Hydrostatics

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3] books

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC3. Definir el concepto de presión desde una perspectiva microscópica.
- OC4. Enunciar el teorema de Pascal y explicar su significancia.
- OC5. Reconocer las distintas unidades que se usan para medir la presión.
- OC6. Identificar las condiciones físicas necesarias para describir un medio usando la ecuación de estado politrópica.
- OC7. Enunciar el principio de Arquímides y estudiar a partir de él las condiciones de estabilidad de cuerpos que flotan.
- OC8. Definir el concepto de tensión superficial, tanto desde la perspectiva de una densidad de energía superficial como desde las fuerzas por unidad de longitud.

1.2 Objetivos específicos procedimentales:

- OP3. Deducir, a partir del teorema de Bernoulli, la ecuación de equilibrio hidrostático global y local.
- OP4. Describir las condiciones de densidad y presión de fluidos incompresibles y compresibles en equilibrio hidrostático.
- OP5. Deducir las ecuaciones del interior de un cuerpo gravitante esférico en equilibrio hidrostático.
- OP6. Deducir la ecuación de Lane-Emden y aplicarla para estudiar el caso de estrellas hipotéticas que obedecen la ecuación de estado politrópica.
- OP7. Aplicar la definición de tensión superficial para describir el fenómeno de capilaridad.

1.3 Objetivos específicos actitudinales:

- OA2. Distinguir y enumerar algunas de las ventajas más importantes de la aproximación continua frente a la aproximación discreta (estadística) de los medios materiales.

2 Pleliminaries

2.1 Density

Fluid is the generic name for liquids and gases. A gas completely fills a closed container, but a liquid does not. The (matter) density of a fluid is defined as

$$\rho \equiv \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}. \tag{1}$$

In continuum physics, density is understood as a macroscopic average over a sufficiently large number of microscopic constituents. This requires a separation of scales: the microscopic length scale (molecular spacing or mean free path) must be much smaller than the macroscopic scale on which density varies appreciably.

In addition to the usual *mass density* ρ, one often encounters:

- **Number density** n: number of particles per unit volume, with $\rho = nm$ for particles of mass m.
- **Energy density** ρ_E: energy per unit volume, relevant in relativistic systems and cosmology.

Table 1: Densities of some common substances

Material	**Density (kg/m^3)**	**Material**	**Density (kg/m^3)**
Air (1 atm, 20°C)	1.20	Iron ($Z = 26$), steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper ($Z = 29$)	8.9×10^3
Ice	0.92×10^3	Silver ($Z = 47$)	10.5×10^3
Water	1.00×10^3	Lead ($Z = 82$)	11.3×10^3
Seawater	1.03×10^3	Mercury ($Z = 80$)	13.6×10^3
Blood	1.06×10^3	Gold ($Z = 79$)	19.3×10^3
Glycerin	1.26×10^3	Platinum ($Z = 78$)	21.4×10^3
Concrete	2.0×10^3	Iridium ($Z = 77$)	22.56×10^3
Aluminum	2.7×10^3	Osmium ($Z = 76$)	22.59×10^3

2.2 Astrophysical objects

Stars. The Sun has an average density of $\sim 1.4 \times 10^3\,\text{kg/m}^3$, comparable to water, but its core density reaches $\sim 1.5 \times 10^5\,\text{kg/m}^3$. In stellar structure, density profiles $\rho(r)$ enter the equations of hydrostatic equilibrium, balancing gravitational forces against pressure gradients.

Compact objects.

- *White dwarfs:* electron-degenerate matter with densities of order $10^9 - 10^{10}\,\text{kg/m}^3$.
- *Neutron stars:* neutron-degenerate matter with densities up to $10^{18}\,\text{kg/m}^3$, comparable to nuclear matter density.

In these cases, density determines the degeneracy pressure that counteracts gravity.

Galactic environments. In addition to stars and compact objects, density plays a central role in galactic dynamics:

- *Galactic center:* The Milky Way hosts a supermassive black hole, Sgr A*, surrounded by a dense stellar cluster. The average stellar mass density in the inner parsec reaches values of order $\rho \sim 10^9\, M_\odot\, \mathrm{pc}^{-3}$, corresponding to $\sim 4 \times 10^{-11}\, \mathrm{kg/m^3}$. Although extremely dilute compared to ordinary matter, this is enormously high relative to the mean cosmic density.

