Set Theory HW #6

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Part 1

Problem I

Problem: Assume that $f : A \to B$. Define $\sim_f$ to be the set:

$$\{\langle x, y\rangle \mid f(x) = f(y)\}$$

1. Prove that $\sim_f$ is an equivalence relation on A.
2. Prove that if $C \neq \varnothing$ and $g : A \to C$ is a function, then there exists a function $h : B \to C$ such that $h \circ f = g$ if and only if:

$$\forall x \forall y \ (x \sim_f y \implies g(x) = g(y)) \tag{1}$$

Solution: Part 1) Consider an arbitrary set x. We have:

$$\begin{aligned} x \in A &\implies f(x) = f(x) \\ &\implies x \sim_f x && \text{(def. of } \sim_f) \end{aligned}$$

And so $\sim_f$ is reflexive. Next consider an arbitrary ordered pair $\langle x, y\rangle$. We have:

$$\begin{aligned} x \sim_f y &\implies f(x) = f(y) && \text{(def. of } \sim_f) \\ &\implies f(y) = f(x) \\ &\implies y \sim_f x && \text{(def. of } \sim_f) \end{aligned}$$

And so $\sim_f$ is symmetric. Finally, consider two arbitrary ordered pairs $\langle x, y\rangle$ and $\langle y, z\rangle$. We have:

$$\begin{aligned} x \sim_f y \,\&\, y \sim_f z &\implies f(x) = f(y) \,\&\, f(y) = f(z) && \text{(def. of } \sim_f) \\ &\implies f(x) = f(z) && \text{(transitivity of equality)} \\ &\implies x \sim_f z && \text{(def. of } \sim_f) \end{aligned}$$

And so $\sim_f$ is transitive. We have thus proven all 3 necessary conditions for the relation $\sim_f$ to be an equivalence relation over A.

Part 2) Let us assume that such a function h exists. We then have that:

$$\begin{aligned} (\forall x, y)\ x \sim_f y &\implies f(x) = f(y) \\ &\implies h \circ f(x) = h \circ f(y) = \underbrace{g(x) = g(y)} \end{aligned}$$

We can see that this last equality is a contradiction, making our assumption that such a function h exists, false. This is, unless, statement (1) holds. In other words, h existing implies (1). Now we need the other direction. Assume that such a function h exists but that (1) is false. We have:

$$\begin{aligned}
(\forall x, y)\ x \sim_f y &\implies f(x) = f(y) \\
&\implies h \circ f(x) = h \circ f(y) \\
&\implies g(x) = g(y) && \text{(def. of } h \circ f\text{)}
\end{aligned}$$

This last implication is a contradiction as we explicitly assumed that (1) was false. Thus, our initial assumption was false and (1) must be true for h to exist.

Problem II

Problem: Prove that if R is an equivalence relation on a set A then there exist a set B and a function $f : A \to B$ such that $R = \sim_f$.

Solution: Let $B = A/R$ and $f(x) = [x]_R$. We then have:

$$\begin{aligned}
x \sim_f y &\iff f(x) = f(y) \\
&\iff [x]_R = [y]_R \\
&\iff xRy && \text{(def. of equivalence class)}
\end{aligned}$$

Problem III

Problem: If R is an equivalence relation on a set A, and $f : A \times A \to A$, then we say that f is compatible with R if:

$$(\forall x, y, x', y' \in A)\ (xRx' \ \&\ yRy' \implies f(x, y)Rf(x', y'))$$

Prove that if R is an equivalence relation on A and $f : A \times A \to A$, then:

1. there exists a function $\hat{f} : A/R \times A/R \to A/R$ such that:

$$(\forall x \forall y \in A)\ \hat{f}([x]_R, [y]_R) = [f(x, y)]_R$$

   iff f is compatible with R

2. if there exists a function $\hat{f}$ such that the above holds, it is unique.

