Relativistic Cosmology

Cosmology is the study of the Universe as a whole, including its origin, nature, evolution and eventual fate. It has ancient roots in philosophy and religion, but modern scientific cosmology dates from 1917 when Einstein first used general relativity to formulate a mathematical model of the Universe.

General relativity can be applied to the Universe (at large scales)

- The 4d spacetime can be described by an appropiate metric tensor $g_{\mu\nu}$, which can, in principle be determined by solving the field Einstein equations.

- For this goal, we have to provide the distribution of energy and momentum at cosmic (large) scales. In other words, we need to be able to write down a $T^{\mu\nu}$ for the whole Universe.

The idea is to find a simple prescription for the cosmic $T^{\mu\nu}$ that capture the essential large-scale features of the Universe while ignoring the details a small to medium scales.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\kappa T_{\mu\nu} \quad , \quad \kappa = 8\pi G_N / c^4 .$$

✓ Energy-momentum tensor: $T_{\mu\nu} = (\rho + p/c^2) U_\mu U_\nu - p g_{\mu\nu}$.

✓ Equation of state: $p = p(\rho)$

- Radiation (relativistic matter): $p_R = c^2 \rho_R / 3$,
- Dust (non-relativistic matter): $p_M = 0$,
- Vacuum energy (cosmological constant): $p_\Lambda = -c^2 \rho_\Lambda$, $\rho_\Lambda = \Lambda c^2 / 8\pi G_N$.

The applicability of general relativity

It is assumed that Einstein's original (unmodified) field equations of general relativity can be applied to the Universe as a whole, provided that a possible contribution from dark energy is included. We may then speak interchangeably of a Universe characterized by a cosmological constant Λ or one in which there is a dark energy contribution of density ρ_Λ and (negative) pressure $p_\Lambda = -\rho_\Lambda c^2 = -\Lambda c^4 / 8\pi G$.

The cosmological principle

The cosmological principle

At any given time, and on a sufficiently large scale, the Universe is **homogeneous** (i.e. the same everywhere) and **isotropic** (i.e. the same in all directions).

homogeneity: same density and same curvature.

Isotropic: any coordinate system gives the same geometry.

- Large scales: $d \gtrsim 100$ Mpc.

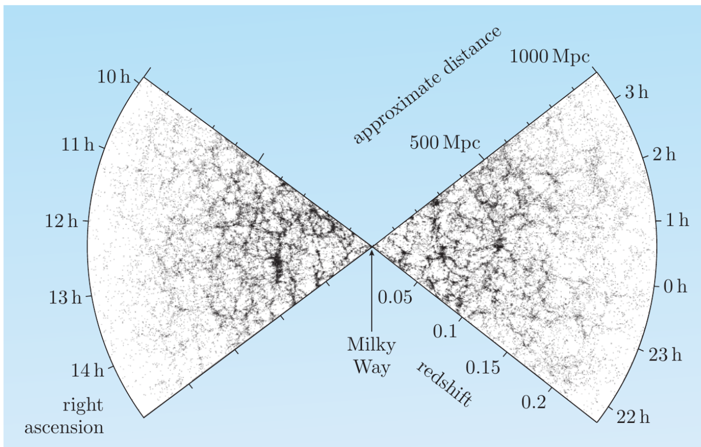


Figure 8.1 The distribution of galaxies reported by the 2dF survey.

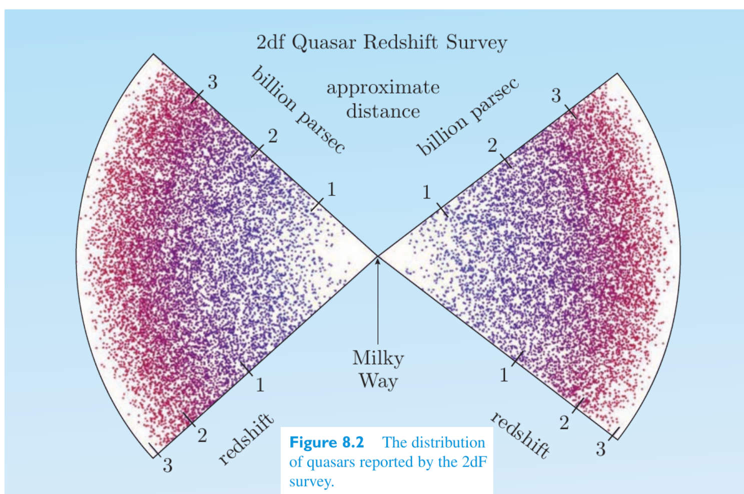
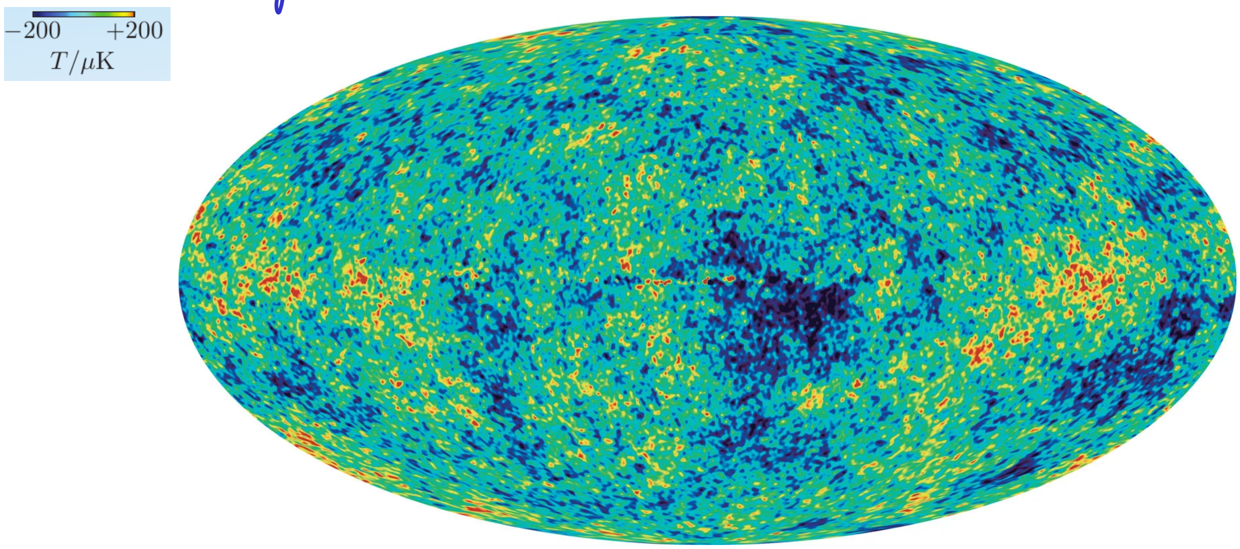


Figure 8.2 The distribution of quasars reported by the 2dF survey.

Large-scale anysotropy in the CMBR



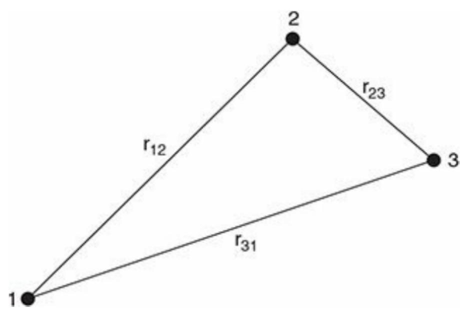
Studies regarding galaxies' motion have confirmed that the large-scale motion, the Hubble flow, is isotropic so it can be characterized by a single rate of expansion at any time: $H = H(t)$, the Hubble constant.

Notice that individual galaxies have their own so-called peculiar motion relative to the large-scale expansion.

$$r_{12} \equiv |\vec{r}_1 - \vec{r}_2|$$

$$r_{23} \equiv |\vec{r}_2 - \vec{r}_3|$$

$$r_{31} \equiv |\vec{r}_3 - \vec{r}_1| .$$



A triangle defined by three galaxies in a uniformly expanding universe.

Homogeneous and uniform expansion means that the shape of the triangle is preserved as the galaxies move away from each other. Maintaining the correct relative lengths for the sides of the triangle requires an expansion law of the form

$$r_{12}(t) = a(t) r_{12}(t_0) \qquad r_{23}(t) = a(t) r_{23}(t_0) \qquad r_{31}(t) = a(t) r_{31}(t_0).$$

$$v_{12}(t) = \frac{dr_{12}}{dt} = \dot{a} r_{12}(t_0) = \frac{\dot{a}}{a} r_{12}(t)$$

$$v_{31}(t) = \frac{dr_{31}}{dt} = \dot{a} r_{31}(t_0) = \frac{\dot{a}}{a} r_{31}(t).$$

Weyl's postulate

to which observers do we demand the universe to appear isotropic?

Fundamental observers: priviliged class of observers who have a particularly view of the Universe, since they move with the Hubble flow.

They find that the Universe around them (including the CMBR) is isotropic.

The proper time measured by each fudamental observer can be correlated with that of every other F.O. so that a value of a single, universally meaningful cosmic time can be associated with every event.

The parameter t is then called the *synchronous time coordinate*.

- We can idetify all the events characterized by any paticular value of cosmic time.
- Such a set of events will form a 3D space (a space-like hypersurface) with geometry properties that are homogeneous and isotropic.

