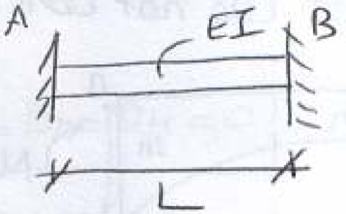
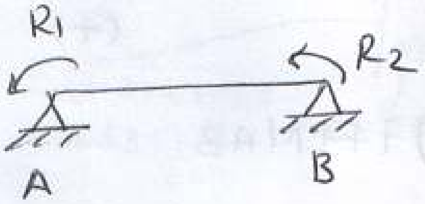
CE 425
HOMEWORK - 1
SOLUTIONS

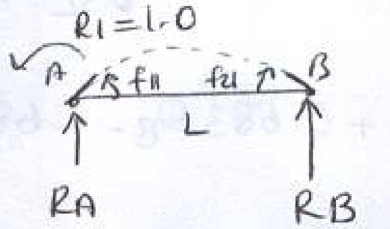
Q1) Calculate the flexibility coefficients of the given fixed-fixed beam. Take rotations at point A and B as your redundant forces. EI is constant through the length of the beam.



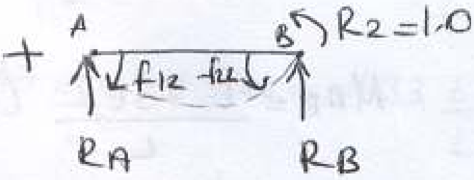
sign convention:



$\Rightarrow$



+



case (1)

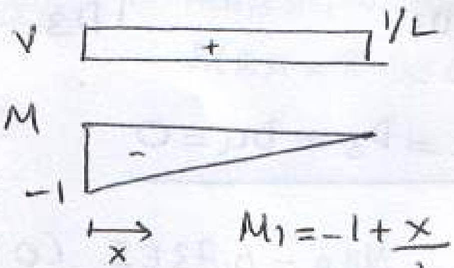
$\Sigma M_B = 0 \Rightarrow RA = 1/L$

$RB = -RA = -1/L$

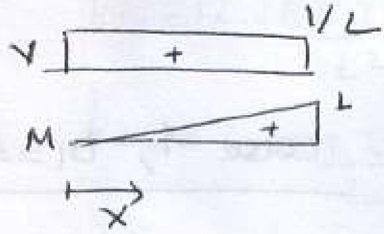
case (2)

$\Sigma M_B = 0 \Rightarrow RA = 1/L$

$RB = -RA = -1/L$



$M_1 = -1 + \frac{x}{L}$



$M_2 = \frac{x}{L}$

$$f_{11} = \frac{1}{EI}\int_0^L M_1^2 . dx = \frac{1}{EI}\int_0^L \left(\frac{x}{L}-1\right)^2 dx = \frac{1}{EI}\int_0^L \left(\frac{x^2}{L^2} - \frac{2x}{L} + 1\right) dx$$

$$= \frac{1}{EI}\left(\frac{x^3}{3L^2} - \frac{2x^2}{2L} + x\right)\Bigg|_0^L = \frac{L}{3EI}$$

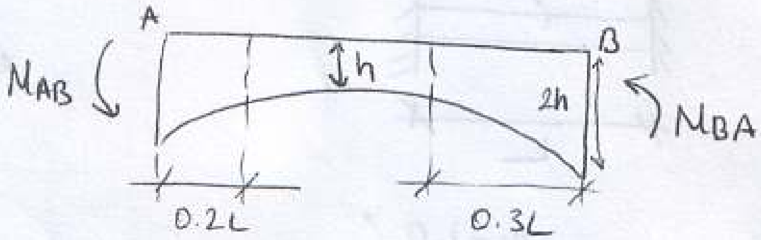
$$f_{22} = \frac{1}{EI}\int_0^L M_2^2 . dx = \frac{1}{EI}\int_0^L \frac{x^2}{L^2} dx = \frac{1}{EI}\frac{x^3}{3L^2}\Bigg|_0^L = \frac{L}{3EI}$$

$$f_{12} = f_{21} = \int_0^L \left(\frac{x}{L}-1\right)\left(\frac{x}{L}\right) dx = \int_0^L \left(\frac{x^2}{L^2} - \frac{x}{L}\right) dx = \frac{1}{EI}\left(\frac{x^3}{3L^2} - \frac{x^2}{2L}\right)\Bigg|_0^L$$

$$= -L/6EI$$

Flexibility Coefficients, $f = \frac{1}{EI}\begin{bmatrix} L/3 & -L/6 \\ -L/6 & L/3 \end{bmatrix}$

Q2) The slope deflections equations for a parabolic haunch beam are given below. Derive the element stiffness matrix for the given beam member (Do not consider axial deformations).

