Solids at Rest

Contents

1 Objectives **3**
1.1 Objetivos específicos conceptuales: 3
1.2 Objetivos específicos procedimentales: 3
1.3 Objetivos específicos actitudinales: 3

2 Stress **4**
2.1 Friction 5
2.2 Stress fields 6
2.2.1 External and internal stresses: 6
2.2.2 Tensile strength and yield stress: 6
2.3 The stress tensor 7
2.3.1 Cauchy's stress hypothesis 7
2.4 Total force 9
2.4.1 Mechanical Pressure 10
2.5 Mechanical equilibrium 11
2.5.1 Symmetry 11
2.5.2 Boundary Conditions 12

3 Strain **13**
3.1 Displacements 13
3.1.1 Uniform scaling 13
3.1.2 Linear displacements 14
3.1.3 Simple Translation 15
3.1.4 Simple Rotation 16
3.1.5 Simple Scaling Along x 16
3.2 Displacement field 17
3.2.1 Displacement Field Representations 17
3.2.2 Local and Infinitesimal Deformation 17
3.2.3 Slowly Varying Displacement Field 18
3.2.4 Cauchy's Strain Tensor 18
3.2.5 Components of the Strain Tensor 18
3.2.6 Principal Axes of Strain 19
3.3 Geometrical Meaning of the Strain Tensor 20
3.3.1 Lengths and Angles 20
3.3.2 Infinitesimal Elements 21
3.4 Thermodynamics of deformation 21
3.4.1 Virtual Displacement Work 22
3.5 Deformation regimes 23
3.5.1 Energy 23
3.6 Helmholtz and Gibbs Free Energies 25
3.6.1 Helmholtz Free Energy 25
3.6.2 Gibbs Free Energy 25

4 Hooke's law 26
4.1 Isotropic bodies: free energy and elastic coefficients . . . 26
4.2 Decomposition of the strain tensor . . . 27
4.3 Stress–strain Relation . . . 27
4.4 Hooke's Law . . . 28
4.5 Homogeneous deformations . . . 28
4.5.1 Hydrostatic Compression . . . 28
4.5.2 Uniform stretching . . . 29
4.5.3 Unilateral Compression . . . 31
4.5.4 Uniform Shear . . . 32

5 Basic elastostatics 33
5.1 Navier–Cauchy Equation . . . 33
5.1.1 The Equations of Equilibrium in a gravitational field . . . 34
5.1.2 Surface-Loaded Bodies . . . 34
5.2 Uniform settling . . . 35
5.2.1 Characteristic length scale . . . 36
5.3 Uniform pure bending . . . 37
5.3.1 Shear-free bending . . . 37
5.3.2 Total force . . . 39
5.3.3 Total moment . . . 39
5.3.4 Moments of inertia of common beam cross-sections . . . 40
5.3.5 Threshold for buckling . . . 40
5.4 Twisting a shaft . . . 41
5.4.1 Pure torsion . . . 41
5.4.2 Strains and stresses . . . 42
5.4.3 Torque . . . 43
5.4.4 Transmitted power . . . 43

6 Applications of the Navier–Cauchy Equation 43
6.1 Elastic Waves and Seismology . . . 44
6.2 Elastic Deformation of Planetary Bodies . . . 44
6.3 Astrophysical Elasticity: Neutron Star Crusts . . . 45
6.4 Worked Examples . . . 45
6.5 Exercises . . . 46

7 Solved problems 48
7.1 Long rod standing vertically in a gravitational field . . . 48
7.2 Hollow sphere . . . 48
7.3 Solid sphere . . . 50
7.4 Cylindrical pipe . . . 50
7.5 Cylinder rotating uniformly . . . 50
7.6 Equations of equilibrium for an isotropic body . . . 51
7.7 General integral of the equations of equilibrium . . . 51

8 Problems 52

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3] books.

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC9. Enumerar los distintos regímenes de deformación a los que está sometido un sólido o un fluido viscoelástico.
- OC10. Definir el campo de desplazamientos, el tensor de deformaciones y el tensor de elasticidad.
- OC11. Enunciar la ley de Hooke en toda su generalidad.

1.2 Objetivos específicos procedimentales:

- OP8. Escribir el tensor de esfuerzos y la ley de Cauchy.
- OP9. Determinar la relación entre las fuerzas experimentadas por un elemento de volumen en un medio continuo, incluyendo la presión, y el tensor de esfuerzos.

1.3 Objetivos específicos actitudinales:

- OA3. Reconocer la importancia de la notación, del álgebra y del cálculo tensorial en la representación de las cantidades físicas asociadas con los medios continuos.

2 Stress

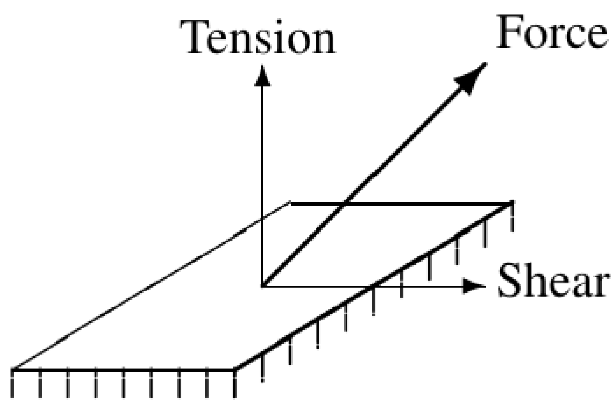
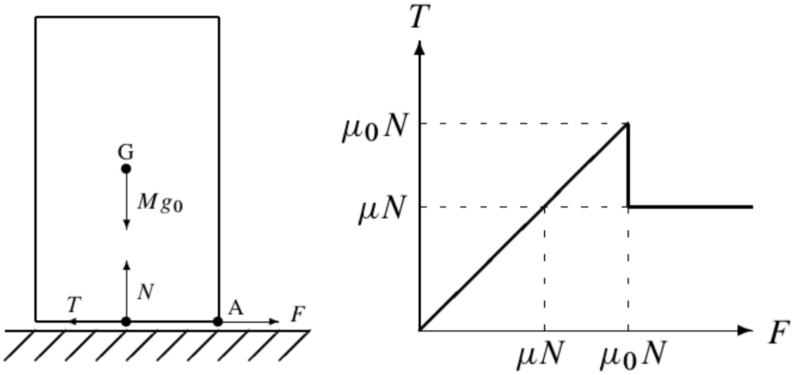


Figure 1: The force on a small piece of a surface can be resolved into a nor- mal pressure force and a tangential shear force. Figure taken from Ref. [3].

- In fluids at rest, the only contact force is pressure, but in solids and viscous fluids in motion, additional *shear stresses* arise—forces acting tangentially to contact surfaces.
- Shear stress is the shear force per unit area and plays a crucial role in maintaining the structural integrity of solids.
- Friction in everyday life is a manifestation of shear stress at contact surfaces between materials.
- Materials are broadly classified as:
  - **Fluids**, which respond to stress by *flowing*.
  - **Solids**, which respond by *deforming*.
- Elastic deformation grows linearly with stress but can lead to permanent plastic deformation or rupture at high stress levels.
- The study of stress applies to all materials—solids, fluids, and intermediate forms—including artificial and exotic substances with special technological uses.
- Contact forces depend not only on spatial position but also on the orientation of the surface; therefore, a more general description using the *stress tensor* is required.
- The stress tensor, consisting of nine components, describes the complete range of contact forces acting within a material.
- Understanding the concept and properties of stress is fundamental to continuum mechanics and central to this chapter.



	μ_0	μ
Glass/glass	0.9	0.4
Rubber/asphalt	0.9	0.7
Steel/steel	0.7	0.6
Metal/metal	0.6	0.4
Wood/wood	0.4	0.3
Steel/ice	0.1	0.05
Steel/teflon	0.05	0.05

Figure 2: Left: Balance of forces on a crate at rest on a horizontal floor. Center: Sketch of tangential reaction (traction) T as a function of applied force F. Up to $F = \mu_0 N$, the traction adjusts itself to the applied force, $T = F$. At $F = \mu_0 N$, the tangential reaction drops abruptly to a lower value, $T = \mu N$, and stays there regardless of the applied force. Right: Typical friction coefficients for various combinations of materials. Figures taken from Ref. [3].

2.1 Friction

- **Friction as a shear force:**
  - Friction between solid bodies is a type of shear force.
  - It enables us to hold, grab, drag, and rub objects without them slipping.
  - Everyday tasks, from stirring coffee to lighting a fire, involve working against friction.
- **Static friction:**
  - Occurs when an object is at rest and resists being set into motion.
  - The static frictional force T balances the applied force F up to a maximum value (see Fig. 2):
    $$T < \mu_0 N,$$
    where μ_0 is the *coefficient of static friction* and N is the normal load.
  - The coefficient μ_0 is dimensionless and typically around 0.5 or higher in everyday materials.
  - Its magnitude depends on the materials in contact and the roughness of their surfaces.
- **Dynamic (kinetic) friction:**
  - Once motion starts, friction still acts but with a smaller magnitude than static friction.
  - The frictional force is proportional to the normal load:
    $$T = \mu N,$$
    where μ is the *coefficient of dynamic friction*, satisfying $\mu < \mu_0$.
  - After motion begins, a smaller force $F = \mu N$ maintains constant speed.

- Work done against dynamic friction is converted into heat:

$$P = FU,$$

  where P is the power and U is the velocity.
- Static and dynamic friction are both independent of the size of the contact area, so that a crate on legs is as hard to drag as one without. The best way to diminish the force necessary to drag the crate is to place it on wheels.

- **Comparison:**
  - Static friction prevents motion; dynamic friction resists motion once it starts.
  - Static friction does not produce work, while dynamic friction continuously dissipates energy as heat.

2.2 Stress fields

- Stress fields describe the local distribution of normal and tangential forces acting on a surface.
- Measured in pascals (Pa = N m^{-2}), analogous to pressure.
- Types of stresses:
  - *Tension stress:* acts along the normal and pulls outward.
  - *Pressure stress:* acts along the normal and pushes inward.
  - *Shear stress:* acts tangentially along the surface.

2.2.1 External and internal stresses:

- *External stresses* act between a body and its environment.
- *Internal stresses* act across imaginary surfaces within the body and exist even in the absence of external forces.
- Internal stresses depend on material composition, geometry, temperature, and external loading.
- Rapid temperature changes (e.g., fast cooling) can permanently "freeze" internal stresses, especially in brittle materials like glass.

2.2.2 Tensile strength and yield stress:

- When external forces grow large, materials may deform plastically or fracture.
- *Tensile strength:* the maximum stress a material can withstand before failure.
- *Yield stress:* the stress at which plastic deformation begins.
- *Ductile materials* (e.g., copper) can stretch without breaking; *brittle materials* (e.g., cast iron) fail suddenly.

- Typical tensile strengths:
  - Metals: several hundred megapascals.
  - Carbon fibers: several gigapascals.
  - Carbon nanotubes: up to 50 GPa.

2.3 The stress tensor

- Stresses at a point are described by nine components σ_{ij} in Cartesian coordinates:

$$\sigma_{ij} = \frac{dF_i}{dS_j}. \tag{1}$$

- These include three normal stresses (σ_{xx}, σ_{yy}, σ_{zz}) and six shear stresses.
- By convention, the sign is chosen such that a positive value of σ_{yy} corresponds to a pull or tension.

2.3.1 Cauchy's stress hypothesis

This asserts that the force

$$d\vec{F} = (dF_x, dF_y, dF_z) \tag{2}$$

on an arbitrary surface element,

$$d\vec{S} = (dS_x, dS_y, dS_z), \tag{3}$$

is of the form

$$\begin{aligned} dF_x &= \sigma_{xx} dS_x + \sigma_{xy} dS_y + \sigma_{xz} dS_z, \\ dF_y &= \sigma_{yx} dS_x + \sigma_{yy} dS_y + \sigma_{yz} dS_z, \\ dF_z &= \sigma_{zx} dS_x + \sigma_{zy} dS_y + \sigma_{zz} dS_z. \end{aligned} \tag{4}$$

The components of the force are expressed compactly as

$$dF_i = \sum_j \sigma_{ij}\, dS_j. \tag{5}$$

- The nine stress components form the *stress tensor field* $\sigma_{ij}(x, t)$.
- The stress tensor is written as

$$\vec{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{6}$$

- The stress acting on a surface element $d\vec{S}$ is given by

$$d\vec{F} = \vec{\sigma} \cdot d\vec{S}. \tag{7}$$

- Writing $d\vec{S} = \hat{n}\, dS$, the stress per unit area becomes

$$\frac{d\vec{F}}{dS} = \vec{\sigma} \cdot \hat{n}, \tag{8}$$

called the *stress vector*.

- Although $\vec{\sigma} \cdot \hat{n}$ behaves like a vector, it is not a true vector field since it depends on the surface orientation.

Proof of Cauchy's stress hypothesis:

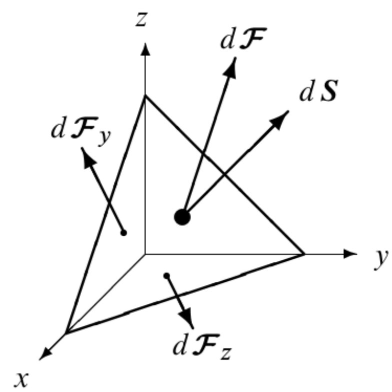


Figure 3: The tiny triangle and its projections form a tetrahedron. The (vector) force acting on the triangle in the xy-plane is called $d\vec{F}_z$, and the force acting on the triangle in the zx-plane is $d\vec{F}_y$. The force $d\vec{F}_x$ acting on the triangle in the yz-plane is hidden from view. The force $d\vec{F}$ acts on the skew triangle. Figure taken from Ref. [3].

Let's take again a surface element in the shape of a tiny triangle (see Fig. 3) with area vector

$$d\vec{S} = (dS_x, dS_y, dS_z).$$

This triangle and its projections on the coordinate planes form together a little body in the shape of a right tetrahedron. Let the external (vector) forces acting on the three triangular faces of the tetrahedron be denoted, respectively,

$$d\vec{F}_x,\ d\vec{F}_y,\ \text{and } d\vec{F}_z.$$

Adding the external force $d\vec{\mathcal{F}}$ acting on the skew face and a possible volume force $\vec{f}\, dV$, Newton's Second Law for the small tetrahedron becomes

$$\vec{a}_{\rm CM}\, dM = \vec{f}\, dV + d\vec{F}_x + d\vec{F}_y + d\vec{F}_z + d\vec{\mathcal{F}}, \tag{9}$$

where $\vec{a}_{\rm CM}$ is the center-of-mass acceleration of the tetrahedron and $dM = \rho\, dV$ its mass.

The volume of the tetrahedron scales like the third power of its linear size, whereas the surface area only scales like the second power. Making the tetrahedron progressively smaller, the body force term and the acceleration term will vanish faster than the surface terms. In the limit of a truly infinitesimal tetrahedron, only the surface terms survive, so that we must have

$$d\vec{F}_x + d\vec{F}_y + d\vec{F}_z + d\vec{F} = 0.$$

Taking into account that the area projections dS_x, dS_y, and dS_z point into the tetrahedron, we define the stress vectors along the coordinate directions as

$$\vec{\sigma}_x = -\frac{d\vec{F}_x}{dS_x}, \quad \vec{\sigma}_y = -\frac{d\vec{F}_y}{dS_y}, \quad \vec{\sigma}_z = -\frac{d\vec{F}_z}{dS_z}.$$

Consequently,

$$d\vec{F} = \vec{\sigma}_x\, dS_x + \vec{\sigma}_y\, dS_y + \vec{\sigma}_z\, dS_z. \tag{10}$$

This shows that the force on an arbitrary surface element may be written as a linear combination of three basic stress vectors along the coordinate axes. Introducing the nine coordinates of the three stress vectors,

$$\vec{\sigma}_x = (\sigma_{xx}, \sigma_{yx}, \sigma_{zx}), \quad \vec{\sigma}_y = (\sigma_{xy}, \sigma_{yy}, \sigma_{zy}), \quad \vec{\sigma}_z = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz}),$$

we arrive at Cauchy's hypothesis.