Solution: Part 1) First we need to extend our definition of compatibility:

$$xRy \ \&\ uRv \implies f(x, y)Rf(x, y)$$

Now let us assume that f is compatible with R, and prove that such a $\hat{f}$ exists:

$$\begin{aligned}
\langle [x], [y] \rangle = \langle [u], [v] \rangle &\implies [x] = [u] \ \&\ [y] = [v] \\
&\implies xRu \ \&\ yRv \\
&\implies f(x, y)Rf(u, v) \\
&\implies [f(x, y)] = [f(u, v)]
\end{aligned}$$

And so $\hat{f}$ is a function, with $\mathrm{dom}\hat{f} = A/R \times A/R$ and $\mathrm{ran}\hat{f} \subseteq A/R$

Now suppose that f is not compatible. By incompatibility, there are some pairs $\langle x, y\rangle, \langle u, v\rangle \in A\times A$ such that the following holds:

$$\begin{array}{rl} xRu \;\&\; yRv & f(x,y) \not R\, f(u,v) \\ [x]=[u] \;\&\; [y]=[v] & [f(x,y)] \neq [f(u,v)] \end{array}$$

Despite both of these conditions needing to be true, the right sides are not equal.

Part 2) The function $\hat{f}$ is unique. Suppose for some $g : A/R \times A/R \to A/R$, the same conditions hold. Then for any $x, y \in A$ we have:

$$g([x],[y]) = [f([x],[y])] = \hat{f}([x],[y])$$

Part 2

The following problems are from pages 61-62 of the textbook.

Exercise 36

Problem: Assume that $f : A \to B$ and that R is an equivalence relation on B. Define Q to be the following set:

$$\{\langle x, y\rangle \in A \times A \mid \langle f(x), f(y)\rangle \in R\}$$

Show that Q us an equivalence relation on A

Solution: Consider an arbitrary set x. We have:

$$\begin{aligned} x \in A &\implies f(x) \in B && \text{(def. of } f\text{)} \\ &\implies \langle f(x), f(x)\rangle \in R && \text{(reflexivity of equiv. relation)} \\ &\implies \langle x, x\rangle \in Q && \text{(def. of } Q\text{)} \end{aligned}$$

And so Q is reflexive. Next consider an arbitrary ordered pair $\langle x, y\rangle$. We have:

$$\begin{aligned} \langle x, y\rangle \in Q &\implies \langle f(x), f(y)\rangle \in R && \text{(def. of } Q\text{)} \\ &\implies \langle f(y), f(x)\rangle \in R && \text{(symmetry of equiv. relation)} \\ &\implies \langle y, x\rangle \in Q && \text{(def. of } Q\text{)} \end{aligned}$$

And so Q is symmetric. Finally, consider two arbitrary ordered pairs $\langle x, y\rangle$ and $\langle y, z\rangle$. We have:

$$\begin{aligned} \langle x, y\rangle \in Q \;\&\; \langle y, z\rangle \in Q &\implies \langle f(x), f(y)\rangle \in R \;\&\; \langle f(y), f(z)\rangle \in R && \text{(def. of } Q\text{)} \\ &\implies \langle f(x), f(z)\rangle \in R && \text{(transitivity of equiv. relation)} \\ &\implies \langle x, z\rangle \in Q && \text{(def. of } Q\text{)} \end{aligned}$$

And so Q is transitive. We have thus proven all 3 necessary conditions for the relation Q to be an equivalence relation over A.

Exercise 37

Problem: Assume Π is a partition of a set A. Define the relation R_Π as follows:

$$xR_\Pi y \iff (\exists B \in \Pi)(x \in B \;\&\; y \in B)$$

Solution: Consider an arbitrary set x. We have:

$$\begin{aligned} x \in A &\implies (\exists B \in \Pi)\, x \in B && \text{(def. of partition)} \\ &\implies xR_\Pi x && \text{(def. of } R_\Pi\text{)} \end{aligned}$$

And so R_Π is reflexive. Next consider an arbitrary ordered pair $\langle x, y \rangle$. We have:

$$\begin{aligned} xR_\Pi y &\implies (\exists B \in \Pi)\, x \in B \;\&\; y \in B && \text{(def. of } R_\Pi\text{)} \\ &\implies (\exists B \in \Pi)\, y \in B \;\&\; x \in B \\ &\implies yR_\Pi x && \text{(def. of } R_\Pi\text{)} \end{aligned}$$