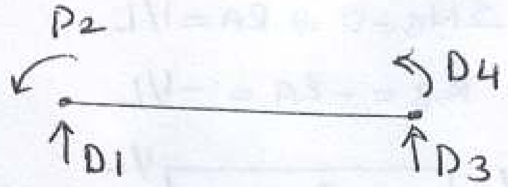


sign conv:

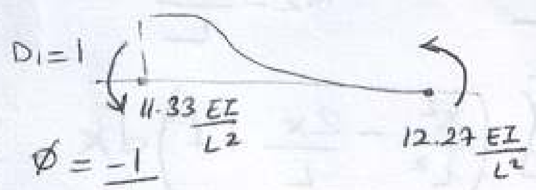
C.C.W (+)

$$M_{AB} = \frac{6.73EI}{L}\left(\theta_A + 0.683\,\theta_B - 1.683\,\phi\right) + FEM_{AB}$$

$$M_{BA} = \frac{7.68EI}{L}\left(0.598\,\theta_A + \theta_B - 1.598\,\phi\right) + FEM_{BA}$$



Case 1, $D_1 = 1$, $D_2 = D_3 = D_4 = 0$



$$M_{AB} = \frac{6.73EI}{L}\left(0 + 0 + 1.683.\frac{1}{L}\right) + \cancel{FEM_{AB}}^{\,0}$$

$$= 11.33EI/L^2$$

$$M_{BA} = \frac{7.68EI}{L}\left(0 + 0 + 1.598.\frac{1}{L}\right) + \cancel{FEM_{BA}}_{\,0}$$

