Hydrostatics

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3] books

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC3. Definir el concepto de presión desde una perspectiva microscópica.

- OC4. Enunciar el teorema de Pascal y explicar su significancia.

- OC5. Reconocer las distintas unidades que se usan para medir la presión.

- OC6. Identificar las condiciones físicas necesarias para describir un medio usando la ecuación de estado politrópica.

- OC7. Enunciar el principio de Arquímides y estudiar a partir de él las condiciones de estabilidad de cuerpos que flotan.

- OC8. Definir el concepto de tensión superficial, tanto desde la perspectiva de una densidad de energía superficial como desde las fuerzas por unidad de longitud.

1.2 Objetivos específicos procedimentales:

- OP3. Deducir, a partir del teorema de Bernoulli, la ecuación de equilibrio hidrostático global y local.

- OP4. Describir las condiciones de densidad y presión de fluidos incompresibles y compresibles en equilibrio hidrostático.

- OP5. Deducir las ecuaciones del interior de un cuerpo gravitante esférico en equilibrio hidrostático.

- OP6. Deducir la ecuación de Lane-Emden y aplicarla para estudiar el caso de estrellas hipotéticas que obedecen la ecuación de estado politrópica.

- OP7. Aplicar la definición de tensión superficial para describir el fenómeno de capilaridad.

1.3 Objetivos específicos actitudinales:

- OA2. Distinguir y enumerar algunas de las ventajas más importantes de la aproximación continua frente a la aproximación discreta (estadística) de los medios materiales.

2 Pleliminaries

2.1 Density

Fluid is the generic name for liquids and gases. A gas completely fills a closed container, but a liquid does not. The (matter) density of a fluid is defined as

$$\rho \equiv \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}. \tag{1}$$

In continuum physics, density is understood as a macroscopic average over a sufficiently large number of microscopic constituents. This requires a separation of scales: the microscopic length scale (molecular spacing or mean free path) must be much smaller than the macroscopic scale on which density varies appreciably.

In addition to the usual *mass density* ρ, one often encounters:

- **Number density** n: number of particles per unit volume, with $\rho = nm$ for particles of mass m.

- **Energy density** ρ_E: energy per unit volume, relevant in relativistic systems and cosmology.

Table 1: Densities of some common substances

Material	Density (kg/m^3)	Material	Density (kg/m^3)
Air (1 atm, 20°C)	1.20	Iron ($Z = 26$), steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper ($Z = 29$)	8.9×10^3
Ice	0.92×10^3	Silver ($Z = 47$)	10.5×10^3
Water	1.00×10^3	Lead ($Z = 82$)	11.3×10^3
Seawater	1.03×10^3	Mercury ($Z = 80$)	13.6×10^3
Blood	1.06×10^3	Gold ($Z = 79$)	19.3×10^3
Glycerin	1.26×10^3	Platinum ($Z = 78$)	21.4×10^3
Concrete	2.0×10^3	Iridium ($Z = 77$)	22.56×10^3
Aluminum	2.7×10^3	Osmium ($Z = 76$)	22.59×10^3

2.2 Astrophysical objects

Stars. The Sun has an average density of $\sim 1.4 \times 10^3 \, \text{kg/m}^3$, comparable to water, but its core density reaches $\sim 1.5 \times 10^5 \, \text{kg/m}^3$. In stellar structure, density profiles $\rho(r)$ enter the equations of hydrostatic equilibrium, balancing gravitational forces against pressure gradients.

Compact objects.

- *White dwarfs:* electron-degenerate matter with densities of order $10^9 - 10^{10} \, \text{kg/m}^3$.

- *Neutron stars:* neutron-degenerate matter with densities up to $10^{18} \, \text{kg/m}^3$, comparable to nuclear matter density.

In these cases, density determines the degeneracy pressure that counteracts gravity.

Galactic environments. In addition to stars and compact objects, density plays a central role in galactic dynamics:

- *Galactic center:* The Milky Way hosts a supermassive black hole, Sgr A*, surrounded by a dense stellar cluster. The average stellar mass density in the inner parsec reaches values of order $\rho \sim 10^9 \, M_\odot \, \mathrm{pc}^{-3}$, corresponding to $\sim 4 \times 10^{-11} \, \mathrm{kg/m^3}$. Although extremely dilute compared to ordinary matter, this is enormously high relative to the mean cosmic density.

- *Local Group:* On megaparsec scales, the mean density of galaxies in the Local Group is of order $\rho \sim 10^{-25} \, \mathrm{kg/m^3}$, only a few times the critical density of the Universe. This low density governs the dynamics of galaxy interactions and the eventual merging of the Milky Way and Andromeda.

Cosmological scales. The mean density of the Universe is extraordinarily low. The critical density is defined as

$$\rho_c = \frac{3H_0^2}{8\pi G}, \tag{2}$$

where $H_0 \approx 68$ km/s/Mpc is the Hubble constant. Numerically, $\rho_c \sim 10^{-26} \, \mathrm{kg/m^3}$. Cosmological structure formation is often described by the density contrast

$$\delta = \frac{\Delta \rho}{\rho},$$

which measures deviations from the average density.

Astrophysical fluids. In accretion disks, stellar winds, and interstellar clouds, density controls emission and cooling processes, and it determines the onset of instabilities such as the Jeans instability, which occurs when self-gravity overcomes pressure support.

