Handout 3 - Fixed Point Iteration

Fixed Point Iteration formula is obtained by rearranging $f(x)=0$ as $x=g(x)$. Consider $f(x)=x-\sqrt{2x}$, by rearranging the function $f(x)$ one can obtain $x=g(x)$ as $x=g(x)=\sqrt{2x}$ with an initial guess of $x=0.1$. A sample MATLAB Code that can be used to find the root of $f(x)$ is given below. Error is defined as: $\text{Error}=\left|\frac{x_{new}-x_{old}}{x_{new}}\right|*100$:

```
clear all
clc
tol=5*10^-3;
err=tol+1;
x=0.1;
while err>=tol
    xnew=sqrt(2*x);
    err=abs((xnew-x)/xnew)*100;
    x=xnew;
end
```

Fixed point iteration is convergent when $|g'(x)|<1$. Type of convergence or divergence varies according to the initial guess and the value of $g'(x)$ in the concerned interval.

Monotonic Convergence
$0<g'(x)<1$

$f(x)=x-\sqrt{2x}=0$

$x=g(x)=\sqrt{2x}$

Initial guess of $x_0=0.5$

$g'(x)=\frac{\sqrt{2}}{2}x^{-0.5}$

$\frac{\sqrt{2}}{2}x^{-0.5}<1$ when $x>0.5$

k	Xk	g(Xk)	Error%	k	Xk	g(Xk)	Error%
0	0.500000	1.000000	1.005000	11	1.998647	1.999323	0.067667
1	1.000000	1.414214	50.000000	12	1.999323	1.999662	0.033839
2	1.414214	1.681793	29.289322	13	1.999662	1.999831	0.016921
3	1.681793	1.834008	15.910358	14	1.999831	1.999915	0.008461
4	1.834008	1.915207	8.299596	15	1.999915	1.999958	0.004231
5	1.915207	1.957144	4.239672				
6	1.957144	1.978456	2.142794				
7	1.978456	1.989199	1.077199				
8	1.989199	1.994592	0.540058				
9	1.994592	1.997294	0.270394				
10	1.997294	1.998647	0.135289				

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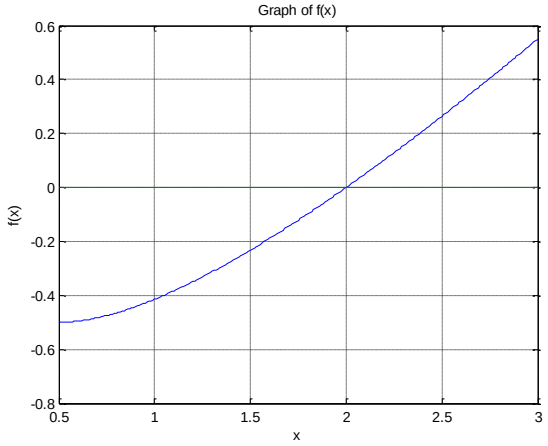