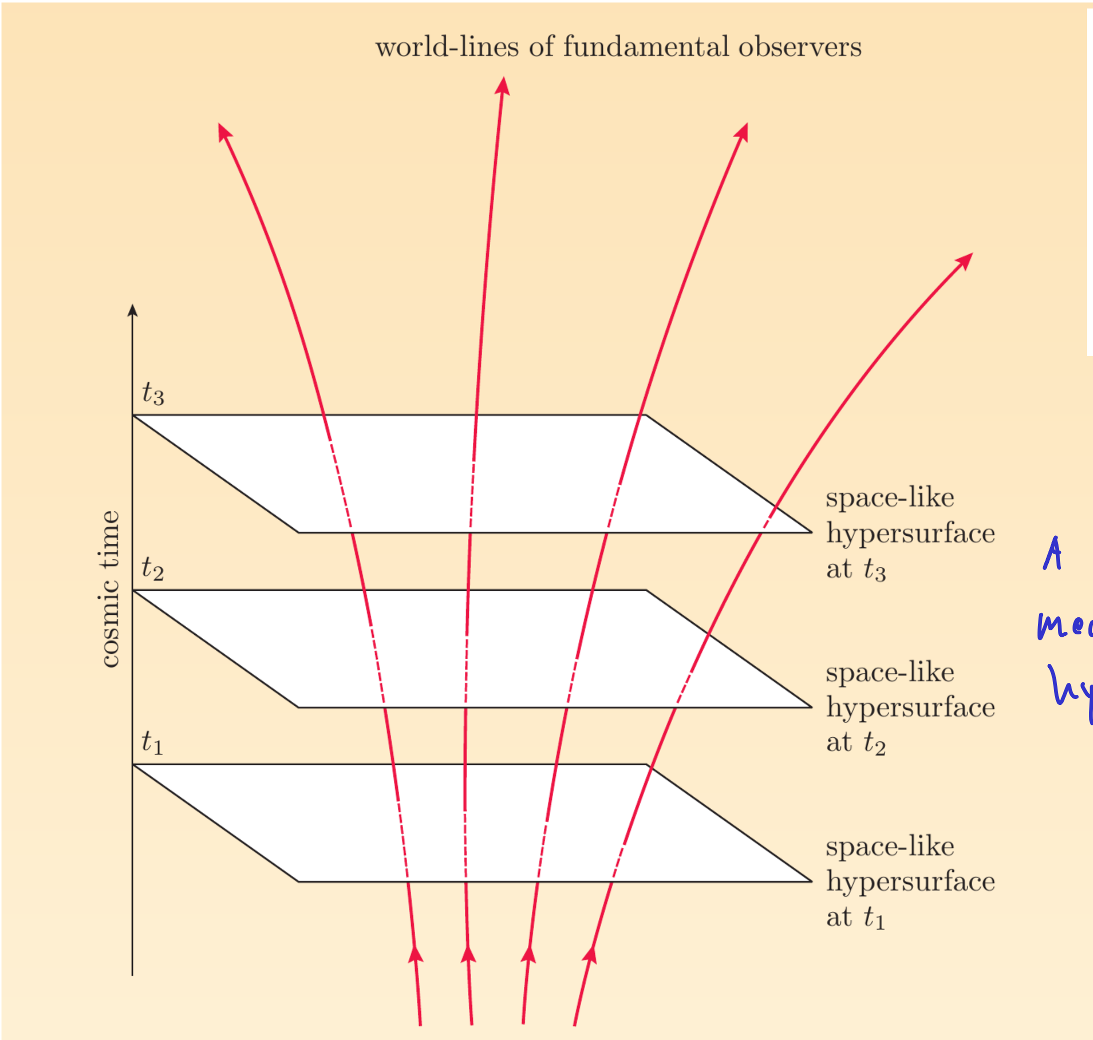


Figure 8.4 The world-lines in cosmic spacetime of the fundamental observers who see the Universe as homogeneous and isotropic. Each world-line can be labelled by fixed co-moving coordinates but intersects successive space-like hypersurfaces at different values of cosmic time.

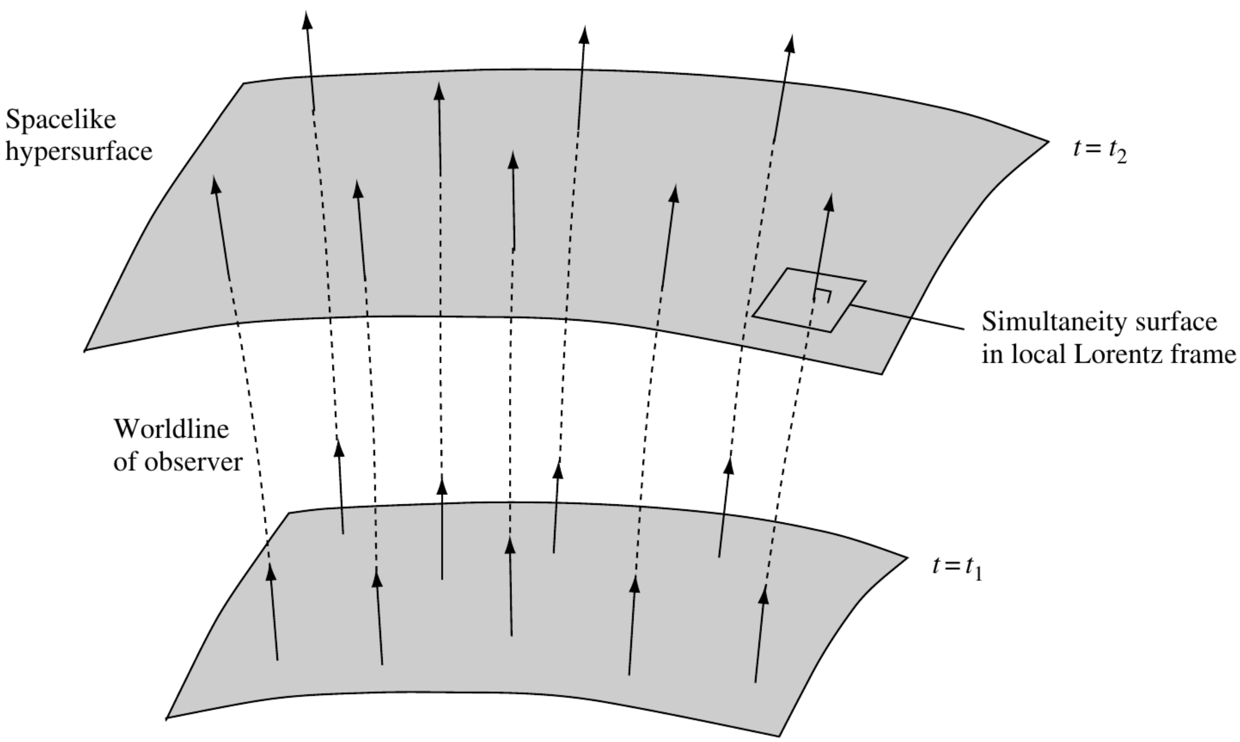
A particular cosmic time means a given spacelike hypersurface.

There is one and only one geodesic passing through each point of spacetime. Therefore, matter at any point possesses a unique velocity: this means that the matter may be taken to be a perfect fluid.

The Weyl's postulate requires that the geodesics are orthogonal to a family of a spacelike hypersurfaces.

The hypersurfaces $t=$ constant may constructed in such a way that the 4-velocity of any fundamental observer is orthogonal to the hypersurface.

Thus, the surface of simultaneity of the local Lorentz frame of any such observer coincides locally with the hypersurface (see Figure 14.1). Each hypersurface may therefore be considered as the 'meshing together' of all the local Lorentz frames of the fundamental observers.



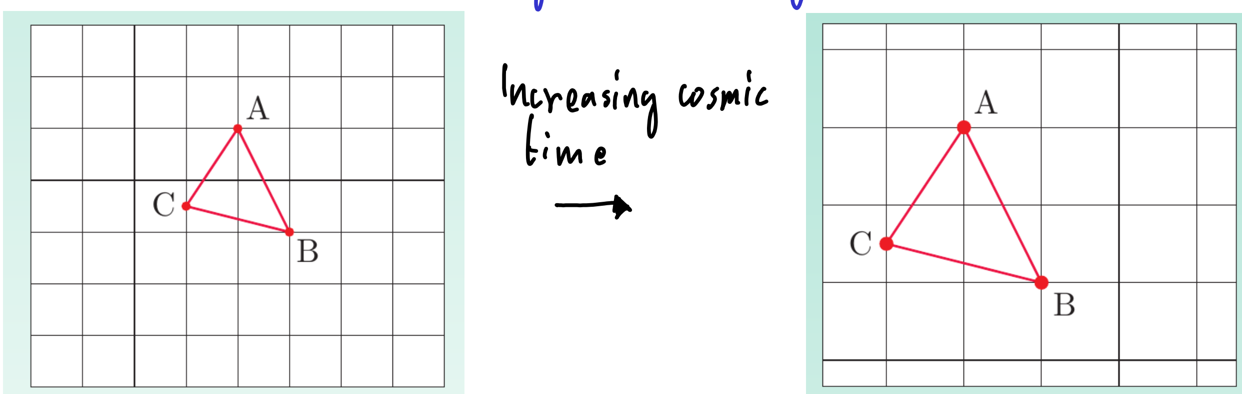
The spacelike hypersurfaces are labelled by t: cosmic time.
let (x^1, x^2, x^3) the spatial coordinates that are constant for any fund. observer.
Since each hypersurface $t=$ constant is orthogonal to the observer's worldline, the line element takes the form

$$ds^2 = c^2 dt^2 - g_{ij}\, dx^i dx^j \,, \quad i,j = 1,2,3, \quad g_{ij} = g_{ij}(x^\mu).$$

Comoving coordinates: the worldline of a fundamental observer is assigned the same values of the 3 spatial coordinates on every space-like hypersurface.

- The grid of comoving coordinates must expand or contract with the space-like hypersurfaces.
- Ignoring the individual peculiar motions, every galaxy will have constant comoving coordinates.

Comoving coordinate grid



If we consider the triangle formed by three nearby galaxies at some particular time t, then isotropy requires that the triangle formed by these same galaxies at some later time must be similar to the original triangle. Moreover, homogeneity requires that the magnification factor must be *independent* of the position of the triangle in the 3-space. It thus follows that time t can enter the g_{ij} only through a common factor, so that the ratios of small distances are the same at all times.

The squared spatial separation on the same hypersurface $t = \text{constant}$ of two nearby galaxies at (x^1, x^2, x^3) and $(x^1 + \Delta x^1, x^2 + \Delta x^2, x^3 + \Delta x^3)$ is

$$d\sigma^2 = g_{ij}\, \Delta x^i \Delta x^j .$$

Friedmann - Lemaitre - Robertson - Walker metric.