And so R_Π is symmetric. Finally, consider two arbitrary ordered pairs $\langle x, y \rangle$ and $\langle y, z \rangle$. We have:

$$\begin{aligned} xR_\Pi y \;\&\; yR_\Pi z &\implies (\exists B \in \Pi)(x \in B \;\&\; y \in B) \;\&\; (\exists C \in \Pi)(y \in C \;\&\; z \in C) && \text{(def. of } R_\Pi\text{)} \\ &\implies (\exists B, C \in \Pi)\, x \in B \;\&\; \underbrace{y \in B \;\&\; y \in C}_{y \in B \cap C} \;\&\; z \in C \end{aligned}$$

Yet recall that, by definition, every element y of a partitioned set A belongs to exactly one set in that partition Π. And so, the sets B and C above must actually be the same set. As such we have:

$$\begin{aligned} &\implies (\exists C \in \Pi)\, x \in C \;\&\; y \in C \;\&\; y \in C \;\&\; z \in C && \text{(B=C)} \\ &\implies (\exists C \in \Pi)\, x \in C \;\&\; z \in C \\ &\implies xR_\Pi z && \text{(def. of } R_\Pi\text{)} \end{aligned}$$

And so R_Π is transitive. We have thus proven all 3 necessary conditions for the relation R_Π to be an equivalence relation over A.

Exercise 38

Problem: Theorem 3P shows that A/R is a partition of A whenever R is an equivalence relation on A. Show that if we start with the equivalence relation R_Π of the preceding exercise, then the partition A/R_Π is just Π.

Solution: Consider an arbitrary element $x \in A$. We have:

$$\begin{aligned} [x] \in A/R_\Pi &\implies (\forall y \in [x])\, xR_\Pi y \\ &\implies (\forall y \in [x])(\exists B \in \Pi)\, x \in B \;\&\; y \in B \end{aligned}$$

Now fix this B and note that for any $z \in [x]$ we have $yR_\Pi z$ (because R_Π is an equiv. relation). From exercise 3, we know that $z \in B$. So since any two elements of $[x]$ are in this fixed set B, we have: $[x] \subseteq B$. And since any $b \in B$ satisfies $bR_\Pi x$, we have the other direction, giving us $[x] = B$. And so every equivalence class in A/R_Π equals some set in the partition Π.

For the other direction, consider an arbitrary set $C \in \Pi$. Note that C is nonempty (since Π is a partition) so consider an arbitrary element $m \in C$. By definition, we know $C \subseteq [m]$. However, via the same reasoning we used in the paragraph above, we also know the other direction giving us $C = [m]$. And so every member of the partition Π equals some equivalence class in A/R_Π.

Putting these two facts together we finally find that:

$$A/R_\Pi = \Pi$$

Exercise 39

Problem: Assume that we start with an equivalence relation R on A and define Π to be the partition A/R. Show that R_Π, as defined in Exercise 37, is just R.

Solution: Consider an arbitrary ordered pair $\langle x, y\rangle$. We have:

$$\begin{aligned} xRy &\iff (\exists B \in A/R)\, x \in B \;\&\; y \in B && \text{(def. quotient set)} \\ &\iff (\exists B \in \Pi)\, x \in B \;\&\; y \in B && \text{(def. } \Pi\text{)} \\ &\iff xR_\Pi y && \text{(def. of } R_\Pi\text{)} \end{aligned}$$

And so the relations are identical.

Exercise 40

Problem: Define an equivalence relation R on the set P of positive integers by:

$$mRn \iff m \text{ and } n \text{ have the same \# of unique prime factors}$$

Is there a function $f : P/R \to P/R$ such that $f([n]_R) = [3n]_R$ for each n?

Solution: Recall from theorem 3Q that, for such a function f to exist the following function $g : P \to P$ must be compatible with R:

$$g(n) = 3n$$

However, consider the following counterexample. Trivially we have $2R3$ Yet note that:

$$\left.\begin{aligned} g(2) = 6 &= \underbrace{2 \cdot 3}_{\text{2 factors}} \\ g(3) = 9 &= \underbrace{3^2}_{\text{1 factor}} \end{aligned}\right\} \implies g(2) \not R\, g(3)$$

And so g isn't compatible with R and the desired function f can't exist.

Part 3

The following problems are from pages 101, 111, 120, and 121 of the textbook.

