M E T U
Department of Mathematics

	Introduction to Differential Equations MidTerm 2	

Code : *Math 219* Acad. Year : *2014-2015* Semester : *Fall* Coordinator: *Özgür Kişisel* Date : *December.20.2014* Time : *13:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 4 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

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Question 1 (25 pts) Find the solution of the initial value problem

$$y''' - 3y'' + 2y' = t + e^t, \quad y(0) = 1, \quad y'(0) = -\frac{1}{4}, \quad y''(0) = -\frac{3}{2}.$$

$$\underbrace{(D^3 - 3D^2 + 2D)}_{D(D-2)(D-1)} y = \underbrace{t + e^t}_{\text{annihilator}: \, D^2(D-1)}$$

$$\Rightarrow D^3 (D-1)^2 (D-2) y = 0$$

roots: 0, 0, 0, 1, 1, 2

$$y = \underbrace{c_1 e^t + c_2 e^{2t} + c_3}_{y_h} + c_4 t + c_5 t^2 + c_6 t e^t$$

$$\Rightarrow D(D-2)(D-1)\left(c_4 t + c_5 t^2 + c_6 t e^t\right) = t + e^t$$

$$(D^3 - 3D^2 + 2D)(c_4 t + c_5 t^2) = t$$
$$-6c_5 + 2c_4 + 4c_5 t = t$$
$$c_5 = 1/4$$
$$2c_4 = 6c_5$$
$$\Rightarrow c_4 = 3/4$$

$$D(D-2)(D-1)(c_6 t e^t) = e^t$$
$$(D^2 - 2D)(c_6 e^t) = e^t$$
$$(c_6 - 2c_6) e^t = e^t$$
$$c_6 = -1$$

$$\Rightarrow y = c_1 e^t + c_2 e^{2t} + c_3 + \frac{3}{4} t + \frac{1}{4} t^2 - t e^t$$
$$y' = c_1 e^t + 2c_2 e^{2t} + \frac{3}{4} + \frac{1}{2} t - t e^t - e^t$$
$$y'' = c_1 e^t + 4c_2 e^{2t} + \frac{1}{2} - t e^t - 2e^t$$

$$y(0) = c_1 + c_2 + c_3 = 1$$
$$y'(0) = c_1 + 2c_2 + \cancel{\frac{3}{4} - 1 = -\frac{1}{4}}$$
$$y''(0) = c_1 + 4c_2 + \cancel{\frac{1}{2} - 2 = -\frac{3}{2}}$$

$c_1 = c_2 = 0$

$c_3 = 1$

$$\Rightarrow \boxed{y(t) = 1 + \frac{3}{4} t + \frac{1}{4} t^2 - t e^t}$$

Question 2 (25 pts) Consider the equation $2x^2y'' + 3xy' - (x^2+1)y = 0$.

(a) Show that $x_0 = 0$ is a regular singular point.

$\frac{3x}{2x^2}$ is not defined at $x_0 = 0 \Rightarrow x_0$ is a singular point.

$$\left.\begin{aligned} &\lim_{x\to 0} x\cdot\frac{3x}{2x^2} = \frac{3}{2} \\ &\lim_{x\to 0} \frac{-(x^2+1)}{2x^2}\cdot x^2 = -\frac{1}{2} \end{aligned}\right\} \text{finite.}$$

$\frac{x\cdot 3x}{2x^2}$ and $-\frac{(x^2+1)}{2x^2}\cdot x^2$ are rational functions with a finite limit at $0 \Rightarrow$ they are analytic.

So, $x_0 = 0$ is a regular singular point.

(b) Find the indicial equation and its roots r_1, r_2.

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = 0$$
$$r^2 + \frac{r}{2} - \frac{1}{2} = 0$$
$$2r^2 + r - 1 = 0$$
$$(2r-1)(r+1) = 0$$

$r_1 = +1/2$, $r_2 = -1$

$r_1 > r_2$

(c) Find the recurrence relation corresponding to the larger root of the indicial equation.

