Consider the following initial value problem (IVP):

$$\frac{dy}{dt} = t^2 - y \quad \text{with} \quad y(0) = 1$$

Solve this problem with

1. Euler
2. Heun's (1 step predictor-corrector method)
3. Runge Kutta (4)

methods for a step size of $h = 0.5$ within the interval [0, 1.5]. Calculate the true error in each step for each method if $y_{\text{true}}(t) = -e^{-t} + t^2 - 2t + 2$.

1. Euler's Method

Given an ordinary differential equation (ODE) in the form, $\frac{dy}{dx} = f(x, y)$, Euler formula predicts the solution according to the following formula:

$$y(x_{i+1}) = y(x_i) + h\ f(x_i, y_i)$$

1st Time Step:

$t_0 = 0$

$y_0 = 1$

$f(t_0, y_0) = t_0^2 - y_0 = 0^2 - 1 = -1$

$y_1 = y_0 + h \times f(t_0, y_0) = 1 + 0.5 \times -1 = 0.5$

$y_{1,t} = -e^{-t_1} + t_1^2 - 2t_1 + 2 = 0.643469$

$\varepsilon_{1,t} = y_{1,t} - y_1 = 0.143469$

2nd Time Step:

$t_1 = 0.5$

$y_1 = 0.5$

$f(t_1, y_1) = t_1^2 - y_1 = 0.5^2 - 0.5 = -0.25$

$y_2 = y_1 + h \times f(t_1, y_1) = 0.5 + 0.5 \times -0.25 = 0.375$

$y_{2,t} = -e^{-t_2} + t_2^2 - 2t_2 + 2 = 0.632121$

$\varepsilon_{2,t} = y_{2,t} - y_2 = 0.257121$

3rd Time Step:

$t_2 = 1$

$y_2 = 0.375$

$f(t_2, y_2) = t_2^2 - y_2 = 1^2 - 0.375 = 0.625$

$y_3 = y_2 + h \times f(t_2, y_2) = 0.375 + 0.5 \times 0.625 = 0.6875$

$y_{3,t} = -e^{-t_3} + t_3^2 - 2t_3 + 2 = 1.026870$

$\varepsilon_{3,t} = y_{3,t} - y_3 = 0.339370$

A table summarizing the iterations is given below:

i	t_i	y_i	$f(t_i,y_i)$	y_t	ε_t
0	0.00	1.000000	-1.000000	1.000000	0.000000
1	0.50	0.500000	-0.250000	0.643469	0.143469
2	1.00	0.375000	0.625000	0.632121	0.257121
3	1.50	0.687500		1.026870	0.339370

2. Heun's Method

Given an ordinary differential equation (ODE) in the form, $\frac{dy}{dx} = f(x, y)$, Euler formula predicts the solution according to the following formula:

$$y(x_{i+1}) = y(x_i) + h\left(\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^*)}{2}\right)$$

where $y_{i+1}^* = y(x_i) + h\, f(x_i, y_i)$.

1st Time Step:

$t_0 = 0$

$y_0 = 1$

$f(t_0, y_0) = t_0^2 - y_0 = 0^2 - 1 = -1$

$y_1^* = y_0 + h \times f(t_0, y_0) = 1 + 0.5 \times -1 = 0.5$

$f(t_1, y_1^*) = t_1^2 - y_1^* = 0.5^2 - 0.5 = -0.25$

$$y_1 = y_0 + h \times \left(\frac{f(t_0, y_0) + f(t_1, y_1^*)}{2}\right) = 1 + 0.5 \times \left(\frac{-1 - 0.25)}{2}\right) = 0.6875$$

$y_{1,t} = -e^{-t_1} + t_1^2 - 2t_1 + 2 = 0.643469$

$\varepsilon_{1,t} = y_{1,t} - y_1 = -0.044031$

2nd Time Step:

