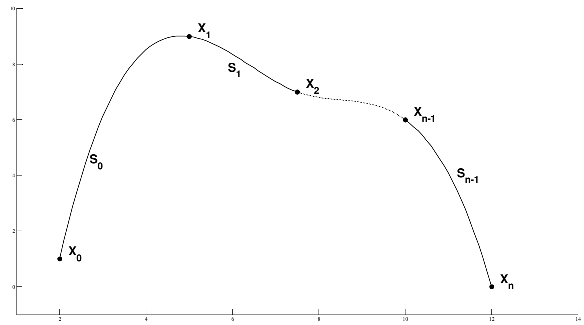
In numerical analysis, a spline function is defined as polynomial pieces on subintervals joined together with certain continuity conditions. Cubic splines are a special form of spline functions where the objective is to derive a third-order polynomial for each interval between knots, as in



$$s_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$$

Formal Definition of Cubic Splines

Suppose that $f \in C[a,b]$ and let $K = \{x_0, x_1, ..., x_m\}$ be a set of $m+1$ knots in the interval $[a,b]$, $a = x_0 < x_1 < ... < x_m = b$. Consider the set S of all functions $s \in C^2[a,b]$ such that

1. $s(x_i) = f(x_i),\ i = 0,1,...,m$
2. s is a cubic polynomial on $[x_{i-1}, x_i], i = 1,2,...,m.$

Conditions Required to Evaluate Unknown Constants of the Cubic Spline Equations

For $n+1$ data points $(i = 0,1,2,3,...,n)$, there are n intervals, thus, n cubic spline equations $(i = 0,1,2,3,...,n-1)$, and $4n$ unknown constants in the cubic spline equations. Consequently, $4n$ conditions are required to evaluate functions. These are:

1. The function values must be equal at the interior knots ($2n-2$ conditions).
$$s_i(x_{i+1}) = y_{i+1} = s_{i+1}(x_{i+1}) \text{ where } i = 0,1,2,...,n-2$$
2. The first and last functions must pass through the end points (2 conditions).
$$s_0(x_0) = y_0 \text{ and } s_{n-1}(x_n) = y_n$$
3. The first derivatives at the interior knots must be equal. ($n-1$ conditions).
$$s_i'(x_{i+1}) = s_{i+1}'(x_{i+1}) \text{ where } i = 0,1,2,...,n-2$$
4. The second derivatives at the interior knots must be equal ($n-1$ conditions).
$$s_i''(x_{i+1}) = s_{i+1}''(x_{i+1}) \text{ where } i = 0,1,2,...,n-2$$
5. The second derivatives at the end knots are zero (2 conditions).
$$s_0''(x_0) = s_{n-1}''(x_n) = 0$$

Derivation Of Unknown Constants of the Cubic Spline Equations for Three Data Points

A cubic spline function $s_i(x)$ can be defined as $s_i(x)=a_ix^3+b_ix^2+c_ix+d_i$ in a sub interval $x\in\left[x_i,x_{i+1}\right]$. Note that second derivative of $s_i(x)$ is a linear function which is $s_i''(x)=6a_ix+2b_i$, thus, it can be represented by a first-order Lagrange interpolating function:

$$s_i''(x)=\frac{x-x_{i+1}}{x_i-x_{i+1}}\,s_i''(x_i)+\frac{x-x_i}{x_{i+1}-x_i}\,s_i''(x_{i+1})\ \text{ for }\ i=0,1,2,\ldots,n-1 \qquad \textbf{[1]}$$

where $s_i''(x)$ is the value of the second derivative at any point x within the $i-th$ interval.

Integrating $s_i''(x)$ twice, Equation [2] can be obtained as follows:

$$s_i(x)=\frac{(x_{i+1}-x)^3}{6(x_{i+1}-x_i)}\,s_i''(x_i)+\frac{(x-x_i)^3}{6(x_{i+1}-x_i)}\,s_i''(x_{i+1})+\alpha_i(x-x_i)+\beta_i(x_{i+1}-x)\ \text{ for }\ i=0,1,2,\ldots,n-1 \qquad \textbf{[2]}$$

where α_i and β_i are constants of integrations.

Equating $s_i(x_i)$ with data points y_i at the knots x_i and x_{i+1}, i.e. $s_i(x_i)=y_i$ and $s_i(x_{i+1})=y_{i+1}$, yields

$$y_i=\frac{1}{6}\,s_i''(x_i)(x_{i+1}-x_i)^2+(x_{i+1}-x_i)\beta_i\ \text{ for }\ i=0,1,2,\ldots,n-1 \qquad \textbf{[3]}$$

$$y_i=\frac{1}{6}\,s_i''(x_{i+1})(x_{i+1}-x_i)^2+(x_{i+1}-x_i)\alpha_i\ \text{ for }\ i=0,1,2,\ldots,n-1 \qquad \textbf{[4]}$$

