Math Statistics
Semiweekly HW 11

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Question 1

Problem: Prove that if $\hat{\theta}$ is a sufficient estimator for parmeter θ then, $\lambda\hat{\theta}$ is also a sufficient estimator of θ. Where $\lambda \in \mathbb{R}^+$.

Solution: Suppose our sample X is taken from the distribution characterized by the joint pdf $f_X(\mathbf{x};\theta)$, where θ is the parmeter. Also consider a statistic $\hat{\theta} : \mathcal{X} \to \mathcal{T}$. We know from the factorization theorem that $\hat{\theta}$ is a sufficient statistic iff there exists functions $h(\mathbf{x})$ and $g(\hat{\theta}(\mathbf{x});\theta)$ such that:

$$f_X(\mathbf{x};\theta) = h(\mathbf{x})g(\hat{\theta}(\mathbf{x});\theta)$$

Now consider a bijection $r : \mathcal{T} \to \mathcal{S}$. Note that r has an inverse r^{-1} since it is a bijection. As a result, we can define the following function:

$$g'(t;\theta) = g(r^{-1}(t);\theta)$$

We can now prove our result:

$$\begin{aligned} f_X(\mathbf{x};\theta) &= h(\mathbf{x})g(\hat{\theta}(\mathbf{x});\theta) && (\hat{\theta}\text{ is sufficient}) \\ &= h(\mathbf{x})g(r^{-1}(r(\hat{\theta}(\mathbf{x})));\theta) \\ &= h(\mathbf{x})g'(r(\hat{\theta}(\mathbf{x}));\theta) \end{aligned}$$

And so by the factorization theorem, we have shown that for any sufficient statistic $\hat{\theta}$ with codomain $\mathcal{T}$ and bijective function r with domain $\mathcal{T}$, the statistic $r(\hat{\theta})$ is also sufficient.

A simple corollary to this is that multiplication by a positive constant λ preserves sufficiency, since $r(t) = \lambda t$ is a bijection with its inverse being $r^{-1}(t) = \frac{t}{\lambda}$.