Early Universe: Large Scale Structure

Disclaimer

Discussions taken from Barbara Ryden [1], Daniel Baumann [2], Bergström & Goobar [3] and Andrew Liddle [4] books

1 The Matthew Effect

$$\bar{\epsilon}(t) = \frac{1}{V}\int_V \epsilon(\vec{r},t)\, d^3r. \tag{1}$$

$$\delta(\vec{r},t) \equiv \frac{\epsilon(\vec{r},t) - \bar{\epsilon}(t)}{\bar{\epsilon}(t)}, \qquad -1 \leq \delta < \infty. \tag{2}$$

1.1 Static and homogeneous matter dominated Universe

In a region of the universe that is approximately static and homogeneous with uniform mass density $\bar{\rho}$, we add a small amount of mass within a sphere of radius R, so that the density within the sphere is $\bar{\rho}(1+\delta)$, with $\delta \ll 1$. If the density excess δ is uniform within the sphere, then the gravitational acceleration at the sphere's surface, due to the excess mass, will be

$$\ddot{R} = -\frac{G_N\,\Delta M}{R^2} = -\frac{G_N}{R^2}\frac{4\pi}{3}R^3\bar{\rho}\,\delta, \tag{3}$$

leading to

$$\frac{\ddot{R}}{R} = -\frac{4\pi\,G_N}{3}\bar{\rho}\,\delta, \tag{4}$$

Thus, a mass excess ($\delta > 0$) will cause the sphere to collapse inward ($\ddot{R} < 0$).

From energy conservation arguments,

$$M = \frac{4\pi}{3}\bar{\rho}[1+\delta(t)]R(t)^3 = \frac{4\pi}{3}\bar{\rho}R_0^3 = \text{constant}. \tag{5}$$

Then

$$R(t) = R_0(1+\delta(t))^{-1/3} \tag{6}$$

$$\approx R_0(1 - \frac{1}{3}\delta(t)), \;\text{ for }\; \delta \ll 1, \tag{7}$$

from which

$$\frac{\ddot{R}}{R} \approx -\frac{1}{3}\ddot{\delta}. \tag{8}$$

It follows that the small overdensity δ evolves as the sphere collapses:

$$\ddot{\delta} = 4\pi\,G_N\,\bar{\rho}\,\delta. \tag{9}$$

The solution of this ODE reads

$$\delta(t) = A_1 \, e^{t/t_{\rm dyn}} + A_2 \, e^{-t/t_{\rm dyn}}, \tag{10}$$

$$t_{\rm dyn} = \frac{1}{\sqrt{4\pi \, G_N \, \bar{\rho}}} \approx 9.6 \, \text{hours} \left(\frac{\bar{\rho}}{1 \, \text{kg/m}^3} \right)^{-1/2}. \tag{11}$$

After a few dynamical times only the exponentially growing term is significant. Thus, gravity tends to make small density fluctuations in a static, pressureless medium grow exponentially with time.

2 Hydrostatic equilibrium: the Jeans Length

If the pressure is nonzero, the attempted collapse will be countered by a steepening of the pressure gradient within the perturbation. The steepening of the pressure gradient, however, doesn't occur instantaneously. Any change in pressure travels at the sound speed. Thus, the time it takes for the pressure gradient to build up in a region of radius R will be

$$t_{\rm pre} \simeq \frac{R}{c_s}, \tag{12}$$

with

$$c_s = c \left(\frac{dP}{d\epsilon} \right)^{1/2} = \sqrt{\omega} \, c; \quad \omega > 0. \tag{13}$$

For hydrostatic equilibrium to be attained, the pressure gradient must build up before the overdense region collapses, implying

$$t_{\rm pre} < t_{\rm dyn}, \tag{14}$$

the, we find that for a density perturbation to be stabilized by pressure against collapse, it must be smaller than some reference size λ_J, given by the relation

$$\lambda_J = c_s \left(\frac{\pi \, c^2}{G_N \, \bar{\epsilon}} \right)^{1/2} = 2\pi \, c_s \, t_{\rm dyn}. \tag{15}$$

