METU
Department of Mathematics

	Introduction to Differential Equations Final	

Code : *Math 219* Acad. Year : *2014-2015* Semester : *Fall* Coordinator: *Özgür Kişisel* Date : *January.08.2014* Time : *17:00* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 7 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	SHOW YOUR WORK	

Question 1 (25pts) Find all solutions of the system of differential equations

$$
\begin{aligned}
x_1' &= x_1 + x_2 + 2x_3 \\
x_2' &= 2x_2 + 2x_3 \\
x_3' &= -x_1 + x_2 + 3x_3
\end{aligned}
$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2 \\ 2-\lambda & 2 \end{vmatrix}$$

$$= (1-\lambda)\big((2-\lambda)(3-\lambda)-2\big) - \underbrace{\big(2 - 2(2-\lambda)\big)}_{2\lambda-2}$$

$$= (1-\lambda)\left[\lambda^2 - 5\lambda + 4 + 2\right] = (1-\lambda)(\lambda-2)(\lambda-3)$$

3 real, distinct eigenvalues

$\lambda_1 = 1$ $\quad (A-I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = k\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$k \neq 0$

$\lambda_2 = 2$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} \cancel{x_1} \\ \cancel{x_2} \\ \cancel{x_3} \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$k \neq 0$

$\lambda_3 = 3$

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = k\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$k \neq 0$

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{t}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

Question 2 (12pts) Find the power series solution of the following initial value problem centered at $x_0 = 0$: $\quad y'' - 2xy' + 10y = 0, \quad y(0) = 0,\ y'(0) = 1.$

$x_0 = 0$ is an ordinary point.

Set $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 10 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 10 a_n x^n = 0$$

$a_0 = y(0) = 0, \quad a_1 = y'(0) = 1.$

$x^0: \quad 2a_2 + 10a_0 = 0 \Rightarrow a_2 = 0.$

$x^n,\ n \geq 1: \quad (n+2)(n+1)a_{n+2} - (2n-10)a_n = 0 \Rightarrow a_{n+2} = \frac{2(n-5)}{(n+2)(n+1)} a_n$

$a_0 = a_2 = a_4 = a_6 = \cdots = 0$

$a_3 = \frac{2\cdot(-4)}{3\cdot 2} a_1 = \frac{-4}{3}, \quad a_5 = \frac{2\cdot(-2)}{5\cdot 4} a_3 = \frac{4}{15}, \quad a_7 = \frac{2\cdot(0)}{7\cdot 6} a_5 = 0$

$a_7 = a_9 = a_{11} = \cdots = 0$

$$\Rightarrow \boxed{y = x - \frac{4x^3}{3} + \frac{4x^5}{15}}$$

Question 3 (13pts) Solve the differential equation $y'' + 2y' + y = e^{-t}\ln t$ for $t > 0$, using variation of parameters.

$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0, \quad r_1 = r_2 = -1$ (repeated, real).

$\Rightarrow y_h = c_1 y_1 + c_2 y_2 = c_1 e^{-t} + c_2 t e^{-t}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = \begin{vmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{vmatrix} = e^{-2t}.$$

$$\begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-t}\ln t \end{bmatrix}$$

$$v_1' = \frac{\begin{vmatrix} 0 & te^{-t} \\ e^{-t}\ln t & e^{-t} - te^{-t} \end{vmatrix}}{W(y_1,y_2)} = \frac{-t\cancel{e^{-2t}}\ln t}{\cancel{e^{-2t}}} = -t\ln t$$

$v_1 = \int -t\ln t\, dt = -\frac{t^2}{2}\ln t - \int -\frac{t^{\cancel{2}}}{2}\frac{1}{\cancel{t}}\, dt = -\frac{t^2}{2}\ln t + \frac{t^2}{4}$

u: $du = \frac{1}{t}dt$

$-t\,dt = dw \Rightarrow w = -\frac{t^2}{2}$

$$v_2' = \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & e^{-t}\ln t \end{vmatrix}}{W(y_1,y_2)} = \frac{\cancel{e^{-2t}}\ln t}{\cancel{e^{-2t}}}$$

$v_2 = \int \ln t\, dt = t\ln t - t$

$$\boxed{y = c_1 e^{-t} + c_2 te^{-t} + e^{-t}\left(-\frac{t^2}{2}\ln t + \frac{t^2}{4}\right) + te^{-t}\left(t\ln t - t\right)}$$

$c_1, c_2 \in \mathbb{R}$

Question 4 (25pts) Consider the partial differential equation $u_t = u_{xx} + 2u_x$ together with the boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$.

