MIDDLE EAST TECHNICAL UNIVERSITY
FACULTY OF ENGINEERING

CE 204 UNCERTAINTY and DATA ANALYSIS
Spring Semester 2012-2013
Homework 3- Date Due: May 2nd, 2013 Thursday till 16.00

IMPORTANT NOTICE:

- ***You are allowed to collaborate with other students (or ask questions to your assistants/ instructors) on homework provided that you stay away from plagiarizing (according to dictionaries "to plagiarize" means to steal and pass off ideas and/or words/ solutions of another as one's own without citing the source). That is, collaboration is accepted if you write and give your own solutions. If you are caught on plagiarizing or cheating by handing in "too similar" homework, you will be graded by zero on this homework.***

1. Let the joint probability density function of the random variables X and Y be modeled as follows:

$f_{XY}(x, y) = k\ (x+y)$ for $1 \leq x \leq 2,\ 2 \leq y \leq 4$

$f_{XY}(x, y) = 0$ elsewhere.

a) Find the constant k, so that $f_{XY}(x, y)$ is a proper joint probability density function of the random variables X and Y.
b) Find the marginal probability density function of the random variable X.
c) Find the marginal probability density function of the random variable Y.
d) Find the conditional mean of Y, given X = x.
e) Are X and Y statistically independent random variables? Explain. If they are not independent find the correlation coefficient ρ_{XY}. What does it imply?

a) $\int_1^2 \int_2^4 k(x+y)\, dy\, dx = \int_1^2 k(xy + y^2/2)\Big|_2^4 dx = \int_1^2 k(4x+8-2x-2)\, dx$
$= k(x^2+6x)\Big|_1^2 = 9k = 1 \Rightarrow k = 1/9.$

b) $f_X(x) = \int_2^4 1/9\,(x+y)\, dy = 1/9\,(xy + y^2/2)\Big|_2^4 = \frac{1}{9}(2x+6), \quad 1 \leq x \leq 2$
$= 0$, elsewhere.

c) $f_Y(y) = \int_1^2 1/9\,(x+y)\, dx = \frac{1}{9}(x^2/2 + xy)\Big|_1^2 = 1/9\,(y + 3/2) \quad 2 \leq y \leq 4$
$= 0$, elsewhere.

d) $E(Y/x) = \int_2^4 y\, f_{Y/x}(x,y)\, dy = \int_2^4 y\, \frac{1/9\,(x+y)}{1/9\,(2x+6)}\, dy = \frac{xy^2/2 + y^3/3}{2(x+3)}\Big|_2^4$
$= \frac{3x + 28/3}{(x+3)}, \quad 1 \leq x \leq 2$

e) $f_{XY}(x,y) = 1/9\,(x+y) \neq \frac{1}{9}(2x+6) \cdot \frac{1}{9}(y + 3/2) \Rightarrow$ X & Y are not independent.
$E(X) = \int_1^2 x\ 1/9\,(2x+6)\, dx = 41/27 = 1.5185$
$E(Y) = \int_2^4 y\, \frac{1}{9}(y+3/2)\, dy = 83/27 = 3.074$
$E(x^2) = \int_1^2 x^2\, 1/9\,(2x+6)\, dx = 2.3889 \Rightarrow V(x) = 2.3889 - (1.5185)^2 = 0.0830$
$E(Y^2) = \int_2^4 y^2\, 1/9\,(y+3/2)\, dy = 9.7778 \Rightarrow V(Y) = 9.7778 - (3.074)^2 = 0.3283$
$E(XY) = \int_1^2 \int_2^4 xy\, \frac{1}{9}(x+y)\, dy\, dx = 4.6667$
$COV(XY) = 4.6667 - 1.5185 \times 3.074 = -0.00117, \quad \rho_{XY} = \frac{-0.00117}{\sqrt{0.083 \times 0.328}}$
$\rho_{XY} \cong -0.0071 \cong -0.01 \Rightarrow$ almost no correlation.

2. An engineer on a field survey observes the damage state (X) and type of buildings (Y). He denotes damage states with 0 (no damage) and 1 (damaged) for the buildings. He designates building types as 1 and 2 for reinforced concrete and masonry structures and records the number of cases for X and Y in the following table

Y \ X	0	1
1	1	3
2	3	1

a) Find the joint probability mass function for X and Y.

b) Find the marginal probability mass functions of X and Y. Are X and Y sataistically independent? Why yes/why no?

c) Find the expected values and standard deviations of X and Y. What do you say about their variations using coefficient of variations.

d) Find the correlation coefficient, ρ_{XY} . What does it imply?

a) $P_{XY}(0,1) = 1/8$, $P_{XY}(0,2) = 3/8$

$P_{XY}(1,1) = 3/8$, $P_{XY}(1,2) = 1/8$.

b) $P_X(x) = \sum_{\forall y} P_{xy}(x,y) \Rightarrow P_X(0) = 4/8$, $P_X(1) = 4/8$.

$P_Y(y) = \sum_{\forall x} P_{xy}(x,y) \Rightarrow P_Y(1) = 4/8$, $P_Y(2) = 4/8$.

