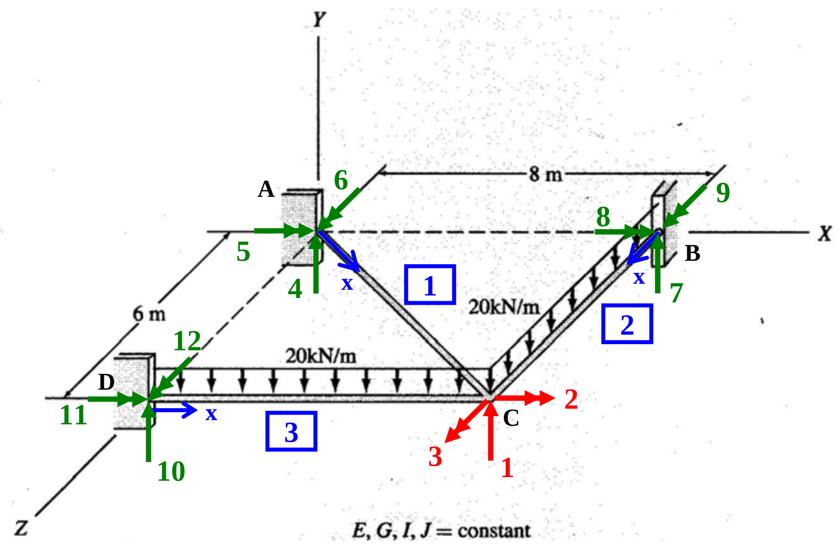
GRID EXAMPLE

Determine the joint displacements, member end forces, and support reactions for the three member grid by using the *Direct Stiffness Method*.
E=200 GPa
G=76 GPa
$I=3.47\times10^{8}$ mm^{4}
$J=1.15\times10^{8}$ mm^{4}



SOLUTION:
There are 3 active dofs (shown in red). Also there exist 9 constrained dofs (shown in green).

First obtain member stiffness matrices in global coordinates (XYZ)

<u>Member 1</u> (From A to C), L=10 m, θ=36.8º
Construct element stiffness matrix in local coordinates ($\underline{K}'_1$)

$$\underline{\mathbf{K}}'_1=\begin{bmatrix} 832.8 & 0 & 4164 & -832.8 & 0 & 4164 \\ 0 & 874 & 0 & 0 & -874 & 0 \\ 4164 & 0 & 27760 & -4164 & 0 & 13880 \\ -832.8 & 0 & -4164 & 832.8 & 0 & -4164 \\ 0 & -874 & 0 & 0 & 874 & 0 \\ 4164 & 0 & 13880 & -4164 & 0 & 27760 \end{bmatrix}$$

Transformation matrix ($\underline{T}_1$) is obtained as

$$\underline{\mathbf{T}}_1=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0.6 & 0 & 0 & 0 \\ 0 & -0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & 0 & 0 & -0.6 & 0.8 \end{bmatrix}$$

Then the element stiffness matrix in global coordinates is obtained through $\underline{\mathbf{K}}_1=\underline{\mathbf{T}}_1^{T}\,\underline{\mathbf{K}}'_1\underline{\mathbf{T}}_1$

$$\underline{\mathbf{K}}_1=\begin{array}{cc} \begin{bmatrix} 832.8 & -2498.4 & 3331.2 & -832.8 & -2498.4 & 3331.2 \\ -2498.4 & 10553 & -12905 & 2498.4 & 4437.4 & -7081.9 \\ 3331.2 & -12905 & 18081 & -3331.2 & -7081.9 & 8568.6 \\ -832.8 & 2498.4 & -3331.2 & 832.8 & 2498.4 & -3331.2 \\ -2498.4 & 4437.4 & -7081.9 & 2498.4 & 10553 & -12905 \\ 3331.2 & -7081.9 & 8568.6 & -3331.2 & -12905 & 18081 \end{bmatrix} & \begin{array}{c} \\ \\ \\ 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{cccccc} & & & 1 & 2 & 3 \end{array} & \end{array}$$

