Intro to Math Reasoning HW 9b

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Problem 1

Problem: Use induction to prove that:

$$(\forall r \in \mathbb{R},\ \forall n, m \in \mathbb{Z}_{\geq 0})\ r^{m+n} = r^m r^n$$

Solution: First we'll state this as a predicate for all reals r and integers m:

$$P(n) \equiv r^{m+n} = r^m r^n$$

We know that $P(0)$ is true:

$$\begin{aligned} P(0) \equiv r^{m+0} &= r^m r^0 \\ r^m &= r^m && \text{(def. of } r^0\text{)} \end{aligned}$$

Now we just need to show that $P(n) \rightarrow P(n+1)$:

$$\begin{aligned} r^{m+n} &= r^m r^n && \text{(given)} \\ r^{m+n}(r) &= r^m r^n (r) \\ r^{m+(n+1)} &= r^m r^{n+1} && \text{(def. of } r^{k+1}\text{)} \end{aligned}$$

And so by induction, the equality holds for all real numbers r and nonnegative integers m, n.

Problem 2

Problem: Use induction to prove that:

$$(\forall r \in \mathbb{R},\ \forall n, m \in \mathbb{Z}_{\geq 0})\ (r^m)^n = r^{mn}$$

Solution: First we'll state this as a predicate for all reals r and integers m:

$$P(n) \equiv (r^m)^n = r^{mn}$$

We know that $P(0)$ is true:

$$
\begin{aligned}
P(0) \equiv (r^m)^0 &= r^{m(0)} \\
(r^m)^0 &= r^0 \\
1 &= 1 && \text{(def. of } r^0\text{)}
\end{aligned}
$$

Now we just need to show that $P(n) \to P(n+1)$:

$$
\begin{aligned}
(r^m)^n &= r^{mn} && \text{(given)} \\
(r^m)^n(r^m) &= r^{mn}(r^m) \\
(r^m)^{n+1} &= r^{mn}r^m && \text{(def. of } r^{k+1}\text{)} \\
&= r^{mn+m} && \text{(problem 1)} \\
&= r^{m(n+1)}
\end{aligned}
$$

And so by induction, the equality holds for all real numbers r and nonnegative integers m, n.

Problem 3

Problem: Prove that given a list of n real numbers a_i:

$$(\forall i,\ 1 \le i < n)\ a_i \ge a_{i+1} \implies a_1 \ge a_n$$

Solution: First we'll establish the following proposition:

$$P(i) \equiv a_1 \ge a_i$$

We know that $P(1)$ is true because $a_1 \ge a_1$ is clearly true. We also know that $P(2)$ is true because letting $i = 1$ our antecedent tells us that $a_1 \ge a_2$.

Now we will prove that $P(i) \to P(i+1)$ assuming $1 \le i < n$:

$$
\begin{aligned}
a_1 &\ge a_i && \text{(given)} \\
a_i &\ge a_{i+1} && \text{(plug } i \text{ into antecedent)} \\
a_1 &\ge a_{i+1} && \text{(transitive property of } \ge\text{)}
\end{aligned}
$$

Notice that we could only do line 2 because we assumed $1 \le i < n$.

And so by induction $(\forall i,\ 1 \le i < n)\ P(i)$.

Problem 4

Problem: Give and prove an explicit formula for the following sequence:

$$
\begin{aligned}
c_1 = 1;\quad c_n &= c_{n-1} + \cdots + c_1 + 1 \\
&= \left(\sum_{i=1}^{n-1}\right) + 1
\end{aligned}
$$

Solution: The explicit formula for this sequence is:

$$2^{n-1}$$

We'll prove it using induction. Consider the predicate:

$$P(n) \equiv \left(\sum_{i=1}^{n-1}\right) + 1 = 2^{n-1}$$

We know that $P(1)$ is true because c_1 is defined to be 1 and:

$$\begin{aligned} P(1) \equiv c_1 &= 2^{1-1} \\ 1 &= 2^0 \\ 1 &= 1 \end{aligned}$$

Now we just need to show that $P(n) \to P(n+1)$

$$\begin{aligned} \left(\sum_{i=1}^{n-1}\right) + 1 &= 2^{n-1} \\ 2\left(\left(\sum_{i=1}^{n-1} c_i\right) + 1\right) &= 2^{n-1}(2) \\ \left(\left(\sum_{i=1}^{n-1} c_i\right) + 1\right) + \left(\left(\sum_{i=1}^{n-1} c_i\right) + 1\right) &= 2^n \\ \left(\left(\sum_{i=1}^{n-1} c_i\right) + 1\right) + c_n &= 2^n && \text{(def. of } c_n\text{)} \\ \left(\sum_{i=1}^{n-1} c_i\right) + c_n + 1 &= 2^n \\ \left(\sum_{i=1}^{n} c_i\right) + 1 &= 2^n && \text{(def. finitary addition)} \end{aligned}$$

And so by induction the explicit formula holds for all $n \geq 1$.

Problem 5

Part a

Problem: Prove that $x + y \in X$ if $x, y \in X$.

Solution: Note that any constant c of the following form (where $c_i \in \mathbb{Z}^n$) is in X by definition:

$$\sum_{i=1}^{n} a_i c_i = c$$

Consider two solutions $x, y \in X$. There must be at least one corresponding list x_i and y_i respectively that when plugged into the function return these constants:

$$\begin{aligned}
x &= \sum_{i=1}^{n} a_i x_i \\
y &= \sum_{i=1}^{n} a_i y_i \\
x + y &= \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} a_i y_i \\
&= \sum_{i=1}^{n} a_i x_i + a_i y_i \\
&= \sum_{i=1}^{n} a_i (x_i + y_i)
\end{aligned}$$

And since that last sum is of the proper form (because the integers are closed under addition), $x + y$ is indeed in X.

Part b

Problem: Prove that $cx \in X$ if $x \in X$.

Solution: Note that any constant c of the following form (where $c_i \in \mathbb{Z}^n$) is in X by definition:

$$\sum_{i=1}^{n} a_i c_i = c$$

Consider a solution $x \in X$. There must be at least one corresponding list x_i that when plugged into the function returns this constant. So if we multiply both sides by some arbitrary $k \in \mathbb{Z}$:

$$\begin{aligned}
x &= \sum_{i=1}^{n} a_i x_i \\
kx &= k \sum_{i=1}^{n} a_i x_i \\
&= \sum_{i=1}^{n} k a_i x_i \\
&= \sum_{i=1}^{n} a_i (k x_i)
\end{aligned}$$

And since that last sum is of the proper form (because the integers are closed under multiplication), kx for any integer k is indeed in X.