Ozaner Hansha

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RUID: 188003484

Honors Calculus III HW 1

Exercise 1.2

$\mathbf{a} := (5,2)$ $\mathbf{b} := (2,-1)$ $\mathbf{c} := (1,1)$

Part 1

Express $\mathbf{a}$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$:

$$\begin{aligned}\mathbf{a} &= s\mathbf{b} + t\mathbf{c} \\ (5,2) &= s(2,-1) + t(1,1) \\ &= (2s,-s) + (t,t) \\ &= (2s+t, t-s)\end{aligned}$$

As the components of the vectors are independent under addition, this implies the following two scalar equations:

$$2s + t = 5 \tag{1st Component}$$
$$t - s = 2 \tag{2nd Component}$$

Which can then be solved via some simple algebraic manipulations:

$$\begin{array}{r} 2s + t = 5 \\ + \;\; 2(t - s = 2) \\ \hline 3t = 9 \\ \rightarrow t = 3 \end{array}$$

Plugging this back in the first equation we find that:

$$2s + 3 = 5 \rightarrow s = 1$$

And so $\mathbf{a}$ can be written as the following linear combination:

$$\mathbf{a} = \mathbf{b} + 3\mathbf{c} \qquad \square$$

Part 2

Express $\mathbf{b}$ as a linear combination of $\mathbf{a}$ and $\mathbf{c}$.

We know the following is true from **Part 1**:

$$\mathbf{a}=\mathbf{b}+3\mathbf{c}$$

Subtracting $-3\mathbf{c}$ from both sides (which is valid as all vectors have an additive inverse) we see that we are done:

$$\mathbf{b}=\mathbf{a}-3\mathbf{c} \tag*{$\square$}$$

Part 3

Express $\mathbf{c}$ as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Again, we can leverage a previous result to prove the above. Here I use **Part 1** once more:

$$\begin{align} &\ \mathbf a=\mathbf b+3\mathbf c \\ \rightarrow&\ 3\mathbf c=\mathbf a-\mathbf b \\ \rightarrow&\ \mathbf c=\frac{1}{3}\mathbf a-\frac{1}{3}\mathbf b \tag*{$\square$}

\end{align}$$

Exercise 1.4

$\mathbf{x} := (4,7,-4,1,2,-2)\ \ \mathbf{y} := (2,1,2,2,-1,-1)$

Part 1

Compute $||\mathbf{x}||$ where $\dim(\mathbf{x})=6$:

$$||\mathbf{x}||=\sqrt{\mathbf{x}\cdot\mathbf{x}}=\sqrt{\sum_{i=1}^{6}x_i^2}$$

$$=\sqrt{4^2+7^2+(-4)^2+1^2+2^2+(-2)^2}=3\sqrt{10}\approx 9.487$$

Part 2

Compute $||\mathbf{y}||$ where $\dim(\mathbf{y})=6$:

$$\|\mathbf{y}\| = \sqrt{\mathbf{y}\cdot\mathbf{y}} = \sqrt{\sum_{i=1}^{6} y_i^2}$$
$$= \sqrt{2^2 + 1^2 + 2^2 + 2^2 + (-1)^2 + (-1)^2} = \sqrt{15} \approx 3.872$$

Part 3

Compute the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

First we compute the dot product between $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{6} x_i y_i$$
$$= 4(2) + 7(1) - 4(2) + 1(2) + 2(-1) - 2(-1) = 9$$

Now, given the geometric definition of the dot product:

$$\mathbf{x}\cdot\mathbf{y} = \|x\|\|y\|\cos\theta$$

We can plug in the value for $\mathbf{x}\cdot\mathbf{y}$ we found above along with the values of $\|x\|$ and $\|y\|$ we found in **Part 1** and **Part 2** respectively:

$$\begin{aligned} 9 &= 3\sqrt{10}\cdot\sqrt{15}\cos\theta \\ &= 15\sqrt{6}\cos\theta \\ \to 3 &= 5\sqrt{6}\cos\theta \to \frac{\sqrt{6}}{10} = \cos\theta \end{aligned}$$

Now we can simply take the $\cos^{-1}$ of both sides to solve for θ:

$$\cos^{-1}\cos\theta = \cos^{-1}\frac{\sqrt{6}}{10}$$
$$\theta \approx 1.3233 \qquad \square$$

Exercise 1.6

$\mathbf{x} := (-5, 2, 5) \;\; \mathbf{y} := (1, 2, 1)$

Is the angle θ between $\mathbf{x}$ and $\mathbf{y}$ acute or obtuse?