Black holes. Although a black hole is not composed of matter in the usual sense, one can define an *effective average density* as the mass divided by the volume of a sphere with the Schwarzschild radius $R_s = 2GM/c^2$. This yields

$$\rho_{\mathrm{BH}} = \frac{3c^6}{32\pi G^3 M^2}. \tag{3}$$

Interestingly, this density decreases with increasing mass:

- A stellar-mass black hole ($M \sim 10 \, M_\odot$) has an effective density of order $10^{19} \, \mathrm{kg/m^3}$, even greater than nuclear matter.

- A supermassive black hole ($M \sim 10^9 \, M_\odot$) has an effective density as low as $\sim 10^3 \, \mathrm{kg/m^3}$, comparable to water.

This illustrates that black holes are not "dense" in the ordinary sense, but are instead defined by the curvature of spacetime.

Table 2 lists characteristic densities of ordinary and astrophysical systems. Density connects microscopic physics (molecular interactions, degeneracy pressure, nuclear matter properties) with macroscopic astrophysical phenomena (stellar equilibrium, gravitational collapse, cosmic expansion). It thus provides a unifying parameter linking laboratory scales to the most extreme environments in the Universe.

Table 2: Characteristic densities of materials and astrophysical systems.

System	Density (kg/m^3)
Air (1 atm, 20°C)	~ 1
Water	1.0×10^3
Average density of the Sun	1.4×10^3
Solar core	1.5×10^5
White dwarf	$10^9 - 10^{10}$
Neutron star	10^{18}
Stellar-mass black hole ($\sim 10 M_\odot$)	$\sim 10^{19}$
Supermassive black hole ($\sim 10^9 M_\odot$)	$\sim 10^3$
Galactic center (Milky Way)	$\sim 4 \times 10^{-11}$
Local Group (galaxies)	$\sim 10^{-25}$
Critical density of Universe	$\sim 10^{-26}$

3 Hydrostatic equilibrium

In the absence of external forces or internal heat sources, fluids eventually reach *hydrostatic equilibrium*, where motion ceases and fields remain constant in time. Examples: atmosphere, oceans, planetary and stellar interiors.
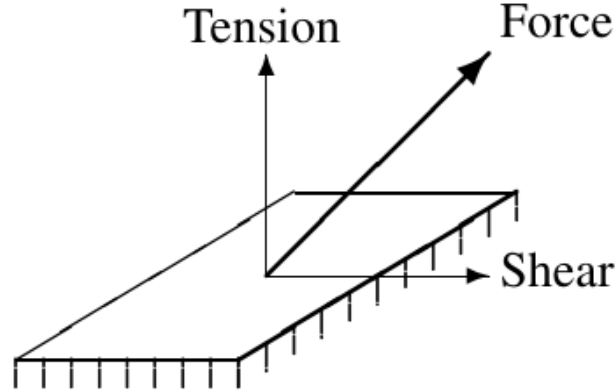

Figure 1: The force acting on the material underneath a small patch of a surface can always be resolved into a perpendicular pressure force and a tangential shear force. The pressure is positive if the force is directed toward the patch, and negative if it (as here) is directed away from it. Figure taken from Ref. [3].

Mechanical equilibrium in a continuum involves a balance between:

- *Contact forces* (short-range, acting through surfaces).

- *Body forces* (long-range, e.g. gravity).

Contact forces act across *contact surfaces* and can be decomposed into (see Fig. 1):

- **Normal component**: aligned with surface normal.

  - Same orientation → *tension force*.

– Opposite orientation → *pressure force.*

- **Tangential component**: parallel to surface → *shear force* or *traction force.*

3.1 Pressure

A fluid at rest cannot sustain shear stresses or tensile stresses. Any attempt to impose a tangential force on the fluid is immediately relieved by flow, so the intermolecular forces in a static fluid can only transmit *normal stresses.* These stresses act equally in all directions and always tend to compress objects immersed in the fluid.

Thus, the force exerted by a static fluid on the surface of an object is always *perpendicular* (normal) to that surface. This isotropic normal stress is what we call the **pressure** of the fluid.

The pressure p in a static fluid is defined as the normal force per unit area:

$$p = \frac{d\mathcal{F}_\perp}{dS}, \tag{4}$$

where $d\mathcal{F}_\perp$ is the component of force acting normal to a surface of area dS. Its SI unit is the newton per square meter (N/m^2), which in 1971 was given the special name *pascal* (Pa). Thus,

$$1\,\text{Pa} = 1\,\text{N/m}^2.$$

Earlier units of pressure that are still in use include:

- The **bar**, defined as $1\,\text{bar} = 10^5\,\text{Pa}$.

- The **standard atmosphere**, defined as

$$1\,\text{atm} = 101{,}325\,\text{Pa},$$

which corresponds approximately to the average air pressure at sea level.

In many practical contexts, other pressure units may appear:

- **Torr**: commonly used in vacuum physics, defined as $1\,\text{Torr} = \frac{1}{760}\,\text{atm} \approx 133.3\,\text{Pa}$.

- **mmHg**: pressure exerted by a column of mercury of height $1\,\text{mm}$, historically used in meteorology and medicine.

Remark. Pressure is a scalar quantity in fluids at rest, because it acts equally in all directions. In fluids in motion, however, stresses may include both pressure and shear contributions, and the concept of a *stress tensor* becomes necessary for a complete description.

