Early Universe: Out of Equilibrium dynamics

1 Disclaimer

Discussions taken from Barbara Ryden [1], Daniel Baumann [2] and Kolb & Turner [3] books

2 Boltzmann equation

In absence of particle interactions

$$\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = \frac{dn_i}{dt} + 3Hn_i = 0, \tag{1}$$

and therefore $n_i \sim a^{-3}$. On the other hand, when the interactions are present

$$\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = \frac{dn_i}{dt} + 3Hn_i = C_i[\{n_j\}], \tag{2}$$

where the form of the collision term C_i depends on the type of interactions. This expression is known as the (integrated) Boltzmann equation. For the reaction

$$1 + 2 \leftrightarrow 3 + 4, \tag{3}$$

it takes the form

$$\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = -\langle\sigma v\rangle_{12\to34}\, n_1\, n_2 + \langle\sigma v\rangle_{34\to12}\, n_3\, n_4. \tag{4}$$

$\langle\sigma v\rangle$ is the thermally averaged cross section. Since in chemical equilibrium the collision term has to vanish, we have the property

$$\langle\sigma v\rangle_{34\to12} = \langle\sigma v\rangle_{12\to34}\frac{n_1^{\rm eq}\, n_2^{\rm eq}}{n_3^{\rm eq}\, n_4^{\rm eq}}, \tag{5}$$

known as the detailed balance. Hence, the Boltzmann equation reads

$$\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = -n_1^{\rm eq}\, n_2^{\rm eq}\langle\sigma v\rangle_{12\to34}\left[\frac{n_1\, n_2}{n_1^{\rm eq}\, n_2^{\rm eq}} - \frac{n_3\, n_4}{n_3^{\rm eq}\, n_4^{\rm eq}}\right], \tag{6}$$

Defining the yield as

$$Y_i(t) \equiv \frac{n_i(t)}{s(t)} \propto n_i(t)\, a(t)^3 = N_i(t), \tag{7}$$

the BE takes the form

$$\frac{d\ln Y_1}{d\ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{Y_1^{\rm eq}\, Y_2^{\rm eq}}{Y_3^{\rm eq} Y_4^{\rm eq}}\right)\left(\frac{Y_3\, Y_4}{Y_1 Y_2}\right)\right] = -\frac{\Gamma_1}{H}\left[1 - \frac{\left(\frac{Y_3\, Y_4}{Y_1 Y_2}\right)}{\left(\frac{Y_3\, Y_4}{Y_1 Y_2}\right)_{\rm eq}}\right], \tag{8}$$

with

$$\Gamma_1 \equiv n_2\, \langle\sigma v\rangle_{34\to12}. \tag{9}$$

The right-hand side of the BE contains a factor describing the interaction efficiency, Γ_1/H, and a factor characterizing the deviation from equilibrium, $[1 - \ldots]$.

3 Dark matter freeze-out

Let's consider the following reaction

$$\chi + \bar{\chi} \leftrightarrow f + \bar{f}, \tag{10}$$

where $\chi, \bar{\chi}$ denotes dark matter particles and antiparticles while $f, \bar{f}$ is a SM light fermion as electrons. To perform the BE we consider the following assumptions:

1. The fermions $f, \bar{f}$ are in equilibrium with the thermal plasma due to their interaction rate with the remaining SM particles (i.e. photons). This means that

$$n_f = n_3 = n_4 = n_f^{\rm eq}. \tag{11}$$

2. There is no any initial difference between the number density of DM particles and antiparticles. That is

$$n_\chi = n_{\bar{\chi}}. \tag{12}$$

3. There are no other annihilation reactions during the freeze-out of the DM such that entropy conservation is largely maintained. That is

$$T \sim a^{-1}. \tag{13}$$

Under this assumption we obtain

$$\frac{d \ln Y_\chi}{d \ln a} = -\frac{\Gamma_\chi}{H}\left[1 - \frac{(Y_\chi^{\rm eq})^2}{Y_\chi^2}\right]. \tag{14}$$

Introducing a new measure of time,

$$x \equiv \frac{m_\chi}{T}, \tag{15}$$

such that $dx/dt = H\,x$, with

$$H(x) = \sqrt{\frac{\epsilon_R}{3M_p^2}} = \sqrt{\frac{\pi^2 g_*}{90}}\frac{T^2}{M_P} = \sqrt{\frac{\pi^2 g_*}{90}}\frac{m_\chi^2}{M_P}x^{-2} = H(m_\chi)\,x^{-2}. \tag{16}$$

it follows that

$$\frac{dY_\chi}{dx} = -\frac{Y_\chi\,\Gamma_\chi}{x\,H}\left[1 - \frac{(Y_\chi^{\rm eq})^2}{Y_\chi^2}\right] = -\frac{s(x)\,\langle\sigma v\rangle_{\chi\bar{\chi}\to f\bar{f}}}{x\,H(x)}\left[Y_\chi^2 - (Y_\chi^{\rm eq})^2\right], \tag{17}$$

which can be cast in the form of Riccati equation

$$\frac{dY_\chi}{dx} = -\frac{\lambda}{x^2}\left[Y_\chi^2 - (Y_\chi^{\rm eq})^2\right], \tag{18}$$

where

$$\lambda = \frac{s(m_\chi)\,\langle\sigma v\rangle_{\chi\bar{\chi}\to f\bar{f}}}{H(m_\chi)} = \frac{s(m_\chi)\,\langle\sigma v\rangle_{\chi\bar{\chi}\to f\bar{f}}}{H(m_\chi)} = \sqrt{\frac{\pi}{45}}\frac{g_s}{\sqrt{g_\rho}}M_P\,m_\chi\,\langle\sigma v\rangle_{\chi\bar{\chi}\to f\bar{f}}. \tag{19}$$