$$y = x^{1/2}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1/2}$$

So, $y' = \sum_{n=0}^{\infty}(n+\frac{1}{2})a_n x^{n-\frac{1}{2}}$, $y'' = \sum_{n=0}^{\infty}(n+\frac{1}{2})(n-\frac{1}{2})a_n x^{n-\frac{3}{2}}$

$$2x^2\left(\sum_{n=0}^{\infty}(n+\tfrac{1}{2})(n-\tfrac{1}{2})a_n x^{n-\frac{3}{2}}\right) + 3x\left(\sum_{n=0}^{\infty}(n+\tfrac{1}{2})a_n x^{n-\frac{1}{2}}\right)$$
$$- x^2\sum_{n=0}^{\infty} a_n x^{n+1/2} - \sum_{n=0}^{\infty} a_n x^{n+1/2} = 0$$

$$\sum_{n=0}^{\infty} 2(n+\tfrac{1}{2})(n-\tfrac{1}{2})a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} 3(n+\tfrac{1}{2})a_n x^{n+\frac{1}{2}} - \sum_{n=2}^{\infty} a_{n-2}x^{n+\frac{1}{2}}$$
$$- \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} = 0$$

$x^{1/2}$: $\underbrace{2\cdot\frac{1}{2}(-\frac{1}{2})a_0 + 3\cdot\frac{1}{2}a_0 - a_0 = 0}_{0=0}$

$x^{3/2}$: $2(\frac{3}{2})(\frac{1}{2})a_1 + 3\cdot(\frac{3}{2})a_1 - a_1 = 0 \}\ a_1 = 0$

$x^{n+1/2}$, $n \geq 2$: $\left[2(n+\frac{1}{2})(n-\frac{1}{2}) + 3(n+\frac{1}{2}) - 1\right]a_n = a_{n-2}$

$$\boxed{a_n = a_{n-2} / (2n^2+3n)}$$

(d) Find the first four nonzero terms of a nontrivial solution corresponding to the larger root of the indicial equation.

Set $a_0 = 1$

$$a_2 = \frac{a_0}{2\cdot 7} = \frac{1}{2\cdot 7}$$
$$a_4 = \frac{a_2}{4\cdot 11} = \frac{1}{2\cdot 4\cdot 7\cdot 11}$$
$$a_6 = \frac{a_4}{6\cdot 15} = \frac{1}{2\cdot 4\cdot 6\cdot 7\cdot 11\cdot 15}$$

$a_1 = 0$

$\Rightarrow a_3 = a_5 = a_7 = \cdots = 0$

$$\Rightarrow \boxed{y = x^{1/2}\left(1 + \frac{x^2}{2\cdot 7} + \frac{x^4}{2\cdot 4\cdot 7\cdot 11} + \frac{x^6}{2\cdot 4\cdot 6\cdot 7\cdot 11\cdot 15} + \cdots\right)}$$

Question 3 (25 pts) By using the Laplace transform, solve the initial value problem $y'' - y = g(t)$, $y(0) = 2$, $y'(0) = -4$, where

$$g(t) = \begin{cases} t, & 0 \le t < 3, \\ 1, & 3 \le t. \end{cases}$$

$$g(t) = u_0(t)\cdot t + u_3(t)\cdot(1-t) = t u_0(t) + u_3(t)\cdot\left(-2-(t-3)\right)$$

$$\Rightarrow \mathcal{L}\{g(t)\} = \frac{1}{s^2} - \frac{2e^{-3s}}{s} - \frac{e^{-3s}}{s^2}$$

$$y'' - y = g(t)$$

$$\Rightarrow s^2\mathcal{L}\{y\} - s\overbrace{y(0)}^{2} - \overbrace{y'(0)}^{-4} - \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$(s^2-1)\mathcal{L}\{y\} = 2s - 4 + \frac{1}{s^2} - \frac{2e^{-3s}}{s} - \frac{e^{-3s}}{s^2}$$

$$\mathcal{L}\{y\} = \frac{2s-4}{(s-1)(s+1)} + (1-e^{-3s})\frac{1}{s^2(s-1)(s+1)} - 2e^{-3s}\cdot\frac{1}{s(s-1)(s+1)}$$

$$\frac{2s-4}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \Rightarrow \left.\begin{aligned} A+B&=2 \\ A-B&=-4 \end{aligned}\right\} A=-1,\ B=3$$

$$\frac{1}{s^2(s-1)(s+1)} = \frac{Cs+D}{s^2} + \frac{E}{s-1} + \frac{F}{s+1} \Rightarrow (Cs+D)(s^2-1) + Es^2(s+1) + Fs^2(s-1) = 1$$

$$\begin{aligned} C+E+F&=0 \\ D+E-F&=0 \end{aligned} \quad \left.\begin{aligned} E+F&=0 \\ E-F&=1 \end{aligned}\right\} \begin{aligned} E&=1/2 \\ F&=-1/2 \end{aligned}$$

