Intro to Real Analysis
HW #1

Ozaner Hansha

January 30, 2021

Let $\mathbb{J}$ be the set of positive integers.

Problem 1

Problem: Prove the following:

$$\forall n \in \mathbb{J},\, 4 \cdot 3^n > (n+2)^2$$

Solution: Consider the predicate $P(n) \equiv 4 \cdot 3^n > (n+2)^2$. We can prove $P(1)$ quite simply:

$$\begin{aligned} P(1) &\equiv 4 \cdot 3 > (1+2)^2 \\ &\equiv 12 > 9 \\ &\equiv \top \end{aligned}$$

Now let us consider the following lemma $\forall n \in \mathbb{J}$:

$$\begin{aligned} 0 &< 2n^2 + 6n + 3 && (n > 0 \implies 2n^2 + 6n + 3 > 0) \\ n^2 + 6n + 9 &< 3n^2 + 12n + 12 \\ (n+3)^2 &< 3(n+2)^2 && \text{(Lemma 1)} \end{aligned}$$

Now we will prove that $P(n) \implies P(n+1)$:

$$\begin{aligned} 4 \cdot 3^{n+1} &= 3(4 \cdot 3^n) && \text{(additive prop. of exponent)} \\ &> 3(n+2)^2 && \text{(assume the antecedent, } P(n)) \\ &> (n+3)^2 && \text{(Lemma 1)} \\ 4 \cdot 3^{n+1} &> (n+1+2)^2 && (P(n+1)) \end{aligned}$$

And so we have shown $P(1)$ and $P(n) \implies P(n+1)$. By the principle of mathematical induction, this implies that $P(n)$ holds for all integers greater than or equal to 1, in other words:

$$\forall n \in \mathbb{J},\, 4 \cdot 3^n > (n+2)^2$$

Problem 2

Problem: For an integer k, let $k\mathbb{J}$ be the set of positive integers which are multiples of k. Prove that $\text{card}(2\mathbb{J}) = \text{card}(3\mathbb{J})$.

Solution: Recall that for two sets to have the same cardinality there must exist a bijection between them. Consider the function $f : 2\mathbb{J} \to 3\mathbb{J}$ defined by $n \to \frac{3n}{2}$. We will first prove that f is injective. Consider arbitrary $x, y \in 2\mathbb{J}$:

$$\begin{aligned} f(x) = f(y) &\implies \frac{3x}{2} = \frac{3y}{2} && \text{(def. of } f) \\ &\implies x = y && \text{(algebra)} \end{aligned}$$

And so we have shown that $f(x) = f(y) \implies x = y$, which is the definition of an injective function.

Next we will show f is surjective. Consider an arbitrary $y \in 3\mathbb{J}$. There exists an x such that $f(x) = y$, namely $x = \frac{2y}{3}$. We prove this below:

$$\begin{aligned}
f(x) &= f\left(\frac{2y}{3}\right) && \text{(def. of } x\text{)} \\
&= \frac{3}{2} \cdot \frac{2y}{3} && \text{(def. of } f\text{)} \\
&= y
\end{aligned}$$

And so we have shown that $(\forall y \in 3\mathbb{J})\,(\exists x \in 2\mathbb{J})\, f(x) = y$, which is the definition of a surjective function.

Since a function that is both injective and surjective is a bijection, we have shown that f is bijective and thus, by the definition of cardinality, that:

$$\text{card}(2\mathbb{J}) = \text{card}(3\mathbb{J})$$

Problem 3

Consider the set of polynomials with integer coefficients $\mathbb{Z}[x]$. Given two polynomials $f, g \in \mathbb{Z}[x]$, we define the relation $\sim$ as:

$$f \sim g \equiv \exists h \in \mathbb{Z}[x],\ f - g = h'$$

Part a: Prove that $f \sim f$.

Solution: Consider an arbitrary polynomial $f \in \mathbb{Z}[x]$. We have $f - f = 0$ and since there exists an $h \in \mathbb{Z}[x]$ such that $h' = 0$, in particular any integer constant C, we have that $f \sim f$.

Part b: Prove that $f \sim g \implies g \sim f$.

Solution: If for two functions $f, g \in \mathbb{Z}$ we have $f \sim g$ this means that:

$$\exists h \in \mathbb{Z}[x],\ \ f - g = h'$$

Now recall that for any function a, the derivative of the negative is equal to the negative of the derivative:

$$(-a)' = -a'$$

Also recall that for any function $a \in \mathbb{Z}[x]$ we also have $-a \in \mathbb{Z}[x]$ since the integers (i.e. its coefficients) are closed under negation. Putting these together we have that:

$$g - f = -h'$$

And since $-h' = (-h)'$ and $-h \in \mathbb{Z}[x]$, since $h \in \mathbb{Z}[x]$, we have that $g \sim f$.

Part c: Prove that $f \sim g \wedge g \sim h \implies f \sim h$.

