Solids at Rest

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3] books.

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC9. Enumerar los distintos regímenes de deformación a los que está sometido un sólido o un fluido viscoelástico.
- OC10. Definir el campo de desplazamientos, el tensor de deformaciones y el tensor de elasticidad.
- OC11. Enunciar la ley de Hooke en toda su generalidad.

1.2 Objetivos específicos procedimentales:

- OP8. Escribir el tensor de esfuerzos y la ley de Cauchy.
- OP9. Determinar la relación entre las fuerzas experimentadas por un elemento de volumen en un medio continuo, incluyendo la presión, y el tensor de esfuerzos.

1.3 Objetivos específicos actitudinales:

- OA3. Reconocer la importancia de la notación, del álgebra y del cálculo tensorial en la representación de las cantidades físicas asociadas con los medios continuos.

2 Stress

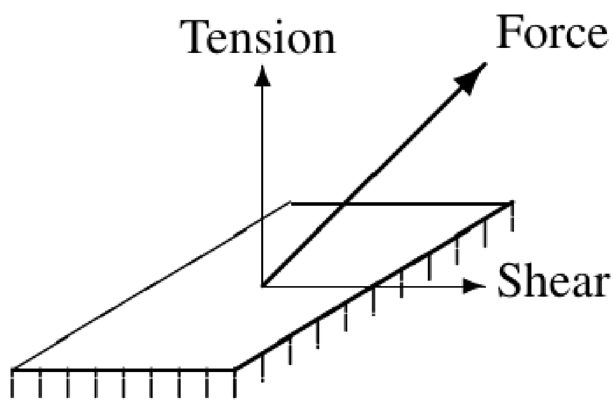
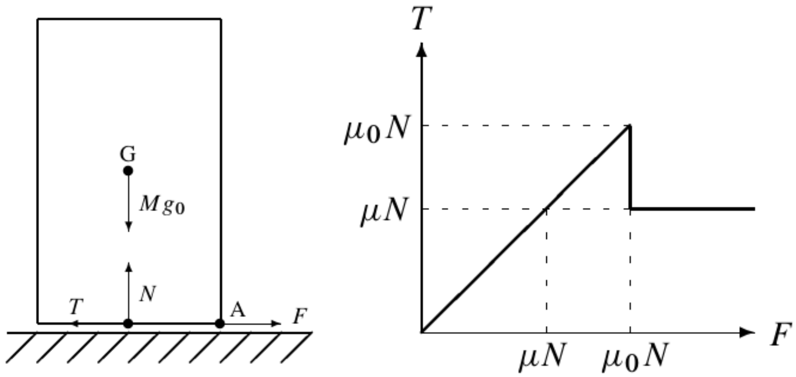


Figure 1: The force on a small piece of a surface can be resolved into a nor- mal pressure force and a tangential shear force. Figure taken from Ref. [3].

- In fluids at rest, the only contact force is pressure, but in solids and viscous fluids in motion, additional *shear stresses* arise—forces acting tangentially to contact surfaces.
- Shear stress is the shear force per unit area and plays a crucial role in maintaining the structural integrity of solids.
- Friction in everyday life is a manifestation of shear stress at contact surfaces between materials.
- Materials are broadly classified as:
  - **Fluids**, which respond to stress by *flowing*.
  - **Solids**, which respond by *deforming*.
- Elastic deformation grows linearly with stress but can lead to permanent plastic deformation or rupture at high stress levels.
- The study of stress applies to all materials—solids, fluids, and intermediate forms—including artificial and exotic substances with special technological uses.
- Contact forces depend not only on spatial position but also on the orientation of the surface; therefore, a more general description using the *stress tensor* is required.
- The stress tensor, consisting of nine components, describes the complete range of contact forces acting within a material.
- Understanding the concept and properties of stress is fundamental to continuum mechanics and central to this chapter.



	μ_0	μ
Glass/glass	0.9	0.4
Rubber/asphalt	0.9	0.7
Steel/steel	0.7	0.6
Metal/metal	0.6	0.4
Wood/wood	0.4	0.3
Steel/ice	0.1	0.05
Steel/teflon	0.05	0.05

Figure 2: Left: Balance of forces on a crate at rest on a horizontal floor. Center: Sketch of tangential reaction (traction) T as a function of applied force F. Up to $F = \mu_0 N$, the traction adjusts itself to the applied force, $T = F$. At $F = \mu_0 N$, the tangential reaction drops abruptly to a lower value, $T = \mu N$, and stays there regardless of the applied force. Right: Typical friction coefficients for various combinations of materials. Figures taken from Ref. [3].

2.1 Friction

- **Friction as a shear force:**
  - Friction between solid bodies is a type of shear force.
  - It enables us to hold, grab, drag, and rub objects without them slipping.
  - Everyday tasks, from stirring coffee to lighting a fire, involve working against friction.
- **Static friction:**
  - Occurs when an object is at rest and resists being set into motion.
  - The static frictional force T balances the applied force F up to a maximum value (see Fig. 2):
    $$T < \mu_0 N,$$
    where μ_0 is the *coefficient of static friction* and N is the normal load.
  - The coefficient μ_0 is dimensionless and typically around 0.5 or higher in everyday materials.
  - Its magnitude depends on the materials in contact and the roughness of their surfaces.
- **Dynamic (kinetic) friction:**
  - Once motion starts, friction still acts but with a smaller magnitude than static friction.
  - The frictional force is proportional to the normal load:
    $$T = \mu N,$$
    where μ is the *coefficient of dynamic friction*, satisfying $\mu < \mu_0$.
  - After motion begins, a smaller force $F = \mu N$ maintains constant speed.

    - Work done against dynamic friction is converted into heat:

$$P = FU,$$

      where P is the power and U is the velocity.
    - Static and dynamic friction are both independent of the size of the contact area, so that a crate on legs is as hard to drag as one without. The best way to diminish the force necessary to drag the crate is to place it on wheels.

- **Comparison:**
    - Static friction prevents motion; dynamic friction resists motion once it starts.
    - Static friction does not produce work, while dynamic friction continuously dissipates energy as heat.

2.2 Stress fields

- Stress fields describe the local distribution of normal and tangential forces acting on a surface.
- Measured in pascals (Pa = N m^{-2}), analogous to pressure.
- Types of stresses:
    - *Tension stress:* acts along the normal and pulls outward.
    - *Pressure stress:* acts along the normal and pushes inward.
    - *Shear stress:* acts tangentially along the surface.

2.2.1 External and internal stresses:

- *External stresses* act between a body and its environment.
- *Internal stresses* act across imaginary surfaces within the body and exist even in the absence of external forces.
- Internal stresses depend on material composition, geometry, temperature, and external loading.
- Rapid temperature changes (e.g., fast cooling) can permanently "freeze" internal stresses, especially in brittle materials like glass.

2.2.2 Tensile strength and yield stress:

- When external forces grow large, materials may deform plastically or fracture.
- *Tensile strength:* the maximum stress a material can withstand before failure.
- *Yield stress:* the stress at which plastic deformation begins.
- *Ductile materials* (e.g., copper) can stretch without breaking; *brittle materials* (e.g., cast iron) fail suddenly.

- Typical tensile strengths:
  - Metals: several hundred megapascals.
  - Carbon fibers: several gigapascals.
  - Carbon nanotubes: up to 50 GPa.

2.3 The stress tensor

- Stresses at a point are described by nine components σ_{ij} in Cartesian coordinates:

$$\sigma_{ij} = \frac{dF_i}{dS_j}. \tag{1}$$

- These include three normal stresses (σ_{xx}, σ_{yy}, σ_{zz}) and six shear stresses.
- By convention, the sign is chosen such that a positive value of σ_{yy} corresponds to a pull or tension.

2.3.1 Cauchy's stress hypothesis

This asserts that the force

$$d\vec{F} = (dF_x, dF_y, dF_z) \tag{2}$$

on an arbitrary surface element,

$$d\vec{S} = (dS_x, dS_y, dS_z), \tag{3}$$

is of the form

$$\begin{aligned} dF_x &= \sigma_{xx}dS_x + \sigma_{xy}dS_y + \sigma_{xz}dS_z, \\ dF_y &= \sigma_{yx}dS_x + \sigma_{yy}dS_y + \sigma_{yz}dS_z, \\ dF_z &= \sigma_{zx}dS_x + \sigma_{zy}dS_y + \sigma_{zz}dS_z. \end{aligned} \tag{4}$$

The components of the force are expressed compactly as

$$dF_i = \sum_j \sigma_{ij}\, dS_j. \tag{5}$$

- The nine stress components form the *stress tensor field* $\sigma_{ij}(x, t)$.
- The stress tensor is written as

$$\vec{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{6}$$

- The stress acting on a surface element $d\vec{S}$ is given by

$$d\vec{F} = \vec{\sigma} \cdot d\vec{S}. \tag{7}$$

- Writing $d\vec{S} = \hat{n}\, dS$, the stress per unit area becomes

$$\frac{d\vec{F}}{dS} = \vec{\sigma} \cdot \hat{n}, \tag{8}$$

called the *stress vector*.

- Although $\vec{\sigma} \cdot \hat{n}$ behaves like a vector, it is not a true vector field since it depends on the surface orientation.

Proof of Cauchy's stress hypothesis:

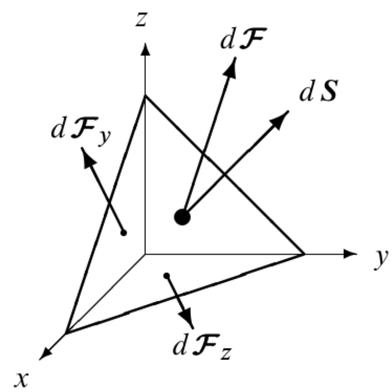


Figure 3: The tiny triangle and its projections form a tetrahedron. The (vector) force acting on the triangle in the xy-plane is called $d\vec{F}_z$, and the force acting on the triangle in the zx-plane is $d\vec{F}_y$. The force $d\vec{F}_x$ acting on the triangle in the yz-plane is hidden from view. The force $d\vec{F}$ acts on the skew triangle. Figure taken from Ref. [3].

