Honors Calculus III HW #2

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1 Problem 1

Problem: Find a right-handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that:

$$\left(\exists k \in \mathbb{R}^+\right) \mathbf{u}_1 = k(4,4,7) \tag{1}$$
$$\mathbf{u}_2 \cdot (1,0,2) = 0 \tag{2}$$

Solution: To find $\mathbf{u}_1$ we must normalize the $(4,4,7)$. This leaves us with:

$$\|(4,4,7)\| = \sqrt{4^2 + 4^2 + 7^2} = 9$$
$$\mathbf{u}_1 = \frac{1}{9}(4,4,7)$$

To find $\mathbf{u}_2$, recall that it must be orthogonal to $(1,0,2)$ (by condition 2) and $\mathbf{u}_1$ (because it's an orthonormal basis). This means that if we take the normalized cross product of these two vectors, we will have a vector that matches both criteria:

$$\begin{aligned}
\mathbf{u}_2 &= \frac{(1,0,2) \times \frac{1}{9}(4,4,7)}{\|(1,0,2) \times \frac{1}{9}(4,4,7)\|} \\
&= \frac{\frac{1}{9}(-8,1,4)}{\left\|\frac{1}{9}(-8,1,4)\right\|} = \frac{\frac{1}{9}(-8,1,4)}{\left|\frac{1}{9}\right| \|(-8,1,4)\|} \\
&= \frac{\frac{1}{9}(-8,1,4)}{\left|\frac{1}{9}\right| \cdot 9} \\
&= \frac{1}{9}(-8,1,4)
\end{aligned}$$

Now we simply take the cross product of $\mathbf{u}_1$ and $\mathbf{u}_2$, both of which we know to be orthonormal, to find a third orthonormal vector:

$$\begin{aligned}
\mathbf{u}_3 &= \mathbf{u}_1 \times \mathbf{u}_2 \\
&= \frac{1}{9}(4,4,7) \times \frac{1}{9}(-8,1,4) \\
&= \frac{1}{9^2}(9,-72,36) \\
&= \frac{1}{9}(1,-8,4)
\end{aligned}$$

Note that the order of the cross product was chosen in accordance to the right-handedness condition.

$$\left\{\frac{1}{9}(4,4,7), \frac{1}{9}(-8,1,4), \frac{1}{9}(1,-8,4)\right\}$$

However, note that when we found the cross product of $(1,0,2) \times \frac{1}{9}(4,4,7)$ we could have switched their order to arrive at a $\mathbf{u}_2$ with a flipped sign that still satisfied the required conditions. And to retain righthandedness, $\mathbf{u}_3$ would also be flipped (in accordance to the anticommutivity of the cross product). This gives us another equally valid basis:

$$\left\{\frac{1}{9}(4,4,7), \frac{-1}{9}(-8,1,4), \frac{-1}{9}(1,-8,4)\right\}$$

And so, there are two valid bases that satisfy the given conditions.

2 Problem 2

Problem: Given two lines ℓ_1 and ℓ_2 parametrized below:

$$\begin{gathered}
\mathbf{x} : \mathbb{R} \to \ell_1 \\
\mathbf{x}(t) = (1,2,2) + t(0,3,3) \\
\mathbf{y} : \mathbb{R} \to \ell_2 \\
\mathbf{y}(s) = s(2,1,2)
\end{gathered}$$

What is the distance between these lines? Also, what values of t and s minimize $\|\mathbf{x}(t) - \mathbf{y}(s)\|$?

Solution: Notice that ℓ_1 is in the direction of $(0,3,3)$ and that ℓ_2 is in the direction of $(2,1,2)$. The shortest path from a line to another line is perpendicular to both of them. As such, we will take the cross product of these vectors (and normalize it to get a vector only describing the direction from ℓ_1 to ℓ_2):

$$\begin{aligned}
\mathbf{u} &= \frac{(2,1,2)\times(0,3,3)}{\|(2,1,2)\times(0,3,3)\|} \\
&= \frac{(-3,-6,6)}{\|(-3,-6,6)\|} = \frac{(-3,-6,6)}{9} \\
&= \frac{1}{3}(-1,-2,2)
\end{aligned}$$

Now we simply take any point on ℓ_1 and any point on ℓ_2, produce the vector that joins them $\mathbf{w}$, and project that vector onto $\mathbf{u}$. The magnitude of this vector is the distance we are looking for:

$$\begin{aligned}
\mathbf{x}(0) &= (1,2,2) \\
\mathbf{y}(0) &= (0,0,0) \\
\mathbf{w} = \mathbf{x}(0) - \mathbf{y}(0) &= (1,2,2)
\end{aligned}$$