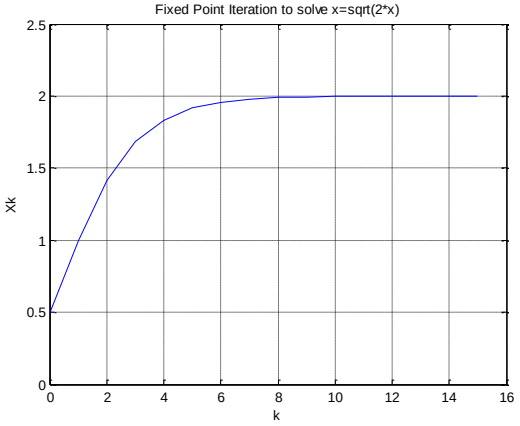

Oscillatory Convergence
$-1 < g'(x) < 0$

$f(x) = x - \cos(x) = 0$

$x = g(x) = \cos(x)$

Initial guess of $x_0 = 0.5$

$g'(x) = -\sin(x)$

$-1 < \sin(x) < 0$ when $0 < x < \pi / 2$

k	Xk	g(Xk)	Error%	k	Xk	g(Xk)	Error%
0	0.500000	0.877583	1.005000	16	0.738705	0.739341	0.127984
1	0.877583	0.639012	43.025304	17	0.739341	0.738912	0.086146
2	0.639012	0.802685	37.334179	18	0.738912	0.739201	0.058059
3	0.802685	0.694778	20.390637	19	0.739201	0.739007	0.039096
4	0.694778	0.768196	15.531158	20	0.739007	0.739138	0.026341
5	0.768196	0.719165	9.557173	21	0.739138	0.739050	0.017741
6	0.719165	0.752356	6.817678	22	0.739050	0.739109	0.011952
7	0.752356	0.730081	4.411518	23	0.739109	0.739069	0.008050
8	0.730081	0.745120	3.050989	24	0.739069	0.739096	0.005423
9	0.745120	0.735006	2.018369	25	0.739096	0.739078	0.003653
10	0.735006	0.741827	1.376047				
11	0.741827	0.737236	0.919381				
12	0.737236	0.740330	0.622704				
13	0.740330	0.738246	0.417912				
14	0.738246	0.739650	0.282211				
15	0.739650	0.738705	0.189782				

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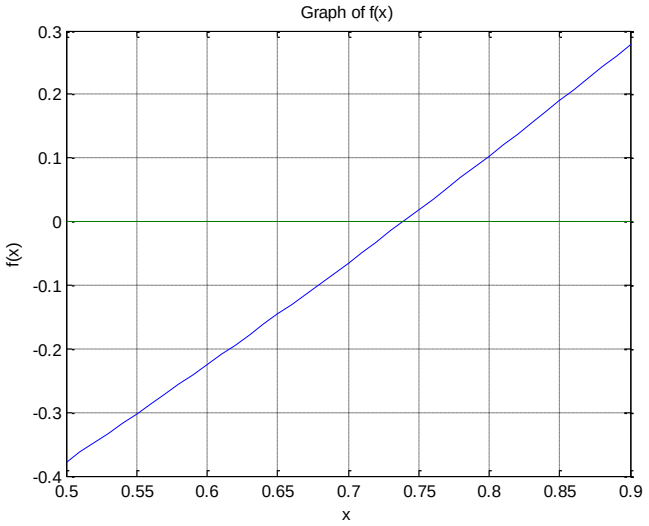

