RULES

1. This is the **version 1.0**. In case there are any corrections for the solutions of Exercise 1, we will post an updated version on our website. You can follow the changes in the exercises by the **Version History** section below.

Version History

v1.0 Solutions of Exercise 1 are released.

1.

a) Rearranging the equations:

$$5\cdot c-7\cdot b+0\cdot a=50$$
$$7\cdot c+4\cdot b+0\cdot a=-30$$
$$-7\cdot c+3\cdot b-4\cdot a=40$$

Now it is easier to construct matrix system:

$$Ax=b \;\rightarrow\; \begin{bmatrix} 5 & -7 & 0 \\ 7 & 4 & 0 \\ -7 & 3 & -4 \end{bmatrix}\begin{bmatrix} c \\ b \\ a \end{bmatrix}=\begin{bmatrix} 50 \\ -30 \\ 40 \end{bmatrix}$$

b) To find the consistency of the system, the rank of A and the rank of A|b should be found. Therefore, the row reduced form of A|b must be found.

$$\left[\begin{array}{ccc|c} 5 & -7 & 0 & 50 \\ 7 & 4 & 0 & -30 \\ -7 & 3 & -4 & 40 \end{array}\right] \xrightarrow{\text{by elementary row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -0.1449 \\ 0 & 1 & 0 & -7.2464 \\ 0 & 0 & 1 & -15.1812 \end{array}\right]$$

Ranks are determined as rank(A)=3 where rank(A|b)=3. Note that there are 3 unknowns.

Since rank(A)=rank(A|b)=3, this system is consistent and has a unique solution.

2)

$$\underline{A}=\begin{bmatrix} 8 & -1 & 3 & 0 \\ 3 & 24 & 5 & -2 \\ 1 & -8 & -16 & -3 \\ 3 & -2 & -1 & 10 \end{bmatrix}=\begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ & u_{22} & u_{23} & u_{24} \\ & & u_{33} & u_{34} \\ & & & u_{44} \end{bmatrix}$$

$u_{11}=A_{11}=8.000000$
$u_{12}=A_{12}=-1.000000$
$u_{13}=A_{13}=3.000000$
$u_{14}=A_{14}=0.000000$

$l_{21}\cdot u_{11}=3 \qquad l_{21}=\dfrac{3}{8}=0.375000$
$l_{21}\cdot u_{12}+u_{22}=24 \qquad u_{22}=24-0.375000\cdot(-1)=24.375000$
$l_{21}\cdot u_{13}+u_{23}=5 \qquad u_{23}=5-0.375000\cdot(3)=3.875000$
$l_{21}\cdot u_{14}+u_{24}=\text{-2} \qquad u_{24}=-2-0.375000\cdot(0)=-2.000000$

$l_{31}\cdot u_{11}=1 \qquad l_{31}=\dfrac{1}{8}=0.125000$
$l_{31}\cdot u_{12}+l_{32}\cdot u_{22}=-8 \qquad l_{32}=\dfrac{-8-0.125000\cdot(-1)}{24.375000}=-0.323077$
$l_{31}\cdot u_{13}+l_{32}\cdot u_{23}+u_{33}=-16 \qquad u_{33}=-16-0.125000\cdot 3-(-0.323077)\cdot 3.875000=-15.123077$
$l_{31}\cdot u_{14}+l_{32}\cdot u_{24}+u_{34}=-3$
$u_{34}=-3-0.125000\cdot 0.000000-(-0.323077)\cdot(-2.000000)=-3.646154$

$l_{41}\cdot u_{11}=3 \qquad l_{41}=\dfrac{3}{8}=0.375000$
$l_{41}\cdot u_{12}+l_{42}\cdot u_{22}=-2 \qquad l_{42}=\dfrac{-2-0.375000\cdot(-1)}{24.375000}=-0.066667$
$l_{41}\cdot u_{13}+l_{42}\cdot u_{23}+l_{43}\cdot u_{33}=-1 \qquad l_{43}=\dfrac{-1-0.375000\cdot 3-(-0.066667)\cdot 3.875000}{-15.123077}=0.123431$
$l_{41}\cdot u_{14}+l_{42}\cdot u_{24}+l_{43}\cdot u_{34}+u_{44}=10$
$u_{44}=10-0.375000\cdot 0-(-0.066667)\cdot(-2.000000)-0.123431\cdot(-3.646154)=10.316718$

$$\underline{U}=\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ & u_{22} & u_{23} & u_{24} \\ & & u_{33} & u_{34} \\ & & & u_{44} \end{bmatrix}=\begin{bmatrix} 8.000000 & -1.000000 & 3.000000 & 0.000000 \\ 0.000000 & 24.375000 & 3.875000 & -2.000000 \\ 0.000000 & 0.000000 & -15.123077 & -3.646154 \\ 0.000000 & 0.000000 & 0.000000 & 10.316718 \end{bmatrix}$$

$$\underline{L}=\begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}=\begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.375000 & 1.000000 & 0.000000 & 0.000000 \\ 0.125000 & -0.323077 & 1.000000 & 0.000000 \\ 0.375000 & -0.066667 & 0.123432 & 1.000000 \end{bmatrix}$$

