Honors Calculus III HW #4

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October 22, 2018

Exercise 1

Problem: Where is the following function continuous?

$$f(x,y) = \begin{cases} \frac{y\sin(xy)}{x^2+y^4} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution: The function above is clearly continuous at all points except possibly $(0,0)$ due to a division by zero. Consider the following limit:

$$\lim_{x\to\infty} f(x^{-3}, x^{-1}) = \lim_{x\to\infty} \frac{x^{-1}\sin(x^{-3})}{x^{-4}+x^{-4}} = \lim_{x\to\infty} \frac{x^{-1}\sin(x^{-3})}{2x^{-4}}$$

We can simplify this further by taking advantage of the small angle approximation:

$$\lim_{x\to 0} \sin(x) = \lim_{x\to 0} x$$

We can rewrite this using infinite limits like so:

$$\lim_{x\to\infty} \sin(x^{-1}) = \lim_{x\to\infty} x^{-1}$$

Using this, our original limit becomes:

$$\lim_{x\to\infty} \frac{x^{-1}x^{-3}}{2x^{-4}} = \lim_{x\to\infty} \frac{x^{-4}}{2x^{-4}} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

Now consider the following limit:

$$\lim_{x\to\infty} f(x^{-1}, 0) = \lim_{x\to\infty} \frac{0}{x^{-2}} = 0$$

And so we are left with two limits whose inputs both approach $(0,0)$ yet their outputs do not approach the same value (i.e. $\frac{1}{2} \neq 0$) and so f is not continuous at $(0,0)$.

Exercise 2

Part a

Problem: Is the following function continuous at $(0,0)$

$$f(x,y) = \begin{cases} \frac{x^2y^3}{x^4+y^6} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution: Consider the following limit:

$$\lim_{x\to\infty} f(x^{-3}, x^{-2}) = \lim_{x\to\infty} \frac{x^{-6}x^{-6}}{x^{-12}+x^{-12}} = \lim_{x\to\infty} \frac{x^{-12}}{2x^{-12}} = \frac{1}{2}$$

Now consider this limit:

$$\lim_{x\to\infty} f(x,0) = \lim_{x\to\infty} \frac{0}{x^4} = 0$$

Even though the inputs both approach $(0,0)$ the limits of the functions do not equal each other, thus f is discontinuous at $(0,0)$.

Part b

Problem: Is the following function continuous at $(0,0)$

$$g(x,y) = \begin{cases} \frac{x^5}{x^4+y^6} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution: Notice that:

$$0 \leq |g(x,y)| = \frac{|x|^5}{x^4+y^6}$$

If we use the polar parametrizations of x and y we get:

$$\frac{|x|^5}{x^4+y^6} = \frac{r^5|\cos^5\theta|}{r^4\cos^4\theta + r^6\sin^6\theta} = \frac{r|\cos^5\theta|}{\cos^4\theta + r^2\sin^6\theta}$$

If we cancel the $\cos^4\theta$ in the numerator with the denominator we get:

$$\frac{r|\cos^5\theta|}{\cos^4\theta + r^2\sin^6\theta} = \frac{r|\cos\theta|}{1 + r^2\sin^2\theta\tan^4\theta} \leq r = x^2+y^2$$

And so we are left with the following chain of inequalities:

$$0 \leq |g(x,y)| \leq x^2+y^2$$

And since it is clear that the left and right functions approach 0 as $(x,y) \to 0$, we know that the middle function must as well due to the squeeze theorem. Thus g is continuous at $(0,0)$.

Exercise 3

Part a

Problem: Is the following function continuous at $(0,0)$

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution: It is continuous at $(0,0)$ in the same way as the above functions, consider this limit:

$$\lim_{x\to\infty} f(x^{-1}, x^{-2}) = \lim_{x\to\infty} \frac{x^{-2}x^{-2}}{x^{-4}+x^{-4}} = \lim_{x\to\infty} \frac{x^{-4}}{2x^{-4}} = \frac{1}{2}$$

This is despite the function approaching 0 from either of the axis, meaning the function is discontinuous. To see that it is bounded note that:

$$|f(x,y)| = \frac{x^2|y|}{x^4+y^2} \leq \frac{x^2|y|}{2x^2y} = \frac{|y|}{2y} \leq \frac{1}{2}$$

And so it's bounded for the entire plane and not just the unit disc.

Part b

Problem: Is the following function continuous at $(0,0)$

$$g(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution: Yes it is, consider the following inequality:

$$0 \leq |g(x,y)| = \frac{x^2y^2}{x^4+y^2} \leq \frac{x^2y^2}{2x^2|y|} = \frac{|y|}{2}$$

Since the left and right sides both have limits at 0 as $(x,y) \to (0,0)$, the squeeze theorem tells us that g is continuous at $(0,0)$. To demonstrate g's boundedness on the unit disc, consider the following inequality:

$$|g(x,y)| \leq \frac{|y|}{2} \leq \frac{x^2+y^2}{2}$$

Since the value x^2+y^2 is always 1 on the unit disc, we have shown that there is a bound of $\frac{1}{2}$ on the unit disc for $|g(x,y)|$.

Exercise 4

Problem: Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0,0) = 0$. If f is continuous, is the function g below also continuous:

$$g(x,y) = \begin{cases} \frac{f(x,y)}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution: The only problem point is the origin $\mathbf{0}$, so we must prove/disprove g is continuous there. Since f is differentiable, we know all the directional derivatives (that is, for any $\mathbf{v}$) exist at $\mathbf{0}$:

$$D_{\mathbf{v}} f(x,y)\Big|_{\mathbf{0}} = \lim_{h \to 0} \frac{f(\mathbf{0} + h\mathbf{v}) - f(\mathbf{0})}{h} = \lim_{h \to 0} \frac{f(h\mathbf{v})}{h} = \nabla f(\mathbf{0}) \cdot \mathbf{v}$$

We can take advantage of this by calling $h := \sqrt{x^2+y^2}$ and $\mathbf{v} = \frac{\mathbf{x}}{h}$. From this we get:

$$g(h\mathbf{v}) = g(\mathbf{x}) = g(x,y) = \frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{f(h\mathbf{v})}{h}$$

If we take the limit of this we find:

$$\lim_{h \to \infty} g(h\mathbf{v}) = \nabla f(\mathbf{0}) \cdot \mathbf{v} = \|\nabla f(\mathbf{0})\| \|\mathbf{v}\| \cos\theta = \|\nabla f(\mathbf{0})\| \cos\theta$$

We defined $\mathbf{v}$ to be a unit vector (since it was divided by $\sqrt{x^2+y^2}$) and so its magnitude was 1. Now we can see when g is continuous at the origin. The continuity of g depends on the angle of approach towards the origin and so it is not continuous. That said, if $\nabla f(\mathbf{0}) = 0$ then g would be continuous regardless of the angle of approach as the limit would always equal 0.