Linear Algebra HW #4

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Problem 1

Problem: Are the following statements true or false. Justify your answers.

a) $\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\alpha^\beta$

b) $\mathbb{F}^{2\times 3} \cong \mathbb{F}^5$

c) $AB = I$ implies that A and B are invertible.

d) If a matrix A is invertible then $(A^{-1})^{-1} = A$

e) Only square matrices have inverses.

Solution: **a)** is false, the correct identity is:

$$\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\beta^\alpha$$

In light of this identity we have that **a)** equates a matrix that takes the basis β to α to one that does the reverse, which clearly cannot be the case in general.

b) is false. Consider the following bases α and β for $\mathbb{F}^{2\times 3}$ and $\mathbb{F}^5$ respectively:

$$\alpha = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Clearly, since α is a basis of $\mathbb{F}^{2\times 3}$ and $|\alpha| = 6$, we have that $\dim(\mathbb{F}^{2\times 3}) = 6$. Similar reasoning leads us to $\dim(\mathbb{F}^5) = 5$. And since $\dim(\mathbb{F}^5) < \dim(\mathbb{F}^{2\times 3})$, they cannot be isomorphic as no linear bijection exists between them.

c) is false. Consider the following matrix product of A and B:

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

While the product $AB = I$, neither A nor B are invertible because they are not square.

d) is true. Note the following two identities:

$$\begin{aligned} A^{-1}A &= I && \text{(def. of inverse)} \\ A^{-1}(A^{-1})^{-1} &= I && \text{(def. of inverse)} \end{aligned}$$

As we can see both A and $(A^{-1})^{-1}$ are inverses of A. But since inverses are unique in a group, and the set of invertible matrices $\mathrm{GL}_n(\mathbb{F})$ is certainly a group, we must have that $A = (A^{-1})^{-1}$.

e) is true, by the definition of invertible matrix.

Problem 2

Problem: Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Solution: Recall that an isomorphism is a linear bijection. For $\mathbf{v}_i \in \beta$ we have:

$$\begin{aligned} a_i = 0 &\iff \sum_{i=1}^{n} a_i \mathbf{v}_i = \mathbf{0} && (\beta \text{ is a basis}) \\ &\iff T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = T(\mathbf{0}) && (T \text{ is bijective}) \\ &\iff \sum_{i=1}^{n} a_i T(\mathbf{v}_i) = \mathbf{0} && (T \text{ is linear}) \end{aligned}$$

And so the vectors $T(\mathbf{v}_i) \in W$ form a linearly independent set. Now note that, since T is surjective, we have that $R(T) = W$. This means that:

$$\mathrm{span}(T(\beta)) = R(T) = W$$

And so $T(\beta)$ spans all of W. Along with its linear independence, this implies that $T(\beta)$ is a basis of W.

Problem 3

Problem: In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Let T be the projection on L along the line perpendicular to L. Find an expression for $T(x, y)$.

Solution: Consider the basis $\beta' = \{\left[\begin{smallmatrix} 1 \\ m \end{smallmatrix}\right], \left[\begin{smallmatrix} -m \\ 1 \end{smallmatrix}\right]\}$. Under this basis, our transformation maps $\mathbf{e}_1$ and $\mathbf{e}_2$ like so:

$$\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Giving us the following matrix representative of T in the basis β':

$$[T]_{\beta'}^{\beta'} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, to express this matrix in our desired basis $\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$, we simply have to compute the change of basis matrix from β' to β and its inverse. This gives us:

$$\begin{aligned}
[\mathrm{id}]_{\beta'}^{\beta} &= [\Phi_\beta(\mathrm{id}(\mathbf{v}_i))] && (\mathbf{v}_i \in \beta') \\
&= [\Phi_\beta(\mathbf{v}_i)] \\
&= \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} && \begin{pmatrix} \mathbf{e}_1 + m\mathbf{e}_2 = \left[\begin{smallmatrix} 1 \\ m \end{smallmatrix}\right] \\ -m\mathbf{e}_1 + \mathbf{e}_2 = \left[\begin{smallmatrix} -m \\ 1 \end{smallmatrix}\right] \end{pmatrix} \\
[\mathrm{id}]_{\beta}^{\beta'} = ([\mathrm{id}]_{\beta'}^{\beta})^{-1} &= \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} && (2 \times 2 \text{ matrix inverse})
\end{aligned}$$