Applications. Understanding pressure units and conversions is essential in astrophysics and geophysics: for example, the pressure in the solar core reaches $\sim 10^{16}\,\text{Pa}$, while the pressure at the center of the Earth is about $3.6 \times 10^{11}\,\text{Pa}$. These extreme values highlight the wide range of conditions described by the same fundamental concept of pressure.

3.2 The pressure field

The (flat) vector surface elements is defined as the product of its area dS and the unit vector $\hat{n}$ in the direction of the normal to the surface,

$$d\vec{S} = \hat{n} \, dS. \tag{5}$$

By universal convention, the normal of a closed surface is taken to point *outward*, so that the enclosed volume lies on the negative side.

A fluid at rest cannot sustain shear forces, so all contact forces on a surface must act along the normal at every point of the surface. The force exerted by the material at the *positive* side of the surface element dS (near $\mathbf{x}$) on the material at the *negative* side is written as

$$d\vec{\mathcal{F}} = -p(\vec{x}) \, d\vec{S}, \tag{6}$$

where $p(\vec{x})$ is the pressure field. Convention dictates that a positive pressure exerts a force directed *toward* the material on the negative side of the surface element, and this explains the minus sign. A negative pressure that pulls at a surface is sometimes called a *tension*. The total pressure force acting on any oriented surface S is obtained by adding all the little vector contributions from each surface element,

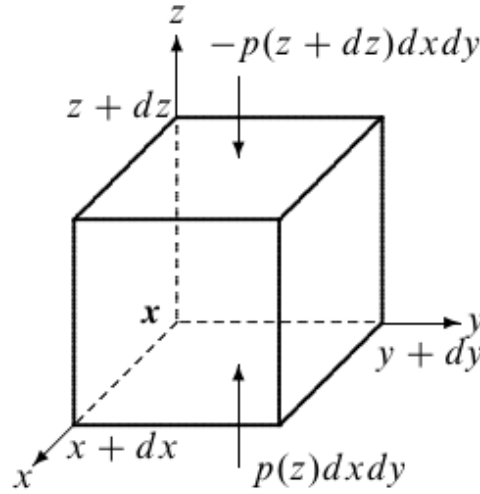$$\vec{\mathcal{F}} = \int_S -p(\vec{x}) \, d\vec{S}. \tag{7}$$

Figure 2: Pressure forces in the z-direction on a material particle in the shape of a rectangular box with sides dx, dy, dz. Figure taken from Ref. [3]

A material particle in the fluid is like any other body subject to pressure forces from all sides, but being infinitesimal it is possible to derive a general expression for the resultant force. Let us consider a material particle in the shape of a small rectangular box with sides dx, dy, and dz, such that it has a volume $dV = dx \, dy \, dz$. Since the pressure is slightly different on opposite sides of the box, the z-component of total pressure force becomes

$$d\mathcal{F}_z = \big(p(x,y,z) - p(x,y,z+dz)\big) \, dx \, dy \approx -\frac{\partial p(x,y,z)}{\partial z} \, dx \, dy \, dz. \tag{8}$$

Doing the same for the other coordinate directions we obtain the total pressure force

$$d\vec{\mathcal{F}} = -\left(\hat{i}\,\frac{\partial p}{\partial x} + \hat{j}\,\frac{\partial p}{\partial y} + \hat{k}\,\frac{\partial p}{\partial z}\right)dV = -\vec{\nabla}p\,dV. \tag{9}$$

3.3   Hydrostatic Equilibrium

In equilibrium, the net force on a differential fluid element vanishes:

$$d\vec{F}_e + d\vec{F}_p = 0, \tag{10}$$

where $d\vec{F}_e$ denotes external (volumetric) and $d\vec{F}_p$ the pressure forces. For fluids, forces are of two kinds:

- **Surface forces**: e.g. pressure and viscous stresses.

- **Volumetric forces**: e.g. gravity, electromagnetic, or inertial (fictitious) forces.

3.3.1   Global hydrostatic equlibrium

Integrating the previous expression gives that the total force in mechanical equilibrium mus vanish,

$$\vec{F}_e + \vec{F}_p = \int_V \vec{f}_e\,dV - \oint_S p\,d\vec{S} = 0. \tag{11}$$

When $\vec{f}_e$ is the force of gravity it corresponds to the weight of the fluid body, and the force due to pressure is called the buoyancy force.

Advantages.

- Provides the overall balance of forces for the entire fluid system.

- Very useful in cases with high symmetry, which simplifies the evaluation of integrals.

- Can be applied directly to obtain macroscopic results such as Archimedes' principle.

- Avoids the need to solve differential equations when the geometry is simple.

Disadvantages.

- Requires explicit knowledge of the force density $\rho\mathbf{g}$ and the pressure field p in order to evaluate the integrals.

- Insufficient for complex geometries where the pressure distribution is not known a priori.

- Does not provide the local pressure at each point in the fluid.

- Less general than the local (differential) form of the hydrostatic equilibrium equation.

3.3.2 Local hydrostatic equilibrium

In static fluids only normal pressures remain, since a fluid at rest cannot sustain shear stresses. Thus,

$$d\vec{F}_e - \vec{\nabla}p\,dV = 0. \tag{12}$$

Defining the volumetric density of external forces as

$$\vec{f}_e = \frac{d\vec{F}_e}{dV}, \tag{13}$$

and the effective force density

$$\vec{f}^* = \vec{f}_e - \vec{\nabla}p, \tag{14}$$

we obtain that in hydrostatic equilibrium $\vec{f}^* = 0$, leading to the basic equation of hydrostatics

$$\vec{f}_e = \vec{\nabla}p. \tag{15}$$

For the case of $\vec{f}_e$ being gravity force,

$$\rho\,\vec{g} = \vec{\nabla}p, \tag{16}$$

where $\vec{g}$ is the gravitational field.

