Model Universes

In a spatially homogeneous and isotropic universe, the relation between the energy density $\varepsilon(t)$, the pressure $P(t)$, and the scale factor $a(t)$ is given by the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2},$$

the fluid equation,

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0,$$

and the equation of state,

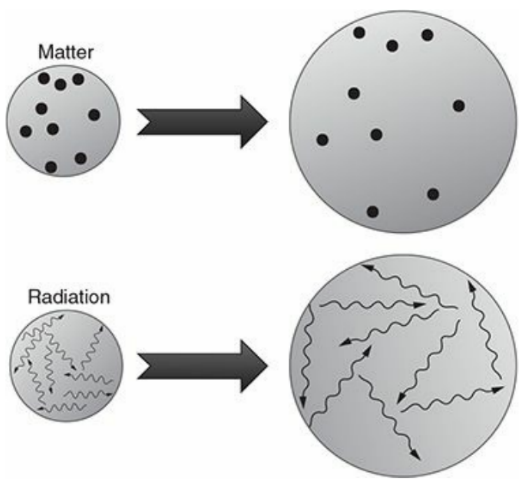
$$P = w\varepsilon.$$

Evolution of Energy Density

$$\varepsilon = \sum_i \varepsilon_i. \qquad P = \sum_i w_i \varepsilon_i. \longrightarrow \dot{\varepsilon}_i + 3\frac{\dot{a}}{a}(\varepsilon_i + P_i) = 0$$

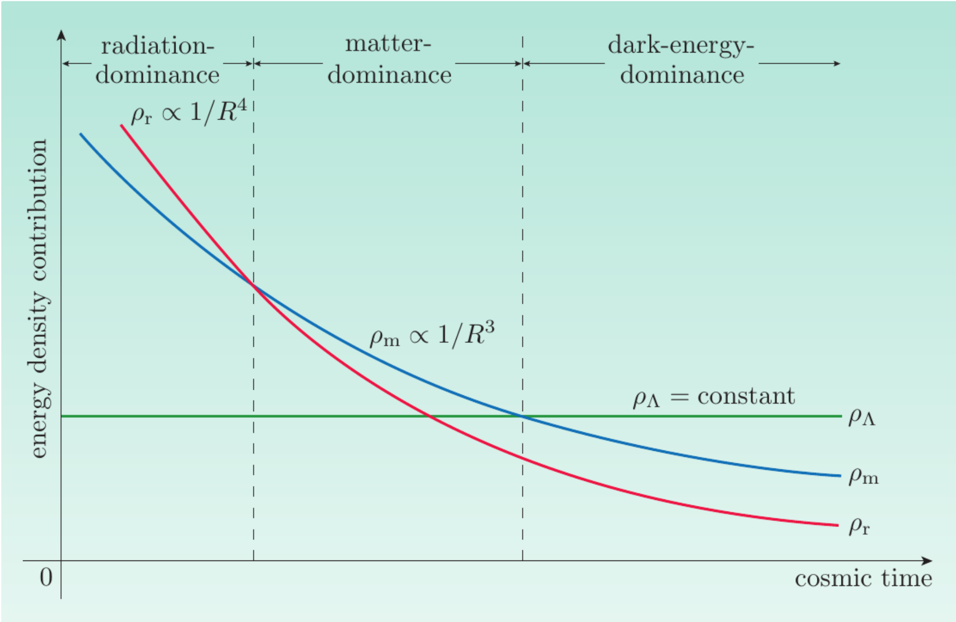
$$\varepsilon_i(a) = \varepsilon_{i,0} a^{-3(1+w_i)}.$$

$$\varepsilon_m(a) = \varepsilon_{m,0}/a^3.$$



$$\varepsilon_r(a) = \varepsilon_{r,0}/a^4.$$

The possible evolution of the density of radiation, matter and dark energy over cosmic time in our Universe.



Cosmic composition

At cosmic time $t = t_0$, the sources of cosmic gravitation are specified by just three values: $\rho_{\rm m,0}$, $\rho_{\rm r,0}$ and ρ_Λ. Given these three values, the cosmic density and pressure at any other cosmic time can be determined, provided that the cosmic scale factor $R(t)$ is known as an explicit function of cosmic time.

Table 5.2 Properties of the Benchmark Model.