We first need to compute $\|x\|$, $\|y\|$, and $\mathbf{x}\cdot\mathbf{y}$:

$$
\begin{aligned}
\|x\| &= \sqrt{(-5)^2 + 2^2 + 5^2} &&= 3\sqrt{6} \\
\|y\| &= \sqrt{1^2 + 2^2 + 1^2} &&= \sqrt{6} \\
\mathbf{x} \cdot \mathbf{y} &= -5(1) + 2(2) + 5(1) &&= 4
\end{aligned}
$$

Now we can plug these into the formula for $\cos\theta$:

$$
\begin{aligned}
\cos\theta &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|x\|\|y\|} \\
&= \frac{4}{3\sqrt{6} \cdot \sqrt{6}} = \frac{2}{9}
\end{aligned}
$$

Now we can get the angle by taking the $\cos^{-1}$ of both sides:

$$
\begin{aligned}
\cos^{-1}\cos\theta &= \cos^{-1}\frac{2}{9} \\
\theta &\approx 1.3467
\end{aligned}
$$

An acute angle is any element of the interval $(0, \frac{\pi}{2})$ and an obtuse angle is any element of $(\frac{\pi}{2}, \pi)$. Notice that $\frac{\pi}{2} \approx 1.571 > 1.347 \approx \theta$ and thus the angle between $\mathbf{x}$ and $\mathbf{y}$ is acute.

$\square$

Exercise 1.8

Prove that for any 3 vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ where $\mathbf{a} \neq \mathbf{0}$, that $\mathbf{b} = \mathbf{c}$ iff $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$

First we use the distributive rules of the dot and cross product, respectively, to establish the following equivalent statements:

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} &\iff \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} = 0 \\
&\iff \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0 \qquad (1) \\
\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} &\iff \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = \mathbf{0} \\
&\iff \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0} \qquad (2)
\end{aligned}
$$

Notice that Eq. 1 implies that $\mathbf{b} - \mathbf{c}$ is either $\mathbf{0}$ or orthogonal to $\mathbf{a}$ (because $\mathbf{a} \neq \mathbf{0}$). However notice that Eq. 2 implies that $\mathbf{b} - \mathbf{c}$ is either $\mathbf{0}$ or parallel to $\mathbf{a}$ (again because $\mathbf{a} \neq \mathbf{0}$).

The only way for both these conditions to be true is if $\mathbf{b} - \mathbf{c} = \mathbf{0}$ which is logically equivalent to the statement $\mathbf{b} = \mathbf{c}$.

We could also phrase it as the zero vector being vacuously orthogonal and parallel to every other vector.

$\square$

Exercise 1.10

$\mathbf{a} := (-1, 1, 2)$ $\mathbf{b} := (2, -1, 1)$

Part 1

Find the set of vectors $\mathbf{x}$ such that $\mathbf{a} \times \mathbf{x} = (-2, 4, -3)$ and $\mathbf{b} \cdot \mathbf{x} = 2$

From the explicit definition of the cross product we find that, given a vector $\mathbf{x} = (x_1, x_2, x_3)$ it's cross product with $\mathbf{a}$ should be in the following form:

$$\mathbf{a} \times \mathbf{x} = (x_3 - 2x_2, 2x_1 + x_3, -x_2 - x_1) = (-2, 4, 3)$$