Let's take again a surface element in the shape of a tiny triangle (see Fig. 3) with area vector

$$d\vec{S} = (dS_x, dS_y, dS_z).$$

This triangle and its projections on the coordinate planes form together a little body in the shape of a right tetrahedron. Let the external (vector) forces acting on the three triangular faces of the tetrahedron be denoted, respectively,

$$d\vec{F}_x, \; d\vec{F}_y, \text{ and } d\vec{F}_z.$$

Adding the external force $d\vec{\mathcal{F}}$ acting on the skew face and a possible volume force $\vec{f}\, dV$, Newton's Second Law for the small tetrahedron becomes

$$\vec{a}_{\text{CM}}\, dM = \vec{f}\, dV + d\vec{F}_x + d\vec{F}_y + d\vec{F}_z + d\vec{\mathcal{F}}, \tag{9}$$

where $\vec{a}_{\text{CM}}$ is the center-of-mass acceleration of the tetrahedron and $dM = \rho\, dV$ its mass.

The volume of the tetrahedron scales like the third power of its linear size, whereas the surface area only scales like the second power. Making the tetrahedron progressively smaller, the body force term and the acceleration term will vanish faster than the surface terms. In the limit of a truly infinitesimal tetrahedron, only the surface terms survive, so that we must have

$$d\vec{F}_x + d\vec{F}_y + d\vec{F}_z + d\vec{F} = 0.$$

Taking into account that the area projections dS_x, dS_y, and dS_z point into the tetrahedron, we define the stress vectors along the coordinate directions as

$$\vec{\sigma}_x = -\frac{d\vec{F}_x}{dS_x}, \quad \vec{\sigma}_y = -\frac{d\vec{F}_y}{dS_y}, \quad \vec{\sigma}_z = -\frac{d\vec{F}_z}{dS_z}.$$

Consequently,

$$d\vec{F} = \vec{\sigma}_x \, dS_x + \vec{\sigma}_y \, dS_y + \vec{\sigma}_z \, dS_z. \tag{10}$$

This shows that the force on an arbitrary surface element may be written as a linear combination of three basic stress vectors along the coordinate axes. Introducing the nine coordinates of the three stress vectors,

$$\vec{\sigma}_x = (\sigma_{xx}, \sigma_{yx}, \sigma_{zx}), \quad \vec{\sigma}_y = (\sigma_{xy}, \sigma_{yy}, \sigma_{zy}), \quad \vec{\sigma}_z = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz}),$$

we arrive at Cauchy's hypothesis.

2.4 Total force

Including a volume force density f_i, the total force on a body of volume V with surface S becomes

$$F_i = \int_V f_i \, dV + \oint_S \sum_j \sigma_{ij} \, dS_j. \tag{11}$$

Using Gauss' theorem, this may be written as a single volume integral,

$$F_i = \int_V f_i^* \, dV, \tag{12}$$

where

$$f_i^* = f_i + \sum_j \nabla_j \sigma_{ij}, \tag{13}$$

is the *effective force density*. The effective force is not just a formal quantity, because the total force on a material particle is $d\vec{F} = \vec{f}^* \, dV$. In matrix notation this reads

$$\vec{f}^* = \vec{f} + \vec{\nabla} \cdot \vec{\sigma}^{\mathrm{T}}, \tag{14}$$

where $\vec{\sigma}^{\mathrm{T}}$ is the transposed matrix, defined as $\sigma^{\mathrm{T}}_{ji} = \sigma_{ij}$.

In other words, the **total contact force** on a small box-shaped material particle (see Fig. 4) is calculated from the differences of the stress vectors acting on the sides. Suppressing the dependence on y and z, the resultant contact force on dS_x becomes

$$d\vec{F} = (\vec{\sigma}_x(x + dx) - \vec{\sigma}_x(x)) \, dS_x \approx \nabla_x \vec{\sigma}_x \, dV.$$

Adding the similar contributions from dS_y and dS_z, one arrives at the total surface force,

$$d\vec{F} = (\nabla_x \vec{\sigma}_x + \nabla_y \vec{\sigma}_y + \nabla_z \vec{\sigma}_z) \, dV = \nabla \cdot \vec{\sigma}^{\mathrm{T}} \, dV.$$

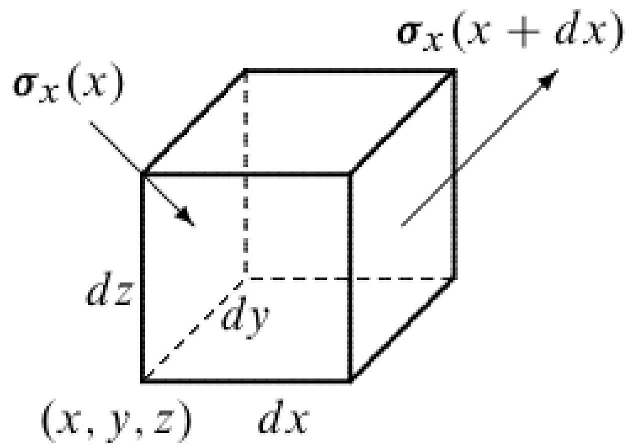


Figure 4: The total contact force on a small box-shaped material particle.

2.4.1 Mechanical Pressure

- **Hydrostatic equilibrium:** When the only contact force is pressure, the stress tensor is isotropic:
$$\boldsymbol{\sigma} = -p\,\mathbf{1}, \qquad \text{or equivalently,} \qquad \sigma_{ij} = -p\,\delta_{ij},$$
where $\mathbf{1}$ is the unit matrix and δ_{ij} is the Kronecker delta.

- **General case:** In real materials, the stress tensor usually has off-diagonal (shear) components. The diagonal components (σ_{xx}, σ_{yy}, σ_{zz}) differ and correspond to the normal stresses, often interpreted as "pressures":
$$p_x = -\sigma_{xx}, \qquad p_y = -\sigma_{yy}, \qquad p_z = -\sigma_{zz}.$$
These components do *not* form a true vector, since they do not transform properly under rotation.

The *mechanical pressure* is defined as the average of the normal stresses:

$$p = \frac{1}{3}(p_x + p_y + p_z) = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}). \tag{15}$$

This definition ensures that pressure is a scalar field, invariant under Cartesian coordinate transformations, because it depends on the *trace* of the stress tensor:

$$\mathrm{Tr}\,\vec{\sigma} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}. \tag{16}$$

The mechanical pressure corresponds to the isotropic part of the stress tensor. It represents the mean normal stress acting equally in all directions and is the only scalar combination of stress components that remains invariant under coordinate changes.

2.5 Mechanical equilibrium

In mechanical equilibrium, the total force on any body must vanish, for if it does not the body will begin to move. So the general condition is that $\vec{F} = 0$ for all volumes V. In particular, requiring that the force on each and every material particle must vanish, we arrive at *Cauchy's equilibrium equation(s)*:

$$f_i^* = f_i + \sum_j \nabla_j \sigma_{ij} = 0. \tag{17}$$

In spite of their simplicity, these partial differential equations govern mechanical equilibrium in all kinds of continuous matter, be it solid, fluid, or otherwise. For a fluid at rest, where pressure is the only stress component, we have $\sigma_{ij} = -p\,\delta_{ij}$, and recover the equation of hydrostatic equilibrium, $f_i - \nabla_i p = 0$.

The three individual equations contained in Cauchy's equilibrium equation read

$$\begin{aligned} f_x + \nabla_x \sigma_{xx} + \nabla_y \sigma_{xy} + \nabla_z \sigma_{xz} &= 0, \\ f_y + \nabla_x \sigma_{yx} + \nabla_y \sigma_{yy} + \nabla_z \sigma_{yz} &= 0, \\ f_z + \nabla_x \sigma_{zx} + \nabla_y \sigma_{zy} + \nabla_z \sigma_{zz} &= 0. \end{aligned} \tag{18}$$

These equations are in themselves not sufficient to determine the state of continuous matter, but must be supplemented by suitable *constitutive equations* connecting stress and state. For fluids at rest, the equation of state serves this purpose by relating hydrostatic pressure to mass density and temperature. In elastic solids, the constitutive equations are more complicated and relate stress to displacement.

2.5.1 Symmetry

There is one very general condition which may normally be imposed, namely the symmetry of the stress tensor:

$$\sigma_{ij} = \sigma_{ji} \quad \text{or} \quad \vec{\sigma}^T = \vec{\sigma}. \tag{19}$$

Symmetry only affects the shear stress components, requiring

$$\sigma_{xy} = \sigma_{yx}, \qquad \sigma_{yz} = \sigma_{zy}, \qquad \sigma_{zx} = \sigma_{xz}, \tag{20}$$

and thus reduces the number of independent stress components from nine to six.

Being thus a symmetric matrix, the stress tensor may be diagonalized. The eigenvectors define the principal directions of stress and the eigenvalues the principal tensions or stresses. In the principal basis, there are no off-diagonal elements, i.e., shear stresses, only pressures. The principal basis is generally different from point to point in space.

There is in fact no general proof of symmetry of the stress tensor, but only some theoretical arguments that allow us to choose the stress tensor to be symmetric in all normal materials. Here we shall present a simple argument, only valid in complete mechanical equilibrium.