$P_{XY}(0,1) \neq P_X(0) \cdot P_Y(1) \Rightarrow$ X & Y are not independent.

c) $E(X) = 0 \times 4/8 + 4/8 \times 1 = 4/8$

$E(X^2) = 0 \times 4/8 + 1 \times 4/8 = 4/8$ $\Rightarrow V(X) = 4/8 - (4/8)^2 = 0.25$

$\sigma_X = 0.5$

$E(Y) = 1 \times 4/8 + 2 \times 4/8 = 12/8$ $\Rightarrow V(Y) = 2.5 - (12/8)^2 = 0.250$

$E(Y^2) = 1 \times 4/8 + 4 \times 4/8 = 2.5$

$\sigma_Y = 0.50$

$\delta_X = 0.5/0.5 = 1.0$, $\delta_Y = \dfrac{0.5}{1.5} = 1/3 \approx 0.33$

Quite large variations exist between mean values and random variables X & Y.

$E(XY) = 1 \times 1 \times 3/8 + 1 \times 2 \times 1/8 = 5/8$

$COV(XY) = 5/8 - 1/2 \times 12/8 = -0.125 = -1/8$

$\rho_{XY} = \dfrac{-0.125}{0.5 \times 0.5} = -0.5$;

X & Y are weakly correlated!

3. The random variable X is to be modeled by the following fuction:

$f_X(x) = k \sin(x/2)$, $0 < x < \pi$
$f_X(x) = 0$, elsewhere.

a) Find the constant k so that $f_X(x)$ is a proper probability density function of X.
b) If $Y = X^2$, find the probability density function of the random variable Y.
c) Plot probability density functions and cumulative distribution functions for X and Y.

a) $\int_0^{\pi} k \sin(x/2)\,dx = 1 \Rightarrow \int_0^{\pi} [2k(-\cos x/2)] = 1$

$2k(-0+1) = 1 \Rightarrow k = 1/2$

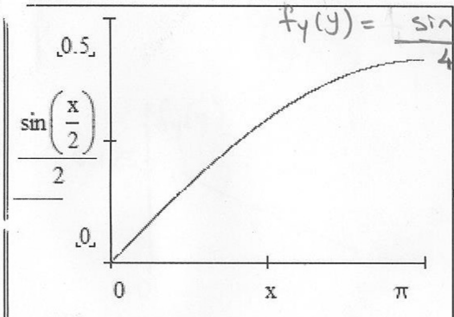
b) $F_X(x) = \int_0^{x} 1/2 \sin r/2 \, dr = -\cos r/2 \Big|_0^{x} = 1 - \cos x/2 \quad 0<x<\pi$

(Check $F_X(0) = 0$ ✓, $F_X(\pi) = 1$ ✓ OK)

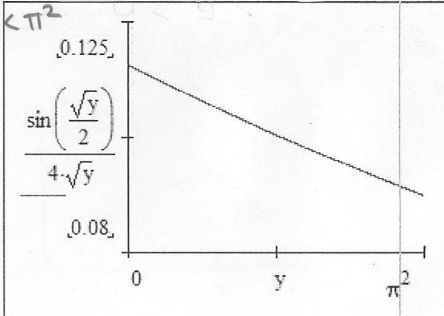
$Y = X^2 \Rightarrow X = \mp\sqrt{Y} \Rightarrow X = \sqrt{Y}$, $0 < Y < \pi^2$

$F_Y(y) = 1 - \cos\sqrt{y}/2$, $0 < y < \pi^2$

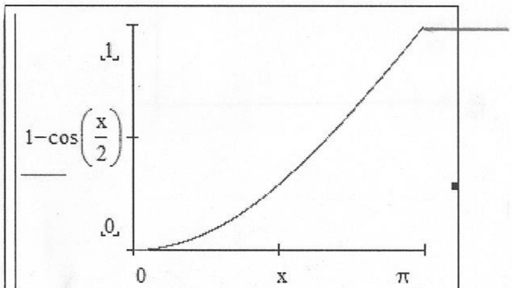
$f_Y(y) = \dfrac{\sin(\sqrt{y}/2)}{4\sqrt{y}} \quad 0<y<\pi^2$



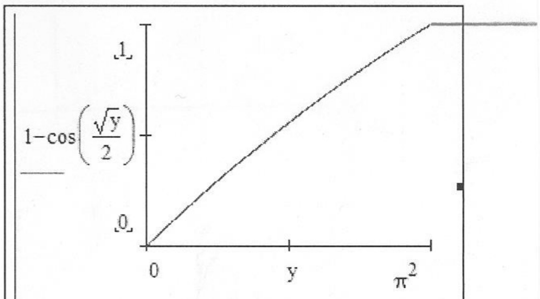
PDF of X



PDF of Y



CDF of X



CDF of Y

4. The surface area S of a cylinder with base and top radii as r and with height h is

$$S = 2\pi r^2 + 2\pi r h$$

If R and H are found to be random variables with known averages ,and coefficient of variations as

	Average Value (in m)	Coefficient of Variation, δ
Radius, r	4.2	0.15
Height, h	6.0	0.20

a) Find the first order average surface area and standard deviation for the cylinder.
b) Find the second order average surface area for the cylinder.