We can also construct another equation using the dot product of $\mathbf{b}$ and $\mathbf{x}$:

$$\mathbf{a} \cdot \mathbf{x} = 2x_1 - x_2 + x_3 = 2$$

Now we can split up the vector equation given by the cross product into 3 separate scalar ones and write it along side the scalar equation given by the dot product condition:

$$x_3 - 2x_2 = -2 \tag{1}$$
$$2x_1 + x_3 = 4 \tag{2}$$
$$-x_2 - x_1 = -3 \tag{3}$$
$$2x_1 - x_2 + x_3 = 2 \tag{4}$$

Notice that we have a set of 4 equations for 3 unknowns, implying that the system has no solution unless one or more of the equations is a linear combination of the others.

We'll just slog through some algebra to show that no contradiction arises when we take all 4 equations to be true. Here we subtract Eq. 2 from Eq. 4:

$$\begin{array}{rr} & 2x_1 + x_3 = 4 \\ - & 2x_1 - x_2 + x_3 = 2 \\ \hline & \boxed{x_2 = 2} \end{array}$$

Plugging x_2 into Eq. 1 we find:

$$x_3 - 2(2) = -2 \rightarrow \boxed{x_3 = 2}$$

Now plugging this into Eq. 2 we arrive at:

$$2x_1 + 2 = 4 \rightarrow \boxed{x_1 = 1}$$

Now to verify that there is no contradiction we plug x_1 and x_2 into Eq. 3:

$$-x_2 - x_1 = -3$$
$$-2 - 1 = -3$$

And so all 4 equations can be simultaneously satisfied by a single vector: $(1, 2, 2)$.

Part 2

Find the set of vectors $\mathbf{x}$ such that $\mathbf{a} \times \mathbf{x} = (2, 4, 3)$ and $\mathbf{b} \cdot \mathbf{x} = 2$

We can set up a set of equations similar to the ones used in part 1:

$$\mathbf{a} \times \mathbf{x} = (x_3 - 2x_2, 2x_1 + x_3, -x_2 - x_1) = (2, 4, 3)$$

$$\mathbf{a} \cdot \mathbf{x} = 2x_1 - x_2 + x_3 = 2$$

Now we just split them into 4 scalar equations:

$$x_3 - 2x_2 = 2 \tag{1}$$
$$2x_1 + x_3 = 4 \tag{2}$$
$$-x_2 - x_1 = 3 \tag{3}$$
$$2x_1 - x_2 + x_3 = 2 \tag{4}$$

Again, we'll assume that all 4 equations are true until we find a contradiction:

$$\begin{array}{rr} & 2x_1 + x_3 = 4 \\ - & 2x_1 - x_2 + x_3 = 2 \\ \hline & \boxed{x_2 = 2} \end{array}$$

Plugging x_2 into Eq. 1 we find:

$$x_3 - 2(2) = 2 \to \boxed{x_3 = 6}$$

Now plugging this into Eq. 2 we arrive at:

$$2x_1 + 6 = 4 \to \boxed{x_1 = -1}$$

Now to verify that there is no contradiction we plug x_1, x_2 into Eq. 3:

$$-x_2 - x_1 = 3$$
$$-2 - (-1) = -1$$
$$-1 \neq 3$$

We've come across a contradiction which means our assumptions that all 4 equations could be simultaneously satisfied was false. Thus the set of vectors that satisfy the given conditions is the null set

$\emptyset$.

Part 3

What vector from the set $\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{a} \times \mathbf{x} = (-2, 4, -3)\}$ is closest to $(1, 1, 1)$?