Inserting α_i and β_i obtained from Equations [3] & [4] into Equation [2], Equation [5] can be obtained:

$$s_i(x)=\frac{(x_{i+1}-x)^3}{6(x_{i+1}-x_i)}\,s''(x_i)+\frac{(x-x_i)^3}{6(x_{i+1}-x_i)}\,s''(x_{i+1})+\left[\frac{s(x_i)}{(x_{i+1}-x_i)}-\frac{s''(x_i)(x_{i+1}-x_i)}{6}\right](x_{i+1}-x)+$$
$$\left[\frac{s(x_{i+1})}{(x_{i+1}-x_i)}-\frac{s''(x_{i+1})(x_{i+1}-x_i)}{6}\right](x-x_i)\ \text{ for }\ i=0,1,2,\ldots,n-1 \qquad \textbf{[5]}$$

Finally, taking first derivative of $s_i(x)$ and employing third condition which is $s_i'(x_{i+1}) = s_{i+1}'(x_{i+1})$ Equation [6] is obtained:

$$\begin{aligned}&(x_{i+1}-x_i)\,s''(x_i) + 2\,(x_{i+2}-x_i)\,s''(x_{i+1}) + (x_{i+2}-x_{i+1})\,s''(x_{i+2}) = \\ &\frac{6}{(x_{i+2}-x_{i+1})}\left[s(x_{i+2})-s(x_{i+1})\right] + \frac{6}{(x_{i+1}-x_i)}\left[s(x_i)-s(x_{i+1})\right]\end{aligned} \qquad \text{for } i=0,1,2,\ldots,n-2 \qquad \textbf{[6]}$$

When Equation [6] is written for $i=0,1,2,\ldots,n-2$; $n-1$ equations will be obtained and there will be $n+1$ unknowns in these equations. By the use of boundary conditions given by $s''(x_0)=s''(x_n)=0$, number of unknowns decrease to $n-1$. These $n-1$ equations construct a system of linear equations which can be denoted by $Ax=b$. The coefficient matrix (A) of the system is tridiagonal matrix. Tridiagonal matrix is a matrix that has nonzero elements only on the main diagonal, the first diagonal below this and the second diagonal above the main diagonal that can be shown as follows:

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_1 & \ddots & \vdots \\ 0 & c_1 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & c_{n-1} & a_n \end{pmatrix}$$

Since the coefficient matrix is tridiagonal, solving the system is much more easier and less time consuming than the full matrix.

Example: Coastal engineers mostly deal with waves usually generated by winds on the sea level. One of the most popular wind velocity database ECMWF (European Centre for Medium-Range Weather Forecasts) provides wind velocity data with 6 hours periods. On the other hand, coastal engineers usually need hourly wind velocity data. To obtain the hourly wind velocity data, cubic splines are used.

Using the following wind data given below and cubic splines, find wind velocity when $x=3,8,12,16$ and 22 hours.

i	**Time – x (hours)**	**Wind velocity – U (m/s)**
0	0	9.84776
1	6	8.86667
2	12	9.84860
3	18	9.17170
4	24	4.94701

Solution:

Equations [7], [8] and [9] can be written by inserting $i=0,1,2$ into Equation [6]:

$i=0$

$$(x_1-x_0)s''(x_0)+2(x_2-x_0)s''(x_1)+(x_2-x_1)s''(x_2)= \frac{6}{(x_2-x_1)}\left[s(x_2)-s(x_1)\right]+\frac{6}{(x_1-x_0)}\left[s(x_0)-s(x_1)\right] \quad \textbf{[7]}$$

$i=1$

$$(x_2-x_1)s''(x_1)+2(x_3-x_1)s''(x_2)+(x_3-x_2)s''(x_3)= \frac{6}{(x_3-x_2)}\left[s(x_3)-s(x_2)\right]+\frac{6}{(x_2-x_1)}\left[s(x_1)-s(x_2)\right] \quad \textbf{[8]}$$

$i=2$

$$(x_3-x_2)s''(x_2)+2(x_4-x_2)s''(x_3)+(x_4-x_3)s''(x_4)= \frac{6}{(x_4-x_3)}\left[s(x_4)-s(x_3)\right]+\frac{6}{(x_3-x_2)}\left[s(x_2)-s(x_3)\right] \quad \textbf{[9]}$$

Inserting the following values and boundary conditions $s''(x_0)=s''(x_4)=0$ into Equations [7], [8] and [9],

$x_0=$	0	$s(x_0)=U_0=$	9.84776	$x_1=$	6	$s(x_1)=U_1=$	8.86667
$x_2=$	12	$s(x_2)=U_2=$	9.84860	$x_3=$	18	$s(x_3)=U_3=$	9.17170
$x_4=$	24	$s(x_4)=U_4=$	4.94701				

the system of linear equations can be arranged as $Ax=b$.

$$\begin{bmatrix} 24 & 6 & 0 \\ 6 & 24 & 6 \\ 0 & 6 & 24 \end{bmatrix}\begin{bmatrix} s''(x_1) \\ s''(x_2) \\ s''(x_3) \end{bmatrix}=\begin{bmatrix} 1.9630 \\ -1.6588 \\ -3.5478 \end{bmatrix}$$

Observe that A is a **tridiagonal matrix**.

