Theory of Probability HW #9

Ozaner Hansha

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Problem 1

Problem: Suppose the moment generating function of a random variable X is given by:

$$M_X(t) = \frac{e^{t^2/2} + e^{t^3/6}}{2}$$

Compute the third moment of Y, where $Y = 2X + 1$.

Solution: The mgf of Y is given by:

$$\begin{aligned}
M_Y(t) &= E[e^{tY}] && \text{(def. of mgf)} \\
&= E[e^{t(2X+1)}] \\
&= E[e^{2tX+t}] \\
&= E[e^{2tX}e^t] \\
&= e^t E[e^{2tX}] \\
&= e^t M_X(2t) \\
&= e^t \left(\frac{e^{(2t)^2/2} + e^{(2t)^3/6}}{2} \right) \\
&= \frac{e^{2t^2+t} + e^{4t^3/3+t}}{2}
\end{aligned}$$

Taking the third derivative and evaluating at $t = 0$ gives us the third moment of Y:

$$\begin{aligned}
M_Y'(t) &= \frac{1}{2}\left((4t+1)e^{2t^2+t} + (4t^2+1)e^{4t^3/3+t}\right) \\
M_Y''(t) &= \frac{1}{2}\left((4t+1)^2e^{2t^2+t} + 4e^{2t^2+t} + (4t^2+1)^2e^{4t^3/3+t} + 8te^{4t^3/3+t}\right) \\
M_Y'''(t) &= \frac{1}{2}\left((4t+1)^3e^{2t^2+t} + 12(4t+1)e^{2t^2+t} + (4t^2+1)^3e^{4t^3/3+t} + 24t(4t^2+1)e^{4t^3/3+t} + 8e^{4t^3/3+t}\right) \\
E[Y^3] = M_Y'''(0) &= \frac{1}{2}(1+12+1+8) = \boxed{11}
\end{aligned}$$

Problem 2

Problem: Compute the moment generating function of X which has a discrete uniform distribution over the integer interval $[1..n]$.

Solution: The mgf of X is given by:

$$\begin{aligned}
M_X(t) &= E[e^{tX}] && \text{(def. of mgf)} \\
&= \sum_{i=1}^{n} e^{it} p_X(i) && \text{(def. of expectation)} \\
&= \frac{1}{n}\sum_{i=1}^{n} e^{it} && \text{(uniform distribution)} \\
&= \frac{1}{n}\left(\frac{e^t(1-e^{tn})}{1-e^t}\right) && \text{(finite geometric series)} \\
&= \boxed{\frac{e^t - e^{t(n+1)}}{n(1-e^t)}}
\end{aligned}$$

Problem 3

Problem: Suppose a random variable X has the following mgf:

$$M_X(t) = \frac{e^{-2t}}{10} + \frac{e^{-t}}{5} + \frac{2}{10} + \frac{e^t}{5} + \frac{e^{2t}}{10}$$

Compute $P(|X| \leq 1)$.

Solution: Consider the random variable $\widetilde{X}$ with the following probability distribution:

$$\begin{aligned}
p_{\widetilde{X}}(-2) &= \frac{1}{10} & p_{\widetilde{X}}(-1) &= \frac{1}{5} & p_{\widetilde{X}}(0) &= \frac{2}{5} \\
p_{\widetilde{X}}(1) &= \frac{1}{5} & p_{\widetilde{X}}(2) &= \frac{1}{10}
\end{aligned}$$

Where the probability of any other value occuring is 0. Note that, as the probabilities sum to 1, this is indeed a valid probability distribution. Also note that the mgf of $\widetilde{X}$ is given by:

$$\begin{aligned}
M_{\widetilde{X}}(t) &= E[e^{t\widetilde{X}}] \\
&= \sum_{i=-2}^{2} e^{ti} p_{\widetilde{X}}(i) \\
&= \frac{e^{-2t}}{10} + \frac{e^{-t}}{5} + \frac{2}{10} + \frac{e^t}{5} + \frac{e^{2t}}{10}
\end{aligned}$$