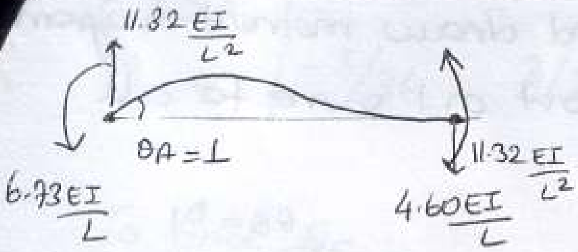
$$= 12.27EI/L^2$$

$$\Sigma M_B = 0$$

$$V_A \cdot L = \frac{(11.33 + 12.27)EI}{L^2} = \frac{23.6EI}{L^3}$$

$$V_B = \frac{-23.6EI}{L^3}$$

Case 2, $D_2 = 1$, $D_1 = D_3 = D_4 = 0$



$M_{AB} = 6.73\,EI/L$

$M_{BA} = 4.60\,EI/L$

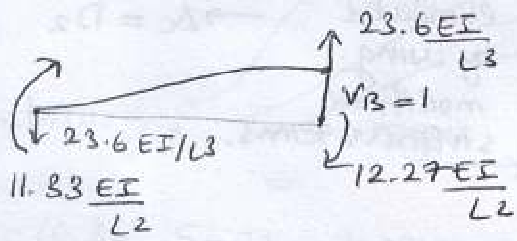
$k_{12} = 11.32\,\dfrac{EI}{L^2}$

$k_{22} = 6.73\,\dfrac{EI}{L}$

$k_{32} = -11.32\,\dfrac{EI}{L^2}$

$k_{42} = 4.60\,\dfrac{EI}{L}$

Case 3, $D_3 = 1$, $D_2 = D_1 = D_4 = 0$



$M_{AB} = -11.33\,\dfrac{EI}{L^2}$

$M_{BA} = -12.27\,\dfrac{EI}{L^2}$

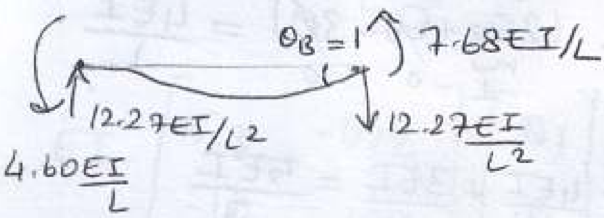
$k_{13} = -23.6\,\dfrac{EI}{L^3}$

$k_{23} = -11.33\,\dfrac{EI}{L^2}$

$k_{33} = +23.6\,\dfrac{EI}{L^3}$

$k_{43} = -12.27\,\dfrac{EI}{L^2}$

Case 4, $D_4 = 1$, $D_1 = D_2 = D_3 = 0$



$M_{AB} = 4.60\,EI/L$

$M_{BA} = 7.68\,EI/L$

$k_{14} = 12.28\,\dfrac{EI}{L^2}$

$k_{24} = 4.60\,\dfrac{EI}{L}$

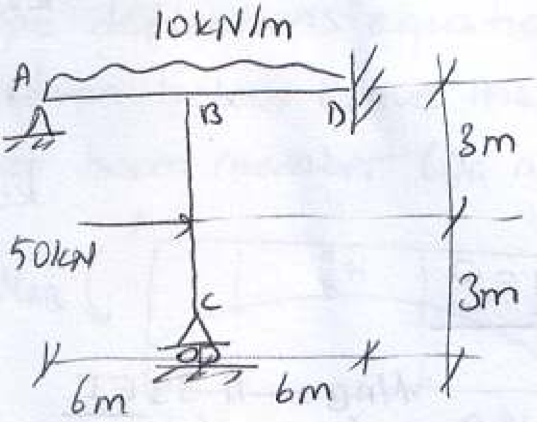
$k_{34} = -12.27\,\dfrac{EI}{L^2}$

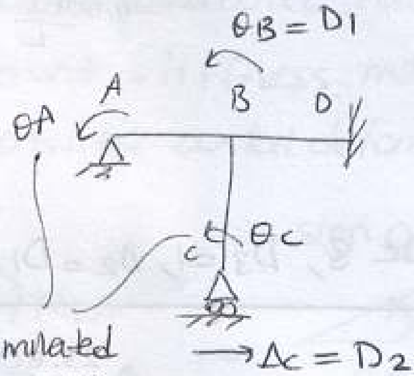
$k_{44} = 7.68\,\dfrac{EI}{L}$

$$K = EI\begin{bmatrix} 23.6/L^3 & 11.32/L^2 & -23.6/L^3 & 12.28/L^2 \\ & 6.73/L & -11.33/L^2 & 4.60/L \\ & \text{symm.} & 23.6/L^3 & -12.27/L^2 \\ & & & 7.68/L \end{bmatrix}$$

Q3) Analyze the given structures using general stiffness method. Calculate the support reactions and draw moment diagram. Assume axial rigidity. EI is constant and same for all members.

a)

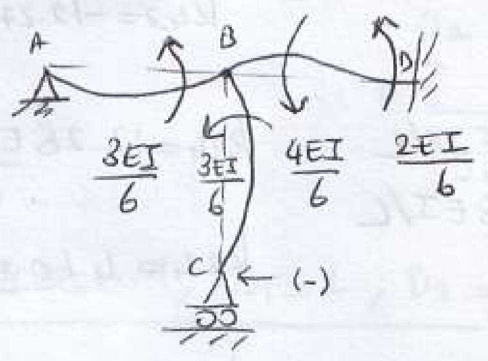




eliminated by using modified stiffness terms.

2.D.O.F ;

Case 1, $D_1 = 1$, $D_2 = 0$



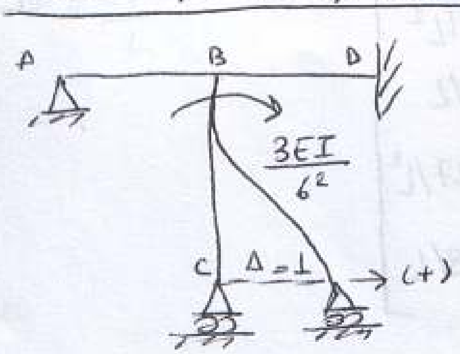
$$M_{BA} = \frac{3EI}{L}(\overset{1}{\theta_i} - \cancel{\phi}^{\,0})$$

$$M_{BD} = \frac{2EI}{L}(2\underbrace{\theta_B}_{1} + \underbrace{\theta_D}_{0} - 3\underbrace{\cancel{\phi}}_{0}) = \frac{4EI}{L}$$

$$k_{11} = \frac{3EI}{6} + \frac{4EI}{6} + \frac{3EI}{6} = \frac{5EI}{3}$$

$$k_{21} = \left(\frac{3EI}{6}\right) \cdot \frac{1}{6} = -\frac{3EI}{36}$$

Case 2, $D_2 = 1$, $D_1 = 0$



$$M_{BC} = \frac{3EI}{L}(\theta_B - \cancel{\phi})$$

$\hookrightarrow \frac{\Delta}{L} = \frac{1}{L}$ (+, c.c.w)