$t_1 = 0.5$

$y_1 = 0.6875$

$f(t_1, y_1) = t_1^2 - y_1 = 0.5^2 - 0.6875 = -0.4375$

$y_2^* = y_1 + h \times f(t_1, y_1) = 0.6875 + 0.5 \times -0.4375 = 0.468750$

$f(t_2, y_2^*) = t_2^2 - y_2^* = 1^2 - 0.468750 = 0.53125$

$$y_2 = y_1 + h \times \left(\frac{f(t_1, y_1) + f(t_2, y_2^*)}{2} \right) = 0.6875 + 0.5 \times \left(\frac{-0.4375 + 0.53125}{2} \right) = 0.710938$$

$y_{2,t} = -e^{-t_2} + t_2^2 - 2t_2 + 2 = 0.632121$

$\varepsilon_{2,t} = y_{2,t} - y_2 = -0.078817$

3rd Time Step:

$t_2 = 1.0$

$y_2 = 0.710938$

$f(t_2, y_2) = t_2^2 - y_2 = 1^2 - 0.710938 = 0.289063$

$y_3^* = y_2 + h \times f(t_2, y_2) = 0.710938 + 0.5 \times 0.289063 = 0.855469$

$f(t_3, y_3^*) = t_3^2 - y_3^* = 1.5^2 - 0.855469 = 1.394531$

$$y_3 = y_2 + h \times \left(\frac{f(t_2, y_2) + f(t_3, y_3^*)}{2} \right) = 0.710938 + 0.5 \times \left(\frac{0.289063 + 1.394531}{2} \right) = 1.131836$$

$y_{3,t} = -e^{-t_3} + t_3^2 - 2t_3 + 2 = 1.026870$

$\varepsilon_{3,t} = y_{3,t} - y_3 = -0.104966$

A table summarizing the above computations is given below:

i	**t_i**	**y_i**	**$f(t_i,y_i)$**	**$y_{i+1,0}$**	**$f(t_{i+1},y_{i+1,0})$**	**y_t**	**ε_t**
0	0.00	1.000000	-1.000000	0.500000	-0.250000	1.000000	0.000000
1	0.50	0.687500	-0.437500	0.468750	0.531250	0.643469	-0.044031
2	1.00	0.710938	0.289063	0.855469	1.394531	0.632121	-0.078817
3	1.50	1.131836				1.026870	-0.104966

3. 4th Order Runge-Kutta Method

Given an ordinary differential equation (ODE) in the form, $\frac{dy}{dx} = f(x, y)$, 4th order Runge-Kutta formula predicts the solution according to the following formula:

$$y(x_{i+1}) = y(x_i) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1)$$
$$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2)$$
$$k_4 = f(x_i + h, y_i + hk_3)$$

1st Time Step:

$t_0 = 0$

$y_0 = 1$

$k_1 = f(t_0, y_0) = t_0^2 - y_0 = 0^2 - 1 = -1$

$$k_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = \left(t_0 + \frac{h}{2}\right)^2 - \left(y_0 + \frac{h}{2}k_1\right) = \left(0 + \frac{0.5}{2}\right)^2 - \left(1 + \frac{0.5}{2} \times -1\right)$$
$$= -0.6875$$

$$k_3 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2) = \left(t_0 + \frac{h}{2}\right)^2 - \left(y_0 + \frac{h}{2}k_2\right) = \left(0 + \frac{0.5}{2}\right)^2 - \left(1 + \frac{0.5}{2} \times -0.6875\right)$$
$$= -0.765625$$

$$k_4 = f(t_0 + h, y_0 + hk_3) = (t_0 + h)^2 - (y_0 + hk_3) = (0 + 0.5)^2 - (1 + 0.5 \times -0.765625)$$
$$= -0.367188$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{0.5}{6}(-1 + 2 \times -0.6875 + 2 \times -0.765625 - 0.367188)$$
$$= 0.643880$$