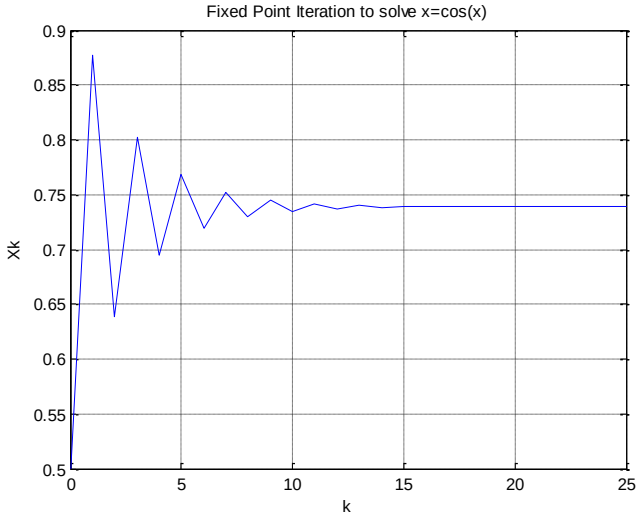

Monotonic Divergence
$1 < g'(x)$

$$f(x) = x - \frac{x^2}{4} - \frac{x}{2}$$

$$x = g(x) = \frac{x^2}{4} + \frac{x}{2}$$

Initial guess of $x_0 = 1.95$

$$g'(x) = \frac{x+1}{2}$$

$$\frac{x+1}{2} > 1 \text{ when } x > 1$$

k	Xk	g(Xk)	Error%	k	Xk	g(Xk)	Error%
0	1.950000	1.925625	1.005000	11	0.406036	0.244234	52.676386
1	1.925625	1.889820	1.265823	12	0.244234	0.137030	66.248576
2	1.889820	1.837766	1.894603	13	0.137030	0.073209	78.234529
3	1.837766	1.763228	2.832511	14	0.073209	0.037944	87.175695
4	1.763228	1.658858	4.227317	15	0.037944	0.019332	92.937608
5	1.658858	1.517381	6.291719	16	0.019332	0.009760	96.276206
6	1.517381	1.334302	9.323741	17	0.009760	0.004904	98.085292
7	1.334302	1.112241	13.720978	18	0.004904	0.002458	99.028788
8	1.112241	0.865391	19.965149	19	0.002458	0.001230	99.510842
9	0.865391	0.619921	28.524747	20	0.001230	0.000616	99.754522
10	0.619921	0.406036	39.597023				

Note that these iterations are the first twenty iterations.

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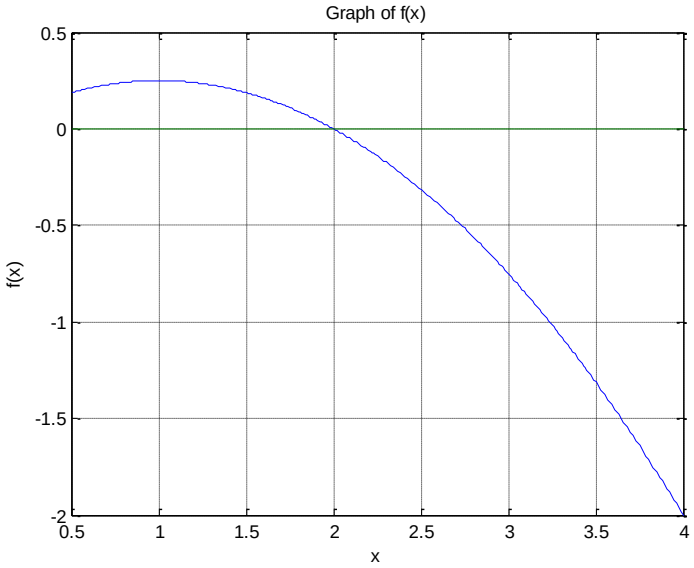

