Introduction

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3], Lai-Rubin-Krempl [4] books

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC1. Definir las propiedades microscópicas básicas de la materia, peso molecular promedio, masa molar, tamaño molar, número de partículas, fracciones molares y de masa.
- OC2. Identificar las condiciones bajo las cuáles la aproximación continua de la materia es suficientemente precisa con respecto a la descripción estadística.

1.2 Objetivos específicos procedimentales:

- OP1. Deducir a partir de los primeros principios y la aplicación de resultados estadísticos básicos, la ecuación de estado de medios continuos elementales, gases perfectos y sólidos simples.
- OP2. Aplicar métodos numéricos para calcular propiedades continuas (densidad, presión, etc.) de sistemas discretos.

1.3 Objetivos específicos actitudinales:

- OA1. Reconocer la aproximación continua como un método poderoso para describir la materia en distintos regímenes.

2 Preliminaries

Fluid mechanics may be seen as a starting point for field theory, both from the mathematical and physical perspectives. This is not only because, historically, the study of fluids created the language with which fields are described, but also because it is one of the first courses in a physics major where field theory can be applied. Hydrodynamics, on the one hand, introduces fundamental concepts such as field, flux, source, sink, divergence, curl, circulation, and vorticity. On the other hand, Euler's dynamical formulation provides the framework to derive and describe results such as Torricelli's and Bernoulli's theorems, as well as the theorems on vortex circulation.

Fluid mechanics is encountered not only in almost every area of our physical lives. Blood flows through our veins and arteries, a ship moves through water, airplanes fly in the air, air flows around wind machines, air is compressed in a compressor, steam flows around turbine blades, a dam holds back water, air is heated and cooled in our homes, and computers require air to cool components.

3 Fundamental blocks of matter

Elementary particles [5][1] are classified according their spin s: bosons have integer $s = 0, 1, 2$ while fermions have half-integer $s = \frac{1}{2}, \frac{3}{2}$.

3.1 Of what is ordinary matter made? Fermions

Fermions are classified accordingly to leptons and quarks, and both are presented in families or generation for a total of 3 of those. Leptons do not undergo the strong interaction while quarks do (see Tables 1 and 2).

Lepton	Spin	Charge (e)	Mass (GeV/c^2)
$e^\pm$	1/2	±1	0.000511
ν_e	1/2	0	$\lesssim 10^{-15}$
$\mu^\pm$	1/2	±1	0.105
ν_μ	1/2	0	$\sim 10^{-12}$
$\tau^\pm$	1/2	±1	1.777
ν_τ	1/2	0	$\sim 10^{-11}$

Table 1: Leptons.

Quark	Spin	Charge (e)	Mass (GeV/c^2)
up	1/2	2/3	0.0023
down	1/2	-1/3	0.0048
charm	1/2	2/3	1.275
strange	1/2	-1/3	0.095
top	1/2	2/3	173
bottom	1/2	-1/3	4.18

Table 2: Quarks.

3.1.1 Composite baryons

Particles made up from quarks are called baryons, and are classified as mesons or hadrons if they are formed by a pair quark-antiquark or a ternary of quarks (anti-quarks), respectively.

3.2 What holds it together? Bosons

The bosons with spin $s = 1$ are dubbed as vector particles and are associated with the mediators of the fundamental interactions: photon with electromagnetism, gluon with strong force and $Z, W^\pm$ with weak force. The Higgs particle is associated with the field responsible for giving mass to all known particles. The properties of the known bosonic particles are displayed in Table 4.

[1] https://revistas.udea.edu.co/index.php/experimenta/article/view/342525.

Meson	Spin	Charge (e)	Mass (GeV/c^2)
$\pi^0\,(u\bar{u}-d\bar{d})$	0	0	0.135
$\pi^+\,(u\bar{d})$	0	1	0.140
$\pi^-\,(d\bar{u})$	0	-1	0.140
Hadron	Spin	Charge (e)	Mass (GeV/c^2)
$p\,(uud)$	1/2	1	0.938
$n\,(udd)$	1/2	0	0.939
$\Lambda^0\,(uds)$	1/2	0	1.116

Table 3: Baryons.

Boson	Spin	Charge	Mass (GeV/c^2)
Photon	1	0	0
Gluon	1	0	0
Z^0	1	0	91.2
$W^\pm$	1	±1	80.4
Higgs	0	0	125

Table 4: Bosonic elementary particles.

3.3 Atoms and molecules

Matter is formed of molecules, which in turn consist of atoms which are form by a nucleus and electrons. All nuclei are form by protons and neutrons. The total number of protons Z plus the number of neutrons $A-Z$ gives the mass number A of an atom's nucleus. The number of neutrons can vary for a given element, leading to different isotopes of such element. Let's briefly comment some examples.

- Deuterium (^{2}H or D), also known as heavy hydrogen is one of two isotopes of hydrogen. The deuterium nucleus (deuteron) contains one proton and one neutron such that $A_{\mathrm{D}} = 2$.
- Tritium (^{3}H or T), is the second isotope of hydrogen. Its nucleus contains one proton and two neutron such that $A_{^3\mathrm{H}} = 3$.
- Helium-3 (^{3}He) is a isotope of helium with $Z = 2$ and $A = 3$. Helium-3 atoms are fermionic and become a superfluid at the temperature of 2.491 mK[2].
- A carbon atom with 6 neutrons (carbon-12) has a mass number of 12 (6 protons + 6 neutrons).

The periodic table of element displayed in Fig. 1 shows the primary source on Earth for each element. In cases where two sources contribute fairly equally, both appear.

[2]Superfluidity is the characteristic property of a fluid with zero viscosity: it flows without any loss of kinetic energy.

