Karaesmen

MIDDLE EAST TECHNICAL UNIVERSITY
FACULTY OF ENGINEERING

CE 204 UNCERTAINTY and DATA ANALYSIS
Spring Semester 2013-14

Homework 2- Date Due: March 27th, 2014 Thursday till 18.00

IMPORTANT NOTICE:

- ***You are allowed to collaborate with other students (or ask questions to your assistants/ instructors) on homework provided that you stay away from plagiarizing (according to dictionaries "to plagiarize" means to steal and pass off ideas and/or words/ solutions of another as one's own without citing the source). That is, collaboration is accepted if you write and give your own solutions. If you are caught on plagiarizing or cheating by handing in "too similar" homework, you will be graded by zero on this homework.***
- ***For late submission of homework, there will be 5 points of deduction for every late day and no homework will be accepted after a week from the due date.***

***NOTE:* The following problems are selected from previous CE 204- Exam questions.**

1. A hole is drilled in a sheet-metal component, and then a shaft is inserted through the hole. The shaft clearance is equal to the difference between the radius of the hole and the radius of the shaft. Let the random variable **X** denote the clearance, in millimeters. The probability density function (pdf) of X is modeled as:

$$f_X(x) = k(1 - x^4) \quad , \quad 0 < x < 1$$
$$= 0 \quad , \quad \text{otherwise}$$

a) Find the constant k, so that $f_X(x)$ is a proper probability density function.
b) Components with clearances larger than **0.8 mm** must be scrapped. What percentage of components is scrapped?
c) Find the cumulative distribution function (cdf), $\mathbf{F_X(x)}$ and plot it.
d) Find the mean, variance, standard deviation and coefficient of variation of the clearance.

SOLUTION

a) $\int_0^1 f_X(x)\,dx = 1 \Rightarrow \int_0^1 k(1-x^4)\,dx = k(x - x^5/5)\Big|_0^1 = 4/5\,k = 1$

$k = 1.25$

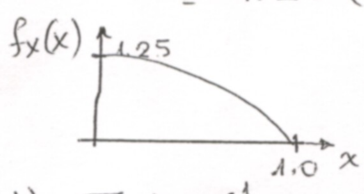
b) $P(x > 0.8) = \int_{0.8}^{1.0} 1.25(1-x^4)\,dx = 1.25(x - x^5/5)\Big|_{0.8}^{1.0}$

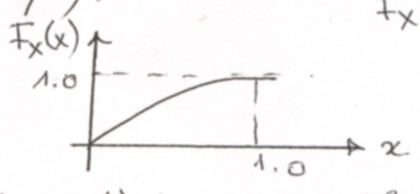
$= 0.08192$

$\left(\text{OR} \quad 1 - \int_0^{0.8} 1.25(1-x^4)\,dx = 0.08192\right)$

c) $F_X(x) = \int_0^x 1.25(1-x^{*4})\,dx^* = 1.25(x^* - x^{*5}/5)\Big|_0^x$

$= 1.25\,(x - x^5/5), \quad 0 \le x \le 1 \quad$ (Check: $F_X(0) = 0$ & $F_X(1) = 1$)





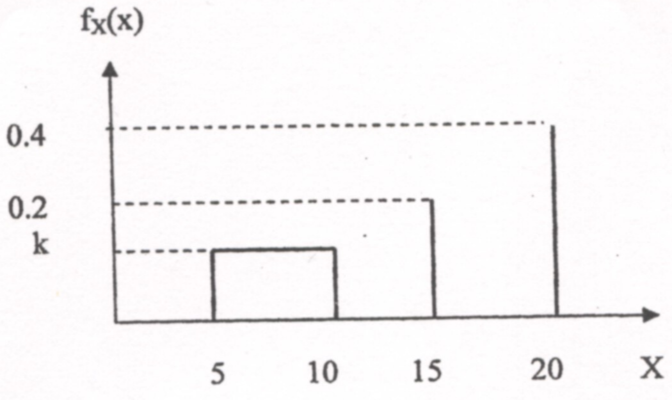
d) $E(x) = \int_0^1 x\, 1.25\,(1-x^4)\,dx = 1.25(x^2/2 - x^6/6)\Big|_0^1 = \frac{1.25}{3} = 0.41667.$

$V(x) = E(x^2) - \mu_X^2 = \int_0^1 x^2\, 1.25\,(1-x^4)\,dx - (0.41667)^2$

$= 1.25(x^3/3 - x^7/7) - 0.41667^2 = 0.06444$

$\sigma_X = 0.25386 \Rightarrow \delta = 0.25386/0.41667 \cong 0.609.$

2. The random variable X is uniformly distributed in the interval between 5 and 10 (i.e. X is chosen at random from the interval 5 to 10), and takes values 15 and 20 with probabilities 0.2 and 0.4, respectively (as shown in the figure below).

