Hydrodynamics

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3] books

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC12. Enumerar los campos que se utilizan típicamente para describir fluidos en movimiento (campos básicos de la hidrodinámica).
- OC13. Reconocer las condiciones necesarias para aproximar el movimiento de un fluido como: incompresible, irrotacional, no viscoso.
- OC14. Distinguir la descripción euleriana y lagrangiana de un fluído.
- OC15. Definir la viscosidad tanto desde el punto de vista microscópico como del punto de vista macroscópico.
- OC16. Definir los conceptos de: vorticidad, potencial de velocidades, campo de corriente.

1.2 Objetivos específicos procedimentales:

- OP10. Deducir las ecuaciones de movimiento de un fluido, mediante la aplicación de la descripción Lagrangiana y la segunda ley de Newton.
- OP11. Enunciar y demostrar el teorema de Bernoulli y estudiar sus aplicaciones.
- OP12. Deducir el tensor de viscosidad para un fluido Newtoniano isotrópico e identificar de allí las viscosidades de corte y volumétrica.
- OP13. Deducir las ecuaciones de Navier-Stokes para fluidos Newtonianos.
- OP14. Resolver analíticamente algunas situaciones hidrodinámicas simples.
- OP15. Discretizar las ecuaciones de Navier-Stokes en situaciones simples (flujo impulsado por presión, flujo impulsado por velocidad, flujo estacionario alrededor de un obstáculo) y resolverlas numéricamente.

1.3 Objetivos específicos actitudinales:

- OA4. Reconocer la importancia de los métodos numéricos en la solución de problemas de hidrodinámica.

2 Continuum dynamics

Continuous matter in motion appears throughout nature: flowing water, atmospheric currents, and even the vibrations of solids such as church bells, trees, or the ground during earthquakes. Although these phenomena are diverse, they are all governed by Newton's equations of motion applied to continuous media. While these equations are easy to formulate, analytic solutions typically exist only in highly idealized situations, and deeper understanding often requires experiments or numerical simulation.

Solids differ from fluids in that they resist changes of shape and tend to retain their form, whereas fluids flow and do not. Both, however, are treated within the same framework of continuum mechanics. Two fundamental principles govern their motion: conservation of mass and Newton's Second Law (momentum balance) applied to a continuous system. With appropriate expressions for internal and external forces, these lead to the equations of motion for the mass density and velocity fields.

In this chapter the analysis will focus on continuous matter in motion, with emphasis on fluids. The same equations of motion will also be applied on cosmological scales, showing that they yield a surprisingly coherent description of the universe. Later chapters will address more terrestrial applications of matter in motion.

The mathematical description of the state of a moving fluid is effected by means of functions which give the distribution of the fluid velocity $\vec{v} = \vec{v}(x, y, z, t)$ and of any two thermodynamic quantities pertaining to the fluid, for instance the pressure $p(x, y, z, t)$ and the density $\rho(x, y, z, t)$. All the thermodynamic quantities are determined by the values of any two of them, together with the equation of state; hence, if we are given five quantities, namely the three components of the velocity $\vec{v}$, the pressure p and the density ρ, the state of the moving fluid is completely determined.

All these quantities are, in general, functions of the coordinates x, y, z and of the time t. We emphasize that $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at a given point (x, y, z) in space and at a given time t, i.e., it refers to fixed points in space and not to specific particles of the fluid; in the course of time, the latter move about in space. The same remarks apply to ρ and p.

2.1 The velocity field

The velocity field $\vec{v}(\vec{x}, t)$ represents the center-of-mass velocity of the collection of molecules in a material particle with center-of-mass position $\vec{x}$ at time t.

For a material particle of mass dM and volume dV, the total momentum is

$$d\vec{P} = \vec{v}(x, t)\, dM = \vec{v}\, \rho\, dV, \tag{1}$$

where $\rho(x, t)$ is the mass density field and $\vec{v}(x, t)$ is the center-of-mass velocity of the material particle. Because ρ and $\vec{v}$ are averages over many molecules, their fluctuations become negligible provided the particle size is larger than the microscopic scale L'_{micro}.

2.1.1 Streamlines

Streamlines are curves everywhere tangent to the velocity field at a fixed time t_0. They satisfy the ordinary differential equation

$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x}, t_0). \tag{2}$$

$\vec{x}_0$.

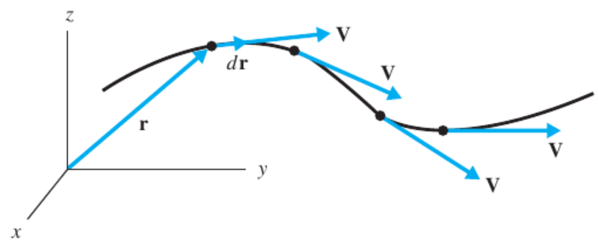

Figure 1: Streamline in a flow field. Figure taken from Ref. [4].

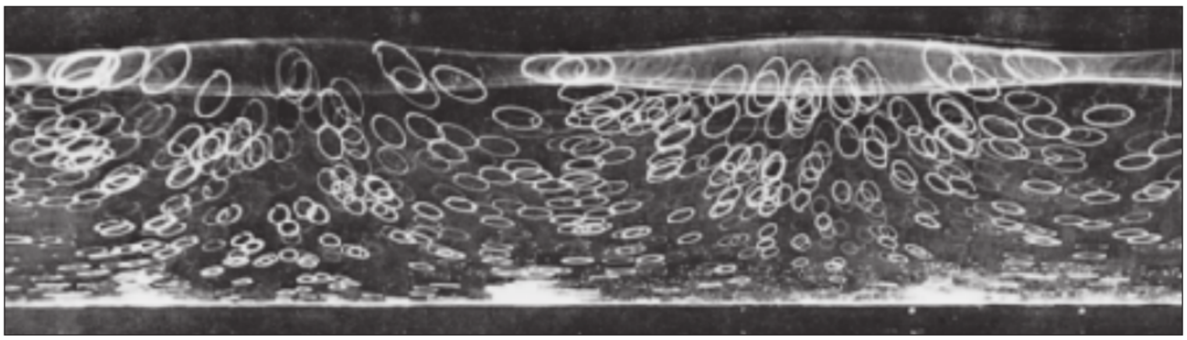

Figure 2: Pathlines underneath a wave in a tank of water. Figure taken from Ref. [4].

A streamline shows the instantaneous direction of flow and depends on the chosen reference frame. At a fixed time, there is one and only one streamline passing through each point in space, and streamlines cannot intersect (see Fig. 1).

2.1.2 Particle Trajectories

A particle trajectory (or particle orbit) is the actual path followed by an infinitesimal tracer particle advected by the flow. It satisfies

$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x}, t). \tag{3}$$

Given initial data $\vec{x}(t_0) = \vec{x}_0$, the trajectory is defined for all times. Unlike streamlines, different trajectories may intersect since they correspond to different times.

A pathline is the locus of points traversed by a given particle as it travels in a field of flow; the path- line provides us with a "history" of the particle's locations. A photograph of a pathline would require a time exposure of an illuminated particle. A photo- graph showing pathlines of particles below a water surface with waves is given in Fig. 2.

2.1.3 Streaklines

A streakline is formed by all particles that have passed through the same fixed point $\vec{x}_0$, but at different emission times. Practically, this is obtained by continuously injecting dye or smoke at $\vec{x}_0$.

A streakline is defined as an instantaneous line whose points are occupied by all particles originating from some specified point in the flow field. Streaklines tell us where the particles are

Figure 3: Streaklines in the unsteady flow around a cylinder. Figure taken from Ref. [4].

"right now." A photograph of a streakline would be a snapshot of the set of illuminated particles that passed a certain point. Figure 3.2 shows streaklines produced by the continuous release of a small diameter stream of smoke as it moves around a cylinder.

2.1.4 Steady flow case

For steady flow, where $\vec{v}(\vec{x}, t) = \vec{v}(\vec{x})$,

- streamlines,
- particle trajectories, and
- streaklines

all coincide.

For unsteady flow, these three types of flow visualization differ significantly, and streaklines may be misleading if interpreted as instantaneous flow directions.

2.2 Mass conservation

Let's consider some volume V_0 of space. The mass of fluid in this volume is $\int \rho\, dV$, where ρ is the fluid density, and the integration is taken over the volume V_0. The mass of fluid flowing in unit time through an element $d\vec{S}$ of the surface bounding this volume is $\rho\,\vec{v} \cdot d\vec{S}$; the magnitude of the vector $d\vec{S}$ is equal to the area of the surface element, and its direction is along the normal. By convention, we take $d\vec{S}$ along the outward normal. Then $\rho\,\vec{v} \cdot d\vec{S}$ is positive if the fluid is flowing out of the volume, and negative if the flow is into the volume. The total mass of fluid flowing out of the volume V_0 in unit time is therefore

$$\oint_S \rho\,\vec{v} \cdot d\vec{S}, \tag{4}$$

where the integration is taken over the whole of the closed surface surrounding the volume in question.

Next, the decrease per unit time in the mass of fluid in the volume V_0 can be written

$$-\frac{\partial}{\partial t}\int \rho\, dV.$$

Equating the two expressions, we have

$$\frac{\partial}{\partial t}\int \rho\, dV \;=\; -\oint \rho\, \vec{v}\cdot d\vec{S}. \tag{5}$$

This is the global equation of mass conservation for an arbitrary fixed control volume.

The surface integral can be transformed by Green's formula to a volume integral:

$$\oint \rho\, \vec{v}\cdot d\vec{S} = \int \nabla\cdot(\rho\vec{v})\, dV. \tag{6}$$

Thus

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right] dV = 0. \tag{7}$$

Since this equation must hold for any volume, the integrand must vanish, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0. \tag{8}$$

This is the local equation of mass conservation, also called the *equation of continuity.* equation of continuity. Although derived from global mass conservation applied to fixed volumes, it is itself a local relation completely without reference to volumes. Expanding the expression $\nabla\cdot(\rho\vec{v})$, we can also write

$$\frac{\partial \rho}{\partial t} + \rho\,\nabla\cdot\,\vec{v} + \vec{v}\cdot\nabla\,\rho = 0. \tag{9}$$

The vector

$$\vec{j} = \rho\,\vec{v} \tag{10}$$

is called the *current density of mass* (or mass flux density). Its direction is that of the motion of the fluid, while its magnitude equals the mass of fluid flowing in unit time through unit area perpendicular to the velocity.

2.3 Incompressible flow

In a great many cases of the flow of liquids (and also of gases), their density may be supposed invariable, i.e. constant throughout the volume of the fluid and throughout its motion. In other words, there is no noticeable compression or expansion of the fluid in such cases. We then speak of incompressible flow.

Most liquids are perceived as incompressible under ordinary circumstances. A fluid is, however, effectively incompressible when flow speeds are much smaller than the velocity of sound.

All materials must nevertheless in principle be compressible. For in truly incompressible matter the sound velocity would be infinite, and that violates the relativistic injunction against any signal moving faster than the speed of light. Incompressibility is always an approximation and should, strictly speaking, be viewed as a condition on the flow rather than an absolute material property.

2.3.1 Global and local forms

Using $\rho(\vec{r}, t) = \rho_0 = constant$ in Eq. (5) we arrive at the global incompressibility condition

$$\oint_S \vec{v} \cdot d\vec{S} = 0, \tag{11}$$

Incompressible matter cannot accumulate anywhere, and equal volumes of incompressible fluid must enter and leave through any fixed surface per unit of time. Since the above condition refers to a single instant of time, it does in fact not matter whether the surface is fixed or moves in any way you may desire. On the other hand, using Eq. 8 we obtain the local incompressibility condition

$$\nabla \cdot \vec{v} = 0. \tag{12}$$

The divergence of the velocity field vanishes in incompressible flow. Notice that this local incompressibility condition does not refer to any volume of matter, only to the velocity field itself. A divergence-free field is sometimes called solenoidal.

2.3.2 Leonardo's Law.

Leonardo da Vinci observed that in an incompressible flow, the product of the cross-sectional area of a canal and the average velocity of the water remains constant. If A_1 and A_2 are two cross-sections of a canal and v_1, v_2 the corresponding average velocities, incompressibility implies

$$A_1 v_1 = A_2 v_2. \tag{13}$$

This follows from the fact that no fluid can pass through the canal walls. Applying the incompressibility condition to the closed surface composed of the two cross-sections and the canal walls gives

$$\oint \vec{v} \cdot d\vec{S} = \int_{A_2} \vec{v} \cdot d\vec{S} - \int_{A_1} \vec{v} \cdot d\vec{S} = 0.$$

The average velocity through any cross-section A is defined as

$$v = \frac{1}{A} \int_A \vec{v} \cdot d\vec{S}.$$

Hence, the quantity Av is constant along the canal for incompressible fluids, regardless of whether the flow is laminar or turbulent.

Leonardo's law, however, does *not* apply to compressible flows. In such cases the product of area and average velocity cannot remain constant, and a stronger law—accounting for density variations—is required.

2.4 Lagrangian and Eulerian derivatives

Fluid motion may be described in two complementary ways.

2.4.1 Lagrangian description

In the *Lagrangian* viewpoint we follow a specific fluid element of mass Δm. Its motion is governed by Newton's law, and the time derivative d/dt represents the change experienced by that material element. This is the *material* or *Lagrangian* derivative. The velocity and acceleration of a fluid particle are written as

$$\vec{v} = \frac{D\vec{r}}{Dt}, \qquad \vec{a} = \frac{D\vec{v}}{Dt}. \tag{14}$$

2.4.2 Eulerian description

In the *Eulerian* viewpoint the fluid is treated as a field $f(\vec{r}, t)$ defined at fixed spatial positions. Changes in time may arise from variations at a fixed point, measured by $\partial f/\partial t$, or from the transport of fluid properties by the motion, represented by spatial derivatives.

According to differential calculus, a function $f(\vec{r}, t)$ has, in Cartesian coordinates, a differential of the form:

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = \frac{\partial f}{\partial t}\, dt + d\vec{r} \cdot \nabla f\, . \tag{15}$$

The temporal derivative of any fluid quantity is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f. \tag{16}$$

Thus the relation between Eulerian and Lagrangian derivatives is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla. \tag{17}$$

Applied to the position vector,

$$\frac{D\vec{r}}{Dt} = \vec{v}, \tag{18}$$

showing that the material derivative of position yields the velocity. Applied to the velocity field,

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}, \tag{19}$$

where the first term is the *local acceleration* and the second term is the *convective acceleration*, arising from velocity gradients in space.

For an incompressible fluid ($\nabla \cdot \vec{v} = 0$), the continuity equation gives

$$\frac{D\rho}{Dt} = 0 \qquad \Rightarrow \qquad \frac{\partial \rho}{\partial t} = -\vec{v} \cdot \nabla \rho.$$

This shows that although density may vary in space, each fluid element preserves its density as it moves.