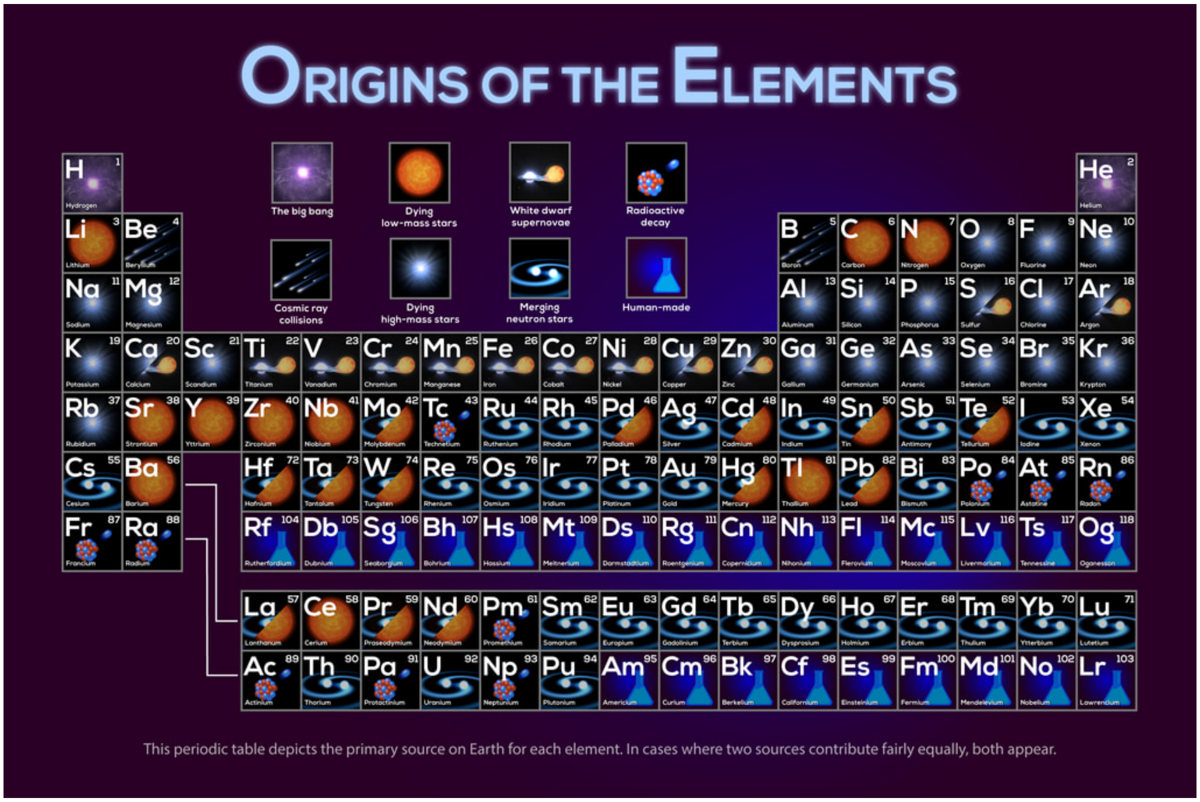

Figure 1: Origins of the elements. Figure taken from Ref. [6].

4 Continuous matter

Matter is formed of molecules, which in turn consist of atoms and subatomic particles. Thus, matter is not continuous. However, beyond length scales greater than 10^{-8} m matter can be described with theories that pay no attention to the molecular structure of materials (see Fig. 2). The theory that aims to describe relationships among gross phenomena, neglecting the structure of material on a smaller scale, is known as continuum theory. The continuum theory regards matter as indefinitely divisible. Thus, within this theory, one accepts the idea of an infinitesimal volume of materials, referred to as a particle in the continuum, and in every neighborhood of a particle there are always neighboring particles.

- It is usually assumed that the physical laws valid at one length scale are not very sensitive to the details of what happens at much smaller scales.
- The microscopic physics affects the macroscopic world almost only through material constants, such as coefficients of elasticity and viscosity, characterizing the interactions between macroscopic amounts of matter. It is, of course, an important task for the physics of materials to derive the values of these constants, but this task lies outside the realm of continuum physics.
- Here we consider the transition from molecules to continuous matter, or mathematically speaking from point particles to fields.
- It is emphasized that the macroscopic continuum description must necessarily be statistical

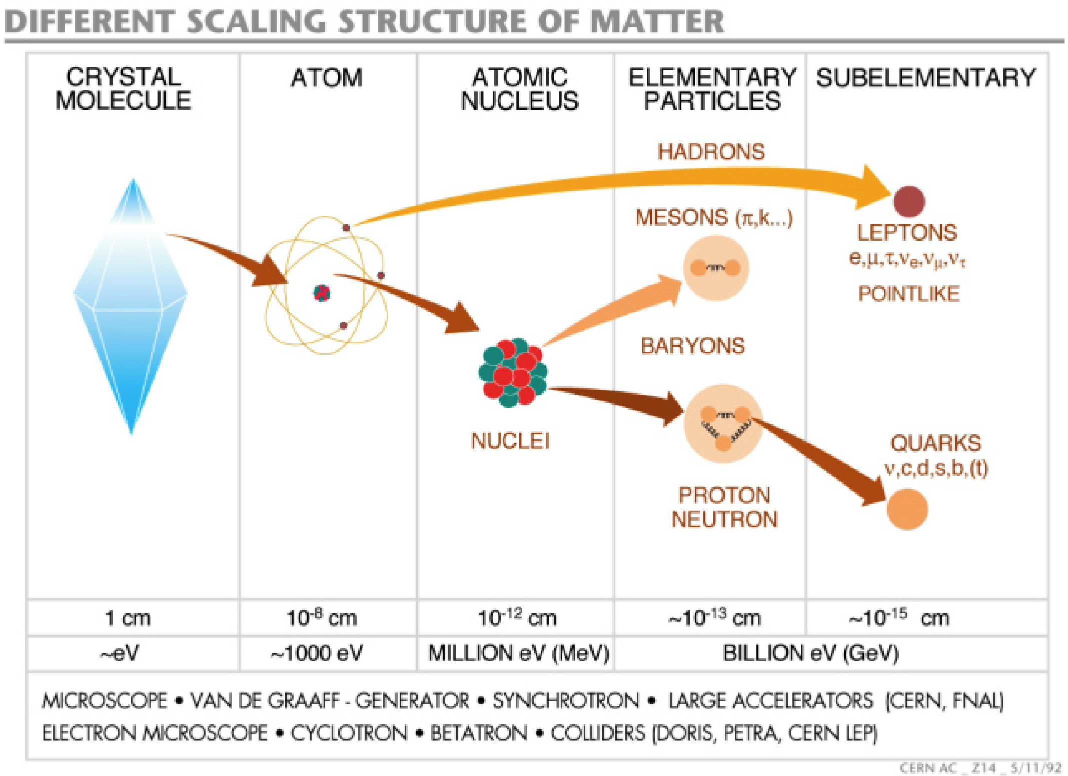

Figure 2: Different scales in the structure of matter. Figure taken from Ref. [7].

in nature, but that random statistical fluctuations are strongly suppressed by the enormity of the number of molecules in any macroscopic material object.

- We will recast Newton's laws for point particles into a systematic theory of continuous matter, and the application of this theory to the macroscopic material world.

4.1 Mole and molar mass

As of May 20, 2019, with the redefinition of SI base units, Avogadro's number is defined as an *exact constant*:

$$N_A = 6.02214076 \times 10^{23} \text{ mol}^{-1}.$$

This means that one mole corresponds to exactly

$$6.02214076 \times 10^{23}$$

specified elementary entities. Unlike the pre-2019 definition, it is no longer based on the mass of 12 g of carbon-12, but rather fixed in the same way that the speed of light c is defined in SI units.

The *molar mass* of a substance is defined as the mass of one mole of its particles (atoms, molecules, or ions). By definition, one mole corresponds to N_A particles. The molar mass is expressed in *grams per mole (g/mol)*. The molar mass M can be expressed as

$$M = \frac{m}{n}, \tag{1}$$

where m is the mass of the sample (in grams) and n is the amount of substance (in moles).

- The molar mass of hydrogen (H) is approximately

$$M(\mathrm{H}) \simeq 1.008 \text{ g/mol.}$$

- The molar mass of water (H_2O) is given by

$$M(\mathrm{H_2O}) = 2 \times 1.008 + 15.999 = 18.015 \text{ g/mol.}$$

In short, the molar mass specifies the mass of 6.022×10^{23} entities of a substance.

4.1.1 The Kilogram

Before May 20, 2019, the kilogram was defined as the mass of the *International Prototype of the Kilogram (IPK)*, a cylinder of platinum–iridium kept at the International Bureau of Weights and Measures (BIPM) in Sèvres, France.