- *Local Group:* On megaparsec scales, the mean density of galaxies in the Local Group is of order $\rho \sim 10^{-25}\, \mathrm{kg/m^3}$, only a few times the critical density of the Universe. This low density governs the dynamics of galaxy interactions and the eventual merging of the Milky Way and Andromeda.

Cosmological scales. The mean density of the Universe is extraordinarily low. The critical density is defined as

$$\rho_c = \frac{3H_0^2}{8\pi G}, \tag{2}$$

where $H_0 \approx 68$ km/s/Mpc is the Hubble constant. Numerically, $\rho_c \sim 10^{-26}\, \mathrm{kg/m^3}$. Cosmological structure formation is often described by the density contrast

$$\delta = \frac{\Delta\rho}{\rho},$$

which measures deviations from the average density.

Astrophysical fluids. In accretion disks, stellar winds, and interstellar clouds, density controls emission and cooling processes, and it determines the onset of instabilities such as the Jeans instability, which occurs when self-gravity overcomes pressure support.

Black holes. Although a black hole is not composed of matter in the usual sense, one can define an *effective average density* as the mass divided by the volume of a sphere with the Schwarzschild radius $R_s = 2GM/c^2$. This yields

$$\rho_{\mathrm{BH}} = \frac{3c^6}{32\pi G^3 M^2}. \tag{3}$$

Interestingly, this density decreases with increasing mass:

- A stellar-mass black hole ($M \sim 10\, M_\odot$) has an effective density of order $10^{19}\, \mathrm{kg/m^3}$, even greater than nuclear matter.

- A supermassive black hole ($M \sim 10^9\, M_\odot$) has an effective density as low as $\sim 10^3\, \mathrm{kg/m^3}$, comparable to water.

This illustrates that black holes are not "dense" in the ordinary sense, but are instead defined by the curvature of spacetime.

Table 2 lists characteristic densities of ordinary and astrophysical systems. Density connects microscopic physics (molecular interactions, degeneracy pressure, nuclear matter properties) with macroscopic astrophysical phenomena (stellar equilibrium, gravitational collapse, cosmic expansion). It thus provides a unifying parameter linking laboratory scales to the most extreme environments in the Universe.

Table 2: Characteristic densities of materials and astrophysical systems.

System	**Density (kg/m^3)**
Air (1 atm, 20°C)	~ 1
Water	1.0×10^3
Average density of the Sun	1.4×10^3
Solar core	1.5×10^5
White dwarf	$10^9 - 10^{10}$
Neutron star	10^{18}
Stellar-mass black hole ($\sim 10 M_\odot$)	$\sim 10^{19}$
Supermassive black hole ($\sim 10^9 M_\odot$)	$\sim 10^3$
Galactic center (Milky Way)	$\sim 4 \times 10^{-11}$
Local Group (galaxies)	$\sim 10^{-25}$
Critical density of Universe	$\sim 10^{-26}$

3 Hydrostatic equilibrium

In the absence of external forces or internal heat sources, fluids eventually reach *hydrostatic equilibrium*, where motion ceases and fields remain constant in time. Examples: atmosphere, oceans, planetary and stellar interiors.

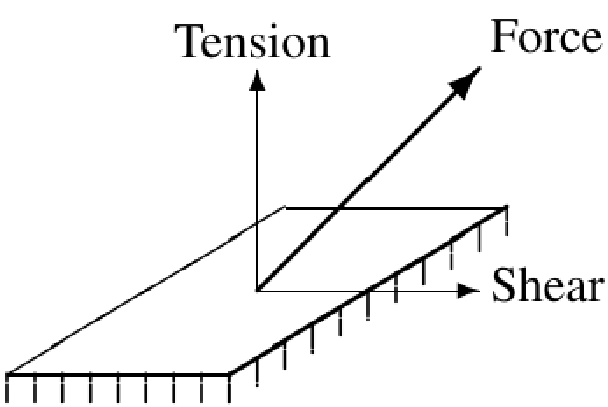

Figure 1: The force acting on the material underneath a small patch of a surface can always be resolved into a perpendicular pressure force and a tangential shear force. The pressure is positive if the force is directed toward the patch, and negative if it (as here) is directed away from it. Figure taken from Ref. [3].

Mechanical equilibrium in a continuum involves a balance between:

- *Contact forces* (short-range, acting through surfaces).
- *Body forces* (long-range, e.g. gravity).

Contact forces act across *contact surfaces* and can be decomposed into (see Fig. 1):

- **Normal component**: aligned with surface normal.
  - Same orientation $\rightarrow$ *tension force*.