Solution: If for functions $f, g, h \in \mathbb{Z}$ we have $f \sim g \wedge g \sim h$ this means that:

$$\begin{gathered}
\exists a \in \mathbb{Z}[x],\ \ f - g = a' \\
\exists b \in \mathbb{Z}[x],\ \ g - h = b'
\end{gathered}$$

Now recall that for any functions a, b, the derivative of their sum is the sum of their derivatives:

$$(a + b)' = a' + b'$$

Also recall that for any functions $a, b \in \mathbb{Z}[x]$ we also have $a + b \in \mathbb{Z}[x]$ since the integers (i.e. their coefficients) are closed under addition. Putting these together we have that:

$$f - h = (f - g) + (g - h) = a' + b'$$

And since $a' + b' = (a+b)'$ and $a + b \in \mathbb{Z}[x]$, since $a, b \in \mathbb{Z}[x]$, we have that $f \sim h$.

Part d: For any $f, g \in \mathbb{Z}[x]$, is $f \sim g$?

Solution: No. To see this consider $f = 2x$ and $g = x$:

$$2x - x = x$$

For $2x \sim x$ it must be the case that their difference x be the derivative of some function $h \in \mathbb{Z}[x]$. Let us solve for all such possible h:

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}x} &= x \\ h_C &= \frac{x^2}{2} + C \end{aligned}$$

Notice that no solution h_C is contained in $\mathbb{Z}[x]$ since the squared term of the solutions have a coefficient of $\frac{1}{2} \notin \mathbb{Z}$. Thus we have shown by counterexample that:

$$\neg \forall f, g \in \mathbb{Z}[x],\ f \sim g$$

Problem 4

Problem: Prove Bernoulli's inequality. That is, prove the following:

$$\forall m, n \in \mathbb{J},\ (1+m)^n \geq 1 + mn$$

Solution: Consider an arbitrary $m \in \mathbb{J}$, we have the following predicate:

$$P(n) \equiv (1+m)^n \geq 1 + mn$$

We can prove $P(1)$ immediately:

$$\begin{aligned} P(1) &\equiv 1 + m \geq 1 + m \\ &\equiv \top \end{aligned}$$

Now we will prove that $P(n) \implies P(n+1)$:

$$\begin{aligned} (1+m)^{n+1} &= (1+m)(1+m)^n && \text{(additive prop. of exponent)} \\ &> (1+m)(1+mn) && \text{(assume the antecedent, } P(n)) \\ &= 1 + mn + m^2 n + m && \text{(algebra)} \\ &\geq 1 + mn + m && (m, n \in \mathbb{J} \implies m^2 n > 0) \\ (1+m)^{n+1} &\geq 1 + m(n+1) && (P(n+1)) \end{aligned}$$

And so we have shown $P(1)$ and $P(n) \implies P(n+1)$. By the principle of mathematical induction, this implies that $P(n)$ holds for all integers greater than or equal to 1, in other words:

$$\forall m, n \in \mathbb{J},\ (1+m)^n \geq 1 + mn$$

Problem 5

Problem: Prove that $P(\mathbb{J})$ is not countable.

Solution: Before we prove the desired statement, let us establish Cantor's theorem. First, consider a set A and its power set $\mathcal{P}(A)$. The function $g : A \to \mathcal{P}(A)$ defined by $x \mapsto \{x\}$ is clearly an injunction from $A \to \mathcal{P}(A)$. Since an injunction exits between these two sets we have, by the definition of cardinality:

$$\operatorname{card}(A) \leq \operatorname{card}(\mathcal{P}(A)) \qquad \text{(Lemma 1)}$$

Next, consider an arbitrary function $f : A \to \mathcal{P}(A)$, and the set $B = \{x \in A \mid x \notin f(x)\}$. Suppose f is surjective. This means that $\forall y \in \mathcal{P}(A)$ there exists an $x \in A$ such that $f(x) = y$, in particular it implies that:

$$\exists c \in A,\ f(c) = B \qquad (f \text{ is surjective } \& \ B \in \mathcal{P}(A))$$

But now we have a contradiction:

$$\begin{aligned} c \in f(c) &\iff c \in B && (f(c) = B \ \& \text{ def. of set equality}) \\ c \in f(c) &\iff c \notin B && (\text{def. of B}) \end{aligned}$$

And so our assumption that f was surjective is false, and thus no map from a set A to its powerset $\mathcal{P}(A)$ can be surjective. By the definition of cardinality this means:

$$\text{card}(A) \ngeq \text{card}(\mathcal{P}(A)) \qquad (\text{Lemma 2})$$

Putting these two lemmas together we have that for an arbitrary set A:

$$\begin{array}{rr} \text{card}(A) \leq \text{card}(\mathcal{P}(A)) & (\text{Lemma 1}) \\ \text{card}(A) \ngeq \text{card}(\mathcal{P}(A)) & (\text{Lemma 2}) \\ \hline \text{card}(A) < \text{card}(\mathcal{P}(A)) & (\text{Cantor's Theorem}) \end{array}$$

Or in other words, the cardinality of a set is strictly smaller than that of its powerset. In the case of $\mathbb{J}$ this implies:

$$\text{card}(\mathbb{J}) < \text{card}(\mathcal{P}(\mathbb{J}))$$

And since, by definition, any cardinality larger than countably infinite (i.e. $\text{card}(\mathbb{J}) = \aleph_0$) is uncountably infinite, we are done.