2.5 Cauchy's equation

Applying Newton's Second Law to any comoving material particle with mass $dM = \rho\, dV$, its equation of motion becomes

$$dM\, \vec{a} = d\vec{\mathcal{F}}, \tag{20}$$

where $d\vec{\mathcal{F}}$ is the total force acting on the particle. Dividing by dV we arrive at the seductively simple equation, $\rho\,\vec{a} = \vec{f}^{\,*}$, where $\vec{f}^{\,*} = d\vec{\mathcal{F}}/dV$ is the *effective force density*. It was defined earlier:

$$f_i^* = f_i + \sum_j \nabla_j \sigma_{ij}, \tag{21}$$

where f_i is the true body force density and σ_{ij} the stress tensor. Finally, inserting the acceleration field we have arrived at Cauchy's equation:

$$\rho\,\frac{D\vec{v}}{Dt} = \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = \vec{f}^{\,*} \tag{22}$$

3 Incompressible frictionless (Euler) fluids

3.1 Euler equation for incompressible ideal flow

In an ideal homogeneous fluid, the only contact force is pressure, but in distinction to hydrostatics where the pressure is in balance with the body forces (like gravity), the effective density of force

$$\vec{f}^{\,*} = \vec{f}_e - \vec{\nabla}p \tag{23}$$

may now be non-vanishing. Inserted into the general dynamic equation we obtain, after dividing by the constant density $\rho(x,t) = \rho_0$,

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{\vec{f}_e}{\rho_0} - \frac{\vec{\nabla}p}{\rho_0}, \tag{24}$$

$$\vec{\nabla}\cdot\vec{v} = 0. \tag{25}$$

These two *Euler equations* govern the dynamics of incompressible ideal fluid, also called *Euler fluid*.

From the identity

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}), \tag{26}$$

with $\vec{A} = \vec{B} = \vec{v}$ it follows that

$$\frac{1}{2}\,\vec{\nabla}v^2 = \vec{v}\cdot\vec{\nabla}\vec{v} + \vec{v}\times(\vec{\nabla}\times\vec{v}), \tag{27}$$

so that, replacing $\vec{v}\cdot\vec{\nabla}\vec{v}$ in Euler's equation, we obtain

$$\rho_0\,\frac{\partial\vec{v}}{\partial t} + \frac{\rho_0}{2}\,\vec{\nabla}v^2 - \rho_0\,\vec{v}\times(\vec{\nabla}\times\vec{v}) + \vec{\nabla}p = \vec{f}_{\mathrm{ext}}. \tag{28}$$

Unlike the hydrostatic case, it is no longer a necessary condition that external forces be derivable from a potential, i.e. that the forces be conservative. External forces may now include, in addition to electrical forces (including time-dependent ones), gravitational and inertial forces (centrifugal, Coriolis and linear acceleration), as well as magnetic forces.

In what follows we restrict ourselves to conservative forces, and more specifically to gravitational forces, centrifugal forces, and constant acceleration. Later, Coriolis forces will be introduced, although no conservative scalar potential exists for them. For the external forces we may then write

$$\vec{f} = -\rho\,\vec{\nabla}\Phi, \quad \text{with} \quad \Phi = \Phi_{\text{grav}} - \frac{\omega^2 r^2}{2} + \vec{a}\cdot\vec{r}.$$

Thus, the equation of motion for non-viscous fluids in the presence of these forces is

$$\rho_0\,\frac{\partial\vec{v}}{\partial t} + \frac{\rho_0}{2}\,\vec{\nabla}v^2 - \rho_0\,\vec{v}\times(\vec{\nabla}\times\vec{v}) + \vec{\nabla}p + \rho_0\,\vec{\nabla}\Phi = 0. \tag{29}$$

3.2 Adiabatic and isentropic flow

The absence of heat exchange between different parts of the fluid (and also, of course, between the fluid and bodies adjoining it) means that the motion is adiabatic throughout the fluid. Thus the motion of an ideal fluid must necessarily be supposed adiabatic.

In adiabatic motion the entropy of any particle of fluid remains constant as that particle moves about in space. Denoting by s the entropy per unit mass, we can express the condition for adiabatic motion as

$$\frac{Ds}{Dt} = 0 = \frac{\partial s}{\partial t} + \vec{v}\cdot\vec{\nabla}s. \tag{30}$$

This is the general equation describing adiabatic motion of an ideal fluid. Using the continuity equation, we may write it as an "equation of continuity" for entropy:

$$\frac{\partial(\rho s)}{\partial t} + \vec{\nabla}\cdot(\rho s\vec{v}) = 0. \tag{31}$$

The product $\rho s\vec{v}$ is the *entropy flux density.*

The adiabatic equation usually takes a much simpler form. If, as usually happens, the entropy is constant throughout the volume of the fluid at some initial instant, it retains everywhere the same constant value at all times and for any subsequent motion of the fluid. In this case we can write the adiabatic equation simply as

$$s = \text{constant}. \tag{32}$$

Such a motion is said to be *isentropic.*

We may use the fact that the motion is isentropic to put the equation of motion in a somewhat different form. To do so, we employ the familiar thermodynamic relation

$$d\mathcal{H} = T\,ds + V\,dp, \tag{33}$$

where $\mathcal{H}$ is the heat function per unit mass (enthalpy), $V = 1/\rho$ is the specific volume, and T is the temperature. Since $s = \text{constant}$, we have simply

$$d\mathcal{H} = V\,dp = \frac{dp}{\rho}. \tag{34}$$

Thus

$$\frac{\vec{\nabla}p}{\rho} = \vec{\nabla}\mathcal{H},$$

and Euler's equation can therefore be written in the form

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{\vec{f}_e}{\rho_0} - \vec{\nabla}\mathcal{H}. \tag{35}$$

Or

$$\rho_0 \frac{\partial \vec{v}}{\partial t} + \frac{\rho_0}{2} \vec{\nabla} v^2 - \rho_0 \vec{v} \times (\vec{\nabla} \times \vec{v}) + \rho_0 \vec{\nabla} \mathcal{H} + \rho_0 \vec{\nabla} \Phi = 0, \tag{36}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \left(\tfrac{1}{2} v^2 + \mathcal{H} + \Phi \right) - \vec{v} \times (\vec{\nabla} \times \vec{v}) = 0. \tag{37}$$

From the isentropic condition $\vec{\nabla}\mathcal{H} = \vec{\nabla}p/\rho$ it follows, by taking the curl, that

$$\vec{\nabla} p \times \vec{\nabla} \rho = 0,$$

so that $p = p(\rho)$. That is, *isentropic flow is associated with a barotropic fluid.* Also, from

$$\rho \, \vec{\nabla} \mathcal{H} = \vec{\nabla} p,$$

taking the curl we obtain

$$\vec{\nabla} \rho \times \vec{\nabla} \mathcal{H} = 0,$$

from which $\mathcal{H} = \mathcal{H}(\rho)$.

The description of the state of a moving fluid is given by $\vec{v}$, p, and ρ, which altogether amount to five quantities. Five equations are therefore required, namely:

- Equation of motion (three equations when expressed in components),
- Continuity equation (one equation),
- Equation of state (one equation).

3.2.1 Boundary conditions

It is also necessary to prescribe boundary conditions. For an ideal fluid (non-viscous, without heat exchange) these are:

(a) The fluid cannot penetrate a solid surface; that is, the component of the velocity perpendicular to the boundary is zero:

$$\vec{v} \cdot \hat{n} \big|_S = 0. \tag{38}$$

The vector $\hat{n}$ is perpendicular to the surface. If the surface moves with velocity $\vec{v}_S$, then

$$\vec{v} \cdot \hat{n} \big|_S = v_S.$$

(b) If there are two immiscible fluids, the pressure and the component of $\vec{v}$ normal to the interface are the same on both sides. If the interface moves, then $\vec{v} \cdot \hat{n}\big|_S = v_S$ again.

3.2.2 irrotational flow

Now, taking the curl of the equation of motion, the pressure gradient term disappears and we obtain an equation involving only the velocity. It describes the time evolution of $\vec{\nabla} \times \vec{v}$, which we shall call the *vorticity*:

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{v}) - \vec{\nabla} \times \left[\vec{v} \times (\vec{\nabla} \times \vec{v}) \right] = 0. \tag{39}$$

A solution—the simplest one—of this equation is

$$\vec{\nabla} \times \vec{v} = 0,$$

which is known as *irrotational flow.*

3.3 Bernoulli's equation

From the equation of motion, specialized to the steady state ($\partial\vec{v}/\partial t = 0$), taking the scalar product with $d\vec{r} = \vec{v}\,dt$ and noting that

$$d\vec{r}\cdot\left[\vec{v}\times(\vec{\nabla}\times\vec{v})\right] = \vec{v}\cdot\left[\vec{v}\times(\vec{\nabla}\times\vec{v})\right]dt = 0,$$

and since $\vec{v}\cdot(\vec{v}\times\vec{\xi}) = 0$, we conclude that

$$d\vec{r}\cdot\vec{\nabla}\left(\frac{1}{2}v^2 + \mathcal{H} + \Phi\right) = \frac{d}{dl}\left(\frac{1}{2}v^2 + \mathcal{H} + \Phi\right) = 0, \tag{40}$$

where d/dl denotes differentiation along the direction of the velocity, that is, along a streamline. It then follows that along a streamline the following condition holds, known as Bernoulli's equation:

$$H \equiv \frac{1}{2}v^2 + \mathcal{H} + \Phi = \text{constant}. \tag{41}$$

This field—which we shall call the Bernoulli field H—may be viewed as an extension of the effective potential to fluid in motion. The constant has the same value at all points on a given streamline, but may differ from one streamline to another. In the incompressible case, we have $\mathcal{H} = p/\rho$.

The tangent to a streamline at any point is in the direction of the velocity at that point. The equation of a streamline is therefore

$$\vec{v}\times d\vec{r} = 0.$$

In steady state these streamlines do not change with time and coincide with the trajectories of the fluid particles.

3.3.1 Flow tube

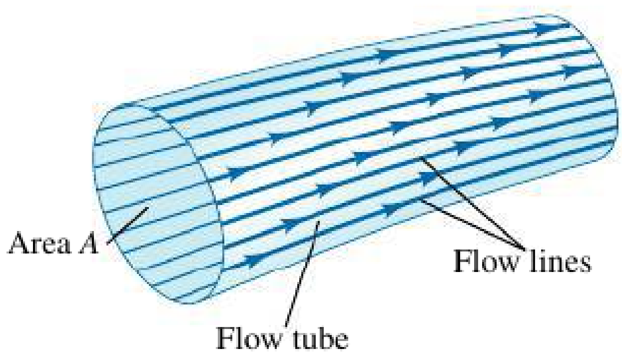

Figure 4: A flow tube bounded by flow lines. In steady flow, fluid cannot cross the walls of a flow tube. Figure taken from Ref. [5].

The path of an individual particle in a moving fluid is called a flow line. The flow lines passing through the edge of an imaginary element of area, such as area A in Fig. 4, form a tube called a flow tube. From the definition of a flow line, in steady flow no fluid can cross the side walls of a given flow tube.

If a fluid of constant density flows in a horizontal tube, the term $\Phi = gz$ remains constant, so that Bernoulli's equation reduces to

$$\frac{1}{2}\rho v^2 + p = \text{constant}. \tag{42}$$

Consequently, in a horizontal tube, the greater the velocity, the lower the pressure, and vice versa.

3.3.2 Lift on an airplane wings

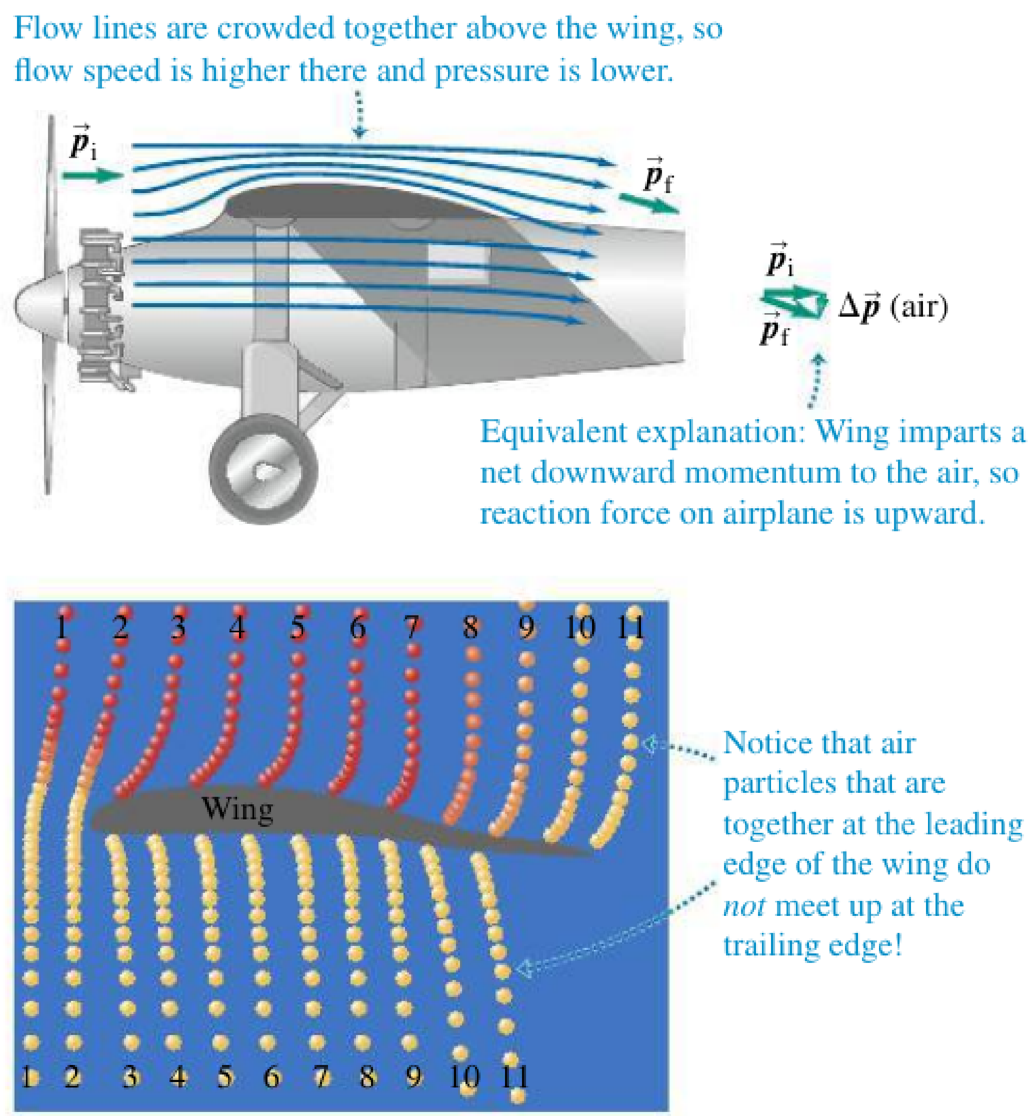

Figure 5: Top: Flow lines around an airplane wing. Bottom: Computer simulation of air parcels flowing around a wing, showing that air moves much faster over the top than over the bottom. Figures taken from Ref. [5].

This effect is responsible for the lift of an airplane, since the shape of the wings is designed so that the curvature is greater on the upper surface (see Fig. 5). This causes the velocity over the wing to be higher and therefore the pressure lower. The resulting upward pressure force lifts the plane.