Combining the metric for a maximally symmetric 3D space with the element line $ds^2 = c^2dt^2 - S^2(t)\, h_{ij}\, dx^i dx^j$, we arrive to the FLRW metric.

$$ds^2 = c^2\,dt^2 - S^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right].$$

Proper distance !! It depends on t!

Assuming that $K \neq 0$, let's define $k = K/|K| = \pm 1$.
Defining $\bar{r} = |K|^{1/2}\, r$

$$ds^2 = c^2\,dt^2 - \frac{S^2(t)}{|K|}\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right].$$

Finally, we define a rescaled scale function $R(t)$ by

$$R(t) = \begin{cases} \dfrac{S(t)}{|K|^{1/2}} & \text{if } K \neq 0, \\ S(t) & \text{if } K = 0. \end{cases}$$

Then, dropping the bars on the radial coordinate, we obtain the standard form for the *Friedmann–Robertson–Walker* (FRW) line element,

$$\boxed{ds^2 = c^2\,dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right],}$$

where k takes the values -1, 0, or 1 depending on whether the spatial section has negative, zero or positive curvature respectively. It is also clear that the coordinates (r, θ, ϕ) appearing in the FRW metric are still *comoving*, i.e. the worldline of a galaxy, ignoring any peculiar velocity, has fixed values of (r, θ, ϕ).

If the scale factor $R(t)$ increases with time, then the fundamental observers become more widely separated with time, the galaxies containing those fundamental observers get further apart, and the Universe is said to be **expanding**. If $R(t)$ decreases with time, then the fundamental observers and their associated galaxies get closer together, and the Universe may be said to be **contracting**. Remember, though, that throughout this process the co-moving coordinates of any fundamental observer remain fixed at all times. Also remember that the space-like hypersurfaces are homogeneous and isotropic, so although the coordinate system will have some particular origin and some particular orientation, any point may be chosen to be the origin, and the chosen orientation of the axes is equally arbitrary.

Geometric properties of the FRW metric

The geometric properties of the homogeneous and isotropic 3-space corresponding to the hypersurface $t =$ constant depend upon whether $k = -1$, 0 or 1. We now consider each of these cases in turn.

Positive spatial curvature: $k=1$. Closed Universe
For $k=1 \Rightarrow g_{rr} \to \infty$. However it is a coordinate singularity.

let $r = \sin\chi \Rightarrow dr = \cos\chi\, d\chi = (1-r^2)^{1/2} d\chi$
Thus $d\sigma^2 = R^2 [\, d\chi^2 + \sin^2\chi \,(d\theta^2 + \sin^2\theta\, d\phi^2)\,]$.

Embedding

$$\begin{aligned} w &= R\cos\chi, \\ x &= R\sin\chi\sin\theta\cos\phi, \\ y &= R\sin\chi\sin\theta\sin\phi, \\ z &= R\sin\chi\cos\theta. \end{aligned}$$

$$w^2+x^2+y^2+z^2 = R^2.$$

$$0 \le \chi \le \pi, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi.$$

$$d\sigma^2 = dw^2+dx^2+dy^2+dz^2 = R^2\left[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\, d\phi^2)\right],$$

This shows that our 3-space can be considered as a three-dimensional sphere in the four-dimensional Euclidean space

For $d\theta = d\phi = 0 \Rightarrow d\sigma = R\,d\chi$: Proper distance along a radial path.
$\Rightarrow \sigma_{max} = R\pi$: Maximum distance that can be apart two observers.

The surfaces $\chi =$ constant are 2-spheres with surface area

$$A = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(R\sin\chi\, d\theta)(R\sin\chi\sin\theta\, d\phi) = 4\pi R^2\sin^2\chi,$$

and (θ, ϕ) are the standard spherical polar coordinates of these 2-spheres. Thus, as χ varies from 0 to π, the area of the 2-spheres increases from zero to a maximum value of $4\pi R^2$ at $\chi = \pi/2$, after which it decreases to zero at $\chi = \pi$. The proper radius of a 2-sphere is $R\chi$, and so the surface area is smaller than that of a sphere of radius $R\chi$ in Euclidean space.

The entire 3-space has a *finite* total volume given by

$$V = \int_{\chi=0}^{\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (R\,d\chi)(R \sin\chi\, d\theta)(R\sin\chi \sin\theta\, d\phi) = 2\pi^2 R^3,$$

which is the reason why, in this case, R is often referred to as the 'radius of the universe'.

Zero spatial curvature: $k=0$; flat Universe

let $r=\chi$, then

$$d\sigma^2 = R^2\left[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right],$$

$$x = R\chi \sin\theta \cos\phi, \qquad y = R\chi \sin\theta \sin\phi, \qquad z = R\chi\cos\theta,$$

$$d\sigma^2 = dx^2 + dy^2 + dz^2.$$

Negative spatial curvature: $k=-1$. Open Universe

$$r = \sinh\chi \qquad \Rightarrow \qquad dr = \cosh\chi\, d\chi = (1+r^2)^{1/2}\, d\chi,$$

$$d\sigma^2 = R^2\left[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right].$$

Embedding in a 4D Minkowski space:

$$w = R\cosh\chi, \qquad 0 \le \chi \le \infty, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi.$$

$$x = R\sinh\chi \sin\theta\cos\phi,$$

$$y = R\sinh\chi\sin\theta\sin\phi,$$

$$z = R\sinh\chi\cos\theta.$$

$$d\sigma^2 = dw^2 - dx^2 - dy^2 - dz^2,$$

$$w^2 - x^2 - y^2 - z^2 = R^2,$$

This shows that the 3-space can be represented as a three-dimensional hyperboloid in the four-dimensional Minkowski space.

The 2-surfaces $\chi =$ constant are 2-spheres with surface area

$$A = 4\pi R^2 \sinh^2\chi,$$

which increases indefinitely as χ increases. The proper radius of such a 2-sphere is $R\chi$, and so the surface area is larger than the corresponding result in Euclidean space. The total volume of the space is infinite.

From the above discussion, we see that a convenient form for the FRW metric is

$$\boxed{ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right],}$$

where the function $r = S(\chi)$ is given by

$$\boxed{S(\chi) = \begin{cases} \sin\chi & \text{if } k = 1, \\ \chi & \text{if } k = 0, \\ \sinh\chi & \text{if } k = -1. \end{cases}}$$

Once again, it is clear that (χ, θ, ϕ) are *comoving coordinates*.

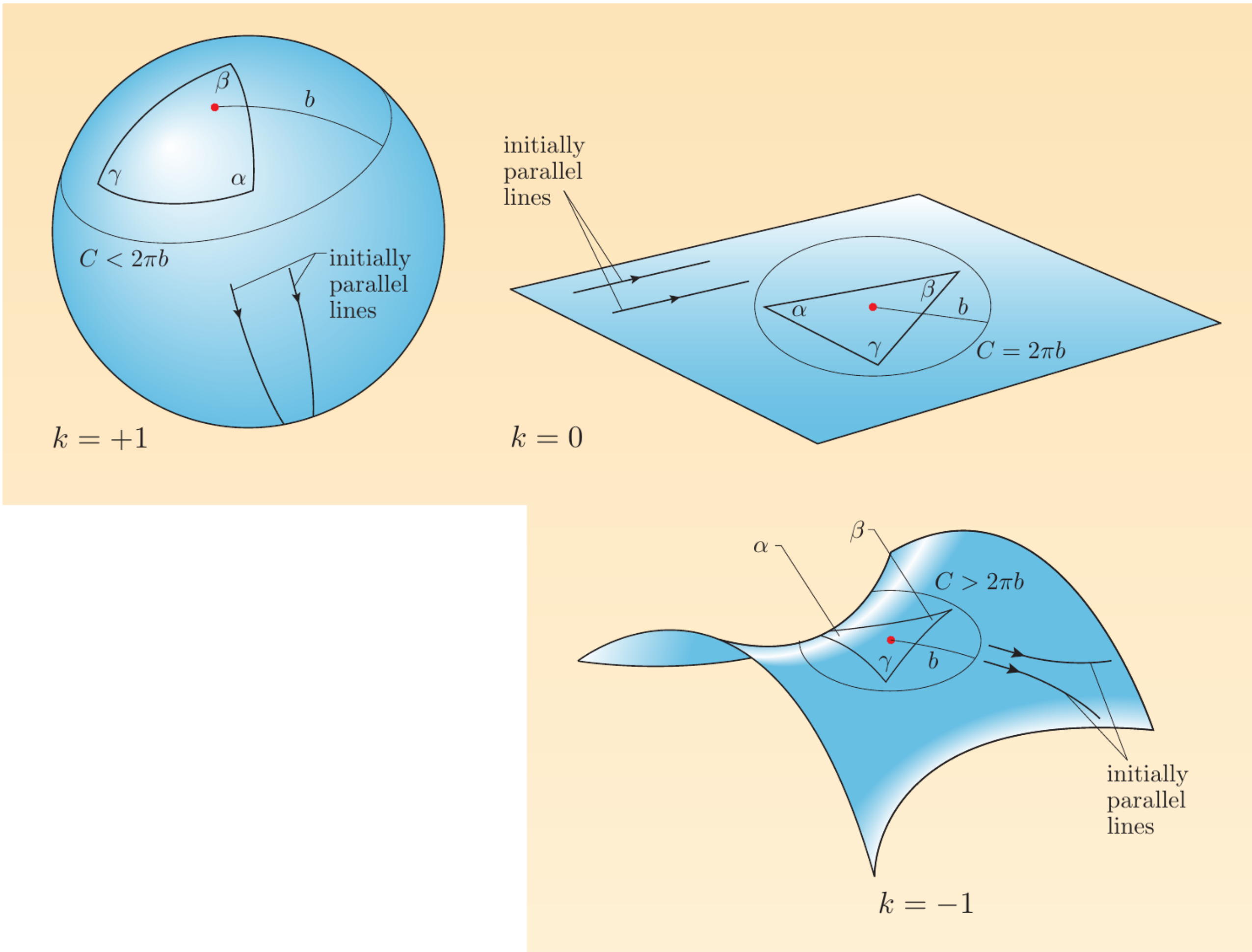
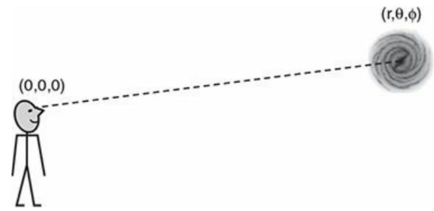


Figure 8.8 Two-dimensional surfaces can provide useful and memorable analogues of the three-dimensional space-like hypersurfaces in the cases $k = +1, 0, -1$. In each case, a circle of proper radius b and proper circumference C is drawn in the surface.

