General Relativity.

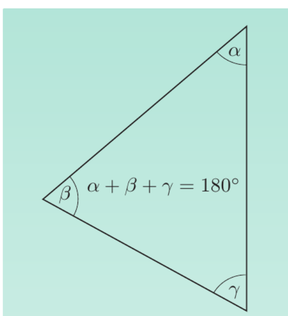
* Special relativity is inconsistent with Newtonian gravitation.

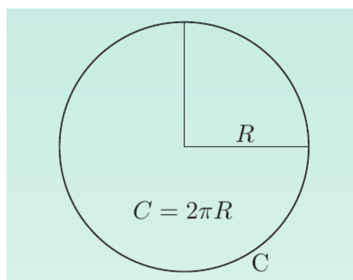
$$\vec{F} = m_G \vec{g} = -m_G \vec{\nabla}\Phi \quad ; \quad \nabla^2\Phi = 4\pi G_N \rho.$$

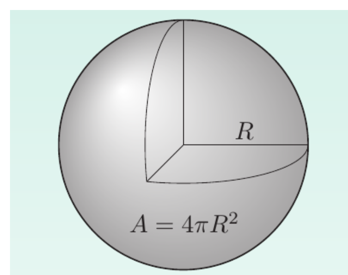
* Einstein realized that a theory of general relative motion (involving accelerated observers) would also shed light on the problem of gravitation.

* Geometry : Minkowski space-time and curved spaces.

Euclidean geometry (Euclid 300 b.c.)







- the internal angles of a triangle add up to 180°
- a circle of radius R has a circumference of length $C = 2\pi R$
- a sphere of radius R has a surface area $A = 4\pi R^2$.

Non Euclidean geometries

Bolyai (1802-1860), Lobachevsky (1792-1856), Gauss (1777-1855), Riemann (1826-1866).

- Metric.
- length of a curve.
- Parallel transport of a vector.
- Geodesics.
- Curvature (tensor).

} Curved spacetime. ⇒ Minkowski ST: flat ST.

Flat spacetime: Euclidean geometry holds true.

Curved spacetime: Euclidean geometry fails.

The distance or interval between P & Q is a function of the coordinates and their differentials: $ds^2 = f(x^a, dx^a)$; $a = 1, \ldots, n$.

The local geometry of the manifold at the point P is determined by ds^2.

Because the principle of equivalence that states

Weak equivalence principle

Within a sufficiently localized region of spacetime adjacent to a concentration of mass, the motion of bodies subject to gravitational effects alone cannot be distinguished by any experiment from the motion of bodies within a region of appropriate uniform acceleration.

we consider intervals of the form

$$\boxed{ds^2 = g_{ab}(x)\, dx^a dx^b.}$$

Riemannian manifolds ($ds^2 > 0$).

Pseudo Riemannian M: ds^2 arbitrary.

Intrinsic and extrinsic curvature

The curvature dictated by $ds^2 = g_{ab} dx^a dx^b$ is an intrinsic property of the manifold itself. In other words, it is independent of whether the manifold is embedded in some higher-dimensional space.

It is, of course, difficult (or impossible) to imagine higher-dimensional curved manifolds, so it is instructive to consider two-dimensional Riemannian manifolds, which can often be visualised as a surface embedded in a three-dimensional Euclidean space. It is important to make a distinction, however, between the extrinsic properties of the surface, which are dependent on how it is embedded into a higher-dimensional space, and properties that are intrinsic to the surface itself.

This distinction is traditionally made clear by considering the viewpoint of some two-dimensional being (called a 'bug') confined exclusively to the two-dimensional surface. Such a being would believe that it is able to look and measure in all directions, whereas it is in fact limited to making measurements of distance, angle etc. only within the surface. For example, it would receive light signals that had travelled within the two-dimensional surface. Properties of the geometry that are accessible to the bug are called intrinsic, whereas those that depend on the viewpoint of a higher-dimensional creature (who is able to see how the surface is shaped in the three-dimensional space) are called extrinsic.