2.4 Total force

Including a volume force density f_i, the total force on a body of volume V with surface S becomes

$$F_i = \int_V f_i\, dV + \oint_S \sum_j \sigma_{ij}\, dS_j. \tag{11}$$

Using Gauss' theorem, this may be written as a single volume integral,

$$F_i = \int_V f_i^*\, dV, \tag{12}$$

where

$$f_i^* = f_i + \sum_j \nabla_j \sigma_{ij}, \tag{13}$$

is the *effective force density*. The effective force is not just a formal quantity, because the total force on a material particle is $d\vec{F} = \vec{f}^*\, dV$. In matrix notation this reads

$$\vec{f}^* = \vec{f} + \vec{\nabla} \cdot \vec{\sigma}^{\mathrm{T}}, \tag{14}$$

where $\vec{\sigma}^{\mathrm{T}}$ is the transposed matrix, defined as $\sigma^{\mathrm{T}}_{ji} = \sigma_{ij}$.

In other words, the **total contact force** on a small box-shaped material particle (see Fig. 4) is calculated from the differences of the stress vectors acting on the sides. Suppressing the dependence on y and z, the resultant contact force on dS_x becomes

$$d\vec{F} = (\vec{\sigma}_x(x + dx) - \vec{\sigma}_x(x))\, dS_x \approx \nabla_x \vec{\sigma}_x\, dV.$$

Adding the similar contributions from dS_y and dS_z, one arrives at the total surface force,

$$d\vec{F} = (\nabla_x \vec{\sigma}_x + \nabla_y \vec{\sigma}_y + \nabla_z \vec{\sigma}_z)\, dV = \nabla \cdot \vec{\sigma}^{\mathrm{T}}\, dV.$$

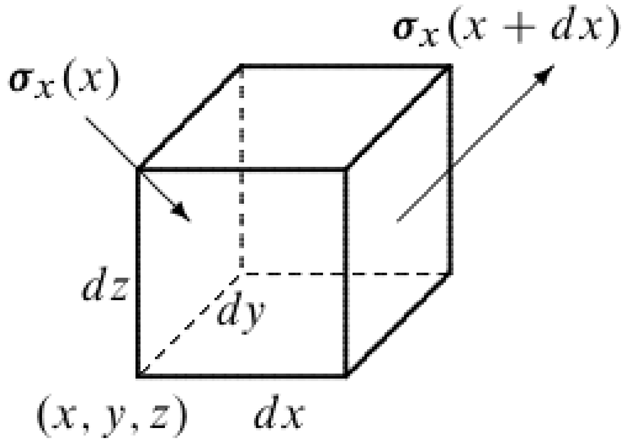


Figure 4: The total contact force on a small box-shaped material particle.

2.4.1 Mechanical Pressure

- **Hydrostatic equilibrium:** When the only contact force is pressure, the stress tensor is isotropic:
$$\boldsymbol{\sigma} = -p\,\mathbf{1}, \qquad \text{or equivalently,} \qquad \sigma_{ij} = -p\,\delta_{ij},$$
where $\mathbf{1}$ is the unit matrix and δ_{ij} is the Kronecker delta.

- **General case:** In real materials, the stress tensor usually has off-diagonal (shear) components. The diagonal components (σ_{xx}, σ_{yy}, σ_{zz}) differ and correspond to the normal stresses, often interpreted as "pressures":
$$p_x = -\sigma_{xx}, \qquad p_y = -\sigma_{yy}, \qquad p_z = -\sigma_{zz}.$$
These components do *not* form a true vector, since they do not transform properly under rotation.

The *mechanical pressure* is defined as the average of the normal stresses:

$$p = \frac{1}{3}(p_x + p_y + p_z) = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}). \tag{15}$$

This definition ensures that pressure is a scalar field, invariant under Cartesian coordinate transformations, because it depends on the *trace* of the stress tensor:

$$\operatorname{Tr}\vec{\sigma} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}. \tag{16}$$

The mechanical pressure corresponds to the isotropic part of the stress tensor. It represents the mean normal stress acting equally in all directions and is the only scalar combination of stress components that remains invariant under coordinate changes.

2.5 Mechanical equilibrium

In mechanical equilibrium, the total force on any body must vanish, for if it does not the body will begin to move. So the general condition is that $\vec{F} = 0$ for all volumes V. In particular, requiring that the force on each and every material particle must vanish, we arrive at *Cauchy's equilibrium equation(s)*:

$$f_i^* = f_i + \sum_j \nabla_j \sigma_{ij} = 0. \tag{17}$$

In spite of their simplicity, these partial differential equations govern mechanical equilibrium in all kinds of continuous matter, be it solid, fluid, or otherwise. For a fluid at rest, where pressure is the only stress component, we have $\sigma_{ij} = -p\,\delta_{ij}$, and recover the equation of hydrostatic equilibrium, $f_i - \nabla_i p = 0$.

The three individual equations contained in Cauchy's equilibrium equation read

$$\begin{aligned} f_x + \nabla_x \sigma_{xx} + \nabla_y \sigma_{xy} + \nabla_z \sigma_{xz} &= 0, \\ f_y + \nabla_x \sigma_{yx} + \nabla_y \sigma_{yy} + \nabla_z \sigma_{yz} &= 0, \\ f_z + \nabla_x \sigma_{zx} + \nabla_y \sigma_{zy} + \nabla_z \sigma_{zz} &= 0. \end{aligned} \tag{18}$$

These equations are in themselves not sufficient to determine the state of continuous matter, but must be supplemented by suitable *constitutive equations* connecting stress and state. For fluids at rest, the equation of state serves this purpose by relating hydrostatic pressure to mass density and temperature. In elastic solids, the constitutive equations are more complicated and relate stress to displacement.

2.5.1 Symmetry

There is one very general condition which may normally be imposed, namely the symmetry of the stress tensor:

$$\sigma_{ij} = \sigma_{ji} \quad \text{or} \quad \vec{\sigma}^T = \vec{\sigma}. \tag{19}$$

Symmetry only affects the shear stress components, requiring

$$\sigma_{xy} = \sigma_{yx}, \qquad \sigma_{yz} = \sigma_{zy}, \qquad \sigma_{zx} = \sigma_{xz}, \tag{20}$$

and thus reduces the number of independent stress components from nine to six.

Being thus a symmetric matrix, the stress tensor may be diagonalized. The eigenvectors define the principal directions of stress and the eigenvalues the principal tensions or stresses. In the principal basis, there are no off-diagonal elements, i.e., shear stresses, only pressures. The principal basis is generally different from point to point in space.

There is in fact no general proof of symmetry of the stress tensor, but only some theoretical arguments that allow us to choose the stress tensor to be symmetric in all normal materials. Here we shall present a simple argument, only valid in complete mechanical equilibrium.

Consider a material particle in the shape of a tiny rectangular box with sides a, b, and c (see Fig. 5). The force acting in the y-direction on a face in the x-plane is $\sigma_{yx}bc$ whereas the force acting in the x-plane on a face in the y-plane is $\sigma_{xy}ac$. On the opposite faces the contact forces have opposite sign in mechanical equilibrium (their difference is, as we have seen, of order abc). Since the total force vanishes, the total moment of force on the box may be calculated around any point we wish. Using the lower left corner, we get

$$\mathcal{M}_z = a\,\sigma_{yx}bc - b\,\sigma_{xy}ac = (\sigma_{yx} - \sigma_{xy})abc.$$

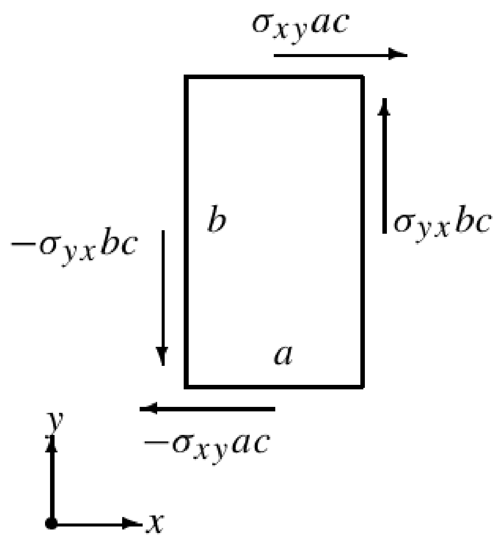


Figure 5: An asymmetric stress tensor will produce a non-vanishing moment of force on a small box (the z- direction not shown). Figure taken from Ref. [3].

This shows that if the stress tensor is asymmetric, $\sigma_{xy} \neq \sigma_{yx}$, there will be a resultant moment on the box. In mechanical equilibrium this cannot be allowed, since such a moment would begin to rotate the box, and consequently the stress tensor must be symmetric. Conversely, when the stress tensor is symmetric, mechanical equilibrium of the forces alone guarantees that all local moments of force will vanish.

2.5.2 Boundary Conditions

- **Cauchy's equation of mechanical equilibrium** requires appropriate boundary conditions on the surface of a body, since it is valid only within the body's volume.
- The **stress tensor** $\vec{\sigma}$ is a local physical quantity that can be assumed continuous in regions where the material properties vary smoothly.
- Across real boundaries or interfaces, where material properties may change abruptly, Newton's Third Law implies that the two sides exert equal and opposite forces on each other (in the absence of surface tension).
- The **stress vector**

$$\vec{\sigma} \cdot \hat{\hat{n}} = \left\{ \sum_j \sigma_{ij} n_j \right\}$$

  must therefore be continuous across a surface with normal $\hat{n}$.
- This condition is expressed as the vanishing of the surface discontinuity of the stress vector:

$$[\vec{\sigma} \cdot \hat{n}] = 0.$$

  where the brackets [] denote the difference between the two sides of the surface.

- This does *not* imply that all stress tensor components are continuous—only three linear combinations (those corresponding to the stress vector) are constrained to be continuous.
- The **mechanical pressure** is, therefore, not necessarily continuous across interfaces. In general continuum mechanics, the mechanical pressure loses the simple, intuitive interpretation it had in hydrostatics.

3 Strain

- All materials deform under external forces, but they respond differently:
  - **Elastic materials** return to their original shape once the force is removed.
  - **Plastic materials** retain permanent deformation.
  - **Viscoelastic materials** behave elastically under quick deformation but flow like viscous fluids over time.
- Elasticity is valid only within a limited range of forces; beyond that, materials become plastic or may fracture.
- Deformation displaces the material from its original position:
  - Small deformations are easier to analyze mathematically than large ones.
  - Large deformations produce complex, non-uniform displacements (e.g., crumpled paper).
  - Coordinate systems deform with the body, becoming curvilinear.
- The theory of finite deformation is mathematically complex, similar to curvilinear coordinate systems. In most engineering applications, deformations are small and can be approximated as linear.
- The description of deformation introduces the **strain tensor**, representing the local deformation or strain of a material.

3.1 Displacements

3.1.1 Uniform scaling

The prime example of deformation is a *uniform scaling* in which the coordinates of all material particles in a body are multiplied with the scale factor κ. A material particle originally situated in the point $\vec{X}$ is thus displaced to the point

$$\vec{x} = \kappa \vec{X}. \tag{21}$$

It is emphasized that *both* $\vec{X}$ and $\vec{x}$ refer to the same coordinate system. Uniform scaling with $\kappa > 1$ is also called uniform *dilatation*, whereas scaling with $0 < \kappa < 1$ is called uniform *compression*. Negative scaling with $\kappa < 0$ is physically impossible.

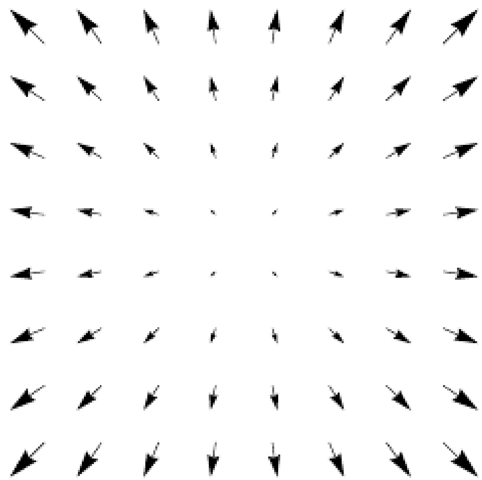


Figure 6: Uniform dilatation. The arrows indicate how material particles are displaced. Figure taken from Ref. [3].

The only point that does not change place during uniform scaling is the origin of the coordinate system. Although it superficially looks as if the origin plays a special role, this is not really the case. All relative positions of material particles scale in the same way, because

$$\vec{x} - \vec{y} = \kappa(\vec{X} - \vec{Y}), \tag{22}$$

independent of the origin of the coordinate system. There is no special center for a uniform scaling, either geometrically or physically. The origin of the coordinate system is simply an *anchor point* for the mathematical description of scaling.

3.1.2 Linear displacements

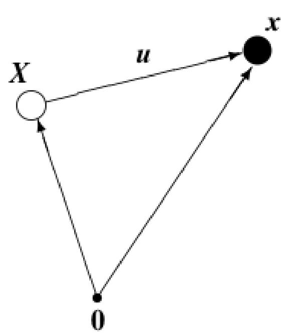


Figure 7: Geometry of displacement. The particle that originally was located at $\vec{X}$ has been displaced to $\vec{x}$ by the displacement vector $\vec{u}$. Figure taken from Ref. [3].

Under a displacement (see Fig. 7), the center–of–mass of a material particle is moved from its original position $\vec{X}$ to its actual position $\vec{x}$. The *displacement vector* is always defined as the difference between the actual and the original coordinates,

$$\vec{u} = \vec{x} - \vec{X}. \tag{23}$$

For the case of uniform scaling, the displacement vector becomes

$$\vec{u} = (\kappa - 1)\vec{X} = \left(1 - \frac{1}{\kappa}\right)\vec{x}. \tag{24}$$

Mathematically we are completely free to express the displacement as a function of either the original position X or the actual position x of the material particle. For scaling, the displacement is in both cases a linear function of the coordinates.

More generally, a linear displacement (and its inverse) takes the form

$$\vec{x} = \mathbf{A} \cdot \vec{X} + \vec{b}, \qquad \vec{X} = \mathbf{A}^{-1} \cdot (\vec{x} - \vec{b}), \tag{25}$$

where $\mathbf{A}$ is a non–singular constant matrix and $\vec{b}$ is a constant vector. As for scaling, the displacement vector may be expressed as a function of either the original or the actual positions:

$$\vec{u} = (\mathbf{A} - 1) \cdot \vec{X} + \vec{b} = (1 - \mathbf{A}^{-1}) \cdot \vec{x} + \mathbf{A}^{-1} \cdot \vec{b}. \tag{26}$$

The general linear displacement may be resolved into simpler types: translation along a coordinate axis, rotation by a fixed angle around a coordinate axis, and scaling by a fixed factor along a coordinate axis (see Fig. 8). Physically impossible reflections are excluded.

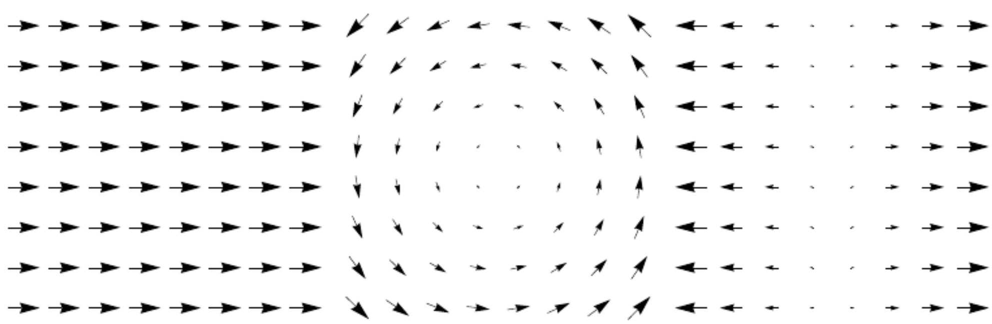


Figure 8: Arrow plots of the displacement fields for simple translation, rotation, and dilatation. Figure taken from Ref. [3].

3.1.3 Simple Translation

A rigid body translation of the material through a distance b along the x–axis is described by

$$x = X + b, \qquad y = Y, \qquad z = Z. \tag{27}$$

The displacement vector becomes

$$u_x = b, \qquad u_y = 0, \qquad u_z = 0. \tag{28}$$

Since the geometric relationships in a body are unchanged, this is not a deformation.