Proper distances and velocities in cosmic spacetime



For two simultaneous events that occur with infinitesimally separated positions, (r, θ, ϕ) and $(r + \mathrm{d}r, \theta + \mathrm{d}\theta, \phi + \mathrm{d}\phi)$, the proper distance separating them can be read directly from the Robertson–Walker line element. Using the symbol $\mathrm{d}\sigma$ to represent that infinitesimal distance, we have

$$\mathrm{d}\sigma = R(t)\left[\frac{(\mathrm{d}r)^2}{1-kr^2} + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta\,(\mathrm{d}\phi)^2\right]^{1/2}.$$

Note that this proper distance element depends on the proper time at which it is measured. This is to be expected in an expanding or contracting Universe since proper separations will change with time even though (co-moving) coordinates don't change their values.

Given two points on such a hypersurface, we can always choose one of them to be the origin of coordinates. The other will then be at some specific co-moving radial coordinate value, $r = \chi$ say, in a fixed direction, specified by particular values of θ and ϕ. In such a case, the two points are linked by a purely radial path that will always be a geodesic (we shall not prove this). Along that radial path $\mathrm{d}\theta = 0$ and $\mathrm{d}\phi = 0$, so the element of proper distance is just $\mathrm{d}\sigma = R(t)\,\mathrm{d}r/(1-kr^2)$. Thus, given two points separated by a fixed radial co-moving coordinate χ, the proper distance between them at time t will be

$$\sigma(t) = \int_0^{\chi} R(t)\,\frac{\mathrm{d}r}{(1-kr^2)^{1/2}}.$$

Proper distance σ related to co-moving coordinate χ

$$\sigma(t) = \begin{cases} R(t)\sin^{-1}\chi & \text{if } k = +1, \\ R(t)\,\chi & \text{if } k = 0, \\ R(t)\sinh^{-1}\chi & \text{if } k = -1. \end{cases}$$

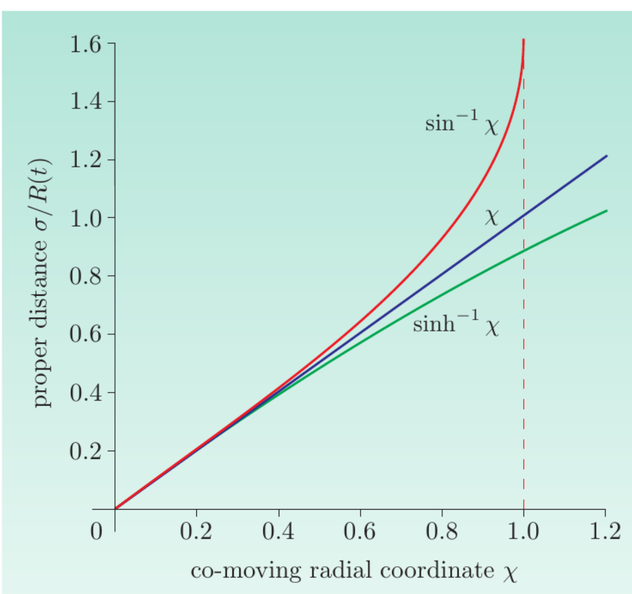


Figure 8.7 The relationship between proper distance and a co-moving radial coordinate χ for the space-like hypersurface corresponding to cosmic time t, in the cases $k = +1, 0, -1$. Note that the proper distance is expressed as a multiple of $R(t)$.

An important point to note concerning co-moving coordinates and their relationship to proper distances involves units and dimensions. The proper distance between two points must be a length. However, the co-moving coordinate is not subject to the same restriction. Since all proper lengths are proportional to the scale factor $R(t)$, it is conventional to treat the co-moving coordinate $r = \chi$ as dimensionless and the scale factor $R(t)$ as having the dimensions of length.

Defining the **proper radial velocity** as the rate of change of proper distance with respect to cosmic time,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \begin{cases} \dfrac{\mathrm{d}R}{\mathrm{d}t}\sin^{-1}\chi & \text{if } k = +1, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t}\chi & \text{if } k = 0, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t}\sinh^{-1}\chi & \text{if } k = -1. \end{cases}$$

In each case we can replace the term involving χ by σ/R. This leads to the same expression for the proper velocity in all three cases:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\sigma.$$

It is conventional to write this relationship in the more memorable form

$$v_{\mathrm{p}} = H(t)\, d_{\mathrm{p}},$$

where d_{p} represents the proper distance between two fundamental observers or their galaxies, v_{p} represents the proper radial velocity at which they are separating (for positive v_{p}) or coming together (for negative v_{p}), and $H(t)$, which is called the **Hubble parameter**, is defined as follows.

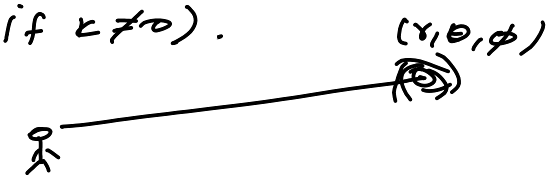
The Hubble parameter

$$H(t) = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}.$$

Equation 8.16 tells us that at any cosmic time t, every fundamental observer is moving radially relative to every other fundamental observer at a proper speed that is proportional to the proper distance that separates them. Note that this is an exact consequence of the nature of Robertson–Walker spacetime. Later we shall re-examine this result in connection with Hubble's observations of cosmic expansion. At that stage we shall relate the proper distance to some other distances that really can be measured and also relate the Hubble parameter to an observable quantity known as the *Hubble constant*.

Cosmological redshift and cosmic expansion

We need to know $a(t_0)$, κ, R_0 (if $\kappa \neq 0$).



Proper distance: (t is fixed)

$$ds^2 = a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2 \right]$$

For Ω: constant $\rightarrow$ $ds = a(t)\,dr$

$$d_p(t) = a(t) \int_0^r dr = a(t)\, r.$$

$$\dot{d}_p(t) = \dot{a} r = \frac{\dot{a}}{a} d_p(t) \Rightarrow v_p(t_0) = H_0 d_p(t_0).$$

If $d_p > d_H \Rightarrow v_p > \dot{d}_p > c$. $\quad d_H = 4380$ Mpc

Along a null geodesic: $ds = 0$. $\quad$ independent of t.

With $d\Omega =$ constant: $\quad c^2 dt^2 = a^2 dr^2 \Rightarrow dr = \frac{c\,dt}{a(t)}$

$$c \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r$$

Next crest is emitted at $t_e + \lambda_e/c$ and observed at time $t_0 + \lambda_0/c$: $c \int_{t_e+\lambda_e/c}^{t_0+\lambda_0/c} \frac{dt}{a(t)} = \int_0^r dr = r$

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\lambda_e/c}^{t_0+\lambda_0/c} \frac{dt}{a(t)}.$$

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} - \int_{t_e+\lambda_e/c}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\lambda_e/c}^{t_0+\lambda_0/c} \frac{dt}{a(t)} - \int_{t_e+\lambda_e/c}^{t_0} \frac{dt}{a(t)}$$

$$\int_{t_e}^{t_e+\lambda_e/c} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\lambda_0/c} \frac{dt}{a(t)}.$$

$$H_0^{-1} \approx 14\,\mathrm{Gyr}.$$

$$\lambda/c \approx 2 \times 10^{-15}\,\mathrm{s} \approx 10^{-32} H_0^{-1}.$$

Friedmann equations and cosmic evolution

For some k, what is the evolution of $R(t)$?
The answer is provided by the Einstein field equations. However we must assume some $T_{\mu\nu}$.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}, \qquad R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - T/2\, g_{\mu\nu} \right).$$

Assuming that the Universe is populated by a perfect fluid, which is characterised at each point by its proper density and pressure in the instantaneous rest frame:

$$T^{\mu\nu} = (\rho + p/c^2)\, U^\mu U^\nu - p\, g^{\mu\nu}.$$

For a homogeneous and isotropic universe: $\rho = \rho(t)$; $p = p(t)$.

- $$\left(\frac{1}{R}\frac{dR}{dt}\right)^2 = \left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G_N}{3}\rho - \frac{c^2 k}{R^2},$$ energy equation.

- $$\frac{1}{R}\frac{d^2R}{dt^2} = \frac{\ddot{R}}{R} = -\frac{4\pi G_N}{3}\left(\rho + 3p/c^2\right).$$ acceleration equation

From $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow$ $$\boxed{\dot{\rho} + \left(\rho + \frac{p}{c^2}\right)\frac{3\dot{R}}{R} = 0,}$$ fluid equation

$$\frac{d(\rho R^3)}{dt} = -\frac{3p\dot{R}R^2}{c^2}. \qquad \frac{d(\rho R^3)}{dR} = -\frac{3pR^2}{c^2}.$$

Introducing the equation of state parameter: $p = w\rho c^2$:

$$\frac{d(\rho R^3)}{dR} = -3w\rho R^2.$$

This equation has the immediate solution

$$\boxed{\rho \propto R^{-3(1+w)},}$$

which gives the evolution of the density ρ as a function of the scale factor $R(t)$. Note that in general ρc^2 is the energy density of the fluid. In particular $w = 0$ for pressureless 'dust', $w = \frac{1}{3}$ for radiation and $w = -1$ for the vacuum (if the cosmological constant $\Lambda \neq 0$;

Multiple-component cosmological fluid

Suppose that the cosmological fluid in fact consists of several distinct components (for example, matter, radiation and the vacuum) that do not interact except through their mutual gravitation. Let us suppose further that each component can be modelled as a perfect fluid, as discussed above.