$y_{1,t} = -e^{-t_1} + t_1^2 - 2t_1 + 2 = 0.643469$

$\varepsilon_{1,t} = y_{1,t} - y_1 = -0.000411$

2nd Time Step:

$$t_1 = 0.5$$

$$y_1 = 0.643880$$

$$\begin{aligned}
k_1 &= f(t_1, y_1) \\
&= t_1^2 - y_1 \\
&= 0.5^2 - 0.643880 \\
&= -0.393880
\end{aligned}$$

$$\begin{aligned}
k_2 &= f(t_1 + \frac{h}{2}, y_1 + \frac{h}{2} k_1) \\
&= \left(t_1 + \frac{h}{2} \right)^2 - \left(y_1 + \frac{h}{2} k_1 \right) \\
&= \left(0.5 + \frac{0.5}{2} \right)^2 - \left(0.643880 + \frac{0.5}{2} \times -0.393880 \right) \\
&= 0.017090
\end{aligned}$$

$$\begin{aligned}
k_3 &= f(t_1 + \frac{h}{2}, y_1 + \frac{h}{2} k_2) \\
&= \left(t_1 + \frac{h}{2} \right)^2 - \left(y_1 + \frac{h}{2} k_2 \right) \\
&= \left(0.5 + \frac{0.5}{2} \right)^2 - \left(0.643880 + \frac{0.5}{2} \times 0.017090 \right) \\
&= -0.085653
\end{aligned}$$

$$\begin{aligned}
k_4 &= f(t_1 + h, y_1 + hk_3) \\
&= (t_1 + h)^2 - (y_1 + hk_3) \\
&= (0.5 + 0.5)^2 - (0.643880 + 0.5 \times -0.085653) \\
&= 0.398946
\end{aligned}$$

$$\begin{aligned}
y_2 &= y_1 + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\
&= 0.643880 + \frac{0.5}{6} (-0.393880 + 2 \times 0.017090 + 2 \times -0.085653 + 0.398946) \\
&= 0.632875
\end{aligned}$$

$$\begin{aligned}
y_{2,t} &= -e^{-t_2} + t_2^2 - 2t_2 + 2 \\
&= 0.632121
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{2,t} &= y_{2,t} - y_2 \\
&= -0.000755
\end{aligned}$$

3rd Time Step:

$$t_2 = 1.0$$

$$y_2 = 0.632875$$

$$k_1 = f(t_2, y_2)$$
$$= t_2^2 - y_2$$
$$= 1^2 - 0.632875$$
$$= 0.367125$$

$$k_2 = f(t_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_1)$$
$$= \left(t_2 + \frac{h}{2}\right)^2 - \left(y_2 + \frac{h}{2}k_1\right)$$
$$= \left(1 + \frac{0.5}{2}\right)^2 - \left(0.632875 + \frac{0.5}{2} \times 0.367125\right)$$
$$= 0.837844$$

$$k_3 = f(t_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_2)$$
$$= \left(t_2 + \frac{h}{2}\right)^2 - \left(y_2 + \frac{h}{2}k_2\right) = \left(1 + \frac{0.5}{2}\right)^2 - \left(0.632875 + \frac{0.5}{2} \times 0.837844\right)$$
$$= 0.720164$$

$$k_4 = f(t_2 + h, y_2 + hk_3)$$
$$= (t_2 + h)^2 - (y_2 + hk_3)$$
$$= (1 + 0.5)^2 - (0.632875 + 0.5 \times 0.720164)$$
$$= 1.257043$$

$$y_3 = y_2 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 0.632875 + \frac{0.5}{6}(0.367125 + 2 \times 0.837844 + 2 \times 0.720164 + 1.257043)$$
$$= 1.027890$$

$$y_{3,t} = -e^{-t_3} + t_3^2 - 2t_3 + 2$$
$$= 1.026870$$

$$\varepsilon_{3,t} = y_{3,t} - y_3$$
$$= -0.001021$$

A table summarizing the above computations is given below:

i	t_i	y_i	k_1	k_2	k_3	k_4	y_t	ε_t
0	0.00	1.000000	-1.000000	-0.687500	-0.765625	-0.367188	1.000000	0.000000
1	0.50	0.643880	-0.393880	0.017090	-0.085653	0.398946	0.643469	-0.000411
2	1.00	0.632875	0.367125	0.837844	0.720164	1.257043	0.632121	-0.000755
3	1.50	1.027890					1.026870	-0.001021

The following graph shows the results of each method besides the true function values.