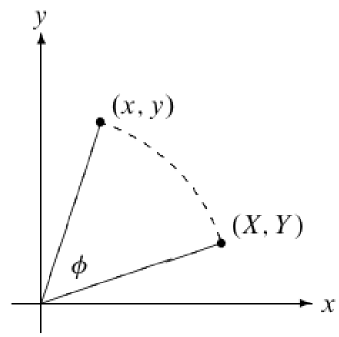


Figure 9: A rigid body rotation through an angle ϕ moves the material particle at (X, Y) to (x, y). Figure taken from Ref. [3].

3.1.4 Simple Rotation

A rigid body rotation (see Fig. 9) through the angle ϕ around the z–axis takes the form

$$x = X\cos\phi - Y\sin\phi, \qquad X = x\cos\phi + y\sin\phi, \tag{29}$$
$$y = X\sin\phi + Y\cos\phi, \qquad Y = -x\sin\phi + y\cos\phi, \tag{30}$$
$$z = Z, \qquad Z = z. \tag{31}$$

The displacement vector components are

$$u_x = -X(1-\cos\phi) - Y\sin\phi = x(1-\cos\phi) - y\sin\phi, \tag{32}$$
$$u_y = X\sin\phi - Y(1-\cos\phi) = x\sin\phi - y(1-\cos\phi), \tag{33}$$
$$u_z = 0 = 0. \tag{34}$$

Since all distances in the body remain unchanged, this is not a deformation.

3.1.5 Simple Scaling Along x

Multiplying all x–coordinates by the factor κ gives

$$x = \kappa X, \qquad y = Y, \qquad z = Z. \tag{35}$$

The displacement vector becomes

$$u_x = (\kappa - 1)X = kx, \tag{36}$$
$$u_y = 0, \tag{37}$$
$$u_z = 0, \tag{38}$$

where $k = 1 - 1/\kappa$. Simple dilatation corresponds to $k > 0$ and simple compression to $k < 0$. Uniform scaling is a combination of three such scalings along the coordinate axes. Scaling is a true deformation.

3.2 Displacement field

3.2.1 Displacement Field Representations

In continuum mechanics, each material point initially at X moves to a new position x under deformation. The **displacement field** is defined as

$$\vec{u}(x) = \vec{x} - \vec{X}(\vec{x}), \tag{39}$$

where $\vec{X}(x)$ denotes the original position of the particle now at $\vec{x}$. This is known as the *Euler representation*. Alternatively, in the *Lagrange representation*, the position is written as $\vec{x} = \vec{x}(\vec{X})$, and

$$\vec{u} = \vec{x}(\vec{X}) - \vec{X}. \tag{40}$$

Both representations are equivalent for small, slowly varying deformations.

3.2.2 Local and Infinitesimal Deformation

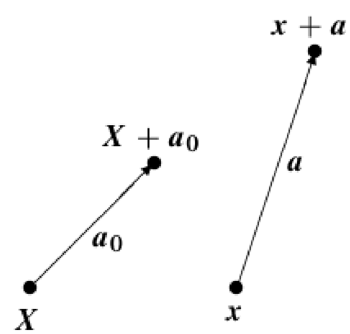


Figure 10: Displacement of a tiny material needle from $\vec{a}_0$ to $\vec{a}$. It may be translated, rotated, and scaled. Only the latter corresponds to a true deformation. Figure taken from Ref. [3].

A true deformation involves local changes in length and angle between neighboring points, not just rigid translations or rotations.

For a small line element or "needle" $\vec{a}_0$ connecting two material points, the deformation changes it to $\vec{a}$. Using $\vec{X}(\vec{x}) = \vec{x} - \vec{u}(\vec{x})$, one finds

$$\vec{a}_0 = \vec{X}(\vec{x} + \vec{a}) - \vec{X}(\vec{x}) = \vec{a} - \vec{u}(\vec{x} + \vec{a}) + \vec{u}(\vec{x}). \tag{41}$$

Expanding $\vec{u}(\vec{x} + \vec{a})$ to first order in $\vec{a}$ gives

$$\vec{u}(\vec{x} + \vec{a}) = \vec{u}(\vec{x}) + (\vec{a} \cdot \nabla)\vec{u}(\vec{x}) + \mathcal{O}(\vec{a}^2), \tag{42}$$

so that the infinitesimal change in the line element is

$$\delta\vec{a} \equiv \vec{a} - \vec{a}_0 = (\vec{a} \cdot \nabla)\vec{u}(\vec{x}). \tag{43}$$

In index notation:

$$\delta a_i = \sum_j a_j \, \nabla_j u_i. \tag{44}$$

This defines the **displacement gradients** $\{\nabla_j u_i\}$, and in dyadic form:

$$\delta\vec{a} = (\vec{a} \cdot \nabla)\vec{u} = \vec{a} \cdot (\nabla\vec{u}) = (\nabla\vec{u})^{\mathrm{T}} \cdot \vec{a}. \tag{45}$$

3.2.3 Slowly Varying Displacement Field

The displacement field is said to be *slowly varying* when

$$|\nabla_j u_i(x)| \ll 1, \tag{46}$$

and small relative to the body size L,

$$|\vec{u}(x)| \ll L. \tag{47}$$

This ensures that deformations are infinitesimal, though not necessarily rigid.

3.2.4 Cauchy's Strain Tensor

The strain tensor quantifies local geometric changes due to deformation. The change in the scalar product of two material line elements $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned}
\delta(\vec{a}\cdot\vec{b}) &= (\delta\vec{a})\cdot\vec{b} + \vec{a}\cdot(\delta\vec{b}) = (\vec{a}\cdot\nabla)\vec{u}\cdot\vec{b} + (\vec{b}\cdot\nabla)\vec{u}\cdot\vec{a} && (48)\\
&= \sum_{ij}(\nabla_i u_j + \nabla_j u_i)a_i b_j && (49)\\
&= 2\sum_{ij} U_{ij} a_i b_j = 2\,\vec{a}\cdot\vec{U}\cdot\vec{b}, && (50)
\end{aligned}$$

where the **Cauchy (infinitesimal) strain tensor** is defined as:

$$U_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i), \tag{51}$$

or, in matrix notation,

$$\vec{U} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^{\mathrm{T}}). \tag{52}$$

A symmetric tensor has six independent component whereas the displacement field has only three independent components. Every strain tensor must consequently satisfy consistency or compatibility conditions that remove three degrees of freedom. These conditions are

$$\nabla_i\nabla_j U_{kl} + \nabla_k\nabla_l U_{ij} = \nabla_i\nabla_l U_{kj} + \nabla_k\nabla_j U_{il}. \tag{53}$$

The strain tensor contains all the information about the local geometric changes caused by the displacement and is accordingly a good measure of local deformation. All bodily translations and rotations have been automatically taken out, and any displacement that is a combination of translations and rotations must consequently yield a vanishing strain tensor. It should, however, be emphasized that Cauchy's expression is only valid for small displacement gradients. When that is not the case, a more complicated expression must be used, involving the square of the displacement gradients .

3.2.5 Components of the Strain Tensor

For a general displacement field:

$$U_{xx} = \nabla_x u_x, \qquad U_{yy} = \nabla_y u_y, \qquad U_{zz} = \nabla_z u_z, \tag{54}$$

$$U_{yz} = U_{zy} = \tfrac{1}{2}(\nabla_y u_z + \nabla_z u_y), \quad U_{zx} = U_{xz} = \tfrac{1}{2}(\nabla_z u_x + \nabla_x u_z), \quad U_{xy} = U_{yx} = \tfrac{1}{2}(\nabla_x u_y + \nabla_y u_x). \tag{55}$$

For a simple linear deformation $\vec{u} = k(x, 0, 0)$,

$$\{U_{ij}\} = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{56}$$

which represents a true deformation (extension along x).

3.2.6 Principal Axes of Strain

The strain tensor is symmetric,

$$U_{ij} = U_{ji}, \tag{57}$$

and can therefore be diagonalized. Its eigenvectors define the **principal axes of strain**, and the corresponding eigenvalues measure the local extension or compression along those directions (see Fig. 11).

Example. Let's calculate the strain tensor for $\vec{u} = \alpha(y, x, 0)$ with $0 < \alpha \ll 1$ and determine the principal directions of strain and the change in length scales along these directions.

For small deformations the (Cauchy) strain tensor is

$$U_{ij} = \tfrac{1}{2}\big(\partial_i u_j + \partial_j u_i\big).$$

With $u_x = \alpha y,\ u_y = \alpha x,\ u_z = 0$, the displacement gradient is

$$\nabla\vec{u} = \begin{pmatrix} \partial_x u_x & \partial_y u_x & \partial_z u_x \\ \partial_x u_y & \partial_y u_y & \partial_z u_y \\ \partial_x u_z & \partial_y u_z & \partial_z u_z \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$U = \tfrac{1}{2}\big(\nabla\vec{u} + (\nabla\vec{u})^T\big) = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the xy-subspace, $U_{2\times 2} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ has eigenvalues and unit eigenvectors

$$\varepsilon_1 = +\alpha, \quad \vec{n}_1 = \tfrac{1}{\sqrt{2}}(1, 1, 0), \qquad \varepsilon_2 = -\alpha, \quad \vec{n}_2 = \tfrac{1}{\sqrt{2}}(1, -1, 0),$$

and a third eigenvalue $\varepsilon_3 = 0$ with $\vec{n}_3 = (0, 0, 1)$.

The normal strain in the direction of a unit vector $\vec{n}$ is $\varepsilon_n = \vec{n}\cdot\vec{U}\,\vec{n}$. Along the principal directions,

$$\left.\frac{\Delta\ell}{\ell}\right|_{\vec{n}_1} = +\alpha \quad \text{(dilation along } x = y\text{)}, \qquad \left.\frac{\Delta\ell}{\ell}\right|_{\vec{n}_2} = -\alpha \quad \text{(contraction along } x = -y\text{)},$$

and $\Delta\ell/\ell = 0$ along $\vec{n}_3 = \hat{z}$.

Thus, the material is stretched along the diagonal $x = y$ by a fractional amount α and compressed by the same amount along the orthogonal diagonal $x = -y$; there is no strain in the z-direction.

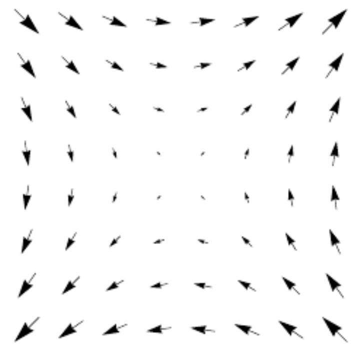


Figure 11: Arrow plot of the two-dimensional Lagrangian linear displacement field $\vec{u} = (y,\, x,\, 0)$ in the square $-1 < x < 1$ and $-1 < y < 1$. The material is dilated along one diagonal and contracted along the other. These are the principal directions of strain everywhere. Figure taken from Ref. [3].

3.3 Geometrical Meaning of the Strain Tensor

The strain tensor contains all the relevant information about local changes in geometric relationships, such as lengths of material elements and angles between them. Other geometric quantities, such as curves, surfaces, and volumes, are also affected under deformation.

3.3.1 Lengths and Angles

To define the projection of a tensor u_{ij} along two directions $\vec{a}$ and $\vec{b}$, we introduce

$$U_{ab} = \hat{a} \cdot \vec{U} \cdot \hat{b} = \frac{\vec{a} \cdot \vec{U} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}. \tag{58}$$

The change in the scalar product is

$$\delta(\vec{a} \cdot \vec{b}) = 2\,|\vec{a}|\,|\vec{b}|\,_{ab}. \tag{59}$$

Setting $\vec{b} = \vec{a}$ gives the change in length:

$$\frac{\delta|\vec{a}|}{|\vec{a}|} = U_{aa}. \tag{60}$$

Thus, the diagonal projection U_{aa} represents the *fractional change of length* in the direction of $\vec{a}$.

For the angle ϕ between $\vec{a}$ and $\vec{b}$, since $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\phi$, one finds

$$\delta\phi = \phi - \phi_0 = \frac{(U_{aa} + U_{bb}) \cos\phi - 2u_{ab}}{\sin\phi}. \tag{61}$$

For orthogonal vectors ($\phi = 90°$),

$$\delta\phi = -2U_{ab}. \tag{62}$$

The off-diagonal projections of the strain tensor thus determine the change in angle between actually orthogonal needles.

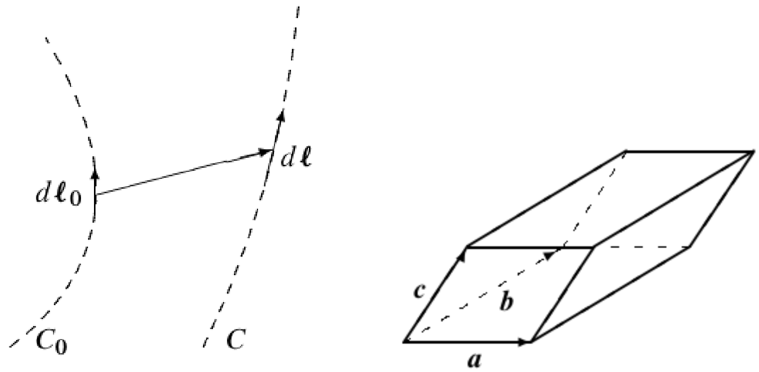


Figure 12: Left: A line element is stretched and rotated by the displacement that changes the curve from C_0 to C. Right : Three infinitesimal needles span a parallelepiped with volume $dV = \vec{a} \times \vec{b} \cdot \vec{c}$. Figure taken from Ref. [3].

3.3.2 Infinitesimal Elements

Curve element. A curve element behaves like a small needle (see Fig. 12). Under a displacement, the change from $d\vec{\ell}_0$ to $d\vec{\ell}$ is

$$\delta(d\vec{\ell}) \equiv d\vec{\ell} - d\vec{\ell}_0 = d\vec{\ell} \cdot \nabla \vec{u} = \nabla \vec{u}^T \cdot d\vec{\ell}. \tag{63}$$

Volume Element. For a tiny parallelepiped spanned by three infinitesimal vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, the volume element is $dV = \vec{a} \times \vec{b} \cdot \vec{c}$ (see Fig. 12). Its variation under displacement is

$$\delta(dV) \equiv dV - dV_0 = \nabla \cdot \vec{u}\, dV \tag{64}$$

$$= dV \sum_i \nabla_i u_i = dV\, U_{ii}. \tag{65}$$

Hence, the divergence of the displacement field which corresponds to the trace of $\vec{U}$, $\nabla \cdot \vec{u} = \mathrm{Tr}(\vec{U})$, gives the fractional change of volume.

The change in density ρ of a material particle, with constant mass dM, satisfies

$$\delta\rho = -\rho\, \nabla \cdot \vec{u}. \tag{66}$$

Thus, when $\nabla \cdot \vec{u} = \mathrm{Tr}(\vec{U}) = 0$, both the volume and density remain unchanged.

Surface Element. For a surface element $d\vec{S} = \vec{a} \times \vec{b}$, we find from the volume relation that

$$\delta(d\vec{S}) = (\nabla \cdot \vec{u}\, \mathbf{1} - \nabla \vec{u}) \cdot d\vec{S}. \tag{67}$$

Both the magnitude and direction of $d\vec{S}$ are modified by the deformation, following a rule different from that of the curve element.

3.4 Thermodynamics of deformation

Deforming a body requires work, part of which is stored as elastic potential energy, while another part is dissipated as heat due to internal friction. No real material is perfectly elastic. For instance, when squeezing a rubber ball, the stored energy is released when the force is removed, but a steel ball bouncing on a floor eventually stops because energy is lost to internal friction and air resistance. In continuum mechanics, energy relations are best derived by *following the work*—that is, tracking the work done by forces within the body.