And now we can express our desired matrix representative $[T]_\beta^\beta$ as the following matrix product:

$$\begin{aligned}
[T]_\beta^\beta &= [\mathrm{id}]_{\beta'}^{\beta} [T]_{\beta'}^{\beta'} [\mathrm{id}]_{\beta}^{\beta'} \\
&= \frac{1}{1+m^2} \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} \\
&= \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}
\end{aligned}$$

And so, using the coorespondence between linear operators and their associated matrix representatives, we can finally express $T(x, y)$ in terms of x and y:

$$\begin{aligned}
T(x, y) &= [T]_\beta^\beta \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \frac{1}{1+m^2} \begin{bmatrix} x + my \\ mx + m^2 y \end{bmatrix}
\end{aligned}$$

Problem 4

Problem: Let $A = \left[\begin{smallmatrix} 1 & 3 \\ 1 & 1 \end{smallmatrix}\right]$ and $\beta = \{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]\}$. Find $[A]_\beta$, and find an invertible matrix Q such that $[A]_\beta = Q^{-1}AQ$.

Solution: Note that A is expressed in the standard basis, so we can denote it $[A]_\alpha$. We then have the following:

$$\begin{aligned}
[A]_\beta &= [\mathrm{id}]_\alpha^\beta [A]_\alpha [\mathrm{id}]_\beta^\alpha \\
&= Q^{-1}AQ && (\text{let } Q = [\mathrm{id}]_\beta^\alpha)
\end{aligned}$$

And now we calculate Q:

$$\begin{aligned}
Q &= [\mathrm{id}]_\beta^\alpha \\
&= [\Phi_\alpha(\mathrm{id}(\beta))] \\
&= [\Phi_\alpha(\beta)] \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
\end{aligned}$$

And its inverse

$$\begin{aligned}
Q^{-1} &= [\mathrm{id}]_{\alpha}^{\beta} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned}$$

With Q and Q^{-1} we can finally calculate $[A]_{\beta}$:

$$\begin{aligned}
[A]_{\beta} &= Q^{-1}AQ && (\text{let } Q = [\mathrm{id}]_{\beta}^{\alpha}) \\
&= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 11 \\ -2 & -4 \end{bmatrix}
\end{aligned}$$

Problem 5

Problem: Verify that the following sets are bases of $\mathcal{P}_2(\mathbb{R})$:

$$\begin{aligned}
\beta &= \{x^2 - x + 1, x + 1, x^2 + 1\} \\
\beta' &= \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}
\end{aligned}$$

And then find the change of coordinate matrix $[\mathrm{id}]_{\beta}^{\beta'}$ from β to β'.

Solution: Recall that $\mathcal{P}_2(\mathbb{R}) \cong \mathbb{R}^3$, and thus we can coordinatize each polynomial under the standard basis $\{1, x, x^2\}$. Putting these coordinatized vectors in a matrix, we can perform Gaussian elimination to determine the matrix's rank and thus the dimension the polynomials span. We start with β:

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] &\xrightarrow{r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \\
&\xrightarrow{r_3 - r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \\
&\xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]
\end{aligned}$$

Once reduced, we see that the matrix has 3 pivot rows and thus is of rank 3. As such, β not only spans all 3 dimensions of $\mathcal{P}_2(\mathbb{R})$ but is a basis of it since $|\beta| = 3$. Now we do the same for β':

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 1 & -3 & 0 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] &\xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -7 & -2 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \\
&\xrightarrow{r_3 - 4r_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -7 & -2 & 0 \\ 0 & -14 & -5 & 0 \end{array}\right] \\
&\xrightarrow{r_3 - 2r_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -7 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]
\end{aligned}$$

