1 b)

Displacement response of an undamped SDOF system to a force function p(t), using Duhamel's Integral is given by,

$$u(t) := \frac{1}{m \cdot \omega_n} \cdot \int_0^t p(\tau) \cdot \sin\left[\omega_n \cdot (t - \tau)\right] d\tau$$

Note that Duhamel's Integral assumes that the initial conditions are zero. If the initial conditions are different than zero, resulting free vibration response should be added to the above displacement response.

$$p(t) := p_0 \cdot \frac{t}{t_d} \qquad t \le t_d \qquad \text{Forced vibration phase}$$

$$p(t) := 0 \qquad t > t_d \qquad \text{Free vibration phase}$$

Forced Vibration Phase

$$u(t) := \frac{1}{m \cdot \omega_n} \cdot \int_0^t p_0 \cdot \frac{\tau}{t_d} \cdot \sin\left[\omega_n \cdot (t - \tau)\right] d\tau$$

$$u(t) := \frac{p_0}{m \cdot \omega_n \cdot t_d} \cdot \int_0^t \tau \cdot \sin\left[\omega_n \cdot (t - \tau)\right] d\tau$$

Using Mathcad,

Symbolics, then Evaluate, then Evaluate Symbolically

$$u(t) := \frac{p_0}{m \cdot \omega_n \cdot t_d} \cdot \left[\frac{1}{\omega_n} \cdot t - \frac{\sin(\omega_n \cdot t)}{(\omega_n)^2}\right]$$

$$u(t) := \frac{p_0}{m \cdot (\omega_n)^2} \cdot \left(\frac{t}{t_d} - \frac{\sin(\omega_n \cdot t)}{\omega_n \cdot t_d}\right)$$

$$u(t) := \frac{p_0}{k} \cdot \left(\frac{t}{t_d} - \frac{\sin(\omega_n \cdot t)}{\omega_n \cdot t_d}\right)$$ which is same as calculated in Part a.

Free Vibration Phase

First Alternative;

$$u(t) := \left[\frac{1}{m\cdot\omega_n}\cdot\int_0^{t_d} p_0\cdot\frac{\tau}{t_d}\cdot\sin\left[\omega_n\cdot(t-\tau)\right]\,d\tau + \frac{1}{m\cdot\omega_n}\cdot\int_{t_d}^{t} 0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]\,d\tau \right]$$

Using Mathcad,

Symbolics, then Evaluate, then Evaluate Symbolically

$$u(t) := \frac{1}{m\cdot\omega_n}\cdot\left[p_0\cdot\frac{-\sin(t_d\cdot\omega_n)\cdot\cos(\omega_n\cdot t) + \cos(t_d\cdot\omega_n)\cdot\sin(\omega_n\cdot t) + t_d\cdot\omega_n\cdot\cos(t_d\cdot\omega_n)\cdot\cos(\omega_n\cdot t) + t_d\cdot\omega_n\cdot\sin(t_d\cdot\omega_n)\cdot\sin(\omega_n\cdot t)}{(\omega_n)^2\cdot t_d} - \frac{\sin(\omega_n\cdot t)}{(\omega_n)^2}\cdot\frac{p_0}{t_d}\right]$$

$$u(t) := \frac{p_0}{m\cdot(\omega_n)^2}\cdot\left(\frac{-\sin(t_d\cdot\omega_n)\cdot\cos(\omega_n\cdot t) + \cos(t_d\cdot\omega_n)\cdot\sin(\omega_n\cdot t) + t_d\cdot\omega_n\cdot\cos(t_d\cdot\omega_n)\cdot\cos(\omega_n\cdot t) + t_d\cdot\omega_n\cdot\sin(t_d\cdot\omega_n)\cdot\sin(\omega_n\cdot t)}{\omega_n\cdot t_d} - \frac{\sin(\omega_n\cdot t)}{\omega_n\cdot t_d}\right)$$

$$u(t) := \frac{p_0}{k}\cdot\left[\frac{\sin\left[\omega_n\cdot(t-t_d)\right] + t_d\cdot\omega_n\cdot\cos\left[\omega_n\cdot(t-t_d)\right]}{\omega_n\cdot t_d} - \frac{\sin(\omega_n\cdot t)}{\omega_n\cdot t_d}\right]$$

$$u(t) := \frac{p_0}{k}\cdot\left[\cos\left[\omega_n\cdot(t-t_d)\right] + \frac{1}{\omega_n\cdot t_d}\cdot\sin\left[\omega_n\cdot(t-t_d)\right] - \frac{1}{\omega_n\cdot t_d}\cdot\sin(\omega_n\cdot t)\right]$$

which is same as calculated in Part a.

Second Alternative;

$$u(t) := \frac{1}{m\cdot\omega_n}\cdot\int_{t_d}^{t} 0\cdot\sin\left[\omega_n\cdot(t-\tau)\right]\,d\tau + u(\text{free vibration})$$

$$u(t) := u(\text{free vibration})$$

u(free vibration) was calculated in 1a, therefore

$$u(t) := \frac{p_0}{k}\cdot\left[\cos\left[\omega_n\cdot(t-t_d)\right] + \frac{1}{\omega_n\cdot t_d}\cdot\sin\left[\omega_n\cdot(t-t_d)\right] - \frac{1}{\omega_n\cdot t_d}\cdot\sin(\omega_n\cdot t)\right]$$