The kilogram is now defined in terms of the *Planck constant*, h, which has been assigned the exact value

$$h = 6.62607015 \times 10^{-34}\,\mathrm{J \cdot s} = \mathrm{kg \cdot m^2/s}\,. \tag{2}$$

From this fixed constant, and using the definitions of the second and the meter, the kilogram is derived. In practice, the realization of the kilogram is achieved through experiments such as the *Kibble balance* or the *x-ray crystal density method.*

4.2 Molecular separation length

Let a sample of a pure substance with volume V and mass M. The molar mass is denoted as $M_{\mathrm{mol}} = n \times M$, where $n = N/N_A$ with N being the number of molecules. The volume per molecule is $V_{\mathrm{mol}} = V/N$, and assuming that $V_{\mathrm{mol}} = L_{\mathrm{mol}}^3$ we have that the molecular separation length becomes

$$L_{\mathrm{mol}} = \left(\frac{V}{N}\right)^{1/3} = \left(\frac{M_{\mathrm{mol}}}{\rho\, N_A}\right)^{1/3}. \tag{3}$$

This length defines the scale at which the discrete molecular nature of matter governs the physics, rendering any continuum description of bulk matter entirely invalid.

For liquids and solids where the molecules touch each other, this length is roughly the size of a molecule. For solid iron $L_{\mathrm{mol}} \approx 0.23$ nm, and for liquid water $L_{\mathrm{mol}} \approx 0.31$ nm. For an ideal gas at normal temperature and pressure ($p = 1$ atm $= 101325$ Pa and $T = 20°\mathrm{C} = 293.15$ K) and with

$$p\,V = n\,R\,T = n\,(N_A\,k_B)\,T = N\,k_B\,T, \tag{4}$$

we obtain

$$L_{\mathrm{mol}} = \left(\frac{k_B\,T}{p}\right)^{1/3} = \left(\frac{1.380649 \times 10^{-23}\,\mathrm{J/K}\,293.15\,\mathrm{K}}{101325\,\mathrm{N\,m^{-2}}}\right)^{1/3} \tag{5}$$

$$\approx 3.42\,\mathrm{nm}\,. \tag{6}$$

There is a lot of vacuum in a volume of gas, in fact about 1000 times the true volume of the molecules at normal temperature and pressure.

4.3 States of matter

The three classic states of neutral matter—solid, liquid, and gas—depend, broadly speaking, on the competition between negative binding energy and positive thermal energy.

4.3.1 Solids

In solid matter the binding is so strong that thermal motion cannot overcome it. The molecules remain bound to each other by largely elastic forces, and constantly undergo small-amplitude thermal motion around their equilibrium positions. If increasing external forces are applied, solids will begin to deform elastically, until they eventually become plastic or even fracture. A solid body retains its shape independently of the shape of a container large enough to hold it, apart from small deformations, for example due to gravity.

4.3.2 Liquids

In liquid matter the binding is weaker than in solid matter, although it is still hard for a molecule on its own to leave the company of the others through an open liquid surface. The molecules stay in contact but are not locked to their neighbors. Molecular conglomerates may form and stay loosely connected for a while, as for example chains of water molecules. Under the influence of external forces, for example gravity, a liquid will undergo bulk motion, called flow, a process that may be viewed as a kind of continual fracturing. A liquid will not expand to fill an empty container completely, but will under the influence of external forces eventually adapt to its shape wherever it touches it.

4.3.3 Gases

In gaseous matter the molecules are bound so weakly that the thermal motion easily overcomes it, and they essentially move around freely between collisions. A gas will always expand to fill a closed empty container completely. Under the influence of external forces, for example a piston pushed into the container, a gas will quickly flow to adapt to the changed container shape.

4.3.4 Superfluids

A superfluid features properties very different from those of an ordinary liquid. For example, if superfluid helium-4 is placed in an open vessel, a thin Rollin film will climb the sides of the vessel, causing the liquid to escape. It is also a property of various other exotic states of matter theorized to exist in astrophysics and high-energy physics.

5 The continuum approximation

5.1 Poisson distribution

The *Poisson distribution* models the probability of observing a certain number of events n in a fixed interval of time, space, or volume under the following conditions:

- Events occur independently of one another.
- The average rate of occurrence is constant.

- The probability of more than one event in an infinitesimal interval is negligible.

The probability mass function is

$$P(n|\lambda) = \frac{\lambda^n}{n!}e^{-\lambda}, \qquad n = 0, 1, 2, \ldots$$

where λ is the expected (mean) number of events in the interval. This distribution is commonly used to describe "rare events," such as the number of radioactive decays in a sample during a given time, the number of molecules found in a small subvolume of a gas, or the number of phone calls received in a fixed time window.

Let's consider a small volume V of a much larger volume of gas, such that the probability for any molecule to be found in V is exceedingly small. If the average number of molecules in V is known to be N, the probability of finding precisely n of the molecules in V is given

$$\Pr(n|N) = \frac{N^n}{n!}e^{-N}, \qquad n = 0, 1, 2, \ldots$$

It is normalized:

$$\sum_{n=0}^{\infty} \Pr(n|N) = e^{-N} \sum_{n=0}^{\infty} \frac{N^n}{n!} = e^{-N}\, e^{N} = 1.$$

The mean value is given by

$$\langle n \rangle = \sum_{n=0}^{\infty} n\, \Pr(n|N) = e^{-N} \sum_{n=1}^{\infty} \frac{n\, N^n}{n!} = e^{-N} \sum_{n=1}^{\infty} \frac{N^n}{(n-1)!}.$$

Let $m = n - 1$:

$$\langle n \rangle = e^{-N} \sum_{m=0}^{\infty} \frac{N^{m+1}}{m!} = e^{-N} N \sum_{m=0}^{\infty} \frac{N^m}{m!} = e^{-N} N e^{N} = N.$$

To calculate the variance it is convenient to compute $\langle n(n-1) \rangle$:

$$\langle n(n-1) \rangle = e^{-N} \sum_{n=2}^{\infty} \frac{n(n-1)N^n}{n!} = e^{-N} \sum_{n=2}^{\infty} \frac{N^n}{(n-2)!}.$$

Let $m = n - 2$:

$$\langle n(n-1) \rangle = e^{-N} \sum_{m=0}^{\infty} \frac{N^{m+2}}{m!} = e^{-N} N^2 \sum_{m=0}^{\infty} \frac{N^m}{m!} = N^2.$$

Now, since $\langle n^2 \rangle = \langle n(n-1) \rangle + \langle n \rangle$, we have

$$\langle n^2 \rangle = N^2 + N.$$

Therefore the variance is

$$\Delta N^2 = \mathrm{Var}(n) = \langle n^2 \rangle - \langle n \rangle^2 = (N^2 + N) - N^2 = N,$$

so that $\Delta N = \sqrt{N}$.

5.2 Density fluctuations

Consider a fixed small volume V of a pure gas with molecules of mass m. If at a given time t the number of molecules in this volume is N, the mass density at this time becomes

$$\rho(t) = \frac{N\,m}{V}. \tag{7}$$

At a time $t' = t + \Delta t$:

$$\rho(t') = \frac{N'\,m}{V}. \tag{8}$$

Since the density is linear in N, the relative fluctuation in density becomes

$$\frac{\Delta\rho}{\rho} = \frac{\Delta N}{N} = \frac{\Delta N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}. \tag{9}$$

For $N \sim N_A$, the relative fluctuation becomes of the order of 10^{-12} and can safely be ignored.

