - 2x + 1\, dx \\
&= \frac{1}{4} \left[\frac{x^3}{3} - x^2 + x \right]_1^5 \\
&= \frac{\left(\frac{125}{3} - 25 + 5\right) - \left(\frac{1}{3} - 1 + 1\right)}{4} = \boxed{\frac{4}{3}}
\end{aligned}
$$