Early Universe

Exercises Week #4

1. Is the total energy of the Universe conserved as it expands?

2. Consider the equation of state $p(\epsilon) = (\gamma - 1)\epsilon$, where γ is a constant in the range $0 < \gamma < 2$. Find the depedence of $\epsilon = \epsilon(a)$, $a = a(t)$ and hence $\epsilon = \epsilon(t)$ for universes containing such matter (assume $k = 0$ in the Friedmann equation).

3. Using your answer to previous problem, what value of γ would be needed so that ϵ has the same dependence as the curvature term k/a^2? Find the solution $a(t)$ to the full Friedmann equation for a fluid with this γ, assuming negative k.

4. Use $R_0 = 1$. Consider the case $k > 0$, with an Universe containing only matter ($p = 0$) so that $\epsilon = \epsilon_0/a^3$. Demonstrate that the parametric solution

$$a(\theta) = \frac{4\pi G\epsilon_0}{3k}(1 - \cos\theta), \quad t(\theta) = \frac{4\pi G\epsilon_0}{3k^{3/2}}(\theta - \sin\theta), \tag{1}$$

solves the Friedmann equation, where θ is a variable which runs form 0 to 2π. Sketch a and t as functions of θ. Describe qualitatively the behaviour of the Universe.

5. Consider the case $k < 0$, with a universe containing only matter ($p = 0$) so that $\epsilon = \epsilon_0/a^3$.

   a) Demonstrate that the parametric solution

   $$a(\eta) = \frac{\Omega_0}{2(1 - \Omega_0)}(\cosh\eta - 1), \quad t(\eta) = \frac{\Omega_0}{2H_0(1 - \Omega_0)^{3/2}}(\sinh\eta - \eta), \tag{2}$$

   solves the Friedmann equation ($H^2 = 8\pi G\epsilon/3 - k/a^2$), where η is a parameter which runs form 0 to ∞. Describe qualitatively the behaviour of the Universe.

   b) What is the solution $a(t)$ in a situation where the curvature term of the Friedmann equation dominates over the density term?

   c) Is domination by the curvature term a stable situation that will continue forever?

6. The most likely cosmology describing our own Universe has a lat geometry with a matter density of $\Omega_0 \sim 0{,}3$ and a cosmological constant with $\Omega_\Lambda(t_0) = 0{,}7$. What will the values of Ω and Ω_Λ be when the Universe has expanded to be five times its present size? Use an approximation suggested by this result to find the late-time solution to the Friedmann equation for our Universe.

7. Suppose that the Universe contains four different contributions to the Friedmann equation, namely radiation, non-relativistic matter, a cosmological constant, and a negatuve (hyperbolic) curvature. Which of them would you expect to dominate the Friedmann equation at early times, and which at late time?

8. By considering both the Friedmann and acceleration equations, and assuming that the radiation fluid is not present ($\epsilon_R = 0$), demonstrate that in order to have a static universe (with constant a) in presence of matter $\epsilon_M \neq 0$ we must have a closed universe with a positive vacuum energy. Using either physical arguments or mathematics, demonstrate that this solution must be unstable.

9. Consider a flat Universe without radiation and cosmological constant, that is, a matter-dominated flat Universe. Give a physical argument explaining why introducing a positive cosmological constant will increase the predicted age of the Universe.

10. For a Universe containing only non-relativistic matter (and no cosmological constant), show that the Friedmann equation can be rewritten as

$$H(z) = H_0(1+z)(1+\Omega_{M,0}z)^{1/2}. \tag{3}$$

11. A hypothesis once used to explain the Hubble relation is the "tired light hypothesis". The tired light hypothesis states that the universe is not expanding, but that photons simply lose energy as they move through space (by some unexplained means), with the energy loss per unit distance being given by the law

$$\frac{dE}{dr} = -\alpha E, \tag{4}$$

where α is a constant.

a) Show that this hypothesis gives a distance–redshift relation that is linear in the limit $z \ll 1$.

b) What must the value of α be in order to yield a Hubble constant of $H_0 = 68$ km/s/Mpc?

12. Suppose you are in a flat, matter-only universe that has a Hubble constant $H_0 = 68$ km/s/Mpc. You observe a galaxy with $z = 1$. How long will you have to keep observing the galaxy to see its redshift change by one part in 10^6?