If we want the relative density fluctuation to be smaller than a given value $\Delta\rho/\rho \lesssim \epsilon$, we must require $N \gtrsim \epsilon^{-2}$. The smallest acceptable number of molecules, ϵ^{-2}, occupies a volume $\epsilon^{-2} L_{\mathrm{mol}}^3$. A cubic cell with this volume has side length

$$L_{\mathrm{micro}} = \epsilon^{-2/3} L_{\mathrm{mol}}.$$

Thus, to secure a relative precision $\epsilon = 10^{-3}$ for the density, the microscopic cell should contain 10^6 molecules and have side length $L_{\mathrm{micro}} = 100 L_{\mathrm{mol}}$. For an ideal gas under normal conditions we find $L_{\mathrm{micro}} \approx 0.34\,\mu$m.

5.3 Velocity

5.3.1 Maxwell–Boltzmann speed distribution $f(v)$

Let's consider a ideal, classical and nonrelativistic gas. For one molecule of mass m, the kinetic energy is $E = \frac{1}{2} m(v_x^2 + v_y^2 + v_z^2)$. In thermal equilibrium at temperature T, the canonical (Boltzmann) weight is $\propto e^{-E/(k_{\mathrm{B}}T)}$. By independence and isotropy,

$$p(\mathbf{v})\, d^3v = A \exp\left[-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_{\mathrm{B}}T}\right] dv_x\, dv_y\, dv_z, \tag{10}$$

with A fixed by normalization $\int_{\mathbb{R}^3} p(\mathbf{v})\, d^3v = 1$. Each component is Gaussian with variance $\sigma^2 = k_{\mathrm{B}}T/m$:

$$\int_{-\infty}^{\infty} e^{-mv_x^2/(2k_{\mathrm{B}}T)}\, dv_x = \sqrt{\frac{2\pi k_{\mathrm{B}}T}{m}}, \tag{11}$$

and similarly for v_y, v_z. Hence

$$1 = \int_{\mathbb{R}^3} p(\mathbf{v})\, d^3v = A\left(\sqrt{\frac{2\pi k_{\mathrm{B}}T}{m}}\right)^3 \quad \Longrightarrow \quad A = \left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}. \tag{12}$$

Thus the *velocity-vector* density is

$$p(\mathbf{v}) = \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \exp\!\left(-\frac{mv^2}{2k_\mathrm{B}T}\right), \qquad v = \|\mathbf{v}\|. \tag{13}$$

The probability that the speed lies in $[v, v+dv]$ equals the probability that $\mathbf{v}$ lies in the spherical shell of radius v and thickness dv: $d^3v = 4\pi v^2\,dv$. Define the *speed* density $f(v)$ by $f(v)\,dv = \int_{\text{shell}} p(\mathbf{v})\,d^3v$. Using (13),

$$f(v) = 4\pi v^2 \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \exp\!\left(-\frac{mv^2}{2k_\mathrm{B}T}\right), \qquad v \ge 0. \tag{14}$$

5.3.2 Speeds from the Maxwell–Boltzmann Distribution

The MB speed distribution for a 3D ideal gas is

$$f(v) = 4\pi \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} v^2 \exp\!\left(-\frac{mv^2}{2k_\mathrm{B}T}\right), \qquad v \ge 0, \tag{15}$$

normalized so that $\int_0^\infty f(v)\,dv = 1$.

- Most probable speed v_mp.
  Maximize $f(v)$ with respect to v:

$$\frac{d}{dv}[\ln f(v)] = \frac{d}{dv}\left[2\ln v - \frac{mv^2}{2k_\mathrm{B}T}\right] = 0 \implies \frac{2}{v} - \frac{m}{k_\mathrm{B}T}v = 0, \tag{16}$$

$$\Rightarrow \quad v_\mathrm{mp}^2 = \frac{2k_\mathrm{B}T}{m}, \qquad \boxed{v_\mathrm{mp} = \sqrt{\frac{2k_\mathrm{B}T}{m}}}. \tag{17}$$

- Mean speed $\langle v\rangle$.
  Compute $\langle v\rangle = \int_0^\infty v\,f(v)\,dv$. Using $\int_0^\infty v^n e^{-\alpha v^2}\,dv = \frac{1}{2}\,\alpha^{-(n+1)/2}\Gamma\!\left(\frac{n+1}{2}\right)$, with $\alpha = \frac{m}{2k_\mathrm{B}T}$:

$$\langle v\rangle = 4\pi \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \int_0^\infty v^3 \exp\!\left(-\alpha v^2\right)\,dv \tag{18}$$

$$= 4\pi \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \cdot \frac{1}{2}\,\alpha^{-2}\Gamma(2) = 2\pi \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \alpha^{-2}. \tag{19}$$

  Since $\alpha^{-2} = (2k_\mathrm{B}T/m)^2$ and $\Gamma(2) = 1$,

$$\langle v\rangle = 2\pi \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \left(\frac{2k_\mathrm{B}T}{m}\right)^2 = \sqrt{\frac{8k_\mathrm{B}T}{\pi m}}, \qquad \boxed{\langle v\rangle = \sqrt{\frac{8k_\mathrm{B}T}{\pi m}}}. \tag{20}$$

- Root-mean-square speed v_rms.
  Compute $\langle v^2\rangle = \int_0^\infty v^2 f(v)\,dv$:

$$\langle v^2\rangle = 4\pi \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \int_0^\infty v^4 \exp\!\left(-\alpha v^2\right)\,dv \tag{21}$$

$$= 4\pi \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \cdot \frac{1}{2}\,\alpha^{-5/2}\Gamma\!\left(\tfrac{5}{2}\right) = 2\pi \left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2} \alpha^{-5/2} \cdot \frac{3\sqrt{\pi}}{4}. \tag{22}$$

With $\alpha = \frac{m}{2k_{\mathrm{B}}T}$, this simplifies to

$$\langle v^2 \rangle = \frac{3k_{\mathrm{B}}T}{m}, \qquad v_{\mathrm{rms}} = \sqrt{\langle v^2 \rangle} = \boxed{\sqrt{\frac{3k_{\mathrm{B}}T}{m}}}. \tag{23}$$

Using $m = M_{\mathrm{mol}}/N_{\mathrm{A}}$ and $R_{\mathrm{mol}} = N_{\mathrm{A}}k_{\mathrm{B}}$:

$$v_{\mathrm{mp}} = \sqrt{\frac{2R_{\mathrm{mol}}T}{M_{\mathrm{mol}}}}, \qquad \langle v \rangle = \sqrt{\frac{8R_{\mathrm{mol}}T}{\pi M_{\mathrm{mol}}}}, \qquad v_{\mathrm{rms}} = \sqrt{\frac{3R_{\mathrm{mol}}T}{M_{\mathrm{mol}}}}. \tag{24}$$