Overdense regions larger than the Jeans length collapse under their own gravity. Overdense regions smaller than the Jeans length merely oscillate in density; they constitute stable standing sound waves. The Jeans length of the Earth's atmosphere, where the sound speed is a third of a kilometer per second and the dynamical time is ten hours, is $\lambda_J \sim 10^5$ km, far longer than the scale height of the Earth's atmosphere. You don't have to worry about density fluctuations in the air undergoing a catastrophic collapse.

2.1 At cosmological scales

Let's consider a spatially flat universe in which the mean density is $\bar{\epsilon}$, but which contains density fluctuations with amplitude $|\delta| \ll 1$. The characteristic time for expansion of such a universe is the Hubble time,

$$H^{-1} = \left(\frac{3c^2}{8\pi G_N \bar{\epsilon}}\right)^{1/2} \tag{16}$$

$$= \left(\frac{3}{2}\right)^{1/2} t_{\rm dyn} = 1.22\, t_{\rm dyn}. \tag{17}$$

The Jeans length in an expanding flat universe will then be

$$\lambda_J = 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{c_s}{H} \tag{18}$$

$$= 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{\sqrt{\omega}\, c}{H} \tag{19}$$

The Jeans length for radiation in an expanding universe is then

$$\lambda_J = 2\pi \left(\frac{2}{9}\right)^{1/2} \frac{c}{H} \approx 3\frac{c}{H}. \tag{20}$$

In order for a universe to have gravitationally collapsed structures much smaller than the Hubble distance, it must have a nonrelativistic component, with $\omega \ll 1$.

Just before decoupling, the Jeans length of the photon–baryon fluid was roughly the same as the Jeans length of a pure photon gas:

$$\lambda_J(\text{before dec}) \approx \frac{3c}{H_{z_{\rm dec}}} \approx 0.66\,\text{Mpc} \approx 2 \times 10^{22}\,\text{m}. \tag{21}$$

The baryonic Jeans mass, M_J, is defined as the mass of baryons contained within a sphere of radius λ_J:

$$M_J(\text{before}) \equiv \rho_B \frac{4\pi}{3} \lambda_J^3. \tag{22}$$

Hence,

$$M_J \equiv \rho_B \frac{4\pi}{3} \lambda_J(\text{before})^3 = 2 \times 10^{49}\,\text{kg} \approx 10^{19}\, M_\odot. \tag{23}$$

This is very large compared to the baryonic mass of the Coma cluster $\sim 10^{14} M_\odot$; it is even large when compared to the baryonic mass of a supercluster $\sim 10^{16} M_\odot$.

Once the photons are decoupled, the photons and baryons form two separate gases, instead of a single photon–baryon fluid. The sound speed in the baryonic gas is

$$c_s(\text{baryon}) = \left(\frac{kT}{mc^2}\right)^{1/2} c. \tag{24}$$

The Jeans mass after decoupling turns to be

$$M_J(\text{after}) \approx 2 \times 10^5\, M_\odot. \tag{25}$$

This is very small compared to the baryonic mass of our own galaxy ($\sim 10^{11} M_\odot$); it is even small when compared to a more modest galaxy like the Small Magellanic Cloud ($\sim 10^9 M_\odot$).

The abrupt decrease of the baryonic Jeans mass at the time of decoupling marks an important epoch in the history of structure formation. Perturbations in the baryon density, from supercluster scales down to the size of the smallest dwarf galaxies, couldn't grow in amplitude until the time of photon decoupling, when the universe had reached the ripe old age of $t_{\rm dec} = 0.37$ Myr. After decoupling, the growth of density perturbations in the baryonic component was off and running.