Moreover, since $\vec{\nabla} \times \vec{\nabla}p = 0$, one has

$$\vec{\nabla} \times \vec{f}_e = 0, \tag{17}$$

which implies that $\vec{f}_e$ derives from a potential,

$$\vec{f}_e = -\vec{\nabla}\mathcal{H}. \tag{18}$$

Hence, hydrostatic equilibrium exists only when the volumetric forces acting on a fluid are conservative. This is the case for gravitational, electrostatic, and inertial forces.

Open Question. What occurs for a charged fluid placed in a magnetic field?

3.3.3 Pressure Force and Gauss' Theorem

The total pressure force on a body of volume V can be obtained in two equivalent ways:

- By integrating the pressure forces acting on its surface S.

- By integrating the forces on all its constituent material particles.

This leads to the relation

$$\oint_S p\,d\vec{S} = \int_V \vec{\nabla}p\,dV. \tag{19}$$

This identity is a direct consequence of **Gauss' theorem**, which relates the surface integral of a scalar field $p(\mathbf{x})$ over a closed surface S to the volume integral of its gradient $\vec{\nabla}p(\vec{x})$ over the enclosed volume V. Hence Gauss' theorem allows us to convert the local equation back into the global one, showing that there is complete mathematical equivalence between the local and global formulations of hydrostatic equilibrium.

3.4 Pascal's principle

Because pressure in a static fluid is the same in all directions, it can be treated as a scalar quantity. This isotropy underlies **Pascal's principle**, which states that any change of pressure applied to a confined fluid is transmitted undiminished throughout the fluid and to the walls of its container. Pascal's principle forms the basis of many hydraulic devices and has wide-ranging applications in both engineering and natural systems.

From Eq. (15), by integrating along a path from point 1 to point 2 inside a fluid, we obtain

$$\int_1^2 \vec{f_e} \cdot d\vec{r} = \int_1^2 \vec{\nabla} p \cdot d\vec{r} = \int_1^2 dp = p_2 - p_1, \tag{20}$$

so that the pressure difference between two points of a fluid in hydrostatic equilibrium depends only on the external forces, such as gravity. If there are no external forces, then $p_2 = p_1$. Moreover,

$$p_2 - p_1 = (p_2 + C) - (p_1 + C) = \int_1^2 \vec{f_e} \cdot d\vec{r}, \tag{21}$$

showing that the addition of a constant C to the pressure at all points of a fluid leaves the difference $p_2 - p_1$ invariant. That is, any pressure shift C applied at one point is accompanied by the same shift at all other points. This statement is known as **Pascal's principle**.

This principle introduces a global gauge freedom in hydrostatics. In differential form, and taking into account that $\vec{\nabla} C = 0$, it follows that $\vec{f_e} = \vec{\nabla} p = \vec{\nabla}(p + C)$.
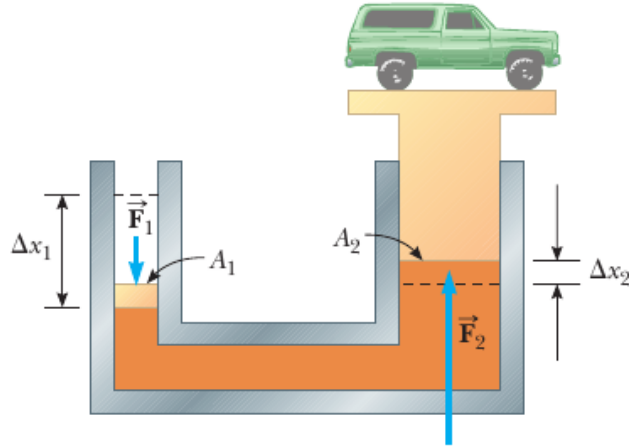
3.4.1 Hydraulic press

Figure 3: Diagram of a hydraulic press. Figure taken from Ref. [4]

An important application of Pascal's law is the hydraulic press (see Fig. 3). A force F_1 is applied to a small piston of surface area A_1. The pressure is transmitted through an incompressible liquid to a larger piston of area A_2. Since the pressure must be equal on both pistons,

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}. \tag{22}$$

Therefore, the output force is amplified by the factor A_2/A_1.

Because the liquid is incompressible, the displaced volumes satisfy

$$A_1 \Delta x_1 = A_2 \Delta x_2,$$

so that

$$\frac{F_2}{F_1} = \frac{\Delta x_1}{\Delta x_2}.$$

Multiplying both sides by displacements shows that the work done is conserved:

$$F_1 \Delta x_1 = F_2 \Delta x_2.$$

The hydraulic press allows a small input force to generate a much larger output force, while conserving energy. This principle is widely used in hydraulic brakes, jacks and car lifts.

