Early Universe: Out of Equilibrium dynamics

Disclaimer

Discussions taken from Barbara Ryden [1], Daniel Baumann [2] and Kolb & Turner [3] books

1 Successes and drawbacks of the Hot Big Bang Model

	z	t
Distant quasars	< 10	$> 0.5\,$Gyr
LSS	1090	0.37 Gyr
Nuc	3×10^8	3 min
Neutron decoupling	4×10^9	1 s

Table 1: Successes of the Hot Big Bang Model

The universe is nearly flat today, and was even flatter in the past
The universe is nearly isotropic and homogeneous today, and was even more so in the past
The universe is apparently free of magnetic monopoles

Table 2: Drawbacks of the Hot Big Bang Model

2 The flatness problem

From the Friedmann equation

$$\frac{H^2}{H_0^2} = \Omega - \frac{k\,c^2}{R_0^2 H_0^2\,a^2}, \tag{1}$$

we have that

$$1 - \Omega_0 = -\frac{k\,c^2}{R_0^2 H_0^2\,a_0^2}. \tag{2}$$

Hence

$$1 - \Omega = \frac{H_0^2(1-\Omega_0)}{H(t)^2\,a(t)^2} \propto \begin{cases} a \propto t, & \text{RD;} \\ a \propto t^{2/3}, & \text{MD.} \end{cases} \tag{3}$$

Experimentally

$$|1 - \Omega_0| \leq 0.005. \tag{4}$$

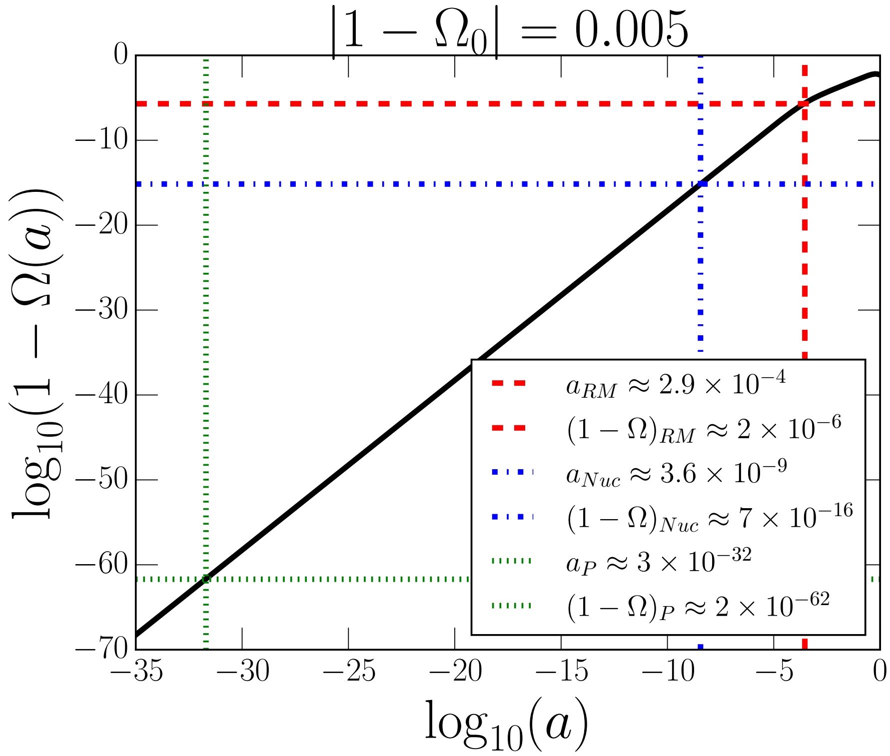


Figure 1: $1-\Omega(a)$ as a function of a.

3 The horizon problem

3.1 Current proper distances

The horizon distance is given by

$$d_{\text{hor}}(t_0) = c\,a(t_0)\int_0^{t_0}\frac{dt}{a(t)} = c\int_0^1 \frac{da}{a^2\,H(a)} \tag{5}$$

$$= c\int_0^{a_0}\frac{da}{a^2\,H_0\sqrt{\Omega_{R,0}/a^4+\Omega_{M,0}/a^3+\Omega_{\Lambda,0}}} \approx 14118\,\text{Mpc}. \tag{6}$$

The current proper distance to the last scattering surface is

$$d_p(t_0) = c\,a(t_0)\int_{t_{ls}}^{t_0}\frac{dt}{a(t)} = c\int_{a_{ls}}^{a_0}\frac{da}{a^2\,H(a)} \approx 0.98\,d_{\text{hor}} \tag{7}$$

Next let's consider two points in opposite sites of the last scattering surface. The are separated by a proper distance of

$$d_p \approx 1.96\,d_{\text{hor}}(t_0), \tag{8}$$

as seen by an observed on Earth.

Since the two points are farther apart than the horizon distance, they are causally disconnected. That is, they haven't had time to send messages to each other, and in particular, haven't had time to come into thermal equilibrium with each other. Nevertheless, the two points have the same temperature to within one part in 10^5. Why? How can two points that haven't had time to swap information be so nearly identical in their properties?