(a) Use separation of variables in order to obtain an ODE in the variable t and a 2 point boundary value problem in the variable x.

$$u(x,t) = X(x)T(t)$$

$$XT' = X''T + 2X'T$$

$$\frac{T'}{T} = \frac{X''+2X'}{X} = -\lambda \quad \to \boxed{T' + \lambda T = 0}$$

$$X(0)T(t) = X(\pi)T(t) = 0 \quad \to T(t) = 0 \text{ for all } t \text{ (trivial soln)}, \quad X(0) = X(\pi) = 0$$

$$\to \boxed{\begin{array}{l} X'' + 2X' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{array}}$$

(b) Show that $X_n(x) = e^{-x}\sin(kx)$ is a solution of the boundary value problem found in part (a) for an appropriate value of the eigenvalue, for each $k = 1, 2, 3, \ldots$

$$X_k(0) = e^{-0}\sin(0) = 0$$

$$X_k(\pi) = e^{-\pi}\sin(k\pi) = 0$$

$$X_k' = -e^{-x}\sin(kx) + ke^{-x}\cos(kx)$$

$$X_k'' = e^{-x}\sin(kx) - ke^{-x}\cos(kx) - ke^{-x}\cos(kx) - k^2e^{-x}\sin(kx)$$

$$X_k'' + 2X_k' + \lambda X_k = e^{-x}\sin(kx)(1 - k^2 - 2 + \lambda) + e^{-x}\cos(kx)\underbrace{(\cancel{-2k+2k})}_{0}$$

$$= 0 \iff \boxed{\lambda = k^2 + 1}$$

(c) Find the solution $u(x,t)$ to the problem above that satisfies $u(x,0) = 2e^{-x}$ for all $0 < x < \pi$.

$$T' + \lambda T = 0 \Rightarrow T = ce^{-\lambda t}$$

Taking $\lambda = k^2+1$, we get $T_n(t) = e^{-(1+n^2)t}$

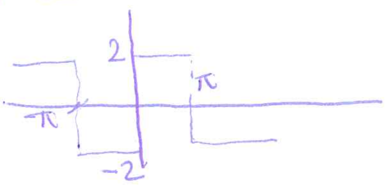
$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} c_n e^{-x}\sin(nx)e^{-(1+n^2)t}$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n e^{-x}\sin(nx) = 2e^{-x}$$

$$\Rightarrow \sum_{n=1}^{\infty} c_n \sin(nx) = 2$$

Need an odd extension of $f(x) = 2$ with period 2π



$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 2\sin(nx)\,dx$$

$$= \frac{-4}{n\pi}\cos(nx)\Big|_0^{\pi} = \frac{-4}{n\pi}(\cos(n\pi) - 1)$$

$$= \begin{cases} 0, & n \text{ even} \\ \dfrac{8}{n\pi}, & n \text{ odd} \end{cases}$$

$$\Rightarrow u(x,t) = \frac{8}{\pi}\left(\frac{e^{-x}\sin x\, e^{-2t}}{1} + \frac{e^{-x}\sin(3x)e^{-10t}}{3} + \frac{e^{-x}\sin(5x)e^{-26t}}{5} + \cdots\right)$$

Question 5 (8pts) Find the largest interval for which the initial value problem

$$(t^2 - 8t + 15)y'' - (t^2 + 1)y' + 2y = 0, \quad y(4) = 6, \quad y'(4) = -1$$

has a unique solution.

$$t^2 - 8t + 15 = (t-5)(t-3)$$

$$y'' - \frac{(t^2+1)}{(t-5)(t-3)} y' + \frac{2}{(t-5)(t-3)} y = 0$$

The eqn. is linear. Coefficients are continuous except at 3 and 5. Therefore, by the existence-uniqueness theorem, there is a unique soln. on the interval $\boxed{(3,5).}$ $(4 \in (3,5))$

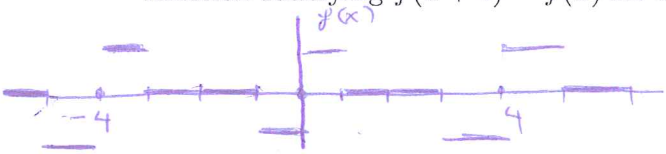
Question 6 (8pts) Suppose that $F(s) = \dfrac{s+1}{s^2+2s+2}$ and $G(s) = \dfrac{1}{s^2+1}$. Express the inverse Laplace transform of $H(s) = F(s)G(s)$ as a convolution integral (do not compute the integral).

$$F(s) = \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1^2} \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t}\cos t$$

$$G(s) = \frac{1}{s^2+1} \Rightarrow g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin t$$

$$\boxed{h(t) = \mathcal{L}^{-1}\{H(s)\} = (f*g)(t) = \int_0^t e^{-(t-\tau)}\cos(t-\tau)\sin\tau \, d\tau}$$

Question 7 (9pts) Suppose that $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 \le x \le 2, \end{cases}$ and that $f(x)$ is an odd function satisfying $f(x+4) = f(x)$ for all $x \in \mathbb{R}$. Find the Fourier series of $f(x)$.



$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right)$$

$$c_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{\cancel{2}}{\cancel{2}}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}\left(-\cos\left(\frac{n\pi}{2}\right) + \cos(0)\right)$$

$$= \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 2/n\pi & \text{if } n \equiv 1 \pmod 4 \\ 4/n\pi & \text{if } n \equiv 2 \pmod 4 \\ 2/n\pi & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

$$\Rightarrow f(x) = \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right) + \frac{4}{2\pi}\sin\left(\frac{2\pi x}{2}\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi x}{2}\right) + \frac{2}{5\pi}\sin\left(\frac{5\pi x}{2}\right) + \cdots$$