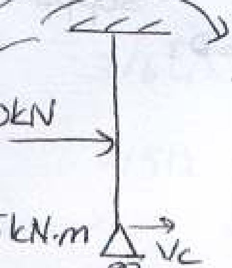
$$= \frac{-3EI}{6^2} = k_{21}$$

$$V_C = \frac{3EI}{63} = k_{22}$$

$$K = EI \begin{bmatrix} 5/3 & -3/36 \\ -3/36 & 3/216 \end{bmatrix} \qquad Q_A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$, no directly applied forces or moments

To find Q_F ;

$\frac{wL^2}{8} = \frac{10.6^2}{8}$ $\qquad \frac{10.6^2}{12}$ $\qquad \frac{10.6^2}{12}$



combining,

$$\frac{P.a.b(L+a)}{2L^2} = \frac{50.3.3(9)}{2.36} = 56,25\,kNm$$

$$\frac{10.36}{12} - \frac{10.36}{8} - 56.25 = -71.25\,kN.m \qquad V_C = \frac{56.25 - 50.3}{6} = -15,63\,kN.m$$

$$Q = Q_A - Q_F = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -71.25 \\ -15.63 \end{bmatrix} = \begin{bmatrix} 71.25 \\ 15.63 \end{bmatrix}$$

$$EI. \begin{bmatrix} 5/3 & -1/12 \\ -1/12 & 1/72 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 71.25 \\ 15.63 \end{bmatrix} \Rightarrow \begin{array}{l} D_1 = 141.45/EI \\ D_2 = 1974.09/EI \end{array}$$

Considering FEM forces, back substitution gives moments & forces,

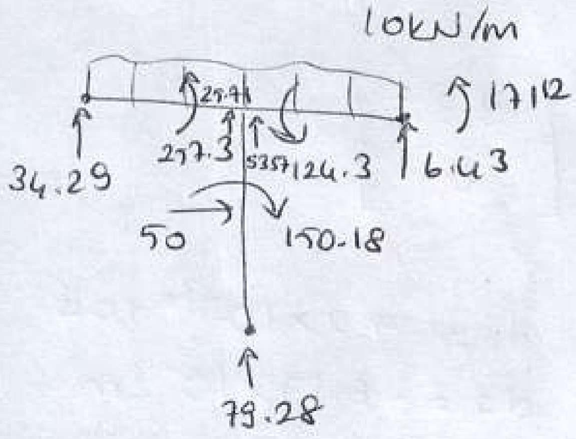
$$M_{BA} = \frac{3EI}{L}.(\underbrace{\theta_i}_{\frac{141.45}{EI}}) + FEM_{BA} = 70.73 - 45 = 25.73\,kN.m$$

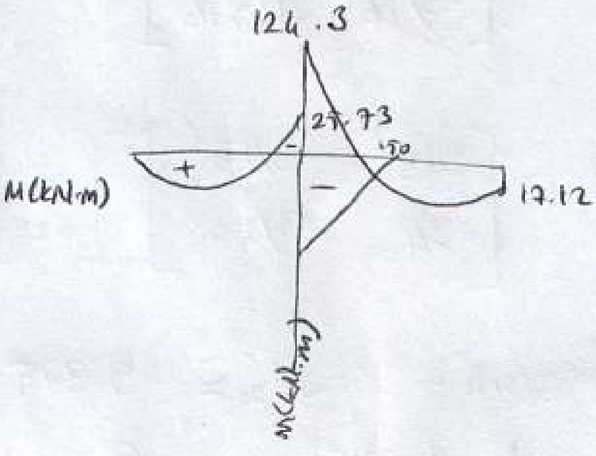
$$M_{BD} = \frac{2EI}{L}\left(2.\frac{141.45}{EI}\right) + 30 = 124.3\,kN.m$$

$$M_{BC} = \frac{3EI}{L}\left(\frac{141.45}{EI} - \frac{1974.09}{EI.6}\right) - 56.25 = -150.18\,kN.m$$