3.2 Proper distances at the time of last scattering

The horizon distance at the time of last scattering was

$$d_{\mathrm{hor}}(t_{ls}) = c\,a(t_{ls}) \int_0^{t_{ls}} \frac{dt}{a(t)} = c\,a(t_{ls}) \int_0^{a_{ls}} \frac{da}{a^2\,H(a)} \approx 0.257\,\mathrm{Mpc}. \tag{9}$$

3.3 How many patches?

In a spatially flat Universe the angular-diameter distance is

$$d_A = \frac{d_p(t_0)}{1+z} \overset{z\to\infty}{\approx} \frac{d_{\mathrm{hor}}(t_0)}{z}. \tag{10}$$

Hence the angular-diameter distance to the last scattering surface is

$$d_A = \frac{14118\,\mathrm{Mpc}}{1090} \approx 12.952\,\mathrm{Mpc}. \tag{11}$$

Thus, points on the last scattering surface that were separated by a horizon distance will have an angular separation equal to

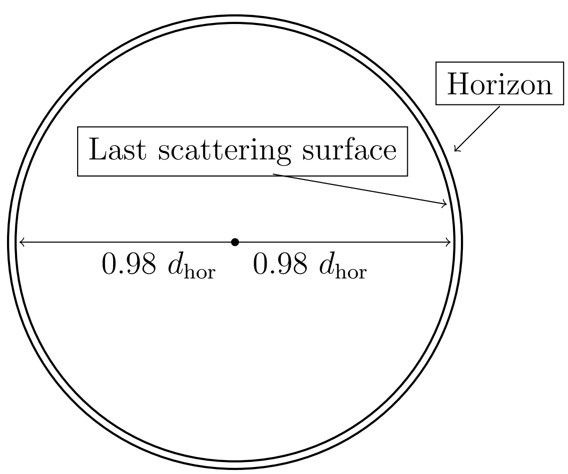
$$\theta_{\mathrm{hor}} = \frac{d_{\mathrm{hor}}(t_{ls})}{d_A} = \frac{0.257\,\mathrm{Mpc}}{12.95\,\mathrm{Mpc}} \approx 0.02\,\mathrm{rad} \approx 1.1^\circ \tag{12}$$

as seen from the Earth today.

Points that are separated by more than 2° on the sky seem never to have been in causal contact, since their past light cones don't overlap. **Notice that 2 degrees in the sky are about four times the angular size of the Moon, seen from Earth**

The size of a causally-connected patch of space is determined by the maximal distance from which light can be received. The surface of last scattering can be divided into some 40 000 patches, each 1.1° across:

$$\#\,\text{of patches} \sim \frac{4\pi}{0.02^2} \approx 40000. \tag{13}$$



In the standard Hot Big Bang scenario, the current proper distance to the last scattering surface is 98% of the current horizon distance.

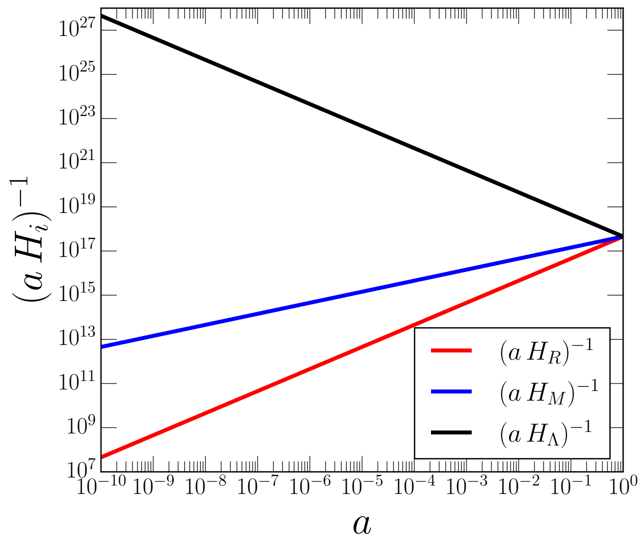


Figure 2: $(aH)^{-1}$ as a function of a.

3.4 Comoving Hubble radius

The Big Bang "started" with the singularity at $t_i = 0$, then the greatest comoving distance from which an observer at time t will be able to receive signals traveling at the speed of light is the (comoving) particle horizon:

$$d_h(a) = \frac{d_{\rm hor}}{a(t)} = \int_{t_i}^{t} \frac{dt}{a(t)} = \int_{a_i}^{a} \frac{da}{a\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1}\, d(\ln a). \tag{14}$$

The causal structure of the spacetime is hence related to the evolution of the comoving Hubble radius, $((aH)^{-1})$. For ordinary matter sources, the comoving Hubble radius is a monotonically increasing function of time (or scale factor), and the integral is dominated by the contributions from late times. This implies that in the standard cosmology $d_h \sim (aH)^{-1}$, which has led to the confusing practice of referring to both the particle horizon and the Hubble radius as the "horizon".