- Opposite orientation $\rightarrow$ *pressure force.*

- **Tangential component**: parallel to surface $\rightarrow$ *shear force* or *traction force.*

3.1 Pressure

A fluid at rest cannot sustain shear stresses or tensile stresses. Any attempt to impose a tangential force on the fluid is immediately relieved by flow, so the intermolecular forces in a static fluid can only transmit *normal stresses.* These stresses act equally in all directions and always tend to compress objects immersed in the fluid.

Thus, the force exerted by a static fluid on the surface of an object is always *perpendicular* (normal) to that surface. This isotropic normal stress is what we call the **pressure** of the fluid.

The pressure p in a static fluid is defined as the normal force per unit area:

$$p = \frac{d\mathcal{F}_\perp}{dS}, \tag{4}$$

where $d\mathcal{F}_\perp$ is the component of force acting normal to a surface of area dS. Its SI unit is the newton per square meter ($\mathrm{N/m^2}$), which in 1971 was given the special name *pascal* (Pa). Thus,

$$1\,\mathrm{Pa} = 1\,\mathrm{N/m^2}.$$

Earlier units of pressure that are still in use include:

- The **bar**, defined as $1\,\mathrm{bar} = 10^5\,\mathrm{Pa}$.
- The **standard atmosphere**, defined as

$$1\,\mathrm{atm} = 101{,}325\,\mathrm{Pa},$$

  which corresponds approximately to the average air pressure at sea level.

In many practical contexts, other pressure units may appear:

- **Torr**: commonly used in vacuum physics, defined as $1\,\mathrm{Torr} = \frac{1}{760}\,\mathrm{atm} \approx 133.3\,\mathrm{Pa}$.
- **mmHg**: pressure exerted by a column of mercury of height $1\,\mathrm{mm}$, historically used in meteorology and medicine.

Remark. Pressure is a scalar quantity in fluids at rest, because it acts equally in all directions. In fluids in motion, however, stresses may include both pressure and shear contributions, and the concept of a *stress tensor* becomes necessary for a complete description.

Applications. Understanding pressure units and conversions is essential in astrophysics and geophysics: for example, the pressure in the solar core reaches $\sim 10^{16}\,\mathrm{Pa}$, while the pressure at the center of the Earth is about $3.6 \times 10^{11}\,\mathrm{Pa}$. These extreme values highlight the wide range of conditions described by the same fundamental concept of pressure.

3.2 The pressure field

The (flat) vector surface elements is defined as the product of its area dS and the unit vector $\hat{n}$in the direction of the normal to the surface,

$$d\vec{S} = \hat{n}\, dS. \tag{5}$$

By universal convention, the normal of a closed surface is taken to point *outward*, so that the enclosed volume lies on the negative side.

A fluid at rest cannot sustain shear forces, so all contact forces on a surface must act along the normal at every point of the surface. The force exerted by the material at the *positive* side of the surface element dS (near $\mathbf{x}$) on the material at the *negative* side is written as

$$d\vec{\mathcal{F}} = -p(\vec{x})\, d\vec{S}, \tag{6}$$

where $p(\vec{x})$ is the pressure field. Convention dictates that a positive pressure exerts a force directed *toward* the material on the negative side of the surface element, and this explains the minus sign. A negative pressure that pulls at a surface is sometimes called a *tension*. The total pressure force acting on any oriented surface S is obtained by adding all the little vector contributions from each surface element,

$$\vec{\mathcal{F}} = \int_S -p(\vec{x})\, d\vec{S}. \tag{7}$$

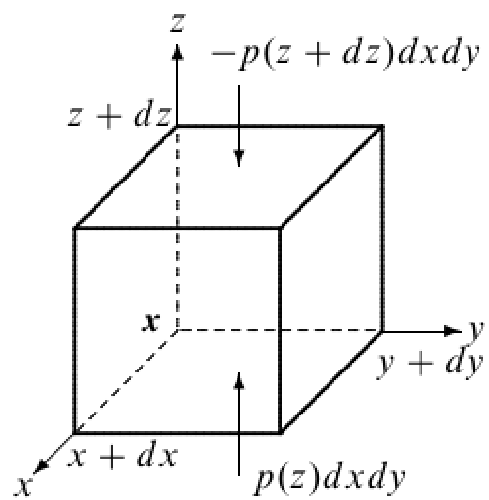

Figure 2: Pressure forces in the z-direction on a material particle in the shape of a rectangular box with sides dx, dy, dz. Figure taken from Ref. [3]