While we could project $\mathbf{w}$ onto $\mathbf{u}$ and take the magnitude of the result, notice that because $\mathbf{u}$ is already a unit vector it suffices to simply take the dot product of the two vectors ($\mathbf{w}$'s component in the $\mathbf{u}$ direction) and take its absolute value:

$$|\mathbf{w}\cdot\mathbf{u}| = \left|(1,2,2)\cdot\frac{1}{3}(-1,-2,2)\right| = \frac{1}{3}$$

So we are done (with the first part) and the distance is $\frac{1}{3}$. To find what t and s actually minimize this we must minimize the following:

$$\|\mathbf{x}(t) - \mathbf{y}(s)\| = \sqrt{(1-2s)^2 + (2+3t-s)^2 + (2+3t-2s)^2}$$

Now we just have to take the partial derivative of the above function with respect to both t and s and set them equal to zero. Also, note that minimizing the norm is the same as minimizing the norm squared, so we will do just that to make the calculus a little easier:

$$\begin{aligned}
\frac{}{\partial t}\partial\|\mathbf{x}(t) - \mathbf{y}(s)\|^2 &= 36t = 18s + 24 = 0 \\
\frac{}{\partial s}\partial\|\mathbf{x}(t) - \mathbf{y}(s)\|^2 &= 18s - 18t - 16 = 0
\end{aligned}$$

We can now solve for t and s via the following system of equations:

$$\begin{aligned}
36t = 18s + 24 &= 0 \\
18s - 18t - 16 &= 0 \\
t = \frac{-4}{9} &\wedge s = \frac{4}{9}
\end{aligned}$$

3 Problem 3

Problem: Let the line ℓ_1 pass through $(1, 2, 2)$ and $(1, 5, 5)$, and let the line ℓ_2 be given by $\mathbf{x}_0 + (2, 1, 2)$. The set of all points $\mathbf{x}_0$ where these lines meet (i.e $\ell_1 \cap \ell_2 \neq \emptyset$) forms a plane. Give this plane in the form $ax + by + cz = d$.

Solution: Call a point that ℓ_1 and ℓ_2 intersect in $\mathbf{p}$. We can now define the plane the lines sit on as the set of all $\mathbf{x}$ such that $\mathbf{a} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$ where $\mathbf{a} \neq \mathbf{0}$. Because ℓ_2 is in this plane, so too is $\mathbf{x}_0$.

Now we'll parameterize the plane using that arbitrary point on the plane $\mathbf{p}$ (and the fact that $(1, 5, 5) - (1, 2, 2) = (0, 3, 3)$ for ℓ_1):

$$\mathbf{r}(t) = \mathbf{p} + t(0, 3, 3)$$
$$\mathbf{q}(s) = \mathbf{p} + s(2, 1, 2)$$

Plugging in $\mathbf{a}(t)$ and $\mathbf{b}(s)$ in for $\mathbf{x}$ into $\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0$ (which they must satisfy given that they parameterize the plane) we find the following:

$$\mathbf{a} \cdot (0, 3, 3) = 0$$
$$\mathbf{a} \cdot (2, 1, 2) = 0$$

This means $\mathbf{a}$ is some scalar multiple of $(0, 3, 3) \times (2, 1, 2) = (3, 6, -6)$. This is equivalent to saying it is a scalar multiple of $(-1, -2, 2)$ (i.e multiply by $\frac{-1}{3}$) and so by taking another point on the plane, say $(1, 2, 2)$ (which is on line ℓ_1) we can compute the standard form of a plane $ax+by+cz = d$ with $(a, b, c) = (-1, -2, 2)$ and $d = (-1, -2, 2) \cdot (1, 2, 2) = -1$:

$$-x - 2y + 2z = -1$$

4 Problem 4

The householder reflection $h_{\mathbf{u}}$ given by $\mathbf{u}$ is defined as $h_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{u})u$. Also note that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ represents the canonical basis of $\mathbb{R}^3$

4.1 Part a

Problem: Find a $\mathbf{u}$ such that $h_{\mathbf{u}}(\mathbf{e}_1) = \frac{1}{9}(4, 4, 7)$

Solution: We have to find a $\mathbf{u}$ such that the following is true:

$$
\begin{aligned}
\mathbf{u} &= \frac{\mathbf{e}_1 - \frac{1}{9}(4,4,7)}{\left\|\mathbf{e}_1 - \frac{1}{9}(4,4,7)\right\|} \\
&= \frac{\frac{1}{9}(5,-4,-7)}{\left\|\frac{1}{9}(5,-4,-7)\right\|} \\
&= \frac{\frac{1}{9}(5,-4,-7)}{\left\|\frac{1}{9}(5,-4,-7)\right\|} \\
&= \frac{\frac{1}{9}(5,-4,-7)}{\frac{10}{\sqrt{3}}} \\
&= \frac{1}{3\sqrt{10}}(5,-4,-7)
\end{aligned}
$$

4.2 Part b

Problem: Using the $\mathbf{u}$ found in part a, compute $h_{\mathbf{u}}(\mathbf{e}_2)$ and $h_{\mathbf{u}}(\mathbf{e}_3)$. Also show that $\{h_{\mathbf{u}}(\mathbf{e}_1), h_{\mathbf{u}}(\mathbf{e}_2), h_{\mathbf{u}}(\mathbf{e}_3)\}$ is a left-handed orthonormal basis of $\mathbb{R}^3$.

