Hydrostatics

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3], Potter et. al [4], White [5] books.

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC3. Definir el concepto de presión desde una perspectiva microscópica.
- OC4. Enunciar el teorema de Pascal y explicar su significancia.
- OC5. Reconocer las distintas unidades que se usan para medir la presión.
- OC6. Identificar las condiciones físicas necesarias para describir un medio usando la ecuación de estado politrópica.
- OC7. Enunciar el principio de Arquímides y estudiar a partir de él las condiciones de estabilidad de cuerpos que flotan.
- OC8. Definir el concepto de tensión superficial, tanto desde la perspectiva de una densidad de energía superficial como desde las fuerzas por unidad de longitud.

1.2 Objetivos específicos procedimentales:

- OP3. Deducir, a partir del teorema de Bernoulli, la ecuación de equilibrio hidrostático global y local.
- OP4. Describir las condiciones de densidad y presión de fluidos incompresibles y compresibles en equilibrio hidrostático.
- OP5. Deducir las ecuaciones del interior de un cuerpo gravitante esférico en equilibrio hidrostático.
- OP6. Deducir la ecuación de Lane-Emden y aplicarla para estudiar el caso de estrellas hipotéticas que obedecen la ecuación de estado politrópica.
- OP7. Aplicar la definición de tensión superficial para describir el fenómeno de capilaridad.

1.3 Objetivos específicos actitudinales:

- OA2. Distinguir y enumerar algunas de las ventajas más importantes de la aproximación continua frente a la aproximación discreta (estadística) de los medios materiales.

2 Pleliminaries

2.1 Density

Fluid is the generic name for liquids and gases. A gas completely fills a closed container, but a liquid does not. The (matter) density of a fluid is defined as

$$\rho \equiv \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}. \tag{1}$$

In continuum physics, density is understood as a macroscopic average over a sufficiently large number of microscopic constituents. This requires a separation of scales: the microscopic length scale (molecular spacing or mean free path) must be much smaller than the macroscopic scale on which density varies appreciably.

In addition to the usual *mass density* ρ, one often encounters:

- **Number density** n: number of particles per unit volume, with $\rho = nm$ for particles of mass m.
- **Energy density** ρ_E: energy per unit volume, relevant in relativistic systems and cosmology.

Table 1: Densities of some common substances

Material	Density (kg/m^3)	Material	Density (kg/m^3)
Air (1 atm, 20°C)	1.20	Iron ($Z = 26$), steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper ($Z = 29$)	8.9×10^3
Ice	0.92×10^3	Silver ($Z = 47$)	10.5×10^3
Water	1.00×10^3	Lead ($Z = 82$)	11.3×10^3
Seawater	1.03×10^3	Mercury ($Z = 80$)	13.6×10^3
Blood	1.06×10^3	Gold ($Z = 79$)	19.3×10^3
Glycerin	1.26×10^3	Platinum ($Z = 78$)	21.4×10^3
Concrete	2.0×10^3	Iridium ($Z = 77$)	22.56×10^3
Aluminum	2.7×10^3	Osmium ($Z = 76$)	22.59×10^3

2.2 Astrophysical objects

Stars. The Sun has an average density of $\sim 1.4 \times 10^3\,\mathrm{kg/m^3}$, comparable to water, but its core density reaches $\sim 1.5 \times 10^5\,\mathrm{kg/m^3}$. In stellar structure, density profiles $\rho(r)$ enter the equations of hydrostatic equilibrium, balancing gravitational forces against pressure gradients.

Compact objects.

- *White dwarfs:* electron-degenerate matter with densities of order $10^9 - 10^{10}\,\mathrm{kg/m^3}$.
- *Neutron stars:* neutron-degenerate matter with densities up to $10^{18}\,\mathrm{kg/m^3}$, comparable to nuclear matter density.

In these cases, density determines the degeneracy pressure that counteracts gravity.

Galactic environments. In addition to stars and compact objects, density plays a central role in galactic dynamics:

- *Galactic center:* The Milky Way hosts a supermassive black hole, Sgr A*, surrounded by a dense stellar cluster. The average stellar mass density in the inner parsec reaches values of order $\rho \sim 10^9\, M_\odot\,\mathrm{pc}^{-3}$, corresponding to $\sim 4 \times 10^{-11}\,\mathrm{kg/m^3}$. Although extremely dilute compared to ordinary matter, this is enormously high relative to the mean cosmic density.

- *Local Group:* On megaparsec scales, the mean density of galaxies in the Local Group is of order $\rho \sim 10^{-25}\,\mathrm{kg/m^3}$, only a few times the critical density of the Universe. This low density governs the dynamics of galaxy interactions and the eventual merging of the Milky Way and Andromeda.

Cosmological scales. The mean density of the Universe is extraordinarily low. The critical density is defined as

$$\rho_c = \frac{3H_0^2}{8\pi G}, \tag{2}$$

where $H_0 \approx 68$ km/s/Mpc is the Hubble constant. Numerically, $\rho_c \sim 10^{-26}\,\mathrm{kg/m^3}$. Cosmological structure formation is often described by the density contrast

$$\delta = \frac{\Delta\rho}{\rho},$$

which measures deviations from the average density.

Astrophysical fluids. In accretion disks, stellar winds, and interstellar clouds, density controls emission and cooling processes, and it determines the onset of instabilities such as the Jeans instability, which occurs when self-gravity overcomes pressure support.

Black holes. Although a black hole is not composed of matter in the usual sense, one can define an *effective average density* as the mass divided by the volume of a sphere with the Schwarzschild radius $R_s = 2GM/c^2$. This yields

$$\rho_{\mathrm{BH}} = \frac{3c^6}{32\pi G^3 M^2}. \tag{3}$$

Interestingly, this density decreases with increasing mass:

- A stellar-mass black hole ($M \sim 10\, M_\odot$) has an effective density of order $10^{19}\,\mathrm{kg/m^3}$, even greater than nuclear matter.

- A supermassive black hole ($M \sim 10^9\, M_\odot$) has an effective density as low as $\sim 10^3\,\mathrm{kg/m^3}$, comparable to water.

This illustrates that black holes are not "dense" in the ordinary sense, but are instead defined by the curvature of spacetime.

Table 2 lists characteristic densities of ordinary and astrophysical systems. Density connects microscopic physics (molecular interactions, degeneracy pressure, nuclear matter properties) with macroscopic astrophysical phenomena (stellar equilibrium, gravitational collapse, cosmic expansion). It thus provides a unifying parameter linking laboratory scales to the most extreme environments in the Universe.

Table 2: Characteristic densities of materials and astrophysical systems.

System	**Density (kg/m^3)**
Air (1 atm, 20°C)	~ 1
Water	1.0×10^3
Average density of the Sun	1.4×10^3
Solar core	1.5×10^5
White dwarf	$10^9 - 10^{10}$
Neutron star	10^{18}
Stellar-mass black hole ($\sim 10M_\odot$)	$\sim 10^{19}$
Supermassive black hole ($\sim 10^9 M_\odot$)	$\sim 10^3$
Galactic center (Milky Way)	$\sim 4 \times 10^{-11}$
Local Group (galaxies)	$\sim 10^{-25}$
Critical density of Universe	$\sim 10^{-26}$

3 Hydrostatic equilibrium

In the absence of external forces or internal heat sources, fluids eventually reach *hydrostatic equilibrium*, where motion ceases and fields remain constant in time. Examples: atmosphere, oceans, planetary and stellar interiors.

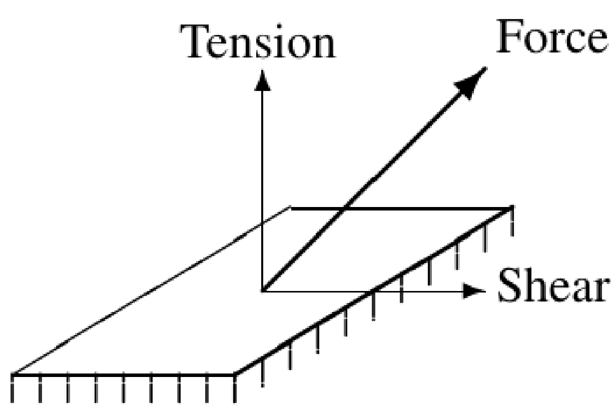


Figure 1: The force acting on the material underneath a small patch of a surface can always be resolved into a perpendicular pressure force and a tangential shear force. The pressure is positive if the force is directed toward the patch, and negative if it (as here) is directed away from it. Figure taken from Ref. [3].

Mechanical equilibrium in a continuum involves a balance between:

- *Contact forces* (short-range, acting through surfaces).
- *Body forces* (long-range, e.g. gravity).

Contact forces act across *contact surfaces* and can be decomposed into (see Fig. 1):

- **Normal component**: aligned with surface normal.
  - Same orientation → *tension force*.

  - Opposite orientation → *pressure force.*

- **Tangential component**: parallel to surface → *shear force* or *traction force.*

3.1 Pressure

A fluid at rest cannot sustain shear stresses or tensile stresses. Any attempt to impose a tangential force on the fluid is immediately relieved by flow, so the intermolecular forces in a static fluid can only transmit *normal stresses.* These stresses act equally in all directions and always tend to compress objects immersed in the fluid.

Thus, the force exerted by a static fluid on the surface of an object is always *perpendicular* (normal) to that surface. This isotropic normal stress is what we call the **pressure** of the fluid.

The pressure p in a static fluid is defined as the normal force per unit area:

$$p = \frac{d\mathcal{F}_\perp}{dS}, \tag{4}$$

where $d\mathcal{F}_\perp$ is the component of force acting normal to a surface of area dS. Its SI unit is the newton per square meter ($\mathrm{N/m^2}$), which in 1971 was given the special name *pascal* (Pa). Thus,

$$1\,\mathrm{Pa} = 1\,\mathrm{N/m^2}.$$

Earlier units of pressure that are still in use include:

- The **bar**, defined as $1\,\mathrm{bar} = 10^5\,\mathrm{Pa}$.
- The **standard atmosphere**, defined as

  $$1\,\mathrm{atm} = 101{,}325\,\mathrm{Pa},$$

  which corresponds approximately to the average air pressure at sea level.

In many practical contexts, other pressure units may appear:

- **Torr**: commonly used in vacuum physics, defined as $1\,\mathrm{Torr} = \frac{1}{760}\,\mathrm{atm} \approx 133.3\,\mathrm{Pa}$.
- **mmHg**: pressure exerted by a column of mercury of height 1 mm, historically used in meteorology and medicine.

Remark. Pressure is a scalar quantity in fluids at rest, because it acts equally in all directions. In fluids in motion, however, stresses may include both pressure and shear contributions, and the concept of a *stress tensor* becomes necessary for a complete description.

Applications. Understanding pressure units and conversions is essential in astrophysics and geophysics: for example, the pressure in the solar core reaches $\sim 10^{16}\,\mathrm{Pa}$, while the pressure at the center of the Earth is about $3.6 \times 10^{11}\,\mathrm{Pa}$. These extreme values highlight the wide range of conditions described by the same fundamental concept of pressure.

3.2 The pressure field

The (flat) vector surface elements is defined as the product of its area dS and the unit vector $\hat{n}$in the direction of the normal to the surface,

$$d\vec{S} = \hat{n}\, dS. \tag{5}$$

By universal convention, the normal of a closed surface is taken to point *outward*, so that the enclosed volume lies on the negative side.

A fluid at rest cannot sustain shear forces, so all contact forces on a surface must act along the normal at every point of the surface. The force exerted by the material at the *positive* side of the surface element dS (near $\mathbf{x}$) on the material at the *negative* side is written as

$$d\vec{\mathcal{F}} = -p(\vec{x})\, d\vec{S}, \tag{6}$$

where $p(\vec{x})$ is the pressure field. Convention dictates that a positive pressure exerts a force directed *toward* the material on the negative side of the surface element, and this explains the minus sign. A negative pressure that pulls at a surface is sometimes called a *tension*. The total pressure force acting on any oriented surface S is obtained by adding all the little vector contributions from each surface element,

$$\vec{\mathcal{F}} = \int_S -p(\vec{x})\, d\vec{S}. \tag{7}$$

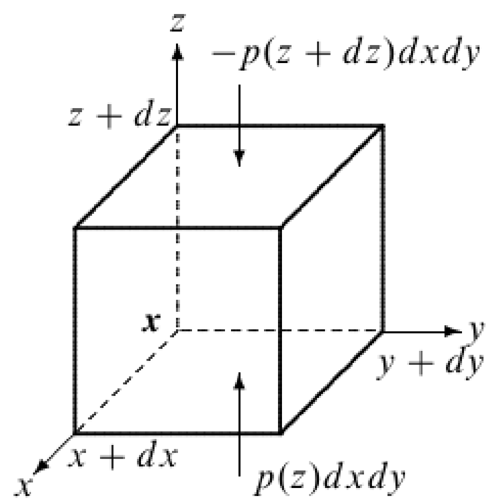


Figure 2: Pressure forces in the z-direction on a material particle in the shape of a rectangular box with sides dx, dy, dz. Figure taken from Ref. [3]

A material particle in the fluid is like any other body subject to pressure forces from all sides, but being infinitesimal it is possible to derive a general expression for the resultant force. Let us consider a material particle in the shape of a small rectangular box with sides dx, dy, and dz, such that it has a volume $dV = dx\, dy\, dz$. Since the pressure is slightly different on opposite sides of the box, the z-component of total pressure force becomes

$$d\mathcal{F}_z = \big(p(x,y,z) - p(x,y,z+dz)\big)\, dx\, dy \;\approx\; -\frac{\partial p(x,y,z)}{\partial z}\, dx\, dy\, dz. \tag{8}$$

Doing the same for the other coordinate directions we obtain the total pressure force

$$d\vec{\mathcal{F}} \; = \; -\left(\hat{\imath}\,\frac{\partial p}{\partial x} + \hat{\jmath}\,\frac{\partial p}{\partial y} + \hat{k}\,\frac{\partial p}{\partial z}\right) dV \; = \; -\vec{\nabla} p\, dV. \tag{9}$$

3.3 Hydrostatic Equilibrium

In equilibrium, the net force on a differential fluid element vanishes:

$$d\vec{F}_e + d\vec{F}_p = 0, \tag{10}$$

where $d\vec{F}_e$ denotes external (volumetric) and $d\vec{F}_p$ the pressure forces. For fluids, forces are of two kinds:

- **Surface forces**: e.g. pressure and viscous stresses.
- **Volumetric forces**: e.g. gravity, electromagnetic, or inertial (fictitious) forces.

3.3.1 Global hydrostatic equlibrium

Integrating the previous expression gives that the total force in mechanical equilibrium mus vanish,

$$\vec{F}_e + \vec{F}_p = \int_V \vec{f}_e\, dV - \oint_S p\, d\vec{S} = 0. \tag{11}$$

When $\vec{f}_e$ is the force of gravity it corresponds to the weight of the fluid body, and the force due to pressure is called the buoyancy force.

Advantages.

- Provides the overall balance of forces for the entire fluid system.
- Very useful in cases with high symmetry, which simplifies the evaluation of integrals.
- Can be applied directly to obtain macroscopic results such as Archimedes' principle.
- Avoids the need to solve differential equations when the geometry is simple.

Disadvantages.

- Requires explicit knowledge of the force density $\rho\mathbf{g}$ and the pressure field p in order to evaluate the integrals.
- Insufficient for complex geometries where the pressure distribution is not known a priori.
- Does not provide the local pressure at each point in the fluid.
- Less general than the local (differential) form of the hydrostatic equilibrium equation.

3.3.2 Local hydrostatic equilibrium

In static fluids only normal pressures remain, since a fluid at rest cannot sustain shear stresses. Thus,

$$d\vec{F}_e - \vec{\nabla} p \, dV = 0. \tag{12}$$

Defining the volumetric density of external forces as

$$\vec{f}_e = \frac{d\vec{F}_e}{dV}, \tag{13}$$

and the effective force density

$$\vec{f}^* = \vec{f}_e - \vec{\nabla} p, \tag{14}$$

we obtain that in hydrostatic equilibrium $\vec{f}^* = 0$, leading to the basic equation of hydrostatics

$$\vec{f}_e = \vec{\nabla} p. \tag{15}$$

For the case of $\vec{f}_e$ being gravity force,

$$\rho \, \vec{g} = \vec{\nabla} p, \tag{16}$$

where $\vec{g}$ is the gravitational field.

Moreover, since $\vec{\nabla} \times \vec{\nabla} p = 0$, one has

$$\vec{\nabla} \times \vec{f}_e = 0, \tag{17}$$

which implies that $\vec{f}_e$ derives from a potential,

$$\vec{f}_e = -\vec{\nabla} \mathcal{H}. \tag{18}$$

Hence, hydrostatic equilibrium exists only when the volumetric forces acting on a fluid are conservative. This is the case for gravitational, electrostatic, and inertial forces.

Open Question. What occurs for a charged fluid placed in a magnetic field?

3.3.3 Pressure Force and Gauss' Theorem

The total pressure force on a body of volume V can be obtained in two equivalent ways:

- By integrating the pressure forces acting on its surface S.
- By integrating the forces on all its constituent material particles.

This leads to the relation

$$\oint_S p \, d\vec{S} = \int_V \vec{\nabla} p \, dV. \tag{19}$$

This identity is a direct consequence of **Gauss' theorem**, which relates the surface integral of a scalar field $p(\mathbf{x})$ over a closed surface S to the volume integral of its gradient $\vec{\nabla} p(\vec{x})$ over the enclosed volume V. Hence Gauss' theorem allows us to convert the local equation back into the global one, showing that there is complete mathematical equivalence between the local and global formulations of hydrostatic equilibrium.

3.4 Pascal's principle

Because pressure in a static fluid is the same in all directions, it can be treated as a scalar quantity. This isotropy underlies **Pascal's principle**, which states that any change of pressure applied to a confined fluid is transmitted undiminished throughout the fluid and to the walls of its container. Pascal's principle forms the basis of many hydraulic devices and has wide-ranging applications in both engineering and natural systems.

From Eq. (15), by integrating along a path from point 1 to point 2 inside a fluid, we obtain

$$\int_1^2 \vec{f}_e \cdot d\vec{r} = \int_1^2 \vec{\nabla} p \cdot d\vec{r} = \int_1^2 dp = p_2 - p_1, \tag{20}$$

so that the pressure difference between two points of a fluid in hydrostatic equilibrium depends only on the external forces, such as gravity. If there are no external forces, then $p_2 = p_1$. Moreover,

$$p_2 - p_1 = (p_2 + C) - (p_1 + C) = \int_1^2 \vec{f}_e \cdot d\vec{r}, \tag{21}$$

showing that the addition of a constant C to the pressure at all points of a fluid leaves the difference $p_2 - p_1$ invariant. That is, any pressure shift C applied at one point is accompanied by the same shift at all other points. This statement is known as **Pascal's principle**. Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

This principle introduces a global gauge freedom in hydrostatics. In differential form, and taking into account that $\vec{\nabla} C = 0$, it follows that $\vec{f}_e = \vec{\nabla} p = \vec{\nabla}(p + C)$.

3.4.1 Hydraulic press

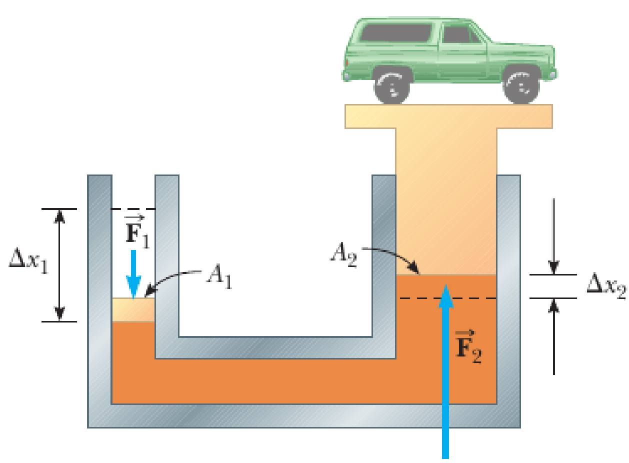


Figure 3: Diagram of a hydraulic press. Figure taken from Ref. [6]

An important application of Pascal's law is the hydraulic press (see Fig. 3). A force F_1 is applied to a small piston of surface area A_1. The pressure is transmitted through an incompressible liquid to a larger piston of area A_2. Since the pressure must be equal on both pistons,

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}. \tag{22}$$

Therefore, the output force is amplified by the factor A_2/A_1.

Because the liquid is incompressible, the displaced volumes satisfy

$$A_1 \Delta x_1 = A_2 \Delta x_2,$$

so that

$$\frac{F_2}{F_1} = \frac{\Delta x_1}{\Delta x_2}.$$

Multiplying both sides by displacements shows that the work done is conserved:

$$F_1 \Delta x_1 = F_2 \Delta x_2.$$

The hydraulic press allows a small input force to generate a much larger output force, while conserving energy. This principle is widely used in hydraulic brakes, jacks and car lifts.

3.5 Hydrostatics in constant gravity

In a flat-Earth coordinate system, the constant field of gravity is $\vec{g}(\vec{r}) = (0, 0, -g_0)$ for all $\vec{r}$. If the external force is gravity, then:

$$\vec{f}_e = \frac{d\vec{F}}{dV} = \frac{dm}{dV}\,\vec{g} = \rho \vec{g},$$

so that,

$$\vec{\nabla} p - \rho \vec{g} = 0. \tag{23}$$

It follows that

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho g_0. \tag{24}$$

The first two equations express that the pressure does not depend on x and y but only on z. It also shows that, independently of the shape of a fluid container, the pressure will always be the same at a given depth (in constant gravity).