A material particle in the fluid is like any other body subject to pressure forces from all sides, but being infinitesimal it is possible to derive a general expression for the resultant force. Let us consider a material particle in the shape of a small rectangular box with sides dx, dy, and dz, such that it has a volume $dV = dx\, dy\, dz$. Since the pressure is slightly different on opposite sides of the box, the z-component of total pressure force becomes

$$d\mathcal{F}_z = \big(p(x,y,z) - p(x,y,z+dz)\big)\, dx\, dy \;\approx\; -\frac{\partial p(x,y,z)}{\partial z}\, dx\, dy\, dz. \tag{8}$$

Doing the same for the other coordinate directions we obtain the total pressure force

$$d\vec{\mathcal{F}} \;=\; -\left(\hat{\imath}\,\frac{\partial p}{\partial x} + \hat{\jmath}\,\frac{\partial p}{\partial y} + \hat{k}\,\frac{\partial p}{\partial z}\right) dV \;=\; -\vec{\nabla} p\, dV. \tag{9}$$

3.3 Hydrostatic Equilibrium

In equilibrium, the net force on a differential fluid element vanishes:

$$d\vec{F}_e + d\vec{F}_p = 0, \tag{10}$$

where $d\vec{F}_e$ denotes external (volumetric) and $d\vec{F}_p$ the pressure forces. For fluids, forces are of two kinds:

- **Surface forces**: e.g. pressure and viscous stresses.
- **Volumetric forces**: e.g. gravity, electromagnetic, or inertial (fictitious) forces.

3.3.1 Global hydrostatic equlibrium

Integrating the previous expression gives that the total force in mechanical equilibrium mus vanish,

$$\vec{F}_e + \vec{F}_p = \int_V \vec{f}_e\, dV - \oint_S p\, d\vec{S} = 0. \tag{11}$$

When $\vec{f}_e$ is the force of gravity it corresponds to the weight of the fluid body, and the force due to pressure is called the buoyancy force.

Advantages.

- Provides the overall balance of forces for the entire fluid system.
- Very useful in cases with high symmetry, which simplifies the evaluation of integrals.
- Can be applied directly to obtain macroscopic results such as Archimedes' principle.
- Avoids the need to solve differential equations when the geometry is simple.

Disadvantages.

- Requires explicit knowledge of the force density $\rho\mathbf{g}$ and the pressure field p in order to evaluate the integrals.
- Insufficient for complex geometries where the pressure distribution is not known a priori.
- Does not provide the local pressure at each point in the fluid.
- Less general than the local (differential) form of the hydrostatic equilibrium equation.

3.3.2 Local hydrostatic equilibrium

In static fluids only normal pressures remain, since a fluid at rest cannot sustain shear stresses. Thus,

$$d\vec{F}_e - \vec{\nabla} p \, dV = 0. \tag{12}$$

Defining the volumetric density of external forces as

$$\vec{f}_e = \frac{d\vec{F}_e}{dV}, \tag{13}$$

and the effective force density

$$\vec{f}^* = \vec{f}_e - \vec{\nabla} p, \tag{14}$$

we obtain that in hydrostatic equilibrium $\vec{f}^* = 0$, leading to the basic equation of hydrostatics

$$\vec{f}_e = \vec{\nabla} p. \tag{15}$$

For the case of $\vec{f}_e$ being gravity force,

$$\rho \, \vec{g} = \vec{\nabla} p, \tag{16}$$

where $\vec{g}$ is the gravitational field.

Moreover, since $\vec{\nabla} \times \vec{\nabla} p = 0$, one has

$$\vec{\nabla} \times \vec{f}_e = 0, \tag{17}$$

which implies that $\vec{f}_e$ derives from a potential,

$$\vec{f}_e = -\vec{\nabla} \mathcal{H}. \tag{18}$$

Hence, hydrostatic equilibrium exists only when the volumetric forces acting on a fluid are conservative. This is the case for gravitational, electrostatic, and inertial forces.

Open Question. What occurs for a charged fluid placed in a magnetic field?

3.3.3 Pressure Force and Gauss' Theorem

The total pressure force on a body of volume V can be obtained in two equivalent ways:

- By integrating the pressure forces acting on its surface S.
- By integrating the forces on all its constituent material particles.