3 Instability in an Expanding Universe

The equation that governs the growth of small amplitude perturbations becomes

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G_N}{c^2}\bar{\epsilon}_M \delta. \tag{26}$$

This form of the equation can be applied to a universe that contains components with non-negligible pressure, such as radiation ($\omega = 1/3$) or a cosmological constant ($\omega = -1$). In multiple-component universes, however, it should be remembered that δ represents the fluctuation in the density of matter alone. That is,

$$\delta = \frac{\epsilon_M - \bar{\epsilon}_M}{\bar{\epsilon}_M}, \qquad -1 \leq \delta < \infty. \tag{27}$$

In term of the matter abundance,

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_M H^2 \delta = 0. \tag{28}$$

- For $\Omega_M \ll 1$ and $H = 1/(2t)$:

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} \approx 0. \tag{29}$$

$$\delta(t) \approx B_1 + B_2 \ln t \tag{30}$$

  During the radiation-dominated epoch, density fluctuations in the dark matter grew only at a logarithmic rate.

- In the far future, if the universe is indeed dominated by a cosmological constant, the density parameter for matter will again be negligibly small, the Hubble parameter will have the constant value:

$$\ddot{\delta} + 2H_\Lambda \dot{\delta} \approx 0. \tag{31}$$

$$\delta(t) \approx C_1 + C_2\, e^{-2H_\Lambda t}. \tag{32}$$

  In a lambda-dominated phase, therefore, fluctuations in the matter density reach a constant fractional amplitude.

- Only when matter dominates the energy density can fluctuations in the matter density grow at a significant rate. In a flat, matter-dominated universe, $\Omega_M = 1, H = 2/(3t)$,

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0, \tag{33}$$

$$\delta(t) \approx D_1 t^{2/3} + D_2 t^{-1}. \tag{34}$$

  When the growing mode is the only survivor, the density perturbations in a flat, matter-only universe grow at the rate

$$\delta \propto t^{2/3} \propto a(t) \propto \frac{1}{1+z}, \tag{35}$$

  as long as $|\delta| \ll 1$.

  Perturbations in cold dark matter can grow during matter domination, whereas the baryons oscillate until decoupling. This means that cold dark matter plays a crucial role for structure formation - they form potential wells in which the baryons can fall as soon as they escape the pressure after decoupling.

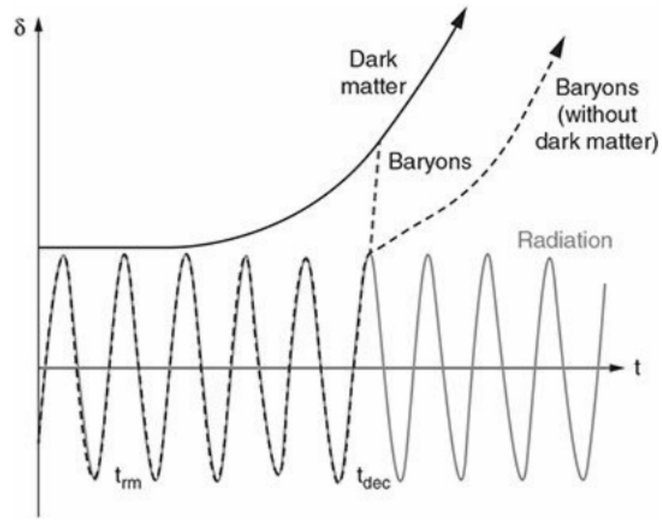

Figure 1: A highly schematic drawing of how density fluctuations in different components of the universe evolve with time.