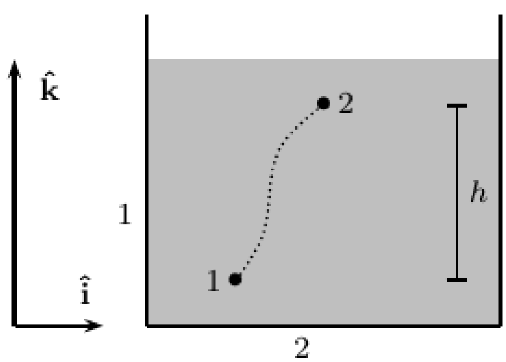


Figure 4: The pressure difference between points 1 and 2 inside a fluid in the Earth's gravity field depends only on their vertical separation. Figure taken from Ref. [1].

For the special case of constant density, $\rho(z) = \rho_0$, the last equation may immediately be integrated. The line integral between points 1 and 2 in Fig. 4, with ρ and $\vec{g}$ constant, is:

$$p_2 - p_1 = \int_1^2 \mathbf{f}_e \cdot d\mathbf{r} = \int_1^2 \rho \mathbf{g} \cdot d\mathbf{r} = -\rho g \int_1^2 dz = -\rho g (z_2 - z_1) = -\rho g h. \tag{25}$$

Thus, the pressure inside a liquid increases with depth:

$$p_1 = p_2 + \rho g h, \tag{26}$$

an expression identical to $(p_1 + C) = (p_2 + C) + \rho g h$. This equation shows that if a force is applied to the free surface of a liquid (for example, with a piston), the pressure increases equally at every point of the fluid. Therefore, the *difference in pressure* between two points depends only on their vertical separation.

3.5.1 Communicating vessels

From the last expression in Eq.4 (24), it follows that $p = \rho g z + C$. At the free surface of the liquid $(z = h)$ the pressure is atmospheric p_0 , so $C = p_0 + \rho g h$. Therefore

$$p = p_0 + \rho\, g(h - z). \tag{27}$$

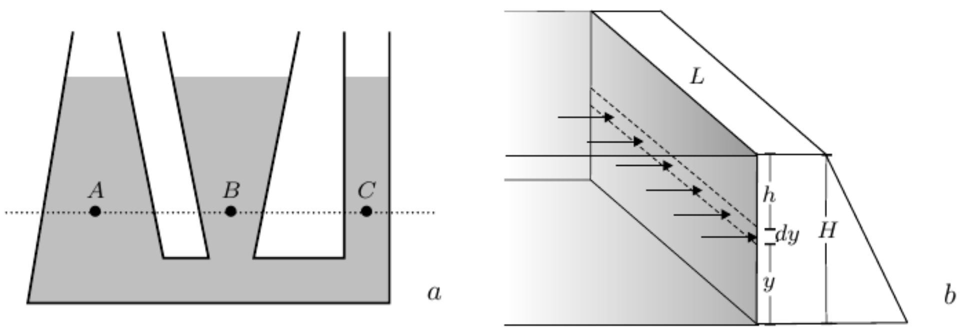


Figure 5: *a*) In communicating vessels, the pressure at points A, B, and C, located at the same height, is the same. *b*) The horizontal hydrostatic force exerted by the water in a dam on a vertical wall differential element of height dy and width L. Figure taken from Ref. [1].

It is therefore true that points at the same depth have the same pressure, independently of the shape of the container (see left panel of Fig. 5). This gives validity to the so-called *principle of communicating vessels* and resolves the so-called *hydrostatic paradox*, according to which—erroneously— the pressure inside a fluid would depend on the shape of the container. As seen here, the principle of communicating vessels is another result derived from the above expression for gravitational external forces $\vec{f}_e$.

According to the erroneous interpretation, the pressure at point B would be greater than at A, since more liquid lies above B than above A. The pressure at C would then be intermediate between those at A and B. The supposed paradox is that in fact

$$P_A = P_B = P_C.$$

This Equation shows that pressure is a scalar quantity, which ensures that *at every point inside the fluid the pressure is the same in all directions.*

3.5.2 The Force on a Dam

Water is filled to a height H behind a dam of width L (see right panel of Fig. 5). Let's determine the resultant force exerted by the water on the dam.

Because pressure increases with depth, the total force on a submerged surface cannot be calculated by simply multiplying the area by a single pressure value. Instead, we must account for the variation of pressure with depth, which requires the use of integration.

Consider a vertical y-axis with $y = 0$ at the bottom of the dam. We divide the face of the dam into narrow horizontal strips located at a height y above the bottom, as illustrated by the dashed strip in the figure.

The atmospheric pressure acting on both sides of the wall does not exert a net force, so it will not be considered. At a height y measured from the bottom of the dam, the pressure has the value

$$p = \rho g h = \rho g(H - y).$$

The horizontal force that the water exerts on the differential portion of wall of area $L\,dy$ is

$$dF = p\,dA = pL\,dy = \rho g L(H - y)\,dy,$$

so that the total horizontal net force, directed to the right, due to the water is

$$F = \rho g L \int_0^H (H - y)\,dy = \tfrac{1}{2}\rho g L H^2.$$

3.5.3 Pressure gauges

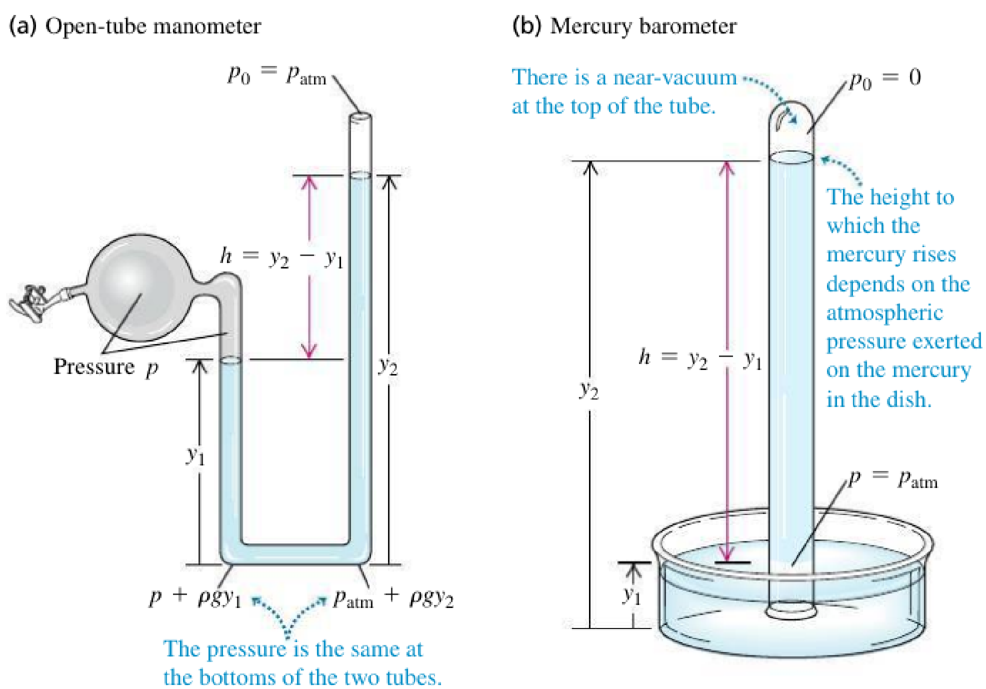


Figure 6: Two types of pressure gauge. Figure taken from Ref. [6]

The simplest pressure gauge is the open-tube *manometer* (left panel of Fig. 6). The U-shaped tube contains a liquid of density ρ, often mercury or water. The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\text{atm}}$. The pressure at the bottom of the tube due to the fluid in the left column is $p + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\text{atm}} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

$$p + \rho g y_1 = p_{\text{atm}} + \rho g y_2,$$

$$p - p_{\text{atm}} = \rho g (y_2 - y_1) = \rho g h.$$

Here, p is the *absolute pressure*, and the difference $p - p_{\text{atm}}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

Another common pressure gauge is the *mercury barometer*. It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (right panel of Fig. 6). . The space above the mercury column contains only mercury vapor; its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. It follows that

$$p_{\text{atm}} = p = 0 + \rho g (y_2 - y_1) = \rho g h.$$

So the height h of the mercury column indicates the atmospheric pressure p_{atm}.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "inches of mercury" or "millimeters of mercury" (abbreviated mm Hg). A pressure of 1 mm Hg is called *1 torr*, after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of g, which varies with location, so the pascal is the preferred unit of pressure.

Open Question A manometer tube is partially filled with water. Oil (which does not mix with water) is poured into the left arm of the tube until the oil-water interface is at the midpoint of the tube as shown in Fig. 7. Both arms of the tube are open to the air. Find a relationship between the heights h_{oil} and h_{water}.

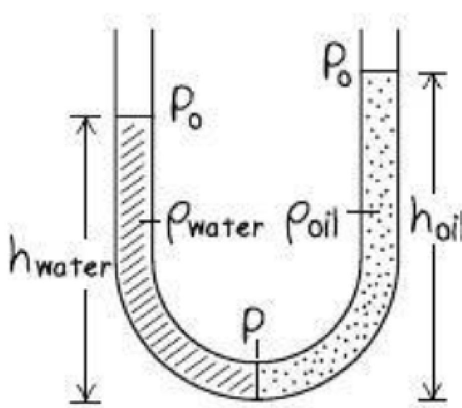


Figure 7: A manometer tube partially filled with water. Figure taken from Ref. [6].

Problems

1. The gate CD in the left panel of Fig. 8 can rotate about C. What horizontal force F, applied at a height h, is required to prevent the gate from rotating under the action of the water on its left, if its width is L?

2. What is the force per unit length required to prevent the cylinder in the right panel of Fig. 8 from rolling under the action of the water located to its left?

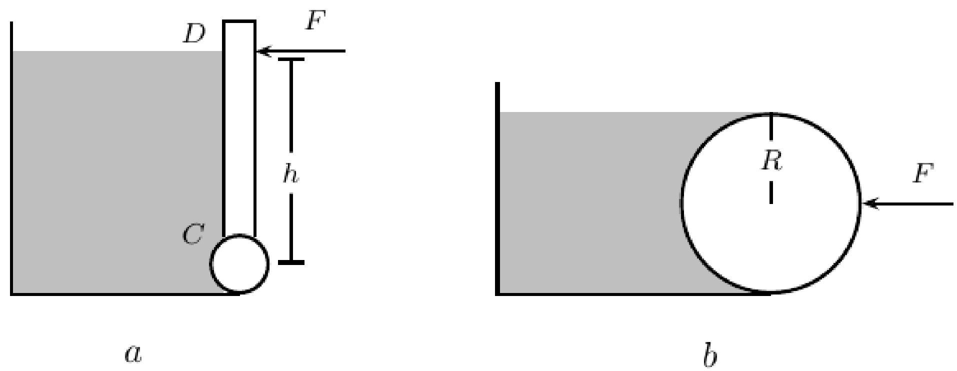


Figure 8: Hydrostatic force on a flat gate in a dam (a) and on the surface of a horizontal cylinder (b). Figure taken from Ref. [1].

3.6 Equation of state

The local equation of hydrostatic equilibrium with an externally given gravitational field is not sufficient by itself; we need a relation between pressure and density. Thermodynamics provides this relationship through the *equation of state*, which relates density ρ, pressure p, and absolute temperature T:

$$F(\rho, p, T) = 0. \tag{28}$$

This relation is valid for any macroscopic amount of homogeneous isotropic fluid in thermodynamic equilibrium. In continuum physics, where conditions vary from point to point, it is assumed that each material particle is in thermodynamic equilibrium with its surroundings, so the equation of state holds locally:

$$F(\rho(x), p(x), T(x)) = 0. \tag{29}$$

3.6.1 The Ideal Gas Law

The *ideal gas law* is the most famous equation of state, often attributed to Clapeyron (1834). In terms of the volume V of a mass M of gas and the number of moles $n = M/M_{\rm mol}$, it is written as

$$pV = nR_{\rm mol}T, \tag{30}$$

where $R_{\rm mol} = 8.31447\,\mathrm{J\,K^{-1}\,mol^{-1}}$ is the *universal molar gas constant*. Since $\rho = M/V = nM_{\rm mol}/V$, the ideal gas law can be expressed in the local form:

$$p = R\rho T, \quad R = \frac{R_{\rm mol}}{M_{\rm mol}}. \tag{31}$$

Remarks. The specific gas constant R varies for different gases (see Table 3). Extensions of the ideal gas law include corrections for excluded molecular volume and intermolecular forces. The ideal gas law also applies to mixtures of gases if the average molar mass is used.

	$10^3 M_{\mathrm{mol}}$ (kg mol^{-1})	$10^{-3} R$ (J K^{-1} kg^{-1})
H_2	2.0	4.157
He	4.0	2.079
Ne	20.2	0.412
N_2	28.0	0.297
O_2	32.0	0.260
Ar	39.9	0.208
CO_2	44.0	0.189
Air	29.0	0.287

Table 3: The molar mass, M_{mol}, and the specific gas constant, $R = R_{\mathrm{mol}}/M_{\mathrm{mol}}$, for a few gases.

The microscopic origin of the ideal gas law is as follows. The equipartition theorem tells us that the average kinetic energy of a molecule of mass m is

$$\frac{1}{2}m\langle v^2\rangle = \frac{1}{2}m\langle v_x^2\rangle + \frac{1}{2}m\langle v_y^2\rangle + \frac{1}{2}m\langle v_z^2\rangle = \frac{3}{2}k_B T, \tag{32}$$

from which

$$p = \frac{1}{3}\rho\langle v^2\rangle = \rho\frac{k_B T}{m} = \rho\frac{N_A k_B T}{N_A m} = \rho\frac{R_{\mathrm{mol}} T}{M_{\mathrm{mol}}}. \tag{33}$$

3.6.2 Case: Isothermal atmosphere

Let us assume a constant temperature, $T(x) = T_0$, and combine the equation of hydrostatic equilibrium with the ideal gas law. Then

$$\frac{dp}{dz} = -\rho g_0 = -\frac{g_0}{RT_0}p. \tag{34}$$

This is an ordinary differential equation for the pressure, and using the initial condition $p = p_0$ for $z = 0$, we find the solution

$$p = p_0 e^{-z/h_0}, \qquad h_0 = \frac{RT_0}{g_0} = \frac{p_0}{\rho_0 g_0}. \tag{35}$$

In the last step we have again used the ideal gas law at $z = 0$ to show that the expression for h_0 is identical to the incompressible atmospheric scale height.

In the isothermal atmosphere the pressure thus decreases exponentially with height on a characteristic length scale again set by $h_0 \approx 8728\,\mathrm{m}$ calculated for 1 atm and 25°C. The pressure at the top of Mount Everest ($z = 8848\,\mathrm{m}$) is now finite and predicted to be 368 hPa.

3.6.3 Barotropic equation of state

Sometimes there exists a so-called *barotropic* relationship between density and pressure,

$$F(\rho(x), p(x)) = 0, \tag{36}$$

which does not depend on the local temperature $T(x)$.

Polytropic relation

The first law of thermodynamics in differential form reads

$$T\,ds = de + p\,d\left(\frac{1}{\rho}\right), \tag{37}$$

where s is the entropy, e the specific internal energy, p the pressure, and ρ the mass density. For an isentropic process we have $ds = 0$, so

$$de = -p\,d\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\,d\rho. \tag{38}$$

For an ideal gas, $e = c_v T$ and $p = \rho R T$. Thus

$$c_v\,dT = \frac{p}{\rho^2}\,d\rho, \qquad T = \frac{p}{\rho R}. \tag{39}$$

Eliminating T we obtain

$$c_v\,d\left(\frac{p}{\rho R}\right) = \frac{p}{\rho^2}\,d\rho, \tag{40}$$

or equivalently

$$\frac{c_v}{R}\left(\frac{dp}{\rho} - \frac{p}{\rho^2}\,d\rho\right) = \frac{p}{\rho^2}\,d\rho. \tag{41}$$

Now, recalling that

$$\gamma \equiv \frac{c_p}{c_v} = 1 + \frac{R}{c_v},$$

we can simplify the relation to

$$\frac{dp}{p} = \gamma\,\frac{d\rho}{\rho}. \tag{42}$$

Integrating both sides gives

$$\ln p = \gamma \ln \rho + \text{const}, \tag{43}$$

which leads to the **polytropic relation**

$$p = C\,\rho^{\gamma}, \tag{44}$$

where C is a constant (the *polytropic constant*), and γ is the *polytropic index* (see Table 4).

3.7 Isobars, Isoclines, and Potentials

For a fluid of density ρ —in general dependent on position— in a gravitational field $\vec{g}$:

$$-\rho\vec{g} + \vec{\nabla}p = 0 \qquad \text{or:} \qquad \rho\vec{\nabla}\Phi + \vec{\nabla}p = 0. \tag{45}$$

Taking the curl of the second equation,

$$\rho\vec{\nabla}\times\vec{\nabla}\Phi + \vec{\nabla}\rho\times\vec{\nabla}\Phi + \vec{\nabla}\times\vec{\nabla}p = 0. \tag{46}$$

Since $\vec{\nabla}\times\vec{\nabla}\Phi \equiv 0$ and $\vec{\nabla}\times\vec{\nabla}p \equiv 0$, it follows that, in a fluid of variable density, this condition holds:

$$\vec{\nabla}\rho\times\vec{\nabla}\Phi = 0 \qquad \text{or:} \qquad \vec{\nabla}\rho\times\vec{g} = 0. \tag{47}$$

	γ	$10^{-3}c_p$
H_2	1.41	14.30
He	1.63	5.38
Ne	1.64	1.05
N_2	1.40	1.04
O_2	1.40	0.91
Ar	1.67	0.52
CO_2	1.30	0.82
Air	1.40	1.00

Table 4: Table of the adiabatic index and the specific heat at constant pressure ($c_p = \gamma R/(\gamma - 1)$) for a few nearly ideal gases. Units: $\mathrm{J\,K^{-1}\,kg^{-1}}$.

This means that, point by point, the direction of the gravitational field lines coincides with that of the density gradient, or, equivalently, that the surfaces of equal density (isoclines) and the gravitational equipotentials are coincident.

Considering

$$\vec{\nabla}\Phi + \frac{\vec{\nabla}p}{\rho} = 0,$$

and its curl is taken, it follows that for a compressible fluid

$$\vec{\nabla}\rho \times \vec{\nabla}p = 0, \tag{48}$$

from which it is concluded that the surfaces of constant pressure (isobars) and those of constant density coincide. From this equation it follows that the pressure is an *exclusive functional* of density: $p = p(\rho)$. Indeed,

$$\vec{\nabla}\rho \times \vec{\nabla}p = \vec{\nabla}\rho \times \left(\frac{dp}{d\rho}\vec{\nabla}\rho\right) = \frac{dp}{d\rho}\vec{\nabla}\rho \times \vec{\nabla}\rho = 0. \tag{49}$$

In an analogous way, it can be shown that Φ is a functional of ρ: $\Phi = \Phi(\rho)$. Thus, the surfaces of Φ, ρ and p constant are the same. This result is valid regardless of the form of the gravitational field. In particular, if $\vec{g} = -\hat{k}g$, the surfaces Φ, ρ and p will be horizontal planes. Outside the Earth's mass, assumed spherical, it is true that

$$\mathbf{g} = -\frac{GM\hat{\mathbf{r}}}{r^2}, \qquad \Phi = -\frac{GM}{r},$$

so the three surfaces are spherical and concentric.

3.8 Pressure potential

Since the gravitational field is conservative and can be obtained from the gradient of the gravitational potential $\Phi(x)$, it follows that

$$\vec{g}(x) = -\vec{\nabla}\Phi(x). \tag{50}$$

In terms of the potential, the hydrostatics equilibrium equation ($\vec{\nabla}p = \rho\vec{g}$) may now be written as

$$\nabla\Phi + \frac{\nabla p}{\rho} = 0. \tag{51}$$

In the case of a constant density, $\rho(x) = \rho_0$, we obtain that

$$\Phi^* \equiv \Phi + \frac{p}{\rho_0} \tag{52}$$

is also constant. In flat-Earth gravity, $\Phi = g_0 z$.

It is always possible to integrate the hydrostatic equation for any barotropic fluid with $\rho = \rho(p)$. Introducing the *pressure potential*,

$$w(p) = \int \frac{dp}{\rho(p)}, \tag{53}$$

it turns out that hydrostatic equilibrium may be written as

$$\Phi^* = \Phi + w(p). \tag{54}$$

Φ^* is dubbed the *effective potential*.

Isothermal gas

Under isothermal conditions, the pressure potential of an ideal gas is calculated by means of the ideal gas law:

$$w = \int \frac{RT_0}{p}\, dp = RT_0 \log p. \tag{55}$$

Polytropic fluid

For fluids obeying a polytropic relation, the pressure potential becomes

$$w = \int C\gamma\rho^{\gamma-1}\frac{d\rho}{\rho} = C\frac{\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}. \tag{56}$$

When the fluid is an ideal gas with $p = R\rho T$, this takes the simpler form

$$w = \frac{\gamma}{\gamma-1}RT = c_p T, \qquad c_p = \frac{\gamma}{\gamma-1}R, \tag{57}$$

where c_p is the specific heat at constant pressure.