List of ingredients		
Photons:	$\Omega_{\gamma,0} = 5.35 \times 10^{-5}$	
Neutrinos:	$\Omega_{\nu,0} = 3.65 \times 10^{-5}$	
Total radiation:	$\Omega_{r,0} = 9.0 \times 10^{-5}$	
Baryonic matter:	$\Omega_{\mathrm{bary},0} = 0.048$	
Nonbaryonic dark matter:	$\Omega_{\mathrm{dm},0} = 0.262$	
Total matter:	$\Omega_{m,0} = 0.31$	
Cosmological constant:	$\Omega_{\Lambda,0} \approx 0.69$	
Important epochs		
Radiation–matter equality:	$a_{rm} = 2.9 \times 10^{-4}$	$t_{rm} = 0.050$ Myr
Matter–lambda equality:	$a_{m\Lambda} = 0.77$	$t_{m\Lambda} = 10.2$ Gyr
Now:	$a_0 = 1$	$t_0 = 13.7$ Gyr

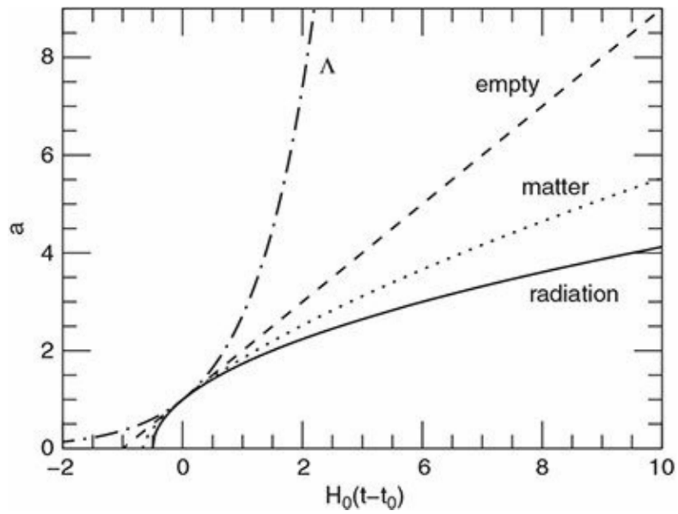


Figure 5.2 Scale factor versus time for an expanding, empty universe (dashed), a flat, matter-dominated universe (dotted), a flat, radiation-dominated universe (solid), and a flat, Λ-dominated universe (dot-dash).

Empty Universes

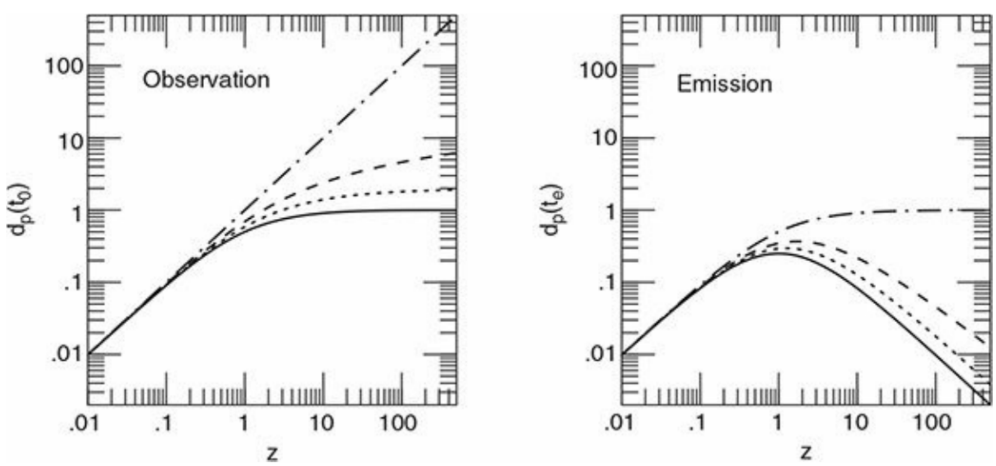


Figure 5.3 The proper distance to an object with observed redshift z, measured in units of the Hubble distance, c/H_0. Left panel: the proper distance at the time the light is observed. Right panel: proper distance at the time the light was emitted. Line types are the same as those of Figure 5.2.

Single-component Universes

$$\dot{a}^2 = \frac{8\pi G\varepsilon_0}{3c^2}a^{-(1+3w)}.$$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}.$$

$$t_0 = \frac{1}{1+w}\left(\frac{c^2}{6\pi G\varepsilon_0}\right)^{1/2}.$$

$$H_0 \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3(1+w)}t_0^{-1}.$$

$$t_0 = \frac{2}{3(1+w)}H_0^{-1}.$$

In a spatially flat universe, if $w > -1/3$, the universe is *younger* than the Hubble time. If $w < -1/3$, the universe is *older* than the Hubble time.

$$\varepsilon(a) = \varepsilon_0 a^{-3(1+w)},$$

$$\varepsilon(t) = \varepsilon_0\left(\frac{t}{t_0}\right)^{-2},$$

$$\varepsilon_0 = \varepsilon_{c,0} = \frac{3c^2}{8\pi G}H_0^2 = \frac{c^2}{6\pi(1+w)^2}t_0^{-2},$$

$$1+z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/(3+3w)}$$

$$t_e = \frac{t_0}{(1+z)^{3(1+w)/2}} = \frac{2}{3(1+w)H_0}\frac{1}{(1+z)^{3(1+w)/2}}.$$