Exercise 1

Problem: Is there a function $F : \mathbb{Z} \to \mathbb{Z}$ satisfying the following:

$$F([\langle m, n\rangle]) = [\langle m + n, n\rangle]$$

Solution: Let $\hat{F} : \mathbb{Z}^2 \to \mathbb{Z}^2$ be defined as:

$$\hat{F}(\langle m, n\rangle) = \langle m + n, n\rangle$$

By theorem 3Q in the textbook, it suffices to show that $\hat{F}$ is not compatible with $\sim$ to show that no such function F can exist. Clearly $\langle 3, 2\rangle \sim \langle 1, 0\rangle$, yet we have:

$$\hat{F}(\langle 3, 2\rangle) = \langle 5, 2\rangle \not\sim \langle 1, 0\rangle = \hat{F}(\langle 1, 0\rangle)$$

And so $\hat{F}$ is not compatible with $\sim$ and thus the function F cannot exist.

Exercise 3

Problem: Is there a function $H : \mathbb{Z} \to \mathbb{Z}$ satisfying the following:

$$H([\langle m, n\rangle]) = [\langle n, m\rangle]$$

Solution: Let $\hat{F} : \mathbb{Z}^2 \to \mathbb{Z}^2$ be defined as:

$$\hat{F}(\langle m, n\rangle) = \langle m + n, n\rangle$$

By theorem 3Q, in proving that there exists such a function H, it suffices to show that $\hat{H}$ is compatible with $\sim$:

$$\begin{aligned}
\langle m, n\rangle \sim \langle m', n'\rangle &\implies m + n' = m' + n \\
&\implies n + m' = n' + m \\
&\implies \langle n, m\rangle \sim \langle n', m'\rangle \\
&\implies \hat{H}(\langle m, n\rangle) \sim \hat{H}(\langle m', n'\rangle)
\end{aligned}$$

And so $\hat{H}$ is compatible and the function H exists.

Exercise 4

Problem: Prove that $+_{\mathbb{Z}}$ is associative.

Solution: Let the integers $a = [\langle m, n\rangle], b = [\langle p, q\rangle]$ and $c = [\langle r, s\rangle]$. We then have:

$$\begin{aligned}
(a +_{\mathbb{Z}} b) +_{\mathbb{Z}} c &= ([\langle m, n\rangle] +_{\mathbb{Z}} [\langle p, q\rangle]) +_{\mathbb{Z}} [\langle r, s\rangle] \\
&= [\langle m + p, n + q\rangle] +_{\mathbb{Z}} [\langle r, s\rangle] \\
&= [\langle (m + p) + r, (n + q) + s\rangle] \\
&= [\langle m + (p + r), n + (q + s)\rangle] && \text{(associativity of } \mathbb{N}) \\
&= [\langle m, n\rangle] +_{\mathbb{Z}} [\langle p + r, q + s\rangle] \\
&= [\langle m, n\rangle] +_{\mathbb{Z}} ([\langle p, q\rangle] +_{\mathbb{Z}} \langle r, s\rangle]) \\
&= a +_{\mathbb{Z}} (b +_{\mathbb{Z}} c)
\end{aligned}$$

Exercise 14

Problem: Show that the ordering of the rationals is dense, i.e. between any two rationals there is a third one:

$$p <_{\mathbb{Q}} \implies (\exists r)\, p <_{\mathbb{Q}} r <_{\mathbb{Q}} s$$

Solution: Let $p = [\langle a, b\rangle]$ and $s = [\langle c, d\rangle]$ with $b, d >_{\mathbb{Z}} 0$. (note that there is no loss of generality because every rational can be expressed in this form i.e. $\langle a, -b\rangle \sim \langle -a, b\rangle$). Also let $p <_{\mathbb{Q}} s$. Now note the following:

$$\begin{aligned}
[\langle a, b\rangle] <_{\mathbb{Q}} [\langle c, d\rangle] &\implies ad <_{\mathbb{Z}} bc \\
&\implies abd <_{\mathbb{Z}} bbc \\
&\implies add <_{\mathbb{Z}} bcd
\end{aligned}$$

Now, define r as the following rational number:

$$\begin{aligned} r &= (p +_{\mathbb{Q}} s) \div [\langle 2, 1\rangle] \\ &= [\langle ad + bc, bd\rangle] \cdot_{\mathbb{Q}} [\langle 1, 2\rangle] \\ &= [\langle ad + bc, 2bd\rangle] \end{aligned}$$

