Hydrodynamics

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3] books

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC12. Enumerar los campos que se utilizan típicamente para describir fluidos en movimiento (campos básicos de la hidrodinámica).

- OC13. Reconocer las condiciones necesarias para aproximar el movimiento de un fluido como: incompresible, irrotacional, no viscoso.

- OC14. Distinguir la descripción euleriana y lagrangiana de un fluído.

- OC15. Definir la viscosidad tanto desde el punto de vista microscópico como del punto de vista macroscópico.

- OC16. Definir los conceptos de: vorticidad, potencial de velocidades, campo de corriente.

1.2 Objetivos específicos procedimentales:

- OP10. Deducir las ecuaciones de movimiento de un fluido, mediante la aplicación de la descripción Lagrangiana y la segunda ley de Newton.

- OP11. Enunciar y demostrar el teorema de Bernoulli y estudiar sus aplicaciones.

- OP12. Deducir el tensor de viscosidad para un fluido Newtoniano isotrópico e identificar de allí las viscosidades de corte y volumétrica.

- OP13. Deducir las ecuaciones de Navier-Stokes para fluidos Newtonianos.

- OP14. Resolver analíticamente algunas situaciones hidrodinámicas simples.

- OP15. Discretizar las ecuaciones de Navier-Stokes en situaciones simples (flujo impulsado por presión, flujo impulsado por velocidad, flujo estacionario alrededor de un obstáculo) y resolverlas numéricamente.

1.3 Objetivos específicos actitudinales:

- OA4. Reconocer la importancia de los métodos numéricos en la solución de problemas de hidrodinámica.

2   Continuum dynamics

Continuous matter in motion appears throughout nature: flowing water, atmospheric currents, and even the vibrations of solids such as church bells, trees, or the ground during earthquakes. Although these phenomena are diverse, they are all governed by Newton's equations of motion applied to continuous media. While these equations are easy to formulate, analytic solutions typically exist only in highly idealized situations, and deeper understanding often requires experiments or numerical simulation.

Solids differ from fluids in that they resist changes of shape and tend to retain their form, whereas fluids flow and do not. Both, however, are treated within the same framework of continuum mechanics. Two fundamental principles govern their motion: conservation of mass and Newton's Second Law (momentum balance) applied to a continuous system. With appropriate expressions for internal and external forces, these lead to the equations of motion for the mass density and velocity fields.

In this chapter the analysis will focus on continuous matter in motion, with emphasis on fluids. The same equations of motion will also be applied on cosmological scales, showing that they yield a surprisingly coherent description of the universe. Later chapters will address more terrestrial applications of matter in motion.

The mathematical description of the state of a moving fluid is effected by means of functions which give the distribution of the fluid velocity $\vec{v} = \vec{v}(x, y, z, t)$ and of any two thermodynamic quantities pertaining to the fluid, for instance the pressure $p(x, y, z, t)$ and the density $\rho(x, y, z, t)$. All the thermodynamic quantities are determined by the values of any two of them, together with the equation of state; hence, if we are given five quantities, namely the three components of the velocity $\vec{v}$, the pressure p and the density ρ, the state of the moving fluid is completely determined.

All these quantities are, in general, functions of the coordinates x, y, z and of the time t. We emphasize that $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at a given point (x, y, z) in space and at a given time t, i.e., it refers to fixed points in space and not to specific particles of the fluid; in the course of time, the latter move about in space. The same remarks apply to ρ and p.

2.1   The velocity field

The velocity field $\vec{v}(\vec{x}, t)$ represents the center-of-mass velocity of the collection of molecules in a material particle with center-of-mass position $\vec{x}$ at time t.

For a material particle of mass dM and volume dV, the total momentum is

$$d\vec{P} = \vec{v}(x, t)\, dM = \mathbf{v}\, \rho\, dV, \tag{1}$$

where $\rho(x, t)$ is the mass density field and $\vec{v}(x, t)$ is the center-of-mass velocity of the material particle. Because ρ and $\vec{v}$ are averages over many molecules, their fluctuations become negligible provided the particle size is larger than the microscopic scale L'_{micro}.

