Math Statistics
Semiweekly HW 12

Ozaner Hansha

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Question 1

Problem: Suppose we have a population of bags of potato chips, each bag containing m chips with each chip having probability p of being tasty. The number of tasty chips in a given bag, then, is given by a binomial distribution $B(m, p)$.

However, note that our sample consists of observations of the number of *not* tasty chips, rather than tasty ones. Thus, our sample is drawn from the binomial distribution $B(m, 1-p)$. Given samples of this form, derive an estimator for both m and p using the method of moments.

Solution: Recall that the method of moments estimator for k parameters is to set the first k moments of the distribution equal to the first k sample moments and solve for each parameter. We do that below, with $\bar{M}_k$ denoting the kth sample moment, and $X \sim B(m, 1-p)$:

$$
\begin{aligned}
\begin{cases} \bar{M}_1 = E(X) \\ \bar{M}_2 = E(X^2) \end{cases}
&\implies \begin{cases} \bar{M}_1 = \frac{d}{dt} M_X(t)\Big|_{t=0} \\ \bar{M}_2 = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} \end{cases} && (k\text{th moment given by mgf}) \\
&\implies \begin{cases} \bar{M}_1 = \frac{d}{dt}(p+(1-p)e^t)^m\Big|_{t=0} \\ \bar{M}_2 = \frac{d^2}{dt^2}(p+(1-p)e^t)^m\Big|_{t=0} \end{cases} && (\text{mgf of binomial RV}) \\
&\implies \begin{cases} \bar{M}_1 = m(1-p) \\ \bar{M}_2 = m(m-1)(1-p)^2 + m(1-p) \end{cases} \\
&\implies \begin{cases} \bar{M}_1 = m(1-p) \\ \frac{\bar{M}_2}{\bar{M}_1} = (m-1)(1-p) + 1 \end{cases} \\
&\implies \begin{cases} \bar{M}_1 = m - mp \\ \frac{\bar{M}_2}{\bar{M}_1} = m - mp + p \end{cases} \\
&\implies \begin{cases} \bar{M}_1 = m(1-p) \\ \frac{\bar{M}_2}{\bar{M}_1} - \bar{M}_1 = p \end{cases} \\
&\implies \begin{cases} \bar{M}_1 = m\left(1 - \left(\frac{\bar{M}_2}{\bar{M}_1} - \bar{M}_1\right)\right) \\ \frac{\bar{M}_2}{\bar{M}_1} - \bar{M}_1 = p \end{cases} \\
&\implies \begin{cases} \frac{\bar{M}_1}{1-\left(\frac{\bar{M}_2}{\bar{M}_1} - \bar{M}_1\right)} = m \\ \frac{\bar{M}_2}{\bar{M}_1} - \bar{M}_1 = p \end{cases}
\end{aligned}
$$

And so we are left with the following estimators, for p we have:

$$
\begin{aligned}
\hat{p}_{MM} &= \frac{\bar{M}_2}{\bar{M}_1} - \bar{M}_1 && (\text{method of moments}) \\
&= \frac{\bar{M}_2}{\bar{M}_1} - \bar{M}_1^2 \\
&= \frac{\frac{1}{n}\sum_{i=1}^n X_i^2}{\frac{1}{n}\sum_{i=1}^n X_i} - \frac{1}{n}\sum_{i=1}^n X_i \\
&= \frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i} - \bar{X}
\end{aligned}
$$

And for m we have:

$$\begin{aligned}
\hat{m}_{MM} &= \frac{\bar{M}_1}{1-\left(\frac{\bar{M}_2}{\bar{M}_1}-\bar{M}_1\right)} && \text{(method of moments)}\\
&= \frac{\bar{M}_1}{1-\left(\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i}-\frac{1}{n}\sum_{i=1}^n X_i\right)} && \text{(from above derivation)}\\
&= \frac{\frac{1}{n}\sum_{i=1}^n X_i}{1-\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i}+\frac{1}{n}\sum_{i=1}^n X_i}\\
&= \frac{\bar{X}}{1-\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i}+\bar{X}} && \text{(def. of sample mean)}
\end{aligned}$$