3.4.1 Virtual Displacement Work

Consider a volume V of material not in mechanical equilibrium. Let the effective force acting on an element of volume dV be $d\vec{\mathcal{F}} = \vec{f}^*\,dV$. To keep all material points in fixed, non-equilibrium positions, we must apply an external distribution of *virtual forces*

$$\vec{f}' = -\vec{f}^*. \tag{68}$$

If the body is then displaced infinitesimally by $\delta\vec{u}(\vec{x})$, the work done by the virtual forces is

$$\delta W = \int_V \vec{f}' \cdot \delta\vec{u}\,dV = -\int_V \vec{f}^* \cdot \delta\vec{u}\,dV \tag{69}$$

$$= -\int_V \vec{f} \cdot \delta\vec{u}\,dV - \int_V (\nabla \cdot \vec{\sigma}^T) \cdot \delta\vec{u}\,dV, \tag{70}$$

where $\vec{f}^* = \vec{f} + \nabla \cdot \vec{\sigma}^T$ was used. Then performing an integration by parts in the second term gives

$$\int_V \sum_{ij} (\nabla_j \sigma_{ij}) \delta u_i\,dV = \int_V \sum_{ij} \nabla_j (\sigma_{ij} \delta u_i)\,dV - \int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i\,dV \tag{71}$$

$$= \int_V \nabla \cdot (\sigma^T \cdot \delta u)\,dV - \int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i\,dV \tag{72}$$

$$= \oint_S (\sigma^T \cdot \delta u) \cdot d\vec{S} - \int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i\,dV. \tag{73}$$

The surface term vanishes because $\delta\vec{u} = 0$ at the boundary S. The second integral can, by virtue of the symmetry of the tensor σ_{ij} , be written as

$$\int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i\,dV = \frac{1}{2} \int_V \sum_{ij} \sigma_{ij} \left(\nabla_j \delta u_i + \nabla_i \delta u_j\right)\,dV \tag{74}$$

$$= \frac{1}{2} \int_V \sum_{ij} \sigma_{ij} \delta \left(\nabla_j u_i + \nabla_i u_j\right)\,dV \tag{75}$$

$$= \int_V \sum_{ij} \sigma_{ij} \delta U_{ij}\,dV \tag{76}$$

$$= \int_V \mathrm{Tr}[\vec{\sigma} \cdot \delta\vec{U}]\,dV \equiv \int_V \vec{\sigma} : \delta\vec{U}\,dV, \tag{77}$$

where

$$\delta U_{ij} = \tfrac{1}{2}\big(\nabla_i \delta u_j + \nabla_j \delta u_i\big) \tag{78}$$

is the infinitesimal change in the strain tensor.

The virtual work becomes

$$\delta W = -\int_V \vec{f} \cdot \delta\vec{u}\, dV + \int_V \vec{\sigma} : \nabla\delta\vec{u}\, dV \tag{79}$$

$$= -\int_V \vec{f} \cdot \delta\vec{u}\, dV + \int_V \vec{\sigma} : \delta\vec{U}\, dV. \tag{80}$$

where $\vec{\sigma} : \nabla\delta\vec{u} = \sum_{ij} \sigma_{ij} \nabla_j \delta u_i$. The first term represents work against body forces, and the second corresponds to work against internal stresses. The portion of virtual work associated with internal stresses defines the infinitesimal work of deformation:

$$\delta W_{\text{deform}} = \int_V \vec{\sigma} : \nabla\delta\vec{u}\, dV. \tag{81}$$

Thus, the deformation work represents the contribution to the body's *deformation energy.*

3.5 Deformation regimes

In solid continuum mechanics, deformation regimes are classified according to the magnitude of the deformation and the material response. Broadly, three regimes are distinguished:

1. **Infinitesimal (small) deformations.**
   In this regime, displacements and strains are very small compared to unity. The linearized strain tensor is valid, geometric nonlinearities such as large rotations or nonlinear displacement gradients are neglected, and the principle of superposition applies. This approximation underlies most problems in classical elasticity, such as stress analysis in beams and plates.

2. **Finite (large) deformations.**
   Here, displacements or strains are not negligible, and nonlinear kinematics must be employed. Quantities such as the deformation gradient tensor and the Green–Lagrange strain are used. Large rotations, shear, and finite strains are fully accounted for, and the governing equations are nonlinear. Applications include rubber elasticity, biomechanics of soft tissues, and geomechanics.

3. **Extreme (highly nonlinear) deformations.**
   This regime involves very large strains, often accompanied by material nonlinearities or instabilities. The material response may be anisotropic or rate-dependent, and phenomena such as plasticity, viscoelasticity, viscoplasticity, damage, or fracture must be considered. Examples include crash simulations, metal forming at large strains, and failure analysis.

In summary: the *small deformation regime* corresponds to linear elasticity, the *finite deformation regime* requires nonlinear kinematics but can still be elastic or elastic–plastic, while the *extreme regime* involves very large strains typically coupled with inelastic effects such as plasticity, fracture, or damage.

3.5.1 Energy

Besides elastic deformations, we shall also suppose that the process of deformation occurs so slowly that thermodynamic equilibrium is established in the body at every instant, in accordance with

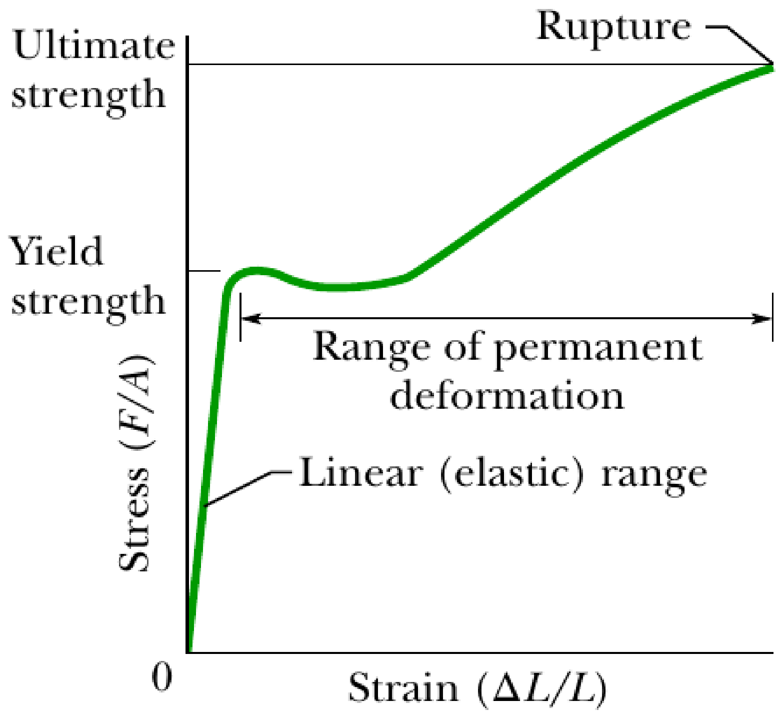


Figure 13: A stress–strain curve for a steel test specimen. The specimen deforms permanently when the stress is equal to the yield strength of the specimen's material. It ruptures when the stress is equal to the ultimate strength of the material. Figure taken from Ref..

the external conditions. This assumption is almost always justified in practice. The process will then be thermodynamically reversible.

In what follows we shall take all such thermodynamic quantities as the entropy $\mathcal{S}$, the internal energy $\mathcal{E}$, etc., relative to unit volume of the body, and not relative to unit mass as in fluid mechanics.

The unit volumes before and after deformation must be distinguished since they generally contain different amounts of matter. The thermodynamic quantities are therefore referred to the unit volume of the **undeformed body**. The total internal energy of the body is obtained by integrating $\mathcal{E}$ over the undeformed volume.

An infinitesimal change in the internal energy is given by the difference between the heat acquired by the unit volume and the work done by internal stresses $\delta R = dW/dV$. For a reversible process, $Td\mathcal{S}$ represents the heat, and δR the work. Thus,

$$d\mathcal{E} = Td\mathcal{S} - \delta R. \tag{82}$$

Using $dW = -\sum_{ij} \sigma_{ij}\delta U_{ij}\, dV$, we obtain

$$\begin{aligned} d\mathcal{E} &= Td\mathcal{S} + \sum_{ij} \sigma_{ij} \nabla_j \delta u_i \qquad (83) \\ &= Td\mathcal{S} + \sum_{ij} \sigma_{ij} \delta U_{ij}. \qquad (84) \end{aligned}$$

This is the **fundamental thermodynamic relation** for deformed bodies.

In the case of hydrostatic compression, $\sigma_{ij} = -p\delta_{ij}$, leading to

$$\sum_{ij} \sigma_{ij} \nabla_j \delta u_i = -p \sum_i \nabla_i \delta u_i = -p \nabla \cdot \delta \vec{u} = -p \frac{\delta(dV)}{dV}. \tag{85}$$

Therefore, this reduces to the familiar form

$$dE = TdS - p\,\delta(dV). \tag{86}$$

3.6 Helmholtz and Gibbs Free Energies

In the thermodynamic description of elastic bodies, it is convenient to introduce thermodynamic potentials whose natural variables correspond to the physical quantities held fixed in a given situation. Two such potentials are the **Helmholtz free energy** and the **Gibbs free energy**.

3.6.1 Helmholtz Free Energy

The Helmholtz free energy is defined as

$$F = \mathcal{E} - TS, \tag{87}$$

where $\mathcal{E}$ is the internal energy per unit volume, T the temperature, and $\mathcal{S}$ the entropy per unit volume. Its differential form at constant temperature is

$$dF = -\mathcal{S}\, dT + \sum_{ik} \sigma_{ik}\, dU_{ik}. \tag{88}$$

Thus $F = F(T, U_{ik})$ is naturally a function of the temperature and the strain tensor. Differentiation with respect to U_{ik} gives the stress tensor:

$$\sigma_{ik} = \left(\frac{\partial \mathcal{E}}{\partial U_{ik}} \right)_{\mathcal{S}} = \left(\frac{\partial \mathcal{F}}{\partial U_{ik}} \right)_T. \tag{89}$$

This makes F particularly useful in elasticity, where one often specifies the strain and seeks the corresponding stress.

3.6.2 Gibbs Free Energy

The Gibbs free energy is defined as

$$\Phi = \mathcal{E} - TS - \sum_{ik} \sigma_{ik} U_{ik} = F - \sum ik \sigma_{ik} U_{ik}. \tag{90}$$

Its differential is

$$d\Phi = -S\, dT - \sum_{ik} U_{ik}\, d\sigma_{ik}, \tag{91}$$

which shows that Φ is naturally a function of temperature and stress:

$$\Phi = \Phi(T, \sigma_{ik}).$$

Differentiating with respect to σ_{ik} yields the strain tensor:

$$U_{ik} = -\left(\frac{\partial \Phi}{\partial \sigma_{ik}}\right)_T. \tag{92}$$

Helmholtz and Gibbs free energies are fundamental thermodynamic potentials that quantify the amount of energy in a system available to do useful work, but they apply under different physical constraints. The Helmholtz free energy is the appropriate potential for systems held at constant temperature and volume, making it useful in contexts such as elasticity where the deformation (and therefore volume change) is controlled. The Gibbs free energy, on the other hand, applies to systems at constant temperature and pressure, and is particularly relevant for processes where the external pressure is fixed, such as phase transitions or chemical reactions. Both potentials encode the balance between internal energy and entropy, but their distinct natural variables determine which one is most convenient for describing a given physical situation.

The two potentials are related by a Legendre transform exchanging the conjugate variables (U_{ik}, σ_{ik}). Both potentials encode the elastic response of the material and provide a systematic way to derive constitutive relations such as Hooke's law from thermodynamic principles.

4 Hooke's law

To apply the general thermodynamic formalism to a specific case, we must express the free energy $\mathcal{F}$ of the body as a function of the strain tensor U_{ik}. Assuming small deformations, the free energy can be expanded in powers of U_{ik}. We mainly restrict our attention to isotropic bodies.

4.1 Isotropic bodies: free energy and elastic coefficients

For a deformed body at constant temperature, we take the undeformed state as the reference configuration, in the absence of external forces and at the same temperature. When $U_{ik} = 0$, the internal stresses vanish, i.e. $\sigma_{ik} = 0$. Hence, since $\sigma_{ik} = \partial F/\partial U_{ik}$, there is no linear term in the expansion of F in powers of U_{ik}.

As $\mathcal{F}$ is a scalar quantity, its expansion must involve scalar combinations of U_{ik}. Two independent second-order scalars can be formed:

$$\left(\sum_i U_{ii}\right)^2 \quad \text{and} \quad \sum_{ik} U_{ik}^2. \tag{93}$$

Thus, retaining terms up to second order, we obtain

$$\mathcal{F} = \mathcal{F}_0 + \tfrac{1}{2}\lambda \left(\sum_i U_{ii}\right)^2 + \mu \sum_{ik} U_{ik}^2 \tag{94}$$

$$= \mathcal{F}_0 + \tfrac{1}{2}\lambda\,(\mathrm{Tr}U)^2 + \mu\,\mathrm{Tr}(U^2) \tag{95}$$

where λ and μ are the **Lamé coefficients**. This is the general expression for the free energy of an isotropic elastic body.

A deformation in which $\sum_i U_{ii} = 0$ corresponds to no change in volume but a change in shape, called a **pure shear**. Conversely, a deformation for which $U_{ik} = \text{constant} \times \delta_{ik}$ represents a change in volume without a change in shape, called a **hydrostatic compression**.

4.2 Decomposition of the strain tensor

Any deformation can be written as the sum of a pure shear and a hydrostatic compression:

$$U_{ik} = \left(U_{ik} - \tfrac{1}{3}\delta_{ik}\sum_l U_{ll}\right) + \tfrac{1}{3}\delta_{ik}\sum_l U_{ll}. \tag{96}$$

The first term represents a pure shear, and the second term a hydrostatic compression. Substituting this decomposition into $\mathcal{F}$, we obtain

$$\mathcal{F} = \mu\sum_{ik}\left(U_{ik} - \tfrac{1}{3}\delta_{ik}\sum_l U_{ll}\right)^2 + \tfrac{1}{2}K\left(\sum_l U_{ll}\right)^2, \tag{97}$$

where K is the **bulk modulus** (modulus of hydrostatic compression), and μ is the **shear modulus** (modulus of rigidity). The bulk modulus is related to the Lamé coefficients by

$$K = \lambda + \tfrac{2}{3}\mu. \tag{98}$$

Thermodynamic stability requires that the free energy be a minimum at equilibrium, i.e.

$$K > 0, \quad \mu > 0. \tag{99}$$

4.3 Stress–strain Relation

From the thermodynamic relation $\sigma_{ik} = (\partial\mathcal{F}/\partial U_{ik})_T$, we compute:

$$d\mathcal{F} = K\sum_l U_{ll}\sum_n dU_{nn} + 2\mu\sum_{ik}(U_{ik} - \tfrac{1}{3}\delta_{ik}\sum_l U_{ll})d(U_{ik} - \tfrac{1}{3}\delta_{ik}\sum_n U_{nn}) \tag{100}$$

$$\begin{aligned}&= K\sum_l U_{ll}\sum_n dU_{nn} + 2\mu\sum_{ik}(U_{ik} - \tfrac{1}{3}\delta_{ik}\sum_l U_{ll})\sum_{ik} dU_{ik}\\ &- 2\mu\sum_{ik}(U_{ik} - \tfrac{1}{3}\delta_{ik}\sum_l U_{ll})\tfrac{1}{3}\delta_{ik}\sum_n dU_{nn}\end{aligned} \tag{101}$$

$$= \left[K\sum_l U_{ll}\sum_{ik}\delta_{ik} + 2\mu\sum_{ik}(U_{ik} - \tfrac{1}{3}\delta_{ik}\sum_l U_{ll})\right] dU_{ik} \tag{102}$$

$$= \left[(K - \tfrac{2}{3}\mu)\sum_l U_{ll}\sum_{ik}\delta_{ik} + 2\mu\sum_{ik} U_{ik}\right] dU_{ik} \tag{103}$$

$$= \left[\lambda\sum_l U_{ll}\sum_{ik}\delta_{ik} + 2\mu\sum_{ik} U_{ik}\right] dU_{ik}. \tag{104}$$

Hence, the stress tensor is

$$\sigma_{ik} = K\sum_l U_{ll}\,\delta_{ik} + 2\mu\left(U_{ik} - \tfrac{1}{3}\delta_{ik}\sum_l U_{ll}\right) \tag{105}$$

$$= \lambda\sum_l U_{ll}\,\delta_{ik} + 2\mu\,U_{ik}. \tag{106}$$

4.4 Hooke's Law

Taking the trace to the first line of the previous expression, we find

$$\sum_i \sigma_{ii} = 3K \sum_l U_{ll}, \qquad \Rightarrow \qquad \sum_l U_{ll} = \frac{\sum_i \sigma_{ii}}{3K}. \tag{107}$$

Substituting back we obtain the inverse relation:

$$U_{ik} = \frac{\delta_{ik}}{9K} \sum_l \sigma_{ll} + \frac{1}{2\mu} \left(\sigma_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l \sigma_{ll} \right), \tag{108}$$

which expresses the strain tensor in terms of the stress tensor. From Eq. (108), we see that U_{ik} is a linear function of σ_{ik}; this proportionality defines **Hooke's law** for small deformations.