Member 2 (From B to C), L=6 m, θ=90º

$$\underline{\mathbf{K}}_2=\underline{\mathbf{T}}_2^{T}\,\underline{\mathbf{K}}'_2\underline{\mathbf{T}}_2=\begin{array}{c}\left|\begin{array}{cccccc}3855.6 & -11567 & 0 & -3855.6 & -11567 & 0\\ -11567 & 46267 & 0 & 11567 & 23133 & 0\\ 0 & 0 & 1456.7 & 0 & 0 & -1456.7\\ -3855.6 & 11567 & 0 & 3855.6 & 11567 & 0\\ -11567 & 23133 & 0 & 11567 & 46267 & 0\\ 0 & 0 & -1456.7 & 0 & 0 & 1456.7\end{array}\right| \\ \begin{array}{cccccc} & & & 1 & 2 & 3\end{array}\end{array}\begin{array}{c}\\ \\ \\ 1\\ 2\\ 3\\ \\ \end{array}$$

Member 3 (From D to C), L=8 m, θ=0º

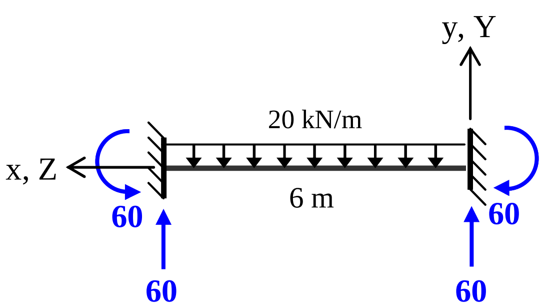
$$\underline{\mathbf{K}}_3=\underline{\mathbf{K}}'_3=\begin{array}{c}\left|\begin{array}{cccccc}1626.6 & 0 & 6506.3 & -1626.6 & 0 & 6506.3\\ 0 & 1092.5 & 0 & 11567 & -1092.5 & 0\\ 6506.3 & 0 & 34700 & -6506.3 & 0 & 17350\\ -1626.6 & 0 & -6506.3 & 1626.6 & 0 & -6506.3\\ 11567 & -1092.5 & 0 & 0 & 1092.5 & 0\\ 6506.3 & 0 & 17350 & -6506.3 & 0 & 34700\end{array}\right| \\ \begin{array}{cccccc} & & & 1 & 2 & 3\end{array}\end{array}\begin{array}{c}\\ \\ \\ 1\\ 2\\ 3\\ \\ \end{array}$$

The structure stiffness matrix is obtained through assebmly, i.e. $\underline{\mathbf{K}}=\underline{\mathbf{K}}_1+\underline{\mathbf{K}}_2+\underline{\mathbf{K}}_3$

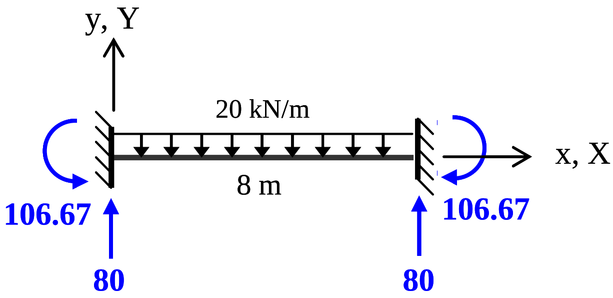
$$\underline{\mathbf{K}}=\begin{array}{c}\begin{array}{ccc}1 & 2 & 3\end{array}\\ \left|\begin{array}{ccc}6315 & 14065 & -9837.5\\ 14065 & 57912 & -12905\\ -9837.5 & -12905 & 54238\end{array}\right|\end{array}\begin{array}{c}\\ 1\\ 2\\ 3\end{array}$$

The joint load vector ($\underline{\mathrm{P}}$) is composed of zeros, i.e. $\underline{\mathbf{P}}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$

Members 2 and 3 are subjected to distributed loads, which should be converted to joint loads.



