Newton Jacobi Method for Non-Linear Systems of Equations

Recall Newton-Raphson Method for a single nonlinear equation $f(x)=0$, that may be derived from Taylor's expansion of $f(x_{k+1})$ about an approximation x_k of the root of $f(x)=0$ at the k'th iteration as:

$$f(x_{k+1}) = f(x_k) + (x_{k+1}-x_k)\frac{df}{dx}\Big|_{x_k} + O\left(\left|x_{k+1}-x_k\right|^2\right) \Rightarrow x_{k+1} \cong x_k - \frac{f(x_k)}{f'(x_k)}$$

Now for n nonlinear equations of the form:

$$\begin{aligned} &f_1(x_1, x_2, x_3, \dots, x_n) = 0 \\ &f_2(x_1, x_2, x_3, \dots, x_n) = 0 \\ &. \\ &. \\ &. \\ &f_n(x_1, x_2, x_3, \dots, x_n) = 0 \end{aligned}$$

or in a more compact form:

$$f_i(x_1, x_2, x_3, \dots, x_n) = 0, i = 1 \text{ to } n.$$

The solution methods for the values of x_i that make f_i zero, $i=1$ to n are mostly open methods; such as fixed point iteration (we may write $x_{i,\,k+1} = g_i(x_{1,\,k},\, x_{2,\,k},\, x_{3,\,k}, \dots,\, x_{n,\,k})$ and with proper initial guesses carry out the iterations till to the required accuracy) and Newton Raphson and some others. Note that plotting curves for obtaining the solutions will not be practical unless we have small number of unknowns and simple curves.
Introducing the notation:

$$x_{i,k+1} - x_{i,k} = \Delta x_{i,k}, i = 1 \text{ to } n \text{ and } k\text{=1 to } N.$$

where "i" denotes the i'th unknown variable and k denotes iteration number let us expand the nonlinear system of equations about $x_{i,k}$ to get approximations for

$f_i(x_{1,k+1}, x_{2,k+1}, x_{3,k+1}, \dots, x_{n,k+1})$:

$$f_1(x_{1,k+1}, x_{2,k+1}, \dots, x_{n,k+1}) \cong f_1(x_{1,k}, x_{2,k}, \dots, x_{n,k}) + \Delta x_{1,k}\frac{\partial f_1}{\partial x_1}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}} + \Delta x_{2,k}\frac{\partial f_1}{\partial x_2}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}} + \dots + \Delta x_{n,k}\frac{\partial f_1}{\partial x_n}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}}$$

$$f_2(x_{1,k+1}, x_{2,k+1}, \dots, x_{n,k+1}) \cong f_2(x_{1,k}, x_{2,k}, \dots, x_{n,k}) + \Delta x_{1,k}\frac{\partial f_2}{\partial x_1}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}} + \Delta x_{2,k}\frac{\partial f_2}{\partial x_2}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}} + \dots + \Delta x_{n,k}\frac{\partial f_2}{\partial x_n}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}}$$

$$f_n(x_{1,k+1}, x_{2,k+1}, \dots, x_{n,k+1}) \cong f_n(x_{1,k}, x_{2,k}, \dots, x_{n,k}) + \Delta x_{1,k} \frac{\partial f_n}{\partial x_1}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}} + \Delta x_{2,k} \frac{\partial f_n}{\partial x_2}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}} + \dots + \Delta x_{n,k} \frac{\partial f_n}{\partial x_n}\Big|_{x_{1,k}, x_{2,k}, \dots, x_{n,k}}$$

To have a more compact form of the above expansions let us use matrix notation as:

Let $\underline{X}_k = (x_{1,k}, x_{2,k}, \dots, x_{n,k})^T$ (that is $\underline{X}_k = \begin{Bmatrix} x_{1,k} \\ x_{2,k} \\ . \\ . \\ . \\ x_{n,k} \end{Bmatrix}$) $\underline{F}_k = (f_{1,k}, f_{2,k}, \dots, f_{n,k})^T$ (i.e $\underline{F}_k = \begin{Bmatrix} f_{1,k} \\ f_{2,k} \\ . \\ . \\ . \\ f_{n,k} \end{Bmatrix}$

and $\Delta \underline{X}_k = (x_{1,k+1} - x_{1,k}, x_{2,k+1} - x_{2,k}, \dots, x_{n,k+1} - x_{n,k})^T$

When $\underline{X}_{k+1}$ is the solution set of $\Delta \underline{F}_{k+1} = \underline{0}$, we can write:

$$\underline{0} = \underline{F}_k + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & . & . & . & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & . & . & . & \frac{\partial f_2}{\partial x_n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & . & . & . & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \Delta \underline{X}_k$$

Noting that the square matrix preceding $\Delta \underline{X}_k$ is the Jacobian matrix of $\underline{F}_k(\underline{X}_k)$ we use the usual notation for it as $J_k(\underline{X}_k)$ and reduce the above equation to a simpler form as:

$$\underline{F}_k(\underline{X}_k) + J_k(\underline{X}_k).\Delta \underline{X}_k = 0 .$$

$\Delta \underline{X}_k$ is the unknown vector (it includes $\underline{X}_{k+1}$) and can be solved as

$$\Delta \underline{X}_k = -J^{-1}{}_k(\underline{X}_k).\underline{F}_k(\underline{X}_k)$$

Thus, we can evaluate $\underline{X}_{k+1} = \underline{X}_k + \Delta \underline{X}_k$

Systematically, the steps of Newton-Jacobi approach are the following:

1) Choose a convenient initial guess vector $\underline{X}_0$.