3.5 Hydrostatics in constant gravity

In a flat-Earth coordinate system, the constant field of gravity is $\vec{g}(\vec{x}) = (0, 0, -g_0)$ for all $\vec{x}$. If the external force is gravity, then:

$$\vec{f}_e = \frac{d\vec{F}}{dV} = \frac{dm}{dV} \vec{g} = \rho \vec{g},$$

so that,

$$\vec{\nabla} p - \rho \vec{g} = 0. \tag{23}$$

It follows that

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho g_0. \tag{24}$$

The first two equations express that the pressure does not depend on x and y but only on z. It also shows that, independently of the shape of a fluid container, the pressure will always be the same at a given depth (in constant gravity).
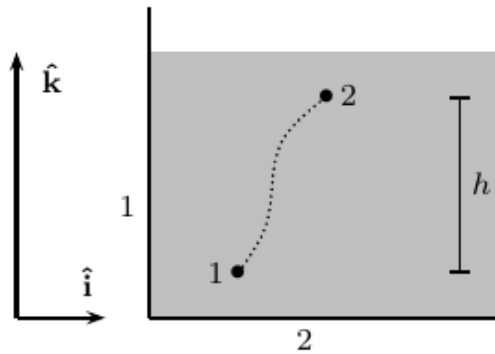

Figure 4: The pressure difference between points 1 and 2 inside a fluid in the Earth's gravity field depends only on their vertical separation. Figure taken from Ref. [1].

For the special case of constant density, $\rho(z) = \rho_0$, the last equation may immediately be integrated. The line integral between points 1 and 2 in Fig. 4, with ρ and $\vec{g}$ constant, is:

$$p_2 - p_1 = \int_1^2 \mathbf{f}_e \cdot d\mathbf{r} = \int_1^2 \rho \mathbf{g} \cdot d\mathbf{r} = -\rho g \int_1^2 dz = -\rho g(z_2 - z_1) = -\rho g h. \tag{25}$$

Thus, the pressure inside a liquid increases with depth:

$$p_1 = p_2 + \rho g h, \tag{26}$$

an expression identical to $(p_1 + C) = (p_2 + C) + \rho g h$. This equation shows that if a force is applied to the free surface of a liquid (for example, with a piston), the pressure increases equally at every point of the fluid. Therefore, the *difference in pressure* between two points depends only on their vertical separation.

3.5.1 Communicating vessels

From the last expression in Eq.4 (24), it follows that $p = \rho g z + C$. At the free surface of the liquid ($z = h$) the pressure is atmospheric p_0 , so $C = p_0 + \rho g h$. Therefore
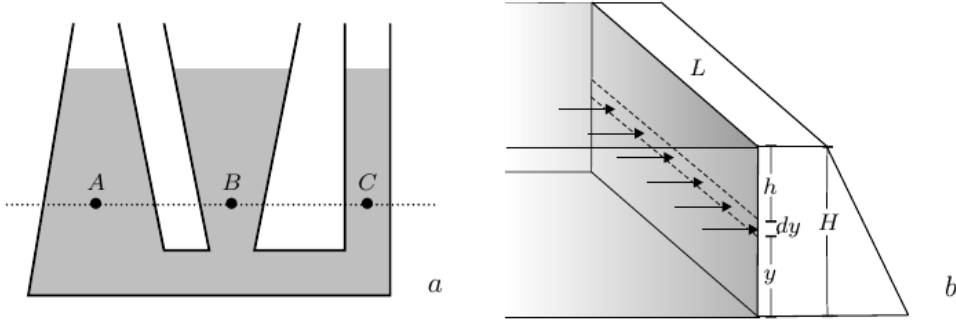
$$p = p_0 + \rho\, g(h - z). \tag{27}$$

Figure 5: *a*) In communicating vessels, the pressure at points A, B, and C, located at the same height, is the same. *b*) The horizontal hydrostatic force exerted by the water in a dam on a vertical wall differential element of height dy and width L. Figure taken from Ref. [1].

It is therefore true that points at the same depth have the same pressure, independently of the shape of the container (see left panel of Fig. 5). This gives validity to the so-called *principle of communicating vessels* and resolves the so-called *hydrostatic paradox*, according to which—erroneously— the pressure inside a fluid would depend on the shape of the container. As seen here, the principle of communicating vessels is another result derived from the above expression for gravitational external forces $\vec{f_e}$.

According to the erroneous interpretation, the pressure at point B would be greater than at A, since more liquid lies above B than above A. The pressure at C would then be intermediate between those at A and B. The supposed paradox is that in fact

$$P_A = P_B = P_C.$$

This Equation shows that pressure is a scalar quantity, which ensures that *at every point inside the fluid the pressure is the same in all directions.*

3.5.2 The Force on a Dam

Water is filled to a height H behind a dam of width L (see right panel of Fig. 5). Let's determine the resultant force exerted by the water on the dam.

Because pressure increases with depth, the total force on a submerged surface cannot be calculated by simply multiplying the area by a single pressure value. Instead, we must account for the variation of pressure with depth, which requires the use of integration.

Consider a vertical y-axis with $y = 0$ at the bottom of the dam. We divide the face of the dam into narrow horizontal strips located at a height y above the bottom, as illustrated by the dashed strip in the figure.

The atmospheric pressure acting on both sides of the wall does not exert a net force, so it will not be considered. At a height y measured from the bottom of the dam, the pressure has the value

$$p = \rho g h = \rho g (H - y).$$

The horizontal force that the water exerts on the differential portion of wall of area $L\,dy$ is

$$dF = p\,dA = pL\,dy = \rho g L (H - y)\,dy,$$

so that the total horizontal net force, directed to the right, due to the water is

$$F = \rho g L \int_0^H (H - y)\,dy = \tfrac{1}{2}\rho g L H^2.$$

3.5.3 Pressure gauges

(a) Open-tube manometer

$p_0 = p_{\text{atm}}$

$h = y_2 - y_1$

Pressure p

y_2

y_1

$p + \rho g y_1 \quad p_{\text{atm}} + \rho g y_2$

The pressure is the same at the bottoms of the two tubes.