$$\underline{\mathbf{P}}_2^{F}=\mathbf{T}_2^{T}\,\mathbf{P}'^{F}_2=\begin{Bmatrix}60\\-60\\0\\60\\60\\0\end{Bmatrix}\begin{array}{c}\\ \\ \\ 1\\ 2\\ 3\end{array}$$



$$\underline{\mathbf{P}}_3^{\,F} = \underline{\mathbf{P'}}_3^{\,F} = \begin{array}{c} \left\{ \begin{array}{c} 80 \\ 0 \\ 106.67 \\ \boxed{\begin{array}{c} 80 \\ 0 \\ -106.67 \end{array}} \end{array} \right\} \end{array} \begin{array}{c} \\ \\ \\ 1 \\ 2 \\ 3 \end{array}$$

Hence the fixed end force vector fort he structure is $\underline{\mathbf{P}}^F = \left\{ \begin{array}{c} 140 \\ 60 \\ -106.67 \end{array} \right\} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$

The structure stiffness equation can be developed as $\underline{\mathbf{P}} - \underline{\mathbf{P}}^F = \underline{\mathbf{K}}\ \underline{\mathbf{D}}$

$$\left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\} - \left\{ \begin{array}{c} 140 \\ 60 \\ -106.67 \end{array} \right\} = \left| \begin{array}{ccc} 6315 & 14065 & -9837.5 \\ 14065 & 57912 & -12905 \\ -9837.5 & -12905 & 54238 \end{array} \right| \left\{ \begin{array}{c} D_1 \\ D_2 \\ D_3 \end{array} \right\}$$

The joint displacements are obtained as $\underline{\mathbf{D}} = \left\{ \begin{array}{c} -55.95 \text{ m} \\ 11.33 \text{ rad} \\ -5.49 \text{ rad} \end{array} \right\} \times 10^{-3}$

Member end displacements and forces (also support reactions) can be determined as follows:

For member 1, local end displacements ($\underline{\mathbf{D}}_1'$) can be calculated as

$$\underline{\mathbf{D'}}_1 = \underline{\mathbf{T}}_1\ \underline{\mathbf{D}}_1 = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ -55.95 \\ 5.77 \\ -11.19 \end{array} \right\} \times 10^{-3}$$

Then, member end forces in local and global coordinates ($\underline{\mathbf{P}}_1'$ and $\underline{\mathbf{P}}_1$) can also be determined.

$$\underline{\mathbf{P'}}_1 = \underline{\mathbf{K'}}_1\underline{\mathbf{D'}}_1 = \begin{array}{l} \left\{\begin{array}{c} 0.015 \\ -5.04 \\ 77.71 \\ -0.015 \\ 5.04 \\ -77.56 \end{array}\right\} \end{array} \begin{array}{l} \text{kN} \\ \text{kNm} \\ \text{kNm} \\ \text{kN} \\ \text{kNm} \\ \text{kNm} \end{array} \qquad \underline{\mathbf{P}}_1 = \underline{\mathbf{T}}_1^{T}\underline{\mathbf{P'}}_1 = \left\{\begin{array}{c} 0.015 \\ -50.66 \\ 59.14 \\ -0.015 \\ 50.57 \\ -59.02 \end{array}\right\} \begin{array}{l} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}$$

End forces for members 2 and 3 can be determined through the following formulations

$$\underline{\mathbf{P'}}_2 = \underline{\mathbf{K'}}_2\underline{\mathbf{D'}}_2 + \mathbf{P'}_2^{F} \qquad \underline{\mathbf{P'}}_3 = \underline{\mathbf{K'}}_3\underline{\mathbf{D'}}_3 + \mathbf{P'}_3^{F}$$

$$\underline{\mathbf{P}}_2 = \mathbf{T}_2^{T}\,\mathbf{P'}_2 \qquad \underline{\mathbf{P}}_3 = \mathbf{T}_3^{T}\,\mathbf{P'}_3$$

$$\underline{\mathbf{P}}_2 = \left\{\begin{array}{c} 144.67 \\ -445.06 \\ 7.99 \\ -24.67 \\ -62.95 \\ -7.99 \end{array}\right\} \begin{array}{l} 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array} \qquad \underline{\mathbf{P}}_3 = \left\{\begin{array}{c} 135.32 \\ -12.38 \\ 375.52 \\ 24.68 \\ 12.38 \\ 67.01 \end{array}\right\} \begin{array}{l} 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array}$$