4.5 Homogeneous deformations

A homogeneous deformation is one in which the strain tensor U_{ik} is constant throughout the body.

Let's recall the boundary condition

$$\vec{\sigma} \cdot \hat{\hat{n}} = \left\{ \sum_j \sigma_{ij} n_j \right\} \tag{109}$$

must therefore be continuous across a surface with normal $\hat{n}$.

4.5.1 Hydrostatic Compression

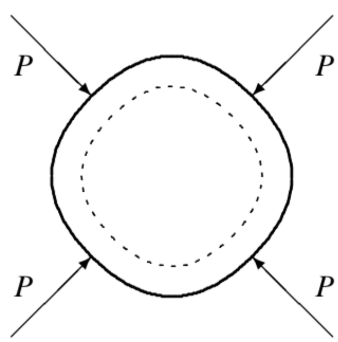


Figure 14: A body made from isotropic, homogenous material subject to a uniform external pressure will be uniformly compressed. Figure taken from Ref. [3].

In a fluid at rest with a constant pressure p, the stress tensor is

$$\sigma_{ik} = -p\delta_{ik} \tag{110}$$

everywhere. If a solid body made from isotropic material is immersed into this fluid, the natural guess is that the pressure will also be p inside the body. Since $\sum_l \sigma_{ll} = -3p$, the strain becomes

$$U_{ik} = -\frac{p}{3K}\delta_{ik}. \tag{111}$$

On the other hand, the displacement field can be obtained from

$$U_{xx} = \nabla_x u_x = -\frac{p}{3K}, \tag{112}$$

$$U_{yy} = \nabla_y u_y = -\frac{p}{3K}, \tag{113}$$

$$U_{zz} = \nabla_z u_z = -\frac{p}{3K}. \tag{114}$$

Integrating directly we are able to find a particular solution for the displacement field:

$$u_x = -\frac{p}{3K}\,x, \tag{115}$$

$$u_y = -\frac{p}{3K}\,y, \tag{116}$$

$$u_z = -\frac{p}{3K}\,z. \tag{117}$$

The most general solution is obtained by adding an arbitrary infinitesimal rigid body displacement to this expression.

We emphasize that this result was obtained by making a reasonable guess for the form of the stress tensor inside the body. Although such a guess could, in principle, be incorrect, it is in fact justified by a uniqueness theorem. The theorem ensures—much like the corresponding uniqueness theorem in electrostatics—that if the mechanical equilibrium equations and the boundary conditions are satisfied by a proposed solution, then the solution is unique (up to an arbitrary small rigid body displacement). Thus, in any elastostatic problem, one may freely add a small rigid body motion to the solution without affecting the stresses.

4.5.2 Uniform stretching

Consider a rod along the z–axis, with forces applied uniformly over its ends (the rod is only pulled (or pushed) from its ends). On the lateral surface (the cylindrical surface), there is no force applied.

Let the force per unit area be p. Since the deformation is homogeneous, σ_{ik} is constant. Since there is no lateral force on the sides of the rod ($n_z = 0$), we have that

$$\sigma \cdot \hat{n} = \sum_k \sigma_{ik}\, n_k = 0, \tag{118}$$

where $\hat{n}$ is the outward normal vector to the side surface. Because the sides are parallel to the z-axis, the outward normal $\hat{n}$ has:

$$\hat{n} = (n_x, n_y, 0) = (n_1, n_2, 0). \tag{119}$$

It follows that

$$\sigma_{11}\, n_1 + \sigma_{12} n_2 = 0, \tag{120}$$

$$\sigma_{12}\, n_1 + \sigma_{22} n_2 = 0, \tag{121}$$

$$\sigma_{13}\, n_1 + \sigma_{23}\, n_2 = 0. \tag{122}$$

Therefore $\sigma_{ik} = 0$ except σ_{zz}. On the end surface we have

$$\sum_k \sigma_{3k}\, n_3 = p, \tag{123}$$

from which $\sigma_{zz} = p$.

From the general strain–stress relation,

$$U_{xx} = U_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right) p, \tag{124}$$

$$U_{zz} = \frac{1}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right) p. \tag{125}$$

The component U_{zz} gives the relative lengthening of the rod. The coefficient of p is called the coefficient of extension, and its reciprocal is the modulus of extension or Young's modulus, E. The longitudinal extension is

$$U_{zz} = \frac{p}{E}, \tag{126}$$

where

$$E = \frac{9K\mu}{3K + \mu} \tag{127}$$

is Young's modulus. On the other hand, the components U_{xx} and U_{yy} give the relative compression of the rod in the transverse direction. The ratio of the transverse compression to the longitudinal extension is called the Poisson's ratio ν:

$$U_{xx} = -\nu\, U_{zz}. \tag{128}$$

Using the Lamé constants,

$$\nu = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}. \tag{129}$$

Since $K > 0$ and $\mu > 0$,

$$-1 \leq \nu \leq \frac{1}{2}. \tag{130}$$

Finally, the relative volume change is

$$U_{ii} = \frac{p}{3K}. \tag{131}$$

The free energy of a stretched rod is

$$F = \frac{p^2}{2E}. \tag{132}$$

The displacement field is then obtained by integrating the relations

$$\nabla_x u_x = U_{xx}, \qquad \nabla_y u_y = U_{yy}, \qquad \nabla_z u_z = U_{zz},$$

which yields the particular solution

$$u_z = \frac{p}{E}\, z, \tag{133}$$

$$u_x = -\nu\, \frac{p}{E}\, x, \tag{134}$$

$$u_y = -\nu\, \frac{p}{E}\, y. \tag{135}$$

This displacement describes a simple dilatation along the z–axis and a corresponding compression toward the z–axis in the xy–plane.

It is often convenient to express the elastic constants in terms of Young's modulus E and Poisson's ratio ν. In these variables, the Lamé coefficients become

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{136}$$

$$\mu = \frac{E}{2(1+\nu)}, \tag{137}$$

$$K = \frac{E}{3(1-2\nu)}. \tag{138}$$

The free energy of a deformed isotropic body can then be written as

$$F = \frac{E}{2(1+\nu)} \left(\sum_{ik} U_{ik}^2 + \frac{\nu}{1-2\nu} (\sum_{l} U_{ll})^2 \right). \tag{139}$$

The stress tensor in terms of the strain tensor takes the form

$$\sigma_{ik} = \frac{E}{1+\nu} \left(U_{ik} + \frac{\nu}{1-2\nu} \sum_{l} u_{ll}\, \delta_{ik} \right). \tag{140}$$

Conversely, the strain tensor in terms of the stress tensor is

$$U_{ik} = \frac{1}{E} \left[(1+\nu)\sigma_{ik} - \nu \sum_{l} \sigma_{ll}\, \delta_{ik} \right]. \tag{141}$$

If the rod-like spring is clamped on the sides by a rigid material, the boundary conditions become

$$u_y = u_x = 0 \quad \text{on the sides.}$$

In this case, the only non-vanishing constant strain is U_{zz}, and the solution is obtained in the same manner as above .

4.5.3 Unilateral Compression

Let us now consider the compression of a rod whose sides are fixed so that they cannot move laterally. The external forces responsible for the compression act on the ends of the rod and along its length, taken to be the z–axis. Such a deformation is called a *unilateral compression*. Since the rod deforms only in the z–direction, the only nonzero component of the strain tensor is U_{zz}.

From Eq. (140) for the stress tensor in terms of the strain tensor, we obtain

$$\sigma_{xx} = \sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)} U_{zz}, \tag{142}$$

$$\sigma_{zz} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} U_{zz}. \tag{143}$$

Denoting the compressive force per unit area (stress) by p so that $\sigma_{zz} = p$ (negative for compression), we solve for the longitudinal strain:

$$U_{zz} = p\, \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}. \tag{144}$$

The components σ_{xx} and σ_{yy} describe the transverse stresses generated by the lateral constraint. Substituting Eq. (144) into Eq. (143) gives

$$\sigma_{xx} = \sigma_{yy} = \frac{p\,\nu}{1-\nu}. \tag{145}$$

Finally, using the general elastic energy formula

$$F = \tfrac{1}{2}\sigma_{ik}u_{ik},$$

and observing that only σ_{zz} and u_{zz} are nonzero, we obtain the free energy of the compressed rod:

$$F = \frac{p^2(1+\nu)(1-2\nu)}{2E(1-\nu)}. \tag{146}$$

4.5.4 Uniform Shear

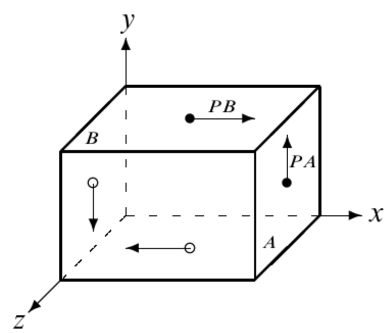


Figure 15: Clamped rectangular slab under constant shear stress $\sigma_{xy} = \sigma_{yx} = P$. The upper clamp is acted upon by an external force $\mathcal{F}_x = PB$, where B is the area of the clamp, while the force on the lower is $\mathcal{F}_x = -PB$. The symmetry of the stress tensor demands a clamp force $\mathcal{F}_y = PA$ on the right-hand side and a force $\mathcal{F}_y = -PA$ on the left-hand side, where A is the area of that clamp. Figure taken from Ref. [3].

Let's consider a clamped rectangular slab of homogeneous, isotropic material subjected to shear stress along one side (here, the x–direction). As argued there, the shear stress

$$\sigma_{xy} = P$$

must be constant throughout the material. What was not appreciated earlier is that the symmetry of the stress tensor requires that

$$\sigma_{yx} = P$$

everywhere as well. Consequently, shearing forces must act on the ends of the slab, while the remaining sides are free (see the margin figure).

Assuming there are no other stresses, the only nonzero strain component is

$$U_{xy} = \frac{P}{2\mu},$$

and using the relation

$$2U_{xy} = \nabla_y u_x + \nabla_x u_y,$$

we obtain a particular solution for the displacement field:

$$u_x = \frac{P}{\mu}\, y, \tag{147}$$

$$u_y = 0, \tag{148}$$

$$u_z = 0. \tag{149}$$

In these coordinates the displacement in the x–direction vanishes for $y = 0$ and increases linearly with y. Each infinitesimal "needle" of the material is not only sheared but also rotated by a small angle

$$\phi = \tfrac{1}{2}\left(\nabla_x u_y - \nabla_y u_x\right) = -\frac{P}{2\mu} \tag{150}$$

about the z–axis.

5 Basic elastostatics

The fundamental equations of elastostatics are obtained from the results of the preceding chapters. They consist of:

$$f_i + \sum_j \nabla_j \sigma_{ij} = 0, \qquad \text{mechanical equilibrium,} \tag{151}$$

$$\sigma_{ij} = 2\mu\, U_{ij} + \lambda\, \delta_{ij} \sum_k U_{kk}, \qquad \text{Hooke's law,} \tag{152}$$

$$U_{ij} = \tfrac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right), \qquad \text{Cauchy's strain tensor.} \tag{153}$$

We shall use these equations with a time-independent external body force. In the presence of gravity, for example, the body force is

$$\vec{f} = \rho\, \vec{g}.$$

Other forces of electromagnetic origin can also appear, such as those generated by an inhomogeneous electric field acting on a dielectric material.

5.1 Navier–Cauchy Equation

Substituting the second and third lines into the equilibrium condition, we obtain in index notation

$$\sum_j \nabla_j \sigma_{ij} = 2\mu \sum_j \nabla_j U_{ij} + \lambda \nabla_i \sum_j U_{jj} \tag{154}$$

$$= \mu \sum_j \nabla_j^2 u_i + (\lambda + \mu) \nabla_i \sum_j \nabla_j u_j. \tag{155}$$

Rewriting the second line in vector notation yields the equilibrium equation in its final form:

$$\vec{f} + \mu\, \nabla^2 \vec{u} + (\lambda + \mu)\, \nabla\big(\nabla \cdot \vec{u}\big) = \vec{0}. \tag{156}$$

This is known as *Navier's equation of equilibrium* or the *Navier–Cauchy equilibrium equation.* Although this equation is compact, it is often more convenient in analytic work to use the original system.

Because the displacement gradients are assumed to be small, all nonlinear terms are ignored. The displacement field is expressed as a function of the original coordinates of the undeformed material, using the Lagrangian representation.

The linearity of the equilibrium equations allows *superposition* of solutions. For instance, if a body is both compressed and stretched uniformly, the total displacement is simply the sum of the two individual displacements.

The boundary conditions are often implicit in the mere posing of an elastostatics problem. Typically, a part of the body surface is "glued" to a hard surface where the displacement has to vanish, and where the environment automatically provides the external reaction forces necessary to balance the surface stresses. On the remaining part of the body surface, explicit external forces implement the "user control" of the deformation. In regions where the external forces vanish, the body surface is said to be free. For the body to remain at rest, the total external force and the total external moment of force must always vanish.

5.1.1 The Equations of Equilibrium in a gravitational field

In terms of the Young's modulus and Poisson's ratio the equation becomes

$$\nabla^2\vec{u} + \frac{1}{1-2\nu}\,\nabla(\nabla\cdot\vec{u}) = -\,\rho\vec{g}\,\frac{2(1+\nu)}{E}. \tag{157}$$

It is sometimes helpful to use the vector identity

$$\nabla(\nabla\cdot\,\vec{u}) = \nabla^2\vec{u} + \nabla\times\,\nabla\times\,\vec{u}. \tag{158}$$

Then

$$\nabla(\nabla\cdot\,\vec{u}) - \frac{1-2\nu}{2(1-\nu)}\nabla\times\,\nabla\times\,\vec{u} = -\,\rho\vec{g}\,\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}. \tag{159}$$

5.1.2 Surface-Loaded Bodies

A particularly important case occurs when the deformation is caused not by body forces but by forces applied only at the boundary. The equilibrium equation then becomes

$$(1-2\nu)\nabla^2\vec{u} + \nabla(\nabla\cdot\vec{u}) = 0, \tag{160}$$

or, using the identity above,

$$2(1-\nu)\nabla(\nabla\cdot\,\vec{u}) - (1-2\nu)\nabla\times\,\nabla\times\,\vec{u} = 0. \tag{161}$$

The external forces appear in the solution only through the boundary conditions. Taking the divergence and using the identity

$$\nabla\cdot\nabla \equiv \nabla^2, \tag{162}$$

we obtain

$$\nabla^2(\nabla\cdot\vec{u}) = 0, \tag{163}$$

i.e. $\nabla\cdot\vec{u}$ (the volumetric strain) is a harmonic function. Taking the Laplacian to Eq. (160) we further obtain

$$\nabla^2\nabla^2\,\vec{u} = 0, \tag{164}$$

demonstrating that, in equilibrium, the displacement vector satisfies the *biharmonic equation.* This remains valid in a uniform gravitational field, but not for general external forces varying in space.

5.2 Uniform settling

An infinitely extended slab of homogeneous and isotropic elastic material placed on a horizontal surface is a kind of "elastic sea", which like the fluid sea may be assumed to have the same properties everywhere in a horizontal plane. In a flat-Earth coordinate system, where gravity is given by $\vec{g} = (0, 0, -g_0)$, we expect a uniformly vertical displacement, which only depends on the z-coordinate,

$$\vec{u} = (0, 0, u_z(z)) = u_z(z)\,\hat{e}_z.$$

In order to realize this "elastic sea" in a finite system, it must be surrounded by fixed, vertical, and slippery walls (see Fig. 16). The vertical walls forbid horizontal but allow vertical displacement, and at the bottom, $z = 0$, we place a horizontal supporting surface that forbids vertical displacement. At the top, $z = h$, the elastic material is left free to move without any external forces acting on it.