They satisfy the universal ordering

$$v_{\mathrm{mp}} < \langle v \rangle < v_{\mathrm{rms}}, \tag{25}$$

with fixed ratios (independent of T and the gas species):

$$\frac{v_{\mathrm{mp}}}{v_{\mathrm{rms}}} = \sqrt{\frac{2}{3}} \approx 0.8165, \qquad \frac{\langle v \rangle}{v_{\mathrm{rms}}} = \sqrt{\frac{8}{3\pi}} \approx 0.9213, \qquad \frac{\langle v \rangle}{v_{\mathrm{mp}}} = \sqrt{\frac{4}{\pi}} \approx 1.1284. \tag{26}$$

For air at room temperature, $T = 300$ K and $M_{\mathrm{mol}} \approx 0.029\ \mathrm{kg\,mol^{-1}}$. Then

$$v_{\mathrm{mp}} = \sqrt{\frac{2R_{\mathrm{mol}}T}{M_{\mathrm{mol}}}} = \sqrt{\frac{2 \times 8.314 \times 300}{0.029}} \approx 4.14 \times 10^2\ \mathrm{m\,s^{-1}}, \tag{27}$$

$$\langle v \rangle = \sqrt{\frac{8R_{\mathrm{mol}}T}{\pi M_{\mathrm{mol}}}} = \sqrt{\frac{8 \times 8.314 \times 300}{\pi \times 0.029}} \approx 4.68 \times 10^2\ \mathrm{m\,s^{-1}}, \tag{28}$$

$$v_{\mathrm{rms}} = \sqrt{\frac{3R_{\mathrm{mol}}T}{M_{\mathrm{mol}}}} = \sqrt{\frac{3 \times 8.314 \times 300}{0.029}} \approx 5.07 \times 10^2\ \mathrm{m\,s^{-1}}. \tag{29}$$

Thus, for air at room temperature,

$$v_{\mathrm{mp}} \approx 414\ \mathrm{m\,s^{-1}}, \quad \langle v \rangle \approx 468\ \mathrm{m\,s^{-1}}, \quad v_{\mathrm{rms}} \approx 507\ \mathrm{m\,s^{-1}}.$$

5.3.3 Derivation of the root-mean-square average of the center-of-mass velocity

Consider a collection of N identical molecules (a "material particle") taken from a large volume of gas. Let the instantaneous molecular velocities be $\mathbf{v}_n$ for $n = 1, 2, \ldots, N$. Collisions with other molecules in the gas at large will randomly change the velocity of each of the selected molecules, but if there is no overall drift in the gas, the velocity of individual molecules should average out to zero, $\langle \mathbf{v}_n \rangle = \mathbf{0}$; the velocities of different molecules should be uncorrelated, $\langle \mathbf{v}_n \cdot \mathbf{v}_m \rangle = 0$ for $n \neq m$; and the average of the square of the velocity should be the same for all molecules, $\langle v_n^2 \rangle = v_0^2$. The center-of-mass velocity is

$$\mathbf{V}_{\mathrm{cm}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{v}_n. \tag{30}$$

Its mean-square value is

$$\langle V_{\mathrm{cm}}^2 \rangle = \left\langle \left(\frac{1}{N} \sum_{n=1}^{N} \mathbf{v}_n \right) \cdot \left(\frac{1}{N} \sum_{m=1}^{N} \mathbf{v}_m \right) \right\rangle = \frac{1}{N^2} \sum_{n,m=1}^{N} \langle \mathbf{v}_n \cdot \mathbf{v}_m \rangle. \tag{31}$$

Using the assumptions, the cross terms vanish for $n \neq m$ and $\langle \mathbf{v}_n \cdot \mathbf{v}_n \rangle = \langle v_n^2 \rangle = v_0^2$, so

$$\left\langle V_{\rm cm}^2 \right\rangle = \frac{1}{N^2} \sum_{n=1}^{N} v_0^2 = \frac{N v_0^2}{N^2} = \frac{v_0^2}{N}. \tag{32}$$

Therefore the root-mean-square center-of-mass speed is

$$V_{\rm rms} = \sqrt{\left\langle V_{\rm cm}^2 \right\rangle} = \frac{v_0}{\sqrt{N}}. \tag{33}$$

5.3.4 Velocity fluctuations

Since the fluctuation scale is set by the molecular velocity, $v_{\rm mol} = v_{\rm rms}$, it takes much larger numbers of molecules to be able to ignore the fluctuations in everyday gas velocities. To maintain a relative precision ϵ in the velocity fluctuations we must require $\Delta v/v \lesssim \epsilon$, implying that the linear size of a gas volume must be larger than

$$L'_{\rm micro} = \left(\frac{v_{\rm mol}}{v} \right)^{2/3} L_{\rm micro}.$$

The velocity fluctuations of a gentle steady wind, say $v \approx 0.5$ m s^{-1}, can be ignored with precision $\epsilon \approx 10^{-3}$ for volumes of linear size larger than $L'_{\rm micro} = 100 L_{\rm micro} \approx 34$ μm. The smoothness scale should similarly be $L'_{\rm macro} = \epsilon^{-1} L'_{\rm micro}$, which in this case becomes $L'_{\rm macro} \approx$ 34 mm. A hurricane wind, $v \approx 50$ m s^{-1}, only requires volumes of linear size $L'_{\rm micro} = 4.6 L_{\rm micro} \approx$ 1.6 μm to yield the desired precision, but in this case fluctuations due to turbulence will anyway completely swamp the molecular fluctuations.

6 Fields

In a mathematical sense, material particles are taken to be truly infinitesimal and all physical properties of the particles as well as other physical observables are described by smooth functions of space and time. Continuum physics is therefore a theory of fields.

Mathematically, a field ϕ is simply a real-valued function.

$$\phi = \phi(\vec{x}, t) = \phi(x_1, x_2, x_3, t) = \phi(x, y, z, t). \tag{34}$$

In general, The fields can be classified as scalar, vector or tensor.

6.1 Scalar field

In this case, the field associates a specific number to every single point:

$$\phi = \phi(\vec{x}, t) = \phi(x_1, x_2, x_3, t) = \phi(x, y, z, t). \tag{35}$$

As examples of scalar fields we have:

- Density field: $\rho(\vec{x}, t)$.
- Temperature field: $T(\vec{x}, t)$.

The generalization to special relativity is

$$\phi = \phi(\vec{x}, ct) = \phi(x_1, x_2, x_3, x_4) = \phi(x^\mu). \tag{36}$$

6.2 Vector fields

In this case, the field associates a specific vector to every single point:

$$\vec{E} = \vec{E}(\vec{x}, t) = \vec{E}(x_1, x_2, x_3, t) = \vec{E}(x, y, z, t). \tag{37}$$

As examples of vector fields we have:

- Gravitational, electric and magnetic fields: $\vec{g}(\vec{x}, t)$, $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$.
- Velocity field: $\vec{v}(\vec{x}, t)$.

The generalization to special relativity is

$$E^\nu = E^\nu(x^\mu). \tag{38}$$

6.3 Tensor fields

In this case, the field associates a specific tensor to every single point:

$$\sigma_{ij} = \sigma_{ij}(\vec{x}, t) = \sigma_{ij}(x_1, x_2, x_3, t) = \sigma_{ij}(x, y, z, t). \tag{39}$$

As examples of tensor fields we have:

- Stress tensor: $\sigma_{ij}(\vec{x}, t)$.