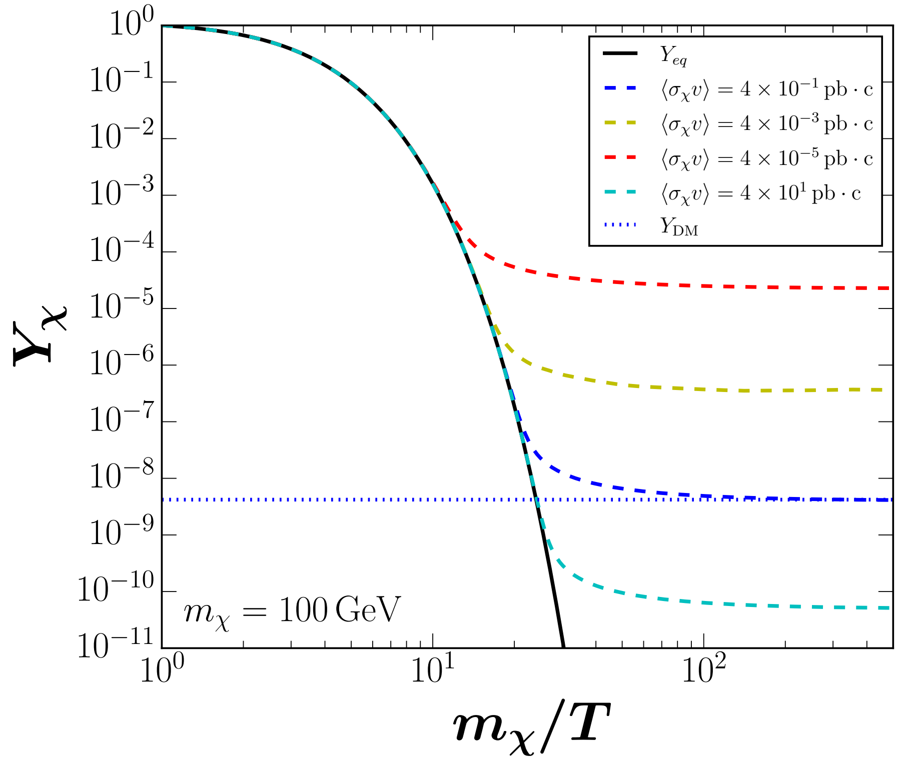

Figure 1: Solution to the BE for WIMPs.

3.1 After decoupling

After the decoupling, we can neglect Y_{eq} in the BE and integrate from T_f, to T_0. We obtain (asumming $g_s = g_\rho$)

$$\begin{aligned}\frac{1}{Y_0} &= \frac{1}{Y_f} + \sqrt{\frac{\pi}{45}} M_P\, m_\chi \int_{x_f}^{x_0} \frac{g_\rho^{1/2}}{x^2} \langle\sigma v\rangle dx \\ &= \frac{1}{Y_f} - \sqrt{\frac{\pi}{45}} M_P \int_{T_f}^{T_0} g_\rho^{1/2} \langle\sigma v\rangle dT. \end{aligned} \tag{20}$$

This is the complete solution of the approximate density equation after freeze-out. In the literature, the term $1/Y_f$, is usually omitted, but its presence is essential in achieving a very good approximation to Y_0 (within a few per cent). Since T_0 is very small compared to the mass of the particles usually under consideration, we will replace it by zero in eq. (20).

Knowing Y_0 we can compute $\Omega_\chi = \rho_\chi^0/\rho_{\mathrm{crit}} = m_\chi s_0 Y_0/\rho_{\mathrm{crit}}$ (for nonrelativistic particles $\rho = mn$), the present density of the species under consideration in terms of the critical density $\rho_{\mathrm{crit}} = 3H^2/8\pi G = 1.05 h^2 \times 10^{-5}\,\mathrm{GeV/cm}^3$ ($H = 100\,h$ km/s/Mpc). By using $s_0 = 2898.45\mathrm{cm}^{-3}$ ($T_0 = 2.726$ K and $g_s(T_0) = 3.915$), the result is

$$\Omega_\chi h^2 = \frac{m_\chi s_0 Y_0}{\rho_c} h^2 = 2.75 \times 10^8 \frac{m_\chi}{\mathrm{GeV}} Y_0. \tag{21}$$

3.2 Instantaneous freeze-out approximation

The instantaneous freeze-out approximation reads

$$\Omega_\chi h^2 \simeq \frac{1.07 \times 10^9\,\mathrm{GeV}^{-1}}{\langle \sigma v \rangle\, g_*(x_f)^{1/2}\, M_{\mathrm{Pl}}}\,, \tag{22}$$

where $x_f = m_\chi / T_f$ and

$$x_f = \ln \frac{0.038\, c\,(c+2)\, g_\rho^{1/2}\, M_P\, m_\chi\, \langle \sigma v \rangle}{x_f^{1/2}}\,. \tag{23}$$

Here c is a constant of order one. Typically, for a WIMP dark matter $x_f \approx 20 - 30$.

