Early Universe
Exercises Week #1

1. Summarize the three strands of evidence that support the cosmological principle.

2. Check the relations for the Planck length, time and mass.

3. Convert the following quantities by inserting the appropriate factors of $c, \hbar$, G_N and k_B:

   a) $T_0 = 2{,}725$ K $\rightarrow$ eV.
   b) $\rho_\gamma = \pi^2 T_0^4/15 \rightarrow$ eV^4 and g/cm^3.
   c) $M_P \equiv 1{,}2 \times 10^{19}$ GeV $\rightarrow$ K.

4. What is the ratio of gravitational interaction energy to Coulomb energy for two protons?

5. The range of length scales involved in cosmology are hard to grasp. The best we can do is to consider relative distances and compare them to something more familiar. In this exercise, we will make some attempt at obtaining a more intuitive understanding of the vastness of the cosmos.

   a) Consider shrinking the Earth to the size of a basketball. What would then be the size of the Moon and its orbit around the Earth?
   b) Now imagine scaling the Earth down to the size of a peppercorn. What would then be the size of the Sun and the Earth's orbit? How far away would the most distant planet in the Solar System be?
   c) The "Solar Neighborhood" is a collection of about fifty nearby stars, spread across about 65 light-years, that travel together with the Sun. Scaling this region down, so that it fits inside a basketball court, what would be the size of the Solar System?.
   d) Shrinking our Galaxy to the size of the basketball court, what would now be the size of the Solar Neighbourhood?
   e) The "Local Group" comprises about fifty nearby galaxies, spread across about 10 million light-years. If we squeezed this region into the size of the basketball court, what would be the size of our Milky Way galaxy?
   f) The largest structures in the universe, like our "Local Supercluster," are about 500 million light-years across. Scaling these superclusters down to the dimensions of the basketball court, what would be the size of our Local Group?
   g) This radius of the observable universe is 46.5 billion light-years. Compressing the observable universe to the size of the basketball court, what would be the size of the largest superclusters?

6. A key parameter in cosmology is the Hubble constant

$$H_0 = 70 \text{ km/s/Mpc}.$$

In the following, you will use the measured value of the Hubble constant to estimate a few fundamental scales of our universe.

a) What is the Hubble time $t_{H_0} \equiv H_0^{-1}$ in years? This is a rough estimate of the age of the universe.

b) What is the Hubble distance $d_{H_0} \equiv cH_0^{-1}$ in meters? This is a rough estimate of the size of the observable universe.

c) The average density of the universe today is

$$\rho_0 = \frac{3H_0^2}{8\pi G_N},$$

where G_N is the NNweton's constant. What is ρ_0 in g/cm^{-3}? How does this compare to the density of water.

7. Let us assume that the universe is filled with only hydrogen atoms. What is then the total number of atoms in the universe? How does this compare to the number of hydrogen atoms in your brain?
Hint: Assume that the brain is mostly water. Use $m_{\mathrm{H}} = 2 \times 10^{-24}$ g and $m_{\mathrm{H_2O}} = 3 \times 10^{-23}$ g.

8. The maximal energy scale probed by the Large Hadron Collider (LHC) is $E_{\max} \sim 1$ TeV. What length scale $\ell_{\min}$ does this correspond to? How does this compare to the size of the universe d_{H_0}?

9. Relate the dimensionless redshift to distance expressed in Mpc.

10. The estimated mass energy density of dark matter in the solar neighbourhood is around 0.3 GeV/cm^3. Suppose that it is made of Wimps of rest energy (mass energy) 100 GeV.

a) How many Wimps are roughly inside your body at any particular time?

b) What is the flux, i.e. the number of particles per cm^2 per second if they move with typical galactic velocities, $v \sim 200$ km/s?

11. For blackbody radiation, the number density of photons with frequencies in the range $[\nu, \nu + d\nu]$ is given by

$$n(\nu, T)d\nu = \frac{8\pi\nu^2}{c^3(e^{h\nu/k_BT} - 1)}d\nu,$$

where T is the 'temperature' of the radiation.

a) Show that the total number density n_γ of photons is

$$n_\gamma(T) = 0{,}224 \left(\frac{2\pi k_B T}{hc} \right)^3 .$$

b) Hence show that the present-day number density of CMB photons in the universe is $n_{\gamma,0} = 411/\mathrm{cm}^3$, and compare this with the present-day number density of protons.

12. A distant galaxy has a redshift $z = \frac{\Delta\lambda}{\lambda} = 0{,}2$. According to Hubble's law, how far away was the galaxy when the light was emitted if the Hubble constant is 70 km/s/Mpc?

