M E T U
Department of Mathematics

Introduction to Differential Equations MidTerm 1		
Code : *Math 219* Acad. Year : *2014-2015* Semester : *Fall* Coordinator: *Özgür Kişisel* Date : *November.15.2014* Time : *13:30* Duration : *120 minutes*	Last Name : Name : Department : Signature :	Student No. : Section :
	4 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4		SHOW YOUR WORK	

Question 1 (12+13=25 pts) This question has two unrelated parts.

(a) Consider the equation $M(x,y)dx+(e^x-\cos x\cos y+xy)dy=0$. Determine all functions $M(x,y)$ such that this equation is exact on $\mathbb{R}^2$.

$\mathbb{R}^2$ is a simply connected domain. So the equation is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial(e^x-\cos x\cos y+xy)}{\partial x} = e^x+\sin x\cos y+y$$

$$\Longleftrightarrow M(x,y) = e^x\cdot y+\sin x\cdot\sin y+\frac{y^2}{2}+h(x)$$

for some $h(x)$.

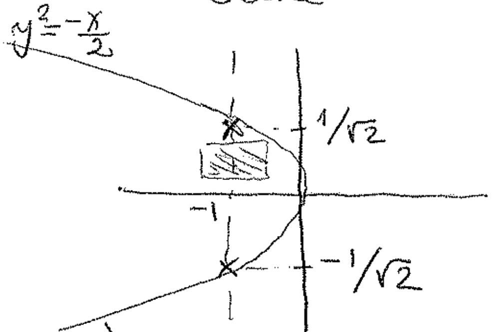
(b) Consider the equation $(3y+2x)dx-(2xy^2+x^2)dy=0$. Determine all values of y_0 such that there is a **unique solution curve** $y(x)$ through $(-1,y_0)$.

$$\frac{dy}{dx} = \frac{3y+2x}{2xy^2+x^2} = f(x,y)$$

$$\frac{\partial f}{\partial y} = \frac{3\cdot(2xy^2+x^2)-4xy(3y+2x)}{(2xy^2+x^2)^2} = \frac{-6xy^2+3x^2-8x^2y}{(2xy^2+x^2)^2}$$

For a given $x \neq 0$, this function is continuous if and only if $2xy^2+x^2\neq 0$, namely $y^2 \neq \frac{-x}{2}$

Same holds for $f(x,y)$.



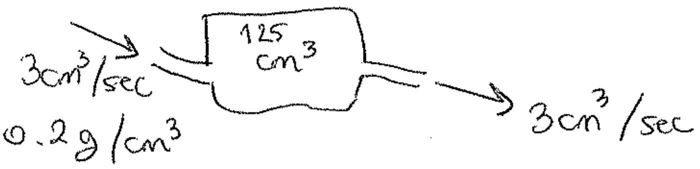
this parabola should be avoided.

Therefore if $y_0 \neq \mp\frac{1}{\sqrt{2}}$ then there is a rectangle around $(-1,y_0)$ in which f and $\partial f/\partial y$ are cont. By the existence - uniqueness thm, there is a unique soln. curve through $(-1,y_0)$.

There is no soln. curve through $(-1,\pm\frac{1}{\sqrt{2}})$ since $(\mp\frac{3}{\sqrt{2}}-2)dx+0dy=0$ can't happen.

Question 2 (20+5=25 pts) Blood carries drug into an organ that has volume $125cm^3$ at a rate $3cm^3/sec$ and leaves at the same rate. Concentration of drug in the blood is $0.2g/cm^3$. Assume that there was no drug in the organ initially.

(a) Find the concentration of the drug in the organ at any time t.



$Q(t)$: Amount of drug in the blood at time t (unit: g)

$$\text{Rate in} = 0.2\,g/cm^3 \cdot 3cm^3/sec = 0.6\,g/sec \quad \Big| \quad \text{Rate out} = \frac{3}{125}Q(t)\ g/sec$$

$$\frac{dQ}{dt} = 0.6 - \frac{3}{125}Q \qquad g/sec$$

$$\int \frac{dQ}{0.6-\frac{3}{125}Q} = \int dt \Rightarrow \frac{-125}{3}\ln\left|0.6-\frac{3}{125}Q\right| = t + c$$

$$0.6 - \frac{3}{125}Q = ce^{\frac{-3}{125}t}$$

$$Q(t) = \frac{125}{3}\left(0.6 - ce^{\frac{-3}{125}t}\right) = 25 - ce^{\frac{-3}{125}t}$$

$$Q(0) = 0 \Rightarrow c = 25$$

$$Q(t) = 25 - 25e^{-3/125\,t} \quad g$$

$$\text{Concentration} = C(t) = \frac{Q(t)}{125} = 0.2 - 0.2e^{\frac{-3}{125}t} \quad g/cm^3$$

(b) What is the limiting concentration when t is very large? How does this agree with the physical expectation?

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty}\left(0.2 - 0.2e^{\frac{-3}{125}t}\right) = 0.2\ g/cm^3.$$

The concentration of drug in the organ approaches asymptotically to the concentration of drug in the blood (but never exceeds it).

Question 3 (15+5+5=25 pts) Consider the system $\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}}_{A} \mathbf{x}$.