a) $$S \approx 2\pi\bar{r}^2 + 2\pi\bar{r}\bar{h} + (4\pi\bar{r} + 2\pi\bar{h})(r-\bar{r}) + (0 + 2\pi\bar{r})(h-\bar{h}) + \frac{1}{2}4\pi(r-\bar{r})^2 + 0 + 2\pi(h-\bar{h})(r-\bar{r})$$

$$E(S)_I = 2\pi(4.2)^2 + 2\pi \times 4.2 \times 6 = 269.17 \ m^2 \checkmark$$

$$V(S)_I = (4\pi \times 4.2 + 2\pi \times 6.0)^2 (4.2 \times 0.15)^2 + (2\pi \times 4.2)^2 (6 \times 0.2)^2 = 4251.94 \, (m^4)$$

$$\sigma_I = 65.2 \ m^2$$

b) $$E(S)_{II} = \underbrace{2\pi(4.2)^2 + 2\pi \times 4.2 \times 6}_{269.17} + 2\pi(4.2 \times 0.15)^2 = 271.66 \text{ MPa} \approx 272 \text{ MPa}$$

5. An engineer is to design a beam. Because of unceratintіes in the strength measurements of materials used, the unceratintіes of the maximum safe strength X is assumed to vary normally with mean 2.0 MPa and standard deviation 0.3 MPa . The maximum stresses Y due to actual load is also assumed to be normally distributed with mean 1.75 MPa and standard deviation 0.35 MPa.

a) Find the probability that the actual maximum stress will exceed the safe maximum (i.e. P (Y > X) if X and Y are assumed to be independent.

b) Find the probability that the actual maximum stress will exceed the safe maximum (i.e. P (Y > X) if X and Y are assumed to be correlated with the correlation coefficient, ρ_{XY} as 0.8.

c) Find the probability that the factor of safety will be less than 2 (i.e. P (X < 2Y)) if X and Y are assumed to be correlated with the correlation coefficient 0.8. (Note that factor of safety is the ratio of maximum strength to that of actual stress).

a) $P(X-Y<0)$? $\quad P\left(z<\frac{0-\mu_{X-Y}}{\sigma_{X-Y}}\right)$

$E(X-Y)=E(X)-E(Y)=2-1.75=0.25$ MPa

$V(X-Y)\overset{ind}{=}V(X)+V(Y)=0.3^2+0.35^2=0.2125\ (MPa)^2$

$\sigma_{X-Y}=\sqrt{0.2125}=0.461$ MPa

$P(z<(0-0.25)/0.461)=P(z<-0.54)=1-0.7054\approx 0.295$

b) $E(X-Y)=0.25$ MPa; $V(X-Y)=\sigma_X^2+\sigma_Y^2-2\rho_{XY}\sigma_X\sigma_Y$

$V(X-Y)=0.3^2+0.35^2-2\times 0.8\times 0.3\times 0.35=0.0445\ (MPa)^2$

$\sigma_{X-Y}=0.2109$ MPa

$P\left(z<\frac{0-0.25}{0.2109}\right)=P(z<-1.185)\approx 1-0.88\cong 0.12$

c) $P(X-2Y<0)\Rightarrow P\left(z<\frac{0-E(X-2Y)}{\sigma_{X-2Y}}\right)$

$E(X-2Y)=2-2\times 1.75=-1.5$ MPa.

$V(X-2Y)=0.3^2+4\times 0.35^2-2\times 0.8\times 2\times 0.3\times 0.35$

$=0.244\ (MPa)^2$

$\sigma_{X-2Y}=0.494$ MPa

$P\left(z<\frac{0-(-1.5)}{0.494}\right)=P(z<3.036)\simeq 0.999$

6. The moment, M (in kNm) at the fixed end of a cantilever due to two concentrated loads X and Y is given as

$$M = 1.2\,X + 2.4\,Y$$

Both X and Y are known to be normally distributed with means 2.0 kN and 1.5 kN, respectively. Both X and Y have the same coefficient of variation, δ as 0.15. Find the probability that the moment M will exceed 5.0 kNm if X and Y are correlated with $\rho_{XY} = 0.6$.

$P(M > 5)$

$P(1.2X + 2.4Y > 5);$

$E(1.2X+2.4Y) = 1.2\times 2 + 2.4\times 1.5 = 6.0$ kN-m

$$V(1.2X+2.4Y) = 1.2^2 V(X) + 2.4^2 V(Y) + 2\rho_{xy}\, 1.2\times 2.4\, \sigma_x \sigma_y$$

$$= 1.2^2 (2\times 0.15)^2 + 2.4^2 (1.5\times 0.15)^2$$

$$+ 2\times 0.6\times 1.2\times 2.4\times (2\times 0.15)\times(1.5\times 0.15)$$

$$= 0.6545\ (\text{kN-m})^2$$

$\sigma_M = 0.809$ kN-m.

$$P\left(Z > \frac{5-6}{0.809}\right) = P(Z > -1.236) \cong 0.89$$