Early Universe

Exercises Week #3

1. Is it possible for a closed universe to evolve to become an open universe? Give a reason for your answer.

2. The curvature of the two-dimensional surface is characterized by what is known as the Gaussian curvature, which may be defined as the limit

$$K = \frac{3}{\pi} \lim_{\mathfrak{s} \to 0} \left(\frac{2\pi\, \mathfrak{s} - C(\mathfrak{s})}{\mathfrak{s}^3} \right) ,$$

where $\mathfrak{s}$ is the measured radius of a circle on the surface and $C(\mathfrak{s})$ the length of the circumference. Suppose a circle is drawn around the North Pole on a sphere with radius a.

a) Show that the curvature of the sphere is equal to $K = 1/a^2$.

b) We may introduce coordinates (r', ϕ) on the sphere with the property that the circumference of a circle around the North Pole has the value exactly equal to $2\pi r'$. With the auxiliary angle $\theta = \mathfrak{s}/a$, we obtain that $r' = a \sin\theta$, so $\mathfrak{s} = a \arcsin(r'/a)$. Show that the line element can be expressed as

$$ds^2 = \frac{1}{1 - K\, r'^2} dr'^2 + r'^2 d\phi^2 .$$

c) For flat space, the same formula applies, with the curvature $K = 0$. A similar construction will show that a two-dimensional space with constant negative curvature $K = -1/a^2$ also obeys this metric equation. Since in the curved case the dimensioned parameter a is used, it may be convenient to measure all lengths in units of a, that is: we introduce the dimensionless variable $r = r'/a$, which gives the line element

$$ds^2 = a^2 \left(\frac{dr^2}{1 - k\, r^2} + r^2 d\phi^2 \right) ,$$

with $k = K\, a^2 = -1, +1, 0$ depending on whether the space is negatively curved, positively curved or flat. Compute the path length s when going from $r = 0$ to a finite value of r along a meridian $d\phi = 0$.

d) Derive the expression for the circumference of a circle on a surface with constant negative curvature, that is $k = -1$.

3. Derive an expression for the line element ds^2 on the two-sphere by using $x^2 + y^2 + z^2 = a^2$ and eliminating z.

4. The line elements of R^3 in Cartesian, cylindrical polar, and spherical polar coordinates are given respectively by

   a) $ds^2 = dx^2 + dy^2 + dz^2$.

   b) $ds^2 = dR^2 + R^2 d\phi^2 + dz^2$.

   c) $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$.

   Expressing ds^2 as $ds^2 = g_{ab}\, dx^a\, dx^b$, find g_{ab}, the inverse g^{ab} and $\det[g]$.

5. In three-dimensional Euclidean space R^3, write down expressions for the change of coordinates from Cartesian coordinates $[x^a] = (x, y, z)$ to spherical polar coordinates $[x'^a] = (r, \theta, \phi)$. Obtain expressions for the transformation and inverse transformation matrices in terms of the primed coordinates. By calculating the Jacobians J and J' for the transformation and its inverse, find where the transformation is non-invertible.

6. Derive an expression for the line element ds^2 on the three-sphere by using $x^2 + y^2 + z^2 + w^2 = a^2$ and eliminating w, to obtain
$$ds^2 = a^2 \left(\frac{dr^2}{1 - r^2} + r^2 d\Omega^2 \right) .$$
   In fact, one can show that every isotropic, homogeneous three-space can be parameterized (perhaps after performing a coordinate transformation) with coordinates of this form giving the metric equation
$$ds^2 = a^2 \left(\frac{dr^2}{1 - k\, r^2} + r^2 d\Omega^2 \right) .$$
   with $k = -1, +1, 0$ depending on whether the space is negatively curved, positively curved or flat.

7. Show that the hyperboloid is indeed a homogeneous and isotropic space. *Hint*: Begin with defining what precisely is a homogeneous and isotropic space.

8. Show that the line element of a 3-sphere of radius a embedded in four-dimensional Euclidean space can be written in the form
$$ds^2 = a^2[d\chi^2 + \sin^2\chi\, d\Omega^2].$$
   Show that the total volume of the 3-sphere is $V = 2\pi^2 a^3$.

9. In a four-dimensional Minkowski space with 'Cartesian' coordinates (w, x, y, z), a 3-hyperboloid is defined by $w^2 - x^2 - y^2 - z^2 = a^2$. Show that the metric on the surface of the 3-hyperboloid can be written in the form

$$ds^2 = a^2[d\chi^2 + \sinh^2\chi\, d\Omega^2].$$

Show that the total volume of the 3-hyperboloid is infinite.

10. Consider the surface of a 2-surface embedded in three-dimensional Euclidean space. In a stereographic projection, one assigns coordinates (ρ, ϕ) to each point on the surface of the sphere. The ϕ-coordinate is the standard azimuthal polar angle. The ρ-coordinate of each point is obtained by drawing a straight line in three dimensions from the south pole of the sphere through the point in question and extending the line until it intersects the tangent plane to the north pole of the sphere; the ρ-coordinate is then the distance in the tangent plane from the north pole to the intersection point. Show that the line element for the surface of the sphere in these coordinates is

$$ds^2 = \frac{d\rho^2}{(1+\rho^2/a^2)^2} + \frac{\rho^2}{1+\rho^2/a^2} d\phi^2\,.$$

11. Consider the surface of the Earth, which we assume for simplicity to be a 2-sphere of radius a. In terms of standard polar coordinates (θ, ϕ), the longitude of a point, in radians, rather than the usual degrees, is simply ϕ (measured eastwards from the Greenwich meridian), whereas its latitude $\lambda = \pi/2 - \theta$ radians. Show that the line element on the Earth's surface in these coordinates is

$$ds^2 = a^2(d\lambda^2 + \cos^2\lambda\, d\phi^2)\,.$$

12. A curve on the surface of a 2-sphere of radius a is defined parametrically by $\theta = u$, $\phi = 2u - \pi$, where $0 \le u \le \pi$. Sketch the curve and show that its total length is

$$L = a\int_0^a \sqrt{1 + 4\sin^4 u}\, du\,.$$

13. By identifying a suitable coordinate transformation, show that the line element

$$ds^2 = (c^2 - a^2t^2)dt^2 - 2at\, dt\, dx - dx^2 - dy^2 - dz^2\,,$$

where a is a constant, can be reduced to the Minkowski line element.

14. From Poisson's equation $\nabla^2\Phi = 4\pi G_N\rho$ how that the gravitational potential outside a spherical object of mass M at a radial distance r from its centre is given by $\Phi(r) = -G_N M/r$. What is the form of $\Phi(r)$ inside a uniform spherical body?

15. A charged object held stationary in a laboratory on the surface of the Earth does not emit electromagnetic radiation. If the object is then dropped so that it is in free fall, it will begin to radiate. Reconcile these observations with the principle of equivalence. *Hint*: Consider the spatial extent of the electric field of the charge.