Consider the simple truss system given in Figure 1. Find the internal forces F_{AB}, F_{AC}, F_{BC} and reactions R_1, R_2, and R_3 respectively at B and C by Gauss elimination and LU decomposition methods for the following load cases:

a. $P_1 = 30\ kN$, $P_2 = 0$

b. $P_1 = 0$, $P_2 = 40\ kN$

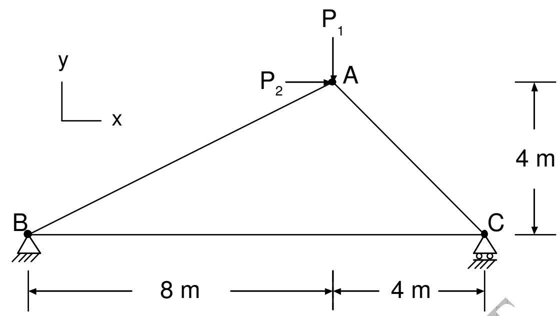
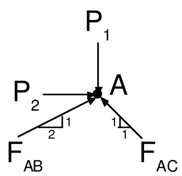
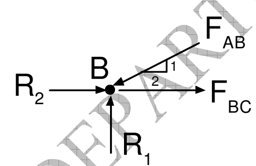


Figure 1 Simple truss sytem

Hint: It can be shown that the equilibrium equations for this truss constitute a 6 by 6 linear system:



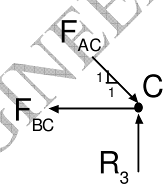


$$\sum F_x = 0 \Rightarrow \frac{2}{\sqrt{5}} F_{AB} - \frac{1}{\sqrt{2}} F_{AC} + P_2 = 0 \ldots (1)$$

$$\sum F_y = 0 \Rightarrow \frac{1}{\sqrt{5}} F_{AB} + \frac{1}{\sqrt{2}} F_{AC} - P_1 = 0 \ldots (2)$$

$$\sum F_x = 0 \Rightarrow F_{BC} - \frac{2}{\sqrt{5}} F_{AB} + R_2 = 0 \ldots (3)$$

$$\sum F_y = 0 \Rightarrow -\frac{1}{\sqrt{5}} F_{AB} + R_1 = 0 \ldots\ldots\ldots (4)$$



$$\sum F_x = 0 \Rightarrow -F_{BC} + \frac{1}{\sqrt{2}} F_{AC} = 0 \ldots (5)$$

$$\sum F_y = 0 \Rightarrow -\frac{1}{\sqrt{2}} F_{AC} + R_3 = 0 \ldots (6)$$

Letting $x_1 = F_{AB}$, $x_2 = F_{AC}$, $x_3 = F_{BC}$, $x_4 = R_1$, $x_5 = R_2$ and $x_6 = R_3$, the equilibrium equations at nodes A, B and C can be written in the following matrix form:

$$\underbrace{\begin{bmatrix} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 \\ 0.447214 & 0.707107 & 0 & 0 & 0 & 0 \\ -0.894427 & 0 & 1 & 0 & 1 & 0 \\ -0.447214 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.707107 & -1 & 0 & 0 & 0 \\ 0 & -0.707107 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}}_{\mathbf{x}} = \underbrace{\begin{Bmatrix} -P_2 \\ P_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\mathbf{b}}$$

<u>GAUSS ELIMINATION</u>

PART A:

(1)

$$\left[\begin{array}{rrrrrrr} \mathbf{0.894427} & -0.707107 & 0 & 0 & 0 & 0 & 0 \\ 0.447214 & 0.707107 & 0 & 0 & 0 & 0 & 30 \\ -0.894427 & 0 & 1 & 0 & 1 & 0 & 0 \\ -0.447214 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.707107 & -1 & 0 & 0 & 0 & 0 \\ 0 & -0.707107 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ row2^{(2)} = row2^{(1)} - 0.5\times row1^{(1)} \\ row3^{(2)} = row3^{(1)} - (-1)\times row1^{(1)} \\ row4^{(2)} = row4^{(1)} - (-0.5)\times row1^{(1)} \\ row5^{(2)} = row5^{(1)} - 0\times row1^{(1)} \\ row6^{(2)} = row6^{(1)} - 0\times row1^{(1)} \end{array}$$

(2)

$$\left[\begin{array}{rrrrrrr} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1.060660} & 0 & 0 & 0 & 0 & 30 \\ 0 & -0.707107 & 1 & 0 & 1 & 0 & 0 \\ 0 & -0.353553 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.707107 & -1 & 0 & 0 & 0 & 0 \\ 0 & -0.707107 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ row3^{(3)} = row3^{(2)} - (-0.666667)\times row2^{(2)} \\ row4^{(3)} = row4^{(2)} - (-0.333333)\times row2^{(2)} \\ row5^{(3)} = row5^{(2)} - 0.666667\times row2^{(2)} \\ row6^{(3)} = row6^{(2)} - (-0.666667)\times row2^{(2)} \end{array}$$

(3)

$$\left[\begin{array}{rrrrrrr} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.060660 & 0 & 0 & 0 & 0 & 30 \\ 0 & 0 & \mathbf{1} & 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & 0 & 0 & 0 & -20 \\ 0 & 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right] \begin{array}{l} \\ \\ \\ row4^{(4)} = row4^{(3)} - 0\times row3^{(3)} \\ row5^{(4)} = row5^{(3)} - (-1)\times row3^{(3)} \\ row6^{(4)} = row6^{(3)} - 0\times row3^{(3)} \end{array}$$

(4)

$$\left[\begin{array}{llrrrrr} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.060660 & 0 & 0 & 0 & 0 & 30 \\ 0 & 0 & 1 & 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right]$$