The generalization to special relativity is

$$F^{\alpha\beta} = F^{\alpha\beta}(x^\mu). \tag{40}$$

6.4 Field equations

The dynamics of physical fields are governed by *field equations*, typically formulated as partial differential equations in both space and time. In continuum mechanics, the fundamental equations of motion arise from the local balance laws of mass, momentum, and energy, obtained by applying Newton's Second Law and the First Law of Thermodynamics to each material particle within the continuum. The resulting system comprises the continuity equation, the Navier–Stokes equations for momentum, and an energy balance, supplemented by constitutive relations for stresses and heat fluxes.

For a Newtonian fluid of density $\rho(\mathbf{x}, t)$, velocity field $\mathbf{v}(\mathbf{x}, t)$, and specific internal energy $e(\mathbf{x}, t)$, the governing equations are:

- Continuity (mass conservation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{41}$$

- Navier–Stokes (momentum balance):

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \left(\zeta + \tfrac{1}{3}\mu \right) \nabla(\nabla \cdot \mathbf{v}) + \rho \mathbf{f}, \tag{42}$$

- Energy (first law of thermodynamics):

$$\rho\left(\frac{\partial e}{\partial t}+\mathbf{v}\cdot\nabla e\right)=-p\,\nabla\cdot\mathbf{v}+\Phi+\nabla\cdot(k\nabla T)+\rho q, \tag{43}$$

where:

- p is the pressure,
- μ the shear viscosity,
- ζ the bulk viscosity,
- $\mathbf{f}$ external body forces (e.g. gravity),
- Φ the viscous dissipation function,
- k the thermal conductivity,
- T the absolute temperature,
- q a possible volumetric heat source.

In contrast to linear field theories such as Maxwell's equations of electromagnetism, Eqs. (41)–(43) are intrinsically nonlinear. The principal source of this nonlinearity is the advective term in the momentum and energy equations, e.g.

$$(\mathbf{v}\cdot\nabla)\mathbf{v},$$

which represents the self-interaction of the velocity field. This quadratic coupling makes the velocity field simultaneously the transported quantity and the agent of transport. As a consequence, the mathematical structure of continuum mechanics is profoundly more complex than that of linear systems.

Historically, the inviscid limit was first formulated by Euler in the mid-18th century, leading to what are now called the Euler equations of fluid motion. In the early 19th century, Navier and Stokes independently extended this framework by incorporating viscous stresses, arriving at the modern Navier–Stokes equations. This refinement established the foundation of fluid mechanics as a field theory and revealed the essential role of viscosity in damping and dissipating motion.

The nonlinear character of the Navier–Stokes equations underlies a range of rich dynamical phenomena, including the emergence of instabilities, bifurcations, and the transition to turbulence. Turbulent flows, in particular, are characterized by strongly nonlinear interactions across a hierarchy of scales and remain one of the central unsolved problems in classical physics. The mathematical challenge of establishing the global existence and smoothness of solutions to the three-dimensional Navier–Stokes equations is recognized as one of the seven Millennium Prize Problems of the Clay Mathematics Institute.

6.4.1 From a linear chain to the Euler–Lagrange field equation

Discrete model. Consider an infinite 1D chain of identical masses m at positions $x_n = na$ (lattice spacing a), with displacements $u_n(t)$ and nearest-neighbor springs of constant k. A natural Lagrangian is

$$L[u_n] = \sum_{n\in\mathbb{Z}} \left[\frac{m}{2}\,\dot{u}_n^2 - \frac{k}{2}\,(u_{n+1} - u_n)^2 \right]. \tag{44}$$

The discrete Euler–Lagrange equations give the equations of motion

$$m\,\ddot{u}_n + k\,(2u_n - u_{n+1} - u_{n-1}) = 0. \tag{45}$$

Continuum limit. Introduce a smooth field $u(x,t)$ with $u_n(t) \approx u(x_n, t)$ and expand

$$u_{n\pm1}(t) = u(x_n \pm a, t) = u \pm a\,\partial_x u + \tfrac{a^2}{2}\,\partial_x^2 u + \mathcal{O}(a^3). \tag{46}$$

Then

$$2u_n - u_{n+1} - u_{n-1} \;=\; -a^2\,\partial_x^2 u + \mathcal{O}(a^4). \tag{47}$$

Assume a finite linear mass density ρ via $m = \rho a$. Passing from sums to integrals, $\sum_n a \to \int dx$, we obtain in the $a \to 0$ limit

$$\rho\,\partial_t^2 u - \tau\,\partial_x^2 u = 0, \qquad \text{with}\;\; \tau \equiv ka, \quad c^2 \equiv \frac{\tau}{\rho}. \tag{48}$$

This is the 1D wave equation with speed c.

Field-theoretic form. The continuum Lagrangian corresponding to the chain is

$$L[u] = \int_{\mathbb{R}} dx\,\mathcal{L}(u, \partial_t u, \partial_x u), \qquad \mathcal{L} = \frac{\rho}{2}\,(\partial_t u)^2 - \frac{\tau}{2}\,(\partial_x u)^2. \tag{49}$$

The field Euler–Lagrange equation follows from $\delta S = 0$ for the action $S = \int dt\, L[u]$:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t u)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x u)}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0 \;\Longrightarrow\; \rho\,\partial_t^2 u - \tau\,\partial_x^2 u = 0. \tag{50}$$

With on-site potential and higher dimensions. If the springs are attached to a substrate or there is a local restoring term, add a potential $V(u)$:

$$\mathcal{L} = \frac{\rho}{2}\,(\partial_t u)^2 - \frac{\tau}{2}\,|\nabla u|^2 - V(u) \quad\Longrightarrow\quad \rho\,\partial_t^2 u - \tau\,\nabla^2 u + V'(u) = 0. \tag{51}$$

In relativistic $d{+}1$ form for a real scalar field ϕ,

$$S[\phi] = \int d^{d+1}x\,\mathcal{L}(\phi, \partial_\mu \phi), \qquad \mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi), \tag{52}$$

the Euler–Lagrange equations read

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \iff \Box\phi + V'(\phi) = 0, \qquad \Box \equiv \partial_\mu\partial^\mu. \tag{53}$$

Remarks on parameters. For an elastic rod (cross-section A, Young's modulus E), one identifies $\tau = EA$ and ρ as mass per unit length; then $c = \sqrt{E/\rho_{3\mathrm{D}}}$ with $\rho_{3\mathrm{D}}$ the volumetric mass density.

6.5 Is matter really discrete or continuous?

In relativistic quantum field theory, the fundamental constituents of matter—electrons, protons, neutrons, nuclei, atoms, and beyond—are described as quantum excitations of underlying fields. Developed between the 1920s and 1970s, this framework has become one of the most successful and far-reaching achievements of modern physics. Nevertheless, it does not incorporate gravitation, and a consistent unification of quantum field theory with general relativity remains an open challenge.

6.6 What is continuum mechanics

Although often used interchangeably, the terms have slightly different emphases:

- *Continuum mechanics*: A formal branch of mechanics that models matter as a continuous medium, ignoring its atomic structure. Physical quantities such as density, velocity, stress, and strain are treated as continuous functions of space and time. It provides a theoretical framework that includes elasticity, fluid mechanics, plasticity, and other fields.