$$\left.\begin{aligned} -C&=0 \\ -D&=1 \end{aligned}\right\} \begin{aligned} C&=0 \\ D&=-1 \end{aligned}$$

$$\frac{1}{s(s-1)(s+1)} = \frac{G}{s} + \frac{H}{s-1} + \frac{I}{s+1} \Rightarrow G(s^2-1) + Hs(s+1) + Is(s-1) = 1$$

$$G+H+I=0$$

$$H-I=0 \rightarrow H=I$$

$$-G = 1 \Rightarrow G=-1 \qquad 2H=1$$

$$H=I=1/2$$

$$\mathcal{L}\{y\} = \frac{-1}{s-1} + \frac{3}{s+1} + (1-e^{-3s})\left(\frac{-1}{s^2} + \frac{1/2}{s-1} + \frac{-1/2}{s+1}\right) - 2e^{-3s}\left(\frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1}\right)$$

$$\boxed{\begin{aligned} y(t) = {}& -e^t + 3e^{-t} + \left(-t + \tfrac{1}{2}e^t - \tfrac{1}{2}e^{-t}\right) - u_3(t)\left(-(t-3) + \tfrac{1}{2}e^{(t-3)} - \tfrac{1}{2}e^{-(t-3)}\right) \\ & -2u_3(t)\left(-1 + \tfrac{1}{2}e^{(t-3)} + \tfrac{1}{2}e^{-(t-3)}\right) \end{aligned}}$$

Question 4 (25 pts) **This question has 4 unrelated parts.**

(a) The differential equation $2y'' + by' + ky = 10^{10}\cos(2t)$ describes a spring mass system without damping. Assume that the forcing function causes resonance. Find b and k.

No damping $\Rightarrow \boxed{b=0}$

Resonance $\Leftrightarrow \omega = \omega_0 = 2$

$\Leftrightarrow \sqrt{\frac{k}{2}} = 2$

$\Leftrightarrow \boxed{k=8}$

(b) Suppose that $xe^x \sin x$ and x are among the solutions of a constant coefficient, linear, homogenous, ordinary differential equation. Determine the minimum possible order for this equation and find the general solution for this minimum possible order.

x is a solution $\Rightarrow \lambda = 0$ is a double root

$xe^x \sin x$ is a solution $\Rightarrow \lambda = 1 \mp i$ are double roots

$\Rightarrow$ characteristic equation has at least 6 roots.

$\Rightarrow$ minimum possible order is $\boxed{6}$.

$(\lambda \cdot \lambda \cdot (\lambda - (1+i))^2 (\lambda - (1-i))^2 = \lambda^2(\lambda^2 - 2\lambda + 2)^2)$

$$\boxed{y = c_1 + c_2 x + c_3 e^x \cos x + c_4 e^x \sin x + c_5 x e^x \cos x + c_6 x e^x \sin x \quad c_1, \dots, c_6 \in \mathbb{R}}$$

(c) If $y'' + y = \delta(t)$, $y(0) = 1$, $y'(0) = 0$, then find the Laplace transform $Y(s)$ of $y(t)$.

$s^2 \mathcal{L}\{y\} - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t)\} = 1$

$\Rightarrow (s^2+1)\mathcal{L}\{y\} = 1 + s$

$$\boxed{Y(s) = \mathcal{L}\{y\} = \frac{1+s}{s^2+1}}$$

(d) Find the inverse Laplace transform of $F(s) = -\dfrac{1}{s(s^2+2s+4)}$.

$-\dfrac{1}{s(s^2+2s+4)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+2s+4}$

$-1 = A(s^2+2s+4) + (Bs+C)s$

$\left.\begin{array}{l} A+B=0 \\ 2A+C=0 \\ 4A=-1 \end{array}\right\} \begin{array}{l} A = -1/4 \\ B = 1/4 \end{array} \quad C = 1/2$

$-\dfrac{1}{s(s^2+2s+4)} = \dfrac{-1/4}{s} + \dfrac{\frac{1}{4}s + \frac{1}{2}}{s^2+2s+4} = \dfrac{-1/4}{s} + \dfrac{\frac{1}{4}(s+1) + \frac{1}{4}}{(s+1)^2 + (\sqrt{3})^2}$

$$\Rightarrow \boxed{\mathcal{L}^{-1}(F(s)) = -\frac{1}{4} + \frac{1}{4}e^{-t}\cos(\sqrt{3}t) + \frac{1}{4\sqrt{3}}e^{-t}\sin(\sqrt{3}t)}$$