(a) Find all real valued solutions of the system.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda)+5 = \lambda^2+2\lambda+2$$
$$= (\lambda+1)^2+1$$

$(\lambda+1)^2+1=0 \Rightarrow \lambda = -1 \mp i$ are the eigenvalues.

eigenvectors: $\lambda_1 = -1+i$ $\left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right] \xrightarrow{(2+i)R_1+R_2\to R_2} \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$(2-i)v_1 - v_2 = 0 \Rightarrow$ eigenvectors are $k\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{(-1+i)t} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)(e^{-t}\cos t + i e^{-t}\sin t)$$

$$= \underbrace{e^{-t}\begin{bmatrix} \cos t \\ 2\cos t+\sin t \end{bmatrix}}_{\vec{y}^{(1)}} + i \underbrace{e^{-t}\begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}}_{\vec{y}^{(2)}}$$

$\vec{y}^{(1)}, \vec{y}^{(2)}$ are two independent solns. since $\lambda_1 = -1+i \neq -1-i = \lambda_2$.

general solution: $c_1 e^{-t}\begin{bmatrix} \cos t \\ 2\cos t+\sin t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \sin t \\ 2\sin t-\cos t \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$

(b) Find a fundamental matrix $\Phi(t)$ such that $\Phi(0) = I$.

$$\Psi(t) = \left[\vec{y}^{(1)} \,\middle|\, \vec{y}^{(2)}\right] = e^{-t}\begin{bmatrix} \cos t & \sin t \\ 2\cos t+\sin t & 2\sin t-\cos t \end{bmatrix}$$

is a fundamental matrix, but $\Psi(0) = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \neq I$

So set $\Phi(t) = \Psi(t)\Psi(0)^{-1}$
$$= e^{-t}\begin{bmatrix} \cos t & \sin t \\ 2\cos t+\sin t & 2\sin t-\cos t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = e^{-t}\begin{bmatrix} \cos t+2\sin t & -\sin t \\ 5\sin t & \cos t-2\sin t \end{bmatrix}$$

(c) Find a solution of the system such that $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$\vec{x}(t) = \Phi(t)\vec{c}$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{x}(0) = \Phi(0)\vec{c}$
$= \vec{c}$

$$\Rightarrow \vec{x}(t) = e^{-t}\begin{bmatrix} \cos t+2\sin t & -\sin t \\ 5\sin t & \cos t-2\sin t \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$= e^{-t}\begin{bmatrix} \cos t - \sin t \\ 3\cos t - \sin t \end{bmatrix}$$

Question 4 (13+12=25 pts) Consider the system $\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}}_{A} \mathbf{x} + \underbrace{\begin{bmatrix} e^{-3t} \\ 0 \end{bmatrix}}_{\vec{b}(t)}$.

(a) Find a fundamental matrix $\Psi(t)$ for the homogenous system associated to this system.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = (1-\lambda)(-7-\lambda)+16 = \lambda^2+6\lambda+9 = (\lambda+3)^2$$

So, eigenvalues are $\lambda_1 = \lambda_2 = -3$.

eigenvectors: $\left[\begin{array}{cc|c} 4 & -4 & 0 \\ 4 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{v}^{(1)} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We can't find two indep. eigenvectors $\Rightarrow$ there must be a generalized eigenvector: $(A+3I)\vec{v}^{(2)} = \vec{v}^{(1)} \underset{\text{say}}{=} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 1/4 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} k+1/4 \\ k \end{bmatrix}, \text{ say } \vec{v}^{(2)} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1/4 \\ 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \qquad e^{Jt} = \begin{bmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$\Psi = Pe^{Jt} = \begin{bmatrix} 1 & 1/4 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{bmatrix} = e^{-3t}\begin{bmatrix} 1 & t+\frac{1}{4} \\ 1 & t \end{bmatrix}$$

(or $\Phi = Pe^{Jt}P^{-1}$)

$\det \Psi = \frac{-1}{4} e^{-6t}$

(b) Find all solutions of the nonhomogenous system using variation of parameters.

$$\Psi^{-1} = \frac{1}{\frac{-1}{4}e^{-6t}} e^{-3t}\begin{bmatrix} t & -t-1/4 \\ -1 & 1 \end{bmatrix} = -4e^{3t}\begin{bmatrix} t & -t-1/4 \\ -1 & 1 \end{bmatrix}$$

$$\vec{v} = \int \Psi^{-1}\vec{b}\, dt = \int -4\cancel{e^{3t}}\begin{bmatrix} t & -t-1/4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \cancel{e^{-3t}}\ 1 \\ 0 \end{bmatrix} dt$$

$$= \int -4\begin{bmatrix} t \\ -1 \end{bmatrix} dt = \begin{bmatrix} -2t^2 \\ 4t \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x} = \Psi\vec{v} = e^{-3t}\begin{bmatrix} 1 & t+1/4 \\ 1 & t \end{bmatrix}\left(\begin{bmatrix} -2t^2 \\ 4t \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)$$

$$= e^{-3t}\begin{bmatrix} 2t^2+t \\ 2t^2 \end{bmatrix} + c_1e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} t+1/4 \\ t \end{bmatrix}$$