2.1.1   Streamlines

Streamlines are curves everywhere tangent to the velocity field at a fixed time t_0. They satisfy the ordinary differential equation

$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x}, t_0). \tag{2}$$

A streamline shows the instantaneous direction of flow and depends on the chosen reference frame. At a fixed time, there is one and only one streamline passing through each point in space, and streamlines cannot intersect.

2.1.2 Particle Trajectories

A particle trajectory (or particle orbit) is the actual path followed by an infinitesimal tracer particle advected by the flow. It satisfies

$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x}, t). \tag{3}$$

Given initial data $\vec{x}(t_0) = \vec{x}_0$, the trajectory is defined for all times. Unlike streamlines, different trajectories may intersect since they correspond to different times.

2.1.3 Streaklines

A streakline is formed by all particles that have passed through the same fixed point $\vec{x}_0$, but at different emission times. Practically, this is obtained by continuously injecting dye or smoke at $\vec{x}_0$.

For steady flow, where $\vec{v}(\vec{x}, t) = \vec{v}(\vec{x})$,

- streamlines,

- particle trajectories, and

- streaklines

all coincide.

For unsteady flow, these three types of flow visualization differ significantly, and streaklines may be misleading if interpreted as instantaneous flow directions.

2.2 Mass conservation

Let's consider some volume V_0 of space. The mass of fluid in this volume is $\int \rho \, dV$, where ρ is the fluid density, and the integration is taken over the volume V_0. The mass of fluid flowing in unit time through an element $d\vec{S}$ of the surface bounding this volume is $\rho \vec{v} \cdot d\vec{S}$; the magnitude of the vector $d\vec{S}$ is equal to the area of the surface element, and its direction is along the normal. By convention, we take $d\vec{S}$ along the outward normal. Then $\rho \vec{v} \cdot d\vec{S}$ is positive if the fluid is flowing out of the volume, and negative if the flow is into the volume. The total mass of fluid flowing out of the volume V_0 in unit time is therefore

$$\oint_S \rho \vec{v} \cdot d\vec{S}, \tag{4}$$

where the integration is taken over the whole of the closed surface surrounding the volume in question.

Next, the decrease per unit time in the mass of fluid in the volume V_0 can be written

$$-\frac{\partial}{\partial t} \int \rho \, dV.$$

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Equating the two expressions, we have

$$\frac{\partial}{\partial t} \int \rho \, dV = -\oint \rho \, \vec{v} \cdot d\vec{S}. \tag{5}$$

This is the global equation of mass conservation for an arbitrary fixed control volume.

The surface integral can be transformed by Green's formula to a volume integral:

$$\oint \rho \, \vec{v} \cdot d\vec{S} = \int \nabla \cdot (\rho \vec{v}) \, dV. \tag{6}$$

Thus

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] dV = 0. \tag{7}$$

Since this equation must hold for any volume, the integrand must vanish, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \tag{8}$$

This is the local equation of mass conservation, also called the *equation of continuity.* equation of continuity. Although derived from global mass conservation applied to fixed volumes, it is itself a local relation completely without reference to volumes. Expanding the expression $\nabla \cdot (\rho \vec{v})$, we can also write

$$\frac{\partial \rho}{\partial t} + \rho \, \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \rho = 0. \tag{9}$$

The vector

$$\vec{j} = \rho \, \vec{v} \tag{10}$$

is called the *current density of mass* (or mass flux density). Its direction is that of the motion of the fluid, while its magnitude equals the mass of fluid flowing in unit time through unit area perpendicular to the velocity.

2.3   Incompressible flow

In a great many cases of the flow of liquids (and also of gases), their density may be supposed invariable, i.e. constant throughout the volume of the fluid and throughout its motion. In other words, there is no noticeable compression or expansion of the fluid in such cases. We then speak of incompressible flow.

Most liquids are perceived as incompressible under ordinary circumstances. A fluid is, however, effectively incompressible when flow speeds are much smaller than the velocity of sound.

All materials must nevertheless in principle be compressible. For in truly incompressible matter the sound velocity would be infinite, and that violates the relativistic injunction against any signal moving faster than the speed of light. Incompressibility is always an approximation and should, strictly speaking, be viewed as a condition on the flow rather than an absolute material property.