13. Consider a FLRW Universe dominated by a perfect fluid with pressure $p = \omega\epsilon$, with $\omega = const.$ Taking into account that the time-dependent density parameter is defined as $\Omega(t) = \epsilon(t)/\epsilon_c(t)$, where the time-dependent critical density is $\epsilon_c(t) = 3H^2/(8\pi G_N)$. Show that

$$\frac{d\Omega}{d\ln a} = (1+3w)\Omega(\Omega-1). \tag{5}$$

Discuss the evolution of $\Omega(a)$.

14. Consider the Universe filled with matter whose equation of state is that of Chaplygin gas,

$$p = -\frac{A}{\epsilon}, \qquad A = \text{constant}. \tag{6}$$

a) Find the dependence of the Hubble parameter on the scale factor.

b) Find the law of evolution $a = a(t)$ at small and large scale factors in all three cases, $k = 0, \pm 1$.

c) Find the complete evolution $a = a(t)$ in the case of spatially flat Universe.

d) What values of k admit static solutions to the Einstein equations?

e) What can be said about the future of the Universe, if it is known that at some moment of time the expansion of the Universe accelerates? Consider all three cases, $k = 0, \pm 1$.

15. At what z did the contributions to energy density due to non-relativistic matter and dark energy become equal to each other?

16. At what z does the transition from deceleration to acceleration occur for dark energy with equation of state $p = \omega\epsilon$, $\omega = \text{const}$? For what value of the parameter ω this transition would occur now? Give numerical estimate using the values given in class.

17. Consider open model without dark energy (this model in fact is excluded by CMB data), in which $\Omega_{M,0} \neq 0$, $k \neq 0$, $\Omega_\Lambda = 0$ and $\Omega_{R,0} = 0$. Find the present age of the Universe at given value of H_0. Calculate the numerical value for $\Omega_{M,0} = 0{,}3$ (estimated from the studies of clusters of galaxies) and $h = 0{,}7$.

18. Find the present age of the Universe for dark energy with equation of state $p = \omega\epsilon$, $\omega = \text{const}$. Give numerical estimates for $\omega = -1{,}1$ and $\omega = -0{,}9$ with $\Omega_{M,0} = 0{,}3$, $\Omega_{\Lambda,0} = 0{,}7$.

19. Einstein introduced the cosmological constant into his field equations to avoid the conclusion that the universe is expanding. In this problem, you will see that this was misguided.

a) Show that for a perfect fluid with positive density and pressure there is no static solution to the Einstein equations.

b) Consider now a universe with pressureless matter and a cosmological constant. Show that it is possible to obtain a static solution–called the Einstein static universe–if

$$\Lambda = 4\pi G\epsilon_{M,0}. \tag{7}$$

What is the spatial curvature of this solution?

c) Show that the Einstein static universe is unstable to small perturbations. *Hint*: Consider a small perturbation around the static solution

$$\epsilon_M(t) = \epsilon_{M,0}[1 + \delta(t)], \qquad a(t) = 1 + \alpha(t), \tag{8}$$

with $|\delta| \ll 1$ and $|\alpha| \ll 1$. Show that the perturbations δ and α are related to each other and that they grow exponentially with time.

20. Dark energy is most likely a cosmological constant with equation of state $\omega = -1$. Radical alternatives have nevertheless been considered. In this problem, you will study the dynamics of "phantom dark energy," with $\omega < -1$.

a) Consider a flat universe with matter (M) and phantom dark energy (X). Show that the energy density of the dark energy increases with time. If the scale factor today is $a_0 \equiv 1$, show that in the future

$$\Omega_X \equiv \frac{\epsilon_X(a)}{\epsilon_c(a)} = \left(1 + \frac{\Omega_{M,0}}{\Omega_{X,0}} a^{3\omega_X}\right)^{-1}. \tag{9}$$

b) If $\Omega_{X,0} = 0{,}75$ and $\omega_X = -2$, at what scale factor is 99.9 % of the energy density in dark energy?

c) If the dark energy density dominates the matter density at a time t_* , show that $a \to \infty$ in a finite time Δt. This divergence of the scale factor has been called a "Big Rip." Find Δt in terms of ω_X and the Hubble parameter H_* at the time t_*.

d) What would happen to the observed wavelength of CMB photons as the Big Rip is approached?