Set Theory HW #4

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The following problems are from pages 52-54 of the textbook.

Exercise 11

Problem: Prove the following version (for functions) of the extensionality principle: Assume that F and G are functions, dom $F =$ dom G, and $F(x) = G(x)$ for all x in the common domain. Then $F = G$.

Solution: Consider an arbitrary $x \in \text{dom } F$, and now consider $(x, F(x))$:

$$\begin{aligned}
&(x, F(x)) \in F && \text{(def. of } F) \\
\implies &(x, G(x)) \in F && (F(x) = G(x) \text{ \& dom } F = \text{dom } G) \\
\implies &(x, G(x)) \in G && \text{(def. of } G) \\
\implies &(x, F(x)) \in G && (F(x) = G(x) \text{ \& dom } F = \text{dom } G)
\end{aligned}$$

And since all elements of F are of the form $(x, F(x))$ for some $x \in \text{dom } F$, this proves that $F \subseteq G$. A symmetric argument (found by switching the F's with the G's) proves the other direction, giving us $F = G$.

Exercise 13

Problem: Assume that f and g are functions with $f \subseteq g$ and dom $g \subseteq$ dom f. Show that $f = g$.

Solution:

$$\begin{aligned}
x \in \text{dom } g \implies & \underbrace{x \in \text{dom} f}_{\text{dom } g \subseteq \text{dom } f} \wedge \underbrace{(x, g(x)) \in g}_{\text{def. of function}} \\
\implies & (x, f(x)) \in f \wedge (x, g(x)) \in g && \text{(def. of funtion)} \\
\implies & (x, f(x)) \in g \wedge (x, g(x)) \in g && (f \subseteq g) \\
\implies & f(x) = g(x) && \text{(right-uniqueness of functions)}
\end{aligned}$$

And so we have that for all members of dom g, $f(x) = g(x)$. So we can say:

$$\begin{aligned} x \in \text{dom } g &\implies x \in \text{dom } f && \text{(subset)} \\ &\implies (x, f(x)) \in f && \text{(def. of function)} \\ &\implies (x, g(x)) \in f && \text{(previous result)} \end{aligned}$$

And since every element of g is of the form $(x, g(x))$ where $x \in \text{dom } g$, we have shown that $g \subseteq f$. This combined with the assumption that $f \subseteq g$ gives us $f = g$.

Exercise 15

Problem: Let A be a set of functions such that for any f and g in A, either $f \subseteq g$ or $g \subseteq f$. Show that $\bigcup A$ is a function.

Solution: We have:

$$(x, y_1), (x, y_2) \in \bigcup A \implies (\exists f, g \in A)\ (x, y_1) \in f \wedge (x, y_2) \in g$$

Now w.l.o.g, suppose $f \subseteq g$. This means that $(x, y_1) \in g$. And so we have $y_1 = g(x) = y_2$, thus $\bigcup A$ is right-unique aka a function.

Exercise 21

Problem: Show that $(R \circ S) \circ T = R \circ (S \circ T)$ for any sets R, S and T.

Solution: We have the following:

$$\begin{aligned} (x, y) \in (R \circ S) \circ T &\implies (\exists t)\ x(R \circ S)t \wedge tTy \\ &\implies (\exists t, s)\ xRs \wedge sSt \wedge tTy \\ &\implies (\exists s)\ xRs \wedge s(S \circ T)y \\ &\implies xR \circ (S \circ T)y \\ &\implies (x, y) \in R \circ (S \circ T) \end{aligned}$$

This is only one direction. The other direction follows a very similar argument. Putting both directions together we have $(R \circ S) \circ T = R \circ (S \circ T)$.

Exercise 22

Problem: Show that the following are correct for any sets.

a) $A \subseteq B \implies F[\![A]\!] \subseteq F[\![B]\!]$

b) $(F \circ G)[\![A]\!] = F[\![G[\![A]\!]]\!]$

c) $Q \upharpoonright (A \cup B) = (Q \upharpoonright A) \cup (Q \upharpoonright B)$

Solution: For **a)** we have the following:

$$
\begin{aligned}
y \in F[\![A]\!] &\implies (\exists x \in A)\ (x,y) \in F && \text{(def. of image)}\\
&\implies (\exists x \in B)\ (x,y) \in F && \text{(assume } A \subseteq B\text{)}\\
&\implies y \in F[\![B]\!] && \text{(def. of image)}
\end{aligned}
$$