(b) Mercury barometer

There is a near-vacuum at the top of the tube.

$p_0 = 0$

The height to which the mercury rises depends on the atmospheric pressure exerted on the mercury in the dish.

$h = y_2 - y_1$

y_2
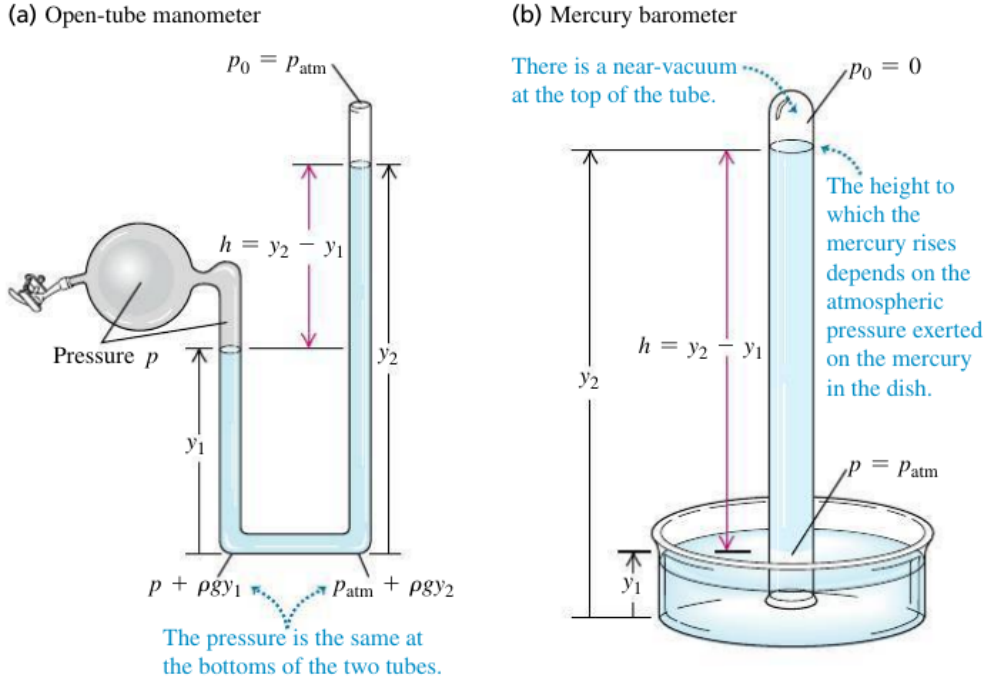
$p = p_{\text{atm}}$

y_1

Figure 6: Two types of pressure gauge. Figure taken from Ref. [4]

The simplest pressure gauge is the open-tube *manometer* (left panel of Fig. 6). The U-shaped tube contains a liquid of density ρ, often mercury or water. The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\text{atm}}$. The pressure at the bottom of the tube due to the fluid in the left column is $p + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\text{atm}} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

$$p + \rho g y_1 = p_{\text{atm}} + \rho g y_2,$$

$$p - p_{\text{atm}} = \rho g(y_2 - y_1) = \rho g h.$$

Here, p is the *absolute pressure*, and the difference $p - p_{\text{atm}}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

Another common pressure gauge is the *mercury barometer*. It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (right panel of Fig. 6). . The space above the mercury column contains only mercury vapor; its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. It follows that

$$p_{\text{atm}} = p = 0 + \rho g(y_2 - y_1) = \rho g h.$$

So the height h of the mercury column indicates the atmospheric pressure p_{atm}.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "inches of mercury" or "millimeters of mercury" (abbreviated mm Hg). A pressure of $1\,\text{mm}\,\text{Hg}$ is called *1 torr*, after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of g, which varies with location, so the pascal is the preferred unit of pressure.

3.6   Equation of state

The local equation of hydrostatic equilibrium with an externally given gravitational field is not sufficient by itself; we need a relation between pressure and density. Thermodynamics provides this relationship through the *equation of state*, which relates density ρ, pressure p, and absolute temperature T:

$$F(\rho, p, T) = 0. \tag{28}$$

This relation is valid for any macroscopic amount of homogeneous isotropic fluid in thermodynamic equilibrium. In continuum physics, where conditions vary from point to point, it is assumed that each material particle is in thermodynamic equilibrium with its surroundings, so the equation of state holds locally:

$$F(\rho(x), p(x), T(x)) = 0. \tag{29}$$

3.6.1   The Ideal Gas Law

The *ideal gas law* is the most famous equation of state, often attributed to Clapeyron (1834). In terms of the volume V of a mass M of gas and the number of moles $n = M/M_{\text{mol}}$, it is written as

$$pV = nR_{\text{mol}}T, \tag{30}$$

where $R_{mol} = 8.31447\,\mathrm{J\,K^{-1}\,mol^{-1}}$ is the *universal molar gas constant*. Since $\rho = M/V = nM_{mol}/V$, the ideal gas law can be expressed in the local form:

$$p = R\rho T, \quad R = \frac{R_{mol}}{M_{mol}}. \tag{31}$$

Remarks. The specific gas constant R varies for different gases (see Table43). Extensions of the ideal gas law include corrections for excluded molecular volume and intermolecular forces. The ideal gas law also applies to mixtures of gases if the average molar mass is used.