3.9 Bulk modulus

All fluids compress if the pressure increases, resulting in a decrease in volume or an increase in density. A common way to describe the compressibility of a fluid is by the bulk modulus of elasticity. It is defined the *bulk modulus* as the pressure increase dp per *fractional decrease* in volume, $-dV/V$, or

$$K = \frac{dp}{-dV/V} = \frac{dp}{d\rho/\rho} = \rho\frac{dp}{d\rho}. \tag{58}$$

In the second step we have used the constancy of the mass $M = \rho V$ of the fluid in the volume to derive that $dM = \rho dV + V d\rho = 0$, from which we get $-dV/V = d\rho/\rho$.

The above definition makes immediate sense for a barotropic fluid, where $p = p(\rho)$ is a function of density. For general fluid states it is necessary to specify the conditions under which the bulk modulus is defined, for example whether the temperature is held constant (isothermal) or whether there is no heat transfer (adiabatic or isentropic). Thus, the equation of state for an ideal gas implies that the isothermal bulk modulus is

$$K_T = \left(\rho \frac{\partial p}{\partial \rho}\right)_T = p, \tag{59}$$

where the index—as commonly done in thermodynamics—indicates that the temperature T is held constant. Similarly, for an isentropic ideal gas obeying the polytropic relation that the *isentropic bulk modulus* becomes

$$K_S = \left(\rho \frac{dp}{d\rho}\right)_S = \gamma p, \tag{60}$$

where the index indicates that the entropy S is held constant. It is larger than the isothermal bulk modulus by a factor of $\gamma > 1$, because adiabatic compression also increases the temperature of the gas, which further increases the pressure.

The definition of the bulk modulus (and the above equation) shows that it is measured in the same units as pressure, for example pascals, bars, or atmospheres. The bulk modulus is actually a measure of *incompressibility*, because the larger it is, the greater is the pressure increase that is needed to obtain a given fractional increase in density. The inverse bulk modulus $\beta = 1/K$ may be taken as a measure of *compressibility.*

The bulk modulus for water at standard conditions is approximately 2100 MPa (310,000 Phi), or 21 000 times the atmospheric pressure. For air at standard conditions, B is equal to 1 atm. In general, B for a gas is equal to the pressure of the gas. To cause a 1% change in the density of water a pressure of 21 MPa (210 atm) is required. This is an extremely large pressure needed to cause such a small change; thus liquids are often assumed to be incompressible. For gases, if significant changes in density occur, say 4%, they should be considered as compressible; for small density changes under 3% they may also be treated as incompressible. This occurs for atmospheric airspeeds under about 100 m/s (220 mph), which includes many airflows of engineering interest: air flow around automobiles, landing and take-off of aircraft, and air flow in and around buildings.

Small density changes in liquids can be very significant when large pressure changes are present. For example, they account for "water hammer," which can be heard shortly after the sudden closing of a valve in a pipeline; when the valve is closed an internal pressure wave propagates down the pipe, producing a hammering sound due to pipe motion when the wave reflects from the closed valve or pipe elbows.

The bulk modulus can also be used to calculate the speed of sound in a liquid:

$$c = \sqrt{\left(\frac{dp}{d\rho}\right)_T} = \sqrt{\frac{K_T}{\rho}} \tag{61}$$

This yields approximately 1450 m/s (4800 ft/s) for the speed of sound in water at standard conditions.

3.10 Earth's homentropic atmosphere

A process that takes place without exchange of heat between a system and its environment is said to be adiabatic. If furthermore the process is reversible, it will conserve the entropy and is called isentropic.

3.10.1 Homentropic equation of state

From the first law:

$$dU = dQ - p\,dV. \tag{62}$$

Adiabaticity implies $dQ = 0$, and for an ideal gas $U = nC_V T$:

$$nC_V\,dT = -p\,dV. \tag{63}$$

Using the ideal–gas law $pV = nRT \Rightarrow dT = \frac{1}{nR}\big(p\,dV + V\,dp\big)$, it follows that

$$\frac{C_V}{R}\big(p\,dV + V\,dp\big) = -p\,dV. \tag{64}$$

Rearrange and use $C_P = C_V + R$:

$$\frac{C_V}{R}\,V\,dp + \frac{C_P}{R}\,p\,dV = 0. \tag{65}$$

Dividing by pV and using $\gamma \equiv C_P/C_V$:

$$\frac{dp}{p} + \gamma\,\frac{dV}{V} = 0. \tag{66}$$

Finally

$$\ln p + \gamma \ln V = \text{const} \quad\Rightarrow\quad p\,V^{\gamma} = \text{const}. \tag{67}$$

In other words,

$$p\,V^{\gamma} = p_0\,V_0^{\gamma}. \tag{68}$$

Expressed in terms of the density $\rho = M/V$, an isentropic process that locally changes density ρ_0 and pressure p_0 to ρ and p must obey the local *polytropic* relation:

$$p\,\rho^{-\gamma} = p_0\,\rho_0^{-\gamma}. \tag{69}$$

3.10.2 Homentropic solution

The hydrostatic equilibrium implies

$$\Phi^* = \Phi + w(p) = g_0\,z + c_p T = \text{const}. \tag{70}$$

Defining this constant as $\Phi^* = c_p T_0$ (with T_0 the sea-level temperature), the temperature profile is

$$T(z) = T_0 - \frac{g_0}{c_p}\,z. \tag{71}$$

Introducing the scale height for the isothermal atmosphere $h_0 = p_0/(\rho_0 g_0)$, this expression takes the form

$$T = T_0\left(1 - \frac{z}{h_2}\right), \qquad h_2 = \frac{c_p T_0}{g_0} = \frac{\gamma}{\gamma - 1}\,h_0, \tag{72}$$

For $\gamma = 7/5$ and $T_0 = 25°\text{C}$, the isentropic scale height is $h_2 \approx 31$ km.

Combining the ideal-gas law with the homentropic (isentropic) relation gives that $T\,\rho^{1-\gamma}$ and $p^{1-\gamma}T^{\gamma}$ are constants, so the density and pressure become

$$\rho = \rho_0\left(1 - \frac{z}{h_2}\right)^{1/(\gamma-1)}, \qquad p = p_0\left(1 - \frac{z}{h_2}\right)^{\gamma/(\gamma-1)}, \tag{73}$$

where ρ_0 and p_0 are the density and pressure at sea level. Note that T, ρ, and p formally vanish at $z = h_2$, which is unphysical; the real atmosphere is more complicated than this simple model.

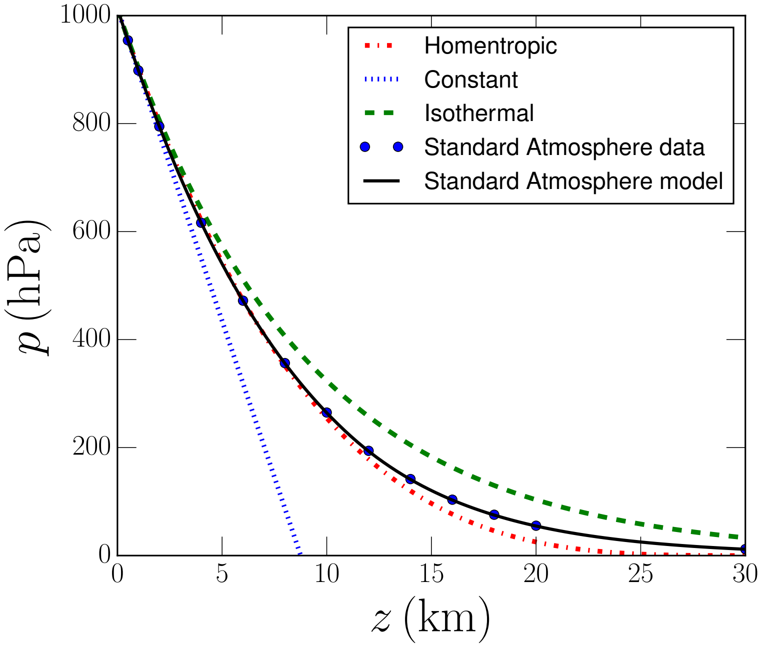


Figure 9: The pressure as a function of altitude (for $T_0 = 25°\text{C}$) in three different atmospheric models analyzed in this chapter: constant density (dotted), isothermal (dashed), and homentropic (dot–dashed). The solid curve is the Standard Atmosphere (1976) model.

3.10.3 The atmospheric temperature lapse rate

The negative vertical temperature gradient, called the *atmospheric temperature lapse rate*, is

$$-\frac{dT}{dz} = \frac{g_0}{c_p}\,. \tag{74}$$

Its value is 9.8 $\text{K}\,\text{km}^{-1}$ for dry air. At the top of Mount Everest ($z = 8.848$ km), the model with $T_0 = 25°\text{C}$ predicts a temperature of $-61°\text{C}$, a density of 0.50 $\text{kg}\,\text{m}^{-3}$, and a pressure of 306 hPa.

Water vapor is always present in the atmosphere and condenses in rising air. The latent heat released during condensation warms the air, making the lapse rate smaller than in dry air, typically 6–7 $\text{K}\,\text{km}^{-1}$, which corresponds to an effective adiabatic index $\gamma \approx 1.23$. Using this value with $T_0 = 25°\text{C}$, the scale height increases to 46 km, and the predicted conditions at the summit of Mount Everest are $p \approx 329$ hPa and $T \approx -33°\text{C}$, close to the measured average of about $-25°\text{C}$.

3.11 Standard Atmosphere

The atmosphere is divided into four layers (see Fig. 10):

1. the *troposphere* (nearest Earth),
2. the *stratosphere*,
3. the *mesosphere*, and
4. the *ionosphere*.

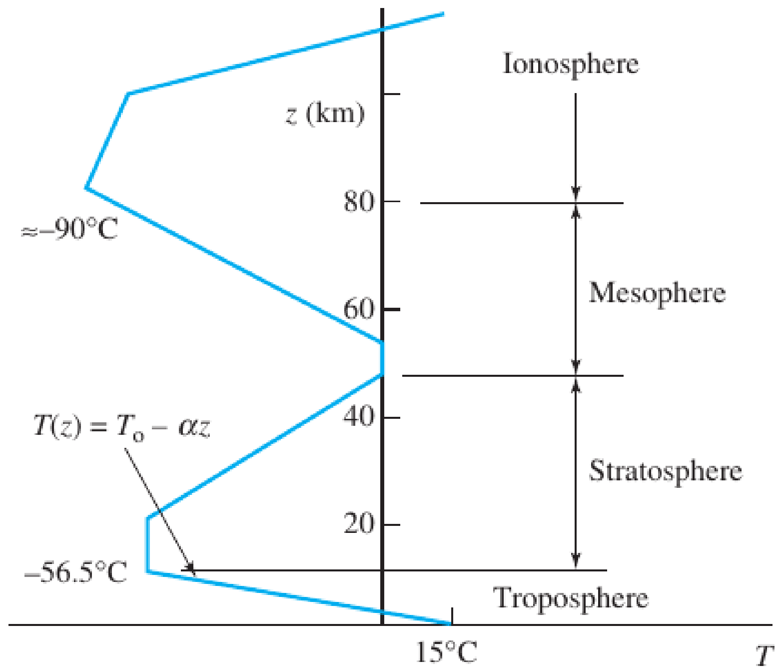


Figure 10: Standard atmosphere. Figure taken from Ref. [4].

Because conditions vary with time and latitude and the layers are thicker at the equator and thinner at the poles, calculations are based on the *standard atmosphere*, taken at 40° latitude.

In the standard atmosphere the temperature in the troposphere varies linearly with elevation,

$$T(z) = T_0 - \alpha z, \tag{75}$$

where the lapse rate is $\alpha = 0.0065$ K/m and $T_0 = 288$ K. In the portion of the stratosphere between 11 and 20 km the temperature is constant at about $-56.5°$C. Above this, the temperature increases again and reaches a maximum near 50 km, then decreases toward the edge of the ionosphere. Because the air density in the ionosphere is so low, it is possible for satellites to orbit the Earth within this layer.

To determine the pressure variation in the troposphere we combine the ideal–gas law $p = \rho RT$ with the hydrostatic relation $dp = -\rho g\, dz$. Eliminating ρ gives

$$\frac{dp}{p} = -\frac{g}{RT}\, dz. \tag{76}$$

With the standard tropospheric lapse $T(z) = T_0 - \alpha z$, integrating from sea level ($z = 0,\ p = p_{\text{atm}}$) to an elevation z

$$\int_{p_{\text{atm}}}^{p} \frac{dp}{p} = -\frac{g}{R}\int_0^z \frac{dz}{T_0 - \alpha z}, \tag{77}$$

leads to

$$\ln\left(\frac{p}{p_{\text{atm}}}\right) = \frac{g}{\alpha R}\ln\left(\frac{T_0 - \alpha z}{T_0}\right), \tag{78}$$

or, equivalently,

$$p = p_{\text{atm}}\left(\frac{T_0 - \alpha z}{T_0}\right)^{g/(\alpha R)}. \tag{79}$$

In the lower stratosphere the temperature is (approximately) constant, $T = T_s$. Integrating $\frac{dp}{p} = -\frac{g}{RT_s}\,dz$ from the tropopause ($z = z_s$, $p = p_s$) to altitude z gives

$$\int_{p_s}^{p} \frac{dp}{p} = -\frac{g}{RT_s}\int_{z_s}^{z} dz \;\;\Rightarrow\;\; \ln\!\left(\frac{p}{p_s}\right) = -\frac{g}{RT_s}\,(z - z_s), \tag{80}$$

or

$$p \;=\; p_s \exp\!\left[\frac{g}{RT_s}\,(z_s - z)\right]. \tag{81}$$

Altitude h(m)	T(K)	p(kPa)	ρ(kg/m^3)	v_s(m/s)
0	288.2	101.3	1.225	340
500	284.9	95.43	1.167	338
1000	281.7	89.85	1.112	336
2000	275.2	79.48	1.007	333
4000	262.2	61.64	0.8194	325
6000	249.2	47.21	0.6602	316
8000	236.2	35.65	0.5528	308
10000	223.3	26.49	0.4136	300
12000	216.7	19.40	0.3119	295
14000	216.7	14.17	0.2278	295
16000	216.7	10.35	0.1665	295
18000	216.7	7.563	0.1216	295
20000	216.7	5.528	0.0889	295
30000	226.5	1.196	0.0184	302
40000	250.4	0.287	4.00×10^{-3}	317
50000	270.7	0.0798	1.03×10^{-3}	330
60000	255.8	0.0225	3.06×10^{-4}	321
70000	219.7	0.00551	8.75×10^{-5}	297
80000	180.7	0.00103	2.00×10^{-5}	269

Table 5: Properties of the Standard Atmosphere. Data taken from Ref. [4].

3.12 The Sun's convective envelope

Stars, like the Sun, are self–gravitating, gaseous, and almost perfectly spherical bodies that generate heat by thermonuclear processes in a fairly small region close to the center. The heat is transported to the surface by radiation, conduction, and convection and is eventually released into space as radiation. Like planets, stars also have a complex structure with several layers differing in chemical composition and other physical properties.

Our Sun consists—like the rest of the universe—of roughly 75% hydrogen and 25% helium (by mass), plus small amounts of other elements (usually called "metals"). The radius of its photosphere is a = 700,000 km, its mass $M_\odot = 2 \times 10^{30}$ kg, and its total luminosity (power) 3.8×10^{26} W. The gases making up the Sun are almost completely ionized, forming a plasma consisting of positively charged hydrogen ions H^+, doubly charged helium ions He^{2+}, and negatively charged electrons e^-. The mean molar mass of the plasma is $M_{\text{mol}} = 0.59\ \text{g}\,\text{mol}^{-1}$. It is smaller than unity because the nearly massless electrons make up about 52% of all the particles in the plasma.

As an exercise, ignore heat production and model only the convective outer layer as a homentropic ideal gas—in other words, model a Sun that does not shine.

3.13 Questions

1. A rubber hose is attached to a funnel, and the free end is bent around to point upward. When water is poured into the funnel, it rises in the hose to the same level as in the funnel, eventhough the funnel has a lot more water in it than the hose does. Why? What supports the extra weight of the water in the funnel?

2. Suppose the door of a room makes an airtight but frictionless fit in its frame. Do you think you could open the door if the air pressure on one side were standard atmospheric pressure and the air pressure on the other side differed from standard by 1%? Explain.

3. At a certain depth in an incompressible liquid, the absolute pressure is p. At twice this depth, will the absolute pressure be equal to $2p$, greater than $2p$, or less than $2p$? Justify your answer.

4. A meteorologist states that the barometric pressure is 28.5 inches of mercury. Convert this pressure to kilopascals.

   (A) 98.6 kPa (B) 97.2 kPa (C) 96.5 kPa (D) 95.6 kPa

5. If the pressure in the air shown in Fig. 11 is increased by 10 kPa, the magnitude of H will be nearest (initially $H = 16$ cm):

   (A) 8.5 cm (B) 10.5 cm (C) 16 cm (D) 24.5 cm

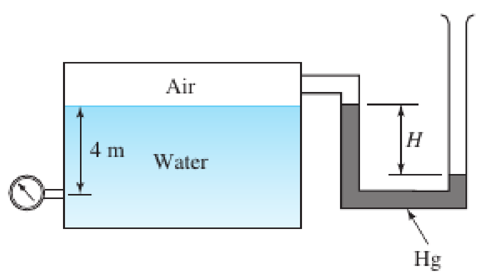


Figure 11: Question 5.

6. The rectangular gate shown in Fig. 12 is 3 m wide. The force P needed to hold the gate in the position shown is nearest:

   (A) 24.5 kN (B) 32.7 kN (C) 98 kN (D) 147 kN

7. A force $P = 300$ kN is needed to just open the gate of Fig. 13 with $R = 1.2$ m and $H = 4$ m. How wide is the gate?

   (A) 2.98 m (B) 3.67 m (C) 4.32 m (D) 5.16 m

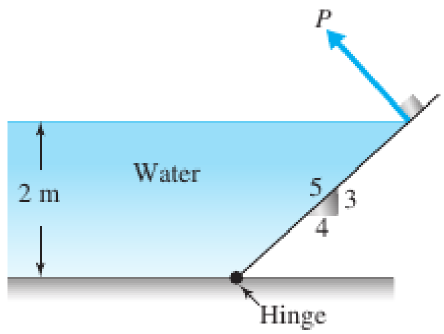


Figure 12: Question 6.

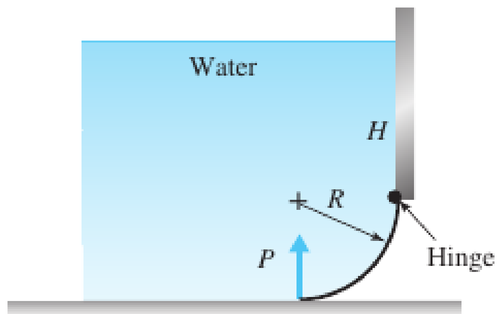


Figure 13: Question 7.

3.14 Problems

1. What is the volume change of $2\,\mathrm{m}^3$ of water at 20°C due to an applied pressure of 10 MPa?

2. Two engineers wish to estimate the distance across a lake. One pounds two rocks together under water on one side of the lake and the other submerges his head and hears a small sound 0.62 s later, as indicated by a very accurate stopwatch. What is the distance between the two engineers?

3. A pressure is applied to 20 L of water. The volume is observed to decrease to 18.7 L. Calculate the applied pressure.

4. **Black Smokers.** Black smokers are hot volcanic vents that emit smoke deep in the ocean floor. Many of them teem with exotic creatures, and some biologists think that life on earth may have begun around such vents. The vents range in depth from about 1500 m to 3200 m below the surface. What is the gauge pressure at a 3200-m deep vent, assuming that the density of water does not vary? Express your answer in pascals and atmospheres.

5. **Oceans on Mars.** Scientists have found evidence that Mars may once have had an ocean 0.500 km deep. The acceleration due to gravity on Mars is 3.71 $\mathrm{m/s^2}$. (a) What would be the gauge pressure at the bottom of such an ocean, assuming it was freshwater? (b) To what depth would you need to go in the earth's ocean to experience the same gauge pressure?

6. (a) Calculate the difference in blood pressure between the feet and top of the head for a person who is 1.65 m tall. (b) Consider a cylindrical segment of a blood vessel 2.00 cm

long and 1.50 mm in diameter. What additional outward force would such a vessel need to withstand in the person's feet compared to a similar vessel in her head?

7. In intravenous feeding, a needle is inserted in a vein in the patient's arm and a tube leads from the needle to a reservoir of fluid (density $1050\,\mathrm{kg/m^3}$) located at height h above the arm. The top of the reservoir is open to the air. If the gauge pressure inside the vein is 5980 Pa, what is the minimum value of h that allows fluid to enter the vein? Assume the needle diameter is large enough that you can ignore the viscosity of the fluid.

8. You are designing a diving bell to withstand the pressure of seawater at a depth of 250 m. a) What is the gauge pressure at this depth? (You can ignore changes in the density of the water with depth.) b) At this depth, what is the net force due to the water outside and the air inside the bell on a circular glass window 30.0 cm in diameter if the pressure inside the diving bell equals the pressure at the surface of the water? (Ignore the small variation of pressure over the surface of the window.)

9. There is a maximum depth at which a diver can breathe through a snorkel tube because as the depth increases, so does the pressure difference, which tends to collapse the diver's lungs. Since the snorkel connects the air in the lungs to the atmosphere at the surface, the pressure inside the lungs is atmospheric pressure. What is the external–internal pressure difference when the diver's lungs are at a depth of 6.1 m (about 20 ft)? Assume that the diver is in freshwater. (A scuba diver breathing from compressed air tanks can operate at greater depths than can a snorkeler, since the pressure of the air inside the scuba diver's lungs increases to match the external pressure of the water.)