- *Continuous mechanics*: A less common expression, often used synonymously with continuum mechanics. In physics-oriented contexts it may emphasize the study of systems made of continuous matter (e.g., fluids, solids, plasmas), rather than the abstract mathematical framework.

7 Review questions

1. Plot the quark and charged lepton masses in some way which suggest regularities, i.e, family number versus mass. If there us a fourth family, what masses might you predict for the members?. What could you infer about the neutrino masses in comparison with the rest of masses?

8 Problems

1. Convert the following quantities by inserting the appropriate factors of $c, \hbar$, G_N and k_B:

   a) $T_0 = 2.725$ K $\to$ eV.

   b) $\rho_\gamma = \pi^2 T_0^4/15 \to$ eV^4 and g/cm^3.

   c) $M_P \equiv 1.2 \times 10^{19}$ GeV $\to$ K.

2. The range of length scales involved in cosmology are hard to grasp. The best we can do is to consider relative distances and compare them to something more familiar. In this exercise, we will make some attempt at obtaining a more intuitive understanding of the vastness of the cosmos.

a) Consider shrinking the Earth to the size of a basketball. What would then be the size of the Moon and its orbit around the Earth?

b) Now imagine scaling the Earth down to the size of a peppercorn. What would then be the size of the Sun and the Earth's orbit? How far away would the most distant planet in the Solar System be?

c) The "Solar Neighborhood" is a collection of about fifty nearby stars, spread across about 65 light-years, that travel together with the Sun. Scaling this region down, so that it fits inside a basketball court, what would be the size of the Solar System?.

d) Shrinking our Galaxy to the size of the basketball court, what would now be the size of the Solar Neighbourhood?

e) The "Local Group" comprises about fifty nearby galaxies, spread across about 10 million light-years. If we squeezed this region into the size of the basketball court, what would be the size of our Milky Way galaxy?

f) The largest structures in the universe, like our "Local Supercluster," are about 500 million light-years across. Scaling these superclusters down to the dimensions of the basketball court, what would be the size of our Local Group?

g) This radius of the observable universe is 46.5 billion light-years. Compressing the observable universe to the size of the basketball court, what would be the size of the largest superclusters?

3. A key parameter in cosmology is the Hubble constant

$$H_0 = 70 \text{ km/s/Mpc}.$$

In the following, you will use the measured value of the Hubble constant to estimate a few fundamental scales of our universe.

a) What is the Hubble time $t_{H_0} \equiv H_0^{-1}$ in years? This is a rough estimate of the age of the universe.

b) What is the Hubble distance $d_{H_0} \equiv cH_0^{-1}$ in meters? This is a rough estimate of the size of the observable universe.

c) The average density of the universe today is

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G_N},$$

where G_N is the Nweton's constant. What is $\rho_{c,0}$ in g/cm^{-3}? How does this compare to the density of water.

d) Let us assume that the universe is filled with only hydrogen atoms. What is then the total number of atoms in the universe? How does this compare to the number of hydrogen atoms in your brain?
Hint: Assume that the brain is mostly water. Use $m_\text{H} = 2 \times 10^{-24}$ g and $m_{\text{H}_2\text{O}} = 3 \times 10^{-23}$ g.

4. The typical mass of a galaxy is given by $10^{12}\,\text{M}_\odot$. Assuming that the matter in form of nucleons (that, is in protons and neutrons) contribute 1/5 to the mass of a galaxy, estimate the typical number of nucleons in a galaxy.

5. Suppose that the Milky Way galaxy is a typical size, containing say 10^{11} stars, and that galaxies are typically separated by a distance of one Mpc. Estimate the density of the Universe in SI units. How does this compare with the density of the Earth?

A Units used in Cosmology

- Astronomical unit (AU), equal to the mean distance between the Earth and Sun:

$$1\,\mathrm{AU} = 1.5 \times 10^{11}\,\mathrm{m}. \tag{54}$$

  - Average distance from the Sun to Neptune: 30.1 A.U.
  - Distance from the Sun to Voyager 1 in October 2024: 165 A.U.
  - Distance light travels in one day: 173 A.U.
  - Distance light travels in one Julian year (365.25 days): 63241 A.U.
  - Distance of the outer limit of Oort cloud from the Sun: 75000 A.U.
  - Distance to Proxima Centauri (Sun's nearest neighboring star): 268000 A.U.

- Parsec (pc), equal to the distance at which 1 AU subtends an angle of 1 arcsecond:

$$1\,\mathrm{pc} = 3.09 \times 10^{16}\,\mathrm{m}. \tag{55}$$

  - Distance to Proxima Centauri: 1.3 pc. The distance to Arcturus and Capella is more than 10 pc, the distances to Canopus and Betelgeuse are about 100 pc and 200 pc, respectively; Crab Nebula - the remnant of supernova seen by naked eye - is 2 kpc away from us.
  - Distance to the center of our galaxy: 8.5 kpc. Our Galaxy is of spiral type, the diameter of its disc is about 30 kpc and the thickness of the disc is about 250 pc.
  - The distance to the nearest dwarf galaxies, satellites of our Galaxy, is about 30 kpc. Fourteen of these satellites are known; the largest of them — Large and Small Magellanic Clouds — are 50 kpc away. Search for new, dimmer satellite dwarfs is underway; we note in this regard that only eight of Milky Way satellites were known by 1994.
  - The nearest "usual" galaxy –the spiral galaxy M31 in Andromeda constellation — is 760 kpc away from the Milky Way. Despite the large distance, it occupies a sizeable area on the celestial sphere: its angular size is larger than that of the Moon! Another nearby galaxy is in the Triangulum constellation. Our Galaxy together with the Andromeda and Triangulum galaxies, their satellites and other 35 smaller galaxies constitute the Local Group, the gravitationally bound object consisting of more than 50 galaxies.
  - Distance to the Virgo cluster (the nearest big cluster of galaxies): 16.5 Mpc. Its angular size is about 5 degrees. Clusters of galaxies are the largest gravitationally bound systems in the Universe.
  - About 20 superclusters are known by now. The Local Group belongs to a supercluster with the center in the direction of Virgo constellation. The size of this supercluster is about 30 Mpc, and besides the Virgo cluster and Local Group it contains about a hundred groups and clusters of galaxies. The nearest to Virgo is the supercluster in the Hydra and Centaurus constellations; its distance to the Virgo supercluster is about half a hundred megaparsec. Distance to the Coma cluster: 99 Mpc.

- Distance to QSO J0313-1806 (the most distant, and hence also the oldest known quasar at $z = 7.64$): 900 Mpc.
- Radius of observable Universe: $\sim$14000 Mpc.

- Solar Mass ($\mathrm{M}_\odot$):

$$1\,\mathrm{M}_\odot = 1.99 \times 10^{30}\,\mathrm{kg}. \tag{56}$$

  - The most massive confirmed exoplanet is Iota Draconis b, which has 16.4 times the mass of Jupiter, that is $16.4 \times 0.00095\,\mathrm{M}_\odot \sim 0.01558\,\mathrm{M}_\odot$.
  - The largest black hole merger detected by LIGO to date was GW190521, which created a black hole with a mass of 142 solar masses.
  - R136a1 is one of the most massive and luminous stars known: $\sim 200\,\mathrm{M}_\odot$.
  - The Milky Way's central black hole, Sagittarius A*, has a mass of $4.3 \times 10^6\,\mathrm{M}_\odot$.
  - The M87* supermassive black hole has a mass of $6.5 \times 10^9\,\mathrm{M}_\odot$.
  - Mass of the Milky Way galaxy: $M_{\mathrm{MW}} \sim 10^{12}\,\mathrm{M}_\odot$.
  - Mass of the Coma cluster: $M_{\mathrm{Coma}} \sim 7 \times 10^{14}\,\mathrm{M}_\odot$.
  - The Laniakea Supercluster encompasses approximately 100,000 galaxies stretched out over 160 Mpc. It has the approximate mass of $10^{17}\,\mathrm{M}_\odot$.