Any vector $\mathbf{x}$ in the above set will satisfy the following equations (which we got from part 1):

$$
\begin{align}
x_3 - 2x_2 &= 2 \tag{1}\\
2x_1 + x_3 &= 4 \tag{2}\\
-x_2 - x_1 &= -3 \tag{3}
\end{align}
$$

Now we solve for x_1, x_2 and x_3 in terms of a single variable. Here I'll use x_1

$$\boxed{x_1 = x_1}$$

$$-x_2 - x_1 = -3 \rightarrow \boxed{x_2 = 3 - x_1}$$

$$2x_1 + x_3 = 4 \rightarrow \boxed{x_3 = 4 - 2x_1}$$

And so the vector $\mathbf{x}$ has one degree of freedom and we have parameterized it to the variable x_1:

$$\mathbf{x} = (x_1, 3 - x_1, 4 - 2x_1)$$

Now to find the vector closest to $(1, 1, 1)$ we simply find the norm of $\mathbf{x} - (1, 1, 1)$:

$$
\begin{align}
\|\mathbf{x} - (1,1,1)\| &= \sqrt{(x_1 - 1)^2 + (3 - x_1 - 1)^2 + (4 - 2x_1 - 1)^2}\\
&= \sqrt{(x_1 - 1)^2 + (2 - x_1)^2 + (3 - 2x_1)^2}\\
&= \sqrt{6x_1^2 - 18x_1 + 14} \tag{foiling}
\end{align}
$$

Now our problem is equivalent to minimizing the function defined above. We can rename it $f(x)$ and remove the subscript for clarity:

$$f(x) = \sqrt{6x^2 - 18x + 14}$$

To find the minimum we must take the derivative of the function and set it equal to 0 to find it's critical points:

$$\begin{aligned}
0 &= \frac{d}{dx}\sqrt{6x^2 - 18x + 14} \\
&= \frac{1}{2\sqrt{6x^2 - 18x + 14}} \cdot \frac{d}{dx}(6x^2 - 18x + 14) \\
&= \frac{1}{2\sqrt{6x^2 - 18x + 14}} \cdot (12x - 18) \\
&= \frac{6x - 9}{\sqrt{6x^2 - 18x + 14}}
\end{aligned}$$

Note that a fraction can only equal 0 if it's numerator does, thus:

$$\begin{aligned}
\frac{6x - 9}{\sqrt{6x^2 - 18x + 14}} = 0 &\to 6x - 9 = 0 \\
&\to x = \frac{3}{2}
\end{aligned}$$

Testing for values after and before $x = \frac{3}{2}$ will verify that it is indeed a minimum:

even though we know that $x = \frac{3}{2}$ must be a minimum given that the equation models the distances of vectors, which certainly have a minimum but not a maximum

$$\begin{aligned}
f(1) &= \sqrt{14} \approx 3.741 \\
f\left(\frac{3}{2}\right) &= \frac{1}{2} = 0.5 \\
f(2) &= \sqrt{2} \approx 1.414
\end{aligned}$$

And so $x = \frac{3}{2}$ is indeed a minimum. The vector is then given by the parametrization shown above:

$$\mathbf{x} = (x_1, 3 - x_1, 4 - 2x_1) = \boxed{\left(\frac{3}{2}, \frac{3}{2}, 1\right)}$$

Exercise 1.12

Show that for any 4 vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x} \in \mathbb{R}^3$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-zero, that the following is true:

$$\mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{x})) = \mathbf{0} \leftrightarrow \mathbf{b} \perp \mathbf{c} \wedge (\exists \lambda \in \mathbb{R})\ \mathbf{a} = \lambda \mathbf{c}$$

To simplify this proof we'll first split this proposition P into 3 propositions:

$$\begin{aligned}
P_1 &\equiv \mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{x})) = \mathbf{0} \\
P_2 &\equiv \mathbf{b} \perp \mathbf{c} \\
P_3 &\equiv (\exists \lambda \in \mathbb{R})\ \mathbf{a} = \lambda \mathbf{c}
\end{aligned}$$

Now we have to prove the following:

$$P_1 \leftrightarrow P_2 \land P_3$$

Which is equivalent to proving the following two statements:

$$P_1 \rightarrow P_2 \land P_3 \tag{Lemma 1}$$
$$P_1 \leftarrow P_2 \land P_3 \tag{Lemma 2}$$

Which we'll call lemma 1 and 2 respectively.