	$10^3 M_{mol}$ (kg mol^{-1})	$10^{-3}R$ (J K^{-1} kg^{-1})
H_2	2.0	4.157
He	4.0	2.079
Ne	20.2	0.412
N_2	28.0	0.297
O_2	32.0	0.260
Ar	39.9	0.208
CO_2	44.0	0.189
Air	29.0	0.287

Table 3: The molar mass, M_{mol}, and the specific gas constant, $R = R_{mol}/M_{mol}$, for a few gases (in SI units).

The microscopic origin of the ideal gas law is as follows. The equipartition theorem tells us that the average kinetic energy of a molecule of mass m is

$$\frac{1}{2}m\langle v^2\rangle = \frac{1}{2}m\langle v_x^2\rangle + \frac{1}{2}m\langle v_y^2\rangle + \frac{1}{2}m\langle v_z^2\rangle = \frac{3}{2}k_B T, \tag{32}$$

from which

$$p = \frac{1}{3}\rho\langle v^2\rangle = \rho\frac{k_B T}{m} = \rho\frac{N_A k_B T}{N_A m} = \rho\frac{R_{mol}T}{M_{mol}}. \tag{33}$$

3.6.2 Case: Isothermal atmosphere

Let us assume a constant temperature, $T(x) = T_0$, and combine the equation of hydrostatic equilibrium with the ideal gas law. Then

$$\frac{dp}{dz} = -\rho g_0 = -\frac{g_0}{RT_0}p. \tag{34}$$

This is an ordinary differential equation for the pressure, and using the initial condition $p = p_0$ for $z = 0$, we find the solution

$$p = p_0 e^{-z/h_0}, \qquad h_0 = \frac{RT_0}{g_0} = \frac{p_0}{\rho_0 g_0}. \tag{35}$$

In the last step we have again used the ideal gas law at $z = 0$ to show that the expression for h_0 is identical to the incompressible atmospheric scale height.

In the isothermal atmosphere the pressure thus decreases exponentially with height on a characteristic length scale again set by $h_0 \approx 8728\,\mathrm{m}$ calculated for 1 atm and 25°C. The pressure at the top of Mount Everest ($z = 8848\,\mathrm{m}$) is now finite and predicted to be 368 hPa.

3.6.3 Barotropic equation of state

Sometimes there exists a so-called *barotropic* relationship between density and pressure,

$$F(\rho(x), p(x)) = 0, \tag{36}$$

which does not depend on the local temperature $T(x)$.

Polytropic relation

The first law of thermodynamics in differential form reads

$$T \, ds = dU + p \, d\left(\frac{1}{\rho}\right), \tag{37}$$

where s is the entropy, U the specific internal energy, p the pressure, and ρ the mass density. For an isentropic process we have $ds = 0$, so

$$dU = -p \, d\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2} \, d\rho. \tag{38}$$

For an ideal gas, $U = c_v T$ and $p = \rho R T$. Thus

$$c_v \, dT = \frac{p}{\rho^2} \, d\rho, \qquad T = \frac{p}{\rho R}. \tag{39}$$

Eliminating T we obtain

$$c_v \, d\left(\frac{p}{\rho R}\right) = \frac{p}{\rho^2} \, d\rho, \tag{40}$$

or equivalently

$$\frac{c_v}{R}\left(\frac{dp}{\rho} - \frac{p}{\rho^2} \, d\rho\right) = \frac{p}{\rho^2} \, d\rho. \tag{41}$$

Now, recalling that

$$\gamma \equiv \frac{c_p}{c_v} = 1 + \frac{R}{c_v},$$

we can simplify the relation to

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho}. \tag{42}$$

Integrating both sides gives

$$\ln p = \gamma \ln \rho + \text{const}, \tag{43}$$

which leads to the **polytropic relation**

$$p = C \, \rho^\gamma, \tag{44}$$

where C is a constant (the *polytropic constant*), and γ is the *polytropic index* (see Table 4).

	γ	$10^{-3}c_p$
H_2	1.41	14.30
He	1.63	5.38
Ne	1.64	1.05
N_2	1.40	1.04
O_2	1.40	0.91
Ar	1.67	0.52
CO_2	1.30	0.82
Air	1.40	1.00

Table 4: Table of the adiabatic index and the specific heat at constant pressure ($c_p = \gamma R/(\gamma - 1)$) for a few nearly ideal gases. Units: $J\,K^{-1}\,kg^{-1}$.

3.7 Isobars, Isoclines, and Potentials

For a fluid of density ρ —in general dependent on position— in a gravitational field $\vec{g}$:

$$-\rho\vec{g} + \vec{\nabla}p = 0 \qquad \text{or:} \qquad \rho\vec{\nabla}\Phi + \vec{\nabla}p = 0. \tag{45}$$

Taking the curl of the second equation,

$$\rho\vec{\nabla} \times \vec{\nabla}\Phi + \vec{\nabla}\rho \times \vec{\nabla}\Phi + \vec{\nabla} \times \vec{\nabla}p = 0. \tag{46}$$

Since $\vec{\nabla} \times \vec{\nabla}\Phi \equiv 0$ and $\vec{\nabla} \times \vec{\nabla}p \equiv 0$, it follows that, in a fluid of variable density, this condition holds:

$$\vec{\nabla}\rho \times \vec{\nabla}\Phi = 0 \qquad \text{or:} \qquad \vec{\nabla}\rho \times \vec{g} = 0. \tag{47}$$

This means that, point by point, the direction of the gravitational field lines coincides with that of the density gradient, or, equivalently, that the surfaces of equal density (isoclines) and the gravitational equipotentials are coincident.