The energy–momentum tensor of a multiple-component fluid is given simply by

$$T^{\mu\nu} = \sum_i (T^{\mu\nu})_i\,,$$

where i labels the various fluid components. Since each component is modelled as a perfect fluid, we have

$$\begin{aligned} T^{\mu\nu} &= \sum_i \left[\left(\rho_i + \frac{p_i}{c}\right) u^\mu u^\nu - p_i g^{\mu\nu}\right] \\ &= \sum_i \left(\rho_i + \frac{p_i}{c}\right) u^\mu u^\nu - (\sum_i p_i) g_{\mu\nu}. \end{aligned}$$

Thus, the multicomponent fluid can itself be modelled as a single perfect fluid with

$$\boxed{\rho = \sum_i \rho_i \qquad \text{and} \qquad p = \sum_i p_i,}$$

Moreover, since we are assuming that the fluid components are non-interacting, conservation of energy and momentum requires that the condition

$$\nabla_\mu (T^{\mu\nu})_i = 0$$

holds separately for each component. Then each fluid will obey an energy equation of the form (14.39). Thus, if $w_i = p_i/(\rho_i c^2)$ then the density of each fluid evolves independently of the other components as

$$\boxed{\rho_i \propto R^{-3(1+w_i)}.}$$

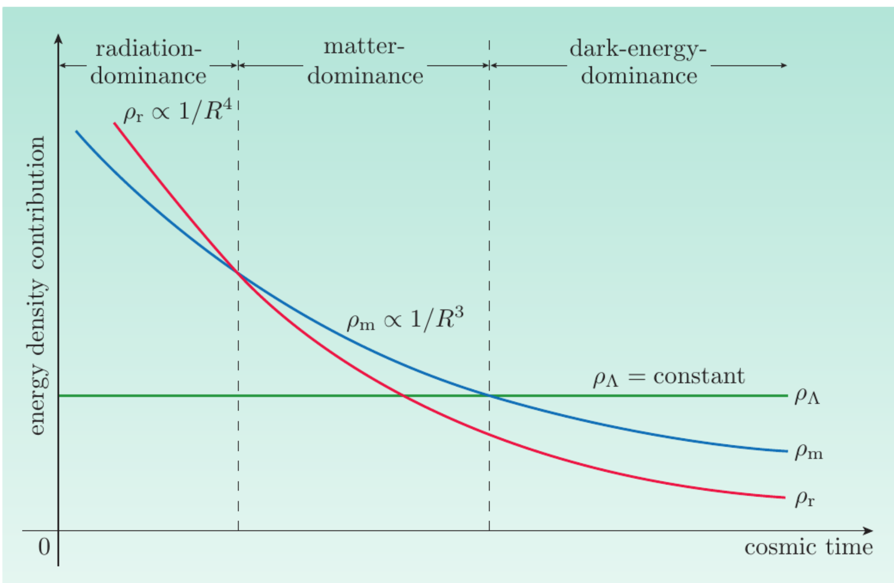


Figure 8.10 The possible evolution of the density of radiation, matter and dark energy over cosmic time in our Universe.

Ω parameter

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{kc^2}{R^2} \longrightarrow \frac{kc^2}{R^2H^2} = \frac{\rho}{(3H^2/8\pi G_N)} - 1$$

Defining $\rho_c(t) = \frac{3H(t)^2}{8\pi G_N}$: critical density. $\rho_{0,c} = \rho_c(t_0) = 1.87h^2 \times 10^{-26} \frac{kg}{m^3}$

Let $\Omega(t) = \rho(t)/\rho_c(t)$, abundance

then $\frac{kc^2}{R^2H^2} = \Omega(t) - 1$.

$$\Rightarrow k = \begin{cases} 1 & \text{if } \Omega(t) > 1, \Rightarrow \text{closed}, \\ -1 & \text{if } \Omega(t) < 1, \Rightarrow \text{Open}, \\ 0 & \text{if } \Omega(t) = 1, \Rightarrow \text{flat}. \end{cases}$$

Cosmological models

$\boxed{k=0: \text{ flat space.}}$ $\rho(t) = \rho_M(t) + \rho_R(t) + \rho_\Lambda(t)$; $P_i = w_i \rho_i c^2$.

$\rho(t) = \rho_i(t)$; $i = M, R, \Lambda$.

Single components.

The de Sitter model (1917) : $\rho = \rho_\Lambda = \text{constant}$. $\rho_\Lambda = \Lambda c^2 / 8\pi G_N$

$H^2 = \frac{\dot{R}^2}{R^2} = \frac{8\pi G_N}{3} \rho_\Lambda = \text{constant}$. $\Rightarrow H(t) = (8\pi G_N \rho_\Lambda / 3)^{1/2} = H(t_0) \equiv H_0$

$\Rightarrow \frac{dR}{R} = H_0 dt \quad \Rightarrow R(t) = R_0 \, e^{H_0(t-t_0)}$.

This model applies to:

* Inflationary epoch.
* Far future of our Universe.

Radiation dominated model : $\rho(t) = \rho_R$.

Since $\rho_R = \rho_{R,0} \left(\frac{R_0}{R}\right)^4$ it follows that

$H^2 = \frac{8\pi G_N \rho_R}{3} = \frac{8\pi G_N \rho_{R,0}}{3} \left(\frac{R_0}{R}\right)^4 = H_0^2 \left(\frac{R_0}{R}\right)^4$; $H_0^2 = 8\pi G_N \rho_{R,0}/3$.

$\Rightarrow \frac{dR}{dt} = H_0 \left(\frac{R_0}{R}\right)^2 R \quad \Rightarrow R\,dR = H_0 R_0^2\, dt \quad \Rightarrow \frac{R^2}{2}\Big|_{R_i}^{R} = H_0 R_0^2 t \Big|_{t_i}^{t}$

Choosing t_i such that $R(t_i) = R(0) = 0 \Rightarrow R = R_0 (2H_0 t)^{1/2} \rightarrow R \sim t^{1/2}$

Since

$H^2 = H_0^2 (R_0/R)^4 \Rightarrow H^2 = H_0^2 \left(1/2H_0 t\right)^2 \Rightarrow H(t) = \frac{1}{2t}$.

Einstein-de Sitter model (1932): $\rho(t) = \rho_M(t)$.

Since $\rho_M = \rho_{M,0}\left(\frac{R_0}{R}\right)^3$ it follows that

$$H^2 = \frac{8\pi G N}{3}\rho_M = \frac{8\pi G N \rho_{M,0}}{3}\left(\frac{R_0}{R}\right)^3 = H_0^2\left(\frac{R_0}{R}\right)^3 \quad ; \quad H_0^2 = 8\pi G_N \rho_{M,0}/3 .$$

$$\Rightarrow \frac{dR}{dt} = H_0\left(\frac{R_0}{R}\right)^{3/2} R \Rightarrow R^{1/2}\, dR = H_0 R_0^{3/2}\, dt \Rightarrow \frac{2}{3} R^{3/2}\Big|_{R_i}^{R} = H_0 R_0^{3/2} t\Big|_{t_i}^{t}$$

Choosing t_i such that $R(t_i) = R(0) = 0$, $\Rightarrow R = R_0\left(\frac{3H_0}{2}t\right)^{2/3} \rightarrow R \sim t^{2/3}$.

Since

$$H^2 = H_0^2 (R_0/R)^3 \Rightarrow H^2 = H_0^2\left(2/3H_0 t\right)^2 \Rightarrow H(t) = \frac{2}{3t} .$$

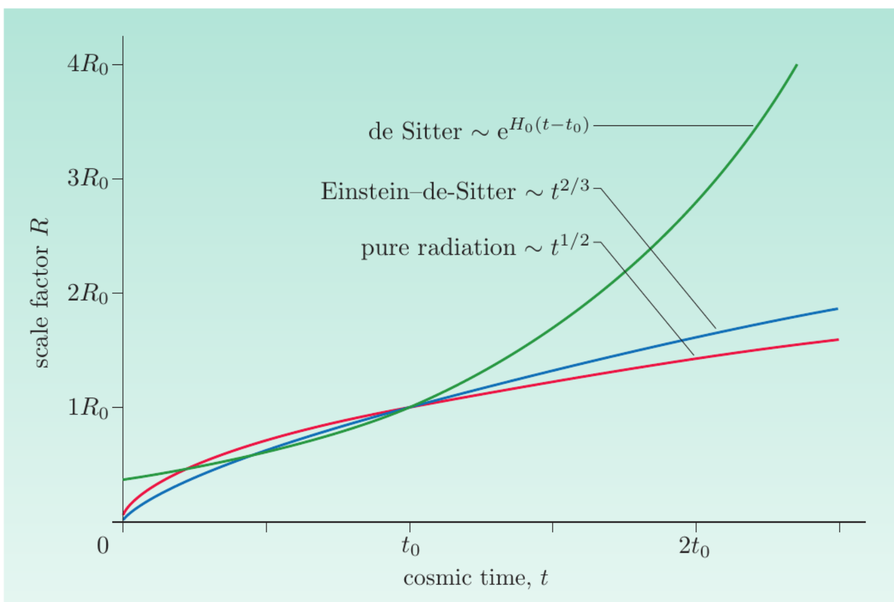


Table 8.1 Spatially flat ($k = 0$) single-component models.