Now that the coefficient matrix A has been reduced to an upper triangular form, the system can be solved by backward substitution.

$$\begin{aligned} & x_6 = 20 \\ & x_5 = 0 \\ & x_4 = 10 \\ & x_3 + x_5 = 20 \Rightarrow x_3 = 20 \\ & 1.060660 x_2 = 30 \Rightarrow x_2 = 28.28 \\ & 0.894427 x_1 - 0.707107 x_2 \Rightarrow x_1 = 22.36 \end{aligned} \quad \Longrightarrow \quad \begin{Bmatrix} F_{AB} \\ F_{AC} \\ F_{BC} \\ R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} 22.36 \\ 28.28 \\ 20 \\ 10 \\ 0 \\ 20 \end{Bmatrix}$$

PART B:

$$\begin{bmatrix} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 & -40 \\ 0 & 1.060660 & 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & 1 & 0 & -26.666667 \\ 0 & 0 & 0 & 1 & 0 & 0 & -13.333333 \\ 0 & 0 & 0 & 0 & 1 & 0 & -40 \\ 0 & 0 & 0 & 0 & 0 & 1 & 13.333333 \end{bmatrix} \Rightarrow \begin{Bmatrix} F_{AB} \\ F_{AC} \\ F_{BC} \\ R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -29.81 \\ 18.86 \\ 13.33 \\ -13.33 \\ -40 \\ 13.33 \end{Bmatrix}$$

<u>LU DECOMPOSITION</u>

Let us first decompose the coefficient matrix A into a lower and upper triangular matrix:

$$A = LU$$

$$\begin{bmatrix} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 \\ 0.447214 & 0.707107 & 0 & 0 & 0 & 0 \\ -0.894427 & 0 & 1 & 0 & 1 & 0 \\ -0.447214 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.707107 & -1 & 0 & 0 & 0 \\ 0 & -0.707107 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 & 0 \\ l_{41} & l_{42} & l_{43} & 1 & 0 & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & 1 & 0 \\ l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} & u_{26} \\ 0 & 0 & u_{33} & u_{34} & u_{35} & u_{36} \\ 0 & 0 & 0 & u_{44} & u_{45} & u_{46} \\ 0 & 0 & 0 & 0 & u_{55} & u_{56} \\ 0 & 0 & 0 & 0 & 0 & u_{66} \end{bmatrix}$$

$$1 \times u_{11} = 0.894427 \Rightarrow u_{11} = 0.894427$$
$$1 \times u_{12} + 0 \times u_{22} = -0.707107 \Rightarrow u_{12} = -0.707107$$
$$1 \times u_{13} + 0 \times u_{23} + 0 \times u_{33} = 0 \Rightarrow u_{13} = 0$$
$$\vdots$$
$$l_{21} \times u_{11} + 1 \times 0 = 0.447214 \Rightarrow l_{21} = 0.5$$
$$l_{21} \times u_{12} + 1 \times u_{22} = 0.707107 \Rightarrow u_{22} = 1.060660$$
$$l_{21} \times u_{13} + 1 \times u_{23} = 0 \Rightarrow u_{23} = 0$$
$$\vdots$$

If the computations are carried out in a similar fashion, the following L and U matrices are obtained:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ -1 & -0.666667 & 1 & 0 & 0 & 0 \\ -0.5 & -0.333333 & 0 & 1 & 0 & 0 \\ 0 & 0.666667 & -1 & 0 & 1 & 0 \\ 0 & -0.666667 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 \\ 0 & 1.060660 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, the LU decomposition can be used to solve the given system of linear equations as follows:

$Ax = b \Rightarrow LUx = b \Rightarrow Ux = y,\ Ly = b$

PART A:

Solve the linear system $Ly = b$ ($b = \{0 \quad 30 \quad 0 \quad 0 \quad 0 \quad 0\}^T$) by forward substitution:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ -1 & -0.666667 & 1 & 0 & 0 & 0 \\ -0.5 & -0.333333 & 0 & 1 & 0 & 0 \\ 0 & 0.666667 & -1 & 0 & 1 & 0 \\ 0 & -0.666667 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \\ 20 \\ 10 \\ 0 \\ 20 \end{Bmatrix}$$

Now, solve the linear system $Ux = y$ by backward substitution:

$$\begin{bmatrix} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 \\ 0 & 1.060660 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \\ 20 \\ 10 \\ 0 \\ 20 \end{Bmatrix} \Rightarrow \begin{Bmatrix} F_{AB} \\ F_{AC} \\ F_{BC} \\ R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} 22.36 \\ 28.28 \\ 20 \\ 10 \\ 0 \\ 20 \end{Bmatrix}$$

PART B:

Solve the linear system $Ly = b$ ($b = \{-40 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^T$) by forward substitution:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ -1 & -0.666667 & 1 & 0 & 0 & 0 \\ -0.5 & -0.333333 & 0 & 1 & 0 & 0 \\ 0 & 0.666667 & -1 & 0 & 1 & 0 \\ 0 & -0.666667 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} -40 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} -40 \\ 20 \\ -26.666667 \\ -13.333333 \\ -40 \\ 13.333333 \end{Bmatrix}$$

Now, solve the linear system $Ux = y$ by backward substitution:

$$\begin{bmatrix} 0.894427 & -0.707107 & 0 & 0 & 0 & 0 \\ 0 & 1.060660 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} -40 \\ 20 \\ -26.666667 \\ -13.333333 \\ -40 \\ 13.333333 \end{Bmatrix} \Rightarrow \begin{Bmatrix} F_{AB} \\ F_{AC} \\ F_{BC} \\ R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} -29.81 \\ 18.86 \\ 13.33 \\ -13.33 \\ -40 \\ 13.33 \end{Bmatrix}$$