Solution: By doing the computations with $\mathbf{u} = \frac{1}{3\sqrt{10}}(5,-4,-7)$ we find:

$$
\begin{aligned}
h_{\mathbf{u}}(\mathbf{e}_2) &= \frac{1}{45}(20,29,-28) \\
h_{\mathbf{u}}(\mathbf{e}_3) &= \frac{1}{45}(35,-28,-4)
\end{aligned}
$$

Of course $h_{\mathbf{u}}(\mathbf{e}_1)$, $h_{\mathbf{u}}(\mathbf{e}_2)$, and $h_{\mathbf{u}}(\mathbf{e}_3)$ are orthonormal because both length and orthogonality are preserved by the householder transformation. To check if they form a lefthanded basis the following must be true:

$$h_{\mathbf{u}}(\mathbf{e}_1) \times h_{\mathbf{u}}(\mathbf{e}_2) = -h_{\mathbf{u}}(\mathbf{e}_3)$$

And indeed when we do the calculations we find:

$$\frac{1}{9}(4,4,7) \times \frac{1}{45}(20,29,-28) = \frac{-1}{45}(35,-28,-4)$$

5 Problem 5

Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be any vectors in $\mathbb{R}^3$ such that $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) \neq 0$

5.1 Part a

Problem: Prove that $|\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)| = |\mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{v}_1)| = |\mathbf{v}_3 \cdot (\mathbf{v}_2 \times \mathbf{v}_1)|$.

Solution: Recall triple product identity, which states that any cyclic permutation of the vectors a, b, and c in the form $a \cdot (b \times c)$ is equivalent to each other. So, simply call $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ a, b, and c respectively and then take the absolute value of the quantity inside (just a weaker statement than the triple product identity) and we're done.

5.2 Part b

Problem: Call the scalar triple product referenced in part a D. Define the 3 vectors:

$$\begin{aligned}\mathbf{w}_1 &= \frac{1}{D}\mathbf{v}_2 \times \mathbf{v}_3 \\ \mathbf{w}_2 &= \frac{1}{D}\mathbf{v}_3 \times \mathbf{v}_1 \\ \mathbf{w}_3 &= \frac{1}{D}\mathbf{v}_1 \times \mathbf{v}_2\end{aligned}$$

Show that for all $1 \leq i$ and $j \leq 3$ the following is true: $\mathbf{v}_i \cdot \mathbf{w}_j = \delta_{ij}$

Solution: Note that the triple product of any two vectors is 0 if any two of them are equal. This is a result of being able to cyclically permutate the vectors until the two equivalent vectors are together in the cross product part of the triple product. And so:

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{w}_1 &= \left(\frac{1}{D}\right)\mathbf{v}_1 \cdot \mathbf{v}_2 \times \mathbf{v}_3 = 1 \\ \mathbf{v}_2 \cdot \mathbf{w}_1 &= \left(\frac{1}{D}\right)\mathbf{v}_2 \cdot \mathbf{v}_2 \times \mathbf{v}_3 = 0 \\ \mathbf{v}_3 \cdot \mathbf{w}_1 &= \left(\frac{1}{D}\right)\mathbf{v}_3 \cdot \mathbf{v}_2 \times \mathbf{v}_3 = 0\end{aligned}$$

The first one being true because the triple product is D and the second and third being true because of the property noted above. The same argument follows for $\mathbf{w}_2$ and $\mathbf{w}_3$.

6 Problem 6

Let v_1, v_2, v_3, w_1, w_2 and $w, 3$ be the same ones from problem 5.

6.1 Part a

Problem: Show that $\text{span}(\{v_1, v_2, v_3\}) = \mathbb{R}^3$

Solution: We know that span$(\{v_1, v_2, v_3\})$ forms a subspace of $\mathbb{R}^3$ because it is closed under scalar multiplication and vector addition. Also note that any subspace of $\mathbb{R}^3$ must either be a line or plane through the origin, or all of $\mathbb{R}^3$ itself. So we just have to show that there exists no plane that can contain these 3 vectors.