Consider a material particle in the shape of a tiny rectangular box with sides a, b, and c (see Fig. 5). The force acting in the y-direction on a face in the x-plane is $\sigma_{yx}bc$ whereas the force acting in the x-plane on a face in the y-plane is $\sigma_{xy}ac$. On the opposite faces the contact forces have opposite sign in mechanical equilibrium (their difference is, as we have seen, of order abc). Since the total force vanishes, the total moment of force on the box may be calculated around any point we wish. Using the lower left corner, we get

$$\mathcal{M}_z = a\,\sigma_{yx}bc - b\,\sigma_{xy}ac = (\sigma_{yx} - \sigma_{xy})abc.$$

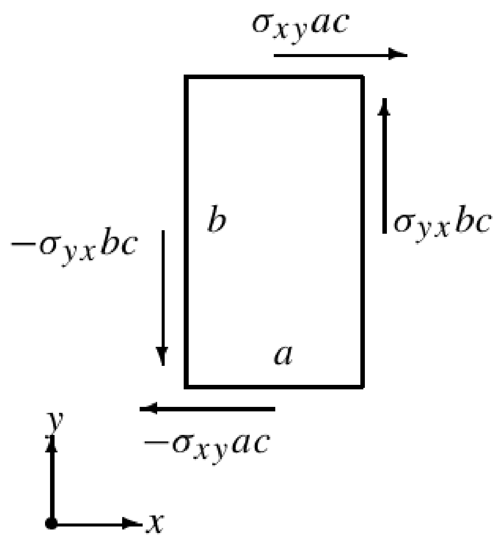


Figure 5: An asymmetric stress tensor will produce a non-vanishing moment of force on a small box (the z- direction not shown). Figure taken from Ref. [3].

This shows that if the stress tensor is asymmetric, $\sigma_{xy} \neq \sigma_{yx}$, there will be a resultant moment on the box. In mechanical equilibrium this cannot be allowed, since such a moment would begin to rotate the box, and consequently the stress tensor must be symmetric. Conversely, when the stress tensor is symmetric, mechanical equilibrium of the forces alone guarantees that all local moments of force will vanish.

2.5.2 Boundary Conditions

- **Cauchy's equation of mechanical equilibrium** requires appropriate boundary conditions on the surface of a body, since it is valid only within the body's volume.

- The **stress tensor** $\vec{\sigma}$ is a local physical quantity that can be assumed continuous in regions where the material properties vary smoothly.

- Across real boundaries or interfaces, where material properties may change abruptly, Newton's Third Law implies that the two sides exert equal and opposite forces on each other (in the absence of surface tension).

- The **stress vector**
$$\vec{\sigma} \cdot \hat{\hat{n}} = \left\{ \sum_j \sigma_{ij} n_j \right\}$$
must therefore be continuous across a surface with normal $\hat{n}$.

- This condition is expressed as the vanishing of the surface discontinuity of the stress vector:
$$[\vec{\sigma} \cdot \hat{n}] = 0.$$
where the brackets [] denote the difference between the two sides of the surface.

- This does *not* imply that all stress tensor components are continuous—only three linear combinations (those corresponding to the stress vector) are constrained to be continuous.
- The **mechanical pressure** is, therefore, not necessarily continuous across interfaces. In general continuum mechanics, the mechanical pressure loses the simple, intuitive interpretation it had in hydrostatics.

3 Strain

- All materials deform under external forces, but they respond differently:
  - **Elastic materials** return to their original shape once the force is removed.
  - **Plastic materials** retain permanent deformation.
  - **Viscoelastic materials** behave elastically under quick deformation but flow like viscous fluids over time.
- Elasticity is valid only within a limited range of forces; beyond that, materials become plastic or may fracture.
- Deformation displaces the material from its original position:
  - Small deformations are easier to analyze mathematically than large ones.
  - Large deformations produce complex, non-uniform displacements (e.g., crumpled paper).
  - Coordinate systems deform with the body, becoming curvilinear.
- The theory of finite deformation is mathematically complex, similar to curvilinear coordinate systems. In most engineering applications, deformations are small and can be approximated as linear.
- The description of deformation introduces the **strain tensor**, representing the local deformation or strain of a material.

3.1 Displacements

3.1.1 Uniform scaling

The prime example of deformation is a *uniform scaling* in which the coordinates of all material particles in a body are multiplied with the scale factor κ. A material particle originally situated in the point $\vec{X}$ is thus displaced to the point

$$\vec{x} = \kappa \vec{X}. \tag{21}$$

It is emphasized that *both* $\vec{X}$ and $\vec{x}$ refer to the same coordinate system. Uniform scaling with $\kappa > 1$ is also called uniform *dilatation*, whereas scaling with $0 < \kappa < 1$ is called uniform *compression*. Negative scaling with $\kappa < 0$ is physically impossible.

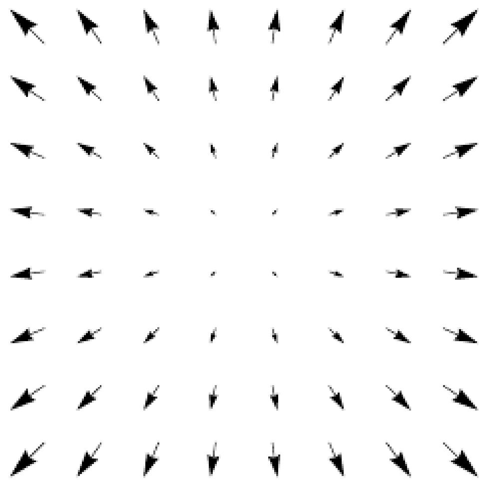


Figure 6: Uniform dilatation. The arrows indicate how material particles are displaced. Figure taken from Ref. [3].

The only point that does not change place during uniform scaling is the origin of the coordinate system. Although it superficially looks as if the origin plays a special role, this is not really the case. All relative positions of material particles scale in the same way, because

$$\vec{x} - \vec{y} = \kappa(\vec{X} - \vec{Y}), \tag{22}$$

independent of the origin of the coordinate system. There is no special center for a uniform scaling, either geometrically or physically. The origin of the coordinate system is simply an *anchor point* for the mathematical description of scaling.

3.1.2 Linear displacements

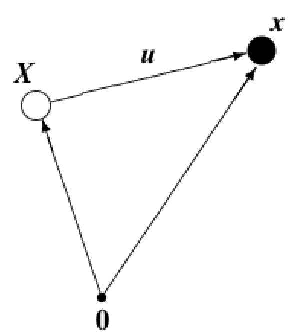


Figure 7: Geometry of displacement. The particle that originally was located at $\vec{X}$ has been displaced to $\vec{x}$ by the displacement vector $\vec{u}$. Figure taken from Ref. [3].

Under a displacement (see Fig. 7), the center–of–mass of a material particle is moved from its original position $\vec{X}$ to its actual position $\vec{x}$. The *displacement vector* is always defined as the difference between the actual and the original coordinates,

$$\vec{u} = \vec{x} - \vec{X}. \tag{23}$$

For the case of uniform scaling, the displacement vector becomes

$$\vec{u} = (\kappa - 1)\vec{X} = \left(1 - \frac{1}{\kappa}\right)\vec{x}. \tag{24}$$

Mathematically we are completely free to express the displacement as a function of either the original position X or the actual position x of the material particle. For scaling, the displacement is in both cases a linear function of the coordinates.

More generally, a linear displacement (and its inverse) takes the form

$$\vec{x} = \mathbf{A} \cdot \vec{X} + \vec{b}, \qquad \vec{X} = \mathbf{A}^{-1} \cdot (\vec{x} - \vec{b}), \tag{25}$$

where $\mathbf{A}$ is a non–singular constant matrix and $\vec{b}$ is a constant vector. As for scaling, the displacement vector may be expressed as a function of either the original or the actual positions:

$$\vec{u} = (\mathbf{A} - 1) \cdot \vec{X} + \vec{b} = (1 - \mathbf{A}^{-1}) \cdot \vec{x} + \mathbf{A}^{-1} \cdot \vec{b}. \tag{26}$$

The general linear displacement may be resolved into simpler types: translation along a coordinate axis, rotation by a fixed angle around a coordinate axis, and scaling by a fixed factor along a coordinate axis (see Fig. 8). Physically impossible reflections are excluded.

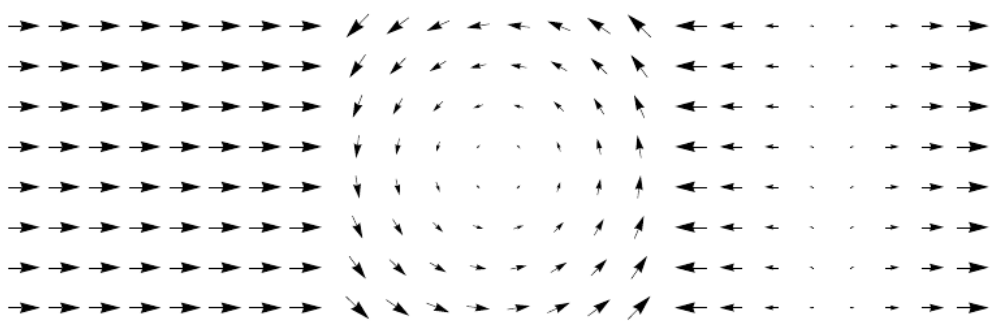


Figure 8: Arrow plots of the displacement fields for simple translation, rotation, and dilatation. Figure taken from Ref. [3].

3.1.3 Simple Translation

A rigid body translation of the material through a distance b along the x–axis is described by

$$x = X + b, \qquad y = Y, \qquad z = Z. \tag{27}$$

The displacement vector becomes

$$u_x = b, \qquad u_y = 0, \qquad u_z = 0. \tag{28}$$

Since the geometric relationships in a body are unchanged, this is not a deformation.

