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Exercise Sheet 3

Due date: Monday, May 08 until 13:00

- Please upload your solutions to Moodle. The upload will be possible from Tuesday, May 02 13:00.
- Hand in your solutions in groups of **two to three students**.
- Hand in the solutions of your group as a single PDF file.
- The solutions for this exercise sheet will be published after the deadline.
- A discussion regarding this exercise sheet will take place on **Friday, May 12 14:30** in room AH II.

Exercise 1 (Sample Size Bound for Decision Tree Learning) — 3+2=5 points

We consider a Boolean classification problem with $a = \mathcal{A}$ features, where each feature can attain at most v possible values. Our hypothesis class $\mathcal{H}$ is the class of Boolean functions represented by decision trees over this feature set.

a) Define a suitable description scheme Δ for the hypothesis class, that is, for decision trees with this feature set.

Suppose $h \in \mathcal{H}$ is a hypothesis that can be represented by a decision tree with n nodes. In terms of n, a and v, give an upper bound for $|h|_\Delta$.

Hint: Your bound needs not be as sharp as possible. You are allowed to make slightly rougher estimates in order to obtain a nicer expression.

b) Consider again the Boolean classification problem from chapter 1 of the lecture whose objective was to decide under which circumstances students attend lectures (slide 1.32). The input features, and their possible values are as follows:

Feature	Possible Values
weather	sunny, cloudy, rainy, snowy
day of week	Monday, Tuesday, Wednesday, Thursday, Friday
party the night before	yes, no
topic	DB, ML, Alg, Logic
lecturer	Codd, Karp, Rabin, Valiant

Assume that the unknown target function can be represented by a decision tree with 10 nodes. Use your description scheme from the first part of the exercise, and give a lower bound m on the number of training examples such that any hypothesis that is consistent with m randomly drawn examples has a generalisation error of at most 5 % with probability greater than 99 %. Justify your answer.

Exercise 2 (VC Dimension) **4 points**

Determine the VC dimension $\mathrm{VC}(\mathcal{H})$ of the following class of functions, and prove that your claim is correct.
Let $\mathbb{X} = \mathbb{R}^2$ and let $\mathcal{H}$ be the class of all functions $h_{a,b}\colon \mathbb{R}^2 \to \{0,1\}$ with

$$h_{a,b}(\mathbf{x}) = \begin{cases} 1 & \text{if } x_1 = a \text{ or } x_2 = b \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

where $a, b \in \mathbb{R}$.

Hint: It may be more convenient to view $\{0,1\}$-valued functions h over $\mathbb{X}$ as subsets S_h of $\mathbb{X}$, where $S_h = \{x \in \mathbb{X} \mid h(x) = 1\}$. Then a set $Y \subseteq \mathbb{X}$ is shattered by $\mathcal{H}$ if (and only if) for every subset Y' of Y there exists $h \in \mathcal{H}$ such that $Y' = S_h \cap Y$.

Exercise 3 (Bandit Learning) **4+1=5 points**

Consider the bandit learning scenario with three slot machines (or actions) $a \in \{1,2,3\}$. We want to execute the Exp3 algorithm over 3 rounds using the parameter $\gamma = \frac{1}{2}$.

a) In order to reenact the execution algorithm, for this exercise we assume that the sequence of actions that gets drawn is

$$\mathbf{a} = (a^{(1)}, a^{(2)}, a^{(3)}) = (1,2,3)$$

(regardless of the probability distribution). Also assume that the rewards of these actions in the respective rounds are

$$q_1^{(1)} = 3\ln(2) \quad q_2^{(2)} = 5\ln(2) \quad q_3^{(3)} = 3\ln(2).$$

Give the weight vectors $(w_1^{(s)}, w_2^{(s)}, w_3^{(s)})$ and the probabilities $(p_1^{(s)}, p_2^{(s)}, p_3^{(s)})$ for $s = 1,2,3,4$ as computed by the Exp3 algorithm. Give numeric results for $s = 4$ that are rounded off to 2 decimal places.

b) In the given action sequence, both action 1 and 3 yielded the same reward when they were chosen. Yet, $w_1^{(4)}$ and $w_3^{(4)}$ are different. Discuss why.

Exercise 4 (MWU with Payoffs) **6 points**

We consider the multiplicative weight update algorithm. In some situations it is easier to model the problem using *payoffs* (also called *gains* or *rewards*) instead of costs. For this we use a *payoff matrix* ($n \times t$, with entries $r_i^{(s)} \in [0,1]$) and the weight update rule

$$w_i^{(t+1)} := w_i^{(t)}\left(1 + \alpha \cdot r_i^{(t)}\right)$$

where $r_i^{(t)}$ is the *reward* of following expert i in round t. The choice which expert to follow is done randomly according to

$$p_i^{(t)} := \frac{w_i^{(t)}}{\sum_{j=1}^n w_j^{(t)}}.$$

Show that, if one extends the MWU algorithm as described above, the following bound on the *expected payoff in round* t holds:

$$\sum_{s=1}^{t}\sum_{i=1}^{n} r_i^{(s)} p_i^{(s)} \geq -\frac{\ln n}{\alpha} + (1-\alpha)\sum_{s=1}^{t} r_j^{(s)}$$

for all $t \geq 1$ and all $j \in [n]$.

Hint: You can follow the general idea of the proof of Theorem 4.2 in the lecture. The following inequalities may be useful for achieving the desired bound (you may use them without showing them to hold):

$$1 + \alpha x \geq (1+\alpha)^x \qquad \text{for all } x \in [0,1] \text{ and } \alpha > -1. \tag{1}$$

$$\ln(1+\alpha) \geq \alpha - \alpha^2 \qquad \text{for all } \alpha \geq 0. \tag{2}$$

$$1 + x \leq e^x \qquad \text{for all } x \in \mathbb{R}. \tag{3}$$