It is important to properly establish the intrinsic curvature because:

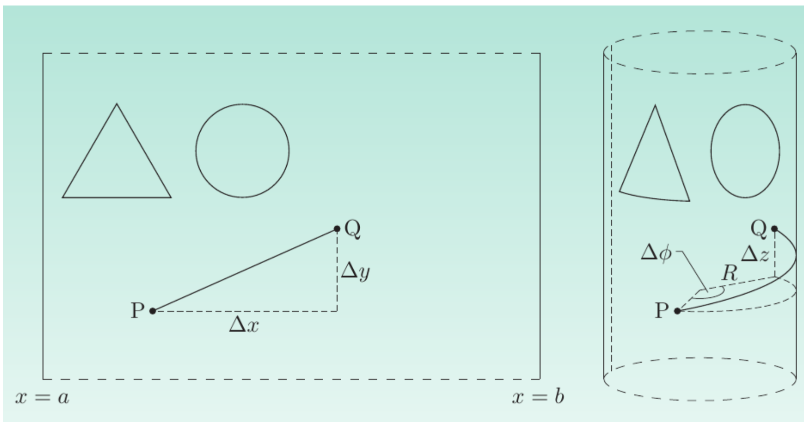
- Not all surfaces can be embedded in 3D Euclidean space.
- Curvature of 4D spacetime is not easy to imagine that it might successfully visualize in spacetime of higher dimension.
- Not everything that appears curved in 3D really is curved in mathematical sense.

$\mathbb{R}^2$ surface

Consider a two-dimensional plane surface, such as a flat sheet of paper, in our three-dimensional Euclidean space. A bug can label the entire sheet using rectangular Cartesian coordinates, so that the distance ds measured over the surface between any pair of points whose coordinate separations are dx and dy is given by

$$ds^2 = dx^2 + dy^2.$$

If this sheet is then rolled up into a cylinder, the bug would not be able to detect any differences in the geometrical properties of the surface.



To the bug, the angles of a triangle still add up to 180°, the circumference of a circle is still $2\pi r$ etc. The proof of this fact is simple – the surface can simply be *un*rolled back to a flat surface without buckling, tearing or otherwise distorting it. A more mathematical approach is to note that if one parameterises the surface of the cylinder (of radius a) using cylindrical coordinates (z, ϕ), the distance ds measured over the surface between any two points whose coordinate separations are dz and $d\phi$ is given by

$$ds^2 = dz^2 + a^2\, d\phi^2.$$

By making the simple change of variables $x = z$ and $y = a\phi$ we recover the expression $ds^2 = dx^2 + dy^2$, which is valid over the whole surface, and so the intrinsic geometry is identical to that of a flat plane. Thus the surface of a cylinder is not intrinsically curved; its curvature is extrinsic and a result of the way it is embedded in three-dimensional space. Even if one were to crumple up the sheet of paper (without tearing it), so that its extrinsic geometry in three-dimensional space was very complicated, its intrinsic geometry would still be that of a plane.

S^2 surface

The situation is somewhat different for a 2-sphere, i.e. a spherical surface, embedded in three-dimensional Euclidean space. Once again the surface is manifestly curved extrinsically on account of its embedding. Additionally, however, it cannot be formed from a flat sheet of paper without tearing or deformation. Its intrinsic geometry – based on measurements within the surface – differs from the intrinsic (Euclidean) geometry of the plane. This problem is well known to cartographers. Mathematically, if we parameterise a sphere (of radius a) by the usual angular coordinates (θ, ϕ) then

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

which cannot be transformed to the Euclidean form $ds^2 = dx^2 + dy^2$ *over the whole surface* by any coordinate transformation. Thus the surface of a sphere is intrinsically curved.

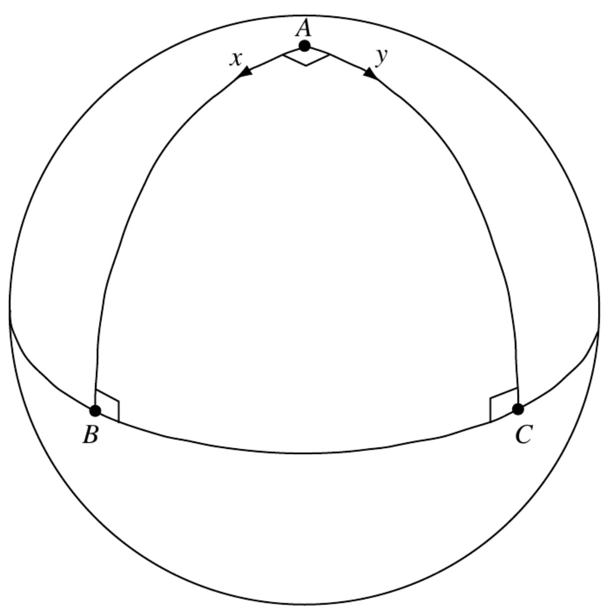


Figure 2.2 A two-dimensional spherical surface.