If A is the area of the wing and P_1 and P_2 are the pressures below and above the wing, respectively, the lift force is

$$F = A(P_1 - P_2) = \frac{A\rho}{2}(v_1^2 - v_2^2). \tag{43}$$

A useful approximation for the lift force is

$$F = \frac{A\rho}{2}(v_1^2 - v_2^2) = \frac{A\rho}{2}(v_1 + v_2)(v_1 - v_2) = A\rho\,\bar{v}\,(v_1 - v_2), \tag{44}$$

where the average velocity $\bar{v}$ is very close to the speed of the airplane relative to the air.

3.3.3 The Venturi effect

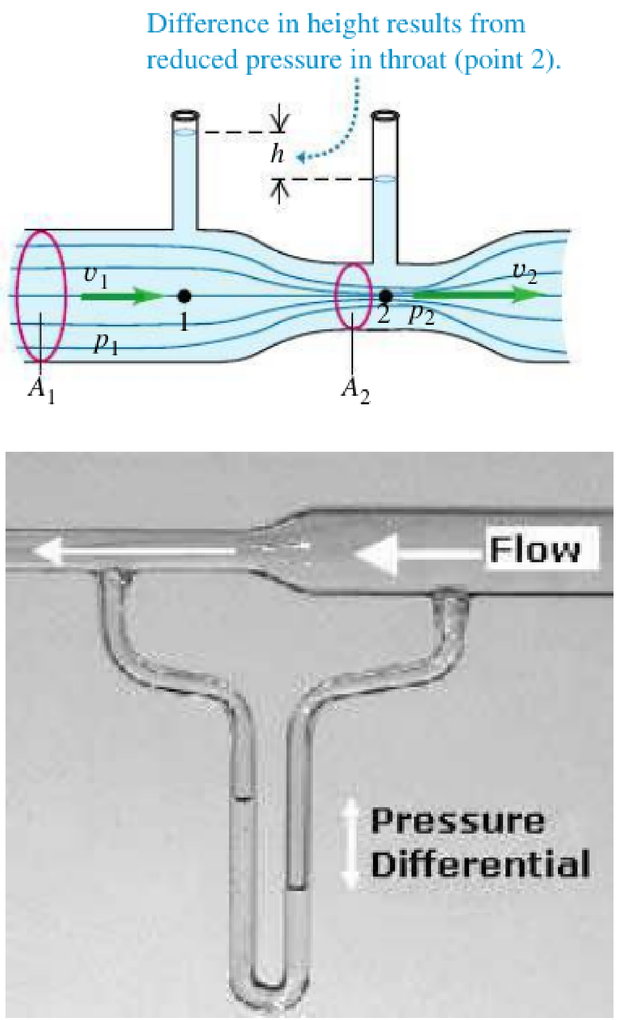

Figure 6: Top: The Venturi meter is used to measure flow speed in a pipe. Bottom: Venturi effect demonstration. Air streams from the right into a narrower tube where it speeds up. The shift in water level in the shunt tube underneath can be used to determine the pressure drop and thus the flow speed.

A simple duct with gently varying cross-section carries a constant volume flow rate Q of incompressible fluid (see Fig. 6). For simplicity we assume the duct is horizontal, such that gravity can be disregarded. For a streamline running through the duct, Bernoulli's theorem tells us that

$$H = \frac{1}{2}v^2 + \frac{p}{\rho_0} \tag{45}$$

takes the same value everywhere along the streamline. Approximating the velocity by its average value $U = Q/A$ over the cross-section A, the pressure becomes

$$p = \rho_0 \left(H - \frac{Q^2}{2A^2} \right). \tag{46}$$

This demonstrates the Venturi effect: the pressure rises when the duct cross-section increases, and conversely. The Venturi effect is used in many technological devices, from carburetors in car engines to perfume atomizers, as well as pumps and flow meters.

3.3.4 Torricelli's law

A barrel of wine has a little spout close to the bottom. If the plug in the spout is suddenly removed, the hydrostatic pressure accelerates the wine to stream out with considerable speed. Provided the spout is narrow compared to the size of the barrel, a nearly steady flow will soon establish itself.

Consider a streamline starting near the top of the barrel and running down through the middle of the spout. Near the top at height $z = h$, the fluid is almost at rest, $v \approx 0$. The pressure is atmospheric, $p = p_0$, and the gravitational potential may be taken to be gz. Thus,

$$H_{\text{top}} = gh + \frac{p_0}{\rho_0}. \tag{47}$$

Just outside the spout, the fluid has horizontal velocity v and the pressure is also atmospheric. Hence,

$$H_{\text{bottom}} = \frac{1}{2}v^2 + \frac{p_0}{\rho_0}. \tag{48}$$

Equating the two expressions gives

$$\frac{1}{2}v^2 = gh, \tag{49}$$

with the solution

$$v = \sqrt{2gh}. \tag{50}$$

This result is called *Torricelli's law.* Surprisingly, the outflow speed equals the speed acquired by a freely falling particle over the same height.

3.3.5 Quasi-stationary emptying

If the barrel has constant cross-section A_0 (see Fig. 7), the average vertical flow velocity is $v_0 = vA/A_0$, where A is the cross-section of the spout and $v = \sqrt{2gz}$ the outflow velocity when the water level is z. Since $dz/dt = -v_0$, we obtain

$$\frac{dz}{dt} = -\frac{A}{A_0}\sqrt{2gz}. \tag{51}$$

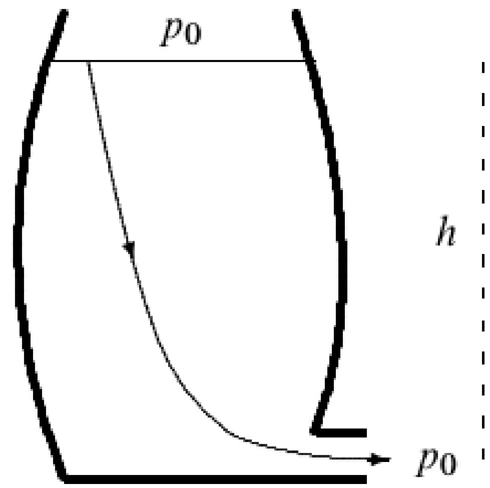

Figure 7: Wine running out of a barrel. The wine emerges with the same speed as it would have obtained by falling freely through the height h of the fluid in the barrel.

Integration gives the emptying time

$$t_0 = \int_0^h \frac{dz}{\sqrt{2gz}} \frac{A_0}{A} = \frac{A_0}{A}\sqrt{\frac{2h}{g}}. \tag{52}$$

3.3.6 Case: The Pitot tube

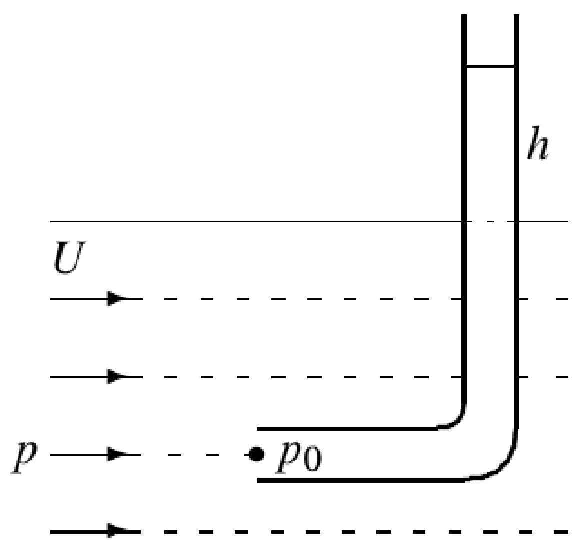

Figure 8: The principle of the Pitot tube. The pressure increase along the stagnating streamline must equal the weight of the raised water column.

The Pitot tube measures flow speed by converting kinetic pressure into static pressure. In steady state, water rises to height h so that

$$\frac{p_0}{\rho_0} = \frac{1}{2}U^2 + \frac{p}{\rho_0}, \tag{53}$$

which gives

$$\Delta p = \frac{1}{2}\rho_0 U^2 = \rho_0 g h. \tag{54}$$

Hence,

$$U = \sqrt{2gh}. \tag{55}$$

At a speed of $1\,\mathrm{m\ s^{-1}}$ the water rises $5\,\mathrm{cm}$. Even if people do not know Bernoulli's theorem, farmers know better than to leave the barn door open in a storm: a strong gust reduces the pressure above the roof and can blow it off.

3.3.7 Rotating fluid in steady state

This problem has often been solved in a reference frame rotating with the liquid. Here we solve it in the inertial frame, by applying directly the equation of motion in steady state, with constant density ρ and in the presence of a uniform gravitational field.

In steady state it is true that

$$\vec{v} = \vec{\omega} \times \vec{r},$$

and therefore

$$\vec{\nabla} \times \vec{v} = 2\,\vec{\omega}, \qquad \vec{v} \times (\vec{\nabla} \times \vec{v}) = 2(\vec{\omega} \times \vec{r}) \times \vec{\omega}.$$

A straightforward calculation then shows that the equation of motion reduces to

$$-\omega^2(x\,\hat{e}_x + y\,\hat{e}_y) + \frac{\vec{\nabla} p}{\rho} + g\,\hat{e}_z = 0. \tag{56}$$

From this, the three component equations follow:

$$\frac{\partial p}{\partial x} - \rho\omega^2 x = 0, \tag{57}$$

$$\frac{\partial p}{\partial y} - \rho\omega^2 y = 0, \tag{58}$$

$$\frac{\partial p}{\partial z} + \rho g = 0. \tag{59}$$

Consequently,

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = \rho\omega^2(x\,dx + y\,dy) - \rho g\,dz, \tag{60}$$

from which

$$p = p_0 + \frac{\rho\omega^2}{2}(x^2 + y^2) + \rho g(z_0 - z), \tag{61}$$

in agreement with the result obtained in Hydrostatic chapter.

3.4 Vorticity

The value of the Bernoulli field $H(x)$ at a point x depends only on the streamline passing through that point. Different streamlines will in general have different values of H. It is, however, often possible to relate the values of H for bundles of streamlines.

3.4.1 The vorticity field

Using the Euler equation for steady incompressible non-viscous flow, the gradient of the Bernoulli field including gravity becomes (in index notation)

$$\nabla_i H = \frac{1}{2}\nabla_i v^2 + \nabla_i \Phi + \frac{1}{\rho_0}\nabla_i p = v_j \nabla_i v_j - v_j \nabla_j v_i = \left(\vec{v} \times (\vec{\nabla} \times \vec{v})\right)_i. \tag{62}$$

Defining the *vorticity field* as the curl of the velocity field,

$$\vec{\omega} = \vec{\nabla} \times \vec{v}, \tag{63}$$

we obtain for steady flow

$$\vec{\nabla} H = \vec{v} \times \vec{\omega}. \tag{64}$$

The vorticity field is a quantitative measure of the local circulation in the fluid. In any region where the vorticity vanishes we have $\vec{\nabla} H = 0$, so the Bernoulli field must take the same value everywhere in that region, that is $H(x) = H_0$. A flow completely free of vorticity is said to be *irrotational*.

3.4.2 Vortex lines

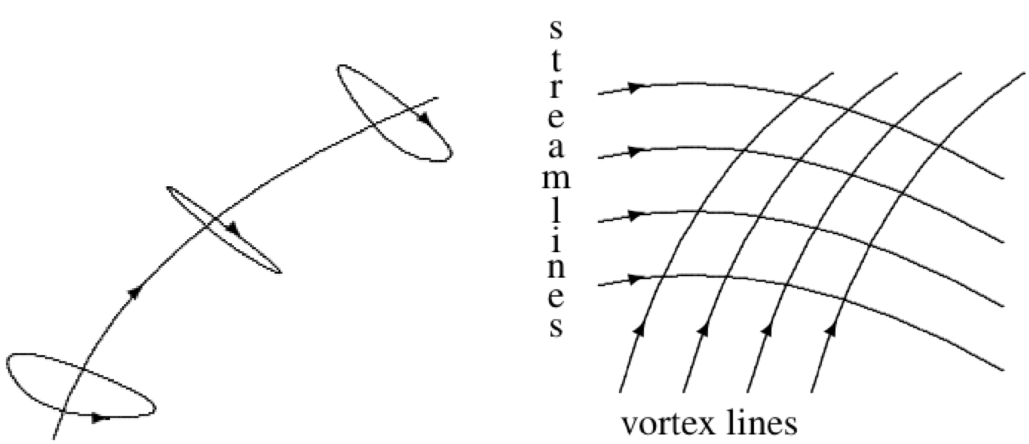

Figure 9: Left: Around a vortex line there is local circulation of fluid. Right: The Bernoulli field is constant on surfaces made from vortex lines and streamlines.

The field lines of the vorticity field are called *vortex lines* (see Fig. 9), and are defined as curves that are everywhere tangent to the vorticity field. Like streamlines, they are solutions of the differential equation at a fixed time t_0,

$$\frac{d\vec{x}}{ds} = \vec{\omega}(\vec{x}, t_0), \tag{65}$$

where s is a running parameter along the curve. Unlike streamlines, the parameter is not time; here s has the dimensions of length×time.

Like streamlines, vortex lines cannot cross each other, and since the vorticity field is a curl, it satisfies

$$\vec{\nabla} \cdot \vec{\omega} = 0.$$

Therefore, vortex lines can neither start nor end anywhere in the fluid; they must continue until they reach the boundary of the flow (possibly at infinity) or close upon themselves.

In steady flow, vortex lines are independent of the chosen time t_0. Bernoulli's theorem, $(\vec{v} \cdot \vec{\nabla})H = 0$, follows immediately by dotting $\vec{\nabla}H = \vec{v} \times \vec{\omega}$ with $\vec{v}$. Similarly, dotting with $\vec{\omega}$ gives

$$(\vec{\omega} \cdot \vec{\nabla})H = 0,$$

showing that the Bernoulli field is also constant along vortex lines. Together these results show that the Bernoulli field is constant on the two-dimensional surfaces formed by combining vortex lines and streamlines. These are sometimes called *Lamb surfaces* or *Bernoulli surfaces*.

3.4.3 Equation of motion for vorticity

Since the vorticity field is derived from the velocity field, its equation of motion must follow from Euler's equation for the velocity field. Retracing the steps leading to $\vec{\nabla}H = \vec{v} \times \vec{\omega}$ but now allowing for time dependence, we obtain

$$\frac{\partial \vec{v}}{\partial t} = \vec{v} \times \vec{\omega} - \vec{\nabla}H. \tag{66}$$

This reduces to the steady case when $\partial \vec{v}/\partial t = 0$.

Taking the curl of both sides and using the identity that the curl of a gradient vanishes, we arrive at the vorticity equation

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega}). \tag{67}$$

A major lesson follows: if the vorticity vanishes initially, $\vec{\omega}(\vec{x}, t) = 0$ throughout a region V at time t, then $\partial \vec{\omega}/\partial t = 0$ there. Consequently, the vorticity field will not change, and if it is zero initially, it will remain zero forever. *Vorticity cannot be generated by the motion of an ideal fluid; it must be present from the outset.*

3.5 Circulation

Vorticity is a local property of a fluid. The corresponding global concept is called *circulation*, defined as the line integral of the velocity field along a closed curve C (see Fig. 10),

$$\Gamma(C, t) = \oint_C \vec{v}(\vec{x}, t) \cdot d\vec{\ell}. \tag{68}$$

If C encircles a whirling region, the circulation measures how strongly the fluid rotates around the loop. The sign depends on whether the circulation is taken with or against the whirling motion.

