# Formula Sheet

## Basics

$$\frac{d}{dx}e^{kx} = k \cdot e^{kx} \qquad \frac{d}{dx}\arctan(kx) = \frac{k}{1+(kx)^2}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-cb} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## Error

$$\varepsilon_{RA} = \left|\frac{x_{k+1}-x_k}{x_{k+1}}\right| \qquad \varepsilon_{RA\%} = \left|\frac{x_{k+1}-x_k}{x_{k+1}}\right| \times 100$$

$$\varepsilon_{AA} = |x_{k+1}-x_k| \qquad RMS = \sqrt{\frac{\sum_{i=1}^{n}\left(f(x_i)-y_i\right)^2}{n}}$$

## Taylor Series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2!} + \frac{h^3 f^{(3)}(x)}{3!} + \ldots + \frac{h^n f^{(n)}(x)}{n!} + R_n \quad \text{where } R_n = h^{n+1}\frac{f^{(n+1)}(\xi)}{(n+1)!},$$

$\xi$ lies in the interval $[x, x+h]$

## Newton-Raphson

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

## Secant Formula

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

## Fixed Point Iteration

$x_{i+1} = g(x_i)$ such that $f(x) = 0$

## Newton-Jacobi

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad F = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{bmatrix} \qquad J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad X_{k+1} = X_k - J_k^{-1} \cdot F_k$$

## Lagrange's Polynomial

$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$ where

$$L_i(x) = \prod_{j=0; i\neq j}^{n} \frac{(x-x_j)}{(x_i - x_j)}$$

$$E_n(x) = \prod_{i=0}^{n}(x-x_i)\frac{f^{(n+1)}(\xi)}{(n+1)!}; \; x_0 < \xi < x_n$$

## Least Squares Fit

For fitting an $n^{th}$ degree polynomial of the form

$$y = C_0 x^0 + C_1 x^1 + C_2 x^2 + \ldots + C_n x^n,$$

$$\begin{bmatrix} \sum x^0 & \sum x^1 & \cdots & \sum x^n \\ \sum x^1 & \sum x^2 & \cdots & \sum x^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum x^n & \sum x^{n+1} & \cdots & \sum x^{2n} \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} \sum x^0 y \\ \sum x^1 y \\ \vdots \\ \sum x^n y \end{bmatrix}$$

## Numerical Integration

### *Trapezoidal Rule*

$$I = \int_a^b f(x)dx = (b-a)\frac{f(a)+f(b)}{2}$$

### *Simpson's Rule*

$$I = \int_a^b f(x)dx = (b-a)\frac{f(x_0)+4f(x_1)+f(x_2)}{6}$$

### *Composite Trapezoidal Rule*

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

*Truncation Error:*

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i) \text{ where } \xi_i \in (x_{i-1}, x_i), \text{ i.e.}$$

$\xi_i$ is any value within the segment.

### *Composite Simpson's Rule*

$$I = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,\cdots}^{n-1} f(x_i) + 2\sum_{j=2,4,\cdots}^{n-2} f(x_j) + f(x_n)\right]$$

*Truncation Error:*

$$E_t = -\frac{(b-a)^5}{180n^5}\sum_{i=1}^{n} f^{(4)}(\xi_i) \text{ where } \xi_i \in (x_{i-1}, x_i), \text{ i.e.}$$

$\xi_i$ is any value within the segment.

## ***Gauss Quadrature***

$$\int_a^b g(t)dt = \int_{-1}^1 f(x)dx,\ \ t = \frac{b+a}{2} + \frac{b-a}{2}x,\ \ dt = \frac{b-a}{2}dx,\ \ \int_{-1}^1 f(x)dx \approx \sum_{i=0}^n w_i f_i$$

| ***N*** | **Abscissas, $x_{N,k}$ ($x_i$)** | **Weights, $w_{N,k}$ ($w_i$)** | **Truncation Error** |
|---|---|---|---|
| 2 | -0.5773502692<br>0.5773502692 | 1.0000000000<br>1.0000000000 | $\frac{f^{(4)}(\xi)}{135}$ |
| 3 | ±0.7745966692<br>0.0000000000 | 0.5555555556<br>0.8888888888 | $\frac{f^{(6)}(\xi)}{15,750}$ |
| 4 | ±0.8611363116<br>±0.3399810436 | 0.3478548451<br>0.6521451549 | $\frac{f^{(8)}(\xi)}{3,472,875}$ |
| 5 | ±0.9061798459<br>±0.5384693101<br>0.0000000000 | 0.2369268851<br>0.4786286705<br>0.5688888888 | $\frac{f^{(10)}(\xi)}{1,237,732,650}$ |

## **Finite Difference**

| | **Forward** | **Central** | **Backward** |
|---|---|---|---|
| $\frac{dy}{dx}$ | $\frac{(y_{i+1}-y_i)}{\Delta x}$ | $\frac{(y_{i+1}-y_{i-1})}{2\Delta x}$ | $\frac{(y_i-y_{i-1})}{\Delta x}$ |
| $\frac{d^2y}{dx^2}$ | $\frac{(y_{i+2}-2y_{i+1}+y_i)}{(\Delta x)^2}$ | $\frac{(y_{i+1}-2y_i+y_{i-1})}{(\Delta x)^2}$ | $\frac{(y_i-2y_{i-1}+y_{i-2})}{(\Delta x)^2}$ |
| Error | $O(h)$ | $O(h^2)$ | $O(h)$ |

## **Ordinary Differential Equations**

| | | |
|---|---|---|
| $\frac{dy}{dx} = f(x,y)$ | ***Euler's Method*** | $y(x_{i+1}) = y(x_i) + hf(x_i, y_i)$ |
| | ***Heun's Method*** | $y(x_{i+1}) = y(x_i) + h\left(\frac{f(x_i,y_i) + f(x_{i+1}, y^*_{i+1})}{2}\right)$<br>where $y^*_{i+1} = y(x_i) + h\, f(x_i, y_i)$ |
| | ***Runge-Kutta 4 Method*** | $y(x_{i+1}) = y(x_i) + \frac{h}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$<br>where $k_1 = f(x_i, y_i)$<br>$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1)$<br>$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2)$<br>$k_4 = f(x_i + h, y_i + hk_3)$ |