10. The lower end of a long plastic straw is immersed below the surface of the water in a plastic cup. An average person sucking on the upper end of the straw can pull water into the straw to a vertical height of 1.1 m above the surface of the water in the cup. a) What is the lowest gauge pressure that the average person can achieve inside his lungs? b) Explain why your answer in part (a) is negative.

11. An electrical short cuts off all power to a submersible diving vehicle when it is 30 m below the surface of the ocean. The crew must push out a hatch of area $0.75\,\mathrm{m^2}$ and weight 300 N on the bottom to escape. If the pressure inside is 1.0 atm, what downward force must the crew exert on the hatch to open it?

12. A tall cylinder with a cross-sectional area $12.0\,\mathrm{cm^2}$ is partially filled with mercury; the surface of the mercury is 8.00 cm above the bottom of the cylinder. Water is slowly poured in on top of the mercury, and the two fluids don't mix. What volume of water must be added to double the gauge pressure at the bottom of the cylinder?

13. A closed container is partially filled with water. Initially, the air above the water is at atmospheric pressure (1.01×10^5 Pa) and the gauge pressure at the bottom of the water is 2500 Pa. Then additional air is pumped in, increasing the pressure of the air above the water by 1500 Pa. (a) What is the gauge pressure at the bottom of the water? (b) By how much must the water level in the container be reduced, by drawing some water out through a valve at the bottom of the container, to return the gauge pressure at the bottom of the water to its original value of 2500 Pa? The pressure of the air above the water is maintained at 1500 Pa above atmospheric pressure.

14. **Exploring Venus.** The surface pressure on Venus is 92 atm, and the acceleration due to gravity there is $0.894g$. In a future exploratory mission, an upright cylindrical tank of benzene is sealed at the top but still pressurized at 92 atm just above the benzene. The tank has a diameter of 1.72 m, and the benzene column is 11.50 m tall. Ignore any effects due to the very high temperature on Venus. (a) What total force is exerted on the inside surface of the bottom of the tank? (b) What force does the Venusian atmosphere exert on the outside surface of the bottom of the tank? (c) What total inward force does the atmosphere exert on the vertical walls of the tank?

15. **The Great Molasses Flood.** On the afternoon of January 15, 1919, an unusually warm day in Boston, a 17.7 m-high, 27.4 m-diameter cylindrical metal tank used for storing molasses ruptured. Molasses flooded into the streets in a 5 m-deep stream, killing pedestrians and horses and knocking down buildings. The molasses had a density of $1600\,\mathrm{kg/m^3}$. If the tank was full before the accident, what was the total outward force the molasses exerted on its sides? (Hint: Consider the outward force on a circular ring of the tank wall of width dy and at a depth y below the surface. Integrate to find the total outward force. Assume that before the tank ruptured, the pressure at the surface of the molasses was equal to the air pressure outside the tank.)

16. What is the pressure in the water pipe shown in the figure Fig. 14?

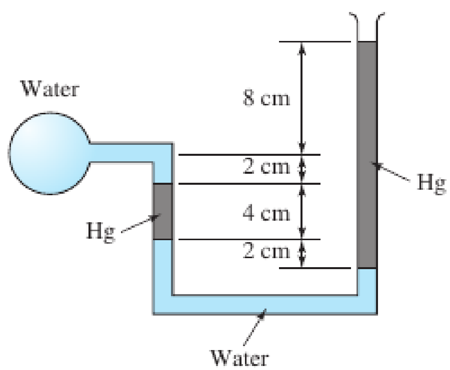


Figure 14: Problem.

17. Determine the force P needed to hold the 4-m wide gate in the position shown in Fig. 15.

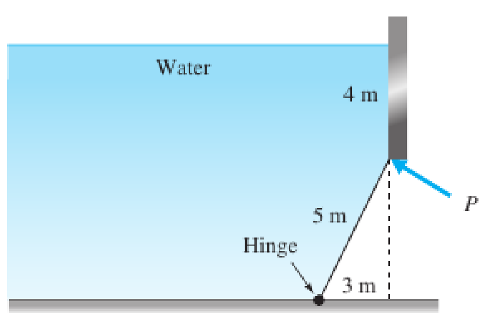


Figure 15: Problem.

18. Find the force P needed to hold the 10-m-long cylindrical object in position as shown in Fig. 16. Here $S \equiv \rho/\rho_{\text{water}}$.

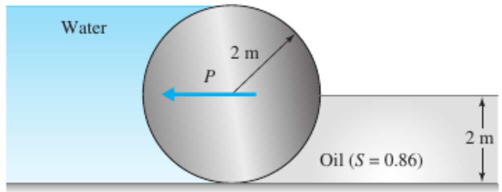


Figure 16: Problem.

19. Find the force P if the parabolic gate shown in Fig. 17 has a width w.

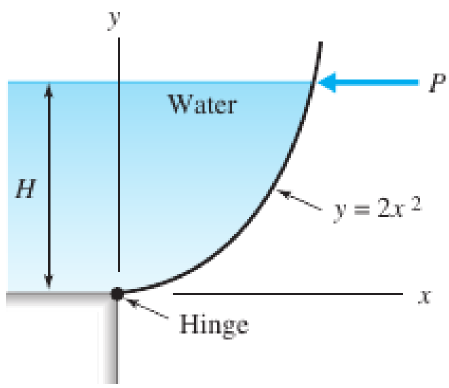


Figure 17: Problem.

20. **Hydrostatic atmospheres** For this exercise you will need the following constants: Boltzmann's constant $k_B = 1.380658 \times 10^{-16}\,\text{erg K}^{-1}$, the atomic mass unit $m = 1.6605 \times 10^{-24}\,\text{g}$, the solar radius $R_\odot = 6.9599 \times 10^{10}\,\text{cm}$, the solar mass $M_\odot = 1.98892 \times 10^{33}\,\text{g}$, the gravitational constant $G = 6.6743 \times 10^{-8}\,\text{cm}^3\,\text{g}^{-1}\,\text{s}^{-2}$, the earth radius $R_\oplus = 6.378 \times 10^8\,\text{cm}$ and the earth mass $M_\oplus = 5.974 \times 10^{27}$. You will calculate characterizations of the sun and earth atmospheres, assuming that the ideal gas law to be valid $p = \frac{\rho k_B T}{\mu m}$.

    i) In addition to the ideal gas assumption, we further assume that pressure and density are related according to

    $$p = K\rho^{1+\frac{1}{n}},$$

    where K and n are two constants. If $n \to \infty$, what does this imply for the temperature profiles in the atmospheres?

    ii) Let's start with the case of the sun's photosphere. You can assume a planar atmosphere and constant T. For the purpose of computing the effect of gravity, you may further assume that the atmosphere has a total mass content that is negligible relative to the mass of the sun and an equally negligible radial extent relative to the radius of the sun (which also justifies the planar assumption). Starting from momentum conservation in the hydrostatic case,

    $$\nabla p = f,$$

show that the pressure profile obeys

$$p = p_0 \exp\left[-(z - z_0)\frac{\mu m g}{k_B T}\right],$$

where g is gravitational acceleration.

iii) Identify a measure of "scale height" in this expression for pressure and interpret the meaning of a "scale height" in an atmosphere.

iv) Compute the scale heights for the solar photosphere at temperature $T_\odot = 5770\,\mathrm{K}$ and the earth at $T_\oplus = 300\,\mathrm{K}$. The solar photosphere is highly ionized and has a mean molecular weight $\mu = 0.6$, while the earth atmosphere consists largely of dinitrogen (N_2) and has a mean molecular weight $\mu = 28$. The scale heights for sun and earth should come out as $H = 2.9 \times 10^7\,\mathrm{cm}$ and $9.0 \times 10^5\,\mathrm{cm}$ respectively.

v) Mauna Kea in Hawaii hosts a number of the world's most powerful telescopes requiring astronomers to work at an altitude of 4.2 km. What is the implication for the atmospheric *density* at this elevation relative to sea level?

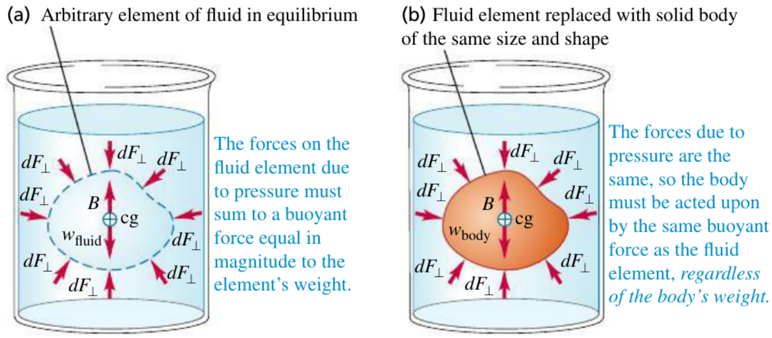


Figure 18: Archimedes's principle

4 Buoyancy and stability

4.1 Archimedes' principle

The external force on a volume element inside a fluid is

$$\vec{F}_e = \int_V \vec{f}_e \, dV \tag{82}$$

and the buoyancy due to pressure acting on its surface,

$$\vec{F}_p = -\int_V \vec{\nabla} p \, dV = -\oint_S p \, d\vec{S} = -\oint_S p \, \hat{n} \, dS, \tag{83}$$

If the total $\vec{F} = \vec{F}_e + \vec{F}_p$ does not vanish, an unrestrained body will accelerate in the direction of $\vec{F}$ according to Newton's Second Law. Therefore, in mechanical equilibrium, weight and buoyancy must precisely cancel each other at all times to guarantee that the body will remain in place.

This equation for the net pressure force holds whether the region is occupied by fluid or by some other body. In equilibrium the fluid volume is at rest, so

$$\vec{F}_p + \vec{F}_e = \vec{0}.$$

If the external force is gravity, $\vec{F}_e = \int_V \rho \, \vec{g} \, dV$ then the Archimedes' principle follows (see Fig. 18): *the force of buoyancy is equal and opposite to the weight of the displaced fluid.* That is,

$$\vec{F}_B = -\int_V \rho_{\text{fluid}} \, \vec{g} \, dV. \tag{84}$$

If a body submerged in the fluid has a weight greater than the weight of the displaced fluid, it sinks; otherwise it rises. The total force on the body may now be written

$$\vec{F} = \vec{F}_G + \vec{F}_B = \int_V \left(\rho_{\text{body}} - \rho_{\text{fluid}}\right) \vec{g} \, dV, \tag{85}$$

explicitly confirming that when the body is made from the same fluid as its surroundings —so that $\rho_{\text{body}} = \rho_{\text{fluid}}$— the resultant force vanishes automatically. In general, however, the distributions of mass in the body and in the displaced fluid will be different.

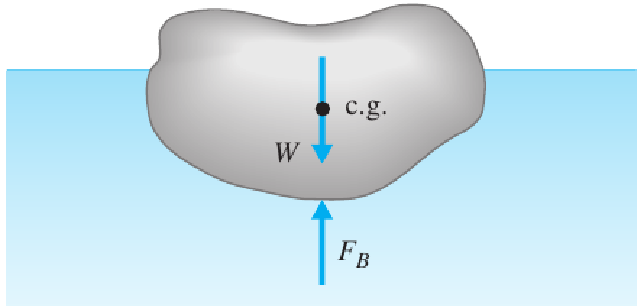


Figure 19: Forces on a floating object. Figure taken from Ref. [4].

An equivalent derivation of Archimedes' principle follows from considering a small vertical cylinder of height L and base area A within the fluid. The pressure on the lower face (p_2) exceeds that on the upper face (p_1), producing a net upward force

$$F = (p_2 - p_1)A. \tag{86}$$

Using the hydrostatic relation $p_2 - p_1 = \rho g L$, one obtains

$$F = \rho g(AL) = \rho g V = mg, \tag{87}$$

which is the weight of the displaced liquid (since ρ is the fluid density and $V = AL$ is the displaced volume).

4.1.1 Hydrometer

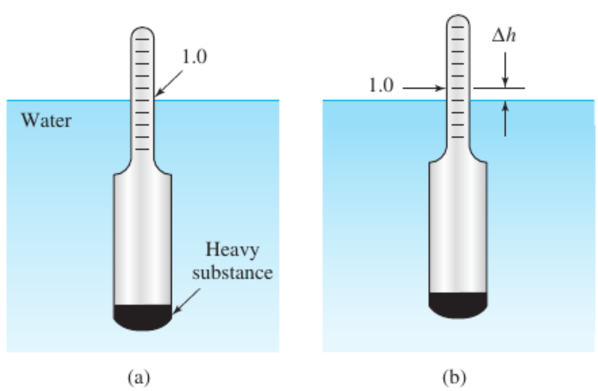


Figure 20: Hydrometer: (a) in water; (b) in an unknown liquid. Figure taken from Ref. [4].

A hydrometer, an instrument used to measure the specific gravity of liquids ($\gamma = \rho g$), operates on the principle of buoyancy. A sketch is shown in Fig. 20. The upper part (the stem) has a constant diameter. When placed in pure water the specific gravity is marked to read $S = \rho/\rho_{\text{water}} = 1.0$. The force balance is

$$W = \gamma_{\text{water}} V, \tag{88}$$

where W is the weight of the hydrometer and V is the submerged volume below the $S = 1.0$ line. In an unknown liquid of specific weight γ_x, a force balance gives

$$W = \gamma_x\big(V - A\,\Delta h\big), \tag{89}$$

where A is the cross–sectional area of the stem. Equating these expressions yields

$$\Delta h = \frac{V}{A}\left(1 - \frac{1}{S_x}\right), \tag{90}$$

where $S_x = \gamma_x/\gamma_{\text{water}}$ is the specific gravity of the unknown liquid relative to water. For a given hydrometer, V and A are fixed, so Δh depends only on S_x; thus the stem can be calibrated to read S_x directly. Hydrometers are used, for example, to measure the amount of antifreeze in an automobile radiator or the charge in a lead–acid battery, since the density of the fluid changes as H_2SO_4 is consumed or produced.

4.1.2 Review questions

1. In hot-air ballooning, a large balloon is filled with air heated by a gas burner at the bottom. Why must the air be heated? How does the balloonist control ascent and descent?

2. You drop a solid sphere of aluminum in a bucket of water that sits on the ground. The buoyant force equals the weight of water displaced; this is less than the weight of the sphere, so the sphere sinks to the bottom. If you take the bucket with you on an elevator that accelerates upward, the apparent weight of the water increases and the buoyant force on the sphere increases. Could the acceleration of the elevator be great enough to make the sphere pop up out of the water? Explain.

3. A rigid, lighter-than-air dirigible filled with helium cannot continue to rise indefinitely. Why? What determines the maximum height it can attain?

4. Which has a greater buoyant force on it: a 25 cm^3 piece of wood floating with part of its volume above water or a 25 cm^3 piece of submerged iron? Or, must you know their masses before you can answer? Explain.

5. A cargo ship travels from the Atlantic Ocean (salt water) to Lake Ontario (freshwater) via the St. Lawrence River. The ship rides several centimeters lower in the water in Lake Ontario than it did in the ocean. Explain.

6. You push a piece of wood under the surface of a swimming pool. After it is completely submerged, you keep pushing it deeper and deeper. As you do this, what will happen to the buoyant force on it? Will the force keep increasing, stay the same, or decrease? Why?

7. A piece of iron is glued to the top of a block of wood. When the block is placed in a bucket of water with the iron on top, the block floats. The block is now turned over so that the iron is submerged beneath the wood. Does the block float or sink? Does the water level in the bucket rise, drop, or stay the same? Explain.

8. You take an empty glass jar and push it into a tank of water with the open mouth of the jar downward, so that the air inside the jar is trapped and cannot get out. If you push the jar deeper into the water, does the buoyant force on the jar stay the same? If not, does it increase or decrease? Explain.

9. An ice cube floats in a glass of water. When the ice melts, will the water level in the glass rise, fall, or remain unchanged? Explain.

4.1.3 Problems

1. You are preparing some apparatus for a visit to a newly discovered planet Caasi having oceans of glycerine and a surface acceleration due to gravity of $5.40\,\mathrm{m/s^2}$. If your apparatus floats in the oceans on Earth with 25.0% of its volume submerged, what percentage will be submerged in the glycerine oceans of Caasi?

2. A rock with density $1200\,\mathrm{kg/m^3}$ is suspended from the lower end of a light string. When the rock is in air, the tension in the string is 28.0 N. What is the tension in the string when the rock is totally immersed in a liquid with density $750\,\mathrm{kg/m^3}$?

3. A cubical block of wood, 10.0 cm on a side, floats at the interface between oil and water with its lower surface 1.50 cm below the interface (Fig. 21). The density of the oil is $790\,\mathrm{kg/m^3}$. (a) What is the gauge pressure at the upper face of the block? (b) What is the gauge pressure at the lower face of the block? (c) What are the mass and density of the block?

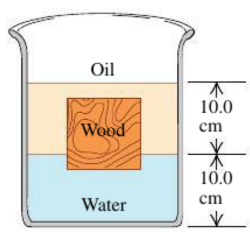


Figure 21: Problem.

4. A rock is suspended by a light string. When the rock is in air, the tension in the string is 39.2 N. When the rock is totally immersed in water, the tension is 28.4 N. When the rock is totally immersed in an unknown liquid, the tension is 21.5 N. What is the density of the unknown liquid?

5. A block composed of two materials with densities ρ_1 and ρ_2 (both less than that of water) and heights h_1 and h_2 is placed in a container with liquid (see left panel of Fig. 22). By what distance a is the block submerged?

6. A uniform cylindrical log of length L and radius R floats in water (see right panel of Fig. 22; the log is viewed along its axis). The density of the log is $0.509\ \mathrm{g\,cm^{-3}}$. Determine h.

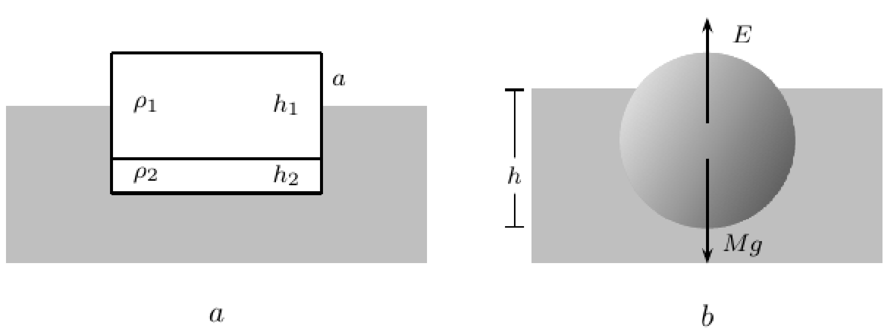


Figure 22: Problem.

7. A piece of wood is 0.600 m long, 0.250 m wide, and 0.080 m thick. Its density is 700 kg/m^3. What volume of lead must be fastened underneath it to sink the wood in calm water so that its top is just even with the water level? What is the mass of this volume of lead?

8. A closed and elevated vertical cylindrical tank with diameter 2.00 m contains water to a depth of 0.800 m. A worker accidentally pokes a circular hole with diameter 0.0200 m in the bottom of the tank. As the water drains from the tank, compressed air above the water in the tank maintains a gauge pressure of 5.00×10^3 Pa at the surface of the water. Ignore any effects of viscosity.

   (a) Just after the hole is made, what is the speed of the water as it emerges from the hole? What is the ratio of this speed to the efflux speed if the top of the tank is open to the air?

   (b) How much time does it take for all the water to drain from the tank? What is the ratio of this time to the time it takes for the tank to drain if the top of the tank is open to the air?

9. A cubical block of density ρ_B and with sides of length L floats in a liquid of greater density ρ_L.

   (a) What fraction of the block's volume is above the surface of the liquid?

   (b) The liquid is denser than water (density ρ_W) and does not mix with it. If water is poured on the surface of that liquid, how deep must the water layer be so that the water surface just rises to the top of the block? Express your answer in terms of L, ρ_B, ρ_L, and ρ_W.

   (c) Find the depth of the water layer in part (b) if the liquid is mercury, the block is made of iron, and $L = 10.0$ cm.

10. The hydrometer shown in Fig. 23 with no mercury has a mass of 0.01 kg. It is designed to float at the midpoint of the 12 cm-long stem in pure water.

    (a) Calculate the mass of mercury needed.

    (b) What is the specific gravity of the liquid if the hydrometer is just submerged?

    (c) What is the specific gravity of the liquid if the stem of the hydrometer is completely exposed?

11. The uniform 5 m-long round wooden rod in Fig. 24 is tied to the bottom by a string. Determine (a) the tension in the string and (b) the specific gravity of the wood. Is it possible, with the given information, to determine the inclination angle θ? Explain.

12. The uniform rod in Fig. 25 is hinged at point B on the waterline and is in static equilibrium as shown when 2 kg of lead (SG = 11.4) are attached to its end. What is the specific gravity of the rod material? What is peculiar about the rest angle $\theta = 30°$?

13. A uniform wooden beam (SG = 0.65) is 10 cm $\times$ 10 cm $\times$ 3 m and is hinged at A, as in Fig. 26. At what angle θ will the beam float in the 20°C water?

