Foundations of QM
HW 2

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For the following questions consider the wavefunction ψ_s given by:

$$\psi_s(x) = \frac{1}{4\sqrt{3}} x^2 e^{-|x|/2}, \quad x \in \mathbb{R}$$

And the wave function ψ_a given by:

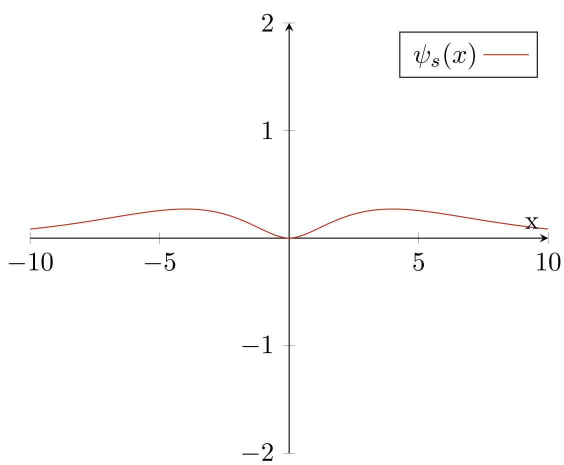
$$\psi_a(x) = \begin{cases} \psi_s(x), & x \geq 0 \\ -\psi_s(x), & x < 0 \end{cases}$$

Also, let X_n be a random variable that gives the position of a particle with wave function ψ_n when measured.

Question 1

Problem: Sketch $\psi_s(x)$:

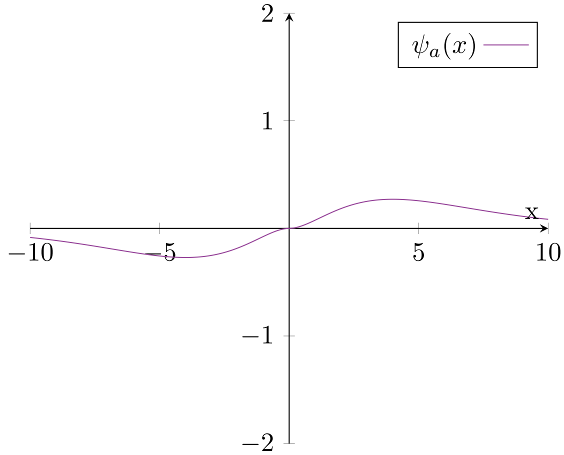
Solution:



Question 2

Problem: Sketch $\psi_a(x)$:

Solution:



Question 3

Problem: What is the relationship between the probability distribution of X_s and X_a?

Solution: These two random variables share the same distribution, that is:

$$X_s \sim X_a$$

To see this, let us denote the pdfs of X_s and X_a by p_s and p_a respectively. We now consider two cases:

- Case 1, $x \geq 0$:

$$\begin{aligned} p_a(x) &= |\psi_a(x)|^2 && \text{(pdf of a wavefunction)} \\ &= |\psi_s(x)|^2 && \text{(def. of } \psi_a(x) \text{ for } x \geq 0) \\ &= p_s(x) && \text{(pdf of a wavefunction)} \end{aligned}$$

- Case 2, $x < 0$:

$$\begin{aligned} p_a(x) &= |\psi_a(x)|^2 && \text{(pdf of a wavefunction)} \\ &= |-\psi_s(x)|^2 && \text{(def. of } \psi_a(x) \text{ for } x < 0) \\ &= |\psi_s(x)|^2 && \text{(modulus is invariant to sign)} \\ &= p_s(x) && \text{(pdf of a wavefunction)} \end{aligned}$$

And so we have shown that $\forall x \in \mathbb{R},\ p_a(x) = p_s(x)$. And since the pdfs of X_a and X_s are equivalent, they must share the same distribution.

Question 4

Problem: What is the probability that $X_s < 0$?

Solution: First note that $\psi_s(x)$ is an even function:

$$\begin{aligned}
\psi_s(-x) &= \left|\frac{1}{4\sqrt{3}}(-x)^2 e^{-|-x|/2}\right|^2 && \text{(def. of } \psi_s\text{)} \\
&= \left|\frac{1}{4\sqrt{3}}x^2 e^{-|-x|/2}\right|^2 && (x^2 = (-x)^2) \\
&= \left|\frac{1}{4\sqrt{3}}x^2 e^{-|x|/2}\right|^2 && (|x| = |-x|) \\
&= \psi_s(x) && \text{(def. of } \psi_s\text{)}
\end{aligned}$$

Now consider the following chain of equalities

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi_s(x)|^2 \, dx && \text{(integral of a pdf over support is 1)} \\
&= \int_{-\infty}^{0} |\psi_s(x)|^2 \, dx + \int_{0}^{\infty} |\psi_s(x)|^2 \, dx \\
&= \int_{-\infty}^{0} |\psi_s(x)|^2 \, dx - \int_{0}^{-\infty} |\psi_s(-x)|^2 \, dx && \text{(expansion of interval of integration by } -1\text{)} \\
&= \int_{-\infty}^{0} |\psi_s(x)|^2 \, dx + \int_{-\infty}^{0} |\psi_s(-x)|^2 \, dx && \text{(reverse interval of integration)} \\
&= \int_{-\infty}^{0} |\psi_s(x)|^2 \, dx + \int_{-\infty}^{0} |\psi_s(x)|^2 \, dx && (\psi_s \text{ is an even function}) \\
&= 2\int_{-\infty}^{0} |\psi_s(x)|^2 \, dx \\
\frac{1}{2} &= \int_{-\infty}^{0} |\psi_s(x)|^2 \, dx \\
&= P(X_s < 0)
\end{aligned}$$

And so, due to the symmetry of $\psi_s(x)$ about the y-axis, the desired probability is $\frac{1}{2}$.

Question 5

Problem: For $\psi_{s+a} = \frac{\psi_s + \psi_a}{\sqrt{2}}$, what is the probability that $X_{s+a} < 0$?

Solution: The desired probability is given by:

$$\begin{aligned}
P(X_{s+a} < 0) &= \int_{-\infty}^{0} |\psi_{s+a}(x)|^2 \, dx \\
&= \int_{-\infty}^{0} \left|\frac{\psi_s + \psi_a}{\sqrt{2}}\right|^2 \, dx && \text{(def. of } \psi_{s+a}\text{)} \\
&= \int_{-\infty}^{0} \left|\frac{\psi_s - \psi_s}{\sqrt{2}}\right|^2 \, dx && (\forall x < 0,\ \psi_a(x) = -\psi_s(x)) \\
&= \int_{-\infty}^{0} |0|^2 \, dx \\
&= 0
\end{aligned}$$

And so, due to the two wave functions destructively interfering with each other, the desired probability is 0.