4 The power spectrum

The density field $\delta(\vec{r}, t_0)$ at a given time is generally assumed to be a Gaussian random field. In fact, this is what is predicted in models of inflation, where quantum fluctuations in the inflaton field are created by the exponen- tial expansion, and could be the 'seeds' of cosmic structure. Defining the rms fluctuations by

$$\sigma^2 = \langle \delta(\vec{r}) \rangle, \tag{36}$$

the autocorrelation function $\xi(|\vec{r}_2 - \vec{r}_1|) \equiv \xi(r)$ is defined as

$$\xi(r) = \frac{1}{\sigma^2} \langle \delta(\vec{r}_2) \delta(\vec{r}_1) \rangle. \tag{37}$$

The Fourier transform is

$$\delta(\vec{r}, t) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{r}} \delta(\vec{k}, t) d^3k. \tag{38}$$

In the random phase approximation, all the Fourier modes $\delta(\vec{k})$ are uncorrelated, which means that

$$\langle \delta^*(\vec{k}) \delta(\vec{k}') \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P(\vec{k}). \tag{39}$$

The function $P(\vec{k})$ is the power spectrum of the fluctuations. In most models $P(\vec{k}) = k^n$, with the spectral index n varying between 0.7 and 1.3. The value $n = 1$ corresponds to so-called scale-invariant fluctuations (Harrison–Zel'dovich spectrum), since it can be shown that for such a spectrum the fluctuations δ are of the same amplitude for all length scales. Inflation predicts almost scale-invariant perturbations.

Observationally, the value of n can be deduced from the temperature fluctuations of the cosmic microwave background on large scales ($\theta > 1.1°$). The temperature correlation function measured by the Planck satellite is consistent with

$$n = 0.97 \pm 0.01. \tag{40}$$

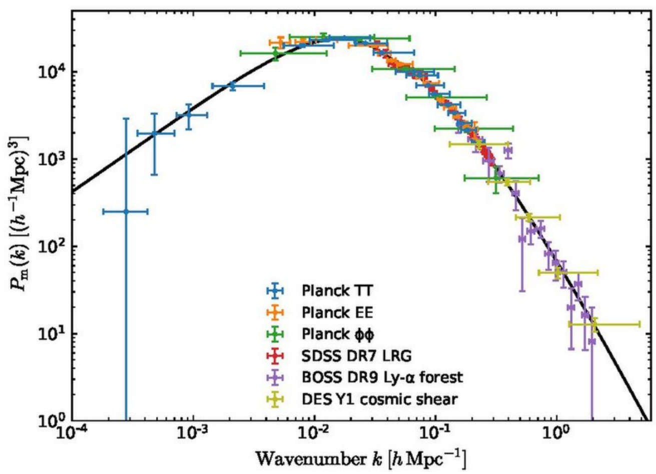

Figure 2: Matter power spectrum inferred from various cosmological probes.

Thus, the observed inflationary power spectrum is only slightly "tilted" relative to a Harrison–Zel'dovich spectrum.

Cold dark matter fluctuations that entered the horizon after matter and radiation equality should not have been modified much from the primordial spectrum. The comoving length scale corresponding to the horizon at equality is roughly

$$\lambda_{\rm eq} \approx \frac{13}{\Omega_T h^2}\,\mathrm{Mpc}. \tag{41}$$

For scales smaller than this, it can be shown that there is a suppression $\sim k^{-2}$. For hot dark matter, the suppression at small scales is much larger, since relativistic particles 'free-stream' out of small density enhancements. Recent data from the distribution of galaxies measured by the 2dF collaboration agree excellently with cold dark matter. This has been used to put an upper limit on the relic density of neutrinos corresponding to around 2 eV for the sum of the masses of neutrinos.

5 Primordial Density Fluctuations

$$n = 1 - 6\epsilon + 2\eta. \tag{42}$$

References

[1] B. Ryden, *Introduction to cosmology.* Cambridge University Press, 1970.

[2] D. Baumann, *Cosmology.* Cambridge University Press, 7, 2022.

[3] L. Bergström and A. Goobar, *Cosmology and Particle Astrophysics.* Springer Praxis Books. Springer, 2004.

[4] A. R. Liddle, *An introduction to modern cosmology.* 1998.