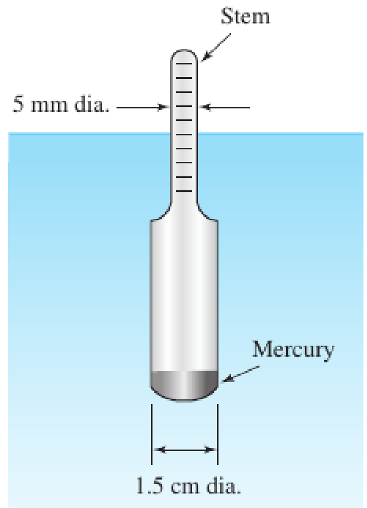


Figure 23: Problem.

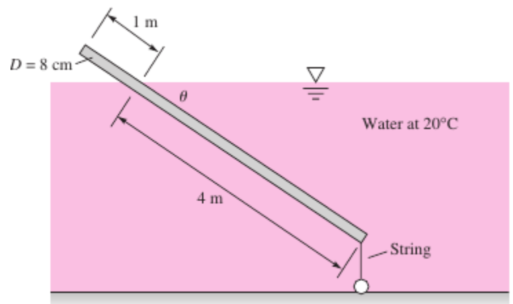


Figure 24: Problem.

4.2 Stability of floating bodies

4.2.1 Vertical stability

The notion of stability can be demonstrated by considering the vertical stability of a floating object. If the object is raised a small distance, the buoyant force decreases and the object's weight returns the object to its original position. Conversely, if a floating object is lowered slightly, the buoyant force increases and the larger buoyant force returns the object to its original position. Thus a floating object has vertical stability, since a small departure from equilibrium results in a restoring force.

4.2.2 Rotational stability

Consider now the rotational stability of a submerged body, shown in Fig. 27. In part (a) the center of gravity G of the body is above the centroid C (also referred to as the *center of buoyancy*) of the displaced volume, and a small angular rotation results in a moment that will continue to increase the rotation; hence the body is unstable and overturning would result. If the center of gravity is below the centroid, as in part (c), a small angular rotation provides a restoring moment and the body is stable. Part (b) shows neutral stability for a body in which the center of gravity and the centroid coincide, a situation encountered whenever the density is constant throughout

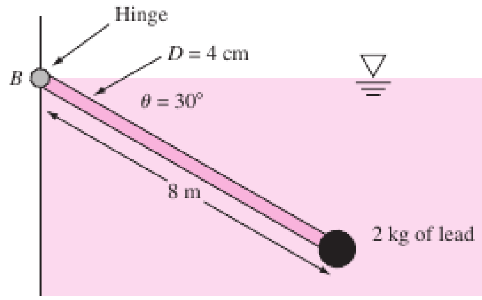


Figure 25: Problem.

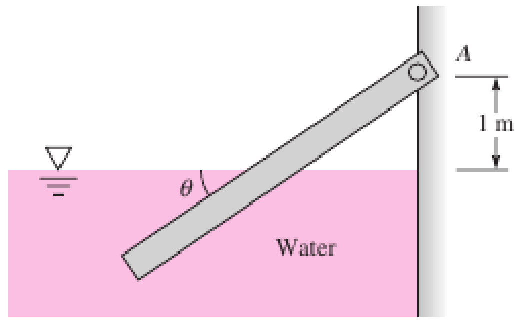


Figure 26: Problem.

the submerged body.

Next, consider the rotational stability of a floating body. If the center of gravity is below the centroid, the body is always stable, as with the submerged body of Fig. 27 (c). The body may be stable, though, even if the center of gravity is above the centroid, as sketched in Fig. 28 (a). When the body rotates the centroid of the volume of displaced liquid moves to the new location C', shown in part (b). If the centroid C' moves sufficiently far, a restoring moment develops and the body is stable, as shown. This is determined by the *metacentric height* GM defined as the distance from G to the point of intersection of the buoyant force before rotation with the buoyant force after rotation. If $\overline{GM}$ is positive, as shown, the body is stable; if $\overline{GM}$ is negative (M lies below G), the body is unstable.

4.3 Moments of gravity and buoyancy

The total moment is like the total force, a sum of two terms,

$$\vec{M} = \vec{M}_G + \vec{M}_B, \tag{91}$$

with one contribution from gravity,

$$\vec{M}_G = \int_V \vec{r} \times \rho_{\text{body}}\, \vec{g}\, dV, \tag{92}$$

and the other from pressure, called the moment of buoyancy,

$$\vec{M}_B = \oint_S \vec{r} \times \left(-p\, d\vec{S}\right). \tag{93}$$

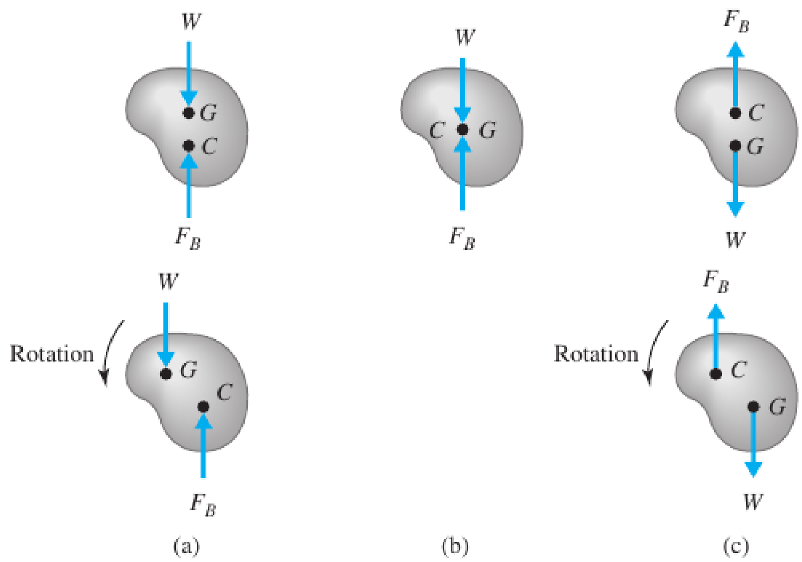


Figure 27: Stability of a submerged body: (a) unstable; (b) neutral; (c) stable. Figure taken from Ref. [4].

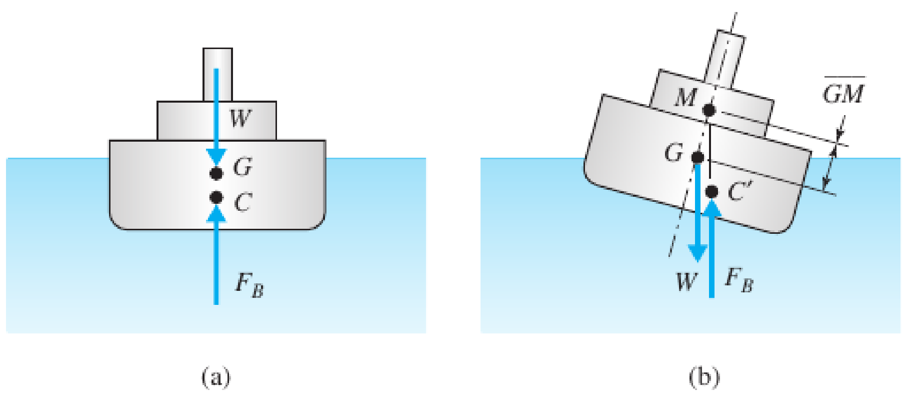


Figure 28: Stability of a floating body: (a) equilibrium position; (b) rotated position. Figure taken from Ref. [4].

If the total force vanishes, $\vec{F} = 0$, the total moment will be independent of the origin of the coordinate system, as may be easily shown.

Assuming again that the presence of the body does not change the local hydrostatic balance in the fluid, the moment of buoyancy will be independent of the nature of the material inside V. If the actual body is replaced by an identical volume of the ambient fluid, this fluid volume must be in total mechanical equilibrium, such that both the total force as well as the total moment acting on it have to vanish. Using that $\vec{M}_G^{\text{fluid}} + \vec{M}_B = 0$, we get

$$\vec{M}_B = -\vec{M}_G^{\text{fluid}} = -\int_V \vec{r} \times \rho_{\text{fluid}}\, \vec{g}\, dV. \tag{94}$$

In other words, we have shown that *The moment of buoyancy is equal and opposite to the moment of the weight of the displaced fluid.*

4.3.1 Constant gravity and mechanical equilibrium.

Let's assume that gravity is constant, $\vec{g}(\vec{x}) = \vec{g}_0$, and that the body is in buoyant equilibrium so that it displaces exactly its own mass of fluid, $M_{\text{fluid}} = M_{\text{body}} = M$. The density distributions in the body and in the displaced fluid will in general be different, $\rho_{\text{body}}(\vec{r}) \neq \rho_{\text{fluid}}(\vec{r})$ for nearly all points.

The moment of gravity may be expressed in terms of the center of mass $\vec{x}_G$ of the body, here called the *center of gravity*:

$$\vec{M}_G = \vec{r}_G \times M\,\vec{g}_0, \qquad \vec{r}_G = \frac{1}{M}\int_V \vec{r}\,\rho_{\text{body}}\,dV. \tag{95}$$

Similarly, the moment of buoyancy may be written

$$\vec{M}_B = -\,\vec{r}_B \times M\,\vec{g}_0, \qquad \vec{r}_B = \frac{1}{M}\int_V \vec{r}\,\rho_{\text{fluid}}\,dV, \tag{96}$$

where $\vec{r}_B$ is the center of mass of the displaced fluid, also called the *center of buoyancy*. Although each of these moments depends on the choice of origin of the coordinate system, the total moment,

$$\vec{M} = (\vec{r}_G - \vec{x}_B) \times M\,\vec{g}_0 \tag{97}$$

is independent of the origin.

As long as the total moment is nonzero, an unrestrained body is not in complete mechanical equilibrium and will rotate toward an orientation with vanishing moment. Except for the trivial case $\vec{r}_G = \vec{r}_B$, the last equation implies the total moment can vanish only if the centers lie on the same vertical line, i.e. $\vec{r}_G - \vec{r}_B \propto \vec{g}_0$. Evidently, there are two possible orientations satisfying this condition: one with the center of gravity below the center of buoyancy and another with it above. At least one of these must be stable, otherwise the body would never come to rest.

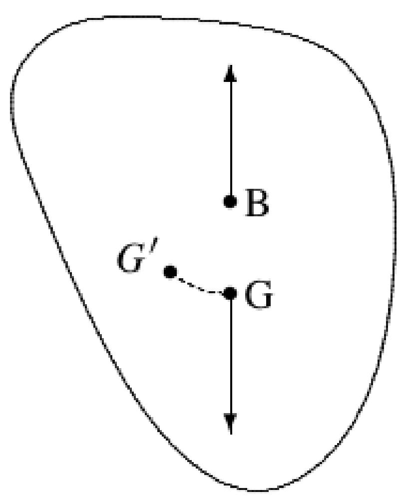


Figure 29: A fully submerged rigid body (a submarine) in stable equilibrium must have the center of gravity situated directly below the center of buoyancy. If G is moved to G', for example by rotating the body, a restoring moment is created that sticks out of the plane of the paper. Figure taken from Ref. [3].

4.3.2 Fully submerged body

A fully submerged rigid body (a submarine) in stable equilibrium must have the center of gravity situated directly below the center of buoyancy. If G is moved to G', for example by rotating the body, a restoring moment is created that sticks out of the plane of the paper.

4.3.3 Body floating on the surface

A floating body may, like a submerged body, possess a stable orientation with the center of gravity directly *below* the center of buoyancy. A heavy keel is, for example, used to lower the center of gravity of a sailing ship so much that this orientation becomes the only stable equilibrium. In that case it becomes virtually impossible to capsize the ship, even in a very strong wind. The stable orientation for most floating objects, such as ships, will in general have the center of gravity situated directly *above* the center of buoyancy. This happens whenever an object of constant mass density floats on top of a liquid of constant mass density, for example an iceberg on water. Since ice, like water, is homogeneous, the part of the iceberg that lies below the waterline must have its center of buoyancy in exactly the same place as its center of gravity. The part of the iceberg lying above the water cannot influence the center of buoyancy, whereas it will always shift the center of gravity upward.

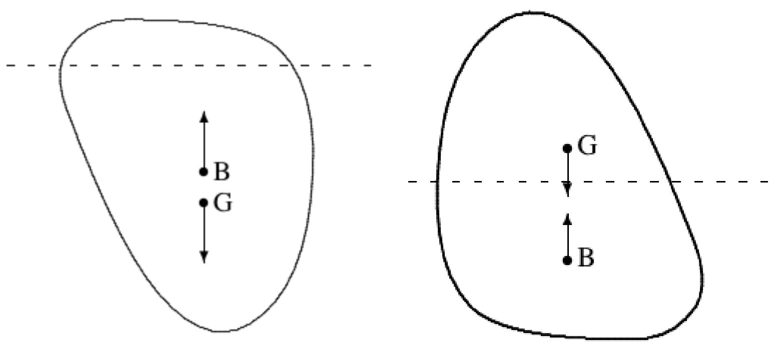


Figure 30: Left: A floating body may have a stable equilibrium with the center of gravity directly below the center of buoyancy. Right: a floating body generally has a stable equilibrium with the center of gravity directly above the center of buoyancy. Figure taken from Ref. [3].

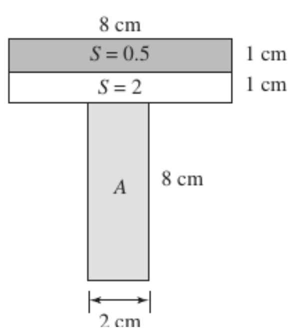


Figure 31: Problem.

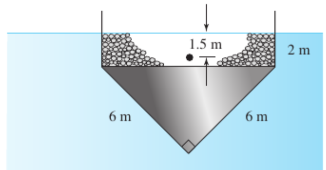


Figure 32: Problem.

4.3.4 Problems

1. Over what range of specific weights will a circular cylinder with uniform specific weight γ_x float in water with ends horizontal if its height equals its diameter?

2. Over what range of specific weights will a homogeneous cube float with sides horizontal and vertical?

3. For the object shown in Fig. 31, calculate S_A for neutral stability when submerged.

4. Is the symmetrically loaded barge shown in Fig. 32 stable? The center of gravity of the barge and load is located as shown.

5. Consider a cylinder of specific gravity $S < 1$ floating vertically in water ($S = 1$), as in Fig. 33. Derive a formula for the stable values of D/L as a function of S and apply it to the case $D/L = 1.2$.

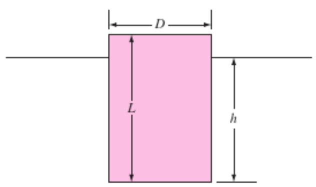


Figure 33: Problem.

6. An iceberg can be idealized as a cube of side length L, as in Fig. 34. If seawater is denoted by $S = 1.0$, then glacier ice (which forms icebergs) has $S = 0.88$. Determine if this "cubic" iceberg is stable for the position shown in Fig. 34.

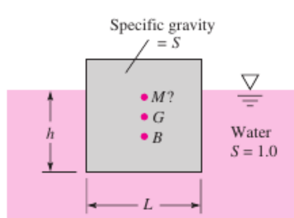


Figure 34: Problem.

5 Self-gravitating continuous media

5.1 Newtonian gravitation

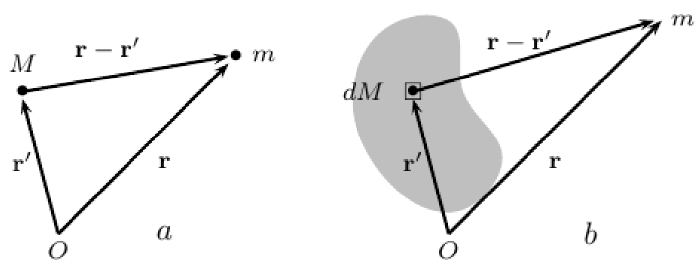


Figure 35: Geometry for describing gravitational attraction, a. between point masses; b. between a mass distribution and a point mass

According to Newton, the gravitational force that a point mass M exerts on another point mass m (see Fig. 35) is given by

$$\vec{F}_{M\to m} = -\frac{G\,M\,m\left(\vec{r}-\vec{r}\,'\right)}{\left|\vec{r}-\vec{r}\,'\right|^3}. \tag{98}$$

The gravitational force that a mass distribution with density $\rho(\vec{r}\,')$ exerts on a point mass m located at $\vec{r}$ is computed as the force produced by the collection of point masses making up the distribution:

$$\vec{F}(\vec{r}) = -\,G\,m\int \rho(\vec{r}\,')\,\frac{\vec{r}-\vec{r}\,'}{\left|\vec{r}-\vec{r}\,'\right|^3}\,d^3\vec{r}\,'\,. \tag{99}$$

The corresponding gravitational fields are

$$\vec{g} = \vec{F}_{M\to m}/m = -\frac{G\,M\left(\vec{r}-\vec{r}\,'\right)}{\left|\vec{r}-\vec{r}\,'\right|^3}. \tag{100}$$

and

$$\vec{g}(\vec{r}) = \vec{F}(\vec{r})/m = -\,G\int \rho(\vec{r}\,')\,\frac{\vec{r}-\vec{r}\,'}{\left|\vec{r}-\vec{r}\,'\right|^3}\,d^3\vec{r}\,'\,. \tag{101}$$

The corresponding gravitational potetials are

$$\Phi = -\frac{G\,M}{\left|\vec{r}-\vec{r}\,'\right|}. \tag{102}$$

and

$$\Phi(\vec{r}) = -\,G\int \rho(\vec{r}\,')\,\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}\,d^3\vec{r}\,'\,. \tag{103}$$

The gravitational potential satisfies

$$\nabla^2\Phi(\vec{r}) = 4\pi G\rho(\vec{r}), \tag{104}$$

while the gravitational field

$$\vec{\nabla} \times \vec{g} = 0. \tag{105}$$

The Poisson's equation tells us what the local impact is of a self–gravitating system on its gravitational potential. It is valid in general, even for the isothermal atmosphere. However, for that case Poisson's equation collapses into the trivial $0 = 0$. After all, g was approximated as constant, so $\nabla^2\Phi = 0$. At the same time, ρ on the right–hand side of Poisson's equation was taken to be zero: the particles of the atmosphere were assumed to only *experience* gravity and not to be counted as mass charges in their own right.

5.2 Self-gravitating mass

Gravitation, in this case, is that of a body acting on itself. This is the case for the Earth, stars, galaxies, and clusters of galaxies—objects whose global behavior, to some extent, can be approximated by that of a fluid.

From hydrostatics equilibrium condition and taking the divergence, we obtain

$$\vec{\nabla}\cdot\left(\frac{\vec{\nabla}p}{\rho}\right) + \nabla^2\Phi = 0, \tag{106}$$

and using the Poisson's equation it follows that

$$\vec{\nabla}\cdot\left(\frac{\vec{\nabla}p}{\rho}\right) + 4\pi G\,\rho = 0. \tag{107}$$

In this equation, in general, $\rho = \rho(\vec{r})$ and $p = p(\vec{r})$, so it is valid even for compressible fluids. With the above equation and an equation of state one can build simple models of the Earth, stars, or galaxies.

5.3 An isothermal slab model

The hydrostatic plane–parallel atmosphere model discussed above involves a number of simplifying assumptions that will not be generally applicable. It assumes the atmosphere to be (1) isothermal, (2) plane–parallel, (3) with negligible gravitational pull of its own, and (4) obeying a single equation of state (both on account of being isothermal and consisting of the same particles throughout). Devising a stellar–structure model will quickly prompt us to drop at least assumptions 1–3, even while retaining the still massively simplifying assumption (4), and we will return to this later. Meanwhile, let us relax only assumption (3), as a stepping stone to stellar–structure modelling, and construct a slightly more sophisticated atmospheric profile model (the "isothermal slab").

We consider an infinite (in x and y) static *isothermal slab*, symmetric about $z = 0$, supported by gas pressure and by its own self–gravity. No other forces act. Because we deal with an isothermal ideal gas, the equation of state $pV = nRT = (N/N_A)RT = Nk_BT$ can be written as

$$p = \frac{N}{V}k_BT = \frac{N\mu}{V}\frac{k_BT}{\mu} = \frac{M}{V}\frac{k_BT}{\mu} = \rho\frac{k_BT}{\mu}. \tag{108}$$

This means that for constant T it may be written as

$$p = A\rho, \tag{109}$$

where A is a constant. The geometry is plane–parallel, so all quantities depend only on z and $\nabla = \partial/\partial z$. Hydrostatic equilibrium then gives

$$A\,\frac{1}{\rho}\,\nabla\rho = -\,\nabla\Phi. \tag{110}$$

Since the variation is only with z,

$$A\,\frac{d}{dz}\ln\rho = -\,\frac{d\Phi}{dz} \qquad \Rightarrow \qquad \Phi = -A\ln\!\left(\frac{\rho}{\rho_0}\right) + \Phi_0, \tag{111}$$

with $\rho_0 = \rho|_{z=0}$ and $\Phi_0 = \Phi|_{z=0}$. Hence

$$\rho(z) = \rho_0 \exp\!\left[-\,\frac{\Phi-\Phi_0}{A}\right]. \tag{112}$$

The profile for $\Phi(z)$ for plane-parallel layers can be determined now that we have a complementary expression for density in terms of gravitational potential that we can apply to Poisson's equation:

$$\frac{d^2\Phi}{dz^2} = 4\pi G\rho_0 \exp\!\left[-\,\frac{\Phi-\Phi_0}{A}\right]. \tag{113}$$

Introduce the dimensionless variables

$$\chi \equiv -\,\frac{\Phi-\Phi_0}{A}, \qquad Z \equiv \left(\frac{2\pi G\rho_0}{A}\right)^{1/2} z, \tag{114}$$

to obtain the ODE

$$\frac{d^2\chi}{dZ^2} = -\,2\,e^{\chi}, \tag{115}$$

with boundary conditions $\chi(0) = 0$ and, by symmetry, $d\chi/dZ|_0 = 0$.