Check the results with MATLAB

```
>> [L U]=lu(A)
L =

   1.000000000000000                   0                   0                   0
   0.375000000000000   1.000000000000000                   0                   0
   0.125000000000000  -0.323076923076923   1.000000000000000                   0
   0.375000000000000  -0.066666666666667   0.123431671753137   1.000000000000000
U =

   8.000000000000000  -1.000000000000000   3.000000000000000                   0
                   0  24.375000000000000   3.875000000000000  -2.000000000000000
                   0                   0 -15.123076923076923  -3.646153846153846
                   0                   0                   0  10.316717531366566
```

To Solve

$\underline{A}\underline{x}=\underline{b}$

Where:

$$\underline{A}=\begin{bmatrix} 8 & -1 & 3 & 0 \\ 3 & 24 & 5 & -2 \\ 1 & -8 & -16 & -3 \\ 3 & -2 & -1 & 10 \end{bmatrix} \quad \underline{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \underline{\text{b}}=\begin{bmatrix} 25 \\ 87 \\ -108 \\ -9 \end{bmatrix}$$

$\underline{L}\underline{d} = \underline{b}$ solve the system for $\underline{d}$

$$\begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.375000 & 1.000000 & 0.000000 & 0.000000 \\ 0.125000 & -0.323077 & 1.000000 & 0.000000 \\ 0.375000 & -0.066667 & 0.123432 & 1.000000 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 25 \\ 87 \\ -108 \\ -9 \end{bmatrix}$$

$d_1 = 25.000000$

$d_2 = 87 - 0.375000 \cdot d_1 = 87 - 0.375000 \cdot 25.000000 = 77.625000$

$d_3 = -108 - 0.125000 \cdot d_1 - (-0.323077 \cdot d_2)$
$d_3 = -108 - 0.125000 \cdot 25.000000 + 0.323077 \cdot 77.625000 = -86.046148$

$d_4 = -9 - 0.375000 \cdot d_1 - (-0.066667 \cdot d_2) - 0.123432 \cdot d_3$
$d_4 = -9 - 0.375000 \cdot 25.000000 + 0.066667 \cdot 77.625000 - 0.123432 \cdot (-86.046148) = -2.579126$

$\underline{U}\underline{x} = \underline{d}$ solve the system for $\underline{x}$

$$\begin{bmatrix} 8.000000 & -1.000000 & 3.000000 & 0.000000 \\ 0.000000 & 24.375000 & 3.875000 & -2.000000 \\ 0.000000 & 0.000000 & -15.123077 & -3.646154 \\ 0.000000 & 0.000000 & 0.000000 & 10.316718 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 25.000000 \\ 77.625000 \\ -86.046148 \\ -2.579126 \end{bmatrix}$$

$$x_4 = \frac{-2.579126}{10.316718} = -0.250000$$

$$x_3 = \frac{-86.046148 - (-3.646154 \cdot x_4)}{-15.123077} = \frac{-86.046148 - (-3.646154 \cdot (-0.250000))}{-15.123077} = 5.750000$$

$$x_2 = \frac{77.625000 - (-2.000000 \cdot x_4) - 3.875000 \cdot x_3}{24.375000}$$

$$x_2 = \frac{77.625000 - (-2.000000 \cdot (-0.250000)) - 3.875000 \cdot 5.750000}{24.375000} = 2.250000$$

$$x_1 = \frac{25.000000 - 0.000000 \cdot x_4 - 3.000000 \cdot x_3 - (-1.000000 \cdot x_2)}{8.000000}$$

$$x_1 = \frac{25.000000 - 0.000000 \cdot (-0.250000) - 3.000000 \cdot 5.750000 - (-1.000000 \cdot 2.250000)}{8.000000} = 1.250000$$

$$\underline{x} = \begin{bmatrix} 1.250000 \\ 2.250000 \\ 5.750000 \\ -0.250000 \end{bmatrix}$$

3. Rearranging the equations, we can get:

$$x_1 = \frac{6 - x_3 + x_2}{3} \qquad x_2 = \frac{-4 + x_1 + 2x_3}{3} \qquad x_3 = \frac{6 - 2x_2}{3}$$

a) By Gauss-Jacobi method, the results can be obtained as:

Iteration #	x_1	x_1	x_1	Error
0	1.4000	0.4000	1.9000	-
1	1.5000	0.4000	2.0667	0.1667
2	1.4444	0.5444	2.0667	0.1444
3	1.4926	0.5259	1.9704	0.0963
4	1.5185	0.4778	1.9827	0.0481
5	1.4984	0.4947	2.0148	0.0321
6	1.4933	0.5093	2.0036	0.0147
7	1.5019	0.5001	1.9938	0.0098

b) By Gauss-Seidel method, the results can be obtained as:

Iteration #	x_1	x_1	x_1	Error
0	1.4000	0.4000	1.9000	-
1	1.5000	0.4333	2.0444	0.1444
2	1.4630	0.5173	1.9885	0.0840
3	1.5096	0.4955	2.0030	0.0466
4	1.4975	0.5012	1.9992	0.0121
5	1.5006	0.4997	2.0002	0.0031

c) It is obvious that Gauss-Seidel is faster than Gauss-Jacobi method because of the algorithm.