3.5 Conformal time

$$d\eta = \frac{dt}{a(t)}. \tag{15}$$

Hence

$$a(\eta) \propto \begin{cases} \eta^{\frac{2}{3(1+\omega_i)}}, & \omega_i \neq 0 \to \eta^2\,(\eta^1), \text{ for MD (RD).} \\ (-\eta)^{-1}, & w_i = -1. \end{cases} \tag{16}$$

The comoving distance to a source is simply equal to the difference in conformal time between the moments when the light was emitted and when it was received:

$$d_h(\eta) = \int_{t_i}^{t} \frac{dt}{a(t)} = \eta - \eta_i. \tag{17}$$

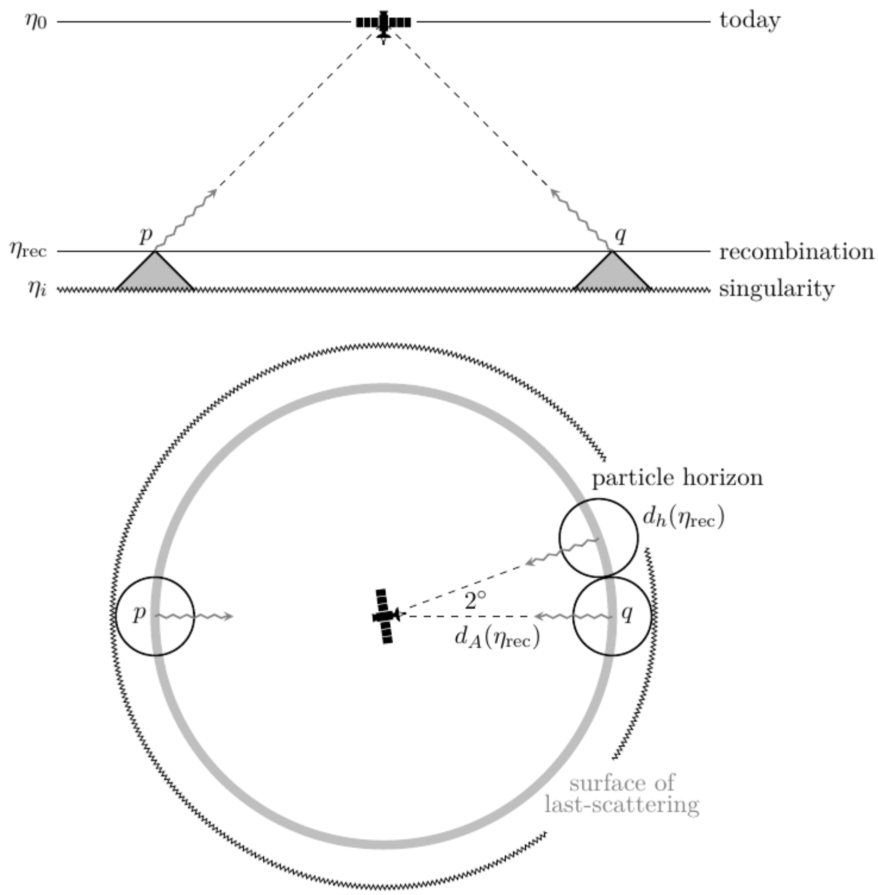


Illustration of the horizon problem in the conventional Big Bang model. All events that we currently observe are on our past light cone. The intersection of our past light cone with the spacelike slice at the time of recombination is the surface of last-scattering. Points that are separated by more than 2° on the sky seem never to have been in causal contact, since their past light cones don't overlap.

Figure 3: Caption

It follows that the amount of conformal time between the initial singularity and the formation of the CMB (or, equivalently, the comoving horizon at the time of recombination) was much smaller than the conformal age of the universe today (or, equivalently, the comoving distance to the last-scattering surface), $\eta_{\rm rec} \ll \eta_0$.

This implies a serious problem: it means that most parts of the CMB have non-overlapping past light cones and hence never were in causal contact.

3.6 Superhorizon and subhorizon

Let us consider the fluctuations that we observe in the cosmic microwave background and in the large-scale structure of the universe. Any fluctuation that is inside the Hubble radius today was outside the Hubble radius at sufficiently early times. For the standard hot Big Bang, the Hubble radius is approximately equal to the particle horizon, so we call these regimes "subhorizon" and "superhorizon." Above we showed that the particle horizon at recombination was about 265 Mpc.

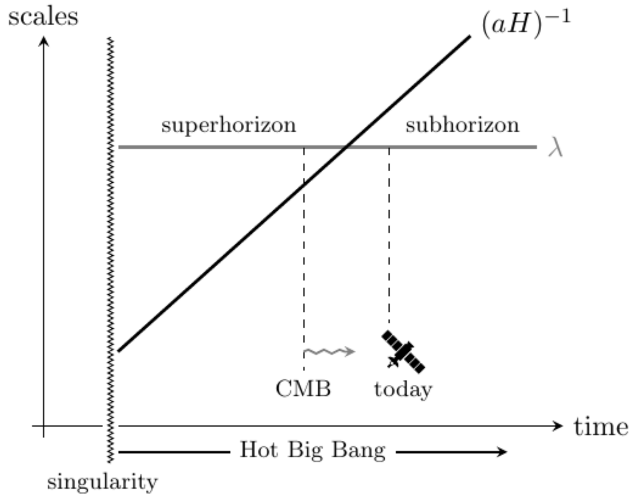


Figure 4: Evolution of a representative fluctuation of fixed (comoving) wavelength λ relative to the Hubble radius $(aH)^{-1}$.