Note that $M_{\widetilde{X}}(t) = M_X(t)$, implying that $\widetilde{X} \sim X$. That is, they share the same probability distribution. And so, the desired probability is given by:

$$\begin{aligned}
P(|X| \leq 1) &= P(|\widetilde{X}| \leq 1) \\
&= P(-1 \leq \widetilde{X} \leq 1) \\
&= p_{\widetilde{X}}(-1) + p_{\widetilde{X}}(0) + p_{\widetilde{X}}(1) \\
&= \frac{1}{5} + \frac{2}{5} + \frac{1}{5} = \boxed{\frac{4}{5}}
\end{aligned}$$

Problem 4

Problem: Consider the following random variables:

$$
\begin{aligned}
\Lambda &\sim \text{Exp}(\mu) \\
X &\sim \text{Poisson}(s)
\end{aligned}
$$

Find the mgf of X.

Solution: The mgf of X is given by:

$$
\begin{aligned}
M_X(t) &= E_X[e^{tX}] && \text{(def. of mgf)} \\
&= E_\Lambda[E_X[e^{tX} \mid \Lambda = s]] \\
&= E_\Lambda\left[\sum_{k=0}^{\infty} \frac{s^k e^{-s}}{k!}\right] && \text{(def. of expectation)} \\
&= \int_0^\infty \sum_{k=0}^{\infty} \frac{s^k e^{-s}}{k!} (\mu e^{-\mu s})\, ds && \text{(def. of expectation)} \\
&= \sum_{k=0}^{\infty} \frac{\mu}{k!} \int_0^\infty s^k e^{-s(1+\mu)}\, ds && \text{(linearity of sum \& integral)}
\end{aligned}
$$

At this point we perform a change of variables, letting $t = s(1+\mu)$ and computing the relevant bounds and differentials:

$$
\begin{aligned}
s &= \frac{t}{1+\mu} \\
ds &= \frac{dt}{1+\mu} \\
t(s) &= s(1+\mu) \\
t(0) &= 0 \\
\lim_{s\to\infty} t(s) &= \infty
\end{aligned}
$$

We can now continue our chain of equalities:

$$\begin{aligned}
M_X(t) &= \sum_{k=0}^{\infty} \frac{\mu}{k!} \int_0^{\infty} s^k e^{-s(1+\mu)}\, ds \\
&= \sum_{k=0}^{\infty} \frac{\mu}{k!} \int_{t(0)}^{t(\infty)} \left(\frac{t}{1+\mu}\right)^k e^{-t} \frac{dt}{1+\mu} && \text{(change of variables)} \\
&= \sum_{k=0}^{\infty} \frac{\mu}{k!(1+\mu)^{k+1}} \int_0^{\infty} t^k e^{-t}\, dt \\
&= \sum_{k=0}^{\infty} \frac{\mu\Gamma(k+1)}{k!(1+\mu)^{k+1}} && \text{(def. of gamma function)} \\
&= \sum_{k=0}^{\infty} \frac{\mu}{(1+\mu)^{k+1}} && ((\forall n \in \mathbb{N})\ \Gamma(n+1) = n!) \\
&= \frac{\mu}{1+\mu} \sum_{k=0}^{\infty} \frac{1}{(1+\mu)^k} \\
&= \frac{\mu}{1+\mu} \left(\frac{1}{1 - \frac{1}{1+\mu}}\right) && \text{(geometric series)} \\
&= \frac{\mu}{1+\mu} \left(\frac{1+\mu}{\mu}\right) = \boxed{1}
\end{aligned}$$

Note that $\mu > 0$ for any exponential distribution. This implies that $\left|\frac{1}{1+\mu}\right| < 1$, thus justifying the second to last step (i.e. the geometric series).