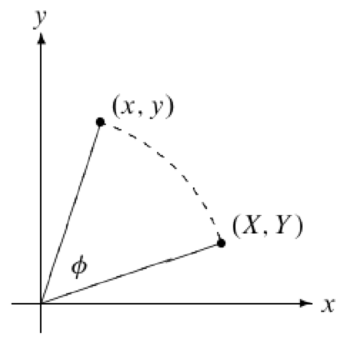


Figure 9: A rigid body rotation through an angle ϕ moves the material particle at (X, Y) to (x, y). Figure taken from Ref. [3].

3.1.4 Simple Rotation

A rigid body rotation (see Fig. 9) through the angle ϕ around the z–axis takes the form

$$x = X\cos\phi - Y\sin\phi, \qquad X = x\cos\phi + y\sin\phi, \tag{29}$$
$$y = X\sin\phi + Y\cos\phi, \qquad Y = -x\sin\phi + y\cos\phi, \tag{30}$$
$$z = Z, \qquad Z = z. \tag{31}$$

The displacement vector components are

$$u_x = -X(1-\cos\phi) - Y\sin\phi = x(1-\cos\phi) - y\sin\phi, \tag{32}$$
$$u_y = X\sin\phi - Y(1-\cos\phi) = x\sin\phi - y(1-\cos\phi), \tag{33}$$
$$u_z = 0 = 0. \tag{34}$$

Since all distances in the body remain unchanged, this is not a deformation.

3.1.5 Simple Scaling Along x

Multiplying all x–coordinates by the factor κ gives

$$x = \kappa X, \qquad y = Y, \qquad z = Z. \tag{35}$$

The displacement vector becomes

$$u_x = (\kappa - 1)X = kx, \tag{36}$$
$$u_y = 0, \tag{37}$$
$$u_z = 0, \tag{38}$$

where $k = 1 - 1/\kappa$. Simple dilatation corresponds to $k > 0$ and simple compression to $k < 0$. Uniform scaling is a combination of three such scalings along the coordinate axes. Scaling is a true deformation.

3.2 Displacement field

3.2.1 Displacement Field Representations

In continuum mechanics, each material point initially at X moves to a new position x under deformation. The **displacement field** is defined as

$$\vec{u}(x) = \vec{x} - \vec{X}(\vec{x}), \tag{39}$$

where $\vec{X}(x)$ denotes the original position of the particle now at $\vec{x}$. This is known as the *Euler representation*. Alternatively, in the *Lagrange representation*, the position is written as $\vec{x} = \vec{x}(\vec{X})$, and

$$\vec{u} = \vec{x}(\vec{X}) - \vec{X}. \tag{40}$$

Both representations are equivalent for small, slowly varying deformations.

3.2.2 Local and Infinitesimal Deformation

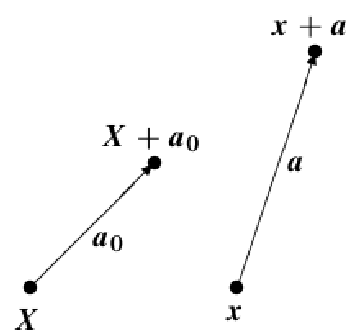


Figure 10: Displacement of a tiny material needle from $\vec{a}_0$ to $\vec{a}$. It may be translated, rotated, and scaled. Only the latter corresponds to a true deformation. Figure taken from Ref. [3].

A true deformation involves local changes in length and angle between neighboring points, not just rigid translations or rotations.

For a small line element or "needle" $\vec{a}_0$ connecting two material points, the deformation changes it to $\vec{a}$. Using $\vec{X}(\vec{x}) = \vec{x} - \vec{u}(\vec{x})$, one finds

$$\vec{a}_0 = \vec{X}(\vec{x} + \vec{a}) - \vec{X}(\vec{x}) = \vec{a} - \vec{u}(\vec{x} + \vec{a}) + \vec{u}(\vec{x}). \tag{41}$$

Expanding $\vec{u}(\vec{x} + \vec{a})$ to first order in $\vec{a}$ gives

$$\vec{u}(\vec{x} + \vec{a}) = \vec{u}(\vec{x}) + (\vec{a} \cdot \nabla)\vec{u}(\vec{x}) + \mathcal{O}(\vec{a}^2), \tag{42}$$

so that the infinitesimal change in the line element is

$$\delta\vec{a} \equiv \vec{a} - \vec{a}_0 = (\vec{a} \cdot \nabla)\vec{u}(\vec{x}). \tag{43}$$

In index notation:

$$\delta a_i = \sum_j a_j \, \nabla_j u_i. \tag{44}$$

This defines the **displacement gradients** $\{\nabla_j u_i\}$, and in dyadic form:

$$\delta\vec{a} = (\vec{a} \cdot \nabla)\vec{u} = \vec{a} \cdot (\nabla\vec{u}) = (\nabla\vec{u})^{\mathrm{T}} \cdot \vec{a}. \tag{45}$$

3.2.3 Slowly Varying Displacement Field

The displacement field is said to be *slowly varying* when

$$|\nabla_j u_i(x)| \ll 1, \tag{46}$$

and small relative to the body size L,

$$|\vec{u}(x)| \ll L. \tag{47}$$

This ensures that deformations are infinitesimal, though not necessarily rigid.

3.2.4 Cauchy's Strain Tensor

The strain tensor quantifies local geometric changes due to deformation. The change in the scalar product of two material line elements $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned} \delta(\vec{a}\cdot\vec{b}) &= (\delta\vec{a})\cdot\vec{b} + \vec{a}\cdot(\delta\vec{b}) = (\vec{a}\cdot\nabla)\vec{u}\cdot\vec{b} + (\vec{b}\cdot\nabla)\vec{u}\cdot\vec{a} && (48)\\ &= \sum_{ij}(\nabla_i u_j + \nabla_j u_i)a_i b_j && (49)\\ &= 2\sum_{ij} U_{ij} a_i b_j = 2\,\vec{a}\cdot\vec{U}\cdot\vec{b}, && (50) \end{aligned}$$

where the **Cauchy (infinitesimal) strain tensor** is defined as:

$$U_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i), \tag{51}$$

or, in matrix notation,

$$\vec{U} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^{\mathrm{T}}). \tag{52}$$

A symmetric tensor has six independent component whereas the displacement field has only three independent components. Every strain tensor must consequently satisfy consistency or compatibility conditions that remove three degrees of freedom. These conditions are

$$\nabla_i\nabla_j U_{kl} + \nabla_k\nabla_l U_{ij} = \nabla_i\nabla_l U_{kj} + \nabla_k\nabla_j U_{il}. \tag{53}$$

The strain tensor contains all the information about the local geometric changes caused by the displacement and is accordingly a good measure of local deformation. All bodily translations and rotations have been automatically taken out, and any displacement that is a combination of translations and rotations must consequently yield a vanishing strain tensor. It should, however, be emphasized that Cauchy's expression is only valid for small displacement gradients. When that is not the case, a more complicated expression must be used, involving the square of the displacement gradients .

3.2.5 Components of the Strain Tensor

For a general displacement field:

$$U_{xx} = \nabla_x u_x, \qquad U_{yy} = \nabla_y u_y, \qquad U_{zz} = \nabla_z u_z, \tag{54}$$

$$U_{yz} = U_{zy} = \tfrac{1}{2}(\nabla_y u_z + \nabla_z u_y), \quad U_{zx} = U_{xz} = \tfrac{1}{2}(\nabla_z u_x + \nabla_x u_z), \quad U_{xy} = U_{yx} = \tfrac{1}{2}(\nabla_x u_y + \nabla_y u_x). \tag{55}$$

For a simple linear deformation $\vec{u} = k(x, 0, 0)$,

$$\{U_{ij}\} = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{56}$$

which represents a true deformation (extension along x).

3.2.6 Principal Axes of Strain

The strain tensor is symmetric,

$$U_{ij} = U_{ji}, \tag{57}$$

and can therefore be diagonalized. Its eigenvectors define the **principal axes of strain**, and the corresponding eigenvalues measure the local extension or compression along those directions (see Fig. 11).

Example. Let's calculate the strain tensor for $\vec{u} = \alpha(y, x, 0)$ with $0 < \alpha \ll 1$ and determine the principal directions of strain and the change in length scales along these directions.

For small deformations the (Cauchy) strain tensor is

$$U_{ij} = \tfrac{1}{2}\big(\partial_i u_j + \partial_j u_i\big).$$

With $u_x = \alpha y,\ u_y = \alpha x,\ u_z = 0$, the displacement gradient is

$$\nabla\vec{u} = \begin{pmatrix} \partial_x u_x & \partial_y u_x & \partial_z u_x \\ \partial_x u_y & \partial_y u_y & \partial_z u_y \\ \partial_x u_z & \partial_y u_z & \partial_z u_z \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$U = \tfrac{1}{2}\big(\nabla\vec{u} + (\nabla\vec{u})^T\big) = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the xy-subspace, $U_{2\times 2} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ has eigenvalues and unit eigenvectors

$$\varepsilon_1 = +\alpha, \quad \vec{n}_1 = \tfrac{1}{\sqrt{2}}(1, 1, 0), \qquad \varepsilon_2 = -\alpha, \quad \vec{n}_2 = \tfrac{1}{\sqrt{2}}(1, -1, 0),$$

and a third eigenvalue $\varepsilon_3 = 0$ with $\vec{n}_3 = (0, 0, 1)$.

The normal strain in the direction of a unit vector $\vec{n}$ is $\varepsilon_n = \vec{n}\cdot\vec{U}\,\vec{n}$. Along the principal directions,

$$\left.\frac{\Delta\ell}{\ell}\right|_{\vec{n}_1} = +\alpha \quad \text{(dilation along } x = y\text{)}, \qquad \left.\frac{\Delta\ell}{\ell}\right|_{\vec{n}_2} = -\alpha \quad \text{(contraction along } x = -y\text{)},$$

and $\Delta\ell/\ell = 0$ along $\vec{n}_3 = \hat{z}$.