13. When a charged π^- meson with very low velocity reacts with a proton, a neutron and a neutral π^0 meson are produced, $\pi^- + p \to \pi^0 + n$. Suppose that the proton, neutron and π^- masses are known: $m_p = 938.3$ MeV, $m_n = 939.6$ MeV, $m_{\pi^-} = 139.6$ MeV. Determine the mass of the π^0, if the neutron kinetic energy is measured to be $T_n \equiv E_n - m_n = 0{,}4$ MeV.

14. Show that a free electron and a free positron, both with mass m_e, cannot annihilate into a single freely propagating photon.

15. The cosmic microwave background radiation (CMBR) consists of photons of typical energy 3×10^{-4} eV. How high an energy must a cosmic gamma photon γ_c have if pair production on this background $\gamma_c + \gamma_{\mathrm{CMB}} \to e^+ + e^-$ is to be kinematically allowed?

16. Suppose that the Milky Way galaxy is a typical size, containing say 10^{11} stars, and that galaxies are typically separated by a distance of one Mpc. Estimate the density of the Universe in SI units. How does this compare with the density of the Earth?

17. In the real Universe the expansion is not completely uniform. Rather, galaxies exhibit some random motion relative to the overall Hubble expansion, known as their peculiar velocity and caused by the gravitational pull of their near neighbours. Supposing that a typical (e.g. root mean square) galaxy peculiar velocity is 600 km/s, how far away would a galaxy have to be before it could be used to determine the Hubble constant to ten per cent accuracy, supposing $H_0 = 70$ km/s/Mpc. Assume in your calculation that the galaxy distance and redshift could be measured exactly. Unfortunately, that is not true of real observations.

18. What evidence can you think of to support the assertion that the Universe is charge neutral, and hence contains an equal number of protons and electrons?

19. The binding energy of the electron in a hydrogen atom is 13.6 eV. What is the frequency of a photon with this energy? At what temperature does the mean photon energy equal this energy?

20. Energy content due to starlight. By assuming the stars have been shining with the same intensity since the beginning of the universe and always had the luminosity density n_*, estimate the density ratio $\Omega_* \equiv \rho_*/\rho_c$ for starlight. For this rough calculation you can take the age of universe to be the Hubble time t_{H_0}.

21. Night sky as bright as day Olbers' paradox is solved in our expanding universe because the age of the universe is not infinite $t_0 \simeq t_{H_0}$ and, having a horizon length $\simeq c\,t_{H_0}$, it is effectively not infinite in extent. Given the present luminosity density n_*, with the same approximation as previous exercise, estimate the total flux due to starlight. Compare your result with the solar flux $f_\odot = L_\odot/(4\pi(\mathrm{AU})^2)$. We can increase the star light flux by increasing the age of the universe t_0. How much older does the universe have to be in order that the night sky is as bright as day?

22. Use Kepler's Third Law to estimate the orbital speed of the solar system at about 8 kpc from the centre of the Milky Way.

23. Use Newtonian gravity to calculate the escape velocity for a particle at the surface of the Earth and the Sun.

24. Show that a galaxy mass distribution $\rho_{\mathrm{DM}} \propto r^{-2}$ is consistent with flat rotation curves.

25. **The steady-state universe**. The conventional interpretation of an ever increasing scale factor (expanding universe) means that all objects must have been closer in the past, leading to a big bang beginning. Besides this, because of an initial overestimate of the Hubble constant (by a factor of seven), there was a "cosmic age problem." To avoid this difficulty, an alternative cosmology, called the steady-state universe (SSU), was proposed by Hermann Bondi, Thomas Gold, and Fred Hoyle. It was suggested that, consistent with the Robertson–Walker description of an expanding universe, all cosmological quantities besides the scale factor (the expansion rate, matter density, etc.) are time independent. A constant mass density means that the universe did not have a big hot beginning; hence there cannot be a cosmic age problem. To have a constant mass density in an expanding universe requires the continuous, energy-nonconserving, creation of matter. To SSU's advocates, this spontaneous mass creation is no more peculiar than the creation of all matter at the instant of big bang. In fact, the name "big bang" was invented by Fred Hoyle as a somewhat disparaging description of the competing cosmology.

    a) Supporters of SSU find this model attractive on theoretical grounds—because it is compatible with the "perfect cosmological principle." From the above outline of SSU and the cosmological principle, can you infer what this "perfect comsmological principle" must be?

    b) It follows that in the SSU the expansion rate $H(t) = H(t_0) = H_0$ is constant. From this, deduce the explicit t-dependence of the scale factor $a(t)$.

c) Since the matter density is a constant $\rho_M(t) = \rho_{M,t_0} = \rho_{M,0} = 0{,}3\rho_{c,0}$ and yet the scale factor increases with time, SSU requires spontaneous matter creation. What must be the rate of this mass creation per unit volume? Express it in terms of the number of hydrogen atoms created per cubic kilometer per year.