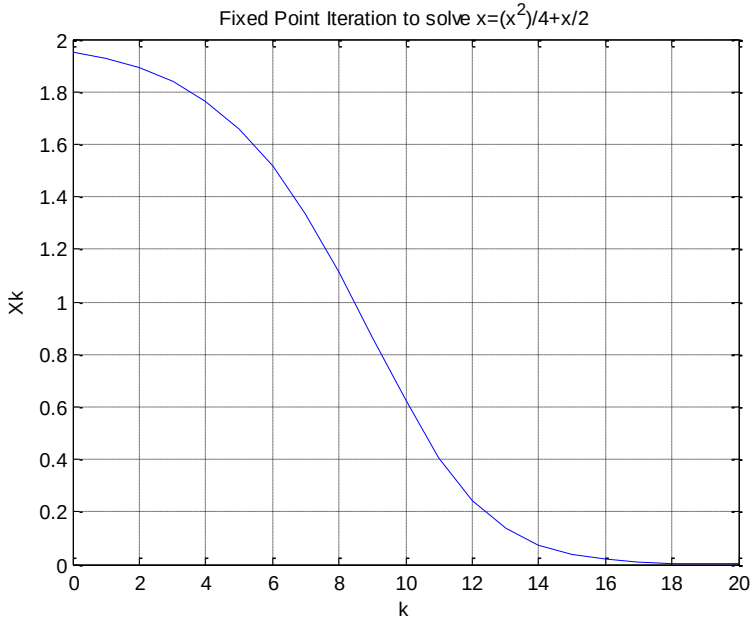

Oscillatory Divergence
$g'(x) < -1$

$f(x) = x - e^{-x} + 5 = 0$

$x = g(x) = e^{-x} - 5$

Initial guess of $x_0 = -2$

$g'(x) = -e^{-x}$

$-e^{-x} < -1$ when $x < 0$

k	Xk	g(Xk)	Error%
0	-2.000000	2.389056	1.005000
1	2.389056	-4.908284	183.715071
2	-4.908284	130.406828	148.673960
3	130.406828	-5.000000	103.763824
4	-5.000000	143.413159	2708.136556
5	143.413159	-5.000000	103.486430
6	-5.000000	143.413159	2968.263182
7	143.413159	-5.000000	103.486430
8	-5.000000	143.413159	2968.263182
9	143.413159	-5.000000	103.486430
10	-5.000000	143.413159	2968.263182

Note that these iterations are the first ten iterations.

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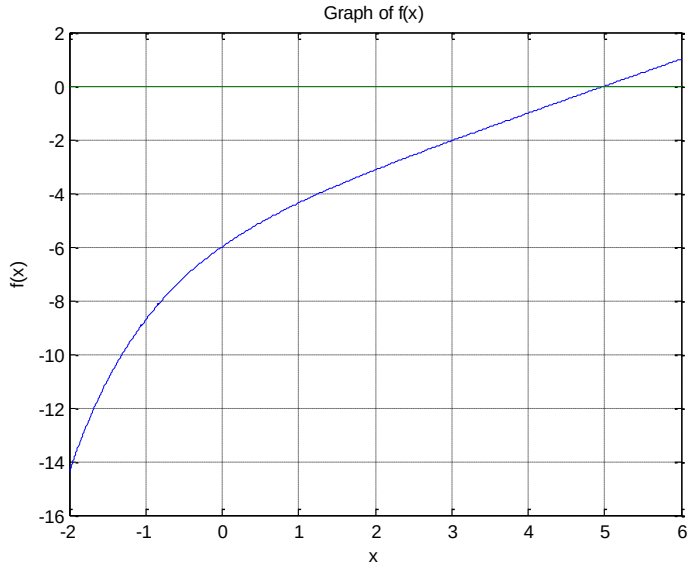

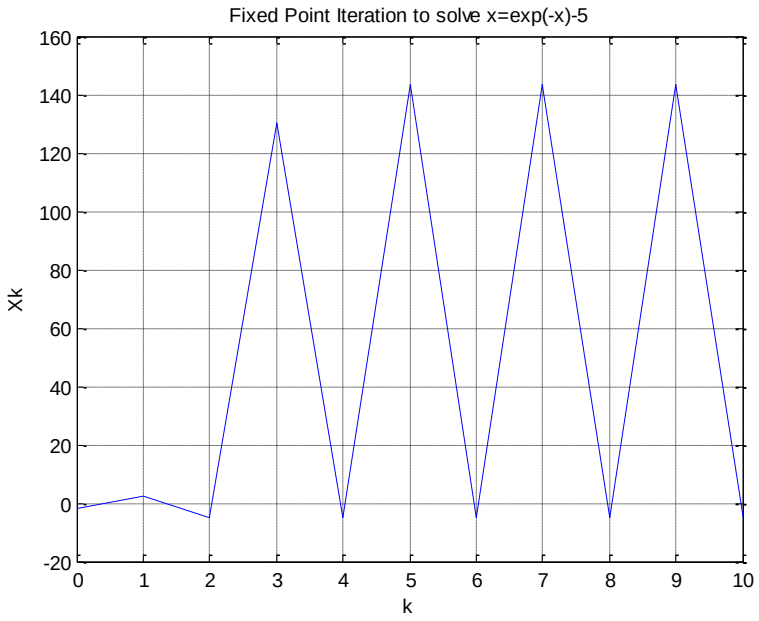

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