Now note that:

$$\begin{aligned} abd &<_{\mathbb{Z}} bbc \\ abd + abd &<_{\mathbb{Z}} abd + bbc \\ 2abd &<_{\mathbb{Z}} b(ad + bc) \end{aligned}$$

Implying that $p <_{\mathbb{Q}} r$. Similarly we have:

$$\begin{aligned} add &<_{\mathbb{Z}} bcd \\ add + bcd &<_{\mathbb{Z}} bcd + bcd \\ (ad + bc)d &<_{\mathbb{Z}} 2bcd \end{aligned}$$

Implying that $r <_{\mathbb{Q}} s$. And so, given any 2 rationals p and s we have constructed a rational number r such that:

$$p <_{\mathbb{Q}} r <_{\mathbb{Q}} s$$

Exercise 15

Problem: In theorem 5RB, show that $\bigcup A$ is closed downward and has no largest element.

Solution: Consider an arbitrary set q. We have:

$$\begin{aligned} q \in \bigcup A &\implies (\exists x \in A)\, q \in x \\ &\implies (\forall r < q)(\exists x \in A)\, r \in x && (x \text{ is closed downwards}) \\ &\implies (\forall r < q)\, r \in \bigcup A \end{aligned}$$

And so $\bigcup A$ is closed downwards. Now consider an arbitrary set p. We have:

$$\begin{aligned} p \in \bigcup A &\implies (\exists x \in A)\, p \in x \\ &\implies (\exists x \in A)\, p \in x \;\&\; (\exists q \in x)\, p < q && (x \text{ has no largest element}) \\ &\implies (\exists x \in A)\, q \in x \;\&\; p < q \\ &\implies q \in \bigcup A \;\&\; p < q \end{aligned}$$

And so $\bigcup A$ has no largest element since for any element p it contains a larger element q.

Exercise 16

Problem: In lemma 5RC, show that $x +_{\mathbb{R}} y$ has no largest element.

Solution: Take any $q + r \in x +_{\mathbb{R}} y$, so that $q \in x$ and $r \in y$. Since neither x nor y has a largest element, there exist a $q' \in x$ and $r' \in y$ such that $q < q'$ and $r < r'$. Since addition preserves order in the rationals, $q + r < q' + r' \in x +_{\mathbb{R}} y$. Hence $x +_{\mathbb{R}} y$ has no largest element.

Exercise 22

Problem:

Solution: Recall that:

$$|x| = x \cup -x$$

Consider two rational numbers q and r such that $q \in |x|$ and $r < q$. We have two cases:

$$\begin{aligned} q \in x \;\&\; r < q &\implies r \in x && (x \text{ is downward closed}) \\ &\implies r \in |x| && (\text{def. of } |x|) \\ q \in -x \;\&\; r < q &\implies r \in -x && (-x \text{ is downward closed}) \\ &\implies r \in |x| && (\text{def. of } |x|) \end{aligned}$$

And so in either case, $|x|$ is downward closed. Now take any rational $p \in |x|$. We again have two cases:

$$\begin{aligned} p \in x &\implies (\exists p' \in x)\, p' > p && (x \text{ has no greatest element}) \\ p \in -x &\implies (\exists p' \in -x)\, p' > p && (-x \text{ has no greatest element}) \end{aligned}$$

And so in either case, $|x|$ has no greatest element. All that's left to prove is that $\varnothing \neq |x| \neq \mathbb{Q}$.

$|x| \neq \varnothing$:
Since no real number is the empty set, and $|x|$ is the union of two real numbers, it can't be the empty set either.

$|x| \neq \mathbb{Q}$:
Assume that $x < 0$. Then we have:

$$(\forall q \in x)\, q < 0_{\mathbb{Q}}$$

and so $0_{\mathbb{Q}} \notin x$, since $0_{\mathbb{Q}} \notin 0$, and consequently not in x. Hence $q \in -x$, and thus $x \subseteq -x$. Hence $|x| = x \cup -x = -x \neq \mathbb{Q}$.

Suppose instead that $x \geq 0$. Suppose $r \geq 0$, then if $s > r$, then $s > 0$, and so $-s < 0$. Thus $s \in x$, so $r \notin -x$. So if $r \in -x$, then necessarily $r < 0$, and so $r \in x$. Hence $-x \subseteq x$, and so:

$$|x| = x \cup -x = x \neq \mathbb{Q}$$