Foundations of QM
HW 5

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Recall that the z and x components of the spin of a spin-1/2 quantum particle are given by the following observables:

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Question 1

Problem: What are the possible values of σ_z?

Solution: After measurement, the possible values the z-component of the spin of a spin-1/2 particle could take are given by the eigenvalues of σ_z. Since this is a diagonal matrix, it is plain to see that these eigenvalues are its diagonal entries $1, -1$.

Question 2

Problem: What are the possible values of σ_x?

Solution: As with problem 1, the possible values are given by the eigenvalues of σ_x. We now solve for them:

$$\begin{aligned} 0 &= \det(\sigma_x - \lambda I_2) && \text{(characteristic equation)} \\ &= \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \\ &= \lambda^2 - 1 && \text{(det. of } 2\times 2 \text{ matrix)} \\ 1 &= \lambda^2 \\ \pm 1 &= \lambda \end{aligned}$$

And so we have that the possible observed values of the x-component of the spin (i.e. the eigenvalues of the observable σ_x) are $1, -1$.

Question 3

Problem: Find a state $|\sigma_x = 1\rangle$ such that $\sigma_x = 1$.

Solution: The state $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ has $\sigma_x = 1$. To see this note the following:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 1 \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

And so we have that $|\sigma_x = 1\rangle = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ since it is a an eigenvector of σ_x with eigenvalue 1. As such, the probability of measuring $\sigma_x = 1$ for a given state ϕ is given by:

$$p_{x=1}(\phi) = |\langle \phi | \sigma_x = 1 \rangle|^2$$

In the case of $\sigma_x = 1$ itself, its probability is given by:

$$p_{x=1}(\phi) = |\langle \sigma_x = 1 | \sigma_x = 1 \rangle|^2 = 1$$

Thus, when the x-component of a 1/2-spin particle's spin is measured in the state $|\sigma_x = 1\rangle = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, it will give the value $\sigma_x = 1$ with 100% certainty.

Question 4

Problem: Suppose σ_x is measured when the particle is in the state $|\sigma_x = 1\rangle$. What is the probability that $\sigma_x = -1$?

Solution: First note that $|\sigma_x = -1\rangle$ is given by $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$:

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = -1 \cdot \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

As such the probability of a particle ϕ to be measured with $\sigma_x = -1$ is given by:

$$p_{x=-1}(\phi) = |\langle\phi|\sigma_x = -1\rangle|^2$$

In the case of $|\sigma_x = 1\rangle$ we have:

$$\begin{aligned}
p_{x=-1}(|\sigma_x = 1\rangle) &= |\langle\sigma_x = 1|\sigma_x = -1\rangle|^2\\
&= \left||\sigma_x = -1\rangle^\dagger|\sigma_x = 1\rangle\right|^2\\
&= \left|\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right|^2\\
&= |0|^2 = 0
\end{aligned}$$

And so the probability that a particle in state $|\sigma_x = 1\rangle$ has its spin's x-component measured to be $\sigma_x = -1$ is 0, as these states are orthogonal.

Question 5

Problem: Suppose σ_z is measured when the particle is in the state $|\sigma_x = 1\rangle$. What is the probability that $\sigma_z = -1$?

Solution: First note that $|\sigma_z = -1\rangle$ is given by $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$:

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = -1 \cdot \begin{bmatrix}0\\1\end{bmatrix}$$

As such the probability of a particle ϕ to be measured with $\sigma_z = -1$ is given by:

$$p_{x=-1}(\phi) = |\langle\phi|\sigma_z = -1\rangle|^2$$

In the case of $|\sigma_x = 1\rangle$ we have:

$$\begin{aligned}
p_{z=-1}(|\sigma_x = 1\rangle) &= |\langle\sigma_x = 1|\sigma_z = -1\rangle|^2\\
&= \left||\sigma_z = -1\rangle^\dagger|\sigma_x = 1\rangle\right|^2\\
&= \left|\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right|^2\\
&= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}
\end{aligned}$$

And so the probability that a particle in state $|\sigma_x = 1\rangle$ has its spin's z-component measured to be $\sigma_z = -1$ is $\frac{1}{2}$.

Question 6

Problem: Consider a particle in state $|\sigma_x = 1\rangle$. Give the probability distribution of the state of the particle immediately after σ_z has been measured.

Solution: First note that, from problem 5, we have that the probability of measuring the particle with $\sigma_z = -1$ is $\frac{1}{2}$. We will now show the same is true for $\sigma_z = 1$ (i.e. it's only other possible value):

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} && (|\sigma_z = 1\rangle = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]) \\
p_{z=1}(|\sigma_x = 1\rangle) &= |\langle \sigma_x = 1 | \sigma_z = 1 \rangle|^2 \\
&= \left| |\sigma_z = 1\rangle^\dagger |\sigma_x = 1\rangle \right|^2 \\
&= \left| \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}
\end{aligned}$$

Now recall that, immediately after a measurement, a quantum particle's state collapses to the eigenvector associated with its measured value. And so after having σ_z measured, our particle ψ can only be one of σ_z's eigenvectors, each with 50% probability:

$$\begin{aligned}
P(\psi = |\sigma_z = 1\rangle) &= P\left(\psi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{1}{2} \\
P(\psi = |\sigma_z = -1\rangle) &= P\left(\psi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{2}
\end{aligned}$$

Question 7

Problem: Suppose we immediately measure σ_x after the situation in problem 6. What is the probability that $\sigma_x = -1$?

Solution: Let ψ_b and ψ_a denote the particle's state before and after the measurement of σ_x respectively. We then have the following:

$$\begin{aligned}
P(\psi_a = |\sigma_x = -1\rangle) &= P(\psi_a = |\sigma_x = -1\rangle \wedge \psi_b = |\sigma_z = 1\rangle) \\
&\quad + P(\psi_a = |\sigma_x = -1\rangle \wedge \psi_b = |\sigma_z = -1\rangle) && \text{(law of total prob.)} \\
&= P(\psi_a = |\sigma_x = -1\rangle | \psi_b = |\sigma_z = 1\rangle) P(\psi_b = |\sigma_z = 1\rangle) \\
&\quad + P(\psi_a = |\sigma_x = -1\rangle | \psi_b = |\sigma_z = -1\rangle) P(\psi_b = |\sigma_z = 1\rangle) && \text{(chain rule)} \\
&= \frac{1}{2} P(\psi_a = |\sigma_x = -1\rangle | \psi_b = |\sigma_z = 1\rangle) \\
&\quad + \frac{1}{2} P(\psi_a = |\sigma_x = -1\rangle | \psi_b = |\sigma_z = -1\rangle) && \text{(problem 6)} \\
&= \frac{1}{2} p_{x=-1}(|\sigma_z = 1\rangle) + \frac{1}{2} p_{x=-1}(|\sigma_z = -1\rangle) \\
&= \frac{1}{2} |\langle \sigma_z = 1 | \sigma_x = -1 \rangle|^2 + \frac{1}{2} |\langle \sigma_z = -1 | \sigma_x = -1 \rangle|^2 \\
&= \frac{1}{2} \left| |\sigma_x = -1\rangle^\dagger |\sigma_z = 1\rangle \right|^2 + \frac{1}{2} \left| |\sigma_x = -1\rangle^\dagger |\sigma_z = -1\rangle \right|^2 \\
&= \frac{1}{2} \left| \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right|^2 + \frac{1}{2} \left| \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right|^2 \\
&= \frac{1}{2} \left| -\frac{1}{\sqrt{2}} \right|^2 + \frac{1}{2} \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}
\end{aligned}$$