CE204 Homework II Solutions

Q.1

① $\mu = 15 \quad \sigma = 8$

a) $\delta = \dfrac{8}{15} = 0.533$

$$\delta > 0.3 \Rightarrow \zeta^2 = \ln\left(1 + \frac{\sigma^2}{\mu^2}\right) = \ln\left(1 + \frac{8^2}{15^2}\right)$$

$$\zeta = 0.5$$

$$\lambda = \ln\mu - \tfrac{1}{2}\zeta^2$$

$$= \ln 15 - \tfrac{1}{2}(0.5)^2$$

$$\lambda = 2.583$$

b) $P(T \leq 5) = ?$

$$P(T \leq 5) = P\left(\frac{\ln T - \lambda}{\zeta} \leq \frac{\ln 5 - 2.583}{0.5}\right)$$

$$= P(z \leq -1.95) = 1 - 0.9744 = 0.0256$$

c) $$P(100 < x < 101 \mid x > 100) = \frac{P(100 < x < 101)}{P(x > 100)}$$

$$= \frac{P\left(\frac{\ln 100 - 2.583}{0.5} < z < \frac{\ln 101 - 2.583}{0.5}\right)}{P\left(z > \frac{\ln 100 - 2.583}{0.5}\right)}$$

$$= \frac{P(4.02 < z < 4.06)}{P(z > 4.06)} = \frac{0}{1} = 0$$

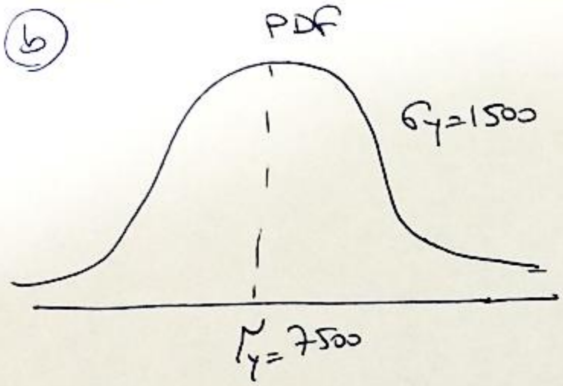
Q.2

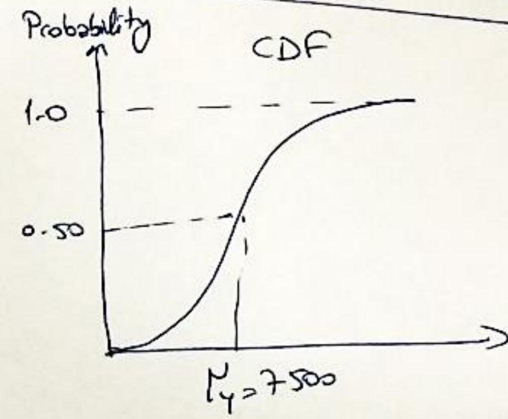
(a) $Y = X/2$, $\mu_X = 15{,}000$, $\sigma_X = 3000$, $P(Y > 9000) = ?$

$$\left.\begin{array}{l} \mu_Y = \mu_X/2 \Rightarrow \mu_Y = 7500 \\ \sigma_Y = \sigma_X/2 \Rightarrow \sigma_Y = 1500 \end{array}\right\} \quad z = \frac{Y-\mu_Y}{\sigma_Y} = \frac{9000-7500}{1500} = 1.0$$

$$P(Y > 9000) = P(z > 1.0) = 1 - P(z < 1) = 1 - 0.84 \; \boxed{= 0.16}$$

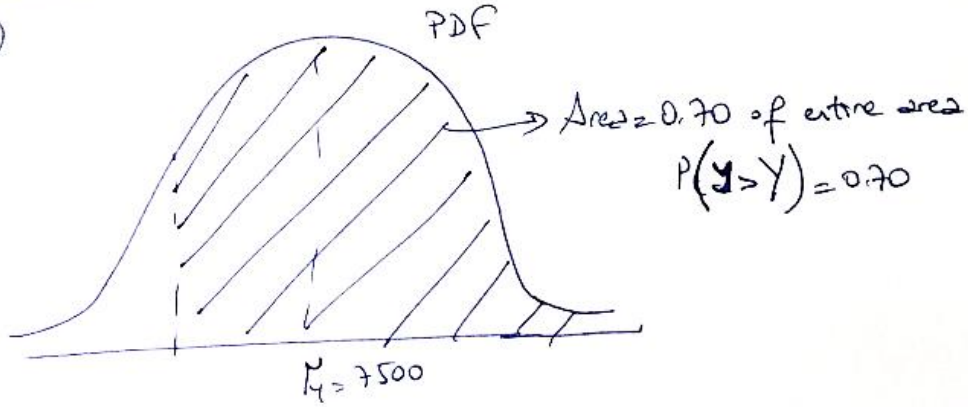
(b)





(c) Mean: average (expected) value
Median: value at which 50% of the data is less than this value
Mode: most likely value of the random variable
for Y, all of mean, median, and mode are the same since Y has Normal distribution.