We note, however, that *locally* at any point A on the spherical surface we can define a set of Cartesian coordinates, so that $ds^2 = dx^2 + dy^2$ is valid in the neighbourhood of A. For example, the street layout of a town can be accurately represented by a flat map, whereas the entire globe can only be represented by performing projections that distort distance and/or angles. As an idea of what can happen to local Cartesian coordinate systems far from the point A where they are defined, consider Figure 2.2. If a bug starts at A and travels in the locally defined x-direction to B, it observes that C still lies in the y-direction. If instead the bug travels from A to C, it finds that B still lies in the x-direction. The non-Euclidean geometry of the spherical surface is also apparent from the fact that the angles of the triangle ABC sum to 270°.

Geometry of the 2-sphere S^2:

In $\mathbb{R}^3$: $ds^2 = dx^2 + dy^2 + dz^2$.

Now, suppose that we have a sphere of radius a with its centre in the origin of our coordinate system.

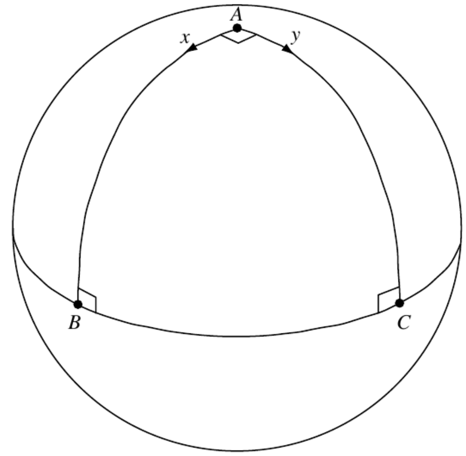


Figure 2.2 A two-dimensional spherical surface.

What is the line element on the surface of the sphere?

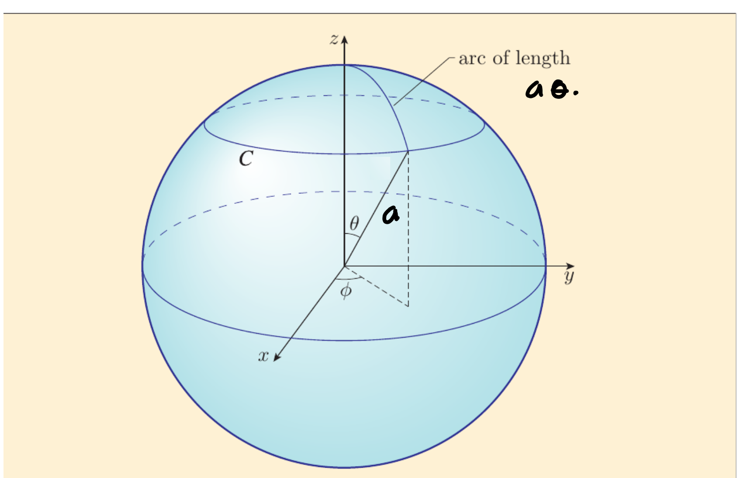
Equation defining S^3: $a^2 = x^2 + y^2 + z^2$.

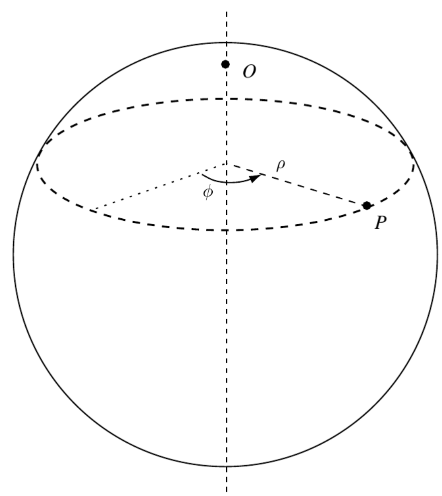
$$\Rightarrow \quad 0 = x\,dx + y\,dy + z\,dz$$

$$\Rightarrow \quad dz = -\frac{x\,dx + y\,dy}{z} = \frac{-(x\,dx + y\,dy)}{(a^2 - (x^2+y^2))^{1/2}}.$$

Substituting in ds^2: $\boxed{ds^2 = dx^2 + dy^2 + \frac{(x\,dx + y\,dy)^2}{a^2 - x^2 - y^2}}$.

ds^2 reduces to $ds^2 = dx^2 + dy^2$ in the neighbourdhood of A.