Multiply by $d\chi/dZ$ and integrate:

$$\frac{1}{2}\frac{d}{dZ}\left(\frac{d\chi}{dZ}\right)^2 = -2\,\frac{d}{dZ}[e^{\chi}] \qquad \Rightarrow \qquad \left(\frac{d\chi}{dZ}\right)^2 = c_1 - 4e^{\chi}. \tag{116}$$

The boundary conditions at $Z = 0$ give $c_1 = 4$, hence

$$\frac{d\chi}{dZ} = 2\sqrt{1-e^{\chi}}. \tag{117}$$

Integrate this by letting $s^2 = e^{\chi}$:

$$\int \frac{d\chi}{\sqrt{1-e^{\chi}}} = \int \frac{2\,ds}{s\sqrt{1-s^2}} = 2\int \frac{d\theta}{\sin\theta} = 2\int \frac{dt}{t}, \tag{118}$$

where $s = \sin\theta$ and $t = \tan(\theta/2)$. Therefore

$$2\ln t = 2Z + c_2. \tag{119}$$

With $\chi(0) = 0 \Rightarrow s(0) = 1 \Rightarrow \theta(0) = \frac{\pi}{2} \Rightarrow t(0) = 1$, we get $c_2 = 0$ and thus $t = e^Z$. Back–substituting,

$$s = \frac{2t}{1+t^2} = \frac{1}{\cosh Z} \qquad \Rightarrow \qquad e^{\chi/2} = \frac{1}{\cosh Z} \qquad \Rightarrow \qquad \chi = -\,2\ln\!\big(\cosh Z\big). \tag{120}$$

Returning to the original variables,

$$\Phi - \Phi_0 = 2A \ln\left[\cosh\left(\sqrt{\frac{2\pi G\rho_0}{A}}\, z\right)\right]. \tag{121}$$

Finally, the isothermal–slab density profile is

$$\rho(z) = \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G\rho_0}{A}}\, z\right)}. \tag{122}$$

5.4 General isothermal slab density profiles

The general solution to the differential equation is

$$\chi \ = \ -2 \ln\left[c_1 \cosh\left(\frac{Z}{c_1} + c_2\right)\right]. \tag{123}$$

with the corresponding density profile being

$$\rho \ = \ \frac{\rho_0}{c_1^2 \cosh^2\left(\sqrt{\frac{2\pi G\rho_0}{A}}\left(\frac{z}{c_1} + c_2\right)\right)}. \tag{124}$$

Problem: show the previous expressions.

5.4.1 A fully self-gravitating plane

Let us consider a fully self-gravitating plane that is not resting on top of a massive foundation (e.g., a galactic disc rather than an atmosphere on top of a planet). Picking $z = 0$ to be the plane of symmetry, we set the gravitational potential at $z = z_0 = 0$ equal $\Phi = \Phi_0$, such that $\chi(Z = 0) = 0$. On account of the symmetry, $z = 0$ represents an extremum of the gravitational potential, so $d\chi/dZ = 0$ at $z = 0$. When implementing these boundary conditions, it turns out that $c_1 = 1$ and $c_2 = 0$

5.4.2 A planetary atmosphere

In the case of a planetary atmosphere, $\chi_0 = 0$ can still be used, while for a planet of mass M and radius R we have

$$\left.\frac{d\chi}{dZ}\right|_0 = -\left(2\pi G\rho_0 A\right)^{-1/2} g = -\left(2\pi G\rho_0 A\right)^{-1/2} \frac{GM}{R^2}. \tag{125}$$

The general expressions for the constants are

$$c_1 = (x^2+1)^{-1/2}, \qquad c_2 = \ln\left[x + \sqrt{x^2+1}\right], \tag{126}$$

with

$$x \equiv \frac{1}{\sqrt{2\pi G\rho_0 A}} \frac{MG}{2R^2}. \tag{127}$$

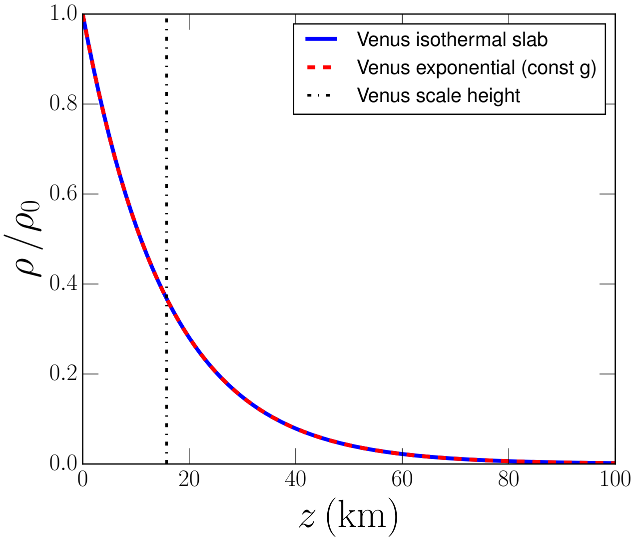


Figure 36: Isothermal slab model applied to Venus. The difference between the full expression and an exponential profile is negligible. In this case, $M \approx 0.815\, M_\oplus$, $R \approx 0.95\, R_\oplus$, and $p_0 \approx 93\times10^5$ Pa, leading to $x \approx 70 \gg 1$.

For a large mass M we have

$$c_1 \approx x^{-1}, \qquad c_2 \approx \ln(2x), \qquad x \gg 1. \tag{128}$$

Substituting these into the density profile and using $\cosh y = (e^y + e^{-y})/2 \sim e^y/2$ for large y, we obtain

$$\rho \approx \rho_0\, x^2\, 4\left[\exp\!\left(\tfrac{zMG}{2AR^2} + \ln(2x)\right) + \exp\!\left(-\tfrac{zMG}{2AR^2} - \ln(2x)\right)\right]^{-2} \tag{129}$$

$$\approx \rho_0\, x^2\, 4\left[\,2x\, \exp\!\left(\tfrac{zMG}{2AR^2}\right)\right]^{-2} \tag{130}$$

$$\approx \rho_0\, \exp\!\left(-\tfrac{zMG}{AR^2}\right). \tag{131}$$

Thus, in the large-M limit we recover the hydrostatic–atmosphere result in which the self-gravity of the atmosphere is negligible (see Fig. 36).

5.4.3 Problems: other atmospheres

We have seen how planetary atmospheres can be modeled when assuming a plane–parallel geometry and when neglecting the gravitational pull of the mass within the atmosphere. Both limiting assumptions are quite reasonable, given, respectively, how small the scale height is relative to the planet's radius and how small the mass contained in the atmosphere relative to the planetary mass. Some permutations in which assumptions are retained and which are relaxed lead to simple solutions. Both cases are addressed as problems.

1. **A curved isothermal atmosphere without self–gravity**. Here we study what happens if we no longer neglect curvature but retain the assumptions of an isothermal atmosphere and no self–gravity.

(a) Assume a gravitational acceleration of magnitude $g = MG/r^2$ and an isothermal atmosphere with $p = A\rho$. Neglect self–gravity (i.e. treat M as a constant throughout the atmosphere). Show that the density profile obeys

$$\rho = \rho_0 \exp\left(\frac{MG}{Ar} - \frac{MG}{Ar_0}\right), \tag{132}$$

where ρ_0 is the mass density at a reference radius r_0.

(b) Show how a series approximation of the previous result reduces (to lowest order) to the case of an isothermal plane–parallel atmosphere without self–gravity when considered only at small distances beyond the reference radius r_0.

2. **A plane–parallel polytropic atmosphere without self–gravity.** This covers the case of a plane–parallel atmosphere without self–gravity but now no longer assuming the atmosphere to be isothermal.

(a) Assume plane–parallel symmetry, gravitational acceleration g independent of height z, and a polytropic gas obeying $p = K\rho^{\hat{\gamma}}$, where K is a constant of proportionality. Assume further that the adiabatic exponent $\hat{\gamma} \neq 1$ (i.e. the atmosphere is not isothermal). Show that the normalized density profile of the atmosphere obeys

$$\frac{\rho}{\rho_0} = \left(1 - \frac{z}{z_{\rm max}}\right)^{1/(\hat{\gamma}-1)}, \tag{133}$$

terminating at a maximum height $z_{\rm max}$ given by

$$z_{\rm max} = \frac{\rho_0^{\hat{\gamma}-1}\,\hat{\gamma}K}{g\,(\hat{\gamma}-1)}. \tag{134}$$

(b) Compare the above density profile to that of an isothermal atmosphere of equal total mass. Show that the scale height H of the latter obeys

$$H = \frac{\hat{\gamma}-1}{\hat{\gamma}}\,z_{\rm max}. \tag{135}$$

(c) Plot profiles of both atmospheric models on a single graph to show their similarity (e.g. by using Python). Take $\hat{\gamma} = 7/5$, representative of the Earth's atmosphere.

5.5 Spherical bodies

If there is spherical symmetry—i.e., density and pressure depend only on the radial coordinate. It follows that

$$\vec{g}(r) = -\frac{G_N M(r)}{r^2}\hat{e}_r, \tag{136}$$

and

$$\vec{\nabla}p(r) = \frac{dp(r)}{dr}\vec{\nabla}\vec{r} = \frac{dp(r)}{dr}\hat{e}_r, \tag{137}$$

from which the hydrostatic equilibrium equation takes the form

$$\frac{dp(r)}{dr} = \rho(r)g(r) = -\rho(r)\frac{G_N M(r)}{r^2}. \tag{138}$$

Multipliying by r^2/ρ and differentiating after with respect to r we arrive to

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) + 4\pi G\,\rho = 0. \tag{139}$$

5.5.1 Boundary conditions

In principle, a second–order differential equation requires two boundary values (or integration constants), for example the central pressure $p_c = p(0)$ and its first derivative $\mathrm{d}p/\mathrm{d}r$ for $r = 0$. We shall make the reasonable assumption that the density ρ_c at the centre of the body is finite. Then for 'small' r we have $M(r) \approx \frac{4}{3}\pi r^3 \rho_c$ and equation (138) becomes for $r \to 0$,

$$\frac{\mathrm{d}p}{\mathrm{d}r} \approx -\frac{4}{3}\pi G\rho_c^2\, r, \tag{140}$$

which integrates to

$$p(r) \approx p_c - \frac{2}{3}\pi G\rho_c^2\, r^2. \tag{141}$$

Thus, under the assumption of finite central density, the pressure is parabolic near the centre with $\mathrm{d}p/\mathrm{d}r = 0$ for $r = 0$. This shows that under reasonable physical assumptions the hydrostatic equation requires in fact only one boundary condition, for example the central pressure. Knowing p_c together with the equation of state (which also determines ρ_c), the pressure may be calculated throughout the body.

The central pressure and density are, of course, not known for planets and stars, objects that are only accessible from the outside. Most such bodies have a well–defined surface radius, $r = a$, at which the pressure vanishes. We shall arbitrarily call a body a planet, if the density jumps abruptly to zero at the surface, and a star if the density vanishes along with the pressure at the surface. Such a convention makes the gaseous giant planets, Jupiter and Saturn, count as stars even though they probably do not burn much hydrogen.

The requirement of zero pressure at $r = a$ will determine the central pressure. The solutions to the hydrostatic equation can be expressed entirely in terms of the radius of the body and the parameters in the equation of state. In particular the mass M_0 of the body is—as we shall see below—calculable in terms of a (and the state parameters). Conversely, if the mass and radius are known, one of the other unknown parameters may be determined.

5.5.2 Planet with constant density

For a planet with constant density, ρ_0, the assumption of finite central density is exactly valid throughout the planet,

$$p = p_c - \frac{2}{3}\pi G\rho_0^2\, r^2. \tag{142}$$

At the surface of the planet where the pressure has to vanish this leads to

$$p_c = \frac{2}{3}\pi G\rho_0^2 a^2. \tag{143}$$

If the mass and radius are known, the density is obtained from

$$M_0 = \frac{4}{3}\pi a^3 \rho_0. \tag{144}$$

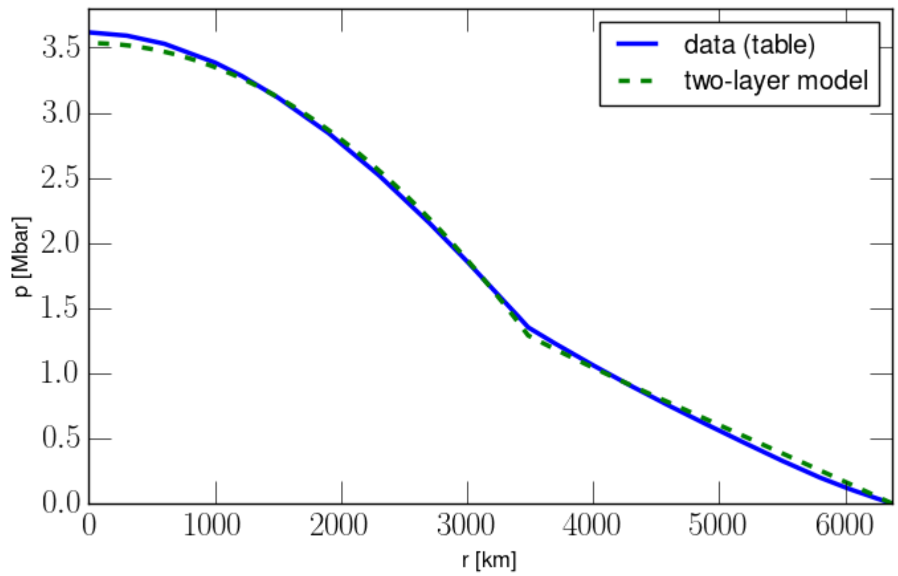


Figure 37: Pressure distribution in the Earth. Solid line data from Table 5.6 and dashed: the two-layer model.

5.5.3 The Moon with constant density

Let us consider the Moon. The Moon's mass is 7.3×10^{22} kg and its radius is 1738 km, making the average density $\rho_0^{\text{Moon}} = 3.34$ g/cm^3. Form this the central pressure is predicted to be 46 500 atm.

5.6 Pressure distribution in the Earth

The four main layers of the Earth, from outermost to innermost, are the **crust**, **mantle**, **outer core**, and **inner core**. The crust is the thin, rocky shell we live on; the mantle is a thick, dense layer of rock; the outer core is a liquid layer of iron and nickel; and the inner core is a solid, extremely hot ball of iron and nickel at Earth's center.

5.6.1 A two-layer model

Let us consider that the Earth consists of two layers with constant mass density,

$$\rho(r) = \begin{cases} \rho_1, & 0 \le r \le a_1, \\ \rho_2, & a_1 < r \le a, \\ 0, & r > a. \end{cases}$$

Let $M(r)$ be the mass enclosed within radius r and $g(r)$ the (inward) gravitational field strength. By spherical symmetry,

$$g(r) = \frac{GM(r)}{r^2}, \qquad \frac{d\Phi}{dr}(r) = -g(r), \qquad \Phi(\infty) = 0,$$

so that $\Phi(r) = -\int_\infty^r g(r')\, dr'$.

Gravitational field and potential. The enclosed mass is

$$M(r) = \frac{4\pi}{3} \times \begin{cases} \rho_1 r^3, & 0 \le r \le a_1, \\ \rho_1 a_1^3 + \rho_2\,(r^3 - a_1^3), & a_1 < r \le a, \\ \rho_1 a_1^3 + \rho_2\,(a^3 - a_1^3) \equiv M, & r > a. \end{cases}$$

Hence

$$g(r) = \frac{GM(r)}{r^2} = \begin{cases} \dfrac{4\pi G}{3}\,\rho_1\, r, & 0 \le r \le a_1, \\ \dfrac{4\pi G}{3}\left(\rho_2 r + \dfrac{(\rho_1 - \rho_2) a_1^3}{r^2}\right), & a_1 < r \le a, \\ \dfrac{GM}{r^2}, & r > a. \end{cases}$$

With a zero potential at infinity $\Phi(\infty) = 0$ one gets the potential

$$\Psi(r) = \begin{cases} -\dfrac{GM}{a} + \dfrac{2\pi G\rho_2}{3}\,(a^2 - a_1^2) + \dfrac{4\pi G}{3}(\rho_1 - \rho_2) a_1^2\left(1 - \dfrac{a_1}{a}\right) + \dfrac{2\pi G\rho_1}{3}\,(a_1^2 - r^2), & 0 \le r \le a_1, \\ -\dfrac{GM}{a} + \dfrac{2\pi G\rho_2}{3}\,(a^2 - r^2) + \dfrac{4\pi G}{3}(\rho_1 - \rho_2) a_1^3\left(\dfrac{1}{r} - \dfrac{1}{a}\right), & a_1 < r \le a, \\ -\dfrac{GM}{r}, & r > a. \end{cases}$$

Where is gravity stronger: at $r = a_1$ or at $r = a$? At the interface $r = a_1$,

$$g(a_1) = \frac{4\pi G}{3}\,\rho_1\, a_1,$$

while at the surface

$$g(a) = \frac{GM}{a^2} = \frac{4\pi G}{3a^2}\Big[\rho_1 a_1^3 + \rho_2\,(a^3 - a_1^3)\Big].$$

The condition $g(a_1) > g(a)$ is therefore equivalent (after cancelling $\frac{4\pi G}{3}$) to

$$\rho_1 a_1 > \frac{\rho_1 a_1^3 + \rho_2 (a^3 - a_1^3)}{a^2} \quad \Longleftrightarrow \quad \frac{\rho_1 - \rho_2}{\rho_2} > \frac{a^2}{a_1(a + a_1)}.$$

Numerical check for Earth. Take $a \simeq 6371$ km, $a_1 \simeq 3480$ km (core radius), $\rho_1 \simeq 11$–13 g cm^{-3} (core), $\rho_2 \simeq 4.5$–5.5 g cm^{-3} (mantle). Then

$$\frac{\rho_1 - \rho_2}{\rho_2} \approx 1.2\text{–}1.9, \qquad \frac{a^2}{a_1(a + a_1)} \approx 1.18,$$

so the inequality is satisfied, and the gravitational field is indeed stronger at the core–mantle boundary than at the surface.

Hydrostatic pressure Hydrostatic balance and the enclosed mass are

$$\frac{dp}{dr} = -\rho\,\frac{GM(r)}{r^2}, \qquad M(r) = 4\pi \int_0^r \rho(r')\,r'^2\,dr'.$$

We use the boundary condition $p(a) = 0$ and integrate inward.

(i) Mantle, $a_1 < r \le a$. Here $\rho = \rho_2$ and

$$M(r) = \frac{4\pi}{3}\left[\rho_1 a_1^3 + \rho_2\left(r^3 - a_1^3\right)\right].$$

Hence

$$p(r) = \int_r^a \rho_2\,\frac{GM(r')}{r'^2}\,dr' = \frac{4\pi G\rho_2}{3}\left[\frac{\rho_2}{2}\left(a^2 - r^2\right) + (\rho_1 - \rho_2)a_1^3\left(\frac{1}{r} - \frac{1}{a}\right)\right].$$

(ii) Core, $0 \le r \le a_1$. Here $\rho = \rho_1$ and $M(r) = \frac{4\pi}{3}\rho_1 r^3$. Demanding continuity at $r = a_1$ gives

$$p(r) = p(a_1) + \int_r^{a_1} \rho_1\,\frac{GM(r')}{r'^2}\,dr' = p(a_1) + \frac{2\pi G}{3}\,\rho_1^2\left(a_1^2 - r^2\right),$$

with

$$p(a_1) = \frac{4\pi G\rho_2}{3}\left[\frac{\rho_2}{2}\left(a^2 - a_1^2\right) + (\rho_1 - \rho_2)a_1^2\left(1 - \frac{a_1}{a}\right)\right].$$

Central pressure. At $r = 0$,

$$\boxed{p_c = \frac{2\pi G}{3}\left[\rho_1^2\,a_1^2 + \rho_2^2\left(a^2 - a_1^2\right) + 2\rho_2(\rho_1 - \rho_2)\,a_1^3\left(\frac{1}{a_1} - \frac{1}{a}\right)\right].}$$

Numerical estimate for Earth. Taking

$$a = 6.371 \times 10^6 \text{ m}, \quad a_1 = 3.48 \times 10^6 \text{ m}, \quad \rho_1 \simeq 1.2 \times 10^4 \text{ kg m}^{-3}, \quad \rho_2 \simeq 5.0 \times 10^3 \text{ kg m}^{-3},$$

the above expression yields

$$p_c \approx 3.9 \times 10^{11} \text{ Pa} \ (\approx 390 \text{ GPa}),$$

consistent with accepted seismological values ($\sim 3.6 \times 10^{11}$ Pa).

5.6.2 Problems

1. Consider a spherically symmetric self-gravitating astrophysical object—for example, a galaxy—with density $\rho(r) = \rho_0\left(1 - \frac{r^2}{a^2}\right)$ for $0 \le r \le a$, where a is the sphere's radius. Compute dp/dr and $p(r)$ by imposing $p = 0$ at $r = a$ and regularity (no divergences). What is the total mass?