Scales larger than this would not have been inside the horizon before the CMB was created. Yet, we find the CMB fluctuations to be correlated on scales that are larger than this apparent horizon. This is the modern version of the horizon problem. Not only is the CMB homogeneous on apparently acausal scales, it also has correlated fluctuations on these scales.

4 The Monopole problem

Monopoles may be originated in the early universe within so called grand unified theories (GUTs). The typical mass-energy of these particles is

$$M_{\mathrm{M}} = E_{\mathrm{GUT}} \sim 10^{15}\,\mathrm{GeV}. \tag{18}$$

Since one monopole per horizon volume, the monopole number density at their formation time, $t_{\mathrm{GUT}} \sim 10^{-36}$ s, is

$$n_{\mathrm{M}}(t_{\mathrm{GUT}}) = \frac{1}{(2\,c\,t_{\mathrm{GUT}})^3} \sim 10^{82}\,\mathrm{m}^{-3}. \tag{19}$$

The corresponding energy density is

$$\epsilon_{\mathrm{M}}(t_{\mathrm{GUT}}) = M_{\mathrm{M}}\,n_{\mathrm{M}}(t_{\mathrm{GUT}}) \sim 10^{97}\,\mathrm{GeV/m^3}. \tag{20}$$

For comparison purposes, the energy density of radiation at t_{GUT} is

$$\epsilon_R(t_{\mathrm{GUT}}) = \frac{\pi^2}{30} g_* T_{\mathrm{GUT}}^4 \approx 10^{57}\,\mathrm{GeV}^4 \tag{21}$$

$$\sim 10^{108}\,\mathrm{GeV/cm^3}, \tag{22}$$

where

$$T_{\mathrm{GUT}} \approx 1.31\times 10^{10}\,\mathrm{K}\left(\frac{1\,\mathrm{s}}{t_{\mathrm{GUT}}}\right)^{1/2} = 1.1341\,\mathrm{MeV}\left(\frac{1\,\mathrm{s}}{t_{\mathrm{GUT}}}\right)^{1/2} \tag{23}$$

$$\approx 10^{15}\,\mathrm{GeV}. \tag{24}$$

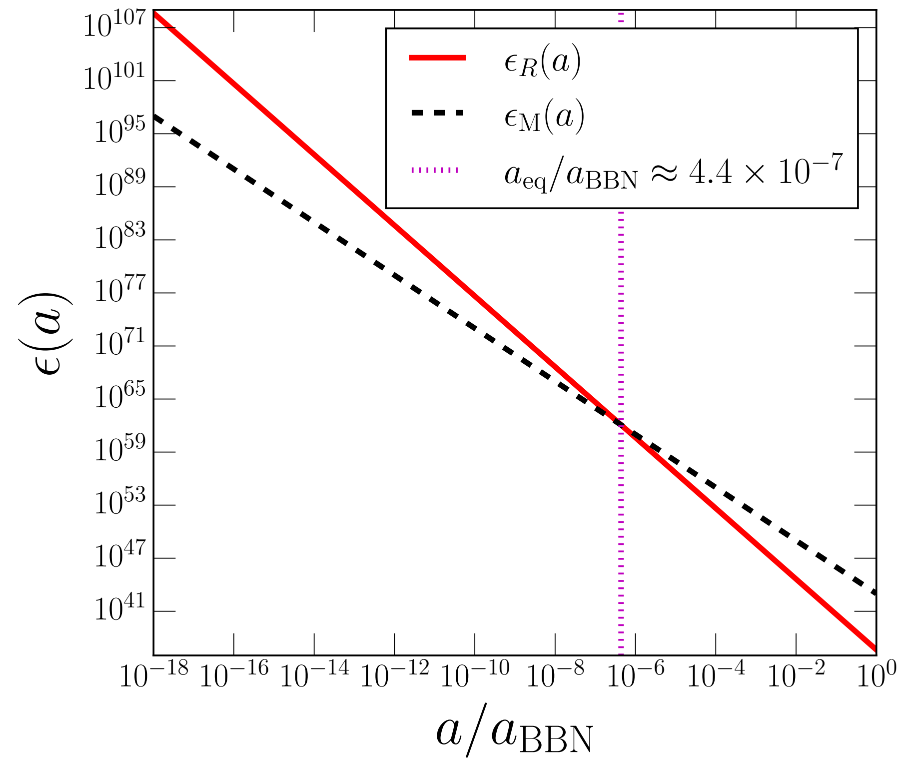


Figure 5: Energy density of radiation and monopole as a function of $a/a_{\rm BBN}$.