Rephrasing the Propositions

Let's start off by rewriting P_1:

$$\begin{aligned} P_1 &\equiv \mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{x})) = \mathbf{0} \\ &\equiv \mathbf{a} \times ((\mathbf{b} \cdot \mathbf{x})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{x}) = \mathbf{0} && \text{(Lagrange's formula)} \\ &\equiv (\mathbf{b} \cdot \mathbf{x})(\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \times \mathbf{x}) = \mathbf{0} && \text{(Distributive Prop.)} \\ &\equiv (\mathbf{b} \cdot \mathbf{x})(\mathbf{a} \times \mathbf{c}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \times \mathbf{x}) \end{aligned}$$

Now notice that we can rephrase P_2 as:

$$\begin{aligned} P_2 &\equiv \mathbf{b} \perp \mathbf{c} \\ &\equiv \mathbf{b} \cdot \mathbf{c} = 0 \end{aligned}$$

And finally, we can express P_3 as:

$$\begin{aligned} P_3 &\equiv (\exists \lambda \in \mathbb{R})\ \mathbf{a} = \lambda \mathbf{c} \\ &\equiv \mathbf{a} \parallel \mathbf{c} \\ &\equiv \mathbf{a} \times \mathbf{c} = \mathbf{0} \end{aligned}$$

Lemma 1

Proving the first half of the proposition is simply a matter of substitution. Given that $\mathbf{b} \cdot \mathbf{c} = 0$ (P_2) and that $\mathbf{a} \times \mathbf{c} = \mathbf{0}$ (P_3):

$$\begin{aligned} P_1 &\equiv (\mathbf{b} \cdot \mathbf{x})\,(\mathbf{a} \times \mathbf{c}) = (\mathbf{b} \cdot \mathbf{c})\,(\mathbf{a} \times \mathbf{x}) \\ &\equiv (\mathbf{b} \cdot \mathbf{x})\,(\mathbf{0}) = (0)\,(\mathbf{a} \times \mathbf{x}) \\ &\equiv \mathbf{0} = \mathbf{0} \\ &\equiv T \end{aligned}$$

And so we have shown that P_1 is true assuming P_2 and P_3. More formally:

$$P_2 \land P_3 \rightarrow P_1$$

Lemma 2

Notice that we can rewrite P_1 by distributing the constants into the cross product:

$$\begin{aligned} P_1 &\equiv (\mathbf{b} \cdot \mathbf{x})(\mathbf{a} \times \mathbf{c}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \times \mathbf{x}) \\ &\equiv (\mathbf{a} \times (\mathbf{b} \cdot \mathbf{x})\,\mathbf{c}) = (\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})\,\mathbf{x}) \end{aligned}$$

We can further simplify this by calling $s := (\mathbf{b} \cdot \mathbf{x})$ and $t := (\mathbf{b} \cdot \mathbf{c})$, subtracting one side from the other and using the distributive property:

$$\begin{aligned} P_1 &\equiv (\mathbf{a} \times s\mathbf{c}) = (\mathbf{a} \times t\mathbf{x}) \\ &\equiv (\mathbf{a} \times s\mathbf{c}) - (\mathbf{a} \times t\mathbf{x}) = 0 \\ &\equiv \mathbf{a} \times (s\mathbf{c} - t\mathbf{x}) = 0 \end{aligned}$$

Notice because $\mathbf{a} \neq \mathbf{0}$, and that $\mathbf{c} - \mathbf{x}$ is coplanar to $s\mathbf{c} - t\mathbf{x}$, then $\mathbf{c} - \mathbf{x}$ is parallel to $\mathbf{a}$. So $(\mathbf{a} \times \mathbf{c}) = 0 \equiv P_3$

This means that $\mathbf{b} \cdot \mathbf{c}$ must equal 0 because there are some $\mathbf{x}$ such that $(\mathbf{a} \times \mathbf{x})$ is not $\mathbf{0}$. Thus for P_1 to be true $\mathbf{b} \cdot \mathbf{c} = 0$ thus P_2.