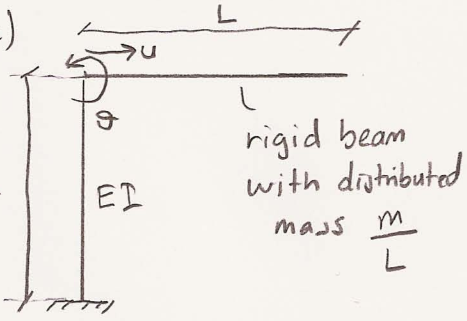
CE 487 HW3 SOLUTIONS (FALL 2006)

Q1) Solution 1

a)



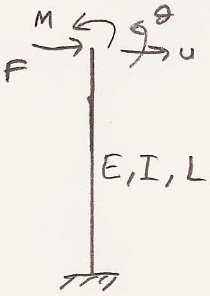
Selected degrees of freedom are the displacement and rotation at the end of the column.

Equation of motion;

$\underline{K}\,\underline{U} + \underline{M}\,\underline{\ddot{U}} = \underline{0}$ where $\underline{U} = \begin{bmatrix} u \\ \theta \end{bmatrix}$ and $\underline{\ddot{U}} = \begin{bmatrix} \ddot{u} \\ \ddot{\theta} \end{bmatrix}$

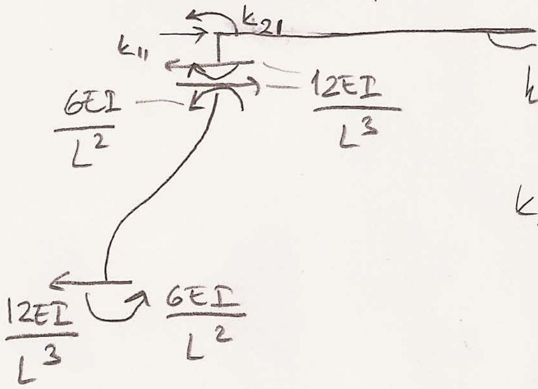
Determination of $\underline{K}$ (stiffness matrix)

Consider the column



$$\begin{bmatrix} F \\ M \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix}$$

For calculation of k_{11} and k_{21} $\Rightarrow u = 1, \theta = 0$

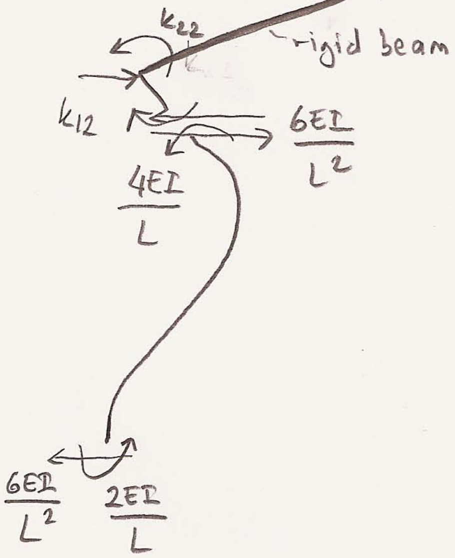


rigid beam

$k_{11} = \frac{12EI}{L^3}$

$k_{21} = \frac{6EI}{L^2}$

For calculation of k_{12} and $k_{22} \Rightarrow u=0,\ \theta=1$



$$k_{12} = \frac{6EI}{L^2}$$

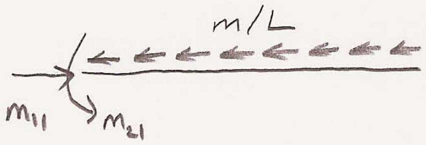
$$k_{22} = \frac{4EI}{L}$$

Hence, $\underline{K} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$

Determination of mass matrix ($\underline{M}$)

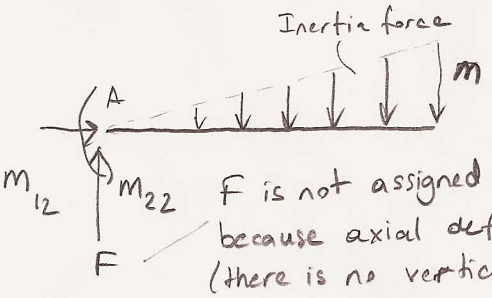
$$\underline{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