Name	de Sitter	Pure radiation	Einstein–de Sitter
Composition	Dark energy only ($w = -1$)	Radiation only ($w = 1/3$)	Matter only ($w = 0$)
Scale factor $R(t)$	$R(t) = R_0 e^{H_0(t-t_0)}$	$R(t) = R_0(2H_0t)^{1/2}$	$R(t) = R_0\left(\frac{3}{2}H_0t\right)^{2/3}$
Hubble parameter $H(t)$	$H(t) = \text{constant}$	$H(t) = \dfrac{1}{2t}$	$H(t) = \dfrac{2}{3t}$
Density at time t_0 ρ_0	$\rho_{\Lambda,0} = \rho_{c,0} = \dfrac{3H_0^2}{8\pi G}$	$\rho_{r,0} = \rho_{c,0} = \dfrac{3H_0^2}{8\pi G}$	$\rho_{m,0} = \rho_{c,0} = \dfrac{3H_0^2}{8\pi G}$
Density at time t $\rho(t) = \rho_c(t)$	$\rho_\Lambda(t) = \rho_{\Lambda,0}$	$\rho_r(t) = \rho_{r,0}\left[\dfrac{t_0}{t}\right]^2$	$\rho_m(t) = \rho_{m,0}\left[\dfrac{t_0}{t}\right]^2$

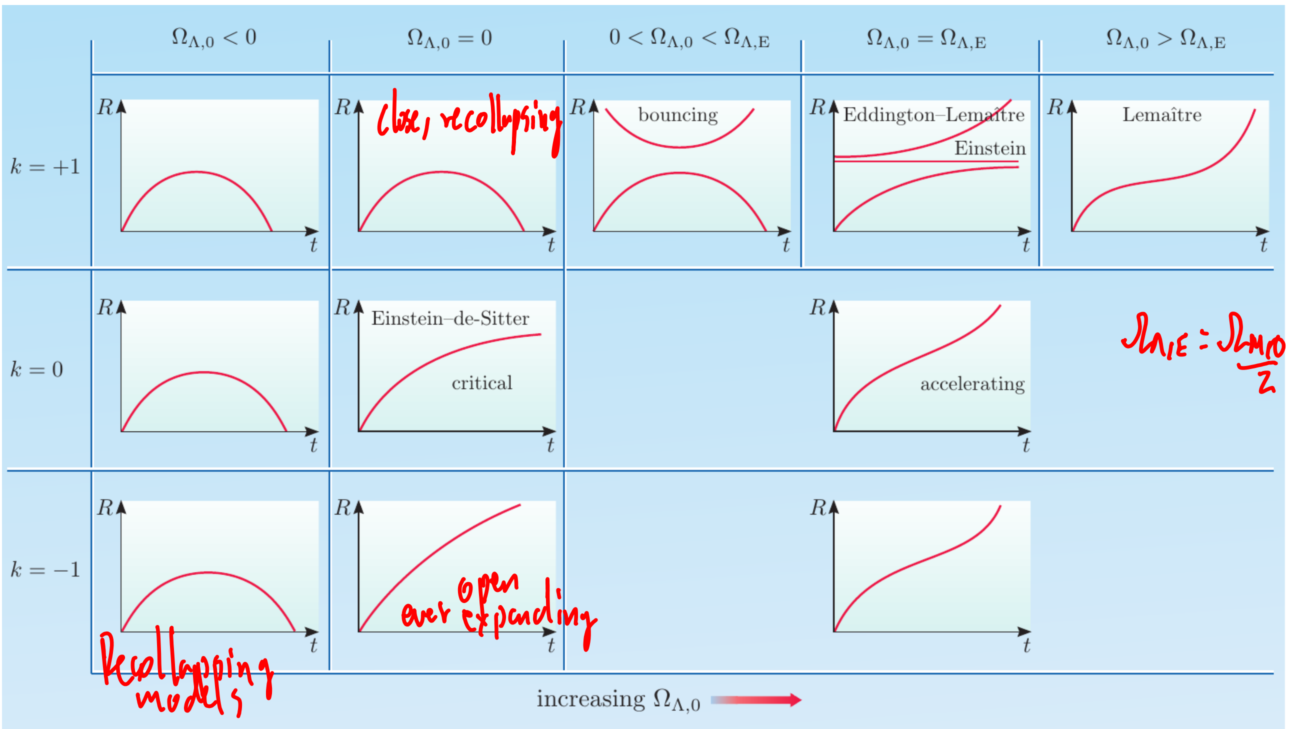
Multicomponent models

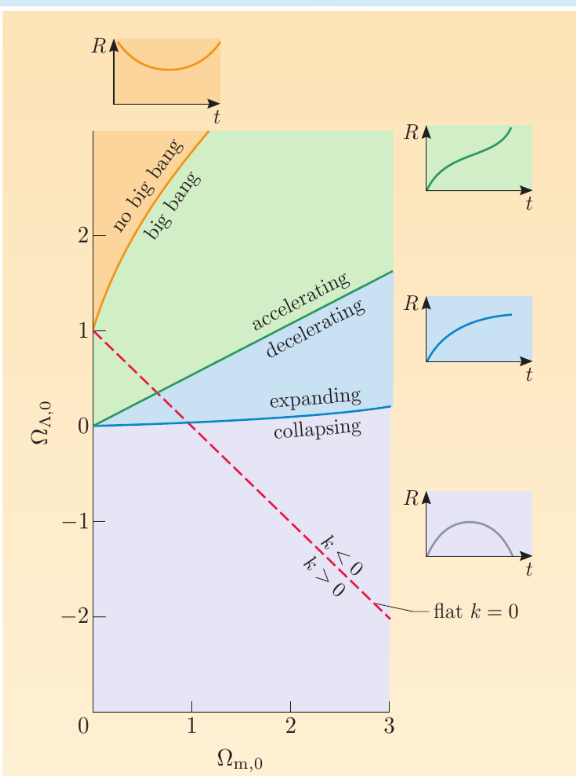
Density parameters

$$\Omega_m(t) = \frac{\rho_m(t)}{\rho_c(t)}, \quad \Omega_r(t) = \frac{\rho_r(t)}{\rho_c(t)}, \quad \Omega_\Lambda(t) = \frac{\rho_\Lambda}{\rho_c(t)}.$$

if $\Omega_m + \Omega_r + \Omega_\Lambda < 1$, then $k < 0$ and space will be open,
if $\Omega_m + \Omega_r + \Omega_\Lambda = 1$, then $k = 0$ and space will be flat,
if $\Omega_m + \Omega_r + \Omega_\Lambda > 1$, then $k > 0$ and space will be closed.

From $1 = \frac{\rho(t)}{\rho_c(t)} - \frac{kc^2}{R^2H^2} = \Omega_r + \Omega_m + \Omega_\Lambda - \frac{kc^2}{R^2H^2}$.





Cosmological observables

t_0: current cosmic time

Cosmological redshift: $z = \dfrac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$ $\begin{cases} z>0: & \text{redshift} \\ z<0: & \text{blueshift.} \end{cases}$

Suppose that a fundamental observer, at the origin of co-moving coordinates in a Robertson–Walker spacetime, observes a light signal emitted from a distant galaxy at a fixed radial co-moving coordinate $r = \chi$. We can take the coordinates of the emission event to be $(t_{\mathrm{em}}, \chi, 0, 0)$ and the coordinates of the observation event to be $(t_{\mathrm{ob}}, 0, 0, 0)$. The light signal will travel along a null geodesic where $(\mathrm{d}s)^2 = 0$, so it follows from the Robertson–Walker line element that all along that null geodesic,

$$0 = c^2(\mathrm{d}t)^2 - R^2(t)\frac{(\mathrm{d}r)^2}{1 - kr^2}.$$

$$\int_{t_{em}}^{t_{obs}} \frac{c\,dt}{R(t)} = \int_0^{\chi} \frac{dr}{(1-kr^2)^{1/2}}$$: χ coordinate of the emitting galaxy.

Now suppose that a second signal is emitted from the same source a short time later, at $t_{\mathrm{em}} + \delta t_{\mathrm{em}}$, and that it is observed a short time after the first signal, at $t_{\mathrm{ob}} + \delta t_{\mathrm{ob}}$. This second signal also travels along a null geodesic, so

$$\int_{t_{em}+\delta t_{em}}^{t_{obs}+\delta t_{obs}} \frac{c\,dt}{R(t)} = \int_0^{\chi} \frac{dr}{(1-kr^2)^{1/2}}$$

Equating both expressions:

$$\int_{t_{em}}^{t_{obs}} \frac{c\,dt}{R(t)} = \int_{t_{em}+\delta t_{em}}^{t_{obs}+\delta t_{obs}} \frac{c\,dt}{R(t)}$$

$\downarrow$

$$\int_{t_{em}}^{t_{em}+\delta t_{em}} \frac{c\,dt}{R(t)} + \int_{t_{em}+\delta t_{em}}^{t_{obs}} \frac{c\,dt}{R(t)} = \int_{t_{em}+\delta t_{em}}^{t_{obs}} \frac{c\,dt}{R(t)} + \int_{t_{obs}}^{t_{obs}+\delta t_{obs}} \frac{c\,dt}{R(t)}.$$

$$\Rightarrow \quad \int_{t_{em}}^{t_{em}+\delta t_{em}} \frac{dt}{R(t)} = \int_{t_{obs}}^{t_{obs}+\delta t_{obs}} \frac{dt}{R(t)}$$

but each of these integrals covers a very short period of time, so the integrand will be effectively constant for the short duration of the integration, and we can write

$$\frac{\delta t_{em}}{R(t_{em})} = \frac{\delta t_{obs}}{R(t_{obs})}$$

If we now let $\delta t_{\rm em}$ be the proper period of oscillation of the emitted light, then $\delta t_{\rm ob}$ will be the period of the observed light and we can use the fact that frequency is inversely proportional to period to replace $\delta t_{\rm em}/\delta t_{\rm ob}$ by $f_{\rm ob}/f_{\rm em}$, giving

$$\frac{f_{obs}}{f_{em}} = \frac{R(t_{em})}{R(t_{obs})}$$

Cosmological redshift related to scale factor

$$1 + z = \frac{R(t_{\rm ob})}{R(t_{\rm em})}.$$

So the redshift of the light is determined by the ratio of the scale factors at the times of observation and emission. In an expanding Universe, $R(t_{\rm ob})$ will be bigger than $R(t_{\rm em})$, so Equation 8.56 predicts that the observed light will be positively redshifted. If the Universe expands monotonically, then the more distant the source of the light, the longer the time the light will spend in transit, and, generally speaking, the greater will be the observed redshift.

- A distant quasar has a redshift $z = 6.0$. By what factor has the Universe expanded since the quasar emitted the light that we receive today?
- Substituting $z = 6.0$ in Equation 8.56 gives $R(t_0)/R(t_{\rm em}) = 7$.