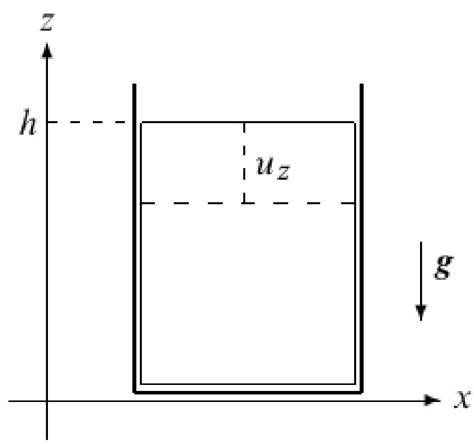


Figure 16: Elastic "sea" of material undergoing a downward displacement because of gravity. The container has fixed, slippery walls. Figure taken from Ref. [3].

The only non-vanishing strain is $u_{zz} = \nabla_z u_z$. From the explicit form of Hooke's law, we obtain the non-vanishing stresses

$$\sigma_{xx} = \sigma_{yy} = \lambda u_{zz}, \qquad \sigma_{zz} = (\lambda + 2\mu)u_{zz}. \tag{165}$$

The first Cauchy's equilibrium equation simplifies in this case to

$$\nabla_z \sigma_{zz} = \rho_0 g_0. \tag{166}$$

where ρ_0 is the constant mass density of the undeformed material. Using the boundary condition $\sigma_{zz} = 0$ at $z = h$, this equation may be integrated immediately to

$$\sigma_{zz} = -\rho_0 g_0 (h - z). \tag{167}$$

The vertical pressure $p_z = -\sigma_{zz} = \rho_0 g_0 (h - z)$ is positive and rises linearly with depth $h - z$, just as in the fluid sea. It balances everywhere the full weight of the material above, but this was expected since there are no shear stresses to distribute the vertical load. The horizontal pressures $p_x = p_y = p_z \lambda/(\lambda + 2\mu)$ are also positive but smaller than the vertical, because both λ and μ are positive in normal materials. The horizontal pressures are eventually balanced by the stiffness of the fixed vertical walls surrounding the elastic sea.

The strain

$$u_{zz} = \nabla_z u_z = \frac{\sigma_{zz}}{\lambda + 2\mu} = -\frac{\rho_0 g_0}{\lambda + 2\mu}(h - z) \tag{9.8}$$

is negative, corresponding to a compression.

5.2.1 Characteristic length scale

The variation in stress over the vertical size L of a body is $|\Delta\sigma_{ij}| \approx \rho g L$. The corresponding variation in strain becomes $|\Delta u_{ij}| \sim L\rho g/E$ for non-exceptional materials. Since u_{ij} is dimensionless, it is convenient to define the *gravitational deformation scale*,

$$D \sim \frac{E}{\rho g}, \tag{168}$$

so that $|\Delta u_{ij}| \sim L/D$. The length D characterizes the scale for major gravitational deformation (of order unity), and small deformations require $L \ll D$. Finally, the gravitationally induced variation in the displacement over a vertical distance L is estimated to be of magnitude $|\Delta u_i| \sim L\,|\Delta u_{ij}| \sim L^2/D \ll L$.

The characteristic length scale for major deformation is in this case chosen to be

$$D = \frac{\lambda + 2\mu}{\rho_0 g_0} = \frac{1-\nu}{(1+\nu)(1-2\nu)}\,\frac{E}{\rho_0 g_0}. \tag{9.9}$$

Integrating the strain with the boundary condition $u_z = 0$ for $z = 0$, we finally obtain

$$u_z = \frac{h^2 - (h-z)^2}{2D}. \tag{9.10}$$

The displacement is always negative, largest in magnitude at the top, $z = h$, and vanishes at the bottom. It varies quadratically with height h (see Fig. 17).

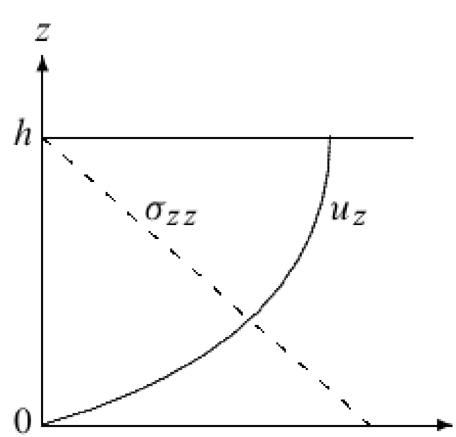


Figure 17: Sketch of the displacement (solid curve) and stress (dashed) for the elastic "elastic sea in a box". Figure taken from Ref. [3].

5.3 Uniform pure bending

Uniform pure bending describes the deformation of a beam into a circular arc under the action of terminal couples (moments) without shear forces or body forces (see Fig. 18). In pure bending, the stresses applied at the ends do not stretch or compress the beam on average, but only produce bending moments. The bending is assumed to be *uniform*, meaning that stresses and strains are identical along every cross-section of the beam.

For this to occur, an initially straight beam of length L must deform into a circular arc of radius R, with each transverse ray also becoming part of this circle. Because every ray follows the same curvature, it is sufficient to study a small slice of the beam to understand the deformation of the entire beam.

Non-uniform bending can later be described by assembling slices with different radii of curvature. By Saint-Venant's principle and the linearity of elasticity, the behavior of beams under various end loads can be constructed by superposition of displacement fields.

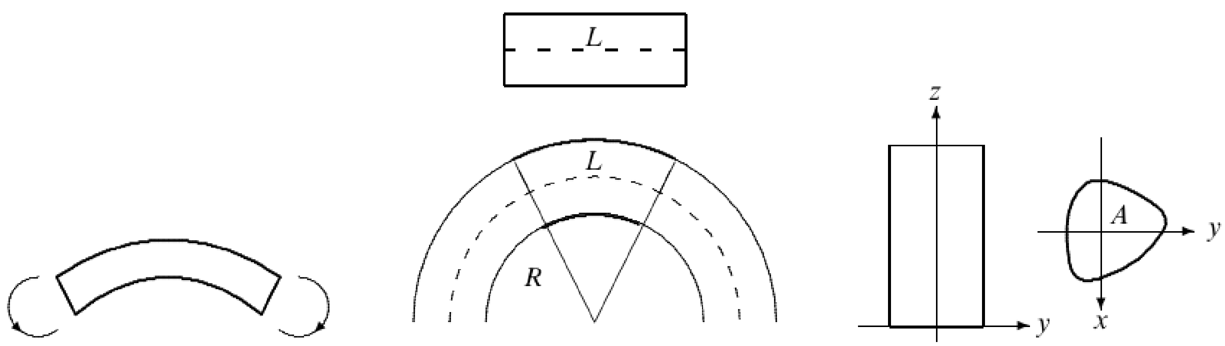


Figure 18: Left: Bending a beam by wrenching it at the ends. Center: A uniformly bent beam is a part of a circular ring. Right: The unbent beam is aligned with the z-axis. Its cross-section, A, in the xy-plane is the same for all z. Figures taken from Ref. [3].

In a Cartesian coordinate system, we align the undeformed beam with the z-axis, and put the terminal cross-sections at $z = 0$ and $z = L$. The length L of the beam may be chosen as small as we please. The cross-section A in the xy-plane may be of arbitrary shape, but we may position the coordinate system in the xy-plane with its origin coinciding with the *area centroid*, such that the area integral over the coordinates vanishes,

$$\int_A x\, dA = \int_A y\, dA = 0.$$

Finally, we require that the central ray after bending becomes part of a circle in the yz-plane with radius R and its center on the y-axis at $y = R$. The radius R is obviously the length scale for major deformation, and must be assumed large compared to the transverse dimensions of the beam.

5.3.1 Shear-free bending

What precisely happens in the beam when it is bent depends on the way the actual stresses are distributed on its terminals, although by Saint-Venant's principle the details should only matter near the terminals. In the simplest case we may view the beam as a loose bundle of thin elastic

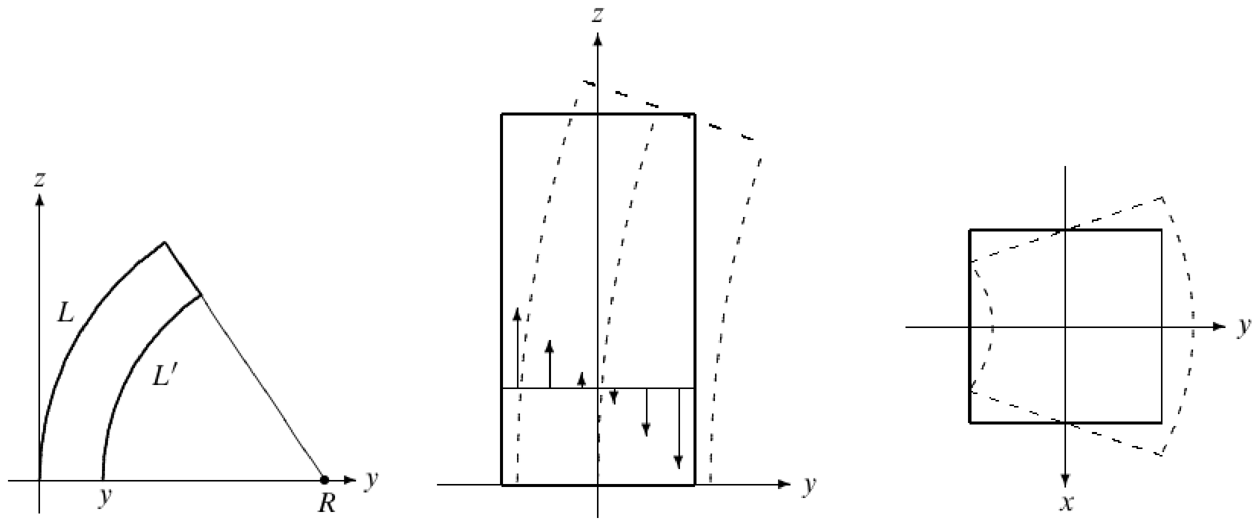


Figure 19: Left: The length of the arc at y must satisfy $L'/(R-y) = L/R$. Center: Sketch of the bending of a beam. The arrows show the strain U_{zz}. Right: Sketch of the deformation in the xy-plane of a beam with quadratic cross-section. This deformation may easily be observed by bending a rubber eraser. Figures taken from Ref. [3].

strings that do not interact with each other, but are stretched or compressed individually according to their position in the beam without generating shear stresses.

Let us fix the central string so that it does not change its length L when bent into a circle of radius R. A simple geometric construction shows that a nearby ray in position y will change its length to L', satisfying $(R-y)/L' = R/L$ (see Fig. 19). Thus the beam experiences a longitudinal strain,

$$U_{zz} = \frac{\delta L}{L} = \frac{L' - L}{L} = -\frac{y}{R}. \tag{169}$$

For negative y the material of the beam is being stretched, while for positive y it is being compressed. For the strain to be small, we must have $|y| \ll R$ everywhere in the beam.

Under the assumption that the bending is done without shear and that there are no forces acting on the sides of the beam, it follows as in the preceding section that $\sigma_{xx} = \sigma_{yy} = 0$. The only non-vanishing stress is $\sigma_{zz} = EU_{zz}$, and the non-vanishing strains are as before found from the inverted Hooke's law,

$$U_{xx} = U_{yy} = -\nu U_{zz} = \nu \frac{y}{R}, \tag{170}$$

where ν is Poisson's ratio. This shows that the material is being stretched horizontally and compressed vertically for $y > 0$ and conversely for $y < 0$.

Using $U_{ij} = \nabla_i u_j$ and requiring that the central ray is only bent, not stretched, a particular solution is found to be

$$u_x = \nu \frac{xy}{R}, \qquad u_y = \frac{z^2}{2R} + \nu \frac{y^2 - x^2}{2R}, \qquad u_z = -\frac{yz}{R}.$$

The second term in u_y is, like in the preceding section, forced upon us by the requirement of no shear stresses (and strains). For displacement gradients to be small, all dimensions of the beam have to be small compared to R. Note that the beam's actual dimensions do not appear in the displacement field, which is therefore a generic solution for pure bending of any beam.

5.3.2 Total force

The only non-vanishing stress component is

$$\sigma_{zz} = Eu_{zz} = -\frac{E}{R}\,y. \tag{171}$$

It is a tension for negative y, and we consequently expect the material of the beam to first break down at the most distant point of the cross-section opposite the direction of bending, as common experience also tells us.

The total force acting on a cross-section vanishes,

$$\mathcal{F}_z = \int_A \sigma_{zz}\,dS_z = -\frac{E}{R}\int_A y\,dA = 0. \tag{172}$$

because the origin of the coordinate system is chosen to coincide with the centroid of the beam cross-section.

5.3.3 Total moment

The moments of the longitudinal stress in any cross-section are

$$\mathcal{M}_x = \int_A y\,\sigma_{zz}\,dS_z = -\frac{E}{R}\int_A y^2\,dA. \tag{173}$$

$$\mathcal{M}_y = -\int_A x\,\sigma_{zz}\,dS_z = \frac{E}{R}\int_A xy\,dA. \tag{174}$$

$$\mathcal{M}_z = 0. \tag{175}$$

The component $\mathcal{M}_x$ orthogonal to the bending plane is called the *bending moment*. The integral

$$I = \int_A y^2\,dA \tag{176}$$

is the 'moment of inertia' (or *area moment*) of the beam cross section around the direction of $\mathcal{M}_x$. The component $\mathcal{M}_y$ vanishes only if

$$\int_A xy\,dA = 0. \tag{177}$$

This will, for example, be the case if the beam has circular cross-section, is mirror symmetric under reflection in either axis, or more generally if the axes coincide with the *principal directions* of the cross-section. There are always two orthogonal principal directions for any shape of cross-section.

The magnitude of the bending moment $\mathcal{M}_b - \mathcal{M}_x$ is known as the *Bernoulli–Euler* law,

$$\frac{1}{R} = \frac{\mathcal{M}}{EI}. \tag{178}$$

The product EI is called the *flexural rigidity* or *bending stiffness* of the beam. The larger it is, the larger is the moment required to bend it with a given radius of curvature. The unit of flexural rigidity is $\mathrm{Pa\,m^4 = N\,m^2}$.

Note that constant shear or normal stresses, if present, do not contribute to the cross-section moment, and in many engineering applications the Bernoulli–Euler law combined with linearity and Saint-Venant's principle is enough to give a reasonable idea of how much a beam is deformed by external loads.

5.3.4 Moments of inertia of common beam cross-sections

Rectangular beam A rectangular beam with sides $2a$ (along x) and $2b$ (along y) has moment of inertia

$$I = \int_{-a}^{a} dx \int_{-b}^{b} y^2\, dy = \frac{4}{3} ab^3. \tag{179}$$

This shows that bending resistance increases much faster with the thickness $2b$ in the bending direction.

Elliptic beam An elliptical beam with major axes $2a$ and $2b$ has moment of inertia

$$I = \int_{-a}^{a} dx \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} y^2\, dy = \frac{4}{3} ab^3 \int_0^1 (1-t^2)^{3/2} dt = \frac{\pi}{4} ab^3. \tag{180}$$

For a circular beam of radius a, this becomes $I = \frac{\pi}{4} a^4$, the same for all bending directions.

Circular pipe For a circular pipe with inner radius a and outer radius b,

$$I = \int_{a \le r \le b} y^2\, dA = \int_{r \le b} y^2\, dA - \int_{r \le a} y^2\, dA = \frac{\pi}{4}(b^4 - a^4). \tag{181}$$

Thus the moment of inertia of a pipe is the difference between two circular beams. Because of the fourth-power dependence on radius, flexural rigidity is highly sensitive to pipe diameter.

5.3.5 Threshold for buckling

Let an undeformed slender beam of length L be placed vertically along the z-axis (see Fig. 20). Applying a constant terminal force $\mathcal{F}$ along the negative z-direction, the moment exerted by the upper part of the deformed beam on a cross-section is $\mathcal{M} = y\mathcal{F}$, where x is the horizontal displacement at height z. There may also be non-vanishing normal and shear forces in the cross-section, but they do not contribute to the moment. Since the moment is not constant, the bending cannot be uniform. We may nevertheless use the Bernoulli–Euler law to relate local moment to local curvature,

$$\frac{1}{R} = \frac{\mathcal{F}}{EI}\, y. \tag{182}$$

This holds for any deformation of the beam and shows that the curvature is largest at mid-span and smallest where y vanishes.