For calculation of m_{11} and $m_{21} \Rightarrow \ddot{u}=1,\ \ddot{\theta}=0$



$$m_{11} = \int_0^L \frac{m}{L}\,dx = m \qquad m_{21} = 0$$

For calculation of m_{12} and $m_{22} \Rightarrow \ddot{u}=0,\ \ddot{\theta}=1$



$$m_{12} = 0 \quad m_{22} = m\frac{L}{2}\frac{2L}{3} = \frac{mL^2}{3}$$

F is not assigned to a mass influence coefficient, because axial deformations of the column are neglected (there is no vertical dof at point A)

$$\underline{M} = \begin{bmatrix} m & 0 \\ 0 & \frac{mL^2}{3} \end{bmatrix}$$

The equation of motion for free vibration becomes;

$$\begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} + \begin{bmatrix} m & 0 \\ 0 & \frac{mL^2}{3} \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) Determination of modal frequencies

$$|\underline{K} - \omega_n^2 \underline{M}| = 0$$

$$\begin{vmatrix} \frac{12EI}{L^3} - m\omega_n^2 & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} - \frac{mL^2}{3}\omega_n^2 \end{vmatrix} = 0$$

$$\left(\frac{12EI}{L^3} - m\omega_n^2\right)\left(\frac{4EI}{L} - \frac{mL^2}{3}\omega_n^2\right) - \frac{36(EI)^2}{L^4} = 0$$

$$\frac{48(EI)^2}{L^4} - \frac{4EI\,m\omega_n^2}{L} - \frac{12EI\,m\omega_n^2}{3L} + \frac{m^2L^2}{3}\omega_n^4 - \frac{36(EI)^2}{L^4} = 0$$

$$\frac{12(EI)^2}{L^4} - \frac{8EI\,m\omega_n^2}{L} + \frac{m^2L^2}{3}\omega_n^4 = 0$$

Since $m^2L^2 = 0$, the equation can be divided by m^2L^2

$$\frac{12(EI)^2}{m^2L^6} - \frac{8EI\,w_n^2}{mL^3} + \frac{w_n^4}{3} = 0$$

$$w_n^2 = t \,, \quad \frac{EI}{mL^3} = a$$

$$12a^2 - 8at + \frac{t^2}{3} = 0$$

$$36a^2 - 24at + t^2 = 0 \Rightarrow t^2 - 24at + 36a^2 = 0$$

$$t_1 = \frac{24a}{2} - \frac{\sqrt{(-24a)^2 - 4*36a^2}}{2} = 1.61a$$

$$t_2 = \frac{24a}{2} + \frac{\sqrt{(-24a)^2 - 4*36a^2}}{2} = 22.39a$$

Therefore

$$w_1^2 = 1.61\frac{EI}{mL^3} \,, \quad w_2^2 = 22.39\frac{EI}{mL^3}$$

$$w_1 = 1.27\sqrt{\frac{EI}{mL^3}} \,, \quad w_2 = 4.73\sqrt{\frac{EI}{mL^3}}$$

Determination of mass normalized modal vectors

First modal vector,

$$\left[\underline{K} - w_1^2\underline{M}\right]\underline{\phi}_1 = \underline{0}$$

$$\begin{bmatrix} \frac{12EI}{L^3} - m(1.61)\frac{EI}{mL^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} - \frac{mL^2}{3}\frac{(1.61)EI}{mL^3} \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10.39\dfrac{EI}{L^3} & \dfrac{6EI}{L^2} \\ \dfrac{6EI}{L^2} & \dfrac{10.39}{3}\dfrac{EI}{L} \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 10.39\frac{EI}{L^3}\phi_{11} + \frac{6EI}{L^2}\phi_{21} = 0 \\ \frac{6EI}{L^2}\phi_{11} + \frac{10.39}{3}\frac{EI}{L}\phi_{21} = 0 \end{aligned}\right\} \quad \frac{1.732}{L}\phi_{11} = -\phi_{21} \Rightarrow \underline{\Phi}_1 = \begin{bmatrix} 1 \\ -\dfrac{1.732}{L} \end{bmatrix}$$