Solving for x, the unknowns can be found as $\begin{bmatrix} s''(x_1) \\ s''(x_2) \\ s''(x_3) \end{bmatrix}=\begin{bmatrix} 0.0968 \\ -0.0601 \\ -0.1328 \end{bmatrix}$.

Finally, cubic spline functions can be written by inserting $i=0,1,2$ and 3 into Equation [5]:

$i=0$

$$s_0(x)=\frac{(x_1-x)^3}{6(x_1-x_0)}s''(x_0)+\frac{(x-x_0)^3}{6(x_1-x_0)}s''(x_1)+\left[\frac{s(x_0)}{(x_1-x_0)}-\frac{s''(x_0)(x_1-x_0)}{6}\right](x_1-x)+\left[\frac{s(x_1)}{(x_1-x_0)}-\frac{s''(x_1)(x_1-x_0)}{6}\right](x-x_0) \quad \textbf{[8]}$$

$i=1$

$$s_1(x)=\frac{(x_2-x)^3}{6(x_2-x_1)}s''(x_1)+\frac{(x-x_1)^3}{6(x_2-x_1)}s''(x_2)+\left[\frac{s(x_1)}{(x_2-x_1)}-\frac{s''(x_1)(x_2-x_1)}{6}\right](x_2-x)+\left[\frac{s(x_2)}{(x_2-x_1)}-\frac{s''(x_2)(x_2-x_1)}{6}\right](x-x_1) \quad [9]$$

$i=2$

$$s_2(x)=\frac{(x_3-x)^3}{6(x_3-x_2)}s''(x_2)+\frac{(x-x_2)^3}{6(x_3-x_2)}s''(x_3)+\left[\frac{s(x_2)}{(x_3-x_2)}-\frac{s''(x_2)(x_3-x_2)}{6}\right](x_3-x)+\left[\frac{s(x_3)}{(x_3-x_2)}-\frac{s''(x_3)(x_3-x_2)}{6}\right](x-x_2) \quad [10]$$

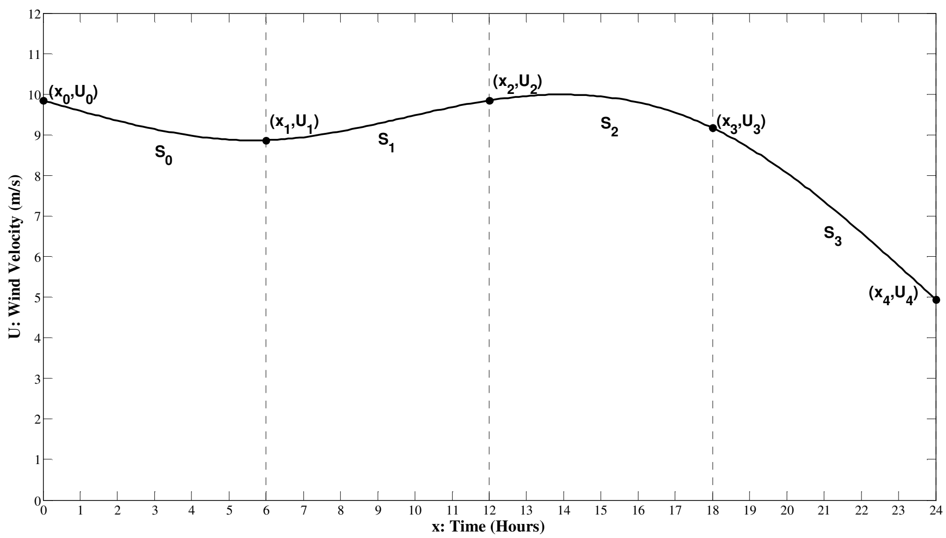
$i=3$

$$s_3(x)=\frac{(x_4-x)^3}{6(x_4-x_3)}s''(x_3)+\frac{(x-x_3)^3}{6(x_4-x_3)}s''(x_4)+\left[\frac{s(x_3)}{(x_4-x_3)}-\frac{s''(x_3)(x_4-x_3)}{6}\right](x_4-x)+\left[\frac{s(x_4)}{(x_4-x_3)}-\frac{s''(x_4)(x_4-x_3)}{6}\right](x-x_3) \quad [11]$$

Using Equations [8]-[11], wind velocity at $x=3,8,12,16$ and 22 can be found as:

Time – x (hours)	Wind velocity – U (m/s)
3	$s_0(3)=9.13936$
8	$s_1(8)=9.08571$
12	$s_1(12)=s_2(12)=9.84860$
16	$s_2(16)=9.79932$
22	$s_3(22)=6.59132$

After finding cubic spline functions, Wind Velocity vs Time graph can be plotted as follows:



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