The surface of a sphere parameterised by the coordinates (ρ, ϕ)

Defining $x = \rho\cos\phi$, $y = \rho\sin\phi$ $\Rightarrow$ $\boxed{ds^2 = \frac{a^2 d\rho^2}{a^2 - \rho^2} + \rho^2 d\phi^2.}$

Since $ds^2 = g_{ab}(x)\, dx^a dx^b$: $x^1 = \rho$, $x^2 = \phi$ $\Rightarrow$ $[g_{ab}] = \begin{pmatrix} \frac{1}{1-\rho^2/a^2} & 0 \\ 0 & \rho^2 \end{pmatrix}$.

Remark: this ds^2 contains a "hidden symmetry": the freedom to choose an arbitrary point on the sphere as the origin $\rho = 0$.

Note that the line elements above look different from the metric we would write down using standard spherical polars. Nonetheless, both are valid line elements for the two-dimensional surface of a sphere.

$$\boxed{ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta\, d\phi^2.}$$

The observant reader will have noticed that the line elements have singularities at $\sqrt{x^2 + y^2} = a$, or, equivalently, $\rho = a$, corresponding to the equator of the sphere (relative to A). From our embedding picture, it is clear why the (x, y) and (ρ, ϕ) coordinates cover the surface of the sphere uniquely only up to this point. We note, however, that there is nothing pathological in the intrinsic geometry of the 2-sphere at the equator. What we have observed is only a *coordinate singularity*, which has resulted simply from choosing coordinates with a restricted domain of validity. Although the embedding picture we have adopted gives both the (x, y) and (ρ, ϕ) coordinate systems a clear geometrical meaning in our three-dimensional Euclidean space, it is important to realise that a bug confined to the two-dimensional surface of the sphere could, if it wished, have defined these coordinate systems to describe the intrinsic geometry without any reference to an embedding in higher dimensions.

S^3: 3-sphere

We can make an analogous construction to find the metric for a *3-sphere* embedded in *four-dimensional* Euclidean space. The metric for the four-dimensional Euclidean space is

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2,$$

Equation of S^3: $$ds^2 = dx^2 + dy^2 + dz^2 + dw^2,$$

$$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz + 2w\,dw = 0,$$

$$\boxed{ds^2 = dx^2 + dy^2 + dz^2 + \frac{(x\,dx + y\,dy + z\,dz)^2}{a^2 - (x^2 + y^2 + z^2)}.}$$

Transforming to spherical polar coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

we obtain an alternative form for the line element:

$$\boxed{ds^2 = \frac{a^2}{a^2 - r^2}\,dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2.}$$

Notice that, in the limit $a \to \infty$, the metric tends to the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2,$$

which is simply the metric of ordinary Euclidean three-dimensional space $ds^2 = dx^2 + dy^2 + dz^2$, rewritten in spherical polar coordinates.

Describes a non-Euclidean 3-space!!.

The singularity at $r = a$ is again a coordinate singularity.

Line element in a plane

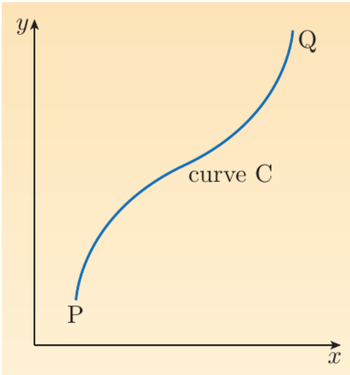


Figure 3.4 A smooth curve C in a Euclidean plane.