4.1 Phase transitions

When the Universe is cooling, there could have been a whole series of phase transitions in the plasma. Hence some 'defects' may form during phase transitions such as

- Monopoles.
- Domain walls.
- Cosmic strings.
- Textures.

They could have had striking effects on structure formation, and have been searched for in balloon-borne and satellite experiments on the microwave background. No signal has been found so far, which means that if cosmic defects exist, they must play a subdominant role. Anyway, the study of cosmological phase transitions is a fascinating branch of cosmology which has strong ties both to particle and condensed matter physics.

The formation of strings has been experimentally verified in superfluid ^{3}He (the subject of the 1996 Nobel prize in physics) where the microscopic description is more complicated (pairs of ^{3}He atoms bind together similar to Cooper pairs in superconductivity) but the effective Hamiltonian contains a vacuum manifold with the same $U(1)$ structure as discussed above. It is reassuring that the experimental results verify the existence of string-like defects and the formation mechanism we have just presented. This research produces very interesting analogies between condensed matter physics and cosmology.

5 Inflation to rescue

The description of the horizon and flatness problems has highlighted the fundamental role played by the growing comoving Hubble radius of the standard Big Bang cosmology. A simple solution to these problems therefore suggests itself: let us conjecture a phase of decreasing Hubble radius in the early universe,

$$\frac{d}{dt}(aH)^{-1} < 0. \tag{25}$$

That is

$$\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -2(\dot{a})^{-2}\ddot{a} < 0, \tag{26}$$

$$\Rightarrow \ddot{a} > 0. \tag{27}$$

The Inflation hypothesis is a period in the early universe when the expansion was accelerating outward. From the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3c^2}(\epsilon + 3p) = -\frac{4\pi G_N}{3c^2}(1 + 3\omega)\epsilon, \tag{28}$$

the inflationary period would have taken place if the universe were temporarily dominated by a component with equation-of-state parameter

$$\omega_i < -\frac{1}{3}. \tag{29}$$

5.1 Early vacuum dominated era Λ_i

The simplest implementation of inflation states that the universe was temporarily dominated by a positive cosmological constant Λ_i ($\omega_i = -1$). It follows that acceleration equation becomes

$$\frac{\ddot{a}}{a} = \frac{\Lambda_i}{3} > 0, \tag{30}$$

meanwhile the Hubble rate reads

$$\left(\frac{\dot{a}}{a}\right) = H_i = \left(\frac{\Lambda_i}{3}\right)^{1/2}. \tag{31}$$

From this we obtain the exponential grow of the scale factor during inflation:

$$a_i \sim e^{H_i t}. \tag{32}$$

To illustrate in a simple way how inflation mechanism operates let's consider the following piecewise function

$$a(t) = \begin{cases} a_i \left(\frac{t}{t_i}\right)^{1/2}, & t < t_i; \\ a_i\, e^{H_i(t-t_i)}, & t_i < t < t_f; \\ a_i e^{H_i(t_f - t_i)} \left(\frac{t}{t_f}\right)^{1/2}, & t > t_F. \end{cases} \tag{33}$$

Defining the number of e-foldings of inflation, N, as

$$N \equiv H_i(t_f - t_i), \tag{34}$$

the increasing of the scale factor is given by

$$\frac{a(t_f)}{a(t_i)} = e^N. \tag{35}$$

Thus, if the duration of inflation, $t_f - t_i$, was long compared to the Hubble time during inflation, H_i^{-1}, then N was large, and the growth in scale factor during inflation was enormous.

Typically

$$t_i = t_{\rm GUT} \sim 10^{-36}\, s, \tag{36}$$

$$H_i = t_{\rm GUT}^{-1} \sim 10^{36}\, s^{-1}, \tag{37}$$

$$t_f = (N+1) t_{\rm GUT}. \tag{38}$$

5.1.1 Solution to the flatness problem

$$|1 - \Omega(t)| = \frac{c^2}{R_0^2\, a(t)^2\, H(t)^2} \propto e^{-2H_i t}. \tag{39}$$

From t_i to t_f

$$|1 - \Omega(t_f)| = |1 - \Omega(t_i)|\, e^{-2N}. \tag{40}$$

For a highly curved universe $|1 - \Omega(t_i)| \sim 1$, the amount of flattening produced by inflation is

$$|1 - \Omega(t_f)| = e^{-2N}. \tag{41}$$

On the other hand, during radiation domination

$$a(t) = \sqrt{2\, \Omega_{R,0}^{1/2}\, H_0\, t}. \tag{42}$$

Using $\Omega_{R,0} = 9 \times 10^{-5}$, $H_0 = 14.4/\mathrm{Gyr}$ and $t_f = (N+1) \times 10^{-36}$ s:

$$a(t_f) \approx 2 \times 10^{-28} \sqrt{N+1}, \tag{43}$$

and noting that

$$1 - \Omega(t_f) = \frac{(1 - \Omega_0) a_f^2}{\Omega_{R,0} + a_f \Omega_{M,0}} \tag{44}$$