Mass normalization

$$\underline{\Phi}_1^T \underline{M}\, \underline{\Phi}_1 = m + \frac{(1.732)^2}{L^2}\frac{mL^2}{3} = 2m$$

$$\left(\underline{\Phi}_1\right)_{\substack{\text{mass} \\ \text{normalized}}} = \frac{\underline{\Phi}_1}{\sqrt{\underline{\Phi}_1^T \underline{M}\, \underline{\Phi}_1}} = \begin{bmatrix} 1 \\ -\dfrac{1.732}{L} \end{bmatrix} * \frac{1}{\sqrt{2m}}$$

$$\left(\underline{\Phi}_1\right)_{\substack{\text{mass} \\ \text{normalized}}} = \begin{bmatrix} \dfrac{0.707}{\sqrt{m}} \\ \dfrac{-1.225}{L\sqrt{m}} \end{bmatrix}$$

Second modal vector:

$$\left[\underline{K} - \omega_2^2 \underline{M}\right] \underline{\Phi}_2 = \underline{0}$$

$$\begin{bmatrix} \dfrac{12EI}{L^3} - m(22.39)\dfrac{EI}{mL^3} & \dfrac{6EI}{L^2} \\ \dfrac{6EI}{L^2} & \dfrac{4EI}{L} - \dfrac{mL^2}{3}(22.39)\dfrac{EI}{mL^3} \end{bmatrix} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

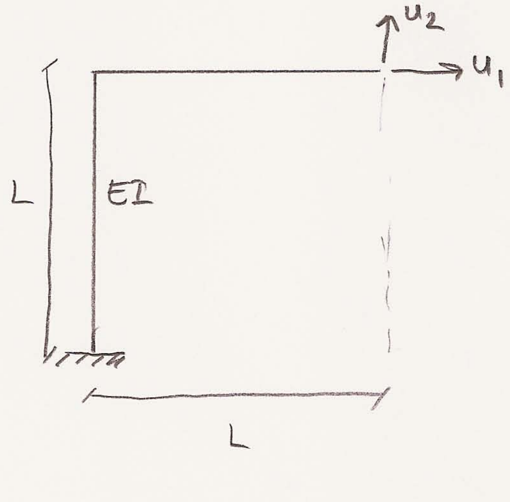
$$\left.\begin{aligned} -10.39\,\frac{EI}{L^3}\,\phi_{12} + \frac{6EI}{L^2}\,\phi_{22} &= 0 \\ \frac{6EI}{L^2}\,\phi_{12} - \frac{10.39}{3}\,\phi_{22} &= 0 \end{aligned}\right\} \quad \frac{1.732}{L}\,\phi_{12} = \phi_{22}$$

$$\underline{\phi}_2 = \begin{bmatrix} 1 \\ \frac{1.732}{L} \end{bmatrix}$$

$$\underline{\phi}_2^T\,\underline{M}\,\underline{\phi}_2 = m + \frac{(1.732)^2}{L^2}\,\frac{mL^2}{3} = 2m$$

$$\left(\underline{\phi}_2\right)_{\substack{mass\\normalized}} = \begin{bmatrix} \frac{1}{\sqrt{2}.\sqrt{m}} \\ \frac{1.732}{\sqrt{2}\sqrt{m}L} \end{bmatrix} = \begin{bmatrix} \frac{0.707}{\sqrt{m}} \\ \frac{1.225}{L\sqrt{m}} \end{bmatrix}$$

Q1) Solution 2



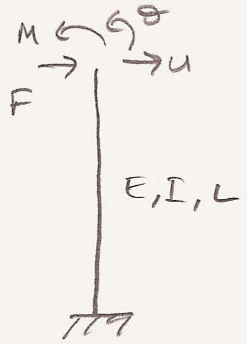
Selected degrees of freedom are the horizontal and vertical displacement at the end of the rigid beam.

