Logic and Theory of Discrete Systems
RWTH AACHEN UNIVERSITY
Prof. Dr. M. Grohe E. Fluck, N. Runde

Exercise Sheet 4

Due date: Monday, May 15 until 13:00

- Please upload your solutions to RWTH Moodle.
- The due date is at Monday, May 15 until 13:00.
- Hand in your solutions in groups of **two to three students**. If you need to change your group, contact algds@lics.rwth-aachen.de.
- Hand in the solutions of your group as a single PDF file.
- A discussion regarding this exercise sheet will take place on **Friday, May 26 14:30** in room AH II, this is one week later than usual due to the holiday.
- Note that Exercise 4 gives 0 points and will not be corrected.

Exercise 1 (Unit Balls of the Manhattan Norm) 2+1+2=5 points

The *Manhattan norm* (or ℓ_1-norm) of a vector $\mathbf{x} = (x_1, \dots, x_\ell)^\intercal \in \mathbb{R}^\ell$ is defined as

$$\|\mathbf{x}\|_1 \coloneqq \sum_{i=1}^{\ell} |x_i| \,.$$

The ℓ-dimensional ℓ_1 *unit ball* is defined as $B_1^\ell \coloneqq \left\{\mathbf{x} \in \mathbb{R}^\ell \mid \|\mathbf{x}\|_1 \leq 1\right\}$.

a) (i) Draw $B_1^2 \subseteq \mathbb{R}^2$ in the plane.

(ii) Describe the shape of $B_1^3 \subseteq \mathbb{R}^3$.

b) Compute $\operatorname{vol}\left(B_1^2\right)$ and $\operatorname{vol}\left(B_1^3\right)$.

c) Prove that $\lim_{\ell\to\infty} \operatorname{vol}\left(B_1^\ell\right) = 0$.

Exercise 2 (Hyperball and Hypercube) 2+5=7 points

Let $\ell \in \mathbb{N}_{>0}$ and let $s \in \mathbb{R}_{>0}$. We define $Q_{\ell,s}$ as the ℓ-dimensional *hypercube* of side length $s \in \mathbb{R}$ that is centred in the origin. That is,

$$Q_{\ell,s} = \left\{ (x_1, \ldots, x_\ell)^\intercal \in \mathbb{R}^\ell \;\middle|\; |x_i| \leq \tfrac{s}{2} \text{ for all } i = 1, \ldots, \ell \right\} = \left[-\tfrac{s}{2}, \tfrac{s}{2} \right]^\ell.$$

Note that $Q_{\ell,s}$ has 2^ℓ corners. We fill $Q_{\ell,s}$ with (Euclidean) hyperballs the following way:

1) We place 2^ℓ hyperballs of radius $\frac{s}{4}$ as close as possible to the 2^ℓ corners of the hypercube, so that their distance to the origin (i. e. the centre of $Q_{\ell,s}$) is maximal while still being completely contained in Q.

2) We place an additional single hyperball $B(Q_{\ell,s})$ in the origin (i. e. the centre of $Q_{\ell,s}$), such that its radius is maximal with the property that it intersects with none of the other hyperballs' interiors.

Solve the following tasks.

a) Sketch the situation (that is, the hypercube and all the hyperballs) for $\ell = 2$ and $s = 2$.

b) Let $s \in \mathbb{R}_{>0}$ be arbitrary but fixed. Find the minimal dimension $\ell \in \mathbb{N}$ for which it holds that $B(Q_{\ell,s}) \nsubseteq Q_{\ell,s}$ (that is, for which the final hyperball contains points outside of the hypercube). Prove that your answer is correct.

Exercise 3 (Power Iteration) **2+2+2+2=8 points**

Power iteration works since the successive multiplication with the same matrix shifts a randomly generated vector towards the eigenvector which belongs to the eigenvalue of the largest magnitude (the dominant eigenvalue). The algorithm starts with a random vector and terminates when this vector does not change anymore.
For the computations of the power iteration, we always start with the appropriate vector $\mathbf{x}$ consisting only of ones. For three dimensions this is $\mathbf{x} = (1, 1, 1)^T$.

Hint: You do not need to hand in any code, if used. It suffices to give the results up to 3 significant digits, for task a)-c), and 4 significant digits for task d).

a) Compute the power iteration on M_1 and M_2 for 5 iterations.

b) Compute three iterations of the power iteration procedure for M_3. Will the Power Iteration converge? If not, why does Power Iteration fail on this matrix? Justify your answer.

c) Observe that, if A is non-singular, then from $A\mathbf{x} = \lambda\mathbf{x}$ we get $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. Use this to compute the eigenvalue with the smallest magnitude and corresponding eigenvector of M_1. What is the exact result?

d) How many iterations does the algorithm need until the eigenvector becomes stable for up to 3 significant digits for the matrices M_4 and M_5? Make sure to give your results with a precision of 4 significant digits and only stop when the first 3 digits (rounded correctly) become stable.

$$M_1 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}; \; M_2 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}; \; M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}; \; M_4 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}; \; M_5 = \begin{pmatrix} -4 & 10 \\ 7 & 5 \end{pmatrix}$$

Exercise 4 (Positive Semi-Definite Matrices) **0 points**

This exercise will not be corrected and awards 0 points.

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called *positive semi-definite* if every eigenvalue of A is non-negative. Solve the following tasks.

a) For $c \in \mathbb{R}$, consider the matrices

$$A_c = \begin{pmatrix} 2 & 0 & c \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

Determine the set $\{c \in \mathbb{R} \mid A_c \text{ is positive semi-definite}\}$. Give the answer as an interval in $\mathbb{R}$ and prove that it is correct.

b) Prove that the following is an equivalent definition to the one given above: A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive semi-definite if and only if there exists $B \in \mathbb{R}^{n\times n}$ such that $A = BB^{\intercal}$.

c) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, let $\lambda_1 \neq \lambda_2$ be two distinct eigenvalues of A and let E_1 and E_2 denote the corresponding eigenspaces. Prove that E_1 and E_2 are orthogonal, that is, for all $\mathbf{v}_1 \in E_1$ and $\mathbf{v}_2 \in E_2$ it holds that $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$.