we obtain that the amount of flattening required by observations (for $|1 - \Omega_0| \leq 0.005$) is

$$1 - \Omega(t_f) \leq 2.2 \times 10^{-54} (N+1). \tag{45}$$

Therefore, the minimum number of e-foldings of inflation needed to match the observations is

$$N \approx 60, \tag{46}$$

such that

$$\frac{a(t_f)}{a(t_i)} = e^{60} \sim 10^{26}. \tag{47}$$

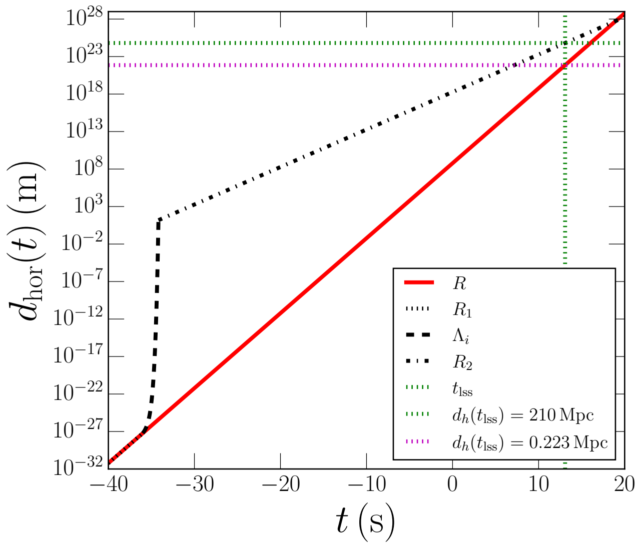


Figure 6: The growth of the horizon distance as a function of time.

5.1.2 Solution to the horizon problem

Let's calculate the horizon distance at $t_i \sim 10^{-36}$ s and t_f. Before inflation

$$d_{\rm hor}(t_i) = a(t_i)\,c\int_0^{t_i}\frac{dt}{a(t)} = a(t_i)\,c\int_0^{t_i}\frac{dt}{a_i(t/t_i)^{1/2}} = 2\,c\,t_i \approx 6\times 10^{-28}\,{\rm m}. \tag{48}$$

At the end of inflation with $N = 65$,

$$d_{\rm hor}(t_i) = a(t_f)\,c\left(\int_0^{t_i}\frac{dt}{a(t)} + \int_{t_i}^{t_f}\frac{dt}{a(t)}\right) \tag{49}$$

$$= a_i\,e^N\,c\left(\int_0^{t_i}\frac{dt}{a_i(t/t_i)^{1/2}} + \int_{t_i}^{t_f}\frac{dt}{a_i\,e^{H_i(t-ti)}}\right) \tag{50}$$

$$= e^N\,c(2t_i + H^{-1}) \approx e^N\,3t_i\,c \approx 15\,{\rm m}. \tag{51}$$

This is, from $t_i \approx 10^{-36}$ s to $t_i \approx 10^{-34}$ s the horizon distance increased from 10^{-27} m to 15 m.

$$d_p(t_0) \approx 14000\,{\rm Mpc}, \tag{52}$$

$$a(t_f) \approx 2\times 10^{-27}, \tag{53}$$

$$d_p(t_f) = a_f\,d_p(t_0) \approx 3\times 10^{-27}\,{\rm Mpc} = 0.9\,{\rm m}, \tag{54}$$

$$d_p(t_i) = e^{-N}\,d_p(t_f) \approx 4\times 10^{-29}\,{\rm m}, \tag{55}$$

$$d_{\rm hor}(t_i) = 2\,c\,t_i \approx 6\times 10^{-28}\,{\rm m}. \tag{56}$$

5.1.3 Solution to the monopole problem

$$
\begin{aligned}
a &\propto e^{H_i t}, &(57)\\
n_{\mathrm{M}} &\propto e^{-3H_i t}, &(58)\\
n_{\mathrm{M}}(t_{\mathrm{GUT}}) &\approx 10^{82}/\mathrm{m}^3, &(59)\\
n_{\mathrm{M}}(t_f) &\approx e^{-195} n_{\mathrm{M}}(t_{\mathrm{GUT}}) \approx 0.002/\mathrm{m}^3, &(60)\\
n_{\mathrm{M}}(t_0) &= n_{\mathrm{M}}(t_f) a_f^3 \approx 2\times 10^{-83}/\mathrm{m}^3 \approx 5\times 10^{-16}/\mathrm{Mpc}^3. &(61)
\end{aligned}
$$

5.2 Λ_i as vaccum energy?