The length of the curve C from P to Q is

$$L_C(P, Q) \approx \sum_{i=1}^{n} \Delta l_i.$$

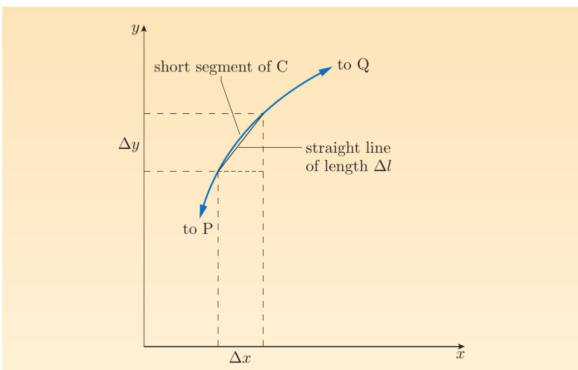


Figure 3.5 Each short segment of a curve C can be approximated by a straight line of length Δl.

$$\Delta l^2 = \Delta x^2 + \Delta y^2 ;$$

$$L_C = \int_P^Q dl = \int_P^Q (dx^2 + dy^2)^{1/2} .$$

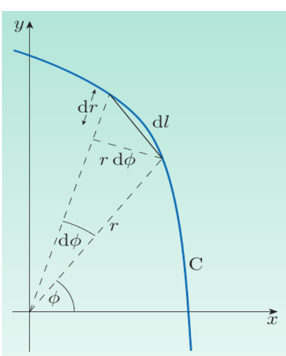


Figure 3.6 A line segment in plane polar coordinates.

Polar coordinates (r, ϕ)

$x = r\cos\varphi$.

$y = r\sin\varphi$.

$dl^2 = dr^2 + r^2 d\varphi^2$.

Parametric curve: $x = x(u)$, $y = y(u)$.

Every point on the curve should be identified with a unique value of some continuosly varying parameter u.

$$L_C = \int_{u_P}^{u_Q} \left[\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 \right]^{1/2} du.$$

Length of a curve in curved space

The interval between two infinitesimally separated points A & B in the manifold is $ds^2 = g_{ab}\,dx^a dx^b$.

If A & B are joined by some path, then the length of the curve is

$$L_{AB} = \int_A^B ds^2 = \int_A^B |g_{ab}\,dx^a\,dx^b|^{1/2},$$

where the integral is evaluated along the curve.

If the equation of the curve is parametrised as $x^a = x^a(u)$

$$\boxed{L_{AB} = \int_{u_A}^{u_B} \left| g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \right|^{1/2} du,}$$

where u_A & u_B are the values of u at the endpoints of the curve.

Orthogonal system of coordinates: $g_{ab}(x) = 0$ for $a \neq b$.

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \cdots + g_{NN}(dx^N)^2.$$

Such a system of coordinates is called *orthogonal* since, at all points in the manifold, any pair of coordinate curves cross at right angles,

Thus, in orthogonal coordinate systems the ideas of area and volume can be built up simply. Consider, for example, an element of area in the (x^1, x^2)-surface defined by $x^a = \text{constant}$ for $a = 3, 4, \ldots, N$. Suppose that the area element is defined by the *coordinate* lengths dx^1 and dx^2 (see Figure 2.3). The *proper lengths* of the two line segments will be $\sqrt{g_{11}}\,dx^1$ and $\sqrt{g_{22}}\,dx^2$ respectively. Thus the element of area is[7]

$$\boxed{dA = \sqrt{|g_{11}g_{22}|}\,dx^1\,dx^2.}$$

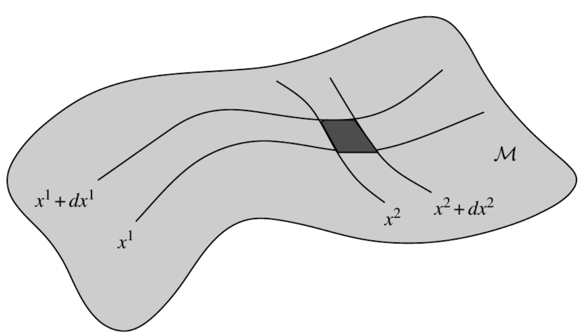


Figure 2.3 An element of area, on a manifold $\mathcal{M}$, defined by the coordinate intervals dx^1 and dx^2. The proper lengths dl^1 and dl^2 of these intervals are related to dx^1 and dx^2 by the metric functions. If the coordinate lines are orthogonal then the area of is $dl^1\,dl^2$.