(d)



Q.3

$$E(Y_A) = 2.E(x) + 4E(Y) + 6E(Z)$$

$$= 2\times2 + 4\times3 + 6\times4 = 40$$

$$V(Y_A) = a^2\sigma_a^2 + b^2\sigma_b^2 + c^2\sigma_z^2 + 2ab\rho_{xy}\sigma_x\sigma_y + 2ac\sigma_x\sigma_z\rho_{xz} + 2bc\rho_{yz}\sigma_y\sigma_z$$

$$= 2\times0.5^2 + 4\times0.6^2 + 6^2\times0.6^2 + \ldots$$

$$= 34.36$$

$$\left.\begin{matrix} Y_A = 40 \\ \sigma_A = 5.86 \end{matrix}\right\} \rightarrow \zeta = \ln\left(1 + \frac{\sigma^2}{\mu^2}\right) = 0.0213 \Rightarrow \zeta = 0.1459$$

$$\lambda = \ln 40 - \frac{1}{2}\zeta^2 = 3.6782$$

$$P(Y_A = 0.50) = 0.3, \quad P(X_A = 45) = 0.7$$

$$P(X>50) = P\left(Z > \frac{\ln 50 - 3.6782}{0.1459}\right) = P(Z>1.60) = 1 - 0.9452 = 0.0548$$

$$P(X>45) = P\left(Z > \frac{\ln 45 - 3.6782}{0.1459}\right) \Rightarrow P(Z>0.88) = 1 - 0.8106 = 0.1894$$

$$0.3\times0.0548 + 0.7\times0.1894 = 0.149$$

Q.4

a) $\lambda = \ln X_m = \ln \mu - \frac{1}{2} \sigma^2$

$= (1.9756 - 1.974) \times 2 = 0.0016 \times 2$
$\quad (\ln \mu) \qquad (\ln X_m)$

$\sigma^2 = 0.0032$

$\sigma = 0.05656$

$\lambda = 1.974$

b) $\frac{\ln 6.5 - 1.974}{0.05656} < x < \frac{\ln 8.0 - 1.974}{0.05656}$

$\frac{1.8718 - 1.974}{0.05656} < z < \frac{2.07944 - 1.974}{0.05656}$

$\frac{-0.10219}{0.05656} < z < \frac{0.10544}{0.05656} \Rightarrow -1.806 < z < 1.864$

$\phi(1.864) - (1 - \phi(1.806)) = 0.9686 + 0.9649 - 1$

$= 0.9335$

c) $p = 0.9335$
$q = 0.0665$

$p^{10} + \binom{10}{1} p^9 q^1 = 0.503 + 0.358 = 0.861$

d) Binominal → distinct events
Poisson → Continuous events

e) Poisson → Any probability
Exponential → Only first time happening

Q.5

a) $\int_0^1 \int_0^x ax^2 dy dx = 1$

$1 = \int_0^1 ax^2 y \Big|_0^x dx = \int_0^1 ax^3 dx = a\frac{x^4}{4}\Big|_0^1$

$1 = \frac{a}{4}$

$\underline{\underline{a = 4}}$

b) $g(x) = \int_{-\infty}^{\infty} f(x,y) dy$

$\Rightarrow g(x) = \int_0^x 4x^2 dy = 4x^2 y\Big|_0^x = \underline{\underline{4x^3}}$

c) $E(x) = \int_{-\infty}^{\infty} x f_x(x) dx$

$\Rightarrow E(x) = \int_0^1 x 4x^3 dx = \frac{4x^5}{5}\Big|_0^1 = 4/5 = \underline{\underline{0.8}}$

$var(x) = E(x^2) - (E(x))^2$

$\Rightarrow Var(x) = \int_0^1 x^2 4x^3 dx - (0.8)^2 = \frac{4x^6}{6}\Big|_0^1 - (0.8)^2 = 0.0267$

$\sigma_x = \sqrt{var(x)} = \sqrt{0.0267} = \underline{\underline{0.163}}$

d) $z = x^2 \quad 0 \le x \le 1$

$g^{-1}(z) = x = \sqrt{z} \quad 0 \le z \le 1$

$f_z(z) = f_x(g^{-1}(z)) \frac{dg^{-1}}{dz} = 4(\sqrt{z})^3 \cdot \frac{1}{2} z^{-1/2} = \underline{\underline{2z}}$

e) $E(z) = \int_0^1 z 2z dz = \frac{2z^3}{3}\Big|_0^1 = \underline{\underline{0.667}}$

$var(z) = E(z^2) - (E(z))^2$

$= \int_0^1 z^2 2z dz - (0.667)^2 = \frac{2z^4}{4}\Big|_0^1 - (0.667)^2 = 0.0556$