Considering

$$\vec{\nabla}\Phi + \frac{\vec{\nabla}p}{\rho} = 0,$$

and its curl is taken, it follows that for a compressible fluid

$$\vec{\nabla}\rho \times \vec{\nabla}p = 0, \tag{48}$$

from which it is concluded that the surfaces of constant pressure (isobars) and those of constant density coincide. From this equation it follows that the pressure is an *exclusive functional* of density: $p = p(\rho)$. Indeed,

$$\vec{\nabla}\rho \times \vec{\nabla}p = \vec{\nabla}\rho \times \left(\frac{dp}{d\rho}\vec{\nabla}\rho\right) = \frac{dp}{d\rho}\vec{\nabla}\rho \times \vec{\nabla}\rho = 0. \tag{49}$$

In an analogous way, it can be shown that Φ is a functional of ρ: $\Phi = \Phi(\rho)$. Thus, the surfaces of Φ, ρ and p constant are the same. This result is valid regardless of the form of the gravitational field. In particular, if $\vec{g} = -\hat{k}g$, the surfaces Φ, ρ and p will be horizontal planes. Outside the Earth's mass, assumed spherical, it is true that

$$\mathbf{g} = -\frac{GM\hat{\mathbf{r}}}{r^2}, \qquad \Phi = -\frac{GM}{r},$$

so the three surfaces are spherical and concentric.

3.8 Pressure potential

Since the gravitational field is conservative and can be obtained from the gradient of the gravitational potential $\Phi(x)$, it follows that

$$\vec{g}(x) = -\vec{\nabla}\Phi(x). \tag{50}$$

In terms of the potential, the hydrostatics equilibrium equation ($\vec{\nabla}p = \rho\vec{g}$) may now be written as

$$\nabla\Phi + \frac{\nabla p}{\rho} = 0. \tag{51}$$

In the case of a constant density, $\rho(x) = \rho_0$, we obtain that

$$\Phi^* = \Phi + \frac{p - p^*}{\rho_0} \tag{52}$$

is also constant. In flat-Earth gravity, $\Phi = g_0 z$.

It is always possible to integrate the hydrostatic equation for any barotropic fluid with $\rho = \rho(p)$. Introducing the *pressure potential*,

$$w(p) = \int \frac{dp}{\rho(p)}, \tag{53}$$

it turns out that hydrostatic equilibrium may be written as

$$\Phi^* = \Phi + w(p). \tag{54}$$

Φ^* is dubbed the *effective potential.*

Isothermal gas

Under isothermal conditions, the pressure potential of an ideal gas is calculated by means of the ideal gas law:

$$w = \int \frac{RT_0}{p} \, dp = RT_0 \log p. \tag{55}$$

Polytropic fluid

For fluids obeying a polytropic relation, the pressure potential becomes

$$w = \int C\gamma\rho^{\gamma-1}\frac{d\rho}{\rho} = C\frac{\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}. \tag{56}$$

When the fluid is an ideal gas with $p = R\rho T$, this takes the simpler form

$$w = \frac{\gamma}{\gamma-1}RT = c_p T, \qquad c_p = \frac{\gamma}{\gamma-1}R, \tag{57}$$

where c_p is the specific heat at constant pressure.

3.9   Bulk modulus

It is defined the *bulk modulus* as the pressure increase dp per *fractional decrease* in volume, $-dV/V$, or

$$K = \frac{dp}{-dV/V} = \frac{dp}{d\rho/\rho} = \rho\frac{dp}{d\rho}. \tag{58}$$

In the second step we have used the constancy of the mass $M = \rho V$ of the fluid in the volume to derive that $dM = \rho dV + V d\rho = 0$, from which we get $-dV/V = d\rho/\rho$.

The above definition makes immediate sense for a barotropic fluid, where $p = p(\rho)$ is a function of density. For general fluid states it is necessary to specify the conditions under which the bulk modulus is defined, for example whether the temperature is held constant (isothermal) or whether there is no heat transfer (adiabatic or isentropic). Thus, the equation of state for an ideal gas implies that the isothermal bulk modulus is

$$K_T = \left(\rho\frac{\partial p}{\partial \rho}\right)_T = p, \tag{59}$$

where the index—as commonly done in thermodynamics—indicates that the temperature T is held constant. Similarly, for an isentropic ideal gas obeying the polytropic relation that the *isentropic bulk modulus* becomes

$$K_S = \left(\rho\frac{dp}{d\rho}\right)_S = \gamma p, \tag{60}$$

where the index indicates that the entropy S is held constant. It is larger than the isothermal bulk modulus by a factor of $\gamma > 1$, because adiabatic compression also increases the temperature of the gas, which further increases the pressure.

The definition of the bulk modulus (and the above equation) shows that it is measured in the same units as pressure, for example pascals, bars, or atmospheres. The bulk modulus is actually a measure of *incompressibility*, because the larger it is, the greater is the pressure increase that is needed to obtain a given fractional increase in density. The inverse bulk modulus $\beta = 1/K$ may be taken as a measure of *compressibility*.

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