The plane through the origin containing $\mathbf{v}_1$ and $\mathbf{v}_2$ is defined by the equation $x \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = 0$. However, recall from problem 5 that v_3 does not satisfy that equation. And so, the three vectors are not contained in a plane (ruling out a line as well) thus they must span the entirety of $\mathbb{R}^3$.

6.2 Part b

Problem: Show that all vectors in $\mathbb{R}^3$ can be expressed as a unique linear combination of the vectors $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3$ and that $t_j = \mathbf{w}_j \cdot \mathbf{x}$.

Solution: Because, as we've shown in part a, these **three** vectors span all of real **three**-space, we can conclude that there exists a unique triplet of scalars that satisfy the following for all $\mathbf{x} \in \mathbb{R}^3$:

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 = \mathbf{x}$$

So we've proved the first statement. Now taking the dot product of both sides with $\mathbf{w}_1$ we get:

$$\mathbf{w}_1 \cdot \mathbf{x} = t_1(\mathbf{w}_1 \cdot \mathbf{v}_1) + t_2(\mathbf{w}_1 \cdot \mathbf{v}_2) + t_3(\mathbf{w}_1 \cdot \mathbf{v}_3) = t_1$$

Remember from problem 5 that because the indices don't match (i.e. orthogonal), the dot product equals 0 for the last two terms. The same argument follows for $\mathbf{w}_2$ and $\mathbf{w}_3$.

7 Problem 7

Define the following three vectors as so:

$$\begin{aligned}
\mathbf{v}_1 &= (1, 0, 1) \\
\mathbf{v}_2 &= (1, 1, 1) \\
\mathbf{v}_3 &= (1, 2, 3)
\end{aligned}$$

7.1 Part a

Problem: Find three vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ such that for all $1 \leq i$ and $j \leq 3$ the following is true: $\mathbf{v}_i \cdot \mathbf{w}_j = \delta_{ij}$

Solution: First we compute the following three vectors:

$$\begin{aligned}\mathbf{v}_2 \times \mathbf{v}_3 &= (1, -2, 1)\\ \mathbf{v}_3 \times \mathbf{v}_1 &= (2, 2, -2)\\ \mathbf{v}_1 \times \mathbf{v}_2 &= (-1, 0, 1)\end{aligned}$$

Now, using the formula from problem 6 part a, we divide them by $D = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) = 2$, and arrive at:

$$\begin{aligned}\mathbf{w}_1 = \mathbf{v}_2 \times \frac{1}{2}\mathbf{v}_3 &= (1, -2, 1)\\ \mathbf{w}_2 = \mathbf{v}_3 \times \mathbf{v}_1 &= (1, 1, -1)\\ \mathbf{w}_3 = \mathbf{v}_1 \times \frac{1}{2}\mathbf{v}_2 &= (-1, 0, 1)\end{aligned}$$

7.2 Part b

Problem: Find three numbers t_1, t_2, and t_3 such that:

$$t_1(1, 0, 1) + t_2(1, 1, 1) + t_3(1, 2, 3) = (12, -7, 19)$$

Solution: Now, using the formula from problem 6 part b, we just solve for the constants:

$$\begin{aligned}t_1 = \mathbf{w}_1 \cdot (12, -7, 19) &= \frac{45}{2}\\ t_2 = \mathbf{w}_2 \cdot (12, -7, 19) &= -14\\ t_3 = \mathbf{w}_3 \cdot (12, -7, 19) &= \frac{7}{2}\end{aligned}$$

8 Problem 8

Problem: Show that for any 3 vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$ that:

$$(\mathbf{b} \times \mathbf{c}) \cdot [(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})] = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|^2$$

Solution: Recall Lagrange's identity, that is for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$:

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$$

Now we define $\mathbf{x} = (\mathbf{c} \times \mathbf{a})$, $\mathbf{y} = \mathbf{a}$, and $\mathbf{z} = \mathbf{b}$. Plugging these into the identity we find:

$$(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) = ((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})\mathbf{a} - ((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{a})\mathbf{b}$$

Because of the fact that any if any 2 vectors in the triple product are equal the product is 0, we can cancel out the second term on the left-hand side to arrive at:

$$(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) = ((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})\mathbf{a}$$

Now we just take the dot product of both sides with $(\mathbf{b} \times \mathbf{c})$:

$$\begin{aligned}
(\mathbf{b} \times \mathbf{c}) \cdot [(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})] &= (\mathbf{b} \times \mathbf{c}) \cdot [((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})\mathbf{a}] \\
&= ((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}) && \text{(distribute)} \\
&= ((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}) && \text{(cyclic permutate)} \\
&= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|^2
\end{aligned}$$

And we are done.