Thus, the material is stretched along the diagonal $x = y$ by a fractional amount α and compressed by the same amount along the orthogonal diagonal $x = -y$; there is no strain in the z-direction.

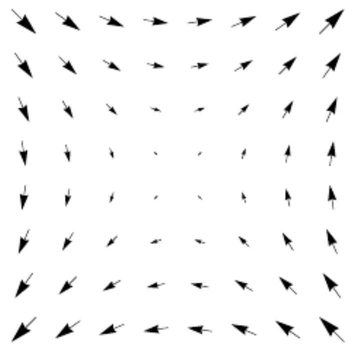


Figure 11: Arrow plot of the two-dimensional Lagrangian linear displacement field $\vec{u} = (y,\, x,\, 0)$ in the square $-1 < x < 1$ and $-1 < y < 1$. The material is dilated along one diagonal and contracted along the other. These are the principal directions of strain everywhere. Figure taken from Ref. [3].

3.3 Geometrical Meaning of the Strain Tensor

The strain tensor contains all the relevant information about local changes in geometric relationships, such as lengths of material elements and angles between them. Other geometric quantities, such as curves, surfaces, and volumes, are also affected under deformation.

3.3.1 Lengths and Angles

To define the projection of a tensor u_{ij} along two directions $\vec{a}$ and $\vec{b}$, we introduce

$$U_{ab} = \hat{a} \cdot \vec{U} \cdot \hat{b} = \frac{\vec{a} \cdot \vec{U} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}. \tag{58}$$

The change in the scalar product is

$$\delta(\vec{a} \cdot \vec{b}) = 2\,|\vec{a}|\,|\vec{b}|_{\,ab}. \tag{59}$$

Setting $\vec{b} = \vec{a}$ gives the change in length:

$$\frac{\delta|\vec{a}|}{|\vec{a}|} = U_{aa}. \tag{60}$$

Thus, the diagonal projection U_{aa} represents the *fractional change of length* in the direction of $\vec{a}$.
For the angle ϕ between $\vec{a}$ and $\vec{b}$, since $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\phi$, one finds

$$\delta\phi = \phi - \phi_0 = \frac{(U_{aa} + U_{bb})\cos\phi - 2u_{ab}}{\sin\phi}. \tag{61}$$

For orthogonal vectors ($\phi = 90°$),

$$\delta\phi = -2U_{ab}. \tag{62}$$

The off-diagonal projections of the strain tensor thus determine the change in angle between actually orthogonal needles.

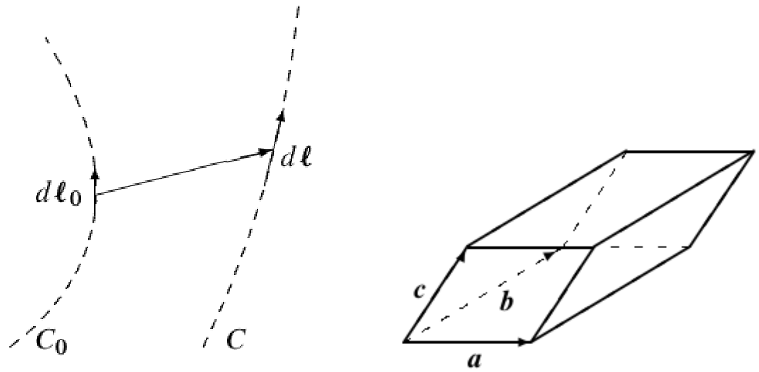


Figure 12: Left: A line element is stretched and rotated by the displacement that changes the curve from C_0 to C. Right : Three infinitesimal needles span a parallelepiped with volume $dV = \vec{a} \times \vec{b} \cdot \vec{c}$. Figure taken from Ref. [3].

3.3.2 Infinitesimal Elements

Curve element. A curve element behaves like a small needle (see Fig. 12). Under a displacement, the change from $d\vec{\ell}_0$ to $d\vec{\ell}$ is

$$\delta(d\vec{\ell}) \equiv d\vec{\ell} - d\vec{\ell}_0 = d\vec{\ell} \cdot \nabla\vec{u} = \nabla\vec{u}^T \cdot d\vec{\ell}. \tag{63}$$

Volume Element. For a tiny parallelepiped spanned by three infinitesimal vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, the volume element is $dV = \vec{a} \times \vec{b} \cdot \vec{c}$ (see Fig. 12). Its variation under displacement is

$$\delta(dV) \equiv dV - dV_0 = \nabla \cdot \vec{u}\, dV \tag{64}$$

$$= dV \sum_i \nabla_i u_i = dV\, U_{ii}. \tag{65}$$

Hence, the divergence of the displacement field which corresponds to the trace of $\vec{U}$, $\nabla \cdot \vec{u} = \mathrm{Tr}(\vec{U})$, gives the fractional change of volume.

The change in density ρ of a material particle, with constant mass dM, satisfies

$$\delta\rho = -\rho\, \nabla \cdot \vec{u}. \tag{66}$$

Thus, when $\nabla \cdot \vec{u} = \mathrm{Tr}(\vec{U}) = 0$, both the volume and density remain unchanged.

Surface Element. For a surface element $d\vec{S} = \vec{a} \times \vec{b}$, we find from the volume relation that

$$\delta(d\vec{S}) = (\nabla \cdot \vec{u}\, \mathbf{1} - \nabla\vec{u}) \cdot d\vec{S}. \tag{67}$$

Both the magnitude and direction of $d\vec{S}$ are modified by the deformation, following a rule different from that of the curve element.

3.4 Thermodynamics of deformation

Deforming a body requires work, part of which is stored as elastic potential energy, while another part is dissipated as heat due to internal friction. No real material is perfectly elastic. For instance, when squeezing a rubber ball, the stored energy is released when the force is removed, but a steel ball bouncing on a floor eventually stops because energy is lost to internal friction and air resistance. In continuum mechanics, energy relations are best derived by *following the work*—that is, tracking the work done by forces within the body.

3.4.1 Virtual Displacement Work

Consider a volume V of material not in mechanical equilibrium. Let the effective force acting on an element of volume dV be $d\vec{\mathcal{F}} = \vec{f}^* \, dV$. To keep all material points in fixed, non-equilibrium positions, we must apply an external distribution of *virtual forces*

$$\vec{f}' = -\vec{f}^*. \tag{68}$$

If the body is then displaced infinitesimally by $\delta\vec{u}(\vec{x})$, the work done by the virtual forces is

$$\delta W = \int_V \vec{f}' \cdot \delta\vec{u} \, dV = -\int_V \vec{f}^* \cdot \delta\vec{u} \, dV \tag{69}$$

$$= -\int_V \vec{f} \cdot \delta\vec{u} \, dV - \int_V (\nabla \cdot \vec{\sigma}^T) \cdot \delta\vec{u} \, dV, \tag{70}$$

where $\vec{f}^* = \vec{f} + \nabla \cdot \vec{\sigma}^T$ was used. Then performing an integration by parts in the second term gives

$$\int_V \sum_{ij} (\nabla_j \sigma_{ij}) \delta u_i \, dV = \int_V \sum_{ij} \nabla_j (\sigma_{ij} \delta u_i) \, dV - \int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i \, dV \tag{71}$$

$$= \int_V \nabla \cdot (\sigma^T \cdot \delta u) \, dV - \int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i \, dV \tag{72}$$

$$= \oint_S (\sigma^T \cdot \delta u) \cdot d\vec{S} - \int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i \, dV. \tag{73}$$

The surface term vanishes because $\delta\vec{u} = 0$ at the boundary S. The second integral can, by virtue of the symmetry of the tensor σ_{ij} , be written as

$$\int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i \, dV = \frac{1}{2} \int_V \sum_{ij} \sigma_{ij} \left(\nabla_j \delta u_i + \nabla_i \delta u_j \right) \, dV \tag{74}$$

$$= \frac{1}{2} \int_V \sum_{ij} \sigma_{ij} \delta \left(\nabla_j u_i + \nabla_i u_j \right) \, dV \tag{75}$$

$$= \int_V \sum_{ij} \sigma_{ij} \delta U_{ij} \, dV \tag{76}$$

$$= \int_V \mathrm{Tr}[\vec{\sigma} \cdot \delta\vec{U}] \, dV \equiv \int_V \vec{\sigma} : \delta\vec{U} \, dV, \tag{77}$$

where

$$\delta U_{ij} = \tfrac{1}{2} \big(\nabla_i \delta u_j + \nabla_j \delta u_i \big) \tag{78}$$

is the infinitesimal change in the strain tensor.

The virtual work becomes

$$\delta W = -\int_V \vec{f} \cdot \delta\vec{u}\, dV + \int_V \vec{\sigma} : \nabla\delta\vec{u}\, dV \tag{79}$$

$$= -\int_V \vec{f} \cdot \delta\vec{u}\, dV + \int_V \vec{\sigma} : \delta\vec{U}\, dV. \tag{80}$$

where $\vec{\sigma} : \nabla\delta\vec{u} = \sum_{ij} \sigma_{ij}\nabla_j \delta u_i$. The first term represents work against body forces, and the second corresponds to work against internal stresses. The portion of virtual work associated with internal stresses defines the infinitesimal work of deformation:

$$\delta W_{\text{deform}} = \int_V \vec{\sigma} : \nabla\delta\vec{u}\, dV. \tag{81}$$

Thus, the deformation work represents the contribution to the body's *deformation energy.*

3.5 Deformation regimes

In solid continuum mechanics, deformation regimes are classified according to the magnitude of the deformation and the material response. Broadly, three regimes are distinguished:

1. **Infinitesimal (small) deformations.**
   In this regime, displacements and strains are very small compared to unity. The linearized strain tensor is valid, geometric nonlinearities such as large rotations or nonlinear displacement gradients are neglected, and the principle of superposition applies. This approximation underlies most problems in classical elasticity, such as stress analysis in beams and plates.