The circulation of the fluid, Γ, is defined as the line integral along the closed curve c:

$$\Gamma = \oint_c \vec{v} \cdot d\vec{l}. \tag{69}$$

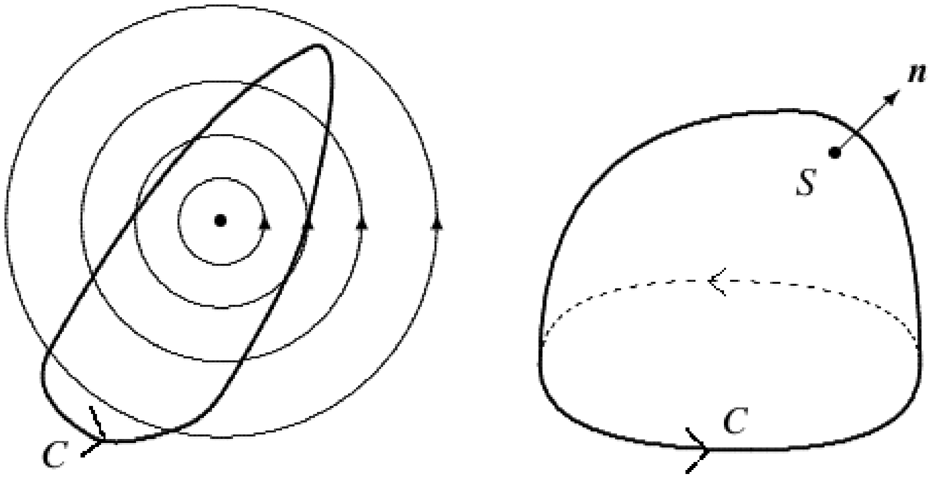

Figure 10: Left: A closed curve C encircling a whirl. Right: A surface S with perimeter C. The normal to the surface is consistent with the orientation of C (here using a right-hand rule).

3.5.1 Conservation of circulation

The Lagrangian time derivative of the circulation reads

$$s\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_C \vec{v}\cdot d\vec{\ell}. \tag{70}$$

When performing the differentiation, one must take into account the change both in $\vec{v}$ and in the shape of the curve, that is,

$$\frac{D\Gamma}{Dt} = \oint_C \frac{D\vec{v}}{Dt}\cdot d\vec{\ell} + \oint_C \vec{v}\cdot\frac{D}{Dt}(d\vec{\ell}). \tag{71}$$

Taking into account equation $d\vec{v} = d\vec{\ell}/dt$, this can be written as

$$\frac{D\Gamma}{Dt} = \oint_C \frac{D\vec{v}}{Dt}\cdot d\vec{\ell} + \oint_C \vec{v}\cdot d\vec{v} = \oint_C \frac{D\vec{v}}{Dt}\cdot d\vec{\ell} + \left.\frac{v^2}{2}\right|_a = \oint_C \frac{D\vec{v}}{Dt}\cdot d\vec{\ell}, \tag{72}$$

that is,

$$\frac{D\Gamma}{Dt} = \oint_C \frac{D\vec{v}}{Dt}\cdot d\vec{\ell}. \tag{73}$$

This expression is known as Kelvin's equation.

Replacing here the equation of motion, whose Lagrangian form is

$$\rho\,\frac{D\vec{v}}{Dt} + \nabla p + \rho\,\nabla\Phi = 0, \tag{74}$$

one obtains

$$\frac{D\Gamma}{Dt} = -\oint_C \left(\frac{\nabla p}{\rho} + \nabla\Phi\right)\cdot d\vec{\ell}. \tag{75}$$

In the isentropic case, $\nabla\mathcal{H} = \nabla p/\rho$, so that

$$\frac{D\Gamma}{Dt} = -\oint_C \nabla(\mathcal{H}+\Phi)\cdot d\vec{\ell} = -\oint_C d(\mathcal{H}+\Phi), \tag{76}$$

which implies that the circulation of the isentropic velocity field of an ideal and barotropic fluid is a constant of motion:

$$\Gamma = \text{constant}. \tag{77}$$

The quantity Γ, also known as the vortex intensity, cannot be created or destroyed in an ideal isentropic fluid. In other words, vorticity is a convective property that is conserved in the flow of the fluid.

This is the Hankel–Kelvin circulation theorem (formulated between 1861 and 1869), also known as the law of conservation of velocity circulation or the law of conservation of vortex intensity.

Kelvin's circulation theorem: In an ideal (inviscid), barotropic, and isentropic fluid, the circulation around any closed material curve moving with the fluid is conserved in time.

Physical interpretation In an ideal, isentropic, and barotropic fluid, the forces acting on a fluid element per unit mass reduce to gradients of scalar potentials. In particular, the pressure force can be written as the gradient of the specific enthalpy, and body forces such as gravity derive from a conservative potential.

As a consequence, these forces are incapable of generating or destroying circulation. Since the line integral of a gradient over a closed material curve vanishes, the circulation around any loop that moves with the fluid remains unchanged in time.

From a physical standpoint, this means that vorticity is a convective property: it is transported and deformed by the flow, but it cannot be created in the absence of viscosity, entropy gradients, or non-conservative forces. Any change in circulation must therefore originate from dissipative effects (viscosity), non-barotropic effects ($p \neq p(\rho)$), or external sources beyond the assumptions of ideal and isentropic flows.

It should be emphasized that this result has been obtained using Euler's equation in the form given previously, and therefore relies on the assumption that the flow is isentropic. The theorem does not hold for flows that are not isentropic.

3.5.2 Stokes' theorem and steady flow

The circulation of a vector field around a closed curve (see Fig. 10) equals the flux of its curl through any surface S bounded by the curve:

$$\oint_C \vec{v} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S}. \tag{78}$$

It does not matter which surface S is chosen, provided it has C as boundary and lies entirely within the fluid.

By applying Kelvin's theorem to an infinitesimal closed contour δC and transforming the line integral according to Stokes' theorem, we obtain

$$\oint_C \vec{v} \cdot d\vec{\ell} = \int (\nabla \times \vec{v}) \cdot d\vec{S} \simeq \delta\vec{S} \cdot (\nabla \times \vec{v}) = \text{constant}, \tag{79}$$

where $d\vec{S}$ is a fluid surface element spanning the contour δC. The vector $\nabla \times \vec{v}$ is the vorticity of the fluid flow at a given point. The constancy of the product above can be intuitively interpreted as meaning that the vorticity is transported with the fluid.

3.6 Potential flow

In the absence of viscosity, a flow that is irrotational in a region will stay irrotational. High Reynolds numbers confine vorticity to thin boundary layers, so that the main body of the flow is nearly irrotational.

From $\vec{\nabla} \times \vec{v} = 0$ it follows that the velocity field is the gradient of a scalar potential,

$$\vec{v} = \vec{\nabla}\Psi. \tag{80}$$

The scalar Ψ is called the *velocity potential.*

3.6.1 Incompressible potential flow

In incompressible flow, $\vec{\nabla} \cdot \vec{v} = 0$ implies that Ψ satisfies Laplace's equation:

$$\nabla^2\Psi = 0. \tag{81}$$

Inserting $\vec{\omega} = 0$ and $\vec{v} = \vec{\nabla}\Psi$ into Euler's equation gives

$$\vec{\nabla}\left(H + \frac{\partial \Psi}{\partial t}\right) = 0,$$

so the expression in parentheses depends only on time. Using the Bernoulli function and solving for the pressure, we obtain

$$p = C(t) - \rho_0\left(\frac{1}{2}v^2 + \Phi + \frac{\partial \Psi}{\partial t}\right), \tag{82}$$

where $C(t)$ is an arbitrary function of time.

Potential flow in incompressible fluids is much simpler than flow with vorticity because Laplace's equation is well understood. Any linear superposition of solutions of Laplace's equation is again a solution, and practical problems reduce to finding the solution that satisfies the boundary conditions.

3.7 Application: Cylinder in uniform crosswind

A circular cylinder with axis along the z-axis and radius a is placed in an asymptotically uniform crosswind U along the x-axis. This does not break the translational invariance along the z-axis so that the velocity potential $\Psi(x, y)$ and the stream function $\psi(x, y)$ are both independent of z. Evidently, the natural choice is cylindrical coordinates (r, ϕ, z).

3.7.1 Velocity potential

Asymptotically, for $r \to \infty$, the velocity potential must approach the field of a constant uniform crosswind, $\Psi \to Ux = Ur\cos\phi$. The linearity of Laplace's equation demands that the velocity potential is linear in the asymptotic values,

$$\Psi = U\cos\phi\, f(r), \tag{83}$$

where $f(r)$ is an unknown function that approaches r as $r \to \infty$.

Inserting this into the cylindrical Laplacian we have

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{\partial^2\Psi}{\partial z^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r}, \tag{84}$$

which leads to

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2} = 0. \tag{85}$$

Looking for power-law solutions $f \sim r^\alpha$, we find $\alpha = \pm 1$. Thus the general solution is

$$f(r) = Ar + \frac{B}{r},$$

with constants A and B.

The asymptotic condition implies $A = 1$, while B is determined by the boundary condition that the radial component of the velocity vanishes at the surface of the cylinder, i.e. $f'(a) = 0$. This yields $B = a^2$, and the solution becomes

$$\Psi = Ur\cos\phi\left(1 + \frac{a^2}{r^2}\right). \tag{86}$$

3.7.2 Velocity and pressure fields

The velocity field is obtained from derivatives of the potential. The non-vanishing components are

$$v_r = \frac{\partial\Psi}{\partial r} = U\cos\phi\left(1 - \frac{a^2}{r^2}\right), \tag{87}$$

$$v_\phi = \frac{1}{r}\frac{\partial\Psi}{\partial\phi} = -U\sin\phi\left(1 + \frac{a^2}{r^2}\right). \tag{88}$$

The radial velocity vanishes at the surface as expected, while the tangential component at $r = a$,

$$v_\phi\Big|_{r=a} = -2U\sin\phi,$$

vanishes only at the stagnation points $\phi = 0, \pi$.

The pressure follows from Bernoulli's theorem. In the absence of gravity and normalized to vanish at infinity, it is

$$p = \frac{1}{2}\rho_0\left(U^2 - v^2\right) = \frac{1}{2}\rho_0 U^2\frac{a^2}{r^2}\left(4\cos^2\phi - 2 - \frac{a^2}{r^2}\right). \tag{89}$$

On the surface $r = a$ this reduces to

$$p_a = \frac{1}{2}\rho_0 U^2(4\cos^2\phi - 3), \tag{90}$$

which is negative for $30^\circ < \phi < 150^\circ$.

The symmetry $p(\phi) = p(-\phi)$ implies that the total force in the y-direction (lift) vanishes, as expected. More surprisingly, the invariance $p(\pi - \phi) = p(\phi)$ implies that the total force in the x-direction (drag) also vanishes. This is known as *d'Alembert's paradox*.

3.7.3 Lift on a half-cylinder

The same solution also describes ideal flow over a cylinder that is half buried in a flat surface, since the normal velocity vanishes in the symmetry plane $y = 0$.

The vertical lift on a stretch of the half-cylinder of length L is

$$\mathcal{L}_y = -\int_0^\pi p_a \, dS_y = -\int_0^\pi p_a \sin\phi \, La \, d\phi = \frac{5}{3}\rho_0 U^2 La. \tag{91}$$

If the average density of the cylinder is $\rho_1 > \rho_0$, the lift-to-weight ratio is

$$\frac{\mathcal{L}_y}{Mg_0} = \frac{5}{3\pi} \frac{\rho_0}{\rho_1 - \rho_0} \frac{U^2}{ag_0}, \tag{92}$$

including buoyancy. There is therefore a critical wind speed beyond which the lift overcomes the weight and the half-buried cylinder can be lifted out of the ground.

3.8 Problems

1. A water tank of fixed height H discharges liquid upward through an outlet at the bottom (Fig. 11). What height does the column reach?

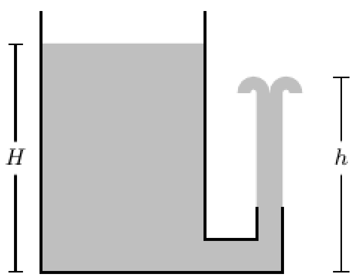

Figure 11: Problem 3.8.1. Figure taken from Ref. [1].

2. Consider the steady flow of an incompressible ideal gas. Using Bernoulli's equation, show that

$$\frac{1}{2}v^2 + \frac{kT}{\mu} \ln\rho + \Phi = \text{constant}. \tag{93}$$

3. An incompressible, non-viscous fluid flows along the interior of a cone with opening angle α, as shown in Fig. 12. Neglecting gravity and considering only pressure differences, study the relation between the pressure and the flow velocity in the steady state.

4. The equation of motion of an isentropic fluid, in a rotating reference frame with constant angular velocity $\boldsymbol{\omega}$, can be written in the form

$$\rho \frac{d\vec{v}}{dt} + \nabla \left[H + \mathcal{G} + \frac{\omega^2 r^2}{2} \right] + 2\rho \, \vec{v} \times \boldsymbol{\omega} = 0. \tag{94}$$

   Show that the circulation is conserved, that is,

$$\frac{d\Gamma}{dt} = 0. \tag{95}$$

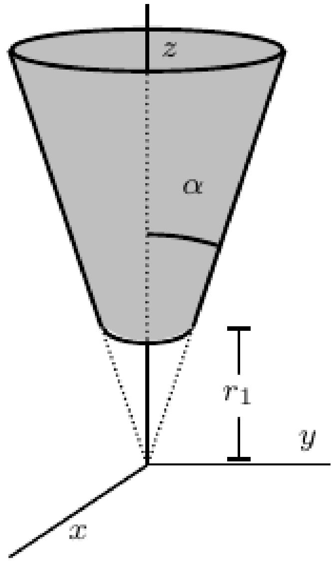

Figure 12: Problem 3.8.3. Figure taken from Ref. [1].

4 Viscosity

Disturbing a fluid at rest slightly, setting it into motion with spatially varying velocity field, will to first order of approximation generate stresses that depend linearly on the spatial derivatives of the velocity field. Fluids with a linear relationship between stress and velocity gradients are said to be Newtonian, and the coefficients in this relationship are material constants that characterize the strength of viscosity.

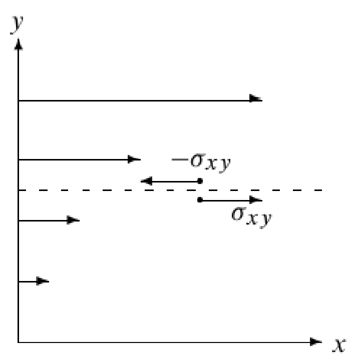

Figure 13: Shear viscosity in laminar (layered) flow. The fluid above the dashed line moves slightly faster than the fluid below and exerts a positive shear stress σ_{xy} on the fluid beneath. By Newton's third law, the fluid below exerts an opposite shear stress $-\sigma_{xy}$ on the fluid above. Figure taken from Ref. [3].

