Intro to Math Reasoning HW 7a

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Problem 1

Problem: Give an example of a relation on a set that is neither reflexive nor anti-reflexive.

Solution: Consider the following graph of a relation on the set $\{1,2\}$:

$$G = \{(1,1)\}$$

This relation is not reflexive because $\neg(2R2)$ and not anti-reflexive because $(1R1)$.

Problem 2

Problem: Give an example of a relation on a set that is neither symmetric nor anti-symmetric.

Solution: Consider the following graph of a relation on the set $\{1,2,3\}$:

$$G = \{(1,2),(2,1),(2,3)\}$$

This relation is not symmetric because $(2R3) \wedge \neg(3R2)$ and not anti-symmetric because $(1R2) \wedge (2R1)$.

Problem 3

Problem: If R is a transitive relation on A, prove that for all $x, y \in A$:

$$xRy \to \mathrm{im}_R(y) \subseteq \mathrm{im}_R(x)$$

Solution: First note that we can rewrite the proposition as the following:

$$xRy \to (\forall a \in A)\, yRa \to xRa$$

Now to prove this we can simply assume the antecedent and prove the consequent:

$$(\forall a \in A)\, yRa \to xRa$$

We can prove this statement by contradiction. Let us assume there is some element $a_0 \in A$ such that:

$$yRa_0 \land \neg(xRa_0)$$

This assumption turns out to be false because, from the transitive property of R and the assumption xRy:

$$\begin{aligned} & xRy \land yRa_0 \to xRa_0 \\ & xRy \\ \therefore\ & xRa_0 \end{aligned}$$

And we are done.

Problem 4

Problem: Suppose R is a transitive relation on A and $\mathrm{im}_R(y) \subseteq \mathrm{im}_R(x)$ for any $x, y \in A$. Prove that these conditions imply xRy.

Solution: This is false. Consider the following graph of a relation on the set $\{1, 2\}$:

$$G = \{(1,1), (2,1)\}$$

The image of 2 is a subset of that of 1:

$$\begin{aligned} \mathrm{im}_R(1) &= \{1\} \\ \mathrm{im}_R(2) &= \{1\} \\ \mathrm{im}_R(2) &\subseteq \mathrm{im}_R(1) \end{aligned}$$

Yet $\neg(1R2)$ and so the proposition does not hold for any transitive relation R.

Problem 5

Problem: Prove that the following relation R on $\mathcal{P}(\mathbb{Z})$ is a partial order:

$$XRY \implies (X = Y) \lor (Y \setminus X \neq \emptyset \land (\forall z \in Y \setminus X, \forall x \in X)\, z > x)$$

Solution: We have to prove this relation is reflexive, anti-symmetric, and transitive.

Reflexivity

Proving reflexivity is trivial:

$$XRY \iff (X = Y) \vee (Y \setminus X \neq \emptyset \wedge (\forall z \in Y \setminus X, \forall x \in X)\ z > x)$$

and so:

$$(X = Y) \implies XRY$$

Anti-symmetry

To prove anti-symmetry we must prove that $XRY \wedge YRX \rightarrow X = Y$. This is equivalent to the contraposition:

$$\begin{aligned} X \neq Y \rightarrow\ & \neg(XRY \wedge YRX) \\ \equiv\ & \neg(XRY) \vee \neg(YRX) \\ \equiv\ & XRY \rightarrow \neg(YRX) \end{aligned}$$

So assuming $X \neq Y$ and XRY we have to show that $\neg(YRX)$. We can prove this by contradiction and assume that indeed YRX. Now notice that:

$$\begin{aligned} & (XRY \wedge X \neq Y) \equiv (\exists y_0 \in Y \setminus X, \forall x \in X)\ y_0 > x \\ & XRY \wedge X \neq Y \\ & Y \setminus X \subset Y \\ \therefore & (\exists y_0 \in Y, \forall x \in X)\ y_0 > x \end{aligned}$$

Now notice the same holds true for YRX:

$$\begin{aligned} & (YRX \wedge Y \neq X) \equiv (\exists x_0 \in X \setminus Y, \forall y \in Y)\ x_0 > y \\ & YRX \wedge Y \neq X \\ & X \setminus Y \subset X \\ \therefore & (\exists x_0 \in X, \forall y \in Y)\ x_0 > y \end{aligned}$$

Those two statements cannot simultaneously be true. This is a contradiction, thus our assumption that YRX was false. This means $\neg(YRX)$ which in turn chains back and proves our initial proposition of anti-symmetry.

Transitivity

Proving this means proving the following for all subsets Z:

$$XRZ \wedge ZRY \rightarrow XRY$$

For convience, I'm using the notation $A > B$ to mean:

$$(\forall a \in A, b \in B)\ a > b$$

If we assume $XRZ \wedge ZRY$ then we can say:

$$(Z \setminus X > X) \wedge (Y \setminus Z > Z)$$

Now note that because $Z \setminus X \subseteq Z$ we can say:

$$Y \setminus Z > Z \setminus X$$

This gives us the following chain of inequalities (which is valid because every element of the right sides are strictly less than that of the elements on the left):

$$Y \setminus Z > Z \setminus X > X$$

And clearly since every element in Z, barring those in X, is bigger than the elements of X and every element in Y, barring those in Z, is bigger than the elements of Z, again barring those in X:

$$Y \setminus X > X$$

That is $Y \setminus Z > Z \setminus X$ and $Z \setminus X > X$ imply the above due to all the element in Y (that aren't also in X) necessarily being bigger than all of those in X. And so the relation is transitive.