Einstein's Field Equation

Einstein's field equation plays a role in general relativity that is analogous to the role played by Poisson's equation in Newtonian dynamics.

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \longleftrightarrow \nabla^2 \Phi = 4\pi G \rho,$$

idealized case (which fortunately is a good approximation for our purposes), the stress-energy tensor $T_{\mu\nu}$ depends only on $\varepsilon(t)$ and $P(t)$. The metric describing the curvature of spacetime, in this case, is the homogeneous and isotropic Robertson–Walker metric (Equation 3.41):

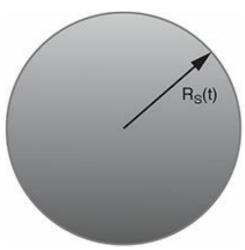
$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2 \right], \tag{4.8}$$

$$S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0) & (\kappa = +1) \\ r & (\kappa = 0) \\ R_0 \sinh(r/R_0) & (\kappa = -1). \end{cases}$$

The Friedmann Equation

To begin, consider a homogeneous sphere of matter, with total mass M_s constant with time (Figure 4.1). The sphere is expanding or contracting isotropically, so that its radius $R_s(t)$ is increasing or decreasing with time. Place a test mass, of infinitesimal mass m, at the surface of the sphere. The gravitational force F experienced by the test mass will be, from Newton's law of gravity,

$$F = -\frac{G M_s m}{R_s(t)^2}.$$



$$\frac{d^2 R_s}{dt^2} = -\frac{G M_s}{R_s(t)^2}.$$

Multiply each side of the equation by dR_s/dt and integrate to find

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{G M_s}{R_s(t)} + U,$$

where U is a constant of integration.

$$\epsilon_{\rm kin} = \frac{1}{2}\left(\frac{dR_s}{dt}\right)^2, \qquad \epsilon_{\rm pot} = -\frac{G M_s}{R_s(t)},$$

Since the mass of the sphere is constant, we may write

$$M_s = \frac{4\pi}{3}\rho(t)R_s(t)^3.$$

Since the expansion or contraction is isotropic about the sphere's center, we may write the radius $R_s(t)$ in the form

$$R_s(t) = a(t)r_s, \qquad \frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}Gr_s^2\rho(t)a(t)^2 + U.$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}.$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}.$$

The Newtonian case with $U < 0$ corresponds to the relativistic case with positive curvature ($\kappa = +1$); conversely, $U > 0$ corresponds to negative curvature ($\kappa = -1$). The Newtonian special case with $U = 0$ corresponds to the relativistic special case where the space is perfectly flat ($\kappa = 0$). Although I have not given the derivation of the Friedmann equation in the general relativistic case, it makes sense that the curvature, given by κ and R_0, the expansion rate, given by $a(t)$, and the energy density ε should be bound up together in the same equation. After all, in Einstein's view, the energy density of the universe determines both the curvature of space and the overall dynamics of the expansion.

$$\varepsilon_c(t) \equiv \frac{3c^2}{8\pi G}H(t)^2.$$

$$\varepsilon_{c,0} = \frac{3c^2}{8\pi G}H_0^2 = (7.8 \pm 0.5) \times 10^{-10}\,\mathrm{J\,m^{-3}} = 4870 \pm 290\,\mathrm{MeV\,m^{-3}}.$$

The critical density is frequently written as the equivalent mass density,

$$\begin{aligned}\rho_{c,0} \equiv \frac{\varepsilon_{c,0}}{c^2} &= (8.7 \pm 0.5) \times 10^{-27}\,\mathrm{kg\,m^{-3}} \\ &= (1.28 \pm 0.08) \times 10^{11}\,\mathrm{M_\odot\,Mpc^{-3}}.\end{aligned} \tag{4.32}$$

The current critical density is roughly equivalent to a density of one proton per 200 liters. This is definitely not a large density, by terrestrial standards; a