4.1 Shear viscosity

Consider a fluid flowing steadily along the x-direction with a velocity field $v_x(y)$ that is independent of x but may vary with y. Such a field could, for example, be created by enclosing a fluid between moving plates, and is an elementary example of *laminar* or layered flow. If the velocity field has no y-dependence, so that the fluid is in uniform motion along the x-axis, there should not be any internal stresses. If, on the other hand, the velocity grows with y, so that $dv_x(y)/dy > 0$, we expect the fluid immediately above a plane $y = \text{const}$ to drag along the fluid immediately below because of friction and thus exert a positive shear stress, $\sigma_{xy}(y) > 0$, on this plane (see Fig. 13).

It seems reasonable to expect that a larger velocity gradient will evoke stronger stress. In Newton's law of viscosity the shear stress is made proportional to the gradient,

$$\sigma_{xy}(y) = \eta \frac{dv_x(y)}{dy}. \tag{96}$$

The proportionality constant η is called the coefficient of *shear viscosity*, the *dynamic viscosity*, or simply the *viscosity*. It is a measure of how strongly the moving layers of fluid are coupled by friction. In incompressible fluids there is also a bulk viscosity coefficient, but in practice it is usually much smaller and often negligible.

The viscosities of real fluids range over many orders of magnitude. Since dv_x/dy has dimension of inverse time, the unit for viscosity is $\text{Pa} \cdot \text{s}$ (pascal–seconds).

4.1.1 Molecular origin of viscosity in gases

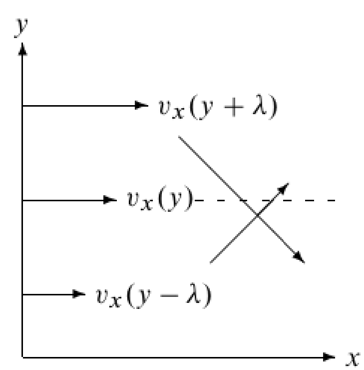

Figure 14: Layers of gas moving with different velocities give rise to shear forces because they exchange molecules with different average velocities. Figure taken from Ref. [3].

In gases where molecules are far apart, internal stress arises from molecular bombardment of surfaces, transferring momentum across them. In liquids the stress is caused partly by molecular motion and partly by intermolecular forces. We therefore restrict the discussion to gases.

Molecules in a gas move nearly randomly at speeds much larger than the macroscopic flow velocity $v(x,t)$. In laminar planar flow with velocity $v_x(y)$ and gradient dv_x/dy, a molecule of mass m crossing a surface element dS_y from above carries more momentum in the x-direction than one crossing from below. The result is a net transfer of momentum downward.

Let λ be the mean free path and τ the typical time between collisions. Disregarding numerical factors of order unity, a layer of thickness λ above dS_y carries an excess momentum

$$dP_x \approx \rho \lambda^2 \frac{dv_x}{dy} \, dS_y. \tag{97}$$

The shear stress follows as momentum transferred per unit time and area:

$$\sigma_{xy} \approx \frac{dP_x}{\tau dS_y},$$

which recovers Newton's law with estimate

$$\eta \approx \rho \frac{\lambda^2}{\tau} \approx \rho \lambda v_{\mathrm{mol}}. \tag{98}$$

Using $\lambda/\tau \approx v_{\mathrm{mol}}$ and the rms molecular speed $v_{\mathrm{mol}} = \sqrt{3RT}$, this explains the magnitude of gas viscosity.

4.1.2 Temperature dependence of viscosity

The viscosity of any material depends on temperature. Most liquids become thinner when heated (viscosity decreases), while gases become more viscous at higher temperatures.

For an ideal gas one finds

$$\eta = \eta_0 \sqrt{\frac{T}{T_0}}. \tag{99}$$

The viscosity of gases is independent of pressure and increases slightly faster than $\sqrt{T}$ because of molecular attraction.

4.1.3 Kinematic viscosity

The estimate above suggests defining the *kinematic viscosity*

$$\nu = \frac{\eta}{\rho}. \tag{100}$$

It has units of $\mathrm{m^2\,s^{-1}}$. In ideal gases we have $\rho \sim p/T$, so that

$$\nu \sim \frac{T^{3/2}}{p},$$

and the kinematic viscosity decreases with temperature for isentropic flow.

Unlike the dynamic viscosity, it is the kinematic viscosity ν that appears in the Navier–Stokes equation. Under velocity-driven flows, air behaves as if it were more viscous than water by an order of magnitude, which explains why air seems "thicker" in certain aerodynamic contexts.

4.1.4 Velocity-driven planar flow

Assume laminar planar flow with velocity $v_x(y,t)$. Since there is no advection term $(\vec{v} \cdot \vec{\nabla})v_x$, Cauchy's equation reduces to

$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial \sigma_{xy}}{\partial y} = \eta \frac{\partial^2 v_x}{\partial y^2}. \tag{101}$$

Dividing by ρ gives

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}. \tag{102}$$

This is a diffusion equation for velocity.

4.1.5 Case: Steady planar flow

In steady state,

$$\frac{\partial^2 v_x}{\partial y^2} = 0,$$

so

$$v_x = A + By.$$

For plates at $y = 0$ and $y = d$ with velocities 0 and U, respectively, boundary conditions yield

$$v_x(y) = \frac{U}{d} y. \tag{103}$$

The shear stress is

$$\sigma_{xy} = \eta \frac{U}{d}. \tag{104}$$

4.1.6 Case: Viscous friction

If the contact area is A and fluid thickness is d, the drag force is

$$D \approx \eta \frac{A}{d} U. \tag{105}$$

For a mass M experiencing only viscous drag,

$$M \frac{dU}{dt} = -\eta \frac{A}{d} U, \tag{106}$$

with solution

$$U = U_0 e^{-t/\tau}, \qquad \tau = \frac{Md}{\eta A}. \tag{107}$$

The stopping distance is

$$L = \int_0^\infty U dt = U_0 \tau = \frac{U_0 M d}{\eta A}. \tag{108}$$

4.1.7 Case: Shear wave (Stokes' second problem)

Let a plate oscillate as $U(t) = U_0 \cos(\omega t)$. The velocity field is

$$v_x(y,t) = U_0 e^{-ky} \cos(\omega t - ky), \qquad k = \sqrt{\frac{\omega}{2\nu}}. \tag{109}$$

This is a damped transverse wave. The wavelength is $\lambda = 2\pi/k = 2\sqrt{\pi \nu \tau}$ and the penetration depth $d = 1/k$.

A 1 kHz shear wave penetrates only 71 μm in air and 18 μm in water.

4.1.8 Case: Free momentum diffusion

The planar flow equation describes momentum diffusion. A Gaussian initial profile

$$v_x(y,0) = Ue^{-y^2/a^2}$$

evolves as

$$v_x(y,t) = U\frac{a}{\sqrt{a^2+4\nu t}}\exp\left(-\frac{y^2}{a^2+4\nu t}\right). \tag{110}$$

The width grows as $\sqrt{a^2+4\nu t}$. At late times,

$$\delta_{\text{front}} = 2\sqrt{\nu t}.$$

4.1.9 Case: Growth of a boundary layer (Stokes' first problem)

A plate at $y = 0$ starts moving at speed U. Seeking a similarity solution,

$$v_x(y,t) = Uf\left(\frac{y}{\sqrt{\nu t}}\right), \tag{111}$$

leads to

$$f''(s) + \frac{1}{2}sf'(s) = 0. \tag{112}$$

The solution satisfying $f(0) = 1$, $f(\infty) = 0$ is

$$f(s) = \frac{1}{\sqrt{\pi}}\int_s^\infty e^{-u^2/4}du. \tag{113}$$

For large s, the decay is Gaussian,

$$v_x \sim \exp\left(-\frac{y^2}{4\nu t}\right),$$

as expected from momentum diffusion.

4.2 Dynamics of incompressible Newtonian fluids

Numerous everyday fluids obey Newton's law of viscosity (Eq. (96)), for example water, air, oil, alcohol, and antifreeze. A number of common fluids are only approximately Newtonian, for example paint and blood, and others are strongly non-Newtonian, for example tomato ketchup, jelly, and putty. There also exist *viscoelastic* materials that—depending on frequency—are both elastic and viscous. They are sometimes used in toys that can be slowly deformed like clay but also bounce like rubber balls when dropped on the floor.

In Newtonian fluids the shear stress σ_{xy} is directly proportional to the velocity gradient $\nabla_y v_x$—also called the *shear strain rate*— with proportionality constant equal to the shear viscosity η. Most non-Newtonian fluids become "thinner" as the shear strain rate increases, meaning the shear stress grows slower than linearly. Even the most Newtonian of fluids, water, is thinner at shear strain rates above $10^{12}\,\text{s}^{-1}$. Only a few fluids (for example some starches stirred in water) appear to thicken with increasing strain rate. The science of the general flow properties of materials is called *rheology*.

We shall in this section establish the general dynamics for incompressible, isotropic, and homogeneous Newtonian fluids.

4.2.1 Isotropic viscous stress

Newton's law of viscosity (15.1) is a linear relation between the shear stress and the velocity gradient, only valid in a particular flow geometry. As for Hooke's law for elasticity, we want a definition of viscous stress that takes the same form in any flow geometry and in any Cartesian coordinate system.

Most fluids are not only Newtonian but also *isotropic*. In an isotropic fluid at rest there are no internal directions, and the stress tensor is determined by the pressure,

$$\sigma_{ij} = -p\,\delta_{ij}.$$

When such a fluid is set in motion, the velocity field $\vec{v}(\vec{x},t)$ defines a direction at every point. Stress is produced by velocity gradients, hence viscous stresses are determined by the tensor $\nabla_i v_j$. In an incompressible fluid,

$$\sum_i \nabla_i v_i = \vec{\nabla}\cdot\vec{v} = 0,$$

so the most general symmetric tensor that can be built from velocity gradients is

$$\sigma_{ij} = -p\,\delta_{ij} + \eta\left(\nabla_i v_j + \nabla_j v_i\right). \tag{114}$$

This is the natural generalization of Newton's law of viscosity for incompressible flow in any Cartesian coordinate system. For steady planar flow $\vec{v} = (v_x(y), 0, 0)$ the only nonzero components are

$$\sigma_{xy} = \sigma_{yx} = \eta\nabla_y v_x,$$

confirming that η is indeed the shear viscosity.

4.3 The Navier–Stokes equations

The right-hand side of Cauchy's equation of motion equals the effective density of force

$$f_i^* = f_i + \sum_j \nabla_j\sigma_{ij}.$$

Using the stress tensor above and $\vec{\nabla}\cdot\vec{v} = 0$,

$$\sum_j \nabla_j\sigma_{ij} = -\nabla_i p + \eta\left(\sum_j \nabla_i\nabla_j v_j + \sum_j \nabla_j^2 v_i\right) = -\nabla_i p + \eta\nabla^2 v_i.$$

Assuming the fluid is homogeneous, i.e. η and ρ are constants, we obtain the Navier–Stokes equations for incompressible Newtonian fluids:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{g} - \frac{1}{\rho_0}\vec{\nabla}p + \nu\nabla^2\vec{v}, \qquad \vec{\nabla}\cdot\vec{v} = 0. \tag{115}$$

Here ρ_0 is the constant density, $\nu = \eta/\rho_0$ the kinematic viscosity, and $\vec{g} = \vec{f}/\rho_0$ the acceleration field of volume forces (e.g. gravity).

Although compact, these equations contain the full complexity of real fluid dynamics and generally cannot be solved in closed form except for highly symmetric geometries and idealized conditions.

4.4 Boundary conditions

The Navier–Stokes equation is first order in time and requires initial values and boundary conditions.

Density. The density may be discontinuous across a material interface and therefore provides no boundary condition.

Velocity. The normal component

$$v_n = \vec{v} \cdot \vec{n}$$

must be continuous across an interface between incompressible fluids. The tangential component

$$\vec{v}_t = \vec{n} \times (\vec{v} \times \vec{n})$$

must also be continuous, leading to the *no-slip condition.* The full velocity field is therefore continuous across any interface,

$$[\vec{v}] = 0. \tag{116}$$

An exception is an interface between a liquid and vacuum, where velocity may be discontinuous.

Stress. Newton's Third Law requires continuity of the stress vector:

$$[\boldsymbol{\sigma} \cdot \hat{n}] = 0. \tag{117}$$

At a vacuum boundary, $\boldsymbol{\sigma} \cdot \hat{n} = 0$, since vacuum cannot exert force.

Pressure. Pressure need not be continuous across all interfaces, but for incompressible viscous flow near a solid boundary it must coincide with the wall pressure.

Proof (sketch). Assume the wall lies locally at $z = Ax^2 + By^2 + 2Cxy$. Expanding the velocity near the wall,

$$\vec{v}(\vec{x}) = x\nabla_x \vec{v}_0 + y\nabla_y \vec{v}_0.$$

From $\vec{v} = 0$ at the wall we get all tangential derivatives zero at the origin. Using $\vec{\nabla} \cdot \vec{v} = 0$ we find $\nabla_z v_z = 0$, and from the stress tensor one obtains

$$\sigma_{zz} = -p.$$

Hence continuity of normal stress implies continuity of pressure at the wall.

4.5 Classification of flows

The most interesting phenomena in fluid dynamics arise from the competition between inertia and viscosity, represented in the Navier–Stokes equation by the advective acceleration $(\vec{v} \cdot \vec{\nabla})\vec{v}$ and the viscous diffusion term $\nu\nabla^2\vec{v}$. Inertia attempts to continue motion, whereas viscosity acts as a brake.

If inertia is dominant, the viscous term may be neglected and Euler's equation for ideal flow is recovered. If viscosity dominates, the advective term may be dropped and the equations reduce to those describing slow *creeping flow.*

4.5.1 The Reynolds number

A dimensionless estimate of whether inertia or viscosity dominates is given by the Reynolds number, defined as the ratio of the advective to viscous terms. For a characteristic velocity scale U and length scale L,

$$|\vec{\nabla}\vec{v}| \sim \frac{U}{L}, \qquad |\nabla^2\vec{v}| \sim \frac{U}{L^2}.$$

Thus,

$$\mathrm{Re} \approx \frac{|(\vec{v}\cdot\vec{\nabla})\vec{v}|}{|\nu\nabla^2\vec{v}|} = \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu}. \tag{118}$$

Alternatively, the Reynolds number may be viewed as the ratio

$$\mathrm{Re} \approx \frac{t_{\text{diff}}}{t_{\text{flow}}},$$

where $t_{\text{diff}} \sim L^2/\nu$ is the diffusion time and $t_{\text{flow}} \sim L/U$ is the flow time.

For $\mathrm{Re} \ll 1$ the flow is slow and viscous (creeping). For $\mathrm{Re} \gg 1$ inertia dominates and the flow is lively. At intermediate values, turbulence may develop.

4.5.2 Examples

Bathtub turbulence Typical values $U \sim 1$ m/s and $L \sim 1$ m give

$$\mathrm{Re} \sim 10^6,$$

so turbulence is inevitable.