To determine the threshold for buckling, assume the beam is only slightly displaced so that the curvature satisfies $1/R \approx -\, d^2y/dz^2$ (an expression that is correct to order $|dy/dz|^2$). The shape equation becomes

$$\frac{d^2y}{dz^2} = -k^2 y, \tag{183}$$

where $k = \sqrt{\mathcal{F}/EI}$. This is the harmonic oscillator equation whose solution is $y = A \sin kz + B \cos kz$. Applying boundary conditions $y = 0$ at $z = 0$ and $z = L$ gives

$$y = A \sin\left(\frac{n\pi z}{L}\right), \tag{184}$$

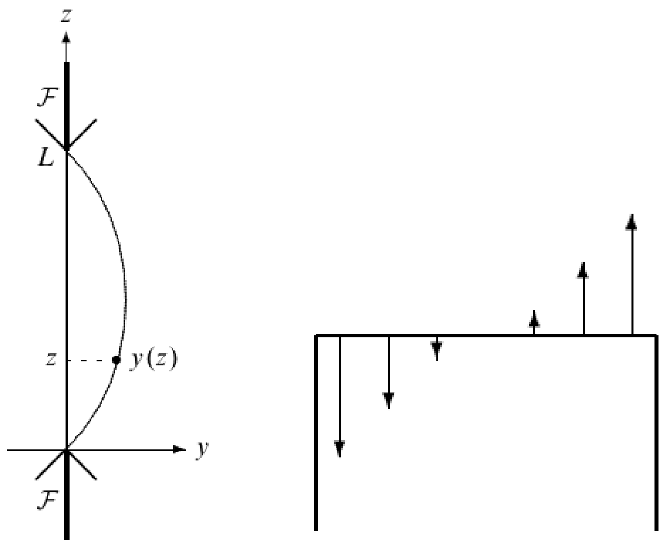


Figure 20: Left: A strut with a longitudinal terminal load may buckle when the force $\mathcal{F}$ exceeds a certain threshold. There is of course an equal and opposite force from the ground at $z = 0$. The local moment exerted in a strut cross section is $\mathcal{M} = y\mathcal{F}$. Right: Sketch of the stress forces that create the bending couple on the piece of the rod below z. The material is expanded away from the center of curvature and compressed toward it (on the left). Figures taken from Ref. [3].

where n is any integer. Thus $k = n\pi/L$, and since $\mathcal{F} = k^2 EI$, Euler's critical load becomes

$$\mathcal{F} = n^2\pi^2 \frac{EI}{L^2}. \tag{185}$$

Although mathematically the beam appears stable only for discrete values of n, everyday experience shows that a walking stick under a small terminal load does not spontaneously adopt one of these shapes. What actually happens is that a small force compresses the beam longitudinally rather than bending it—an effect not included in the calculation above. As the applied force increases, a threshold is eventually reached at $n = 1$ for which the first buckling mode becomes unstable:

$$\mathcal{F}_E = \pi^2 \frac{EI}{L^2}. \tag{186}$$

At this point the longitudinal compression mode becomes unstable, and the first buckling solution takes over at the slightest provocation. In practice, only the lowest mode is seen unless very large forces are applied, in which case the beam may crumble or collapse completely.

5.4 Twisting a shaft

5.4.1 Pure torsion

Let the shaft be a beam with circular cross-section of radius a and axis coinciding with the z-axis (see Fig. 21). The deformation is said to be a *pure torsion* if the shaft's material is rotated by a constant amount τ per unit length, such that a given cross-section at the position z is rotated by an angle τz relative to the cross-section at $z = 0$. The constant τ that measures the rotation angle per unit length of the beam is called the *torsion*. Its inverse, $1/\tau$, is the length of a beam that undergoes a pure torsion through 1 radian.

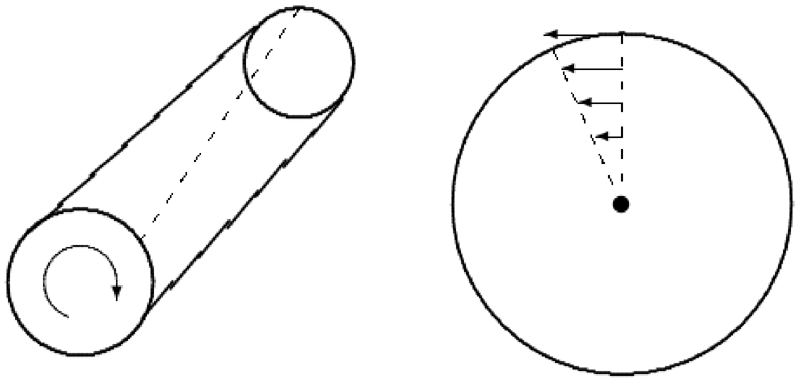


Figure 21: Left: Pure torsion consists of rotating every cross-section by a fixed amount per unit of length. Right: The displacement field for a rotation through a tiny angle τz (exaggerated here) is purely tangential and grows linearly with the radial distance. Figures taken from Ref. [3].

The uniform nature of pure torsion allows us to consider just a small slice of the shaft of length L, which is only twisted through a tiny angle, $\tau L \ll 1$. Since the physical conditions are the same in all such slices, we can later put them together to make a shaft of any length. To lowest order in the vector angle $\vec{\phi} = \tau z\, \hat{e}_z$, the displacement field in the slice becomes

$$\vec{u} = \vec{\phi} \times \vec{x} = \tau z\, \hat{e}_z \times (x, y, 0) = \tau z\, (-y, x, 0). \tag{187}$$

Not surprisingly, it is purely tangential and is always much smaller than the radius of the shaft, $|\vec{u}| = \tau |z|\, |x| \leq \tau L a \ll a$, because $\tau L \ll 1$.

5.4.2 Strains and stresses

The displacement gradient tensor becomes

$$\{\nabla_j u_i\} = \begin{pmatrix} 0 & -\tau z & -\tau y \\ \tau z & 0 & \tau x \\ 0 & 0 & 0 \end{pmatrix}, \tag{188}$$

and for this matrix to be small, we must also require $\tau a \ll 1$, i.e. the twist must be small over a length of the shaft comparable to its radius.

The only non-vanishing strains are

$$U_{xz} = U_{zx} = -\frac{1}{2}\tau y, \qquad U_{yz} = U_{zy} = \frac{1}{2}\tau x, \tag{189}$$

and the corresponding stresses from isotropic Hooke's law are

$$\sigma_{xz} = \sigma_{zx} = -\mu\tau y, \qquad \sigma_{yz} = \sigma_{zy} = \mu\tau x. \tag{190}$$

Inserting these stresses into the equilibrium equation shows that it is trivially fulfilled. In vector notation the equations may be written as

$$\vec{\sigma} \cdot \hat{e}_r = 0, \qquad \vec{\sigma} \cdot \hat{e}_z = \mu\tau\, \hat{e}_z \times \vec{x}, \tag{191}$$

where $\hat{e}_r = (x, y, 0)/r$ is the radial unit vector and $r = \sqrt{x^2 + y^2}$.

To realize a pure torsion, the correct stress distribution must be applied to the ends of the shaft. A different stress distribution generally leads to departures from pure torsion near the ends, but according to Saint-Venant's principle, the pure torsion solution is valid everywhere except within one diameter from the ends.

5.4.3 Torque

At any cross-section we may calculate the total moment of force around the shaft axis (the *torque*). For a surface element dS, the moment is $d\vec{\mathcal{M}} = \vec{x} \times d\vec{F} = \vec{x} \times \vec{\sigma} \cdot d\vec{S}$. Since the cross-section lies in the xy-plane, the z-component of the torque becomes

$$\mathcal{M}_z = \int_A (x\sigma_{yz} - y\sigma_{xz})\, dA = \mu\tau \int_A (x^2 + y^2)\, dA. \tag{192}$$

The torque can always be written analogously to the Euler–Bernoulli bending law:

$$\mathcal{M}_t = \mu J\, \tau, \tag{193}$$

where J for a circular pipe with inner and outer radii a and b is

$$J = \int_{a<r<b} (x^2 + y^2)\, dx\, dy = \frac{\pi}{2}(b^4 - a^4). \tag{194}$$

The quantity μJ is the *torsional rigidity* of the beam. For a pipe with circular cross-section, $J = 2I$ and thus

$$EI = (1 + \nu)\mu J. \tag{195}$$

Knowing the torsional rigidity and the torque $\mathcal{M}_t$, one may calculate the torsion $\tau = \mathcal{M}_t/\mu J$.

5.4.4 Transmitted power

If the shaft rotates with constant angular velocity Ω, a material point (x, y, z) has velocity $\vec{v} = \Omega\hat{e}_z \times \vec{x} = \Omega(-y, x, 0)$. The shear stresses transmit power $dP = \vec{v} \cdot d\vec{F}$. Integrating over the cross-section yields

$$P = \int_A \vec{v} \cdot \vec{\sigma} \cdot d\vec{S} = \int_A \Omega(x\sigma_{yz} - y\sigma_{xz})\, dx\, dy = \Omega\mathcal{M}_z = \vec{\Omega} \cdot \vec{\mathcal{M}}. \tag{196}$$

This shows that the transmitted power depends only on the torque applied, not on the detailed stress distribution, and is the standard expression for the power delivered by a torque acting on a rotating body.

6 Applications of the Navier–Cauchy Equation

The Navier–Cauchy equation is the basic field equation for small deformations of an isotropic elastic solid:

$$\rho\, \ddot{\mathbf{u}} = (\lambda + \mu)\, \nabla(\nabla \cdot \mathbf{u}) + \mu\, \nabla^2 \mathbf{u} + \mathbf{f}, \tag{197}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the displacement field, ρ the mass density, λ and μ the Lamé parameters, and $\mathbf{f}$ a body-force density (e.g. gravity).

In astronomy and planetary science, Eq. (197) is used at a wide range of scales:

- seismic waves and normal modes of the Earth and other planets,
- tidal deformation and lithospheric flexure in rocky planets and icy moons,
- elastic oscillations of dense stars (white dwarfs, neutron stars),
- estimates of crustal breaking and "starquakes" in magnetars.

In this section we present a compact review tailored to third–year astronomy students, followed by worked examples and exercises.

6.1 Elastic Waves and Seismology

If body forces vary slowly in time, or can be neglected on wave time-scales, we set $\mathbf{f} = 0$ in Eq. (197). For a homogeneous material (ρ, λ, μ constant), we look for plane-wave solutions of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A}\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)},$$

where $\mathbf{k}$ is the wavevector, ω the angular frequency and $\mathbf{A}$ a constant polarization vector.

Inserting this ansatz into Eq. (197) (with $\mathbf{f} = 0$) and simplifying, one finds two independent types of waves:

1. **Longitudinal (P) waves:**

$$\mathbf{A} \parallel \mathbf{k}, \qquad v_P = \frac{\omega}{k} = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

2. **Transverse (S) waves:**

$$\mathbf{A} \perp \mathbf{k}, \qquad v_S = \sqrt{\frac{\mu}{\rho}}.$$

The speeds v_P and v_S depend directly on the elastic constants and the density of the material.

In terrestrial seismology these relations are used in two directions:

- Given λ, μ, ρ for a rock, we can predict v_P and v_S.
- Given measured v_P and v_S from seismic waves, we can *invert* for λ, μ and infer the composition and state (solid vs liquid) of the interior.

The absence of S-waves in the Earth's outer core, for example, indicates that this region is fluid.

6.2 Elastic Deformation of Planetary Bodies

In a self-gravitating planet or moon, the body force is due to gravity. For small deformations around a hydrostatic reference state, the static version of Eq. (197) can be written as

$$(\lambda + \mu)\, \nabla(\nabla\cdot\mathbf{u}) + \mu\, \nabla^2\mathbf{u} + \rho\, \nabla\Phi = 0, \tag{198}$$

where Φ is a perturbing potential (e.g. the tidal potential produced by a companion). Solutions of this equation in spherical geometry lead to the *Love numbers* h_l, k_l that quantify how much a planet deforms under tidal forcing. These parameters are routinely used to constrain the thickness and rigidity of icy shells in moons such as Europa and Enceladus.

Another important application is lithospheric flexure: the bending of a planetary lithosphere in response to volcanic loads, impact basins or ice caps. In the thin-shell limit, the Navier–Cauchy equation reduces to a fourth order equation for the deflection $w(x, y)$ of the lithosphere, with an effective flexural rigidity proportional to Eh^3 (where E is Young's modulus and h the plate thickness).

6.3 Astrophysical Elasticity: Neutron Star Crusts

At nuclear densities, matter in the outer layers of neutron stars forms a crystalline solid. The same basic Eq. (197) describes small elastic disturbances of this crust, though with extremely large μ ($\sim 10^{30}\,\mathrm{Pa}$) and extreme densities ($\rho \sim 10^{17}\,\mathrm{kg\,m^{-3}}$).

Shear waves in the crust have speed

$$v_S = \sqrt{\frac{\mu}{\rho}} \sim 10^8 \ \mathrm{m\,s^{-1}}, \tag{199}$$

and global torsional oscillations of the crust are one possible explanation for quasi-periodic oscillations (QPOs) observed in the X-ray tails of magnetar flares. Additionally, slow build-up of elastic strain in the crust is thought to lead to "starquakes", which may be associated with glitches in pulsar timing and bursts of high-energy radiation.

6.4 Worked Examples

Example 1: Inferring elastic moduli from seismic speeds. Suppose seismic observations in a homogeneous layer of a terrestrial planet give

$$v_P = 8.0 \ \mathrm{km\,s^{-1}}, \qquad v_S = 4.5 \ \mathrm{km\,s^{-1}},$$

and the density is $\rho = 3.3 \times 10^3 \ \mathrm{kg\,m^{-3}}$. Estimate the Lamé parameters λ and μ, and the Poisson ratio ν.

Solution. First, from $v_S^2 = \mu/\rho$,

$$\mu = \rho v_S^2 = 3.3 \times 10^3\,(4.5 \times 10^3)^2 \approx 6.7 \times 10^{10} \ \mathrm{Pa}.$$

Next, from $v_P^2 = (\lambda + 2\mu)/\rho$,

$$\lambda + 2\mu = \rho v_P^2 = 3.3 \times 10^3\,(8.0 \times 10^3)^2 \approx 2.1 \times 10^{11} \ \mathrm{Pa},$$

so

$$\lambda \approx 2.1 \times 10^{11} - 2(6.7 \times 10^{10}) \approx 7.7 \times 10^{10} \ \mathrm{Pa}.$$

The Poisson ratio is

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \approx \frac{7.7}{2(7.7 + 6.7)} \approx 0.27,$$

a typical value for silicate rocks.

Example 2: Simple tidal bulge estimate for a homogeneous planet. Consider a homogeneous, incompressible, self-gravitating sphere of radius R, density ρ and shear modulus μ, subjected to a small tidal potential of amplitude U_0 at its surface. Using scaling arguments from Eq. (198), estimate the vertical tidal displacement u_r at the surface.

Solution. Roughly, the elastic-restoring term scales as

$$\mu \frac{u_r}{R^2},$$

while the tidal body force density is $\rho \nabla U \sim \rho U_0/R$. Setting these of the same order at equilibrium,

$$\mu \frac{u_r}{R^2} \sim \rho \frac{U_0}{R} \quad \Rightarrow \quad u_r \sim \frac{\rho R}{\mu} U_0.$$

For Earth-like parameters ($\rho \sim 5 \times 10^3$ kg m^{-3}, $\mu \sim 7 \times 10^{10}$ Pa, $R \sim 6.4 \times 10^6$ m) and a tidal potential of order $U_0 \sim 1$ m^2 s^{-2}, the estimate gives

$$u_r \sim \frac{5 \times 10^3\, 6.4 \times 10^6}{7 \times 10^{10}} \times 1 \sim 0.5 \text{ m},$$

consistent with the order of magnitude of solid Earth tides.