- Sun's luminosity ($\mathrm{L}_\odot$):

$$1\,\mathrm{L}_\odot = 3.83 \times 10^{26}\,\mathrm{watts}. \tag{57}$$

  - The peak luminosity of SN 1987A, the famous supernova observed in the Large Magellanic Cloud in 1987, was approximately $10^9\,\mathrm{L}_\odot$.
  - Approximately luminosity of the Milky Way galaxy: $L_{\mathrm{MW}} \sim 3 \times 10^{10}\,\mathrm{L}_\odot$.
  - The luminosity of QSO J0313–1806 is approximately $3.6 \times 10^{13}\,\mathrm{L}_\odot$.

- Age of the Universe (t_U):

$$t_U = 13.8\,\mathrm{Gyr}. \tag{58}$$

  - Matter-dark energy equality: 5.1 Gy ago, that is, $t_{M\Lambda} = 8.8 \times 10^9$ y.
  - Time of formation of CMB (recombination): 380 ky.
  - Radiation-Matter equality: $t_{RM} = 50 \times 10^3$ y.
  - Time of formation of light nuclei (Big Bang Nucleosynthesis): 1 s.
  - Time of electroweak symmetry breaking: 10^{-12} s.

B Planck units

The Planck system is based on the combination of four universal constants:

$$G_N = 6.67 \times 10^{-11} \, \frac{\mathrm{m}^3}{\mathrm{kg\ sec}^2}, \tag{59}$$

$$c = 299792458 \, \frac{\mathrm{m}}{\mathrm{sec}}, \tag{60}$$

$$\hbar = 1.055 \times 10^{-34} \, \mathrm{J\,s} = 6.582 \times 10^{-25} \, \mathrm{GeV\,s}, \tag{61}$$

$$k_B = 1.38 \times 10^{-23} \, \frac{\mathrm{J}}{\mathrm{K}} = 8.617 \times 10^{-14} \, \frac{\mathrm{GeV}}{\mathrm{K}}, \tag{62}$$

with $1\,\mathrm{eV} = 1.602 \times 10^{-19}$ J. The Planck length, mass and time are defined as

$$\ell_P \equiv \left(\frac{G_N \hbar}{c^3}\right)^{1/2} = 1.62 \times 10^{-35} \, \mathrm{m}, \tag{63}$$

$$M_P \equiv \left(\frac{\hbar c}{G_N}\right)^{1/2} = 2.18 \times 10^{-8} \, \mathrm{kg}, \tag{64}$$

$$t_P \equiv \left(\frac{G_N \hbar}{c^5}\right)^{1/2} = 5.39 \times 10^{-44} \ \mathrm{sec}\,. \tag{65}$$

The Planck energy is defined as

$$E_P \equiv M_P c^2 = 1.96 \times 10^9 \, \mathrm{J} = 1.22 \times 10^{19} \, \mathrm{GeV}, \tag{66}$$

while the Planck temperature becomes

$$T_P \equiv E_P / k_B = 1.42 \times 10^{32} \, \mathrm{K}. \tag{67}$$

C Natural units

When distance, mass, time, and temperature are measured in the appropriate Planck units, then $c = k_B = \hbar = G_N = 1$. For instance, the Einstein's equation for energy, $E = mc^2$ becomes $E = m$.

From $\hbar = 1$ it follows that

$$1\,\mathrm{GeV} = \frac{1}{6.6 \times 10^{-25}} \, \mathrm{s}^{-1}. \tag{68}$$

From $c = 1$ we obtain

$$1\,\mathrm{s} = 3 \times 10^8 \, \mathrm{m}. \tag{69}$$

From this two expressions

$$10^{-15} \, \mathrm{m} \approx 5 \, \mathrm{GeV}^{-1}. \tag{70}$$

Hence

$$1\,\mathrm{mb} \equiv 10^{-31} \, \mathrm{m}^2 \approx 2.56 \, \mathrm{GeV}^{-2}, \tag{71}$$

or

$$1\,\mathrm{GeV}^2 \approx 0.39 \times 10^{-31} \, \mathrm{m}^{-2}. \tag{72}$$

The mass of the electron in natural units can be expressed as

$$m_e \approx 0.5 \, \mathrm{GeV} = 0.25 \times 10^{13} \, \mathrm{m}^{-1} = 0.77 \times 10^{21} \, \mathrm{s}^{-1}. \tag{73}$$

D Recognitions in fluid mechanics and related fields

- 1913 – Heike Kamerlingh Onnes (Nobel Prize in Physics): Investigations of matter at low temperatures, liquefaction of helium, and discovery of superconductivity.
- 1968 – Lars Onsager (Nobel Prize in Chemistry): Reciprocal relations in irreversible processes, fundamental to transport theory and hydrodynamics.
- 1977 – Ilya Prigogine (Nobel Prize in Chemistry): Contributions to nonequilibrium thermodynamics and the theory of dissipative structures.
- 1978 – Pyotr Kapitsa (Nobel Prize in Physics): Basic inventions and discoveries in low-temperature physics, including superfluidity in helium-4.
- 1991 – Pierre-Gilles de Gennes (Nobel Prize in Physics): Theoretical advances in liquid crystals, polymers, and complex fluids.
- 1996 – David M. Lee, Douglas D. Osheroff, Robert C. Richardson (Nobel Prize in Physics): Discovery of superfluidity in helium-3.
- 2001 – Eric A. Cornell, Wolfgang Ketterle, Carl E. Wieman (Nobel Prize in Physics): Realization of Bose–Einstein condensation in dilute atomic gases.
- 2012 – Sylvia Serfaty (Henri Poincaré Prize): Work on vortex dynamics in Ginzburg–Landau models, closely linked to fluid and superconducting systems.
- 2016 – Barbara L. White (APS Fluid Dynamics Prize): Fundamental contributions to complex fluids, turbulence modeling, and polymer flows.
- 2019 – Karen Uhlenbeck (Abel Prize): Groundbreaking contributions to nonlinear PDEs and geometric analysis, with deep implications for mathematical fluid mechanics.
- 2021 – Beverley McKeon (APS Stanley Corrsin Award): Pioneering studies of wall-bounded turbulence and flow control.
- Ellen Kuhl (Stanford University): Recognized for advances in computational fluid dynamics applied to biological systems; elected to the U.S. National Academy of Engineering.
- Elisabeth Guazzelli (CNRS, France): International leader in the physics of particle-laden flows, suspensions, and sedimentation; recipient of multiple international recognitions.
- Helen Czerski (University College London): Widely recognized for studies of bubble physics and ocean surface dynamics, as well as for science communication in fluid mechanics.

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