Water pipe For a pipe of diameter $d = 1.25$ cm and flow rate $Q = 100$ cm^3/s,

$$U = \frac{Q}{\pi a^2} \approx 0.8 \text{ m/s}, \qquad \mathrm{Re} \approx 10^4,$$

which is deep in the turbulent regime. For olive oil under otherwise identical conditions we get $\mathrm{Re} \approx 0.15$, and the flow would be creeping.

System	Fluid	Size (m)	Velocity (m/s)	Reynolds number
Ship (Queen Mary 2)	Water	345	15	5.2×10^9
Submarine (Ohio class)	Water	170	12	2.0×10^9
Jet airplane (Boeing 747)	Air	71	250	1.2×10^9
Blue whale	Water	33	10	3.3×10^8
Car	Air	5	30	9.7×10^6
Swimming human	Water	2	1	2.0×10^6
Jogging human	Air	1	3	1.9×10^5
Herring	Water	0.3	1	3.0×10^5
Golf ball	Air	0.043	40	1.1×10^5
Ping-pong ball	Air	0.040	10	2.6×10^4
Fly	Air	0.01	1	6.5×10^2
Flea	Air	0.001	3	1.9×10^2
Gnat	Air	0.001	0.1	6.5
Bacterium	Water	10^{-6}	10^{-5}	10^{-5}

4.5.3 Hydrodynamic similarity

Two flows are said to be *hydrodynamically similar* if they have the same Reynolds number and congruent geometry.

For steady incompressible flow with $\vec{g} = 0$,

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho_0}\vec{\nabla}p + \nu\nabla^2\vec{v}. \tag{119}$$

Introduce dimensionless variables

$$\vec{v} = U\tilde{\vec{v}}, \qquad \vec{x} = L\tilde{\vec{x}}, \qquad p = \rho_0 U^2 \tilde{p}, \qquad \vec{\nabla} = \frac{1}{L}\tilde{\vec{\nabla}}. \tag{120}$$

Then the Navier–Stokes equation becomes

$$(\tilde{\vec{v}} \cdot \tilde{\vec{\nabla}})\tilde{\vec{v}} = -\tilde{\vec{\nabla}}\tilde{p} + \frac{1}{\mathrm{Re}}\tilde{\nabla}^2\tilde{\vec{v}}. \tag{121}$$

The only remaining parameter is the Reynolds number. Thus, any two flows with equal Reynolds numbers and identical geometry have the same dimensionless solutions.

The viscous force on an object of size L scales as

$$D \sim \sigma L^2 \sim \eta\frac{U}{L}L^2 = \eta U L = \rho\nu U L = \rho U^2 L^2 \frac{1}{\mathrm{Re}}.$$

For stones in air and water with equal Reynolds numbers,

$$\frac{D_{\mathrm{air}}}{D_{\mathrm{water}}} = \left(\frac{\eta\nu}{\eta\nu}\right)_{\mathrm{air/water}} \approx 0.27.$$

4.5.4 Example: Robofly

Using hydrodynamic similarity, one can study insect flight using scaled models. A wing of size $L \approx 4\,\mathrm{mm}$ flapping at $50\,\mathrm{Hz}$ has

$$U \approx \pi f L \approx 1.3 \text{ m/s}, \qquad \mathrm{Re} \approx 160.$$

The same flow can be reproduced using mineral oil of higher viscosity.

4.5.5 Example: High-pressure wind tunnels

In early flight research, Reynolds numbers were increased by raising pressure. Since η is pressure-independent and $\rho \propto p$,

$$\mathrm{Re} \propto \rho.$$

The Variable Density Tunnel operated at 20 atm and enabled realistic testing on small-scale models.

Flows with the same Reynolds number may still differ in geometry. Hence Re predicts general behavior (laminar vs turbulent), not detailed flow structures.

4.6 Dynamics of compressible Newtonian fluids

The velocity divergence, $\nabla \cdot \vec{v}$, is the only scalar that can be formed from linear combinations of the velocity gradients. Therefore, the only additional term allowed in the stress tensor must be proportional to $\delta_{ij} \nabla \cdot \vec{v}$. The proportionality constant is conventionally written as $\zeta - \frac{2}{3}\eta$, where η is the shear viscosity and ζ is the bulk (or expansion) viscosity. The stress tensor for a compressible isotropic Newtonian fluid is then

$$\sigma_{ij} = -p\,\delta_{ij} + \eta\left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\nabla \cdot \vec{v}\,\delta_{ij}\right) + \zeta\,\nabla \cdot \vec{v}\,\delta_{ij}. \tag{122}$$

Interpreting this expression as a first-order expansion in the velocity field, the pressure p is identified with the thermodynamic pressure of the fluid at rest. The tracelessness of the shear term implies that the mechanical pressure is

$$p_{\mathrm{mech}} = p - \zeta\,\nabla \cdot \vec{v}, \tag{123}$$

showing that bulk viscosity generates an additional dynamic pressure proportional to the local expansion or contraction of the fluid.

4.6.1 The Navier–Stokes equations

Inserting the stress tensor into Cauchy's equation of motion, we obtain the most general form of the Navier–Stokes equation (for constant η and ζ),

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = \vec{f} - \nabla p + \eta\nabla^2\vec{v} + \left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\vec{v}). \tag{124}$$

If the viscosities have a spatial variation, for example due to temperature, the expression becomes more complicated. Together with the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0, \tag{125}$$

we have obtained four dynamic equations for the four fields v_x, v_y, v_z, and ρ, while the pressure is determined by the thermodynamic equation of state,

$$p = p(\rho, T).$$

For isothermal or isentropic flow the temperature is given algebraically, whereas in the general case we also need a differential heat equation to specify the dynamics of the temperature field.

4.6.2 Boundary conditions

The principal results of the discussion of boundary conditions for incompressible fluids remain valid for compressible fluids, namely the continuity of the velocity field and the stress vector across a material interface,

$$[\vec{v}] = 0, \qquad [\boldsymbol{\sigma}\cdot\hat{n}] = 0. \tag{126}$$

The proof of the continuity of pressure at a solid wall can, however, not be carried through in this case.

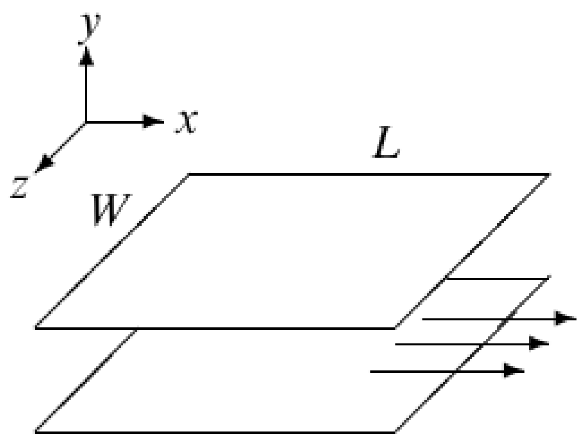

Figure 15: Channel flow between plates of width W parallel and length L. Figure taken from Ref. [3].

5 Steady, incompressible, viscous flow

Most of the fluids encountered in daily life, air, water, gasoline, and oil, are effectively incompressible as long as flow velocities are well below the speed of sound, and often they flow steadily through the channels and pipes that we use to guide them. In looking for exact solutions for viscous flow we shall make the simplifying assumptions that the flow is incompressible and steady, satisfying the Navier–Stokes equations,

$$(\vec{v} \cdot \nabla)\vec{v} = \vec{g} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\vec{v}, \qquad \nabla \cdot \vec{v} = 0, \tag{127}$$

where ρ_0 is the constant density of the fluid and ν the constant kinematic viscosity.

5.1 Pressure-driven channel flow

Here we analyze velocity-driven planar flow between infinitely extended parallel plates where both plates are fixed and fluid is driven between them by a pressure gradient. For simplicity we assume that there is no gravity; gravity-driven flow will be analyzed in the following section.

The coordinate system is chosen with the x-axis pointing along the direction of flow and the y-axis orthogonal to the plates (see Fig. 15). A velocity field respecting the planar symmetry is of the form

$$\vec{v} = \big(v_x(y), 0, 0\big) = v_x(y)\,\hat{e}_x. \tag{128}$$

An infinitely extended flow like this is of course unphysical, but should nevertheless offer an approximation to the real flow between plates of finite extent, provided the dimensions of the plates are sufficiently large compared to their mutual distance.

With the assumed form of the flow, the incompressibility condition is automatically fulfilled,

$$\nabla \cdot \vec{v} = \nabla_x v_x(y) = 0.$$

The advective acceleration vanishes likewise,

$$(\vec{v} \cdot \nabla)\vec{v} = v_x(y)\,\nabla_x v_x(y)\,\hat{e}_x = 0.$$

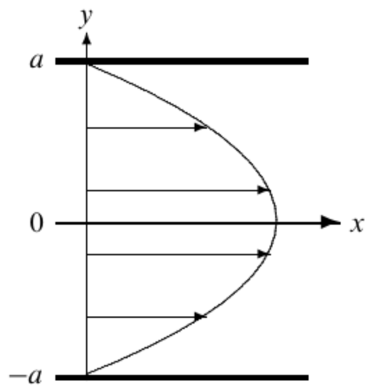

Figure 16: Characteristic parabolic steady velocity profile between the infinitely extended plates with distance $d = 2a$. Figure taken from Ref. [3].

The Navier–Stokes equation (16.1) then takes the form

$$\nabla p = \eta\, \hat{e}_x\, \nabla^2 v_x(y). \tag{129}$$

From the y- and z-components of this equation we get $\nabla_y p = \nabla_z p = 0$, implying that the pressure cannot depend on y and z, or in other words, $p = p(x)$, and the x-component of the above equation becomes

$$\frac{dp(x)}{dx} = \eta\, \frac{d^2 v_x(y)}{dy^2}. \tag{130}$$

The left-hand side depends only on x and the right-hand side only on y, and this is only possible if both sides take the same constant value independent of both x and y. Denoting the common value by $-G$, we may immediately solve each of the equations

$$\frac{dp}{dx} = -G \qquad \text{and} \qquad \eta\, \frac{d^2 v_x}{dy^2} = -G$$

with the result

$$p = p_0 - Gx, \qquad v_x = -\frac{G}{2\eta} y^2 + Ay + B. \tag{131}$$

Here p_0, A, and B are integration constants. The only freedom left in the planar flow problem lies in these constants, which are fixed by the boundary conditions of the specific flow configuration.

boundary conditions. Let the plates be positioned a distance $d = 2a$ apart, for example at $y = -a$ and $y = a$. Applying the no-slip boundary conditions $v_x(-a) = v_x(a) = 0$ to the general solution, we obtain $A = 0$ and $B = Ga^2/2\eta$, and the velocity field becomes (see Fig. 16)

$$v_x = \frac{G}{2\eta}\left(a^2 - y^2\right). \tag{132}$$

It has a characteristic parabolic shape with the maximal velocity

$$v_x^{\max} = \frac{Ga^2}{2\eta}$$

in the middle of the gap, $y = 0$.

Driving pressure and discharge rate. For a real system with finite plates of length L in the direction of flow and width W with a tiny mutual distance, $d \ll L, W$, the flow should be reasonably well described by this solution, except near the edges of the plates (see Figure 16.1).

Two global quantities are immediately measurable for a real system with finite plates. One is the pressure drop,

$$\Delta p = p(0) - p(L),$$

along the length of the plate L. It is called the driving pressure, and may be determined by means of a suitable manometer. The other is the total volume ΔV of (incompressible) fluid discharged during a time T. From these we may calculate the pressure gradient G and the total volumetric discharge rate Q,

$$G = \frac{\Delta p}{L}, \qquad Q = \frac{\Delta V}{T}. \tag{133}$$

Apart from edge effects, both G and Q ought to be constants, independent of the chosen length L or the collection time T.

The solution relates these quantities. Integrating the velocity field over the area $A = Wd$, orthogonal to the flow, we find

$$Q = \int_A \vec{v} \cdot d\vec{S} = \int_{-a}^{a} v_x(y)\, W\, dy = \frac{2GWa^3}{3\eta}. \tag{134}$$

From the discharge rate we may calculate the average velocity of the flow,

$$U = \frac{Q}{A} = \frac{Ga^2}{3\eta}. \tag{135}$$

It is not surprising that for fixed pressure gradient the average velocity grows with the plate distance, $d = 2a$, because the friction from the walls becomes less and less important with increasing plate distance. The maximal flow velocity is 50% higher than the average,

$$v_x^{\max} = \frac{Ga^2}{2\eta} = \frac{3}{2}U.$$

Since the velocity changes from zero at the walls to 50% more than the average in the middle between them, the Reynolds number for pressure-driven flow between parallel plates a distance d apart is conventionally defined as

$$\mathrm{Re} = \frac{Ud}{\nu}. \tag{136}$$

This definition makes sense independent of whether the incompressible flow is laminar or turbulent, because the discharge rate Q, and thus the average velocity $U = Q/A$, is the same everywhere along the channel.

Example. Oil with $\eta = 2 \times 10^{-2}\,\mathrm{Pa\,s}$ and $\rho_0 = 800\,\mathrm{kg\,m^{-3}}$ between plates $d = 1\,\mathrm{cm}$ apart is driven by a pressure drop $\Delta p = 10^3\,\mathrm{Pa}$ over a distance $L = 1\,\mathrm{m}$. The average velocity becomes in this case $U \approx 0.4\,\mathrm{m\,s^{-1}}$, corresponding to a Reynolds number of $\mathrm{Re} \approx 167$.

Drag and power The no-slip condition requires the fluid to be at rest at the bounding walls. It nevertheless exerts a shear stress on the walls, which may be calculated from the planar solution at $y = -a$:

$$\sigma_{\text{wall}} = \sigma_{xy}\big|_{y=-a} = \eta \nabla_y v_x\big|_{y=-a} = Ga. \tag{137}$$

This wall shear stress may be viewed as analogous to static friction between solid bodies.

The total shear force on the walls, called the drag, is obtained by multiplying the wall shear stress by the total wall area of both plates,

$$D = \sigma_{\text{wall}}\, 2LW = GLWd = \Delta p\, A, \tag{138}$$

where $A = Wd$ is the area of the opening through which the fluid flows. That the wall drag should equal the total pressure force on the fluid could have been foreseen because the momentum of the fluid between the plates is constant in steady flow, implying that the total force on the fluid,

$$F_x = -D + \Delta p\, A,$$

must vanish.

The rate of work, or power, of the external pressure forces is similarly obtained:

$$P = \int_A \vec{v} \cdot \Delta p\, d\vec{S} = \Delta p \int_A \vec{v} \cdot d\vec{S} = \Delta p\, Q = \frac{2G^2a^3}{3\eta}\, WL. \tag{139}$$

Since the fluid is at rest at the bounding walls, the wall shear stresses perform no work. The work performed on the fluid by the driving pressure is not lost but dissipated into heat by the viscous friction. If the walls are insulating, this heat is washed along with the flow and leaves the channel through the exit.

5.2 Gravity-driven planar flow

Gravity may also drive the flow between parallel plates if they are inclined an angle θ to the horizon. We choose again a coordinate system with the x-axis in the direction of flow and the y-axis orthogonal to the plates. Assuming constant gravity, the gravitational field is also inclined an angle θ to the negative y-axis. See left panel of Fig. 17.