Again, we see that the matrix is of rank 3. And since $|\beta'| = 3$, it is a basis of $\mathcal{P}_2(\mathbb{R})$.
Now all that's left is to compute the change of basis matrix $[\mathrm{id}]_\beta^{\beta'}$:

$$[\mathrm{id}]_\beta^{\beta'} = [\Phi_{\beta'}(\mathrm{id}(\mathbf{v}_i))] = [\Phi_{\beta'}(v_i)] \qquad (\mathbf{v}_i \in \beta)$$

To do this, we must perform the ardous task of solving 3 systems of equations to find what the coordinatization of the vectors in β are in terms of β'. Of course, just as when we checked if they were bases, we will be operating on their coordinatized form w.r.t. the standard basis $\{1, x, x^2\}$ rather than their polynomial form:

$$[\beta' \mid \mathbf{v}_1] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 1 & -3 & 0 & -1 \\ 4 & 2 & 3 & 1 \end{array}\right] \xrightarrow[r_3-4r_1]{r_2-r_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -7 & -2 & -2 \\ 0 & -14 & -5 & -3 \end{array}\right] \xrightarrow{r_3-2r_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -7 & -2 & -2 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[r_1+2r_3]{r_2-2r_3} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & -7 & 0 & -4 \\ 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow[-r_3]{(-1/7)r_2} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 1 & 0 & 4/7 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{r_1-4r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/7 \\ 0 & 1 & 0 & 4/7 \\ 0 & 0 & 1 & -1 \end{array}\right] \implies \Phi_{\beta'}(\mathbf{v}_1) = \frac{1}{7}\begin{bmatrix} 5 \\ 4 \\ -7 \end{bmatrix}$$

$$[\beta' \mid \mathbf{v}_2] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 1 & -3 & 0 & 1 \\ 4 & 2 & 3 & 1 \end{array}\right] \xrightarrow[r_3-4r_1]{r_2-r_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -7 & -2 & 1 \\ 0 & -14 & -5 & 1 \end{array}\right] \xrightarrow{r_3-2r_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -7 & -2 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[r_1+2r_3]{r_2-2r_3} \left[\begin{array}{ccc|c} 1 & 4 & 0 & -2 \\ 0 & -7 & 0 & 3 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow[-r_3]{(-1/7)r_2} \left[\begin{array}{ccc|c} 1 & 4 & 0 & -2 \\ 0 & 1 & 0 & -3/7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{r_1-4r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2/7 \\ 0 & 1 & 0 & -3/7 \\ 0 & 0 & 1 & 1 \end{array}\right] \implies \Phi_{\beta'}(\mathbf{v}_2) = \frac{1}{7}\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

$$[\beta' \mid \mathbf{v}_3] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 1 & -3 & 0 & 0 \\ 4 & 2 & 3 & 1 \end{array}\right] \xrightarrow[r_3-4r_1]{r_2-r_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -7 & -2 & -1 \\ 0 & -14 & -5 & -3 \end{array}\right] \xrightarrow{r_3-2r_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -7 & -2 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[r_1+2r_3]{r_2-2r_3} \left[\begin{array}{ccc|c} 1 & 4 & 0 & -1 \\ 0 & -7 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow[-r_3]{(-1/7)r_2} \left[\begin{array}{ccc|c} 1 & 4 & 0 & -1 \\ 0 & 1 & 0 & -1/7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{r_1-4r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/7 \\ 0 & 1 & 0 & -1/7 \\ 0 & 0 & 1 & 1 \end{array}\right] \implies \Phi_{\beta'}(\mathbf{v}_3) = \frac{1}{7}\begin{bmatrix} -3 \\ -1 \\ 7 \end{bmatrix}$$

And so we can finally express the change of basis matrix as the following:

$$[\mathrm{id}]_\beta^{\beta'} = [\Phi_{\beta'}(v_i)] = \frac{1}{7}\begin{bmatrix} 5 & -2 & -3 \\ 4 & -3 & -1 \\ -7 & 7 & 7 \end{bmatrix}$$