Math Statistics
Weekly HW 8

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November 23, 2020

Some notation before we continue. Let $Y_{\mu,\sigma^2} \sim \mathcal{N}(\mu, \sigma^2)$, $t_\nu \sim$ student-$t(\nu)$, and $B_{n,p} \sim B(n, p)$.

Question 1

Suppose we have a normal population $X_i \sim \mathcal{N}(\mu, \sigma^2)$. Our null hypothesis H_0 is that $\mu = 10$ and our alternative hypothesis H_1 is that $\mu > 10$. Suppose we have an i.i.d. sample X of $n = 16$.

Part a: Suppose $\sigma^2 = 9$ and that our condition for accepting the null hypothesis is:

$$\hat{H}_0 : \bar{X} \leq 10 + \frac{2\sigma}{\sqrt{n}} = 11.5$$

Give the probability of a type I error and a bound on type II errors.

Solution: For a type I error, i.e. rejecting the null hypothesis given that it's true, we have:

$$\begin{aligned}
P(\text{Type I error}) &= P(\neg\hat{H}_0 \mid H_0) && \text{(def. of type I error)} \\
&= P(\bar{X} > 11.5 \mid \mu = 10) && \text{(from def. of hypotheses)} \\
&= P(Y_{10,9/16} > 11.5) && (\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n) \text{ mean of i.i.d. normals}) \\
&= P\left(Z > \frac{11.5 - 10}{9/16}\right) && \text{(standardize normal RV)} \\
&= 1 - \Phi\left(\frac{8}{3}\right) && \text{(cdf of normal RV)} \\
&\approx 0.0038
\end{aligned}$$

And for a type II error, i.e. accepting a the null hypothesis given that it's false, we have:

$$\begin{aligned}
P(\text{Type II error}) &= P(\hat{H}_0 \mid H_1) && \text{(def. of type II error)} \\
&= P(\bar{X} \leq 11.5 \mid \mu > 10) && \text{(from def. of hypotheses)} \\
&= P(Y_{\mu,9/16} \leq 11.5 \mid \mu > 10) && (\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n) \text{ mean of i.i.d. normals}) \\
&= P\left(Z \leq \frac{11.5 - \mu}{9/16} \mid \mu > 10\right) && \text{(standardize normal RV)} \\
&< P\left(Z \leq \frac{11.5 - 10}{9/16}\right) && (\Phi \text{ is strictly increasing}) \\
&= \Phi\left(\frac{8}{3}\right) \\
&\approx 0.9962
\end{aligned}$$

That is to say, depending on the true value of μ, the probability of a type 2 error could be anything in the interval $(0, 0.9962)$.

Part b: Suppose we don't know what σ^2 is and that our conditions for accepting the null hypothesis is:

$$\hat{H}_0 : \bar{X} \leq 10 + \frac{2S}{\sqrt{n}} = 10 + \frac{S}{2}$$

Given $\bar{X}$ and S^2, give the probability of a type I error and a bound on type II errors.

Solution: For a type I error we have:

$$\begin{aligned}
P(\text{Type I error}) &= P(\neg\hat{H}_0 \mid H_0) && \text{(def. of type I error)}\\
&= P\left(\bar{X} > 10 + \frac{S}{2} \mid \mu = 10\right) && \text{(from def. of hypotheses)}\\
&= P\left(Y_{10,\sigma^2/9} > 10 + \frac{S}{2}\right) && (\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n) \text{ mean of i.i.d. normals})\\
&= P\left(t_{15} > \frac{10 + \frac{S}{2} - 10}{S/\sqrt{16}}\right) && \text{(sample mean to } t\text{-distribution)}\\
&= P\left(t_{15} > 2\right)\\
&= 1 - F_{t_{15}}\left(2\right) && \text{(cdf of } t\text{-distribution)}\\
&\approx 0.03197
\end{aligned}$$

And for a type II error we have:

$$\begin{aligned}
P(\text{Type II error}) &= P(\hat{H}_0 \mid H_1) && \text{(def. of type II error)}\\
&= P\left(\bar{X} \leq 10 + \frac{S}{2} \mid \mu > 10\right) && \text{(from def. of hypotheses)}\\
&= P\left(Y_{\mu,\sigma^2/9} \leq 10 + \frac{S}{2} \mid \mu > 10\right) && (\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n) \text{ mean of i.i.d. normals})\\
&= P\left(t_{15} \leq \frac{10 + \frac{S}{2} - \mu}{S/\sqrt{16}} \mid \mu > 10\right) && \text{(sample mean to } t\text{-distribution)}\\
&< P\left(t_{15} \leq 2\right) && \text{(cdf of } t\text{-distribution is strictly increasing)}\\
&= F_{t_{15}}\left(2\right)\\
&\approx 0.96803
\end{aligned}$$

That is to say, depending on the true value of μ, the probability of a type 2 error could be anything in the interval $(0, 0.96803)$.

Question 2

Suppose we have a Bernoulli population $X_i \sim \text{Bernoulli}(\theta)$. Our null hypothesis H_0 is that $\theta = \frac{1}{2}$ and our condition for accepting it $\hat{H}_0$ is that there is that not all observations X_i are the same. Suppose we have an i.i.d. sample X of $n = 10$

Part a: Suppose our alternative hypothesis H_1 is $\theta \neq \frac{1}{2}$. Give the probability of a type I error and a bound on type II errors.