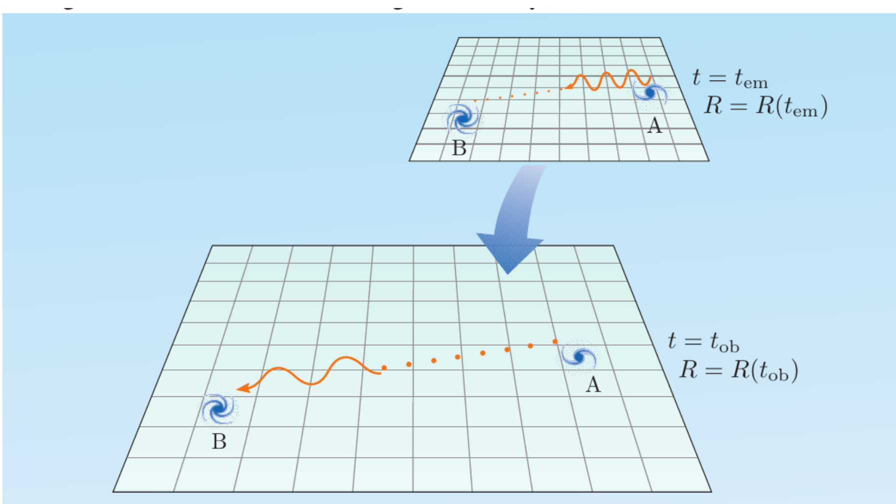


Figure 8.15 A schematic view of the origin of cosmological redshift as a result of the expansion of space.

Luminosity distance d_L $(K=0)$

$F = L/4\pi d_L^2$; L: Luminosity (power) ; F: energy flux

In an FLRW universe at time t_{ob}:

$$F = \frac{L}{4\pi R^2(t_{ob})\chi^2(1+z)^2} \Rightarrow d_L = R(t_{ob})\,\chi\,(1+z) = \overbrace{\sigma(t_{ob})}^{\text{Properdistance}}(1+z)$$

Naively: $F = \dfrac{L}{4\pi R^2(t_{obs})\chi^2}$, but

$$\begin{cases} \dfrac{\delta t_{em}}{\delta t_{obs}} = \dfrac{R(t_{em})}{R(t_{obs})} = 1/1+z \,. & \delta t_{obs} > \delta t_{em} \\ \dfrac{E_{em}}{E_{obs}} = \dfrac{\lambda_{obs}}{\lambda_{em}} = (1+z) & \Rightarrow E_{obs} < E_{em}. \end{cases}$$

Deceleration parameter

Let us expand $R(t)$ around the lookback time $(t_0 - t)$:

$$R(t) = R(t_0) + \dot{R}(t)|_{t_0}(t-t_0) + 1/2\,\ddot{R}(t)|_{t_0}(t-t_0)^2 + \dots$$
$$= R(t_0)\left[1 - H_0(t_0-t) - 1/2\, q_0 H_0^2 (t_0-t)^2 + \dots\right].$$

where

$$q(t) = -\frac{1}{H(t)^2\,R(t)}\ddot{R}(t).$$

Then $1/R(t) \approx 1 + H_0(t_0-t) + (1+q_0/2)H_0^2(t_0-t)^2$.

For $t = t_{em}$

$$\Rightarrow \quad z = H_0(t_0-t_e) + (1+q_0/2)H_0^2(t_0-t_e)^2.$$

$$\Rightarrow \quad t_0 - t_e \simeq H_0^{-1}\left[z - (1+q_0/2)z^2\right]$$

From $\displaystyle\int_{t_{em}}^{t_{obs}} \frac{c\,dt}{R(t)} = \int_0^\chi \frac{dr}{(1-kr^2)^{1/2}} \overset{k=0}{=} \chi = \frac{\sigma(t_0)}{R(t_0)}$. $\quad\sigma(t_0)$ proper distance to the emitting galaxy.

Then

$$\sigma(t_0) = c(t_0 - t_e) + cH_0/2\,(t_0 - t_e)^2 = \frac{cz}{H_0}\left[1 - \frac{1+q_0}{2}z\right].$$

Finally

$$d_L = \sigma(t_0)(1+z) = \frac{c}{H_0}\left(z + \frac{1}{2}(1-q_0)z^2 + \cdots\right).$$

Predicted relation of redshift to luminosity distance for small z

$$d_{\mathrm{L}} = \frac{c}{H_0} z. \quad ($$

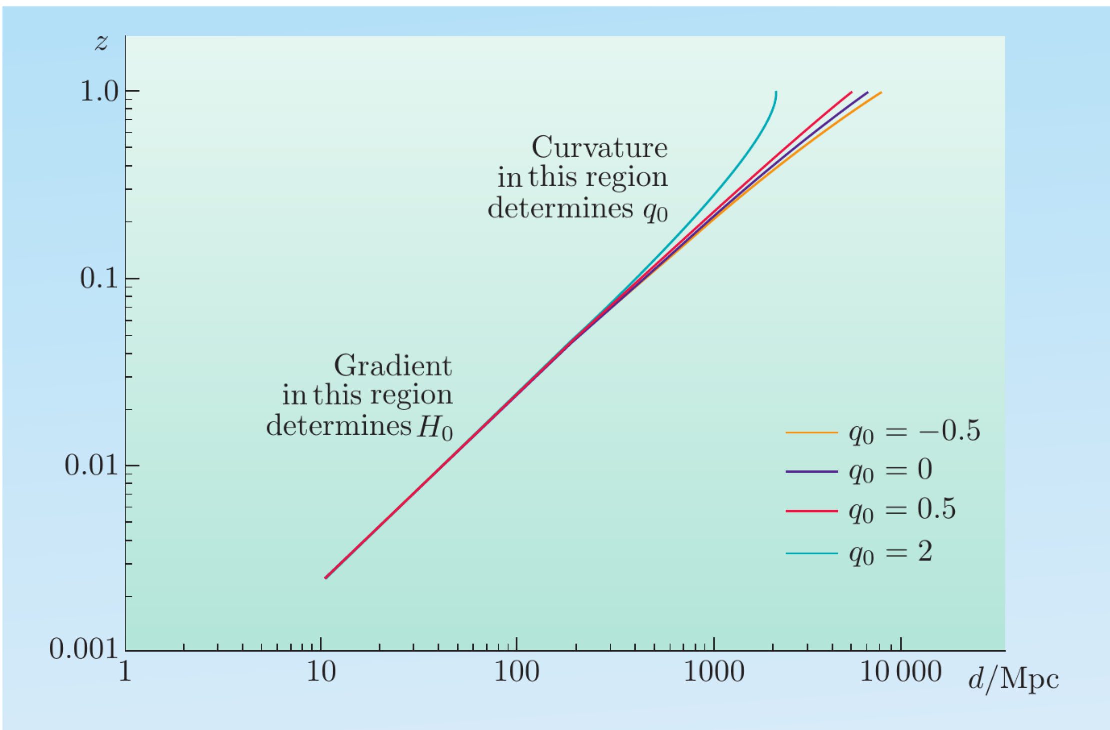


Figure 8.16 The predicted relation between redshift and luminosity distance for various current values of the deceleration parameter q_0.

Age of the universe

Key cosmological parameters

$$\Omega_{\mathrm{m},0} \approx 0.27, \quad \Omega_{\mathrm{r},0} \approx 0.00, \quad \Omega_{\Lambda,0} \approx 0.73,$$
$$H_0 = 74.2 \pm 3.6\,\mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

$$t(z) = \frac{1}{H_0}\int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\sqrt{\Omega_{\Lambda,0} + (\Omega_0 - 1)x^{-2} + \Omega_{\mathrm{m},0}\,x^{-3} + \Omega_{\mathrm{r},0}\,x^{-4}}},$$

so, the current age of the Universe, t_0 (corresponding to $z = 0$), is given by

$$t_0 = \frac{1}{H_0}\int_0^{1} \frac{\mathrm{d}x}{x\sqrt{\Omega_{\Lambda,0} + (\Omega_0 - 1)x^{-2} + \Omega_{\mathrm{m},0}\,x^{-3} + \Omega_{\mathrm{r},0}\,x^{-4}}}.$$

With the currently favoured key values for the various parameters, this indicates a value for t_0 of about 13.7×10^9 years.

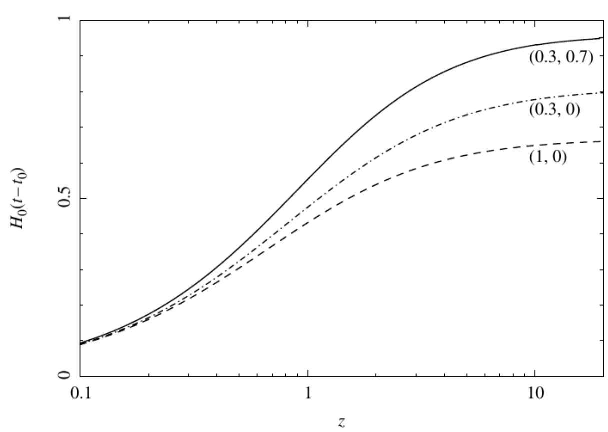


Figure 15.7 The variation in look-back time, in units of the Hubble time, as a function of redshift z for several sets of values $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0})$ as indicated, assuming that $\Omega_{\mathrm{r},0}$ is negligible.