Equation of motion for free vibration:

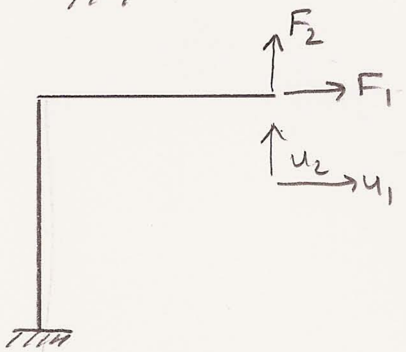
$$\underline{K}\,\underline{U} + \underline{M}\,\underline{\ddot{U}} = 0 \quad \text{where } \underline{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

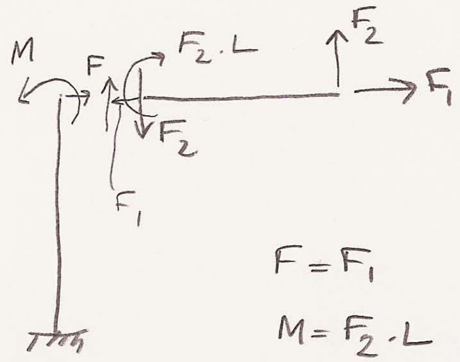
$$\text{and } \underline{\ddot{U}} = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix}$$

Determination of $\underline{K}$ (stiffness matrix)

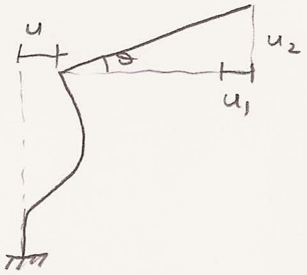


$$\begin{bmatrix} F \\ M \end{bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} \quad \text{----- (1)}$$





$$F = F_1 \qquad \text{-- (2)}$$
$$M = F_2 \cdot L$$



$$u = u_1 \qquad \text{----- (3)}$$
$$\theta = \frac{u_2}{L}$$

Substituting (2) and (3) into (1)

$$\begin{bmatrix} F_1 \\ F_2 L \end{bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2/L \end{bmatrix}$$

$$F_1 = \frac{12EI}{L^3} u_1 + \frac{6EI}{L^3} u_2$$

$$F_2 \cdot L = \frac{6EI}{L^2} u_1 + \frac{4EI}{L^2} u_2$$

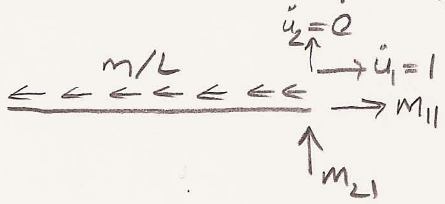
$$F_2 = \frac{6EI}{L^3} u_1 + \frac{4EI}{L^3} u_2$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \underline{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix}$$

Determination of $\underline{M}$ (mass matrix)

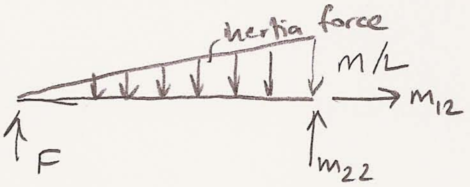
$$\underline{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

For calculation of m_{11} and $m_{21} \Rightarrow \ddot{u}_1 = 1,\ \ddot{u}_2 = 0$



$$m_{11} = \int_0^L \frac{m}{L}\,dx = m \qquad m_{21} = 0$$

For calculation of m_{12} and $m_{22} \Rightarrow \ddot{u}_1 = 0,\ \ddot{u}_2 = 1$



$$m_{12} = 0$$

$$\frac{m}{L}\cdot\frac{L}{2}\cdot\frac{2L}{3} = m_{22}\cdot L$$

F is not assigned to a mass influence coefficient.

