Foundations of QM
HW 3

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Consider a particle in 2D space with the following wavefunction, for some constant A:

$$\psi(x,y) = \begin{cases} Ae^{i(3x+4y)}, & 0<x<2,\, 0<y<3 \\ 0, & \text{otherwise} \end{cases}$$

Also, let the random vector (X,Y) denote the position obtained by measuring the particle given by ψ.

Question 1

Problem: Solve for A.

Solution: First let us establish the following result, call it lemma 1, for any $c \in \mathbb{R}$:

$$\begin{aligned}
|e^{ic}|^2 &= |\cos x + i\sin x|^2 && \text{(Euler's formula)} \\
&= \left(\sqrt{(\cos x)^2 + (\sin x)^2}\right)^2 && \text{(def. of modulus)} \\
&= \left(\sqrt{1}\right)^2 && \text{(trig identity)} \\
&= 1
\end{aligned}$$

With this lemma in hand we can now solve for A. For $\psi(x,y)$ to be a valid wavefunction, we must have:

$$\begin{aligned}
1 &= \iint\limits_{(x,y)\in\mathbb{R}^2} |\psi(x,y)|^2\,dx\,dy && (\psi \text{ is a wavefunction}) \\
&= \int_0^3\int_0^2 \left|Ae^{i(3x+4y)}\right|^2\,dx\,dy && \text{(from def. of } \psi) \\
&= \int_0^3\int_0^2 \left(|A|\left|e^{i(3x+4y)}\right|\right)^2\,dx\,dy && \text{(multiplicativity of modulus)} \\
&= |A|^2\int_0^3\int_0^2 \left|e^{i(3x+4y)}\right|^2\,dx\,dy && \text{(linearity)} \\
&= |A|^2\int_0^3\int_0^2 dx\,dy && \text{(lemma 1)} \\
&= |A|^2\int_0^3 [x]_0^2\,dy \\
&= |A|^2\,[2y]_0^3 \\
&= 6|A|^2 \\
\frac{1}{6} &= |A|^2 \\
\frac{1}{\sqrt{6}} &= |A|
\end{aligned}$$

And so A is any complex number whose modulus is equal to $\frac{1}{\sqrt{6}}$. If we limit ourselves to the positive real numbers, then $A = \frac{1}{\sqrt{6}}$.

Question 2

Problem: What is the pdf of the measured position of the particle?

Solution: The joint pdf of the measured particle (X, Y) is given by:

$$\begin{aligned} f_{X,Y}(x, y) &= |\psi(x, y)|^2 \\ &= \left| \frac{e^{i(3x+4y)}}{\sqrt{6}} \right|^2 \\ &= \frac{\left|e^{i(3x+4y)}\right|^2}{6} \\ &= \frac{1}{6} \end{aligned}$$

With support $[0, 2] \times [0, 3]$. And so the position (X, Y) has a uniform probability distribution over its support.

Question 3

Problem: What is the probability that $X > Y$?

Solution: The desired probability is given by:

$$\begin{aligned} P(X > Y) &= \iint\limits_{x>y} f_{X,Y}(x, y)\, dy\, dx \\ &= \int_0^2 \int_0^x |\psi(x, y)|^2\, dy\, dx \\ &= \frac{1}{6} \int_0^2 \int_0^x dy\, dx \\ &= \frac{1}{6} \int_0^2 [y]_0^x\, dx \\ &= \frac{1}{6} \int_0^2 x\, dx \\ &= \frac{1}{6} \left[\frac{x^2}{2} \right]_0^2 \\ &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$