$\sigma_z = \sqrt{var(z)} = \sqrt{0.0556} = 0.\underline{\underline{236}}$

f) $E(x) = 0.8 \qquad \sigma_x = 0.163$

$E(z) = 0.667 \qquad \sigma_z = 0.333 \qquad W = X e^z$

$E(w) = \mu_x e^{\mu_z} = 0.8\, e^{0.667} = \underline{\underline{1.559}}$

$var(w) = (0.163)^2 (e^{\mu_z})^2 + (0.333)^2 \cdot (\mu_x e^{\mu_z})^2$

$= (0.163)^2 (e^{0.667})^2 + (0.333)^2 (0.8 \cdot e^{0.667})^2$

$var(w) = 0.37$

$\sigma_w = \sqrt{var(w)} = \sqrt{0.37} = 0.608$

$\delta = \frac{\sigma}{\mu} = \frac{0.608}{1.559} = \underline{\underline{0.39}}$

Q.6

a) $T = 17/100 = \underline{\underline{0.17}}$ hours for a round

b) $V = \frac{100}{17} = 5.88$ rounds in an hour

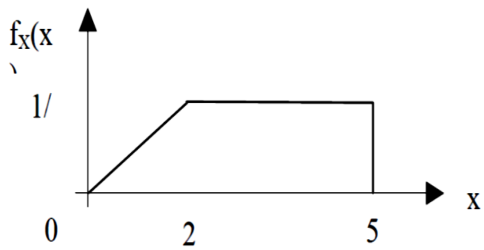
$= 0.098$ rounds in a minute

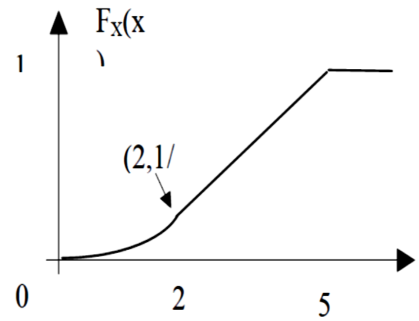
$$P(T_1 > 10) = e^{-0.098 \times 10} = \underline{\underline{0.375}}$$

c) $$P(X=0) = \frac{(0.098 \cdot 30)^0}{0!} e^{-0.098 \times 30} = \underline{\underline{0.053}}$$

Q.7

(a) Integration of the PDF gives $F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/16 & 0 < x \le 2 \\ x/4 - 1/4 & 2 < x \le 5 \\ 1 & x > 5 \end{cases}$; sketches follow:





(b) E(X) = $\int_0^2 x(x/8)dx + \int_2^5 x(1/4)dx = \left[\frac{x^3}{24}\right]_0^2 + \left[\frac{x^2}{8}\right]_2^5 = 71/24 \cong \mathbf{2.96}$ (mm)

(c) P(X < 4) = 1 – P(X > 4)
Where P(X > 4) is easily read off from the PDF as the area (5 – 4)(1/4) = 1/4, hence

P(X < 4) = 1 – 1/4 = 3/4 = **0.75**

(d) A vertical line drawn at the median x_m would divide the unit area under f_X into two equal halves; the right hand rectangle having area

$$0.5 = (5 - x_m)(1/4)$$
$$\Rightarrow x_m = 5 - 4(0.5) = \mathbf{3} \text{ (mm)}$$

(e) Since each of the four cracks has p = 0.25 probability of exceeding 4mm (as calculated in (c)), only one of them exceeding 4mm has the binomial probability (where n = 4, p = 0.25)

$4 \times 0.75^3 \times 0.25 \cong \mathbf{0.422}$

Q.8

(a) $f_X(x)$ is obtained by "integrating out" the independence on y,

$$\therefore f_X(x) = \int_0^1 \frac{6}{5}(x+y^2)dy = \frac{6}{5}\left[xy+\frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{5}(3x+1) \qquad (0 < x < 1)$$

(b) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{(6/5)(x+y^2)}{(2/5)(3x+1)} = 3\dfrac{x+y^2}{3x+1}$

Hence $P(Y > 0.5 \mid X = 0.5) = \int_{0.5}^{1} f_{Y|0.5}(y \mid x=0.5)dy$

$$= 3\int_{0.5}^{1}\frac{0.5+y^2}{1.5+1}dy = (3/2.5)\left[0.5y+\frac{y^3}{3}\right]_{0.5}^{1}$$

$$= \mathbf{0.65}$$

(c) $E(XY) = \int_0^1\int_0^1 xyf_{X,Y}(x,y)dxdy = \frac{6}{5}\int_0^1\int_0^1 (x^2y+xy^3)dxdy$

$$= \frac{2}{5}\int_0^1 ydy + \frac{3}{5}\int_0^1 y^3dy = 1/5 + 3/20 = 7/20 = 0.35$$

$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0.35 - (3/5)(3/5) = -0.01,$

while

$$\sigma_X = \{E(X^2) - [E(X)]^2\}^{1/2} = [(13/30) - (3/5)^2]^{1/2} = 0.271$$

$$\sigma_Y = \{E(Y^2) - [E(Y)]^2\}^{1/2} = [(11/25) - (3/5)^2]^{1/2} = 0.283,$$

Hence the correlation coefficient,

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{-0.01}{0.271 \times 0.283} \cong \mathbf{-\ 0.131}$$