2) Evaluate $\underline{F}_0$ and J_0 for $\underline{X}_0$.

3) Solve $\underline{F}_0(\underline{X}_0) + J_0(\underline{X}_0).\Delta\underline{X}_0 = \underline{0}$ for $\Delta\underline{X}_0$.

(usually by $\Delta\underline{X}_0 = -J^{-1}{}_0(\underline{X}_0).\underline{F}_0(\underline{X}_0)$ or by other methods).

4) Find $\underline{X}_1 = \underline{X}_0 + \Delta\underline{X}_0$

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Repeat 1 to 4 for $\underline{X}_1$, $\underline{X}_2$ up to $\underline{X}_k$ where $\left\|\Delta\underline{X}_k\right\| \leq$ error tolerance, δ (or ε_{tol})

(Usually the norm that will be used will be either max. norm (∞-norm) or l_2 norm (length norm of the vector $\Delta\underline{X}_k$)).

Example:

Solve the following system of nonlinear equations by Newton-Jacobi Method for its roots in the first quadrant. Take error tolerance as 0.00005 (max. norm).

$$4x_1^2 - x_2^2 = 0$$
$$4x_1x_2^2 - x_1 = 1$$

Solution:

$F = \begin{Bmatrix} 4x_1^2 - x_2^2 \\ 4x_1x_2^2 - x_1 - 1 \end{Bmatrix}, J = \begin{bmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1x_2 \end{bmatrix}$ let us take the initial guess as $\underline{X}_0^T = (0 \quad 1)$ for the root in the first quadrant.

No. of it. (k)	$x_{1,(k)}$	$x_{2,(k)}$	$f_{1,(k)}$	$f_{2,(k)}$	$J_{(k)}$	$J^{-1}_{(k)}$	$\Delta\underline{X}_{(k)} = -J^{-1}_{(k)}F_{(k)}$	$Max\|\Delta\underline{X}_{(k)}\|$ $(\varepsilon < 0.00005)$
0	0.00000	1.00000	-1.00000	-1.00000	$\begin{bmatrix} 0.00000 & -2.00000 \\ 3.00000 & 0.00000 \end{bmatrix}$	$\begin{bmatrix} 0.00000 & 0.33333 \\ -0.50000 & 0.00000 \end{bmatrix}$	$\begin{bmatrix} 0.33333 \\ -0.50000 \end{bmatrix}$	0.50000
1	0.33333	0.50000	0.19444	-1.00000	$\begin{bmatrix} 2.66667 & -1.00000 \\ 0.00000 & 1.33333 \end{bmatrix}$	$\begin{bmatrix} 0.37500 & 0.28125 \\ 0.00000 & 0.75000 \end{bmatrix}$	$\begin{bmatrix} 0.20833 \\ 0.75000 \end{bmatrix}$	0.75000
2	0.54167	1.25000	-0.38883	1.84382	$\begin{bmatrix} 4.33334 & -2.50000 \\ 5.25010 & 5.41667 \end{bmatrix}$	$\begin{bmatrix} 0.14800 & 0.06831 \\ -0.14345 & 0.11841 \end{bmatrix}$	$\begin{bmatrix} -0.06839 \\ -0.27410 \end{bmatrix}$	0.27410
3	0.47328	0.97590	-0.05633	0.32975	$\begin{bmatrix} 3.59274 & -1.79639 \\ 2.22700 & 3.22697 \end{bmatrix}$	$\begin{bmatrix} 0.20559 & 0.11398 \\ -0.14293 & 0.22751 \end{bmatrix}$	$\begin{bmatrix} -0.02234 \\ -0.07224 \end{bmatrix}$	0.07224
4	0.45094	0.90366	-0.00322	0.02201	$\begin{bmatrix} 3.60750 & -1.80732 \\ 2.26641 & 3.25996 \end{bmatrix}$	$\begin{bmatrix} 0.20559 & 0.11398 \\ -0.14293 & 0.22751 \end{bmatrix}$	$\begin{bmatrix} -0.00185 \\ -0.00547 \end{bmatrix}$	0.00547
5	0.44909	0.89819	-0.00002	0.00013	$\begin{bmatrix} 3.59274 & -1.79639 \\ 2.22700 & 3.22697 \end{bmatrix}$	$\begin{bmatrix} 0.20693 & 0.11520 \\ -0.14281 & 0.23039 \end{bmatrix}$	$\begin{bmatrix} -0.00001 \\ -0.00003 \end{bmatrix}$	0.00003

Answer: $X^T = (0.4491, 0.8982)$