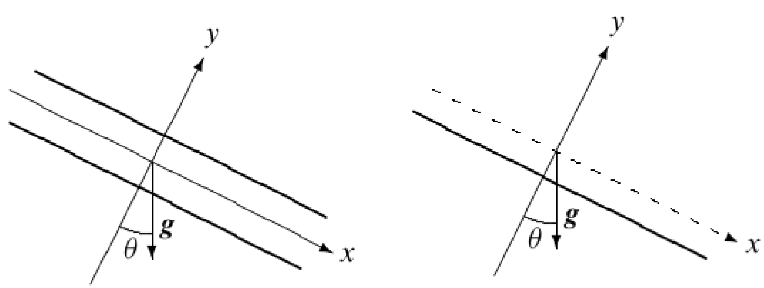

Figure 17: Left: Parallel plates inclined an angle θ to the horizon. Right: Flow with an open surface on a plate inclined an angle θ to the horizon. Figures taken from Ref. [3].

The y- and z-components of the Navier–Stokes equation take the form

$$\nabla_y p = -\rho_0 g_0 \cos\theta, \qquad \nabla_z p = 0. \tag{140}$$

The solution is

$$p = p_0(x) - \rho_0 g_0 y \cos\theta,$$

where $p_0(x)$ is an arbitrary function of x. Inserting this into the x-component of (16.1) we get

$$\frac{1}{\rho_0}\frac{dp_0(x)}{dx} = g_0 \sin\theta + \nu \frac{d^2 v_x(y)}{dy^2}. \tag{141}$$

As in the preceding section it follows that $\nabla_x p_0(x)$ is a constant, and assuming that there is no pressure difference between the ends of the plates at $x = 0$ and $x = L$, it follows that $p_0(x)$ must also be constant, $p_0(x) = p_0$. Applying the no-slip boundary conditions,

$$v_x(-a) = v_x(a) = 0,$$

the complete solution becomes

$$v_x = \frac{g_0 \sin\theta}{2\nu}\left(a^2 - y^2\right), \qquad p = p_0 - \rho_0 g_0 y \cos\theta. \tag{142}$$

Here p_0 is the constant pressure in the central plane $y = 0$. The pressure is for all θ simply the hydrostatic pressure in a constant field of gravity of strength $g_0 \cos\theta$.

5.3 Inclined flow with an open surface

A liquid layer of constant thickness a flowing down an inclined plate with an open surface is another example of purely gravity-driven flow (see right panel of Fig. 17). On the inclined plate, $y = -a$, the no-slip condition again demands $v_x(-a) = 0$, whereas on the open surface, $y = 0$, the pressure must be constant and the stress must vanish,

$$\sigma_{xy} = \eta \frac{dv_x}{dy} = 0.$$

Since these boundary conditions are fulfilled, the velocity and pressure are given by the same expressions as above.

The velocity profile now has its maximum at the open surface. The average velocity is

$$U = \frac{1}{a}\int_{-a}^{0} v_x(y)\, dy = \frac{g_0 a^2 \sin\theta}{3\nu}. \tag{143}$$

The volumetric discharge rate in a swath of width W orthogonal to the flow is

$$Q = UaW.$$

The Reynolds number is naturally defined in terms of the layer thickness,

$$\mathrm{Re} = \frac{Ua}{\nu}. \tag{144}$$

Introducing the convenient velocity parameter

$$U_0 = \left(\tfrac{1}{3}\nu g_0 \sin\theta\right)^{1/3}, \tag{145}$$

we may express all the variables in the convenient form

$$U = \mathrm{Re}^{2/3} U_0, \qquad a = \mathrm{Re}^{1/3}\frac{\nu}{U_0}, \qquad W = \frac{Q}{\nu\,\mathrm{Re}}. \tag{146}$$

Given the inclination angle θ, the viscosity ν, and the discharge rate Q, the Reynolds number Re is the only variable. The length L of the plate is irrelevant as long as $L \gg a$.

Example (Liquid film). A water film of thickness $a = 0.1\,\mathrm{mm}$ flowing down an inclined plate at $\theta = 30°$ has average velocity $U = 16\,\mathrm{mm\,s^{-1}}$ and Reynolds number $\mathrm{Re} = 1.6$. A $10\,\mathrm{mm}$ layer of glycerol flowing down the same slope has much larger average velocity $15\,\mathrm{cm\,s^{-1}}$ but a similar Reynolds number of 1.3.

5.4 Laminar pipe flow

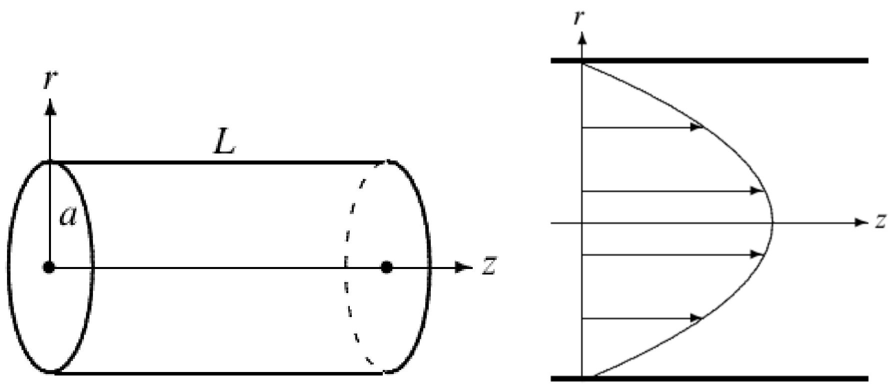

Figure 18: Left: Section of a circular pipe of inner radius a and length L. The pressure is higher at $z = 0$ than at $z = L$, and the pressure drop $\Delta p = p(0) - p(L)$ drives the fluid through the pipe. Right: Velocity profile for laminar flow through a circular pipe. Figures taken from Ref. [3].

An infinitely long circular cylindrical tube is invariant both under translations along its axis and rotations around it (see left panel of Fig. 18). In a coordinate system with the z-axis coincident with the cylinder axis, one velocity field that respects this symmetry is

$$\vec{v} = (0, 0, v_z(r)) = v_z(r)\,\hat{e}_z, \tag{147}$$

where

$$r = \sqrt{x^2 + y^2}$$

is the radial distance from the cylinder axis.

The gradient operator in cylindrical coordinates takes the form

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z}. \tag{148}$$

Applying this operator to the assumed velocity field, we find

$$\nabla \cdot \vec{v} = \hat{e}_r \cdot \partial_r v_z(r) \hat{e}_z = 0, \qquad (\vec{v} \cdot \nabla)\vec{v} = v_z\,\partial_z v_z(r)\,\hat{e}_z = 0.$$

In the absence of gravity, $\vec{g} = 0$, the Navier–Stokes equation now simplifies to

$$\nabla p = \eta\,\hat{e}_z \nabla^2 v_z(r) = \eta\,\hat{e}_z \left(\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr} \right). \tag{149}$$

From the x- and y-components of this equation we get $\nabla_x p = \nabla_y p = 0$, and consequently the pressure can only depend on z, that is,

$$p = p(z).$$

Since the left-hand side depends only on z whereas the right-hand side depends only on r, neither side can depend on r or z. Denoting the common constant value by $-G$, we obtain the ordinary differential equations

$$\frac{dp}{dz} = -G, \qquad \eta \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -G. \tag{150}$$

Integrating, we find

$$p = p_0 - Gz, \qquad v_z = -\frac{G}{4\eta}r^2 + A\log r + B, \tag{151}$$

where p_0, A, and B are integration constants.

Consider now a pipe with inner radius a and diameter $d = 2a$. The fluid velocity cannot be infinite at $r = 0$, implying $A = 0$ in the general solution. The no-slip boundary condition requires that $v_z(a) = 0$, and this fixes the last integration constant to $B = Ga^2/4\eta$, so that the velocity profile becomes

$$v_z = \frac{G}{4\eta}\left(a^2 - r^2\right). \tag{152}$$

See right panel of Fig. 18. As for pressure-driven channel flow, it is parabolic and reaches, as one would expect, its maximal value

$$v_z^{\max} = \frac{Ga^2}{4\eta}$$

at the center of the pipe.

Driving pressure and discharge rate Two global quantities are—as for channel flow—immediately measurable for a pipe of finite length L. One is the pressure drop between entry and exit,

$$\Delta p = p(0) - p(L),$$

and the other is the total volume ΔV of (incompressible) fluid discharged during a time T. From these we may calculate the pressure gradient G and the total volumetric discharge rate Q,

$$G = \frac{\Delta p}{L}, \qquad Q = \frac{\Delta V}{T}. \tag{153}$$

Apart from edge effects, both G and Q should be constant when the pipe is sufficiently long and the discharge is collected over a sufficiently long time. We shall return to this point in the discussion of entry length below.

The volumetric discharge rate may immediately be calculated by integrating the velocity field over the cross-section,

$$A = \pi a^2,$$

of the pipe,

$$Q = \int_A \vec{v}\cdot d\vec{S} = \int_0^a v_z(r)\,2\pi r\,dr = \frac{G}{4\eta}\pi\int_0^a (a^2 - r^2)\,2r\,dr. \tag{154}$$

Carrying out the integral, this becomes the famous *Hagen–Poiseuille law*, relating the driving pressure gradient and the discharge rate,

$$Q = \frac{\pi a^4}{8\eta}G. \tag{155}$$

As could have been expected, the discharge rate grows linearly with the pressure gradient, and inversely with viscosity. The dramatic fourth-power growth with radius could of course have been deduced from dimensional arguments since it is the only missing factor. It expresses the fairly common observation that much more water flows through a wider pipe for a given driving pressure because the shear wall stress becomes less able to hold back the fluid.

The velocity of the flow averaged over the cross-section of the pipe may be calculated from the rate of discharge,

$$U = \frac{Q}{\pi a^2} = \frac{Ga^2}{8\eta}. \tag{156}$$

The maximal velocity at the center of the pipe is twice the average velocity,

$$v_z^{\max} = 2U.$$

The central flow must be faster than for channel flow, because in a pipe there is less room in the central region to compensate for the slow-down of fluid at the pipe wall.

Reynolds number The Reynolds number should be understood as a dimensionless characterization of the ratio of advective to viscous forces in terms of the speed and geometry of the general flow and becomes

$$\mathrm{Re} = \frac{Ud}{\nu}. \tag{157}$$

Empirically pipe flow remains laminar until turbulence sets in at a Reynolds number between 2,000 and 4,000, with 2,300 as a "nominal" value for smooth pipes. At that point the otherwise linear relationship between volume discharge Q and pressure gradient G becomes nonlinear. For a smooth pipe under very carefully controlled conditions, the transition to turbulence can in fact be delayed until a Reynolds number on the order of 100,000. Above that value the flow is so sensitive to disturbances that it becomes practically impossible to avoid turbulence, even if the Poiseuille solution is believed to be formally stable toward infinitesimal perturbations (in the rz-plane) for all Reynolds numbers [**?**].

Example (Aortic flow). Human blood is not a particularly Newtonian fluid, but its viscosity may approximately be taken to be $\eta = 2.7 \times 10^{-3}\,\mathrm{Pa\,s}$ and its density near that of water. The stroke volume of the resting heart is about $70\,\mathrm{cm}^3$, and with a resting heart rate of 60 beats per minute, the aortic blood flow rate (averaged over a heartbeat) is $Q \approx 70\,\mathrm{cm}^3\,\mathrm{s}^{-1}$. Since the aortic root diameter is $2a \approx 25\,\mathrm{mm}$, the average blood velocity becomes $U \approx 14\,\mathrm{cm\,s}^{-1}$ and the Reynolds number $\mathrm{Re} \approx 1{,}300$, well below the turbulent region. The pressure gradient becomes $G \approx 20\,\mathrm{Pa\,m}^{-1}$, showing that the pressure drop in the large arteries is small compared to systolic blood pressure, $p \approx 120\,\mathrm{mmHg} \approx 16{,}000\,\mathrm{Pa}$.

Example (Hypodermic syringe). A hypodermic syringe has a cylindrical chamber with a diameter of about $2b = 1\,\mathrm{cm}$ and a hollow needle with an internal diameter of about $2a = 0.5\,\mathrm{mm}$. During an injection, about $5\,\mathrm{cm}^3$ of the liquid (here assumed to be water) is gently pressed through the needle in a time $\Delta t = 10\,\mathrm{s}$, such that the volume rate is about $Q = 0.5\,\mathrm{cm}^3\,\mathrm{s}^{-1}$. The average fluid velocity in the needle becomes $U \approx 2.5\,\mathrm{m\,s}^{-1}$, corresponding to a Reynolds number $\mathrm{Re} \approx 1{,}300$, which is in the laminar region below the onset of turbulence. One is thus justified in using the Poiseuille solution for the flow through the needle. The pressure gradient necessary to drive this flow is found from (16.32) and becomes rather large, $G \approx 3.3\,\mathrm{bar\,m}^{-1}$. For a needle of length

$L \approx 5\,\mathrm{cm}$, the pressure drop becomes $\Delta p \approx 0.16\,\mathrm{bar} \approx 16{,}000\,\mathrm{Pa}$, about the same as the systolic blood pressure. The pressure drop in the syringe chamber can be completely ignored, because the chamber's diameter is 20 times that of the needle.

Example (Water pipe). Household water supply has to reach the highest floor in apartment buildings with pressure "to spare." Pressures must therefore be of the order of bars when water is not tapped. The typical discharge rate from a kitchen faucet is around $Q \approx 100\,\mathrm{cm}^3\,\mathrm{s}^{-1}$, leading to an average velocity in a half-inch pipe of about $U \approx 0.8\,\mathrm{m\,s}^{-1}$. The Reynolds number becomes about $\mathrm{Re} \approx 10{,}000$, which is well inside the turbulent regime. The pressure gradient calculated from the Hagen–Poiseuille law, $G \approx 160\,\mathrm{Pa\,m}^{-1}$, is for this reason untrustworthy.

5.5 Laminar cylindric flow

Infinitely extended coaxial cylinders are both translationally symmetric along the rotation axis and rotationally symmetric around it. In cylindrical coordinates a maximally symmetric circulating flow takes the form

$$\vec{v} = v_\phi(r)\,\hat{e}_\phi, \tag{158}$$

where $\hat{e}_\phi$ is the tangential unit vector and r the distance from the axis. The field lines are concentric circles.

