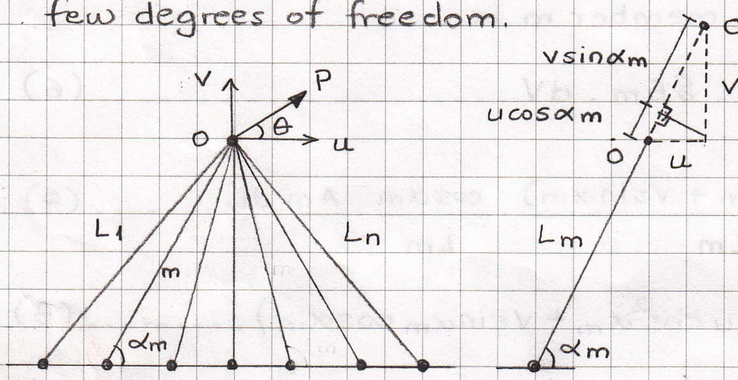
UNIT DUMMY DISPLACEMENT METHOD (UDDM)

This method is used for deformable bodies and can be written in general as

$$\underbrace{P \cdot \cancel{\delta u}_{\,1}}_{\delta W_{ext}} = \underbrace{\int_V \underline{\sigma}^T \cdot \delta \underline{\epsilon} \; dV}_{\delta W_{int}}$$

It is generally applicable to simple indeterminate structures with few degrees of freedom.



Consider the pin connected planar truss with n members. It has two degrees of freedom (u, v).

Relation between member forces & end displacements

A typical member has A_m, L_m and α_m. When the external force P is applied, joint O undergoes true displacements u & v. The member is elongated an amount

$$\Delta L_m = u\cos\alpha_m + v\sin\alpha_m \quad \text{.........} (1)$$

The true strain by definition is $(\varepsilon = \Delta L / L)$

$$\epsilon_m = \frac{u\cos\alpha_m + v\sin\alpha_m}{L_m} \quad \text{.........} (2)$$

The true stress can be defined as $(\sigma = E\varepsilon)$

$$\sigma_m = \frac{E_m (u\cos\alpha_m + v\sin\alpha_m)}{L_m} \quad \text{.........} (3)$$

One step further, the axial force in bar m in terms of end displacements u & v can be calculated as $(F = \sigma . A)$

$$F_m = \frac{E_m A_m}{L_m} \cdot (u\cos\alpha_m + v\sin\alpha_m) \quad \text{.........} (4)$$

Application of UDDM

Since there are two degrees of freedom, the method will be applied twice by emposing a unit virtual displacement once in each degree of freedom direction.

$\delta u = 1 \quad (\delta v = 0)$

$$\delta \varepsilon_m = \frac{\overset{1}{\cancel{\delta u}} \cdot \cos\alpha_m + \overset{0}{\cancel{\delta v}} \cdot \sin\alpha_m}{L_m} = \frac{\cos\alpha_m}{L_m} \quad \text{------------} (5)$$

Internal virtual work done in member m is

$$(\delta W_{int})_m = \int_V \sigma_m \cdot \delta\varepsilon_m \cdot dV \quad \text{------------} (6)$$

or

$$(\delta W_{int})_m = \int_0^{L_m} \frac{E_m (u\cos\alpha_m + v\sin\alpha_m)}{L_m} \cdot \frac{\cos\alpha_m}{L_m} \cdot A_m \, dL \quad \text{------} (7)$$

$$(\delta W_{int})_m = \frac{E_m A_m}{L_m} (u\cos^2\alpha_m + v\sin\alpha_m\cos\alpha_m) \quad \text{------} (8)$$

Total internal virtual work (δW_{int}) becomes

$$\delta W_{int} = u \cdot \sum_{m=1}^{n} \frac{E_m A_m}{L_m} \cos^2\alpha_m + v \sum_{m=1}^{n} \frac{E_m A_m}{L_m} \sin\alpha_m \cos\alpha_m \quad \text{------} (9)$$

External virtual work is equal to $\delta W_{ext} = P\cos\theta \cdot \overset{1}{\cancel{\delta u}}$

Hence $$P\cos\theta = u \sum_{m=1}^{n} \frac{E_m A_m}{L_m} \cos^2\alpha_m + v \sum_{m=1}^{n} \frac{E_m A_m}{L_m} \sin\alpha_m \cos\alpha_m \quad \text{------} (10)$$

$\delta v = 1 \quad (\delta u = 0)$

$$\delta \varepsilon_m = \frac{\overset{0}{\cancel{\delta u}} \cdot \cos\alpha_m + \overset{1}{\cancel{\delta v}} \cdot \sin\alpha_m}{L_m} = \frac{\sin\alpha_m}{L_m} \quad \text{------------} (11)$$

Again writing the internal virtual work equation,

$$(\delta W_{int})_m = \int_0^{L_m} \frac{E_m (u\cos\alpha_m + v\sin\alpha_m)}{L_m} \cdot \frac{\sin\alpha_m}{L_m} \cdot A_m \, dL \quad \text{--------} (12)$$

Hence

$$(\delta W_{int})_m = \frac{E_m A_m}{L_m} \cdot (u\cos\alpha_m \sin\alpha_m + v\sin^2\alpha_m) \quad \text{------------} (13)$$

Total internal virtual work δW_{int} becomes

$$\delta W_{int} = u \sum_{m=1}^{n} \frac{E_m A_m}{L_m} \cos\alpha_m \sin\alpha_m + v \sum_{m=1}^{n} \frac{E_m A_m}{L_m} \sin^2\alpha_m \quad \text{------} (14)$$

and applying UDDM for the second time,

$$P\sin\theta = u\sum_{m=1}^{n}\frac{E_m A_m}{L_m}\cos\alpha_m \sin\alpha_m + v\sum_{m=1}^{n}\frac{E_m A_m}{L_m}\sin^2\alpha_m \quad \text{............ (15)}$$

Now we have a system of two equations with two unknowns (u & v) which can be expressed in vector form as

$$\boxed{\underbrace{P\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}}_{\text{known load vector}} = \underbrace{\begin{bmatrix}\sum\frac{E_m A_m}{L_m}\cos^2\alpha_m & \sum\frac{E_m A_m}{L_m}\sin\alpha_m\cos\alpha_m \\ \sum\frac{E_m A_m}{L_m}\cos\alpha_m\sin\alpha_m & \sum\frac{E_m A_m}{L_m}\sin^2\alpha_m\end{bmatrix}}_{\text{known stiffness matrix}} \cdot \underbrace{\begin{bmatrix}u \\ v\end{bmatrix}}_{\text{unknown displacement vector}}}$$

Note that UDDM is not useful for solving systems with many free joints (i.e. large number of unknowns) but in general good for trusses like the above with many members meeting at one point with one load.