2.3.1   Global and local forms

Using $\rho(\vec{r}, t) = \rho_0 = constant$ in Eq. (5) we arrive at the global incompressibility condition

$$\oint_S \vec{v} \cdot d\vec{S} = 0, \tag{11}$$

Incompressible matter cannot accumulate anywhere, and equal volumes of incompressible fluid must enter and leave through any fixed surface per unit of time. Since the above condition refers to a single instant of time, it does in fact not matter whether the surface is fixed or moves in any way you may desire. On the other hand, using Eq. 8 we obtain the local incompressibility condition

$$\nabla \cdot \vec{v} = 0. \tag{12}$$

The divergence of the velocity field vanishes in incompressible flow. Notice that this local incompressibility condition does not refer to any volume of matter, only to the velocity field itself. A divergence-free field is sometimes called solenoidal.

2.3.2   Leonardo's Law.

Leonardo da Vinci observed that in an incompressible flow, the product of the cross-sectional area of a canal and the average velocity of the water remains constant. If A_1 and A_2 are two cross-sections of a canal and v_1, v_2 the corresponding average velocities, incompressibility implies

$$A_1 v_1 = A_2 v_2. \tag{13}$$

This follows from the fact that no fluid can pass through the canal walls. Applying the incompressibility condition to the closed surface composed of the two cross-sections and the canal walls gives

$$\oint \vec{v} \cdot d\vec{S} = \int_{A_2} \vec{v} \cdot d\vec{S} - \int_{A_1} \vec{v} \cdot d\vec{S} = 0.$$

The average velocity through any cross-section A is defined as

$$v = \frac{1}{A} \int_A \vec{v} \cdot d\vec{S}.$$

Hence, the quantity Av is constant along the canal for incompressible fluids, regardless of whether the flow is laminar or turbulent.

Leonardo's law, however, does *not* apply to compressible flows. In such cases the product of area and average velocity cannot remain constant, and a stronger law—accounting for density variations—is required.

2.4   Lagrangian and Eulerian derivatives

Fluid motion may be described in two complementary ways. In the *Lagrangian* viewpoint we follow a specific fluid element of mass Δm. Its motion is governed by Newton's law, and the time derivative d/dt represents the change experienced by that material element. This is the *material* or *Lagrangian* derivative. The velocity and acceleration of a fluid particle are written as

$$\vec{v} = \frac{D\vec{r}}{Dt}, \qquad \vec{a} = \frac{D\vec{v}}{Dt}. \tag{14}$$

In the *Eulerian* viewpoint the fluid is treated as a field $f(\vec{r}, t)$ defined at fixed spatial positions. Changes in time may arise from variations at a fixed point, measured by $\partial f/\partial t$, or from the transport of fluid properties by the motion, represented by spatial derivatives.

According to differential calculus, a function f(r, t) has, in Cartesian coordinates, a differential of the form:

$$df = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = \frac{\partial f}{\partial t}\,dt + d\vec{r}\cdot\nabla f\,. \tag{15}$$

The temporal derivative of any fluid quantity is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f. \tag{16}$$

Thus the relation between Eulerian and Lagrangian derivatives is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla. \tag{17}$$

Applied to the position vector,

$$\frac{D\vec{r}}{Dt} = \vec{v}, \tag{18}$$

showing that the material derivative of position yields the velocity. Applied to the velocity field,

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}, \tag{19}$$

where the first term is the *local acceleration* and the second term is the *convective acceleration*, arising from velocity gradients in space.

For an incompressible fluid ($\nabla\cdot\vec{v} = 0$), the continuity equation gives

$$\frac{D\rho}{Dt} = 0 \qquad \Rightarrow \qquad \frac{\partial\rho}{\partial t} = -\vec{v}\cdot\nabla\rho.$$

This shows that although density may vary in space, each fluid element preserves its density as it moves.

References

[1] A. Sepúlveda, *Hidrodinámica*. Fondo Editorial EIA, Colombia, 2019.

[2] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, vol. 6 of *Course of Theoretical Physics*. Pergamon Press, Oxford, UK, 2nd ed., 1987.

[3] B. Lautrup, *Physics of Continuous Matter: Exotic and Everyday Phenomena in the Macroscopic World*. CRC Press, Boca Raton, 2 ed., 2011.