$$m_{22} = \frac{m}{3}$$

$$\underline{M} = \begin{bmatrix} m & 0 \\ 0 & \frac{m}{3} \end{bmatrix}$$

Equation of motion for free vibration becomes

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} m & 0 \\ 0 & \frac{m}{3} \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) Determination of modal frequencies

$$|\underline{K} - w_n^2 \underline{M}| = 0$$

$$\begin{vmatrix} \frac{12EI}{L^3} - m w_n^2 & \frac{6EI}{L^3} \\ \frac{6EI}{L^3} & \frac{4EI}{L^3} - \frac{m}{3} w_n^2 \end{vmatrix} = 0$$

$$\left(\frac{12EI}{L^3} - m w_n^2\right)\left(\frac{4EI}{L^3} - \frac{m}{3} w_n^2\right) - \frac{36(EI)^2}{L^6} = 0$$

$$\frac{48(EI)^2}{L^6} - \frac{4EI\, m w_n^2}{L^3} - \frac{12EI\, m w_n^2}{3L^3} + \frac{m^2}{3} w_n^4 - \frac{36(EI)^2}{L^6} = 0$$

$$\frac{12(EI)^2}{L^6} - \frac{8EI\, m w_n^2}{L^3} + \frac{m^2}{3} w_n^4 = 0$$

$$\frac{12(EI)^2}{m^2 L^6} - \frac{8EI\, m w_n^2}{m^2 L^3} + \frac{w_n^4}{3} = 0$$

Same equation in Solution 1 is obtained.

Therefore $w_1^2 = 1.61 \frac{EI}{mL^3}$ $\quad w_2^2 = 22.39 \frac{EI}{mL^3}$

$w_1 = 1.27 \sqrt{\frac{EI}{mL^3}}$ $\quad w_2 = 4.73 \sqrt{\frac{EI}{mL^3}}$

Determination of mass normalized modal vectors

First modal vector

$$\left[\underline{K} - \omega_1^2 \underline{M}\right] \underline{\phi}_1 = \underline{0}$$

$$\begin{bmatrix} \frac{12EI}{L^3} - m(1.61)\frac{EI}{L^3} & \frac{6EI}{L^3} \\ \frac{6EI}{L^3} & \frac{4EI}{L^3} - \frac{m}{3}(1.61)\frac{EI}{mL^3} \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix} 10.39 & 6 \\ 6 & \frac{10.39}{3} \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 10.39\,\phi_{11} + 6\phi_{21} &= 0 \\ 6\,\phi_{11} + \frac{10.39}{3}\phi_{21} &= 0 \end{aligned}\right\} \Rightarrow 1.732\,\phi_{11} = -\phi_{21} \qquad \underline{\phi}_1 = \begin{bmatrix} 1 \\ -1.732 \end{bmatrix}$$

$$\underline{\phi}_1^T \underline{M}\, \underline{\phi}_1 = m\left(1^2 + \frac{(1.732)^2}{3}\right) = 2m$$

$$(\underline{\phi}_1)_{\substack{\text{mass}\\ \text{normalized}}} = \begin{bmatrix} 1 \\ -1.732 \end{bmatrix} * \frac{1}{\sqrt{2}\sqrt{m}} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.707 \\ -1.225 \end{bmatrix}$$

Second modal vector

$$\left[\underline{K} - \omega_2^2 \underline{M}\right] \underline{\phi}_2 = \underline{0}$$

$$\begin{bmatrix} 12\frac{EI}{L^3} - m(22.39)\frac{EI}{mL^3} & \frac{6EI}{L^3} \\ \frac{6EI}{L^3} & \frac{4EI}{L^3} - \frac{m}{3}(22.39)\frac{EI}{mL^3} \end{bmatrix} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix} -10.39 & 6 \\ 6 & \frac{-10.39}{3} \end{bmatrix}\begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -10.39\,\phi_{12} + 6\phi_{22} &= 0 \\ 6\,\phi_{12} - \frac{10.39}{3}\,\phi_{22} &= 0 \end{aligned}\right\} \Rightarrow 1.732\,\phi_{12} = \phi_{22} \qquad \underline{\phi}_2 = \begin{bmatrix} 1 \\ 1.732 \end{bmatrix}$$

$$\underline{\Phi}_2^T\,\underline{M}\,\underline{\Phi}_2 = m + (1.732)^2\,\frac{m}{3} = 2m$$

$$(\underline{\Phi}_2)_{\text{mass normalized}} = \begin{bmatrix} 1 \\ 1.732 \end{bmatrix} * \frac{1}{\sqrt{2m}} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.707 \\ 1.225 \end{bmatrix}$$