Example 3: Fundamental shear mode of a neutron star crust. Model the crust of a neutron star as a homogeneous elastic spherical shell of thickness H on top of a fluid core. For a rough estimate, take a slab of thickness H and assume that the fundamental standing shear wave has wavelength $\lambda \sim 2H$. Show that its frequency is

$$f_0 \sim \frac{v_S}{2H},$$

and evaluate f_0 for $H = 1$ km and $v_S = 10^8$ m s^{-1}.

Solution. A standing shear wave with nodes at the two boundaries satisfies $\lambda = 2H$ and $f_0 = v_S/\lambda$. Thus

$$f_0 \sim \frac{v_S}{2H} = \frac{10^8}{2 \times 10^3} = 5 \times 10^4 \text{ Hz}.$$

More realistic models including spherical geometry and density gradients give frequencies of tens to hundreds of Hz for global torsional modes; the slab estimate above thus slightly overestimates the frequency but gives the correct scale.

6.5 Exercises

1. **Plane waves in an elastic medium.**
   Starting from Eq. (197) with $\mathbf{f} = 0$ and constant λ, μ, ρ, derive the dispersion relations for longitudinal and transverse plane waves and show that the wave speeds are v_P and v_S given above.

2. **Simple seismic travel time.**
   Consider a planet with a homogeneous mantle of thickness R in which v_P and v_S are constant. An earthquake occurs at depth h below the surface, directly "below" a seismometer.

   (a) Derive expressions for the arrival times t_P and t_S.

Elastic waves in a planet

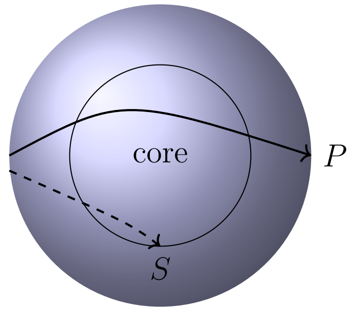


Figure 22: Schematic P and S waves in a layered planet. S-waves do not propagate through liquid regions.

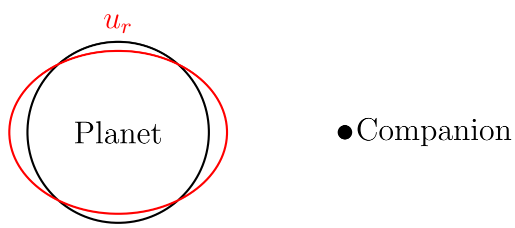


Figure 23: Tidal deformation of a homogeneous planet.

(b) For $R = 3000$ km, $h = 100$ km, $v_P = 8$ km s^{-1} and $v_S = 4.5$ km s^{-1}, compute t_P and t_S and the time delay $\Delta t = t_S - t_P$.

3. **Tidal displacement of a homogeneous sphere (scaling).**
Using Eq. (198), repeat the scaling argument of Example 2 and show that the dimensionless tidal Love number h_2 for a homogeneous incompressible sphere scales roughly as

$$h_2 \sim \frac{\rho g R}{\mu},$$

where g is the surface gravity. Discuss qualitatively what happens to h_2 when μ is very small (fluid planet) or very large (rigid body).

4. **Torsional shear mode in a crustal shell.**
Model a thin spherical elastic shell of thickness H and radius R as supporting purely tangential displacements $\mathbf{u}(\theta, \phi, t)$ with $\nabla \cdot \mathbf{u} = 0$ and no radial component. Show that these obey approximately a shear-wave equation on the sphere with speed v_S, and argue that the fundamental torsional mode has frequency of order

$$f \sim \frac{v_S}{2\pi R}.$$

Evaluate f for a neutron star with $R = 10$ km and $v_S = 10^8$ m s^{-1}.

5. **Energy in a standing shear wave.**
Consider a cube of side L filled with an isotropic elastic material. A standing shear wave in the x-direction has displacement

$$u_y(x, t) = A \sin(kx) \cos(\omega t), \qquad u_x = u_z = 0.$$

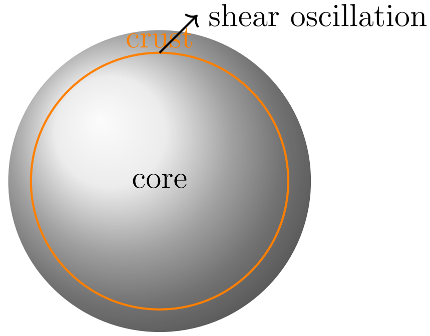


Figure 24: Neutron star with solid crust supporting shear waves.

(a) Compute the shear strain u_{xy} and show that the time-averaged elastic energy stored in the cube is

$$\langle E_{\mathrm{el}} \rangle = \frac{1}{4}\,\mu k^2 A^2 L^3.$$

(b) Using the dispersion relation for shear waves, express this energy in terms of ρ, v_S, ω, A and L.

7 Solved problems

7.1 Long rod standing vertically in a gravitational field

Determine the deformation of a long rod (with length l) standing vertically in a gravitational field.

We take the z-axis along the axis of the rod, and the xy-plane in the plane of its lower end. The equations of equilibrium are

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0, \qquad \frac{\partial \sigma_{iz}}{\partial x_i} = \rho g.$$

On the sides of the rod all components σ_{ik} except σ_{iz} must vanish, and on the upper end ($z = l$) $\sigma_{ix} = \sigma_{iy} = \sigma_{iz} = 0$.

The nonzero components consistent with equilibrium are

$$\sigma_{zz} = -\rho g (l - z), \tag{200}$$

with all other $\sigma_{ik} = 0$. From σ_{iz} we find U_{ik} to be

$$U_{xz} = U_{yz} = \frac{\sigma_{iz}}{E} = 0, \qquad U_{zz} = \frac{\sigma_{zz}}{E}.$$

Integrating, we obtain the components of the displacement vector

$$u_x = \frac{\rho g (l - z) x}{E}, \qquad u_y = \frac{\rho g (l - z) y}{E}, \qquad u_z = \frac{\rho g}{2E}\left[l^2 - (l - z)^2\right] - \sigma(x^2 + y^2). \tag{201}$$

7.2 Hollow sphere

Determine the deformation of a hollow sphere (with external and internal radii R_2 and R_1) with a pressure p_1 inside and p_2 outside.

Using spherical polar coordinates, the displacement vector is radial:

$$\mathbf{u} = u_r(r)\,\hat{\mathbf{e}}_r,$$

and

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2}\frac{d}{dr}(r^2 u_r) = constant \equiv 3a.$$

Thus

$$u_r = ar + \frac{b}{r^2}.$$

The components of the strain tensor are $U_{rr} = a - 2b/r^3$, $U_{\theta\theta} = U_{\phi\phi} = a + b/r^3$. The radial stress is

$$\sigma_{rr} = \frac{E}{(1+\sigma)(1-2\sigma)}\left[(1-\sigma)U_{rr} + 2\sigma U_{\theta\theta}\right] = \frac{E}{1-2\sigma}a - \frac{2E}{1+\sigma}\frac{b}{r^3}. \tag{202}$$

The constants a and b follow from the boundary conditions $\sigma_{rr} = -p_1$ at $r = R_1$ and $\sigma_{rr} = -p_2$ at $r = R_2$:

$$a = \frac{p_1 R_1^3 - p_2 R_2^3}{R_2^3 - R_1^3}\frac{1-2\sigma}{E}, \qquad b = \frac{R_1^3 R_2^3 (p_1 - p_2)}{R_2^3 - R_1^3}\frac{1+\sigma}{2E}. \tag{203}$$

For a thin shell of thickness $h = R_2 - R_1$:

$$u = \frac{pR(1-\sigma)}{2Eh}, \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{pR}{2h}, \qquad \bar{\sigma}_{rr} = \frac{p}{2}. \tag{204}$$

7.3 Solid sphere

Determine the deformation of a solid sphere (with radius R) in its own gravitational field.

With gravitational force $-gr/R$, the displacement satisfies:

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}(r^2u_r)\right)=\rho g\frac{r}{R}. \tag{205}$$

The finite solution satisfying $\sigma_{rr}=0$ at $r=R$ is

$$u_r=-\frac{\rho gR(1-\sigma)(1+\sigma)}{10E(1-\sigma)}\left(\frac{3-\sigma}{1+\sigma}-\frac{r^2}{R^2}\right). \tag{206}$$

It should be noticed that the substance is compressed ($u_r<0$) inside a spherical surface of radius $R\sqrt{(3-\sigma)/3(1+\sigma)}$ and stretched outside it ($u_r>0$). The pressure at the centre of the sphere is

$$\frac{(3-\sigma)\,g\rho R}{10(1-\sigma)}.$$

7.4 Cylindrical pipe

Determine the deformation of a cylindrical pipe (with external and internal radii R_2 and R_1), with pressure p inside and no pressure outside.

Using cylindrical coordinates (r,θ,z) and radial displacement $u_r(r)$:

$$\frac{1}{r}\frac{d}{dr}(ru_r)=constant\equiv 2a.$$

Thus

$$u_r=ar+\frac{b}{r}.$$

The strain tensor gives

$$U_{rr}=a-\frac{b}{r^2},\qquad U_{\phi\phi}=\frac{u_r}{r}=a+\frac{b}{r^2}.$$

Boundary conditions $\sigma_{rr}(R_2)=0$, $\sigma_{rr}(R_1)=-p$ give

$$a=\frac{pR_1^2}{R_2^2-R_1^2}\frac{(1+\sigma)(1-2\sigma)}{E},\qquad b=\frac{pR_1^2R_2^2}{R_2^2-R_1^2}\frac{1+\sigma}{2E}. \tag{207}$$

The stresses become

$$\sigma_{rr}=\frac{pR_1^2}{R_2^2-R_1^2}\left(1-\frac{R_2^2}{r^2}\right),\qquad \sigma_{\phi\phi}=\frac{pR_1^2}{R_2^2-R_1^2}\left(1+\frac{R_2^2}{r^2}\right),\qquad \sigma_{zz}=\frac{2p\sigma R_1^2}{R_2^2-R_1^2}. \tag{208}$$

7.5 Cylinder rotating uniformly

Determine the deformation of a cylinder rotating uniformly about its axis.

The centrifugal force $\rho\Omega^2 r$ acts outward. Displacement $u_r(r)$ satisfies

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru_r)\right)=-\rho\Omega^2 r. \tag{209}$$

Finite at $r=0$ and $\sigma_{rr}=0$ at $r=R$ yields

$$u_r=\frac{\rho\Omega^2(1+\sigma)(1-2\sigma)}{8E(1-\sigma)}\left[r(R^2-\tfrac{2}{3}r^2)\right]. \tag{210}$$

7.6 Equations of equilibrium for an isotropic body

Derive the equations of equilibrium for an isotropic body (in the absence of body forces) in terms of the components of the stress tensor.

The required system of equations contains the three equations

$$\frac{\partial \sigma_{ik}}{\partial x_k} = 0. \tag{211}$$

and also the equations resulting from the fact that the six different components of U_{ik} are not independent quantities. To derive these equations, we first write down the system of differential relations satisfied by the components of the tensor U_{ik}. It is easy to see that the quantities

$$U_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{212}$$

satisfy identically the relations

$$\frac{\partial^2 U_{ik}}{\partial x_l \partial x_m} + \frac{\partial^2 U_{lm}}{\partial x_i \partial x_k} = \frac{\partial^2 U_{il}}{\partial x_k \partial x_m} + \frac{\partial^2 U_{km}}{\partial x_i \partial x_l}. \tag{213}$$

Here there are only six essentially different relations, namely those corresponding to the following values of i, k, l, m : 1122, 1133, 2233, 1123, 2213, 3312. All these are retained if the above tensor equation is contracted with respect to l and m:

$$\Delta U_{ik} + \frac{\partial^2 U_{ll}}{\partial x_i \partial x_k} = \frac{\partial^2 U_{il}}{\partial x_k \partial x_l} + \frac{\partial^2 U_{kl}}{\partial x_i \partial x_l}. \tag{214}$$

Substituting here U_{ik} in terms of σ_{ik} we obtain the required equations:

$$(1+\sigma)\,\Delta\sigma_{ik} + \frac{\partial^2 \sigma_{ll}}{\partial x_i \partial x_k} = 0. \tag{215}$$

These equations remain valid in the presence of external forces constant throughout the body.

Contracting the above equation with respect to the suffixes i and k, we find that $\Delta\sigma_{ll} = 0$, i.e. σ_{ll} is a harmonic function. Taking the Laplacian of equation (3), we then find that $\Delta\Delta\sigma_{ik} = 0$, i.e. the components σ_{ik} are biharmonic functions.

7.7 General integral of the equations of equilibrium

Express the general integral of the equations of equilibrium (in the absence of body forces) in terms of an arbitrary biharmonic vector (B. G. Galerkin 1930).

We try

$$\vec{u} = \Delta\vec{f} + A\,\nabla(\nabla\cdot\vec{f}),$$

so that

$$\operatorname{div}\vec{u} = (1+A)\Delta f.$$

Substituting in the equation of equilibrium gives

$$(1-2\sigma)\Delta\Delta f + [2(1-\sigma)A + 1]\Delta\Delta f = 0. \tag{216}$$

Thus if f is biharmonic ($\Delta\Delta f = 0$):

$$\vec{u} = \Delta\vec{f} - \frac{1}{2(1-\sigma)}\nabla(\nabla\cdot\vec{f}). \tag{217}$$

8 Problems

1. A material is called *incompressible* if there is no change of volume under any and all states of stresses. Show that for an incompressible isotropic linearly elastic solid with finite Young's modulus E, (a) Poisson's ratio $\nu = \frac{1}{2}$, (b) the shear modulus $\mu = E_Y/3$, and (c) $k \to \infty$, $\lambda \to \infty$ and $k - \lambda = 2\mu/3$.

2. If the components of strain at a point of structural steel are

$$U_{11} = 36 \times 10^{-6}, \qquad U_{22} = 40 \times 10^{-6}, \qquad U_{33} = 25 \times 10^{-6},$$
$$U_{12} = 12 \times 10^{-6}, \qquad U_{23} = 0, \qquad U_{13} = 30 \times 10^{-6},$$

   find the stress components.

$$\lambda = 119.2\,\text{GPa}, \qquad \mu = 79.2\,\text{GPa}.$$

3. An isotropic elastic body ($E_Y = 207\,\text{GPa}$, $\mu = 79.2\,\text{GPa}$) has a uniform state of stress given by

$$[\sigma] = \begin{bmatrix} 100 & 40 & 60 \\ 40 & -200 & 0 \\ 60 & 0 & 200 \end{bmatrix} \text{MPa}.$$

   (a) What are the strain components?
   (b) What is the total change of volume for a five-centimeter cube of the material?

4. An isotropic elastic sphere ($E_Y = 207\,\text{GPa}$, $\mu = 79.2\,\text{GPa}$) of radius 5 cm is under the uniform stress field

$$[\sigma] = \begin{bmatrix} 6 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}.$$

   Find the change of volume for the sphere.

5. Given the following displacement field in an isotropic linearly elastic solid:

$$u_1 = kX_3X_2, \qquad u_2 = kX_3X_1, \qquad u_3 = k(X_1^2 - X_2^2), \qquad k = 10^{-4},$$

   (a) Find the stress components.
   (b) In the absence of body forces, is the state of stress a possible equilibrium stress field?

6. Given the following displacement field in an isotropic linearly elastic solid:

$$u_1 = kX_2X_3, \qquad u_2 = kX_1X_3, \qquad u_3 = kX_1X_2, \qquad k = 10^{-4},$$

   (a) Find the stress components.
   (b) In the absence of body forces, is the state of stress a possible equilibrium stress field?

7. Given the following displacement field in an isotropic linearly elastic solid:

$$u_1 = kX_2X_3, \qquad u_2 = kX_1X_3, \qquad u_3 = k(X_1X_2 + X_3^2), \qquad k = 10^{-4},$$

   (a) Find the stress components.
   (b) In the absence of body forces, is the state of stress a possible equilibrium stress field?

References

[1] A. Sepúlveda, *Hidrodinámica*. Fondo Editorial EIA, Colombia, 2019.

[2] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, vol. 7 of *Course of Theoretical Physics*. Elsevier Butterworth-Heinemann, New York, 1986.

[3] B. Lautrup, *Physics of Continuous Matter: Exotic and Everyday Phenomena in the Macroscopic World*. CRC Press, Boca Raton, 2 ed., 2011.