2. Evaluate the pressure distribution $p(r)$ inside a uniform self-gravitating spherical fluid of constant density ρ, radius R, and mass M.

3. Evaluate $p(r)$ for a sphere of self-gravitating gas of radius R and mass M (e.g., a star) if its equation of state is $p(r) = \alpha\,\rho(r)$.

5.7 Idealized stellar structure models

If we wish to model stellar structure from the centre of a star to its outer edge, we cannot assume the profile to be plane–parallel and neither can we neglect self–gravity. A full discussion of stellar structure lies beyond the scope of this course, but isothermal and polytropic stellar–structure models provide us with nice and insightful applications of hydrostatics. Considering a polytropic fluid, the resulting expression, the Lane–Emden equation, is a famous cornerstone of stellar–structure theory.

5.7.1 Stars as self-gravitating polytropes

We assume a spherical system in hydrostatic equilibrium (no rotation that would break spherical symmetry). As always,

$$\nabla p = -\rho \, \nabla \Phi.$$

In spherical coordinates this reads

$$\frac{dp}{dr} = -\rho \, \frac{d\Phi}{dr}.$$

Since $\rho > 0$ within the star, p is a monotonic function of Φ; hence we may write $p = p(\Phi)$. Then

$$\frac{dp}{dr} = \frac{dp}{d\Phi}\frac{d\Phi}{dr} = -\rho \, \frac{d\Phi}{dr} \qquad \Rightarrow \qquad \rho = -\frac{dp}{d\Phi} \qquad \Rightarrow \qquad \rho = \rho(\Phi).$$

Because $p = p(\Phi)$ and $\rho = \rho(\Phi)$, it follows that $p = p(\rho)$; non–rotating stars are therefore *barotropes* (surfaces of constant p, ρ, Φ coincide).

A convenient barotropic equation of state is the *polytrope*

$$p = K \, \rho^{1+\frac{1}{n}},$$

where n is the polytropic index. Substituting this EOS into hydrostatic equilibrium gives

$$-\nabla \Phi = \frac{1}{\rho} \nabla \left(K \rho^{1+\frac{1}{n}} \right) = (n+1) \, \nabla \left(K \rho^{\frac{1}{n}} \right).$$

Therefore

$$\rho = \left(\frac{\Phi_T - \Phi}{(n+1)K} \right)^n,$$

where Φ_T is the value of the potential at the stellar surface ($\rho = 0$).

Using Poisson's equation,

$$\nabla^2 \Phi = 4\pi G \, \rho = 4\pi G \left(\frac{\Phi_T - \Phi}{(n+1)K} \right)^n.$$

Let ρ_c and Φ_c denote the density and potential at the centre. Then

$$\rho = \rho_c \left(\frac{\Phi_T - \Phi}{\Phi_T - \Phi_c} \right)^n, \qquad \nabla^2 \Phi = 4\pi G \, \rho_c \left(\frac{\Phi_T - \Phi}{\Phi_T - \Phi_c} \right)^n.$$

Define the dimensionless function

$$\theta = \frac{\Phi_T - \Phi}{\Phi_T - \Phi_c} \qquad \Rightarrow \qquad \Phi = -(\Phi_T - \Phi_c) \, \theta + \Phi_T.$$

Then

$$\nabla^2\theta = -\frac{4\pi G\rho_c}{\Phi_T - \Phi_c}\,\theta^n.$$

In spherical polars,

$$\nabla^2\theta = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right).$$

Introduce the dimensionless radius

$$\xi = \left(\frac{4\pi G\rho_c}{\Phi_T - \Phi_c}\right)^{1/2} r,$$

to obtain the *Lane–Emden equation of index* n:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

Boundary conditions at the centre:

$$\theta(0) = 1, \qquad \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0.$$

This equation has closed–form solutions for $n = 0, 1, 5$, but generally must be integrated numerically.

5.8 Solutions for the Lane–Emden equation

The Lane–Emden equation of index n is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\,\theta^n. \tag{145}$$

5.8.1 Solution for $n = 0$

For $n = 0$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -1. \tag{146}$$

Integrating once,

$$\xi^2\frac{d\theta}{d\xi} = -\frac{1}{3}\,\xi^3 - C, \tag{147}$$

so

$$\theta = D + \frac{C}{\xi} - \frac{1}{6}\,\xi^2, \tag{148}$$

with constants C, D. Regularity at the origin together with $\theta(0) = 1$ and $d\theta/d\xi|_{\xi=0} = 0$ require $C = 0$, $D = 1$, hence

$$\boxed{\theta_0 = 1 - \frac{\xi^2}{6}} \tag{149}$$

and the surface occurs at the first zero, $\xi_1 = \sqrt{6}$.

5.8.2 Solution for $n = 1$

For $n = 1$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta. \tag{150}$$

Let $\theta = \chi/\xi$. The Lane–Emden equation becomes

$$\frac{d^2\chi}{d\xi^2} = -\,\chi. \tag{151}$$

Thus $\chi = A\sin(\xi + B)$ and

$$\theta = \frac{A\sin(\xi + B)}{\xi}. \tag{152}$$

Regularity at $\xi = 0$ imposes $A = 1$, $B = 0$, giving

$$\boxed{\theta_1 = \frac{\sin\xi}{\xi}} \tag{153}$$

with the first zero at $\xi = \pi$ and θ_1 monotone on $(0, \pi)$.

5.8.3 Solution for $n = 5$

For $n = 5$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^5. \tag{154}$$

Set $x = 1/\xi$. Then the equation can be written

$$x^4\frac{d^2\theta}{dx^2} = -\theta^n. \tag{155}$$

Introduce $t = \ln x$ and

$$\theta = \left[\frac{2(n-3)}{(n-1)^2}\right]^{1/(n-1)} x^{\frac{2}{n-1}}\, z(t). \tag{156}$$

This yields

$$\frac{d^2z}{dt^2} + \frac{5-n}{n-1}\frac{dz}{dt} - \frac{2(n-3)}{(n-1)^2}\, z\,(1 - z^{\,n-1}) = 0. \tag{157}$$

For $n = 5$ this reduces to

$$\frac{d^2z}{dt^2} = \frac{1}{4}\, z\,(1 - z^4). \tag{158}$$

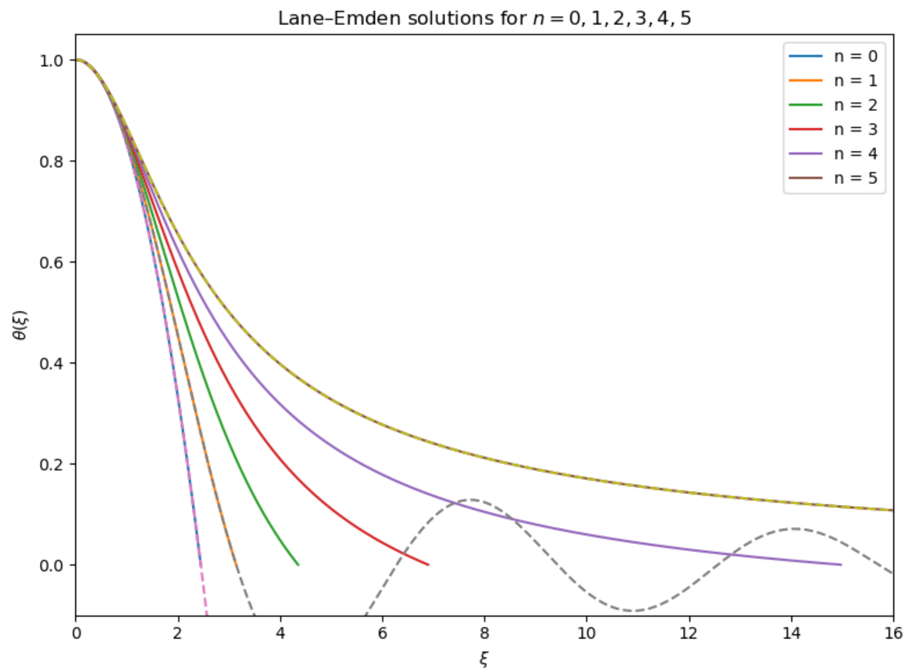
Multiplying by dz/dt and integrating,

$$\frac{1}{2}\frac{d}{dt}\left(\frac{dz}{dt}\right)^2 = \frac{1}{4}\, z(1 - z^4)\frac{dz}{dt} \quad\Rightarrow\quad \frac{1}{2}\left(\frac{dz}{dt}\right)^2 = \frac{1}{8}z^2 - \frac{1}{24}z^6 + D. \tag{159}$$

Boundary conditions imply $D = 0$. With the substitution $\frac{1}{3}z^4 = \sin^2\phi$ one finds $\tan(\phi/2) = Ce^{-t}$, leading finally to

$$\boxed{\theta_5 = \frac{1}{\left(1 + \frac{1}{3}\,\xi^2\right)^{1/2}}} \tag{160}$$

which vanishes only as $\xi \to \infty$; the equilibrium configuration thus extends to infinity.



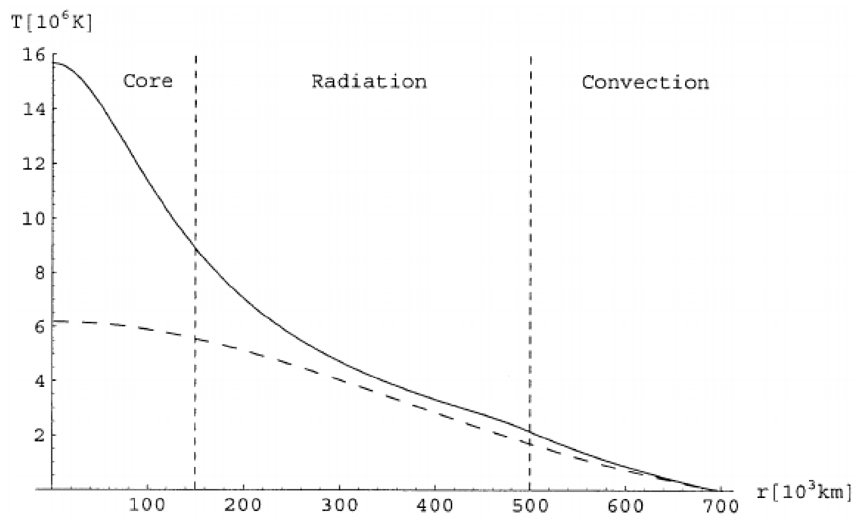


Figure 38: Solutions to the Lane-Emden equation for various values of n. The higher n, the larger the ξ value for which the solution crosses $\theta = 0$. The temperature distribution in the Sun as a function of the distance from the centre. The fully drawn curve is from the 'standard' Sun model and the dashed curve is the Lane–Emden solution for $\gamma = 5/3$.

6 Hydrostatics in non-inertial systems: hydrostatic shapes

Disregarding surface tension, in hydrostatic equilibrium with gravity, an interface between two fluids of different densities, for example the sea and the atmosphere, must coincide with a surface of constant potential, an equipotential surface. Otherwise, if an interface crosses an equipotential surface, there will arise a tangential component of gravity which can only be balanced by shear contact forces which a fluid at rest is unable to supply.

6.1 Fluid interfaces in hydrostatic equilibrium

Consider an equilibrium interface between two fluids with local densities ρ_1 and ρ_2. Since the gravitational field is the same on both sides of the interface, hydrostatic balance holds:

$$(\vec{\nabla} p)_1 = \rho_1 \vec{g}, \qquad (\vec{\nabla} p)_2 = \rho_2 \vec{g}. \tag{161}$$

Because $\rho_1 \neq \rho_2$, there is a jump in the pressure gradient across the interface. Suppose, for the sake of argument, that $\vec{g}$ is not orthogonal to the interface. Then there must be a jump in the *tangential* pressure gradient along the interface. If the pressures on the two sides are equal at one point of the interface, they would necessarily differ a small distance away along the interface. Newton's third law, however, requires the pressure to be the same on both sides of the interface (in the absence of surface tension). The contradiction is avoided only if $\vec{g}$ is everywhere orthogonal to the interface. Therefore the interface coincides with an equipotential surface. **Isobars coincide with equipotential surfaces in hydrostatic equilibrium.**

6.2 Fictitious forces

Newton's Second Law of motion is only valid in *inertial* reference frames, where free particles move on straight lines with constant velocity. In classical mechanics a particle is commonly said to be free if it is not subject to forces caused by identifiable sources, for example the mass or electric charge of a material body. In reference frames that are accelerated or rotating relative to inertial frames, one may formally write the Second Law in its usual form, but the price to pay is the inclusion of certain force-like terms that do not have any obvious sources, but only depend on the motion of the reference frame. Such terms are called *fictitious forces*, although they are by no means pure fiction, as one becomes painfully aware when standing in a bus that suddenly stops. A more reasonable name might be *inertial forces*, since they arise as a consequence of the inertia of material bodies.

6.3 Linearly accelerated containers

Near Earth's surface, a container of liquid is subjected to a constant horizontal acceleration in the y–direction, $\vec{a} = a\,\hat{\jmath}$ (see Fig. 39). The resulting external force density is

$$\vec{f}_e = \rho\,\vec{g} + \vec{f}_{\text{fict}} = \rho\,\vec{g} - \rho\,\vec{a} = \rho\,\vec{\nabla}(-gz - ay) = -\rho\,\vec{\nabla}\Phi_{eff} = -\rho\,\vec{\nabla}\big(gz + ay\big), \tag{162}$$

where $\Phi_{eff} = gz + ay$ is the *effective potential*. Thus the hydrostatic balance in the accelerated frame becomes

$$\vec{\nabla} p + \rho\,\vec{\nabla}\Phi_{eff} = \vec{0}. \tag{163}$$

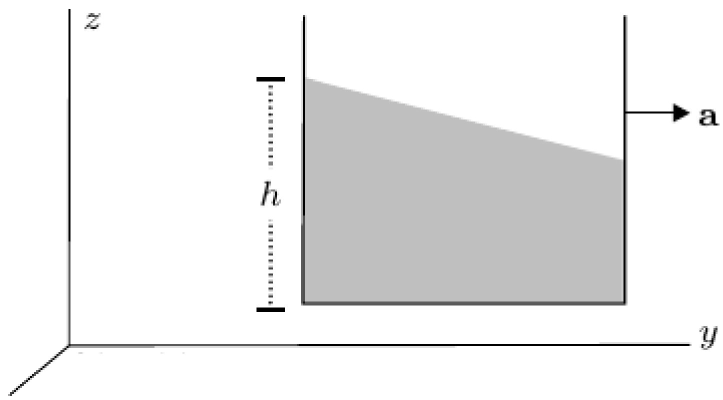


Figure 39: Linearly accelerating container. Figure taken from [1].

6.3.1 Constant–density case

If the density is constant, integrating gives

$$p + \rho\,(gz + ay) = C. \tag{164}$$

If at the point $(y, z) = (0, h)$ the pressure equals p_0, then

$$p = p_0 + \rho g\,(h - z) - \rho a\, y. \tag{165}$$

The free surface $p = p_0$ is therefore an inclined plane,

$$z = h - \frac{a}{g}\, y. \tag{166}$$

6.3.2 Acceleration in a plane

If, instead of a pure y–acceleration, the accelerated frame has components $\vec{a} = \hat{j}\, a_y + \hat{k}\, a_z$, then

$$\Phi_{eff} = \vec{a}\cdot\vec{r} + gz. \tag{167}$$

Taking into account that

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}), \tag{168}$$

it follows that

$$\vec{\nabla}(\vec{a}\cdot\vec{r}) = (\vec{a}\cdot\vec{\nabla})\vec{r} = \left(a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right)(\hat{i}x + \hat{j}y + \hat{k}z) = \hat{j}a_y + \hat{k}a_z. \tag{169}$$

Then

$$\vec{\nabla}\Phi_{eff} = \hat{j}a_y + \hat{k}(a_z + g). \tag{170}$$

The hydrostatic equilibrium condition now reads

$$\vec{\nabla}p + \rho\vec{\nabla}\Phi_{eff} = \vec{\nabla}p + \rho(\hat{j}a_y + \hat{k}(a_z + g)). \tag{171}$$

In components

$$-\rho\,[\hat{k}\,(g+a_z)+\hat{j}\,a_y]=\hat{\imath}\,\frac{\partial p}{\partial x}+\hat{\jmath}\,\frac{\partial p}{\partial y}+\hat{k}\,\frac{\partial p}{\partial z}, \tag{172}$$

$$\therefore \quad \frac{\partial p}{\partial x}=0, \qquad\qquad \frac{\partial p}{\partial y}=-\rho a_y, \qquad \frac{\partial p}{\partial z}=-\rho(g+a_z). \tag{173}$$

Using the total differential then gives

$$dp=\frac{\partial p}{\partial x}\,dx+\frac{\partial p}{\partial y}\,dy+\frac{\partial p}{\partial z}\,dz=-\rho a_y\,dy-\rho(g+a_z)\,dz. \tag{174}$$

Integrating between two arbitrary points 1 and 2 results in

$$p_2-p_1=-\rho a_y\,(y_2-y_1)-\rho(g+a_z)\,(z_2-z_1). \tag{175}$$

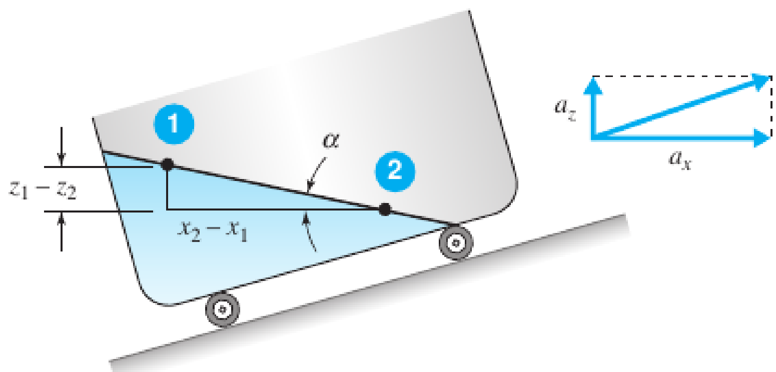


Figure 40: Linearly accelerating container. Figure taken from [4].

If points 1 and 2 lie on a constant–pressure line (e.g., the free surface in Fig. 40), then $p_2-p_1=0$ and

$$\frac{z_1-z_2}{x_2-x_1}=\tan\alpha=\frac{a_x}{g+a_z}, \tag{176}$$

where α is the angle that the constant–pressure line makes with the horizontal.

In solving problems involving liquids, we often use conservation of mass and equate the volumes before and after the acceleration is applied. After the acceleration is first applied, sloshing may occur. Our analysis will assume that sloshing is not present: either sufficient time has passed to damp out time–dependent motions, or the acceleration is applied in such a way that such motions are minimal.

Example. The rectangular, open tank in Fig. 41 (length 2 m, initial water depth 1.2 m, air gap 0.2 m) is accelerated to the right with horizontal acceleration a_x while $a_z=0$. Find:

1. the acceleration a_x required for the free surface in Fig. E2.11b to pass through point A (the right top corner),

2. the pressure p_B at the left bottom corner B,

3. the total hydrostatic force on the tank bottom if the tank width is 1 m.

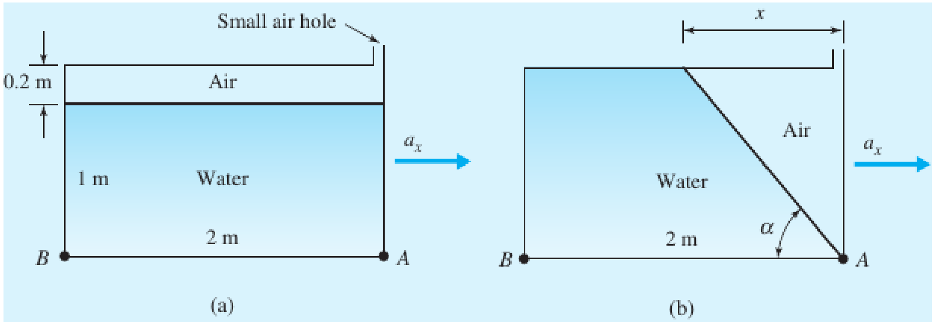


Figure 41: Linearly accelerating tank. Figure taken from [4].