Let's recall that

$$\epsilon_\Lambda = 0.69\epsilon_{c,0} \approx 0.0034\mathrm{TeV}/\mathrm{m}^3. \tag{62}$$

On the other hand,

$$\epsilon_{\Lambda_i} = \frac{H_i^2}{3M_P^2} \sim 10^{105}\mathrm{TeV}/\mathrm{m}^3. \tag{63}$$

6 Field theory of inflation

Let's consider the inflaton field,

$$\phi(\vec{r}, t), \tag{64}$$

with a action in the Minkowski spacetime

$$S = \int dt\, d\vec{r}\, \mathcal{L} = \int dt\, d\vec{r} \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] = \int dt\, d\vec{r} \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]. \tag{65}$$

The Euler-Lagrange equation turns to be Klein-Gordon equation:

$$\ddot{\phi} - \nabla^2\phi = -\frac{\partial V}{\partial\phi}. \tag{66}$$

For a flat FLRW Universe we obtain

$$S = \int dt\, d\vec{r}\, a(t)^3 \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a(t)^2}(\nabla\phi)^2 - V(\phi)\right]. \tag{67}$$

In the case of a homogeneous field configuration, $\phi(\vec{r}, t) = \phi(t)$, the Eulaer-Lagrange equation becomes

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial\phi}. \tag{68}$$

This equation mimics the equation of motion for a particle being accelerated by a force proportional to $-dV/d\phi$ and being impeded by a frictional force proportional to the particle's speed. Thus, the expansion of the universe provides a "Hubble friction" term, which slows the transition of the inflaton field to a value that will minimize the potential V. Just as a skydiver reaches terminal velocity when the downward force of gravity is balanced by the upward force of air resistance, so the inflaton field can reach "terminal velocity" (with $\ddot{\phi} = 0$)

$$3H\dot{\phi} = -\frac{dV}{d\phi}. \tag{69}$$

The energy density associated to the inflaton field is

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{70}$$

while the pressure is

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{71}$$

It follows that for

$$\tfrac{1}{2}\dot{\phi}^2 \ll V(\phi), \tag{72}$$

the inflation equation-of-state becomes

$$\omega_\phi = \frac{P_\phi}{\epsilon_\phi} \approx -1. \tag{73}$$

Thus, an inflaton field can drive exponential inflation if there is a temporary period when its rate of change is small ($\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$), and its potential $V(\phi)$ is large enough to dominate the energy density of the universe.

6.1 Slow-roll inflation

Let's recall the Friedmann equations that rule the dynamics

$$H^2 = \frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right), \tag{74}$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}, \tag{75}$$

which from their combination we obtain

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_P^2}. \tag{76}$$

The ϵ slow-roll parameter is defined as

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{\dot{\phi}^2}{H^2 M_P^2} = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V}. \tag{77}$$

Hence inflation occurs as long as

$$\epsilon \ll 1. \tag{78}$$

The expression for the time derivative of the comoving Hubble radius takes the form

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\epsilon). \tag{79}$$

This shows that a shrinking comoving Hubble radius is associated with $\epsilon < 1$:

$$\frac{d}{dt}(aH)^{-1} < 0 \;\; \Rightarrow \;\; \epsilon < 1. \tag{80}$$

Since inflation must last for a sufficiently long time (usually at least 40 to 60 e-folds), which requires that ϵ remains small for a sufficiently large number of Hubble times. This condition is measured by a second slow-roll parameter

$$\kappa \equiv \frac{\dot{\epsilon}}{H\,\epsilon}. \tag{81}$$

For $|\kappa| < 1$, the fractional change of ϵ per e-fold is small and inflation persists. Moreover, introducing

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}, \tag{82}$$

we obtain that

$$\kappa \equiv 2\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} = 2(\epsilon - \delta). \tag{83}$$

This shows that $\epsilon, |\delta| \ll 1$ implies $\epsilon, |\kappa| \ll 1$. If both the speed and the acceleration of the inflaton field are small, then the inflationary expansion will last for a long time.

6.2 Slow-roll approximation

- The condition $\epsilon \ll 1$ implies $\dot{\phi}^2 \ll V$. Using this in the Friedmann equation

$$H^2 \approx \frac{V}{3M_P^2}. \tag{84}$$

- The condition $\delta \ll 1$ simplifies the Klein-Gordon equation to

$$3H\dot{\phi} = -\frac{dV}{d\phi} \equiv -V_{,\phi}. \tag{85}$$

  This provides a simple relationship between the slope of the potential and the speed of the inflaton.

Using the two expressions one finds the potential slow-roll parameters

$$\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2 M_P^2} = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} \approx \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \equiv \epsilon_V, \tag{86}$$

$$\delta + \epsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} \approx \frac{M_P^2}{2}\left(\frac{V_{,\phi\phi}}{V}\right) \equiv \eta_V. \tag{87}$$