The current proper distance to the galaxy is

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)} = ct_0 \frac{3(1+w)}{1+3w}[1-(t_e/t_0)^{(1+3w)/(3+3w)}],$$

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}[1-(1+z)^{-(1+3w)/2}].$$

$$d_{\rm hor}(t_0) = c\int_0^{t_0} \frac{dt}{a(t)}.$$

$$d_{\rm hor}(t_0) = ct_0 \frac{3(1+w)}{1+3w} = \frac{c}{H_0}\frac{2}{1+3w}.$$

Lambda only

$$\dot{a}^2 = \frac{8\pi G\varepsilon_\Lambda}{3c^2}a^2,$$

$$\dot{a} = H_0 a,$$

$$H_0 = \left(\frac{8\pi G\varepsilon_\Lambda}{3c^2}\right)^{1/2}.$$

$$a(t) = e^{H_0(t-t_0)}.$$

$$d_p(t_0) = c\int_{t_e}^{t_0} e^{H_0(t_0-t)}dt = \frac{c}{H_0}[e^{H_0(t_0-t_e)}-1] = \frac{c}{H_0}z,$$

$$d_p(t_e) = \frac{c}{H_0}\frac{z}{1+z},$$

Spatially flat ($k = 0$) single-component models.

Name	de Sitter	Pure radiation	Einstein–de Sitter
Composition	Dark energy only ($w = -1$)	Radiation only ($w = 1/3$)	Matter only ($w = 0$)
Scale factor $R(t)$	$R(t) = R_0 \mathrm{e}^{H_0(t-t_0)}$	$R(t) = R_0(2H_0 t)^{1/2}$	$R(t) = R_0\left(\frac{3}{2}H_0 t\right)^{2/3}$
Hubble parameter $H(t)$	$H(t) = \text{constant}$	$H(t) = \dfrac{1}{2t}$	$H(t) = \dfrac{2}{3t}$
Density at time t_0 ρ_0	$\rho_{\Lambda,0} = \rho_{\mathrm{c},0} = \dfrac{3H_0^2}{8\pi G}$	$\rho_{\mathrm{r},0} = \rho_{\mathrm{c},0} = \dfrac{3H_0^2}{8\pi G}$	$\rho_{\mathrm{m},0} = \rho_{\mathrm{c},0} = \dfrac{3H_0^2}{8\pi G}$
Density at time t $\rho(t) = \rho_\mathrm{c}(t)$	$\rho_\Lambda(t) = \rho_{\Lambda,0}$	$\rho_\mathrm{r}(t) = \rho_{\mathrm{r},0}\left[\dfrac{t_0}{t}\right]^2$	$\rho_\mathrm{m}(t) = \rho_{\mathrm{m},0}\left[\dfrac{t_0}{t}\right]^2$

Multiple-component Universes

Matter + Curvature

Table 5.1 Curved, matter-dominated universes.

Density	*Curvature*	*Ultimate fate*
$\Omega_0 < 1$	$\kappa = -1$	Big Chill ($a \propto t$)
$\Omega_0 = 1$	$\kappa = 0$	Big Chill ($a \propto t^{2/3}$)
$\Omega_0 > 1$	$\kappa = +1$	Big Crunch

The solution when $\Omega_0 > 1$ is

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos\theta) \qquad t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta), \qquad t_{\rm crunch} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}.$$