Similarly, for 3-volumes in the (x^1, x^2, x^3)-surface defined by $x^a = \text{constant}$ for $a = 4, 5, \ldots, N$, we have

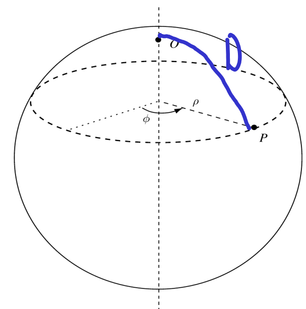
$$\boxed{d^3V = \sqrt{|g_{11}g_{22}g_{33}|}\, dx^1\, dx^2\, dx^3.}$$

We may, of course, define 3-volumes for any other three-dimensional subspace. We can define higher-dimensional 'volume' elements in a similar way until we reach the N-dimensional volume element

$$\boxed{d^NV = \sqrt{|g_{11}g_{22}\cdots g_{NN}|}\, dx^1\, dx^2 \cdots dx^N.}$$

Application: S^2.

$$ds^2 = \frac{a^2 d\rho^2}{a^2 - \rho^2} + \rho^2 d\phi^2,$$



Let's consider a circle defined by $\rho = R$, with R constant.

- The distance in the surface from the centre O along a line of constant ϕ is:

"radius" $D = \int_0^R \frac{a\,d\rho}{(a^2-\rho^2)^{1/2}} = a \sin^{-1}\left(\frac{R}{a}\right)$.

- The circumference of a circle of radius R is:

$C = \int_0^{2\pi} R\, d\phi = 2\pi R.$

$R = a \sin(D/a)$

$\frac{C}{D} = 2\pi \frac{\sin(D/a)}{(D/a)}$.

- The area of the spherical surface enclosed by C is:

$A = \int_0^{2\pi}\int_0^R (|g_{\rho\rho}\, g_{\phi\phi}|)^{1/2} d\rho\, d\phi = \int_0^{2\pi}\int_0^R \frac{a}{(a^2-\rho^2)^{1/2}}\rho\, d\rho\, d\phi = 2\pi a^2\left[1 - \left(1 - \frac{R^2}{a^2}\right)^{1/2}\right].$

Note that if we rewrite the circumference C and area A in terms of the distance D then we obtain

$$C = 2\pi a \sin\left(\frac{D}{a}\right) \qquad \text{and} \qquad A = 2\pi a^2\left[1 - \cos\left(\frac{D}{a}\right)\right].$$

Thus, as D increases, both the circumference and area of the circle increase *until the point when $D = \pi a/2$, after which both C and A become smaller as D increases.*

In fact there is a slight subtlety here. As noted earlier, if we attempt to parameterise points beyond the equator of the sphere using the coordinates (ρ, ϕ), the system becomes degenerate, i.e. there is more than one point in the surface with the same coordinates. The degenerate nature of the (ρ, ϕ) coordinate system means that some care is required, for example, in calculating the total area of the surface. By symmetry this is given by

$$A_{\text{tot}} = 2\int_0^{2\pi}\int_0^a \frac{a}{(a^2-\rho^2)^{1/2}}\rho\, d\rho\, d\phi = 4\pi a^2.$$

Appication: S^3.

$$ds^2 = \frac{a^2}{a^2-r^2}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2,$$

which describes a non-Euclidean three-dimensional space that tends to Euclidean three-dimensional space as $a \to \infty$.

let's consider a 2-sphere of coordinate radius $r=R$.

- The distance from the centre to the surface along a line of $\theta = k$ ^ $\phi = k'$ is:
  $D = \int_0^R (|g_{ab}\, dx^a dx^b|)^{1/2} = \int_0^R \frac{a\, dr}{(a^2-r^2)^{1/2}} = a\sin^{-1}(R/a)$.

- Since the equator of the sphere is the curve $r=R$ ^ $\theta = \pi/2$, then its circumference is: $C = \int_0^{2\pi} R\, d\phi = 2\pi R$.

- Area of the surface $r=R$ is:
  $$A = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta\, d\theta\, d\phi = 4\pi R^2.$$

- the volume enclosed by this surface is:

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^R \frac{ar^2\sin\theta}{(a^2-r^2)^{1/2}}\, dr\, d\theta\, d\phi$$

$$= 4\pi a^3\left\{\frac{1}{2}\sin^{-1}\left(\frac{R}{a}\right) + \frac{R}{a}\left[1-\left(\frac{R}{a}\right)^2\right]^{1/2}\right\}.$$

$$V_{\text{tot}} = 2\int_0^{2\pi}\int_0^{\pi}\int_0^a \frac{ar^2\sin\theta}{(a^2-r^2)^{1/2}}\, dr\, d\theta\, d\phi = 2\pi^2 a^3.$$