2. **Finite (large) deformations.**
   Here, displacements or strains are not negligible, and nonlinear kinematics must be employed. Quantities such as the deformation gradient tensor and the Green–Lagrange strain are used. Large rotations, shear, and finite strains are fully accounted for, and the governing equations are nonlinear. Applications include rubber elasticity, biomechanics of soft tissues, and geomechanics.

3. **Extreme (highly nonlinear) deformations.**
   This regime involves very large strains, often accompanied by material nonlinearities or instabilities. The material response may be anisotropic or rate-dependent, and phenomena such as plasticity, viscoelasticity, viscoplasticity, damage, or fracture must be considered. Examples include crash simulations, metal forming at large strains, and failure analysis.

In summary: the *small deformation regime* corresponds to linear elasticity, the *finite deformation regime* requires nonlinear kinematics but can still be elastic or elastic–plastic, while the *extreme regime* involves very large strains typically coupled with inelastic effects such as plasticity, fracture, or damage.

3.5.1 Energy

Besides elastic deformations, we shall also suppose that the process of deformation occurs so slowly that thermodynamic equilibrium is established in the body at every instant, in accordance with

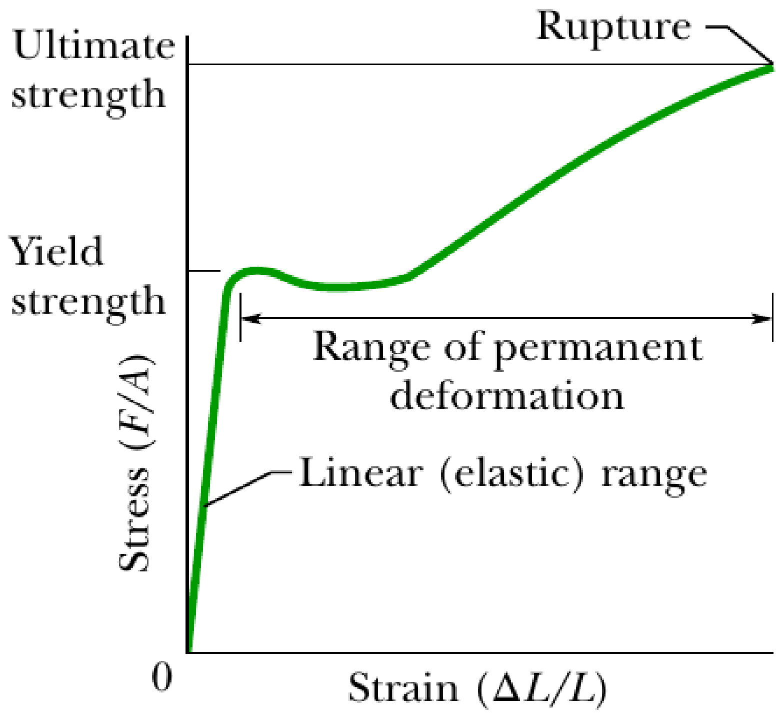


Figure 13: A stress–strain curve for a steel test specimen. The specimen deforms permanently when the stress is equal to the yield strength of the specimen's material. It ruptures when the stress is equal to the ultimate strength of the material. Figure taken from Ref..

the external conditions. This assumption is almost always justified in practice. The process will then be thermodynamically reversible.

In what follows we shall take all such thermodynamic quantities as the entropy $\mathcal{S}$, the internal energy $\mathcal{E}$, etc., relative to unit volume of the body, and not relative to unit mass as in fluid mechanics.

The unit volumes before and after deformation must be distinguished since they generally contain different amounts of matter. The thermodynamic quantities are therefore referred to the unit volume of the **undeformed body**. The total internal energy of the body is obtained by integrating $\mathcal{E}$ over the undeformed volume.

An infinitesimal change in the internal energy is given by the difference between the heat acquired by the unit volume and the work done by internal stresses $\delta R = dW/dV$. For a reversible process, $Td\mathcal{S}$ represents the heat, and δR the work. Thus,

$$d\mathcal{E} = Td\mathcal{S} - \delta R. \tag{82}$$

Using $dW = -\sum_{ij} \sigma_{ij}\delta U_{ij}\, dV$, we obtain

$$d\mathcal{E} = Td\mathcal{S} + \sum_{ij} \sigma_{ij}\nabla_j \delta u_i \tag{83}$$

$$= Td\mathcal{S} + \sum_{ij} \sigma_{ij}\delta U_{ij}. \tag{84}$$

This is the **fundamental thermodynamic relation** for deformed bodies.

In the case of hydrostatic compression, $\sigma_{ij} = -p\delta_{ij}$, leading to

$$\sum_{ij} \sigma_{ij} \nabla_j \delta u_i = -p \sum_i \nabla_i \delta u_i = -p \nabla \cdot \delta \vec{u} = -p \frac{\delta(dV)}{dV}. \tag{85}$$

Therefore, this reduces to the familiar form

$$dE = TdS - p\,\delta(dV). \tag{86}$$

3.6 Helmholtz and Gibbs Free Energies

In the thermodynamic description of elastic bodies, it is convenient to introduce thermodynamic potentials whose natural variables correspond to the physical quantities held fixed in a given situation. Two such potentials are the **Helmholtz free energy** and the **Gibbs free energy**.

3.6.1 Helmholtz Free Energy

The Helmholtz free energy is defined as

$$F = \mathcal{E} - TS, \tag{87}$$

where $\mathcal{E}$ is the internal energy per unit volume, T the temperature, and $\mathcal{S}$ the entropy per unit volume. Its differential form at constant temperature is

$$dF = -\mathcal{S}\,dT + \sum_{ik} \sigma_{ik}\,dU_{ik}. \tag{88}$$

Thus $F = F(T, U_{ik})$ is naturally a function of the temperature and the strain tensor. Differentiation with respect to U_{ik} gives the stress tensor:

$$\sigma_{ik} = \left(\frac{\partial \mathcal{E}}{\partial U_{ik}}\right)_{\mathcal{S}} = \left(\frac{\partial \mathcal{F}}{\partial U_{ik}}\right)_T. \tag{89}$$

This makes F particularly useful in elasticity, where one often specifies the strain and seeks the corresponding stress.

3.6.2 Gibbs Free Energy

The Gibbs free energy is defined as

$$\Phi = \mathcal{E} - TS - \sum_{ik} \sigma_{ik} U_{ik} = F - \sum ik \sigma_{ik} U_{ik}. \tag{90}$$

Its differential is

$$d\Phi = -S\,dT - \sum_{ik} U_{ik}\,d\sigma_{ik}, \tag{91}$$

which shows that Φ is naturally a function of temperature and stress:

$$\Phi = \Phi(T, \sigma_{ik}).$$

Differentiating with respect to σ_{ik} yields the strain tensor:

$$U_{ik} = -\left(\frac{\partial \Phi}{\partial \sigma_{ik}}\right)_T. \tag{92}$$

Helmholtz and Gibbs free energies are fundamental thermodynamic potentials that quantify the amount of energy in a system available to do useful work, but they apply under different physical constraints. The Helmholtz free energy is the appropriate potential for systems held at constant temperature and volume, making it useful in contexts such as elasticity where the deformation (and therefore volume change) is controlled. The Gibbs free energy, on the other hand, applies to systems at constant temperature and pressure, and is particularly relevant for processes where the external pressure is fixed, such as phase transitions or chemical reactions. Both potentials encode the balance between internal energy and entropy, but their distinct natural variables determine which one is most convenient for describing a given physical situation.

The two potentials are related by a Legendre transform exchanging the conjugate variables (U_{ik}, σ_{ik}). Both potentials encode the elastic response of the material and provide a systematic way to derive constitutive relations such as Hooke's law from thermodynamic principles.

4 Hooke's law

To apply the general thermodynamic formalism to a specific case, we must express the free energy $\mathcal{F}$ of the body as a function of the strain tensor U_{ik}. Assuming small deformations, the free energy can be expanded in powers of U_{ik}. We mainly restrict our attention to isotropic bodies.

4.1 Isotropic bodies: free energy and elastic coefficients

For a deformed body at constant temperature, we take the undeformed state as the reference configuration, in the absence of external forces and at the same temperature. When $U_{ik} = 0$, the internal stresses vanish, i.e. $\sigma_{ik} = 0$. Hence, since $\sigma_{ik} = \partial F / \partial U_{ik}$, there is no linear term in the expansion of F in powers of U_{ik}.

As $\mathcal{F}$ is a scalar quantity, its expansion must involve scalar combinations of U_{ik}. Two independent second-order scalars can be formed:

$$\left(\sum_i U_{ii}\right)^2 \quad \text{and} \quad \sum_{ik} U_{ik}^2. \tag{93}$$

Thus, retaining terms up to second order, we obtain

$$\mathcal{F} = \mathcal{F}_0 + \tfrac{1}{2}\lambda \left(\sum_i U_{ii}\right)^2 + \mu \sum_{ik} U_{ik}^2 \tag{94}$$

$$= \mathcal{F}_0 + \tfrac{1}{2}\lambda\, (\mathrm{Tr} U)^2 + \mu\, \mathrm{Tr}(U^2) \tag{95}$$

where λ and μ are the **Lamé coefficients**. This is the general expression for the free energy of an isotropic elastic body.

A deformation in which $\sum_i U_{ii} = 0$ corresponds to no change in volume but a change in shape, called a **pure shear**. Conversely, a deformation for which $U_{ik} = \text{constant} \times \delta_{ik}$ represents a change in volume without a change in shape, called a **hydrostatic compression**.