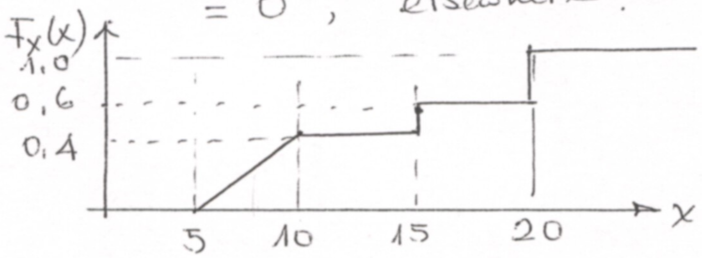


a) Compute the appropriate value of k so that X has a proper probability function. Write down the expressions for the probability and cumulative probability distribution (cdf) functions of the random variable X. Plot the cumulative probability distribution function.
b) Find the mean, median and mode of the random variable X.
c) Find the coefficient of variation of X.
d) What is the probability that X is greater than 8, if it is known that $X \leq 15$.

SOLUTION

a) $(10-5)k + 0.2 + 0.4 = 1.0 \Rightarrow 5k = 0.4 \Rightarrow k = 0.08$

$f_X(x) = 0.08, \quad 5 < x < 10$
$P_X(x) = 0.2, \quad x = 15$
$P_X(x) = 0.4, \quad x = 20$
$= 0, \quad$ elsewhere.



$F_X(x) = 0.08x - 0.4, \quad 5 \leq x \leq 10$
$F_X(x) = 0.4, \quad 10 \leq x < 15$
$F_X(x) = 0.6, \quad 15 \leq x < 20$
$= 1.0, \quad x \geq 20$

b) $E(x) = \int_5^{10} x(0.08)\,dx + 15 \times 0.2 + 20 \times 0.4 = 3 + 3 + 8 = 14$

x_{med} is ≈ 15, $x_{mode} = 20$.

c) $E(x^2) = \int_5^{10} x^2 (0.08)\,dx + 0.2 \times 15^2 + 0.4 \times 20^2 = 228.33$
$V(x) = 228.33 - (14)^2 = 32.333$.
$\sigma_x = 5.686, \quad \delta = 5.686/14 = 0.406$.

d) $P(x > 8 / x \leq 15) = \dfrac{P(8 \leq x \leq 15)}{P(x \leq 15)} = \dfrac{\int_8^{10} 0.08\,dx + 0.2}{0.6} = \dfrac{0.36}{0.60} = 0.60$.

3. The final grades of 2011-2012 CE 204 class were normally distributed with a mean grade of 60 and standard deviation of 10. The instructors decided to give all students the **grade DD** if their final grades were between **50 and 70 (i.e., $\mu \pm 1\sigma$)**, and **38 students** received the grade DD in this way.

- **a)** Calculate the total number of students in the class.
- **b)** How many students received the grade AA if the instructors had given the grade AA only if the student's final grade was above 82?
- **c)** A student was told that he/she had a grade DD or above, what is the probability that he/she got AA if the instructors had given the grade AA only if the student's final grade was above 82?
- **d)** Is using normal distribution model for the grade distribution reasonable? Why yes/why no?

<u>Solution:</u>

N(60,10)

a)

$$P(50 < x < 70) = P\left(\frac{50-60}{10} < z < \frac{70-60}{10}\right) = P(-1 < z < 1)$$
$$= 1 - 2(1 - 0.8413) = 0.6826$$

If 68.26% are 38 students, total number of students will be

$$\frac{38 * 100}{68.26} = 55.67 => 56\ students$$

b)

$$P(x > 82) = P\left(z > \frac{82-62}{10}\right) = P(z > 2.2) = 1 - 0.9861 = 0.0139$$

Number of students graded with AA is

$$56 * 0.0139 \approx 1\ student$$

c)

$$P\left(\frac{x > 82}{x > 50}\right) = \frac{P[(50 > x) \cap (x > 82)]}{P(x > 50)} = P\left(\frac{x > 82}{x > 50}\right)$$
$$= \frac{P\left(z > \frac{82-60}{10}\right)}{P\left(z > \frac{50-60}{10}\right)} = \frac{P(z > 2.2)}{P(z > -1)} = \frac{1 - 0.9861}{0.8413} = 0.0165$$

d)

$$P(x < 0) = P\left(z < \frac{0-60}{10}\right) \cong 0.0$$

So, it is reasonable!