1) Free-surface angle and required acceleration. In a uniformly accelerated container, lines of constant pressure are orthogonal to the effective acceleration $\vec{g}_{\text{eff}} = (g + a_z)\,\hat{k} - a_x\,\hat{i}$, so the free surface (an isobar) satisfies

$$\tan\alpha = \frac{a_x}{g + a_z} \qquad \Rightarrow \qquad \tan\alpha = \frac{a_x}{g}\,.$$

The final free surface must rotate about the fluid volume so that the air *area* (since width is constant) is conserved. Before acceleration the air rectangle is

$$A_{\text{air, ini}} = 0.2 \times 2 = 0.4\ \text{m}^2.$$

After tilting, the air area becomes a right triangle of base 2 and height x at the right wall:

$$A_{\text{air, fin}} = \frac{1}{2}(1.2)\,x.$$

Conservation $A_{\text{air, ini}} = A_{\text{air, fin}}$ gives

$$0.2 \times 2 = \frac{1}{2}(1.2)\,x \qquad \Rightarrow \qquad x = 0.667\ \text{m}.$$

Hence the free-surface slope is

$$\tan\alpha = \frac{1.2}{0.667} = 1.8,$$

and therefore

$$a_x = g\,\tan\alpha = (9.81)\,(1.8) = 1.766 \times 10^1\ \text{m}\,\text{s}^{-2} \approx 17.66\ \text{m}\,\text{s}^{-2}.$$

2) Pressure at B. For hydrostatics in the accelerating frame (with $a_z = 0$) the pressure gradient is

$$\vec{\nabla} p = \rho\,(a_x\,\hat{i} - g\,\hat{k}).$$

Along the bottom ($z = \text{const}$) only the x-variation matters, so between points A (right bottom) and B (left bottom) separated by $\Delta x = x_B - x_A = -2\,\text{m}$,

$$p_B - p'_A = -\rho\,a_x\,(x_B - x_A),$$

where p'_A is the pressure at the bottom point vertically under A. The free surface passes through A, so $p_A = 0$ at A, and by vertical hydrostatic balance $p'_A = 0$ at the bottom under A (same vertical line on an isobaric free surface). Hence, with $\rho = 1000\ \text{kg}\,\text{m}^{-3}$,

$$p_B = -\rho a_x (x_B - x_A) = -1000\,(17.66)\,(-2) = 3.53 \times 10^4\ \text{Pa} = 35.3\ \text{kPa}.$$

3) Resultant force on the bottom (width $= 1\,\mathrm{m}$*).* Pressure varies linearly from $p = 35.3\,\mathrm{kPa}$ at B to $p = 0$ at the bottom point beneath A. The average bottom pressure is thus

$$\bar{p} = \frac{p_B + p'_A}{2} = \frac{35.3\ \mathrm{kPa} + 0}{2} = 17.65\ \mathrm{kPa}.$$

With bottom area $A_{\mathrm{bot}} = 2 \times 1 = 2\ \mathrm{m}^2$, the total force is

$$F = \bar{p}\,A_{\mathrm{bot}} = \left(17.65 \times 10^3\right)(2) = 3.53 \times 10^4\ \mathrm{N} \approx 35\,300\ \mathrm{N}.$$

6.4 Rotating containers

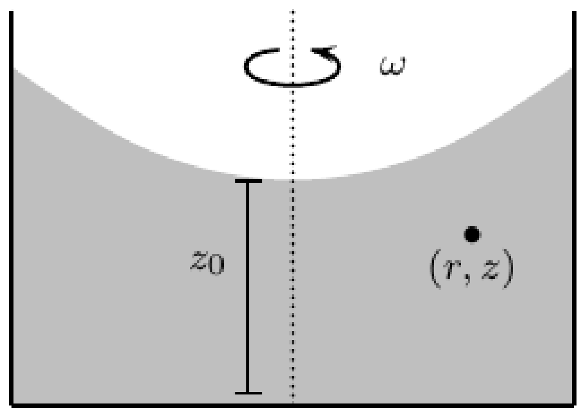


Figure 42: Liquid at rest in a rotating cylindrical container. Figure taken from [1].

To study the statics of a fluid that rotates uniformly, consider the basic hydrostatic equation in a rotating frame. In addition to gravity (when present), the fictitious forces must be included. In cylindrical coordinates, the centrifugal body-force density is

$$\vec{f} \;=\; \frac{d\vec{F}}{dV} \;=\; \frac{dm}{dV}\,\omega^2 r\,\hat{e}_r \;=\; \rho\,\omega^2 r\,\hat{e}_r, \tag{177}$$

so that the effective force density is

$$\vec{f_e} \;=\; \rho\,\vec{g} + \rho\,\omega^2 r\,\hat{e}_r \;=\; \rho\,\vec{\nabla}\left[-\Phi + \frac{\omega^2 r^2}{2}\right] \;=\; \vec{\nabla}p, \tag{178}$$

and therefore the hydrostatic balance can be written as

$$\frac{\vec{\nabla}p}{\rho} \;+\; \vec{\nabla}\left(\Phi - \frac{\omega^2 r^2}{2}\right) \;=\; 0. \tag{179}$$

The quantity $\Phi - \omega^2 r^2/2$ is called the *effective potential* Φ_{eff}. Equivalently, $-\rho\,\vec{\nabla}\Phi_{eff} = \vec{\nabla}p$.

6.4.1 Constant-density liquid

In this case, from the previous relation,

$$\vec{\nabla}\left[\frac{p}{\rho} + \Phi - \frac{\omega^2 r^2}{2}\right] = \vec{0}, \qquad \Rightarrow \qquad p + \rho\, g z - \frac{\omega^2}{2}\,\rho\, r^2 = C, \tag{180}$$

where we used $\Phi = gz$. Taking $p = p_0$ at $z = z_0$ and $r = 0$, one finds $p_0 + \rho g z_0 = C$, and thus

$$= \; p_0 + \rho g\,(z - z_0) + \frac{\omega^2}{2}\,\rho\, r^2. \tag{181}$$

Isobars are therefore paraboloids of revolution. In particular, the free surface (where $p = p_0$, atmospheric pressure) satisfies

$$\omega^2 r^2 \; = \; 2g\,(z - z_0). \tag{182}$$

Note that the shape of the surface does not depend on ρ, provided it is constant. In a bucket of diameter 20 cm rotating once per second the water stands 2 cm higher at the rim than in the centre.

6.4.2 Ideal gas in a closed container

For an ideal gas with constant temperature T and mean molecular weight μ,

$$\frac{\vec{\nabla}p}{\rho} + \nabla\left(\Phi - \frac{\omega^2 r^2}{2}\right) = \vec{0} \qquad \Longleftrightarrow \qquad \vec{\nabla}p\,\frac{kT}{p\mu} + \nabla\left(\Phi - \frac{\omega^2 r^2}{2}\right) = \vec{0}. \tag{183}$$

Hence,

$$\vec{\nabla}\left[\frac{kT}{\mu}\ln p + gz - \frac{\omega^2 r^2}{2}\right] = \vec{0} \quad \Rightarrow \quad \frac{kT}{\mu}\ln p + gz - \frac{\omega^2 r^2}{2} = C. \tag{184}$$

With $p = p_0$ at $r = 0$, $z = 0$, we obtain

$$\frac{kT}{\mu}\ln\left(\frac{p}{p_0}\right) + gz - \frac{\omega^2 r^2}{2} = 0, \qquad \text{or} \qquad p \; = \; p_0 \exp\left[\frac{\mu}{kT}\left(\frac{\omega^2 r^2}{2} - gz\right)\right]. \tag{185}$$

The isobars $z = \omega^2 r^2/(2g)$ are, again, paraboloids of revolution.

6.4.3 The figure of Earth

On a rotating planet or star, centrifugal forces add a cylindrical component of “anti-gravity” to the gravitational acceleration field. A convenient measure of its importance is the ratio of centrifugal acceleration to surface gravity at the equator. With surface gravity $g_0 \approx GM/a^2$ (where M is the mass of the planet and a its equatorial radius), we define

$$q \; = \; \frac{\omega^2 a}{g_0} \; \approx \; \frac{\omega^2 a^3}{GM}. \tag{186}$$

For Earth ($a \approx 6371$ km, $g_0 \approx 9.8$ m s^{-1}, $\omega \approx 73 \times 10^{-6}$ s^{-1}), one finds $q \approx 3.5 \times 10^{-3} = 0.35\%$, confirming that centrifugal effects are small. If Earth rotated as a rigid sphere with a period of 85 min, one would have $q = 1$: objects at the equator would (and could) levitate.

Rotating neutron star (pulsar) Neutron stars are among the most extreme objects in the Universe. A typical neutron star may have radius $a = 10$ km and mass $M \simeq 4 \times 10^{30}$ kg (about twice the Sun's), and can rotate with periods down to 1.4 ms (i.e. $\omega \simeq 4500$ s^{-1}). Then the surface gravity is $g_0 \simeq 2.7 \times 10^{12}$ m s^{-2} and the equatorial centrifugal acceleration is $\omega^2 a \simeq 2 \times 10^{11}$ m s^{-2}. The "levitation parameter" is therefore $q = \omega^2 a/g_0 \simeq 0.076$: about 22 times Earth's value, but still well below unity.

Even on a perfectly spherical Earth, this tiny centrifugal "antigravity" makes the path from pole to equator slightly downhill, creating a shallow centrifugal "valley" centered on the equator. A naive estimate of the valley depth is q times Earth's radius, i.e. 0.35% of a, which is about 22 km. If such a valley were suddenly cut into a spherical Earth, water would flow toward the equator (ignoring the Coriolis force). Since there *is* land at the equator, the true shape is a compromise between mass flow and centrifugal leveling. The observed difference between equatorial and polar radii is in fact 21.4 km, close to the above estimate—roughly the same as the difference between the highest mountain and the deepest ocean trench.

A complete theory of Earth's "figure" is subtle, and the real Earth departs from the simplest oblate-spheroid model. Nonetheless, the parameter q provides a useful first approximation to the shape of a slowly rotating planet.

6.4.4 Centrifugal deformation of a spherical planet

The centrifugal deformation of an originally spherical planet of radius a depends only on the distance $s = a \sin\theta$ to the rotation axis, where θ is the polar angle. If $h(\theta)$ denotes the radial displacement of the surface caused by rotation, the (first–order) total potential evaluated at the surface may be written as

$$\Phi \;=\; g_0\, h(\theta) \;-\; \tfrac{1}{2}\,\Omega^2 a^2 \sin^2\theta \;+\; \Delta\Phi(\theta).$$

The first term is the gravitational potential, the second term is the centrifugal potential at the surface, and $\Delta\Phi(\theta)$ is the contribution from the gravitational potential of the mass that is *redistributed* by the deformation itself (the "self–potential"). Even though the shifted mass is small compared with the total planetary mass, it is located close to the surface, so its effect on the surface potential can be significant and must not be ignored.

For a deformation of the appropriate shape, the self–potential to lowest order is

$$\Delta\Phi \;=\; -\,\frac{3}{5}\,\frac{\rho_1}{\rho_0}\; g_0\, h(\theta),$$

where ρ_1 is the density of the material that is displaced and $\rho_0 = M/\left(\frac{4}{3}\pi a^3\right)$ is the mean density of the planet. Substituting this expression into the surface potential above gives the (deformation) pole–to–equator height range

$$h_0 \;=\; \frac{1}{2}\, a\, q \left(1 \;-\; \frac{3}{5}\,\frac{\rho_1}{\rho_0}\right)^{-1},$$

where $q \equiv \Omega^2 a/g_0$ is the usual "levitation" (rotational) parameter. For Earth, taking representative values $\rho_1 \approx 4.5$ g cm^{-3} for the mantle and $\rho_0 \approx 5.5$ g cm^{-3}, the factor in parentheses is ≈ 0.51, yielding $h_0 \approx 21.9$ km, in good agreement with the observed equatorial–polar radius difference of about 21.4 km. *Caveat:* A fully self–consistent treatment should compute the self–potential from the actual redistribution of mass throughout the planet, including compressibility variations, not just from mean densities.

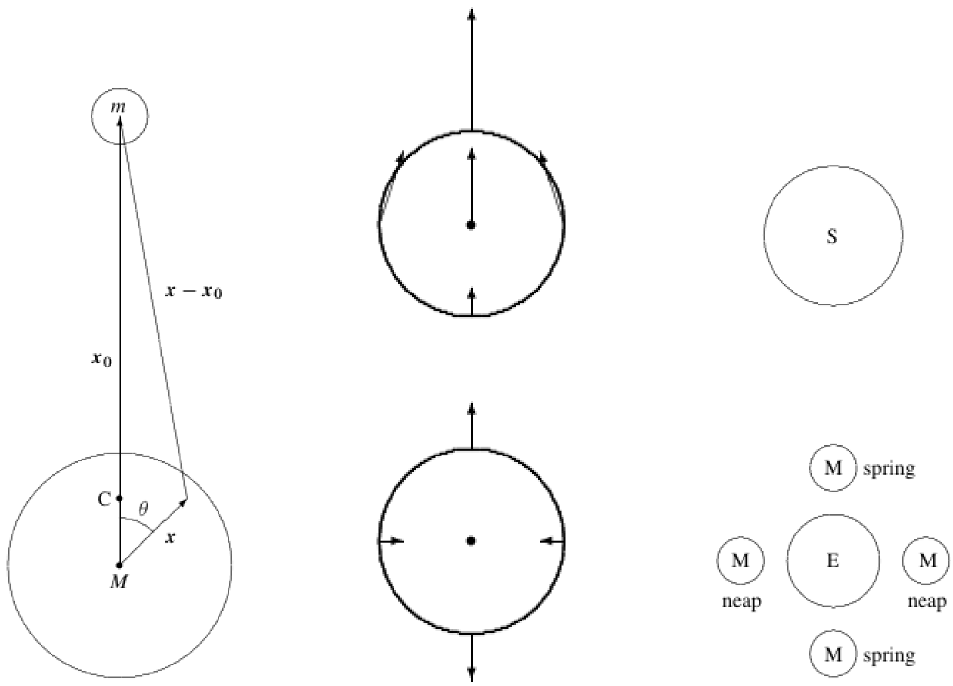


Figure 43: Left: geometry of the Earth and the Moon (not to scale). The point C is the center of mass of the system. Center: (top) how the Moon's gravity varies over the Earth (exaggerated); (bottom) the Moon's gravity with the unobservable constant acceleration canceled. This figure explains why the tides have a semi-diurnal period. Right: Earth and Sun with the Moon in extreme positions corresponding to spring tides and neap tides. The figure is (vastly) not to scale. Figures taken from Ref. [3].

6.5 The Earth, the Moon, and the tides

After Newton we know that the Moon's gravity acts on everything on Earth, including the water in the oceans, and tends to distort it, thereby creating the tides. Yet high tides occur at roughly the same time at antipodal points on Earth—and thus twice per day—so the explanation cannot be merely that the Moon "pulls" the sea toward itself.

The rotation of the Earth, together with the orbital motion of both Earth and Moon about their common center of mass, makes tides an essentially *dynamic* phenomenon that cannot be treated accurately by the hydrostatic methods used in this chapter. Nevertheless, a number of basic features can be revealed with simple approximations (and careful wording of explanations).

6.5.1 The tidal field

The Moon is sufficiently small and distant that its gravitational potential across the Earth can be approximated by that of a point mass m located at position $\vec{x}_0$:

$$\Phi(\vec{x}) = -\frac{G\,m}{|\vec{x} - \vec{x}_0|}\,. \tag{187}$$

Choose coordinates with origin at Earth's center and the z–axis pointing toward the Moon, so that $\vec{x}_0 = (0, 0, D)$, with $D = |\vec{x}_0|$ the Earth–Moon distance (see Fig. 43).

For points with $r = |\vec{x}| \ll D$ the potential can be expanded in powers of r/D. Using

$$|\vec{x} - \vec{x}_0| = \sqrt{x^2 + y^2 + (z-D)^2} = D\sqrt{1 - \frac{2z}{D} + \frac{r^2}{D^2}}\,, \tag{188}$$

and the binomial series to second order,

$$\frac{1}{|\vec{x} - \vec{x}_0|} = \frac{1}{D}\left(1 - \frac{2z}{D} + \frac{r^2}{D^2}\right)^{-1/2} \tag{189}$$

$$\approx \frac{1}{D}\left[1 + \frac{z}{D} + \frac{3z^2 - r^2}{2D^2}\right]. \tag{190}$$

The first term gives a constant $-Gm/D$, which can be dropped. The second term corresponds to a uniform field that keeps the Earth in orbit ($g_z = Gm/D^2 \approx 33\mu\text{m/s}^2$) and is also discarded when focusing on differential ("tidal") effects. The leading nontrivial, spatially varying term is therefore the quadrupolar contribution, which defines the *tidal potential*

$$\Phi_{\text{tidal}} \;=\; -\frac{Gm}{D^3}\,\frac{1}{2}\big(3z^2 - r^2\big). \tag{191}$$

In spherical coordinates (r, θ, ϕ) with θ the angle from the z–axis ($z = r\cos\theta$), this can be written

$$\Phi_{\text{tidal}} \;=\; -\frac{1}{2}\left(3\cos^2\theta - 1\right)\frac{Gm}{D^3}\,r^2. \tag{192}$$

The Moon's tidal *acceleration* is the gradient of the tidal potential. In spherical coordinates,

$$g_r = -\frac{\partial \Phi_{\text{tidal}}}{\partial r} \;=\; \left(3\cos^2\theta - 1\right)\frac{Gm}{D^3}\,r, \tag{193}$$

$$g_\theta = -\frac{1}{r}\frac{\partial \Phi_{\text{tidal}}}{\partial \theta} \;=\; -\frac{3}{2}\sin 2\theta\,\frac{Gm}{D^3}\,r, \tag{194}$$

$$g_\phi = -\frac{1}{r\sin\theta}\frac{\partial \Phi_{\text{tidal}}}{\partial \phi} \;=\; 0. \tag{195}$$

The radial component is largest at $\theta = 0°$ and $180°$ (the sub-Moon and anti-Moon points) and smallest at $\theta = 90°$ (the great circle $90°$ from the Moon), where it is negative and half in magnitude. The tangential component is largest in magnitude at $\theta = 45°$ and $135°$. On Earth's surface the vertical part of the tidal acceleration is $\sim 3\,a\,Gm/D^3 \approx 1.65\,\mu\text{m/s}^2$, where a is Earth's radius.

6.5.2 The tides

If the Earth did not rotate and the Moon stood still above a particular spot, water would rush in to fill up these "valleys", and the sea would come to equilibrium with its open surface at constant total gravitational potential. The total potential near the surface of the Earth is

$$\Phi \;=\; \Phi_{\text{Earth}} + \Phi_{\text{Moon}} \;=\; g_0\,h \;-\; \tfrac{1}{2}\big(3\cos^2\theta - 1\big)\left(\frac{a}{D}\right)^2\frac{Gm}{D}, \tag{196}$$

where g_0 the surface gravity and h a small (signed) height above the spherical surface of the Earth.

Requiring this potential to be constant gives the tidal height

$$h \;=\; h_0 \;+\; \tfrac{1}{2}\big(3\cos^2\theta - 1\big)\Big(\frac{a}{D}\Big)^2 \frac{Gm}{g_0 D}, \tag{197}$$

where h_0 is a constant. Since the average over the sphere of the second term vanishes,

$$\frac{1}{4\pi}\int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \;\sin\theta\,\phi\;\big(3\cos^2\theta - 1\big) \;=\; \frac{1}{2}\int_{-1}^{+1}(3z^2-1)\,\mathrm{d}z \;=\; 0, \tag{198}$$

we conclude that h_0 is the average water depth.

The shape of the equipotential surface corresponding to $\Phi = 0$ becomes

$$h \;=\; \tfrac{1}{2}\big(3\cos^2\theta - 1\big)\Big(\frac{a}{D}\Big)^2 \frac{Gm}{g_0 D}. \tag{199}$$

6.5.3 Tidal range

The maximal difference between high and low tides (the tidal range) occurs between the extreme positions at $\theta = 0°$ and $\theta = 90°$:

$$H_0 \;=\; \frac{3}{2}\Big(\frac{a}{D}\Big)^2 \frac{Gm}{g_0 D} \;=\; \frac{3}{2}\,a\,\frac{m}{M}\Big(\frac{a}{D}\Big)^3, \tag{200}$$

where in the last equality we used $g_0 = GM/a^2$ with M the Earth's mass. Inserting the values for the Moon we get $H_0 \approx 54$ cm. Interestingly, the range of the tides due to the Sun turns out to be half as large, about 25 cm. This makes spring tides when the Sun and the Moon cooperate almost three times higher than neap tides when they do not. For the tides to reach full height, water must move in from huge areas of the Earth as is evident from the shallow shape of the potential. Where this is not possible, for example in lakes and enclosed seas, the tidal range becomes much smaller than in the open oceans. Local geography may also influence tides. In bays and river mouths funnelling can cause tides to build up to huge values. Spring tides in the range of 15 m have been measured in the Bay of Fundy in Canada.

6.6 Exercises

1. Estimate the change in sea level if the air pressure locally rises by $\Delta p = 20$ hPa and stays there (disregarding surface tension).

2. The tides raises the water above the average level at the Moon and anti-Moon positions. (a) How much water is found in each of these tidal bulges?

Appendix

References

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Table 6: Density ρ, pressure p, and gravitational acceleration g as a function of depth within the Earth (after Anderson & Hart, 1976; Carmichael, 1989).

Depth (km)	$\boldsymbol{\rho}$ ($\mathrm{g\,cm^{-3}}$)	$\boldsymbol{p}$ (kbar)	$\boldsymbol{g}$ ($\mathrm{cm\,s^{-2}}$)
Crust			
0	1.02	0	981
3	1.02	3	982
21	2.80	5	983
Mantle (solid)			
21	3.49	5	983
41	3.51	12	983
61	3.52	19	984
81	3.48	26	984
101	3.44	33	984
121	3.40	39	985
171	3.37	56	987
221	3.34	73	989
271	3.37	89	991
321	3.47	106	993
371	3.59	124	994
571	3.95	199	999
871	4.54	328	997
1171	4.67	466	992
1471	4.81	607	991
1771	4.96	752	994
2071	5.12	903	1002
2371	5.31	1061	1017
2671	5.45	1227	1042
2886	5.53	1352	1069
Outer core (liquid)			
2886	9.96	1352	1069
2971	10.09	1442	1050
3371	10.63	1858	953
3671	11.00	2154	874
4071	11.36	2520	760
4471	11.69	2844	641
4871	11.99	3116	517
5156	12.12	3281	427
Inner core (solid)			
5156	12.30	3281	427
5371	12.48	3385	355
5771	12.52	3529	218
6071	12.53	3592	122
6371	12.58	3617	0