Table 15.1 *The age of the universe in* Gyr *for various cosmological models (with* $\Omega_{\mathrm{r},0} = 0$*)*

		H_0 in km $\mathrm{s^{-1}\,Mpc^{-1}}$		
$\Omega_{\mathrm{m},0}$	$\Omega_{\Lambda,0}$	50	70	90
1.0	0.0	13.1	9.3	7.2
0.3	0.0	15.8	11.3	8.8
0.3	0.7	18.9	13.5	10.5

Nota

El problema de la planitud.. El valor del parametro Ω para la materia está suficientemente cerca a 1 como para producir problemas. Una sencilla manipulación algebraica de la ecuación conduce a la expresión:

$$(\Omega^{-1} - 1)\rho a^2 = -\frac{3kc^2}{8\pi G}$$

Si asumimos $\rho = \rho_0(a_0/a)^n$ entonces:

$$(\Omega^{-1} - 1) = -\frac{3kc^2}{8\pi G\rho_0 a_0^n} a^{n-2}$$

Si $n > 2$ (como sucede con todas las formas de materia-energía convencionales - excepto la energía oscura) entonces en el límite:

$$\lim_{a\to 0} (\Omega^{-1} - 1) = 0^{\pm}$$

donde el $\pm$ significa que la expresión se aproxima a cero por encima o por debajo dependiendo del signo inicial de k.
0 bien:

$$\lim_{a\to 0} \Omega = 1^{\pm}$$

Si el universo empieza con un valor ligeramente diferente de 1 con la expansión el valor diferira de esa cantidad muy rápidamente (como a^{n-2}). Esto implica que de tener el universo una geometría abierta o cerrada al principio, hoy el valor del parámetro Ω sería significativamente diferente de 1.
Pero el valor de Ω es muy cercana a 1 (si asumimos que solo hay materia en el universo). Esto lleva a pensar que en realidad el universo comenzó con valor de Ω tan cercano a 1 que no tendría sentido asumir que pudo formarse al azar con un valor así. Por eso algunos asumen que en realidad $\Omega = 1$ y ha sido así desde el principio.
Si este es el caso la única posibilidad es que:

$$\Omega_\Lambda = 1 - \Omega_M = 0{,}6936$$

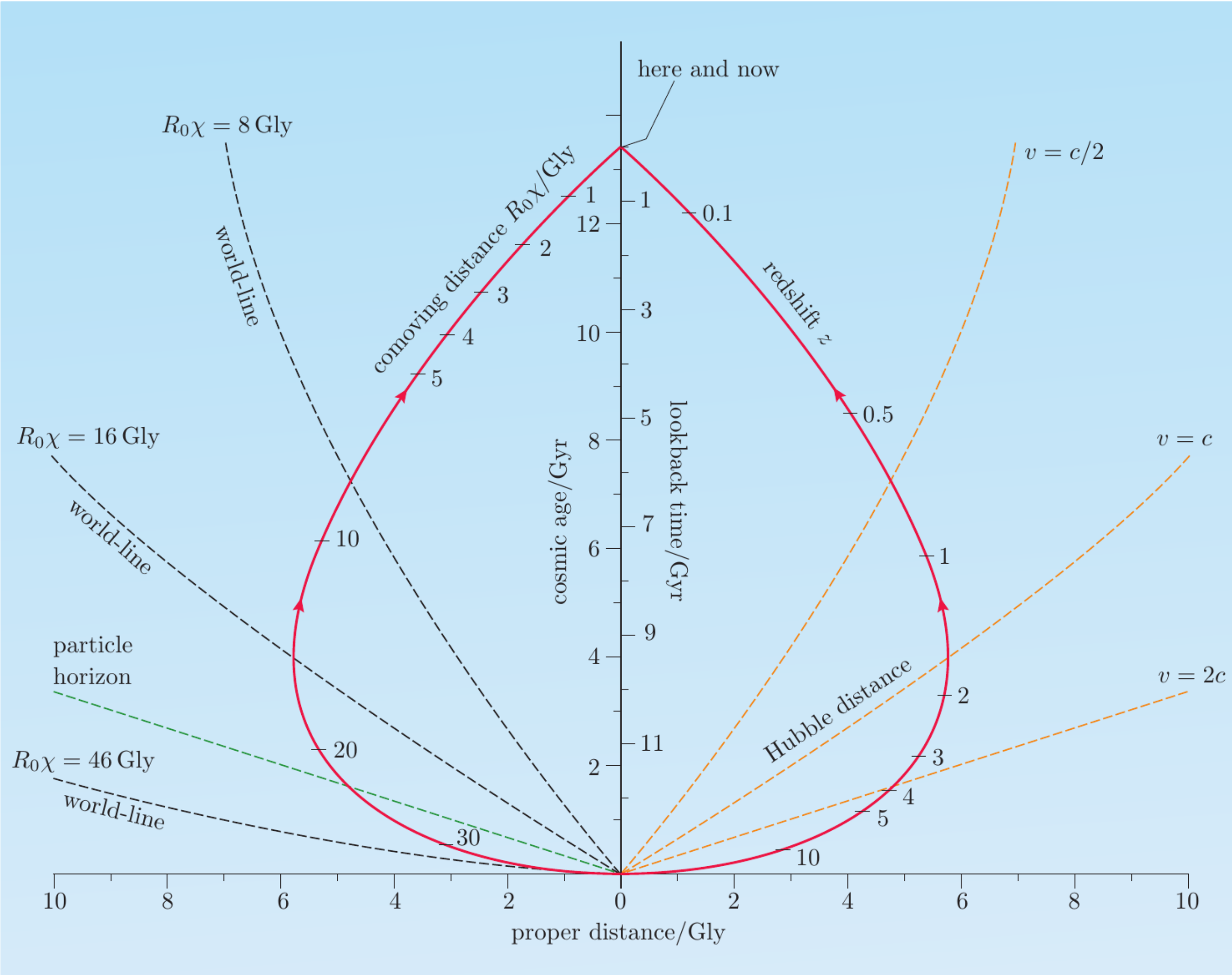


Figure 8.17 A spacetime diagram, with axes showing cosmic time and proper distance, for a Friedmann–Robertson–Walker Universe with $\Omega_{\Lambda,0} = 0.7$, $\Omega_{\mathrm{m},0} = 0.3$ and $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

The curved black lines originating at $(0, 0)$ that cut across the left-hand side of the past lightcone are the world-lines of 'galaxies' (or more accurately, fundamental observers) that travel along geodesics of the Robertson–Walker spacetime as they fall freely under the gravitational influence of the matter and dark energy that shape that spacetime. Each of these world-lines is marked with the co-moving distance of the corresponding 'galaxy'. Also shown cutting across the left half of

Another set of curves cuts across the right-hand half of the past lightcone. These lines connect points at which the Hubble flow has a specific proper radial velocity relative to fundamental observers on the vertical axis (i.e. us). Note in

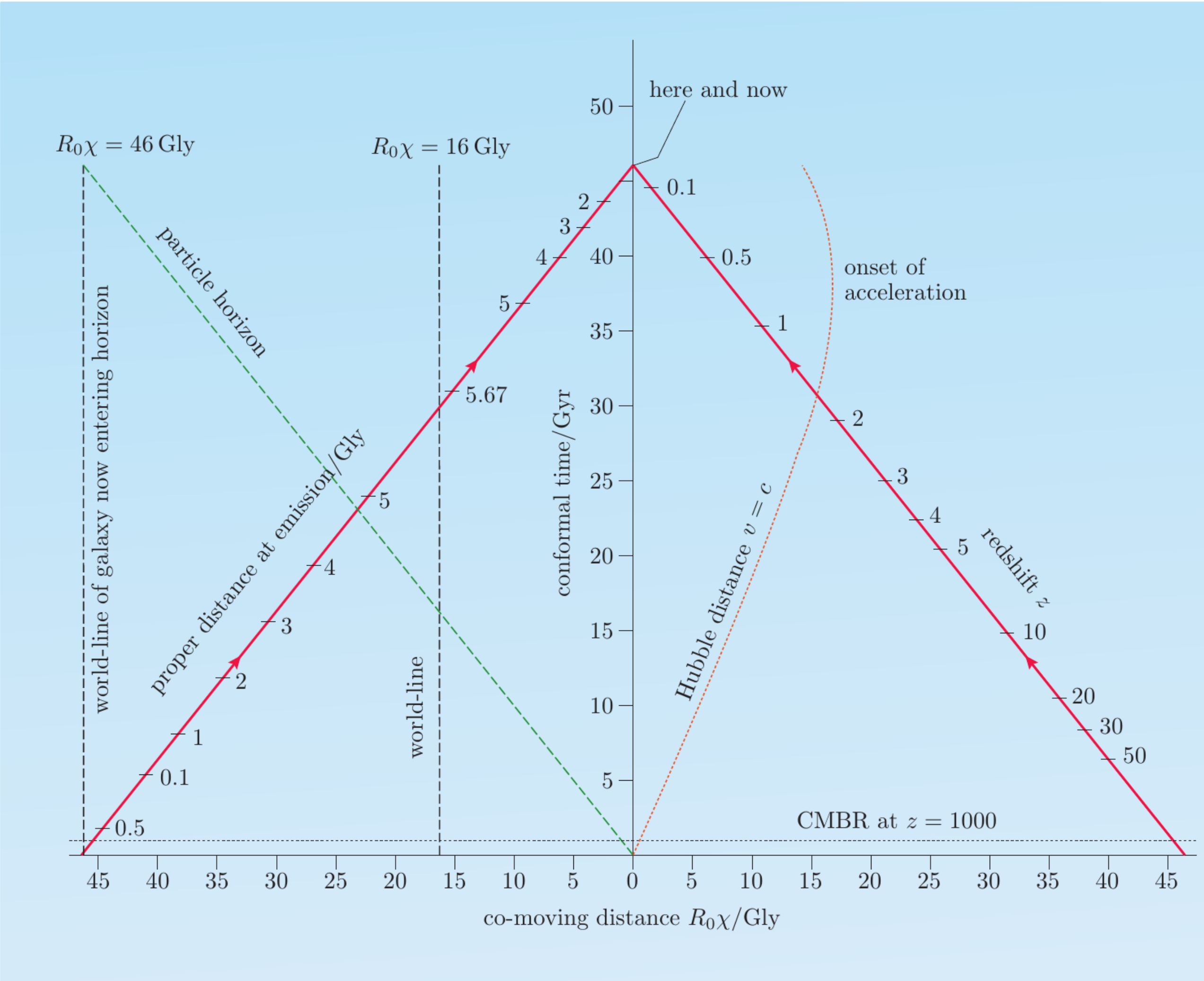


Figure 8.18 A spacetime diagram, with axes showing conformal time and co-moving distance, for a Friedmann–Robertson–Walker Universe with $\Omega_{\Lambda,0} = 0.7$, $\Omega_{\text{m},0} = 0.3$ and $H_0 = 70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$. The past lightcone is shown in red, the particle horizon in green, the Hubble distance in orange and world-lines of fundamental observers (or their galaxies) in black.