Again we turn to the nabla operator in cylindrical coordinates to calculate the tensor gradient of the velocity field. Since there is no z-dependence we find by means of the relation

$$\partial_\phi \hat{e}_\phi = -\hat{e}_r,$$

$$\nabla\vec{v} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}\right)\left(v_\phi(r)\hat{e}_\phi\right) = \hat{e}_r\hat{e}_\phi \frac{dv_\phi}{dr} - \hat{e}_\phi\hat{e}_r \frac{v_\phi}{r}. \tag{159}$$

The trace of this tensor yields the divergence that vanishes,

$$\mathrm{Tr}[\nabla\vec{v}] = \nabla\cdot\vec{v} = 0,$$

because $\nabla\vec{v}$ only has off-diagonal components in the cylindrical basis. This is in agreement with the elementary observation that streamlines neither diverge nor converge in this flow. Dotting the tensor gradient from the left with $\vec{v}$, the advective acceleration becomes

$$(\vec{v}\cdot\nabla)\vec{v} = \vec{v}\cdot(\nabla\vec{v}) = -\hat{e}_r\,\frac{v_\phi^2}{r}. \tag{160}$$

One should not be surprised: the centripetal acceleration in a circular motion with velocity v_ϕ is indeed directed radially inwards and of size v_ϕ^2/r. Finally, dotting the tensor gradient from the left with ∇, we obtain the Laplacian:

$$\nabla^2\vec{v} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}\right)\left(\hat{e}_r\hat{e}_\phi \frac{dv_\phi}{dr} - \hat{e}_\phi\hat{e}_r \frac{v_\phi}{r}\right) = \hat{e}_\phi \frac{d^2 v_\phi}{dr^2} + \hat{e}_\phi \frac{1}{r}\frac{dv_\phi}{dr} - \hat{e}_\phi \frac{v_\phi}{r^2}. \tag{161}$$

In the absence of gravity, the Navier–Stokes equation (16.1) then becomes

$$-\rho_0 \hat{e}_r \frac{v_\phi^2}{r} = -\nabla p + \eta\,\hat{e}_\phi \frac{d}{dr}\left(\frac{1}{r}\frac{d(rv_\phi)}{dr}\right), \tag{162}$$

where we in the last term have rewritten the Laplacian in a convenient way.

Projecting it on the three cylindrical basis vectors, $\hat{e}_r$, $\hat{e}_\phi$, and $\hat{e}_z$, we obtain

$$-\rho_0 \frac{v_\phi^2}{r} = -\frac{\partial p}{\partial r}, \tag{163}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \eta \frac{d}{dr}\left(\frac{1}{r}\frac{d(rv_\phi)}{dr}\right), \tag{164}$$

$$0 = -\frac{\partial p}{\partial z}. \tag{165}$$

The first equation expresses that the radial pressure gradient must deliver the centripetal force required by the circular motion of the fluid. The last equation shows that the pressure is independent of z, and from the second equation we see by differentiation with respect to ϕ that $\partial^2 p/\partial\phi^2 = 0$. This means that p can at most be linear in ϕ, that is, of the form $p = p_0(r) + p_1(r)\phi$. But here we must require $p_1 = 0$, for otherwise the pressure would have different values for $\phi = 0$ and $\phi = 2\pi$, and that is impossible. The pressure does not depend on ϕ but only on r, and thus it disappears completely from (164).

With the pressure out of the way, the integration of (164) has become almost trivial, with the general result

$$v_\phi = Ar + \frac{B}{r}, \tag{166}$$

where A and B are integration constants. Inserting this into (163) and integrating over r, we find the pressure

$$\frac{p}{\rho_0} = C + \frac{1}{2}A^2r^2 - \frac{1}{2}\frac{B^2}{r^2} + 2AB\log r, \tag{167}$$

where C is a third integration constant.

The Couette solution Suppose the fluid is contained between two long coaxial material cylinders with radii a and $b > a$. We shall for simplicity only study the case where the outer cylinder is held fixed and the inner cylinder rotates like a spindle with constant angular velocity Ω. The boundary conditions,

$$v_\phi(a) = a\Omega, \qquad v_\phi(b) = 0,$$

then determine A and B, and the velocity profile becomes

$$v_\phi = \frac{\Omega a^2}{r}\frac{b^2 - r^2}{b^2 - a^2}. \tag{168}$$

The velocity field decreases monotonically from its value $a\Omega$ at the inner cylinder to zero at the outer.

The pressure is found to be

$$p^* = p_0 + \frac{1}{2}\rho_0\left(\frac{\Omega a^2}{b^2 - a^2}\right)^2\left(r^2 - \frac{b^4}{r^2} + 4b^2\log\frac{b}{r}\right), \tag{169}$$

where p_0 is its value at the outer cylinder. The pressure increases monotonically toward p_0.

Stress, torque, and power The velocity gradient shows that the only non-vanishing shear stress component is

$$\sigma_{\phi r} = \eta\left(\hat{e}_\phi \cdot (\nabla \vec{v}) \cdot \hat{e}_r + \hat{e}_r \cdot (\nabla \vec{v}) \cdot \hat{e}_\phi\right) = \eta\left(\frac{dv_\phi}{dr} - \frac{v_\phi}{r}\right), \tag{170}$$

which upon insertion of the Couette solution becomes

$$\sigma_{\phi r} = -2\eta\Omega \frac{a^2 b^2}{b^2 - a^2} \frac{1}{r^2}. \tag{171}$$

It represents the viscous friction between the layers of circulating fluid, and the sign is negative because the fluid *outside* the radius r acts as a brake on the motion of the fluid *inside*.

In order to maintain the steady rotation of the inner cylinder (of length L) it is necessary to act on it with a moment of force or torque (and with an opposite moment on the outer cylinder). More generally, multiplying the shear stress $\sigma_{\phi r}$ with the moment arm r and the area $2\pi r L$ of the cylinder at r, we obtain the torque with which the fluid inside r acts on the fluid outside,

$$\mathcal{M}_z = r(-\sigma_{\phi r})\, 2\pi r L = 4\pi\eta\Omega L \frac{a^2 b^2}{b^2 - a^2}. \tag{172}$$

We could have foreseen that $\mathcal{M}_z$ would be independent of r from angular momentum conservation. In steady flow, the total angular momentum of a layer of fluid contained between any two cylindrical surfaces is constant and there is no transport of angular momentum through the cylindrical surfaces because they are parallel with the velocity. Consequently the total moment of force has to vanish, implying that the moment acting on the inside of the layer must be equal and opposite to the moment that is acting on the outside of the layer.

The rate of work that must be done to keep the inner cylinder rotating is obtained by multiplying the stress on the fluid, $-\sigma_{\phi r}$ at $r = a$, by the velocity $a\Omega$ and the area of the cylinder,

$$P = (-\sigma_{\phi r})\, a\Omega\, 2\pi a L = \mathcal{M}_z \Omega = 4\pi\eta\Omega^2 L \frac{a^2 b^2}{b^2 - a^2}. \tag{173}$$

Since the kinetic energy of the fluid is constant in steady flow and since no fluid enters or leaves the system, this must equal the rate of energy dissipation in the fluid.

6 Creeping flow: the Stokes flow

At low Reynolds number, Re $\ll 1$, the advective acceleration can be left out of the Navier–Stokes equations for incompressible flow, so that they for steady flow become

$$\nabla p = \eta \nabla^2 \vec{v}, \qquad \nabla \cdot \vec{v} = 0. \tag{174}$$

This approximation is usually called *Stokes flow*. Gravity, $\vec{g} = -\nabla\Phi$, has for simplicity been left out but is easy to include by replacing p by $p + \rho_0 \Phi$. From the divergence of the first equation it follows that the pressure must satisfy the Laplace equation,

$$\nabla^2 p = 0.$$

Creeping flow is mathematically (and numerically) much easier to handle than general flow because of the absence of non-linear terms that tend spontaneously to break the natural symmetry of the solutions in time as well as space, with turbulence as the extreme result. The linearity of the creeping flow equations may sometimes be used to express solutions to complicated flow problems as linear superpositions of simpler solutions.

6.1 Creeping flow around a solid ball

A solid spherical ball of radius a moving at constant speed U through a viscous fluid is the centerpiece of creeping flow, going back to Stokes in 1851. Provided the Reynolds number

$$\mathrm{Re} = \frac{2aU}{\nu} \tag{175}$$

is sufficiently small, the pressure and velocity fields must satisfy the Stokes equations in the rest frame of the ball.

The first step in obtaining the solution is to recognize that the perfect spherical symmetry of the sphere is only broken by the asymptotic velocity vector $\vec{U}$, and that the velocity field and the pressure must be linear in $\vec{U}$, as discussed above. Since the only other vector at play in the formulation of the problem is the radius vector $\vec{x}$ from the center of the sphere, the scalar pressure must be proportional to $\vec{U} \cdot \vec{x}$. In spherical coordinates with $\vec{U} = U\hat{e}_z$, we have

$$\vec{U} \cdot \vec{x} = rU \cos\theta,$$

leading to a pressure field of the form

$$p = \eta U \cos\theta \, q(r), \tag{176}$$

where $q(r)$ is a function of only the radial distance $r = |\vec{x}|$. The explicit factor η is included because, as argued above, the pressure must be proportional to the viscosity.

The velocity field must similarly be a linear combination,

$$\vec{v} = A(r)\vec{U} + B(r)(\vec{U} \cdot \vec{x})\vec{x},$$

of the vectors $\vec{U}$ and $(\vec{U} \cdot \vec{x})\vec{x}$ with coefficients that can only depend on r. Using that $\vec{x} \cdot \hat{e}_r = r$ and $\vec{x} \cdot \hat{e}_\theta = 0$, together with $\vec{U} \cdot \hat{e}_r = U\cos\theta$ and $\vec{U} \cdot \hat{e}_\theta = -U\sin\theta$, it follows that the spherical field components must be of the form

$$v_r = U\cos\theta \, f(r), \qquad v_\theta = -U\sin\theta \, g(r), \qquad v_\phi = 0, \tag{177}$$

where $f(r)$ and $g(r)$ are functions of r only.

Generic solution

Having determined the general form of the solution, we must now insert it into the Stokes equations and solve the resulting ordinary differential equations. There are several ways of doing this, but since the calculations can easily become messy we shall attempt to follow a fairly direct path of minimal effort.

Using that the Laplacian of the pressure has to vanish, we obtain from the spherical expression for the scalar Laplacian,

$$\nabla^2 p = \eta U \cos\theta \left(\frac{d^2 q}{dr^2} + \frac{2}{r}\frac{dq}{dr} - \frac{2q}{r^2} \right) = 0. \tag{178}$$

Requiring the expression in parentheses to vanish,

$$\frac{d^2 q}{dr^2} + \frac{2}{r}\frac{dq}{dr} - \frac{2q}{r^2} = 0, \tag{179}$$

we have obtained an ordinary second-order differential equation that is homogeneous in r. One may verify that the two linearly independent solutions are $q \sim r$ and $q \sim 1/r^2$, so that

$$q = \frac{C}{r^2} + Dr, \tag{180}$$

where C and D are integration constants.

The divergence of the velocity field must vanish. Then we get

$$\nabla \cdot \vec{v} = \cos\theta \left(\frac{df}{dr} + \frac{2f}{r} - \frac{2g}{r} \right) U = 0, \tag{181}$$

leading to

$$g = f + \frac{1}{2} r \frac{df}{dr}. \tag{182}$$

This leaves only the function $f(r)$ to be determined. From the Stokes equation and the radial part of the vector Laplacian, we obtain

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} + \frac{4g}{r^2} - \frac{4f}{r^2} = \frac{dq}{dr}, \tag{183}$$

and inserting q and g from above, we arrive at

$$\frac{d^2 f}{dr^2} + \frac{4}{r}\frac{df}{dr} = -\frac{2C}{r^3} + D. \tag{184}$$

The general solution to this linear second-order equation for f is obtained by standard methods,

$$f = A + \frac{B}{r^3} + \frac{C}{r} + \frac{1}{10} D r^2, \tag{185}$$

where A and B are integration constants. Inserting this into the expression for g, we find

$$g = A - \frac{B}{2r^3} + \frac{C}{2r} + \frac{1}{5} D r^2. \tag{186}$$

The framed formulas constitute the complete generic solution for this problem.

The field

Finally we must apply boundary conditions to determine the four unknown integration constants A, B, C, and D. For $r \to \infty$, the field must approach the homogeneous field $\vec{U}$, implying $f \to 1$ and $g \to 1$, or $A = 1$ and $D = 0$. At the surface of the sphere, $r = a$, the field must vanish because of the no-slip condition, leading to

$$B = \tfrac{1}{2} a^3, \qquad C = -\tfrac{3}{2} a.$$

Putting it all together, the spherical velocity components and the pressure become

$$v_r = \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3} \right) U \cos\theta, \tag{187}$$

$$v_\theta = -\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3} \right) U \sin\theta, \tag{188}$$

$$v_\phi = 0, \tag{189}$$

$$p = -\frac{3}{2} \eta \frac{a}{r^2} U \cos\theta. \tag{190}$$

The pressure is forward–backward asymmetric, highest on the part of the sphere that turns toward the incoming flow ($\theta = \pi$). This asymmetry is in marked contrast to the potential flow solution, for which the pressure is symmetric. The flow pattern is independent of the viscosity of the fluid, whereas the pressure is proportional to η, as expected for creeping flow problems where the boundary conditions do not involve the pressure.

Stokes' law

The asymmetric pressure distribution and the viscous shear stresses create a drag on the sphere in the direction of the asymptotic flow. To find it we first calculate the stress components σ_{rr} and $\sigma_{\theta r}$ from the standard Newtonian stress tensor. From the velocity gradient matrix in spherical coordinates, we obtain

$$\sigma_{rr} = -p + 2\eta\frac{\partial v_r}{\partial r} = \frac{3a}{2r^2}\left(3 - \frac{2a^2}{r^2}\right)\eta U\cos\theta, \tag{191}$$

$$\sigma_{\theta r} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right) = -\frac{3a^3}{2r^4}\eta U\sin\theta. \tag{192}$$

The only other non-vanishing stress components are

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{2a^3}{2r^4}\eta U\cos\theta.$$

At the surface of the sphere, $r = a$, the stress components become

$$\sigma_{rr}\Big|_{r=a} = \frac{3\eta U}{2a}\cos\theta, \qquad \sigma_{\theta r}\Big|_{r=a} = -\frac{3\eta U}{2a}\sin\theta. \tag{193}$$

From these we get the surface stress vector

$$\boldsymbol{\sigma}\cdot\hat{e}_r\Big|_{r=a} = \hat{e}_r\sigma_{rr} + \hat{e}_\theta\sigma_{\theta r} = \frac{3\eta U}{2a}\hat{e}_z. \tag{194}$$

Surprisingly it is of constant magnitude and points everywhere in the direction of the asymptotic flow across the entire surface.

The total reaction force is obtained by multiplying the constant stress vector by the area $4\pi a^2$ of the sphere. Since the force is proportional to U, it is a pure drag,

$$D = 6\pi\eta a U. \tag{195}$$

This is the famous *Stokes' law* from 1851.

Terminal velocity

Although Stokes' law has been derived in the rest frame of the sphere, it is also valid in the rest frame of the asymptotic fluid. The terminal velocity of a falling solid sphere may be obtained by equating the force of gravity (minus buoyancy) with the Stokes drag,

$$(\rho_1 - \rho_0)\frac{4}{3}\pi a^3 g_0 = 6\pi\eta a U, \tag{196}$$

where g_0 is the gravitational acceleration and ρ_1 is the average density of the sphere. Solving for U, we find

$$U = \frac{2}{9}\left(\frac{\rho_1}{\rho_0} - 1\right)\frac{a^2 g_0}{\nu}, \tag{197}$$

where $\nu = \eta/\rho_0$ is the kinematic viscosity.

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