And so by the definition of subset we have $A \subseteq B \implies F[\![A]\!] \subseteq F[\![B]\!]$. For **b)** We have the following:

$$
\begin{aligned}
y \in (F \circ G)[\![A]\!] &\implies (\exists x \in A)\ x(F \circ G)y && \text{(def. of image)}\\
&\implies (\exists t, \exists x \in A)\ xGt \wedge tFy && \text{(def. of composition)}\\
&\implies (\exists t)\ t \in G[\![A]\!] \wedge tFy && \text{(def. of image)}\\
&\implies y \in F[\![G[\![A]\!]]\!] && \text{(def. of image)}
\end{aligned}
$$

In the other direction we have:

$$
\begin{aligned}
y \in F[\![G[\![A]\!]]\!] &\implies (\exists x \in G[\![A]\!])\ xFy && \text{(def. of image)}\\
&\implies (\exists z \in A)(\exists x \in G[\![A]\!])\ zGx \wedge xFy && \text{(def. of image)}\\
&\implies (\exists z \in A)\ z(G \circ F)y && \text{(def. of composition)}\\
&\implies (F \circ G)[\![A]\!] && \text{(def. of image)}
\end{aligned}
$$

Putting these two together we have $(F \circ G)[\![A]\!] = F[\![G[\![A]\!]]\!]$. And finally for **c)** we have the following:

$$
\begin{aligned}
y \in Q \restriction (A \cup B) &\iff (\exists x \in A \cup B)\ xQy && \text{(def. of restriction)}\\
&\iff (\exists x)\ (x \in A \vee x \in B) \wedge xQy && \text{(def. of union)}\\
&\iff y \in (Q \restriction A) \vee y \in (Q \restriction B) && \text{(def. of restriction)}\\
&\iff y \in (Q \restriction A) \cup (Q \restriction B) && \text{(def. of union)}
\end{aligned}
$$

And so by extensionality we have $Q \restriction (A \cup B) = (Q \restriction A) \cup (Q \restriction B)$.

Exercise 24

Problem: Show that for a function F:

$$F^{-1}[\![A]\!] = \{x \in \text{dom } F \mid F(x) \in A\}$$

Solution: We have the following:

$$
\begin{aligned}
x \in F^{-1}[\![A]\!] &\iff (\exists y \in A)\ (y,x) \in F^{-1} && \text{(def. of image)}\\
&\iff (\exists y \in A)(x,y) \in F && \text{(def. of inverse)}\\
&\iff (\exists y \in A)(x \in \text{dom } F) \wedge (y = F(x) \in A)
\end{aligned}
$$

And so by extensionality we have $F^{-1}[\![A]\!] = \{x \in \text{dom } F \mid F(x) \in A\}$.

Exercise 28

Problem: Assume that f is a one-to-one function from A into B, and that G is the function with dom $G = \mathcal{P}(A)$ defined by the equation $G(X) = f[\![X]\!]$. Show that G is a bijective map from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

Solution: First we show surjectivity, consider an arbitrary Y:

$$\begin{aligned}
Y \in \mathcal{P}(B) &\implies Y \subseteq B && \text{(def. of powerset)}\\
&\implies (\exists X \subseteq A)\ f[\![X]\!] = Y && \text{(def. of bijective + image)}\\
&\implies (\exists X \in \mathcal{P}(A))\ f[\![X]\!] = Y && \text{(def. of powerset)}\\
&\implies (\exists X \in \mathcal{P}(A))\ G(X) = Y && \text{(def. of } G(X))
\end{aligned}$$

And so G is surjective function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. Now we show injectivity. Consider two arbitrary sets $X, Y \in \mathcal{P}(A)$:

$$\begin{aligned}
G(X) = G(Y) &\implies f[\![X]\!] = f[\![Y]\!] && \text{(assumption)}\\
&\implies (\forall x \in X)\ f(x) \in f[\![X]\!] && (X \subseteq A = \text{dom } f)\\
&\implies (\forall x \in X)\ f(x) \in f[\![Y]\!] && (f[\![X]\!] = f[\![Y]\!])\\
&\implies (\forall x \in X)(\exists y \in Y)\ (y, f(x)) \in f \wedge (x, f(x)) \in f && \text{(def. of image)}\\
&\implies (\forall x \in X)(\exists y \in Y)\ x = y && \text{(injectivity of } f)\\
&\implies (\forall x \in X)\ x \in Y && (y \in Y)\\
&\implies X \subseteq Y && \text{(def. of subset)}
\end{aligned}$$

And so we have shown that if two sets X, Y map to the same output, $X \subseteq Y$. A symmetric argument (switch the X and Y around) shows that $Y \subseteq X$ as well. And so we have that if $G(X) = G(Y)$ then $X = Y$, satisfying injectivity. This combined with the surjectivity shown earlier proves that G is a one-to-one correspondence.