4.2 Decomposition of the strain tensor

Any deformation can be written as the sum of a pure shear and a hydrostatic compression:

$$U_{ik} = \left(U_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l U_{ll} \right) + \tfrac{1}{3}\delta_{ik} \sum_l U_{ll}. \tag{96}$$

The first term represents a pure shear, and the second term a hydrostatic compression. Substituting this decomposition into $\mathcal{F}$, we obtain

$$\mathcal{F} = \mu \sum_{ik} \left(U_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l U_{ll} \right)^2 + \tfrac{1}{2} K \left(\sum_l U_{ll} \right)^2, \tag{97}$$

where K is the **bulk modulus** (modulus of hydrostatic compression), and μ is the **shear modulus** (modulus of rigidity). The bulk modulus is related to the Lamé coefficients by

$$K = \lambda + \tfrac{2}{3}\mu. \tag{98}$$

Thermodynamic stability requires that the free energy be a minimum at equilibrium, i.e.

$$K > 0, \quad \mu > 0. \tag{99}$$

4.3 Stress–strain Relation

From the thermodynamic relation $\sigma_{ik} = (\partial \mathcal{F}/\partial U_{ik})_T$, we compute:

$$\begin{aligned}
d\mathcal{F} &= K \sum_l U_{ll} \sum_n dU_{nn} + 2\mu \sum_{ik} (U_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l U_{ll}) d(U_{ik} - \tfrac{1}{3}\delta_{ik} \sum_n U_{nn}) && (100) \\
&= K \sum_l U_{ll} \sum_n dU_{nn} + 2\mu \sum_{ik} (U_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l U_{ll}) \sum_{ik} dU_{ik} \\
&\quad - 2\mu \sum_{ik} (U_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l U_{ll}) \tfrac{1}{3}\delta_{ik} \sum_n dU_{nn} && (101) \\
&= \left[K \sum_l U_{ll} \sum_{ik} \delta_{ik} + 2\mu \sum_{ik} (U_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l U_{ll}) \right] dU_{ik} && (102) \\
&= \left[(K - \tfrac{2}{3}\mu) \sum_l U_{ll} \sum_{ik} \delta_{ik} + 2\mu \sum_{ik} U_{ik} \right] dU_{ik} && (103) \\
&= \left[\lambda \sum_l U_{ll} \sum_{ik} \delta_{ik} + 2\mu \sum_{ik} U_{ik} \right] dU_{ik}. && (104)
\end{aligned}$$

Hence, the stress tensor is

$$\begin{aligned}
\sigma_{ik} &= K \sum_l U_{ll}\, \delta_{ik} + 2\mu \left(U_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l U_{ll} \right) && (105) \\
&= \lambda \sum_l U_{ll}\, \delta_{ik} + 2\mu\, U_{ik}. && (106)
\end{aligned}$$

4.4 Hooke's Law

Taking the trace to the first line of the previous expression, we find

$$\sum_i \sigma_{ii} = 3K \sum_l U_{ll}, \qquad \Rightarrow \qquad \sum_l U_{ll} = \frac{\sum_i \sigma_{ii}}{3K}. \tag{107}$$

Substituting back we obtain the inverse relation:

$$U_{ik} = \frac{\delta_{ik}}{9K} \sum_l \sigma_{ll} + \frac{1}{2\mu} \left(\sigma_{ik} - \tfrac{1}{3}\delta_{ik} \sum_l \sigma_{ll} \right), \tag{108}$$

which expresses the strain tensor in terms of the stress tensor. From Eq. (108), we see that U_{ik} is a linear function of σ_{ik}; this proportionality defines **Hooke's law** for small deformations.

4.5 Homogeneous deformations

A homogeneous deformation is one in which the strain tensor U_{ik} is constant throughout the body.

Let's recall the boundary condition

$$\vec{\sigma} \cdot \hat{\hat{n}} = \left\{ \sum_j \sigma_{ij} n_j \right\} \tag{109}$$

must therefore be continuous across a surface with normal $\hat{n}$.

4.5.1 Hydrostatic Compression

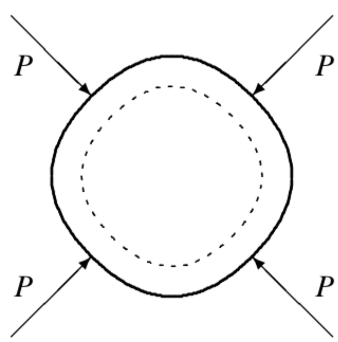


Figure 14: A body made from isotropic, homogenous material subject to a uniform external pressure will be uniformly compressed. Figure taken from Ref. [3].

In a fluid at rest with a constant pressure p, the stress tensor is

$$\sigma_{ik} = -p\delta_{ik} \tag{110}$$

everywhere. If a solid body made from isotropic material is immersed into this fluid, the natural guess is that the pressure will also be p inside the body. Since $\sum_l \sigma_{ll} = -3p$, the strain becomes

$$U_{ik} = -\frac{p}{3K}\delta_{ik}. \tag{111}$$

On the other hand, the displacement field can be obtained from

$$U_{xx} = \nabla_x u_x = -\frac{p}{3K}, \tag{112}$$

$$U_{yy} = \nabla_y u_y = -\frac{p}{3K}, \tag{113}$$

$$U_{zz} = \nabla_z u_z = -\frac{p}{3K}. \tag{114}$$

Integrating directly we are able to find a particular solution for the displacement field:

$$u_x = -\frac{p}{3K}\,x, \tag{115}$$

$$u_y = -\frac{p}{3K}\,y, \tag{116}$$

$$u_z = -\frac{p}{3K}\,z. \tag{117}$$

The most general solution is obtained by adding an arbitrary infinitesimal rigid body displacement to this expression.

We emphasize that this result was obtained by making a reasonable guess for the form of the stress tensor inside the body. Although such a guess could, in principle, be incorrect, it is in fact justified by a uniqueness theorem. The theorem ensures—much like the corresponding uniqueness theorem in electrostatics—that if the mechanical equilibrium equations and the boundary conditions are satisfied by a proposed solution, then the solution is unique (up to an arbitrary small rigid body displacement). Thus, in any elastostatic problem, one may freely add a small rigid body motion to the solution without affecting the stresses.

4.5.2 Uniform stretching

Consider a rod along the z–axis, with forces applied uniformly over its ends (the rod is only pulled (or pushed) from its ends). On the lateral surface (the cylindrical surface), there is no force applied.

Let the force per unit area be p. Since the deformation is homogeneous, σ_{ik} is constant. Since there is no lateral force on the sides of the rod ($n_z = 0$), we have that

$$\sigma \cdot \hat{n} = \sum_k \sigma_{ik}\, n_k = 0, \tag{118}$$

where $\hat{n}$ is the outward normal vector to the side surface. Because the sides are parallel to the z-axis, the outward normal $\hat{n}$ has:

$$\hat{n} = (n_x, n_y, 0) = (n_1, n_2, 0). \tag{119}$$

It follows that

$$\sigma_{11}\, n_1 + \sigma_{12} n_2 = 0, \tag{120}$$

$$\sigma_{12}\, n_1 + \sigma_{22} n_2 = 0, \tag{121}$$

$$\sigma_{13}\, n_1 + \sigma_{23}\, n_2 = 0. \tag{122}$$

Therefore $\sigma_{ik} = 0$ except σ_{zz}. On the end surface we have

$$\sum_k \sigma_{3k}\, n_3 = p, \tag{123}$$

from which $\sigma_{zz} = p$.

From the general strain–stress relation,

$$U_{xx} = U_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right) p, \tag{124}$$

$$U_{zz} = \frac{1}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right) p. \tag{125}$$

The component U_{zz} gives the relative lengthening of the rod. The coefficient of p is called the coefficient of extension, and its reciprocal is the modulus of extension or Young's modulus, E. The longitudinal extension is

$$U_{zz} = \frac{p}{E}, \tag{126}$$

where

$$E = \frac{9K\mu}{3K + \mu} \tag{127}$$

is Young's modulus. On the other hand, the components U_{xx} and U_{yy} give the relative compression of the rod in the transverse direction. The ratio of the transverse compression to the longitudinal extension is called the Poisson's ratio ν:

$$U_{xx} = -\nu\, U_{zz}. \tag{128}$$

Using the Lamé constants,

$$\nu = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}. \tag{129}$$

Since $K > 0$ and $\mu > 0$,

$$-1 \leq \nu \leq \frac{1}{2}. \tag{130}$$

Finally, the relative volume change is

$$U_{ii} = \frac{p}{3K}. \tag{131}$$

The free energy of a stretched rod is

$$F = \frac{p^2}{2E}. \tag{132}$$

The displacement field is then obtained by integrating the relations

$$\nabla_x u_x = U_{xx}, \qquad \nabla_y u_y = U_{yy}, \qquad \nabla_z u_z = U_{zz},$$

which yields the particular solution

$$u_z = \frac{p}{E}\, z, \tag{133}$$

$$u_x = -\nu\, \frac{p}{E}\, x, \tag{134}$$

$$u_y = -\nu\, \frac{p}{E}\, y. \tag{135}$$

This displacement describes a simple dilatation along the z–axis and a corresponding compression toward the z–axis in the xy–plane.