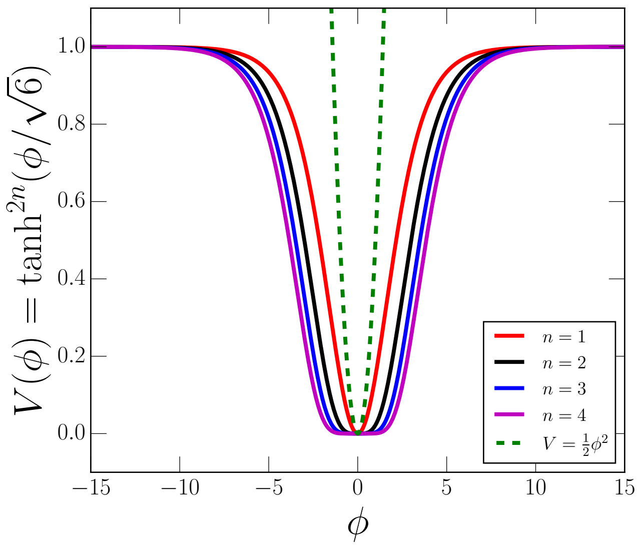


Figure 7: Inflaton potentials

$$\left(\frac{dV}{d\phi}\right)^2 \ll 9H^2V. \tag{88}$$

Successful inflation occurs when these parameters are much smaller than unity.

The total number of e-foldings of accelerated expansion is

$$N = \int_{a_i}^{a_f} d\ln a = \int_{t_i}^{t_f} H\,dt = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}}\,d\phi \tag{89}$$

$$\approx \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon_V}M_P}\,d\phi. \tag{90}$$

6.2.1 Quadratic inflation

The simplest model of inflation is

$$V(\phi) = \frac{1}{2}m^2\phi^2, \tag{91}$$

which implies

$$\epsilon_V = \eta_V = 2\frac{M_P^2}{\phi^2}. \tag{92}$$

From this we need super-Plankian values for the inflaton to satisfy slow-roll conditions:

$$\phi > \phi_f \equiv \sqrt{2}M_P. \tag{93}$$

The number of e-fold is

$$N \approx \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon_V}M_P}\,d\phi = \frac{\phi_i^2}{4M_P^2} - \frac{1}{2}. \tag{94}$$

Hence for $N = 60$, $\phi_i > 2\sqrt{60}M_P \sim 15M_P$.

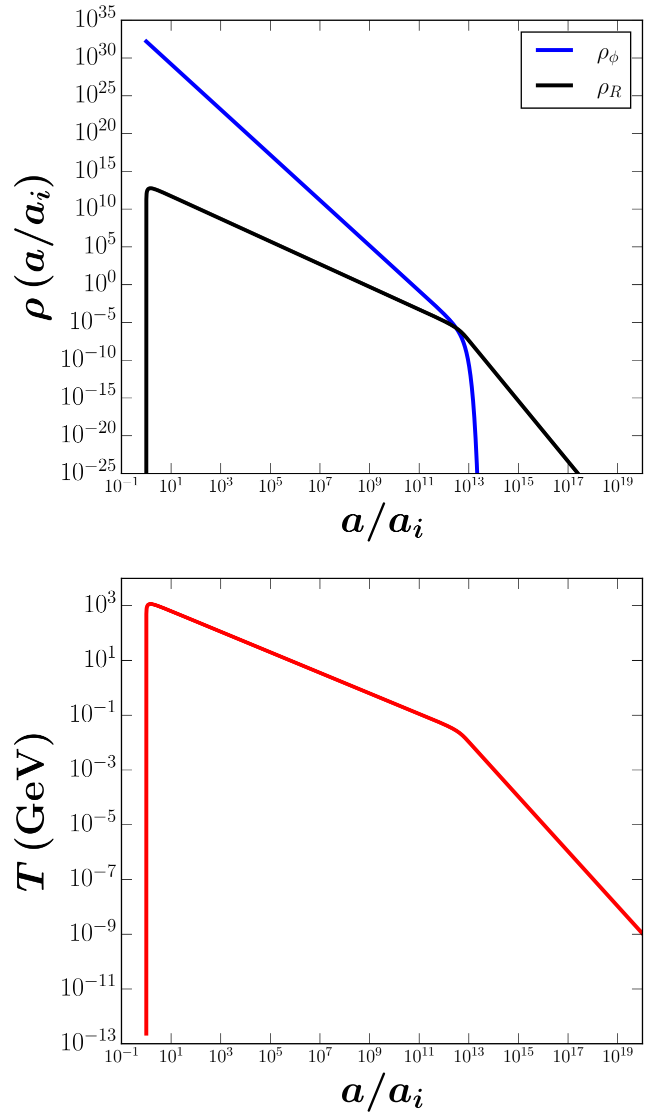


Figure 8: Energy densities and temperature as a function of the scale factor during reheating.

6.3 Reheating

$$\frac{d\rho_\phi}{dt} + 3H\rho_\phi = -\Gamma_\phi \frac{d\rho_\phi}{dt}\,, \tag{95}$$

$$\frac{d\rho_R}{dt} + 4H\rho_R = \Gamma \frac{d\rho_\phi}{dt}\,, \tag{96}$$

$$H = \sqrt{\frac{\rho_\phi + \rho_R}{3\,M_P^2}}\,. \tag{97}$$

Here, Γ_ϕ is the total decay width of the inflaton into radiation, an the SM energy density is defined as a function of the temperature T of the SM photons as

$$\rho_R = \frac{\pi^2}{30} g_* T^4. \tag{98}$$

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