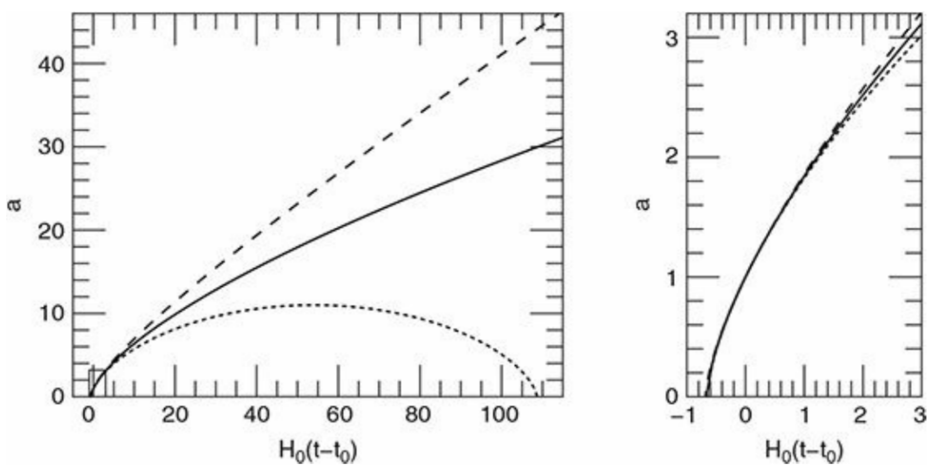


Figure 5.4 Scale factor versus time for universes containing only matter. Solid line: $a(t)$ for a universe with $\Omega_0 = 1$ (flat). Dashed line: $a(t)$ for a universe with $\Omega_0 = 0.9$ (negatively curved). Dotted line: $a(t)$ for a universe with $\Omega_0 = 1.1$ (positively curved). The right panel is a blow-up of the small rectangle near the lower left corner of the left panel.

for the case $\Omega_0 < 1$

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{1 - \Omega_0}(\cosh\eta - 1) \qquad t(\eta) = \frac{1}{2H_0}\frac{\Omega_0}{(1 - \Omega_0)^{3/2}}(\sinh\eta - \eta),$$

Matter + Lambda

For a flat, $\Omega_{\Lambda,0} < 0$ universe,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0}).$$

$$a_{\max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0} - 1}\right)^{1/3},$$

$$t_{\text{crunch}} = \frac{2\pi}{3H_0}\frac{1}{\sqrt{\Omega_{m,0} - 1}}.$$

$$H_0 t = \frac{2}{3\sqrt{\Omega_{m,0} - 1}} \sin^{-1}\left[\left(\frac{a}{a_{\max}}\right)^{3/2}\right].$$

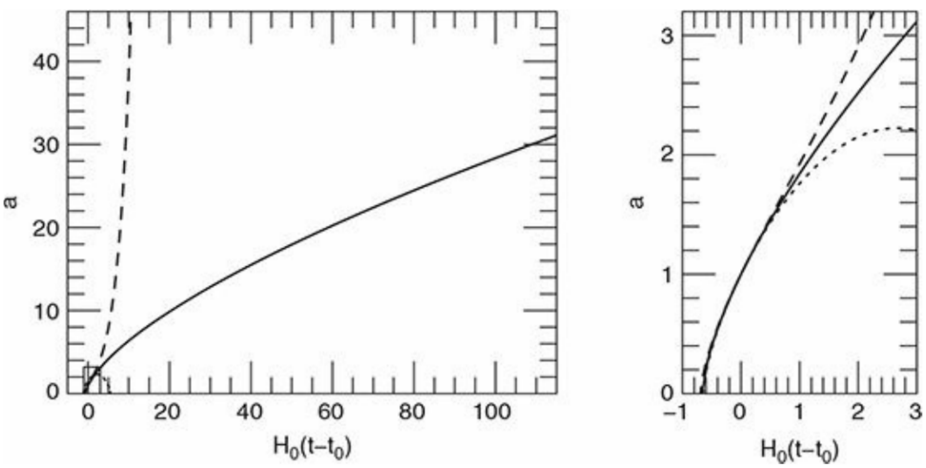


Figure 5.5 Scale factor versus time for flat universes containing both matter and a cosmological constant. Solid line: $a(t)$ for a universe with $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$. Dashed line: $a(t)$ for a universe with $\Omega_{m,0} = 0.9$, $\Omega_{\Lambda,0} = 0.1$. Dotted line: $a(t)$ for a universe with $\Omega_{m,0} = 1.1$, $\Omega_{\Lambda,0} = -0.1$. The right panel is a blow-up of the small rectangle near the lower left corner of the left panel.

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}}\right)^{1/3}.$$

For a flat, $\Omega_{\Lambda,0} > 0$ universe, the Friedmann equation can be integrated to yield the analytic solution

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1 + \left(\frac{a}{a_{m\Lambda}}\right)^3}\right].$$

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1 - \Omega_{m,0}}} \ln\left[\frac{\sqrt{1 - \Omega_{m,0}} + 1}{\sqrt{\Omega_{m,0}}}\right].$$

If we approximate our own universe as having $\Omega_{m,0} = 0.31$ and $\Omega_{\Lambda,0} = 0.69$, we find that its current age is

$$t_0 = 0.955 H_0^{-1} = 13.74 \pm 0.40\,\text{Gyr},$$

Radiation + Matter

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}.$$

$$H_0 dt = \frac{a da}{\Omega_{r,0}^{1/2}} \left[1 + \frac{a}{a_{rm}}\right]^{-1/2}.$$

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2}\right].$$

$$t_{rm} = \frac{4}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}} H_0^{-1} \approx 0.391 \frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2} H_0^{-1}.$$

$$t_{rm} = 3.47 \times 10^{-6} H_0^{-1} = 50\,000\,\mathrm{yr}.$$