It is often convenient to express the elastic constants in terms of Young's modulus E and Poisson's ratio ν. In these variables, the Lamé coefficients become

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{136}$$

$$\mu = \frac{E}{2(1+\nu)}, \tag{137}$$

$$K = \frac{E}{3(1-2\nu)}. \tag{138}$$

The free energy of a deformed isotropic body can then be written as

$$F = \frac{E}{2(1+\nu)} \left(\sum_{ik} U_{ik}^2 + \frac{\nu}{1-2\nu} (\sum_l U_{ll})^2 \right). \tag{139}$$

The stress tensor in terms of the strain tensor takes the form

$$\sigma_{ik} = \frac{E}{1+\nu} \left(U_{ik} + \frac{\nu}{1-2\nu} \sum_l u_{ll}\, \delta_{ik} \right). \tag{140}$$

Conversely, the strain tensor in terms of the stress tensor is

$$U_{ik} = \frac{1}{E} \left[(1+\nu)\sigma_{ik} - \nu \sum_l \sigma_{ll}\, \delta_{ik} \right]. \tag{141}$$

If the rod-like spring is clamped on the sides by a rigid material, the boundary conditions become

$$u_y = u_x = 0 \quad \text{on the sides.}$$

In this case, the only non-vanishing constant strain is U_{zz}, and the solution is obtained in the same manner as above .

4.5.3 Unilateral Compression

Let us now consider the compression of a rod whose sides are fixed so that they cannot move laterally. The external forces responsible for the compression act on the ends of the rod and along its length, taken to be the z–axis. Such a deformation is called a *unilateral compression*. Since the rod deforms only in the z–direction, the only nonzero component of the strain tensor is U_{zz}.

From Eq. (140) for the stress tensor in terms of the strain tensor, we obtain

$$\sigma_{xx} = \sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)} U_{zz}, \tag{142}$$

$$\sigma_{zz} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} U_{zz}. \tag{143}$$

Denoting the compressive force per unit area (stress) by p so that $\sigma_{zz} = p$ (negative for compression), we solve for the longitudinal strain:

$$U_{zz} = p\, \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}. \tag{144}$$

The components σ_{xx} and σ_{yy} describe the transverse stresses generated by the lateral constraint. Substituting Eq. (144) into Eq. (143) gives

$$\sigma_{xx} = \sigma_{yy} = \frac{p\,\nu}{1-\nu}. \tag{145}$$

Finally, using the general elastic energy formula

$$F = \tfrac{1}{2}\sigma_{ik}u_{ik},$$

and observing that only σ_{zz} and u_{zz} are nonzero, we obtain the free energy of the compressed rod:

$$F = \frac{p^2(1+\nu)(1-2\nu)}{2E(1-\nu)}. \tag{146}$$

4.5.4 Uniform Shear

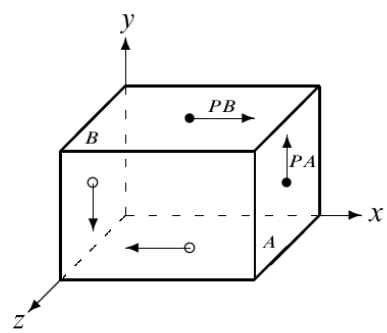


Figure 15: Clamped rectangular slab under constant shear stress $\sigma_{xy} = \sigma_{yx} = P$. The upper clamp is acted upon by an external force $\mathcal{F}_x = PB$, where B is the area of the clamp, while the force on the lower is $\mathcal{F}_x = -PB$. The symmetry of the stress tensor demands a clamp force $\mathcal{F}_y = PA$ on the right-hand side and a force $\mathcal{F}_y = -PA$ on the left-hand side, where A is the area of that clamp. Figure taken from Ref. [3].

Let's consider a clamped rectangular slab of homogeneous, isotropic material subjected to shear stress along one side (here, the x–direction). As argued there, the shear stress

$$\sigma_{xy} = P$$

must be constant throughout the material. What was not appreciated earlier is that the symmetry of the stress tensor requires that

$$\sigma_{yx} = P$$

everywhere as well. Consequently, shearing forces must act on the ends of the slab, while the remaining sides are free (see the margin figure).

Assuming there are no other stresses, the only nonzero strain component is

$$U_{xy} = \frac{P}{2\mu},$$

and using the relation

$$2U_{xy} = \nabla_y u_x + \nabla_x u_y,$$

we obtain a particular solution for the displacement field:

$$u_x = \frac{P}{\mu}\, y, \tag{147}$$

$$u_y = 0, \tag{148}$$

$$u_z = 0. \tag{149}$$

In these coordinates the displacement in the x–direction vanishes for $y = 0$ and increases linearly with y. Each infinitesimal "needle" of the material is not only sheared but also rotated by a small angle

$$\phi = \tfrac{1}{2}\left(\nabla_x u_y - \nabla_y u_x\right) = -\frac{P}{2\mu} \tag{150}$$

about the z–axis.

5 Basic elastostatics

The fundamental equations of elastostatics are obtained from the results of the preceding chapters. They consist of:

$$f_i + \sum_j \nabla_j \sigma_{ij} = 0, \qquad \text{mechanical equilibrium,} \tag{151}$$

$$\sigma_{ij} = 2\mu\, U_{ij} + \lambda\, \delta_{ij} \sum_k U_{kk}, \qquad \text{Hooke's law,} \tag{152}$$

$$U_{ij} = \tfrac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right), \qquad \text{Cauchy's strain tensor.} \tag{153}$$

We shall use these equations with a time-independent external body force. In the presence of gravity, for example, the body force is

$$\vec{f} = \rho\, \vec{g}.$$

Other forces of electromagnetic origin can also appear, such as those generated by an inhomogeneous electric field acting on a dielectric material.

5.0.1 Navier–Cauchy Equation

Substituting the second and third lines into the equilibrium condition, we obtain in index notation

$$\sum_j \nabla_j \sigma_{ij} = 2\mu \sum_j \nabla_j U_{ij} + \lambda \nabla_i \sum_j U_{jj} \tag{154}$$

$$= \mu \sum_j \nabla_j^2 u_i + (\lambda + \mu) \nabla_i \sum_j \nabla_j u_j. \tag{155}$$

Rewriting the second line in vector notation yields the equilibrium equation in its final form:

$$\vec{f} + \mu\, \nabla^2 \vec{u} + (\lambda + \mu)\, \nabla\big(\nabla \cdot \vec{u}\big) = \vec{0}. \tag{156}$$

This is known as *Navier's equation of equilibrium* or the *Navier–Cauchy equilibrium equation.* Although this equation is compact, it is often more convenient in analytic work to use the original system.

Because the displacement gradients are assumed to be small, all nonlinear terms are ignored. The displacement field is expressed as a function of the original coordinates of the undeformed material, using the Lagrangian representation.

The linearity of the equilibrium equations allows *superposition* of solutions. For instance, if a body is both compressed and stretched uniformly, the total displacement is simply the sum of the two individual displacements.

The boundary conditions are often implicit in the mere posing of an elastostatics problem. Typically, a part of the body surface is "glued" to a hard surface where the displacement has to vanish, and where the environment automatically provides the external reaction forces necessary to balance the surface stresses. On the remaining part of the body surface, explicit external forces implement the "user control" of the deformation. In regions where the external forces vanish, the body surface is said to be free. For the body to remain at rest, the total external force and the total external moment of force must always vanish.

5.0.2 The Equations of Equilibrium in a gravitational field

Substituting into the general equilibrium equations,

$$\sum_k \frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0, \tag{157}$$

the stress–strain relation, we have

$$\sum_k \frac{\partial \sigma_{ik}}{\partial x_k} = \frac{E\nu}{(1+\nu)(1-2\nu)} \frac{\sum_l \partial U_{ll}}{\partial x_i} + \frac{E}{1+\nu} \sum_k \frac{\partial u_{ik}}{\partial x_k}. \tag{158}$$

Using the strain definition

$$U_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \tag{159}$$

we obtain the equilibrium equations in the form

$$\frac{E}{2(1+\nu)} \sum_k \frac{\partial^2 u_i}{\partial x_k^2} + \frac{E}{2(1+\nu)(1-2\nu)} \sum_l \frac{\partial^2 u_l}{\partial x_i \partial x_l} + \rho g_i = 0. \tag{160}$$

These equations can be written more compactly in vector notation. The quantities $\partial^2 u_i/\partial x_k^2$ are the components of $\nabla^2 \vec{u}$, and $\sum_l \partial u_l/\partial x_l = \nabla \cdot \vec{u}$. Thus the equations become

$$\nabla^2 \vec{u} + \frac{1}{1-2\nu} \nabla(\nabla \cdot \vec{u}) = -\rho \vec{g}\, \frac{2(1+\nu)}{E}. \tag{161}$$

It is sometimes helpful to use the vector identity

$$\nabla(\nabla \cdot \vec{u}) = \nabla^2 \vec{u} + \nabla \times \nabla \times \vec{u}.$$

Then

$$\nabla(\nabla \cdot \vec{u}) - \frac{1-2\nu}{2(1-\nu)} \nabla \times \nabla \times \vec{u} = -\rho \vec{g}\, \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}. \tag{162}$$

5.0.3 Surface-Loaded Bodies

A particularly important case occurs when the deformation is caused not by body forces but by forces applied only at the boundary. The equilibrium equation then becomes

$$(1-2\nu)\nabla^2\vec{u} + \nabla(\nabla\cdot\vec{u}) = 0, \tag{163}$$

or, using the identity above,

$$2(1-2\nu)\nabla(\nabla\cdot\,\vec{u}) - (1-2\nu)\nabla\times\,\nabla\times\,\vec{u} = 0. \tag{164}$$

The external forces appear in the solution only through the boundary conditions. Taking the divergence and using the identity

$$\nabla\cdot\nabla \equiv \nabla^2,$$

we obtain

$$\nabla^2(\nabla\cdot\vec{u}) = 0, \tag{165}$$

i.e. $\nabla\cdot\vec{u}$ (the volumetric strain) is a harmonic function.

Taking the Laplacian we further obtain

$$\nabla^2\nabla^2\,\vec{u} = 0, \tag{166}$$

demonstrating that, in equilibrium, the displacement vector satisfies the *biharmonic equation*. This remains valid in a uniform gravitational field, but not for general external forces varying in space.

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