Solution: For a type I error we have:

$$
\begin{aligned}
P(\text{Type I error}) &= P(\neg\hat{H}_0 \mid H_0) && \text{(def. of type I error)} \\
&= P\left((\forall i)\, X_i = 1 \vee (\forall j)\, X_j = 0 \mid \theta = \frac{1}{2}\right) && \text{(from def. of hypotheses)} \\
&= P\left(B_{10,1/2} \in \{0, 10\}\right) && (\textstyle\sum X_i \sim B(n,\theta)) \\
&= \binom{10}{0}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^{10-0} + \binom{10}{10}\left(\frac{1}{2}\right)^{10}\left(1-\frac{1}{2}\right)^{10-10} && \text{(pmf of binomial RV)} \\
&= \frac{1}{2^{10}} + \frac{1}{2^{10}} = \frac{1}{2^9}
\end{aligned}
$$

And for a type II error we have:

$$
\begin{aligned}
P(\text{Type II error}) &= P(\hat{H}_0 \mid H_1) && \text{(def. of type II error)} \\
&= P\left((\exists i)\, X_i \neq 1 \wedge (\exists j)\, X_j \neq 0 \mid \theta \neq \frac{1}{2}\right) && \text{(from def. of hypotheses)} \\
&= P\left((\exists i)\, X_i = 0 \wedge (\exists j)\, X_j = 1 \mid \theta \neq \frac{1}{2}\right) && (X_i \in \{0,1\}) \\
&= P\left(B_{10,\theta} \notin \{0, 10\} \mid \theta \neq \frac{1}{2}\right) && (\textstyle\sum X_i \sim B(n,\theta)) \\
&= 1 - P\left(B_{10,\theta} \in \{0, 10\} \mid \theta \neq \frac{1}{2}\right) && \text{(complement)} \\
&= 1 - \binom{10}{0}\theta^0(1-\theta)^{10-0} - \binom{10}{10}\theta^{10}(1-\theta)^{10-10} \text{ s.t. } \theta \neq \frac{1}{2} && \text{(pmf of binomial RV)} \\
&= 1 - (1-\theta)^{10} - \theta^{10} \text{ s.t. } \theta \neq \frac{1}{2}
\end{aligned}
$$

Note that $f(x) = 1-(1-x)^{10} - x^{10}$ is maximized when $x = \frac{1}{2}$ and 0 when $x \in \{0, 1\}$. As such, depending on the value of θ, the probability of type II errors is given by the bound:

$$
\begin{aligned}
P(\text{Type II error}) &\in \left[0, f\left(\frac{1}{2}\right)\right) && \text{(exclusive right bound since } \theta \neq \tfrac{1}{2}) \\
&\in \left[0, 1 - \left(1-\frac{1}{2}\right)^{10} - \left(\frac{1}{2}\right)^{10}\right) \\
&\in \left[0, 1 - \frac{1}{2^9}\right)
\end{aligned}
$$

Part b: Suppose our alternative hypothesis H_1 is that $|\theta - 0.5| > .3$. Give the probability of a type I error and a bound on type II errors.

Solution: Our type I error probability is the same as in part a as we haven't changed our null hypothesis H_0 nor its acceptance condition $\hat{H}_0$:

$$
\begin{aligned}
P(\text{Type I error}) &= P(\neg\hat{H}_0 \mid H_0) && \text{(def. of type I error)} \\
&= P\left((\forall i)\, X_i = 1 \vee (\forall j)\, X_j = 0 \mid \theta = \frac{1}{2}\right) && \text{(from def. of hypotheses)} \\
&= \frac{1}{2^9} \approx 0.001953125 && \text{(part a)}
\end{aligned}
$$

For a type II error we have:

$$
\begin{aligned}
P(\text{Type II error}) &= P(\hat{H}_0 \mid H_1) && \text{(def. of type II error)} \\
&= P\left((\exists i)\, X_i \neq 1 \wedge (\exists j)\, X_j \neq 0 \mid \theta > .8 \vee \theta < .2\right) && \text{(from def. of hypotheses)} \\
&= P\left((\exists i)\, X_i = 0 \wedge (\exists j)\, X_j = 1 \mid \theta > .8 \vee \theta < .2\right) && (X_i \in \{0,1\}) \\
&= P\left(B_{10,\theta} \notin \{0, 10\} \mid \theta > .8 \vee \theta < .2\right) && \left(\textstyle\sum X_i \sim B(n,\theta)\right) \\
&= 1 - P\left(B_{10,\theta} \in \{0, 10\} \mid \theta > .8 \vee \theta < .2\right) && \text{(complement)} \\
&= 1 - (1-\theta)^{10} - \theta^{10} \text{ s.t. } \theta > .8 \vee \theta < .2
\end{aligned}
$$

Recall from part a that we know $f(x) = 1 - (1-x)^{10} - x^{10}$ reaches a maximum at $f\left(\frac{1}{2}\right)$ and that it is 0 at $x \in \{0, 1\}$. And so, depending on the value of θ, the probability of type II errors is given by the bound:

$$
\begin{aligned}
P(\text{Type II error}) &\in [0, f\,(.2)) \cup (f(.8), 1] && \text{(exclusive bounds since } \theta > .2 \vee \theta < .8\text{)} \\
&\in [0, f\,(.2)) && (f(x) \text{ is symmetric about } x = .5 \text{ and } .5 - .2 = .8 - .5) \\
&\in \left[0, 1 - \frac{1}{5^9}\right)
\end{aligned}
$$