4. The compressive strength of concrete to be used for a mass dam is modeled by a normal distribution with a mean of 45 MPa and coefficient of variation of 0.20 by the design engineer.

- **a)** If the specifications require the compressive strength to exceed 50 MPa, what is the probability of satisfying the specification limits?
- **b)** Is it reasonable to use normal distribution model for the compressive strength? Why yes/why no?
- **c)** What must be the specification limit of concrete so that probability of exceeding it, is 0.99?
- **d)** If another engineer proposes to model the concrete by a lognormal distribution with the same mean and coefficient of variation as given above, what is the probability of the strength to exceed 50 MPA? Compare your results in parts "a" and "d" and comment on them.

Solution:

$$\mu = 45\ MPa\ \ , \delta = 0.2 => N(45{,}9)$$

a)

$$P(x > 50\ MPa) = P\left(z > \frac{50-45}{9}\right) = 1 - 0.7107 = 0.2893$$

b)

$$P(x < 0) = P\left(z > -\frac{45}{9}\right) \approx 0.0\ \ => So, it\ is\ reasonable$$

c)

$$P(x_{max} < x) = 0.99$$

$$P\left(\frac{x_{max}-45}{9} < \frac{x-45}{9}\right) = 0.99$$

$$P\left(z > \frac{x_{max}-45}{9}\right) = 0.99 => z = -2.326$$

$$\frac{x_{max}-45}{9} = -2.326 => x_{max} = 24.067\ MPa$$

d)

$$\xi^2 = \ln(1+\delta^2) = \ln(1+0.2^2) = 0.03922$$

$$\xi = 0.198$$

$$\lambda = ln45 - \frac{1}{2}(0.03922) = 3.787$$

$$P(x > 50) = P\left(z > \frac{ln50 - 3.787}{0.198}\right) = P(z > 0.6314) = 1 - 0.7361 = 0.2639$$

There is about 3% difference between the results in parts "a" and "d".

5. The time between severe earthquakes, T at a given region follows a lognormal distribution with mean 80 years and standard deviation of 32 years.

a) Determine the parameters of this lognormally distributed recurrence time, T. (Ans. 4.3078, 0.3853).

b) Determine the probability that a severe earthquake will occur within 20 years from the previous one?

c) Suppose the last sever earthquake in the region took place 100 years ago. What is the probability that severe earthquake will occur over the next year?

SOLUTION

$\mu = 80$ years & $\sigma = 32$ years $\Rightarrow \delta = 32/80 = 0.4$

a) $\xi^2 = \ln(1+\delta^2) = \ln(1+0.4^2) = 0.14842$

$\xi = 0.3853$

$\lambda = \ln\mu - 1/2\,\xi^2 = \ln 80 - 1/2 \times 0.14842$

$= 4.3078$

b) $P(x \leq 20) = P\left(\frac{\ln x - \lambda}{\xi} \leq \frac{\ln 20 - 4.3078}{0.3853}\right)$

$= P(z < -3.4054) \cong 1 - 0.9997$

$\cong 0.0003$.

c) $P\left(100<x<101 / x>100\right) = \frac{P(100<x<101)}{P(x>100)}$

$P(x>100) = P\left(z > \frac{\ln 100 - 4.3078}{0.3853}\right) = P(z > 0.7719)$

$= 1 - 0.7799 = 0.2201$

$P(100 < x < 101) = P\left(\underbrace{\frac{\ln 100 - 4.3078}{0.3853}}_{0.7719} < z < \underbrace{\frac{\ln 101 - 4.3078}{0.3853}}_{0.7976}\right)$

$= 0.7874 - 0.7799 \cong 0.0075$

$P\left(100<x<101 / x>100\right) = 0.0075/0.2201 \cong 0.034$.