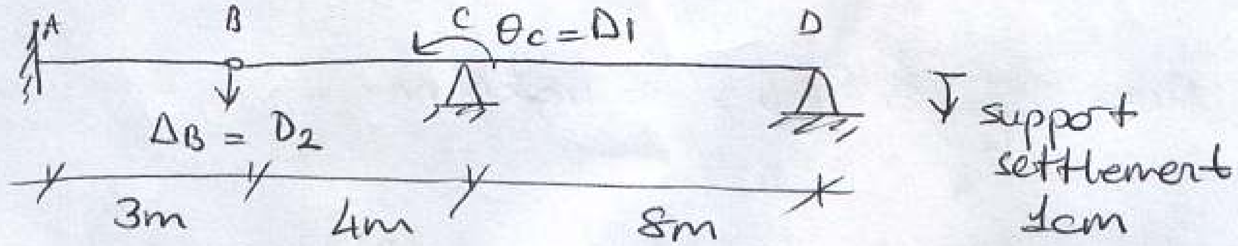
$$V_A = \frac{10.6.3 + 25.72}{6} = 34.29\,kN \qquad V_D = \frac{10.6.3 - 17.14 - 124.29}{6} = 6.43\,kN$$

$Y_C = 10.12 - 34.29 - 6.43 = 79.28 \text{ kN}$

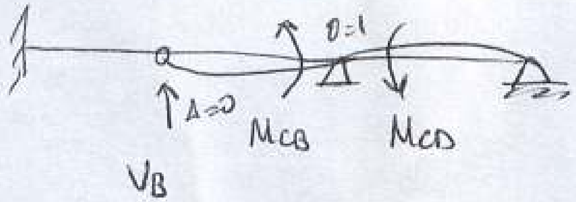




b)



case 1, $D_1 = 1$, $D_2 = 0$



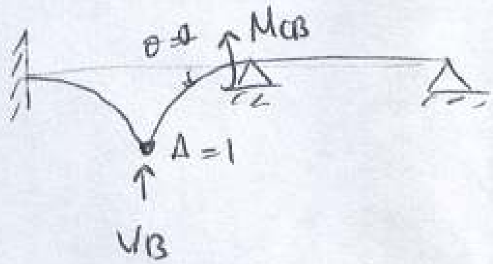
$$M_{CB} = \frac{3EI}{L} = \frac{3EI}{4}$$

$$M_{CD} = \frac{3EI}{L} = \frac{3EI}{8}$$

$$k_{11} = \frac{3EI}{4} + \frac{3EI}{8} = \frac{9EI}{8}$$

$$V_B = \frac{3EI}{16}, \quad k_{12} = \frac{-3EI}{16}$$

case 2, $D_2 = 1$, $D_1 = 0$

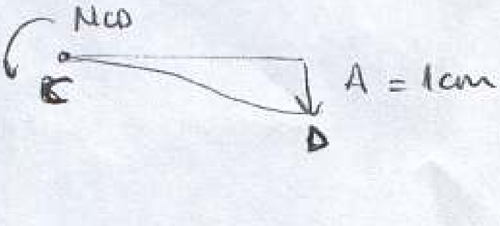


$$M_{CB} = \frac{3EI}{L}\left(\theta_C - \frac{\Delta}{L}\right)$$

$\frac{\Delta}{L} \rightarrow 1/4$ (+, c.c.w)

$$= \frac{-3EI}{16} = k_{21}, \quad M_{AB} = \frac{EI}{3}$$

$$V_B = \frac{3EI}{16} \cdot \frac{1}{4} + \frac{EI}{3} \cdot \frac{1}{3} = \frac{91\,EI}{576} = k_{22}$$



$$M_{CD} = \frac{3EI}{8} \cdot \left(+\frac{0{,}01}{8}\right) = \frac{3EI}{6400}$$

$$= \frac{3 \cdot 20.000}{6400}$$

$$= 9.375 \text{ kN·m}$$

$$K = EI \begin{bmatrix} 9/8 & -3/16 \\ -3/16 & 91/576 \end{bmatrix} \qquad Q_A - Q_F = \begin{bmatrix} -9.375 \\ 0 \end{bmatrix}$$

$$20.000 \begin{bmatrix} 9/8 & -3/16 \\ -3/16 & 91/576 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -9.375 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 22500 d_1 - 3750 d_2 &= -9.375 \\ -3750 d_1 + 3160 d_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} d_1 &= -5.2 \times 10^{-4} \text{ rad} \\ d_2 &= -6.17 \times 10^{-4} \text{ m} \end{aligned}$$

$$M_{CB} = \underset{4}{\frac{3EI}{L}} \left(\overset{-5.2\times10^{-4}}{\theta_C} - \underset{4}{\frac{\overset{-6.17\times10^{-4}}{\Delta}}{L}} \right) = -5.48 \text{ kN.m}$$

$$M_{AB} = \frac{3EI}{3} \left(0 + \frac{6.17 \times 10^{-4}}{3} \right) = 4.13 \text{ kN.m}$$