4 Non-equilibrium recombination

Let the reaction

$$e^- + p^+ \leftrightarrow \mathrm{H} + \gamma. \tag{24}$$

$$\frac{1}{a^3}\frac{d(n_e a^3)}{dt} = -\langle \sigma v \rangle \left[n_e^2 - \left(\frac{n_e^2}{n_{\mathrm{H}}} \right)_{\mathrm{eq}} n_{\mathrm{H}} \right], \tag{25}$$

where we have used $n_e = n_p$ and $n_\gamma = n_\gamma^{\mathrm{eq}}$. Notice that

$$\left(\frac{n_e^2}{n_{\mathrm{H}}} \right)_{\mathrm{eq}} = \left(\frac{2\pi}{m_e T} \right)^{-3/2} e^{-E_I/T}. \tag{26}$$

Since $n_e = X_e\, n_B$ and $n_{\mathrm{H}} = (1 - X_e) n_B$, it follows

$$\frac{dX_e}{dt} = \langle \sigma v \rangle \left(\frac{2\pi}{m_e T} \right)^{-3/2} e^{-E_I/T}(1 - X_e) - \langle \sigma v \rangle\, n_B\, X_e^2 \tag{27}$$

$$= \beta(T)(1 - X_e) - \alpha(T)\, n_B\, X_e^2 \tag{28}$$

The parameter $\alpha(T)$ characterizes the recombination rate, while $\beta(T)$ is associated with the ionization rate. When α is large, the right-hand side must vanish and the evolution of X_e is given by Saha equilibrium. The function α depends on the precise way that the electrons get captured into the ground state of a hydrogen atom. When recombination occurs directly into the ground state, it releases photons with energy larger than 13.6 eV. These photons will then quickly ionize other atoms, leading to no net recombination.

To avoid the instantaneous re-ionization of the hydrogen atoms, recombination must first occur to an excited state, which then decays to the ground state. The photons created during this multi-step recombination have lower energy and are therefore less likely to ionize the plasma. The details were worked out by Peebles in 1968.

References

[1] B. Ryden, *Introduction to cosmology.* Cambridge University Press, 1970.

[2] D. Baumann, *Cosmology.* Cambridge University Press, 7, 2022.

[3] E. W. Kolb, *The Early Universe*, vol. 69. Taylor and Francis, 5, 2019.

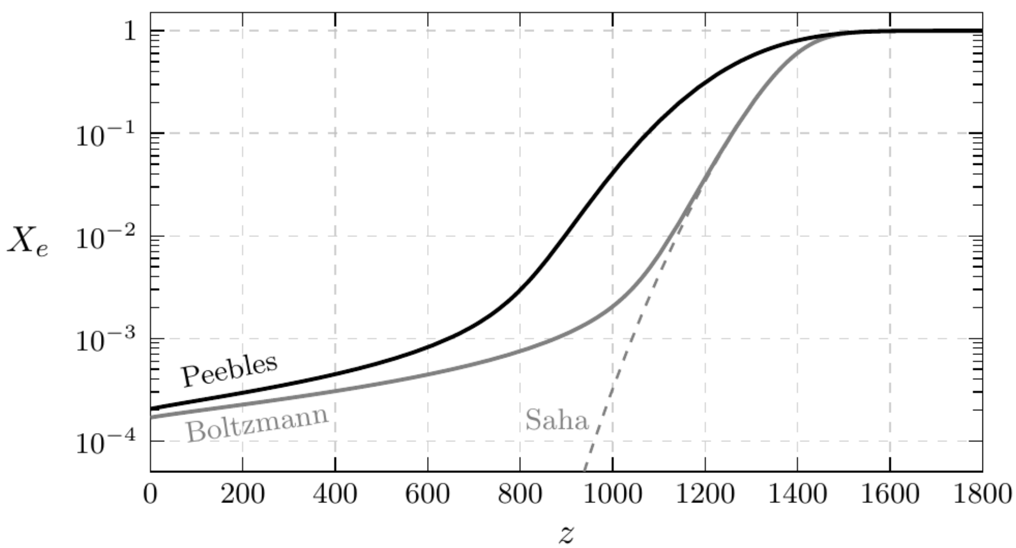

Figure 2: Free electron fraction as a function of redshift. The dashed curved labelled “Saha” is the prediction which assumes that equilibrium holds throughout. The curves labelled “Boltzmann” and “Peebles” are numerical solutions to the Boltz-mann equation. We see that recombination is delayed relative to the Saha prediction, and that the electron abundance freezes out at a constant value. Figure taken from [2].