This leads to the relation

$$\oint_S p \, d\vec{S} = \int_V \vec{\nabla} p \, dV. \tag{19}$$

This identity is a direct consequence of **Gauss' theorem**, which relates the surface integral of a scalar field $p(\mathbf{x})$ over a closed surface S to the volume integral of its gradient $\vec{\nabla} p(\vec{x})$ over the enclosed volume V. Hence Gauss' theorem allows us to convert the local equation back into the global one, showing that there is complete mathematical equivalence between the local and global formulations of hydrostatic equilibrium.

3.4 Pascal's principle

Because pressure in a static fluid is the same in all directions, it can be treated as a scalar quantity. This isotropy underlies **Pascal's principle**, which states that any change of pressure applied to a confined fluid is transmitted undiminished throughout the fluid and to the walls of its container. Pascal's principle forms the basis of many hydraulic devices and has wide-ranging applications in both engineering and natural systems.

From Eq. (15), by integrating along a path from point 1 to point 2 inside a fluid, we obtain

$$\int_1^2 \vec{f}_e \cdot d\vec{r} = \int_1^2 \vec{\nabla} p \cdot d\vec{r} = \int_1^2 dp = p_2 - p_1, \tag{20}$$

so that the pressure difference between two points of a fluid in hydrostatic equilibrium depends only on the external forces, such as gravity. If there are no external forces, then $p_2 = p_1$. Moreover,

$$p_2 - p_1 = (p_2 + C) - (p_1 + C) = \int_1^2 \vec{f}_e \cdot d\vec{r}, \tag{21}$$

showing that the addition of a constant C to the pressure at all points of a fluid leaves the difference $p_2 - p_1$ invariant. That is, any pressure shift C applied at one point is accompanied by the same shift at all other points. This statement is known as **Pascal's principle**. Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

This principle introduces a global gauge freedom in hydrostatics. In differential form, and taking into account that $\vec{\nabla} C = 0$, it follows that $\vec{f}_e = \vec{\nabla} p = \vec{\nabla}(p + C)$.

3.4.1 Hydraulic press

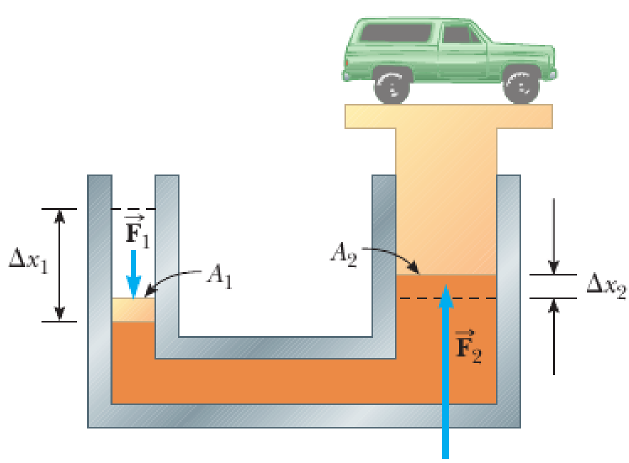

Figure 3: Diagram of a hydraulic press. Figure taken from Ref. [4]

An important application of Pascal's law is the hydraulic press (see Fig. 3). A force F_1 is applied to a small piston of surface area A_1. The pressure is transmitted through an incompressible liquid to a larger piston of area A_2. Since the pressure must be equal on both pistons,

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}. \tag{22}$$

Therefore, the output force is amplified by the factor A_2/A_1.

Because the liquid is incompressible, the displaced volumes satisfy

$$A_1 \Delta x_1 = A_2 \Delta x_2,$$

so that

$$\frac{F_2}{F_1} = \frac{\Delta x_1}{\Delta x_2}.$$

Multiplying both sides by displacements shows that the work done is conserved:

$$F_1 \Delta x_1 = F_2 \Delta x_2.$$

The hydraulic press allows a small input force to generate a much larger output force, while conserving energy. This principle is widely used in hydraulic brakes, jacks and car lifts.

3.5 Hydrostatics in constant gravity

In a flat-Earth coordinate system, the constant field of gravity is $\vec{g}(\vec{r}) = (0, 0, -g_0)$ for all $\vec{r}$. If the external force is gravity, then:

$$\vec{f}_e = \frac{d\vec{F}}{dV} = \frac{dm}{dV}\vec{g} = \rho\vec{g},$$

so that,

$$\vec{\nabla}p - \rho\vec{g} = 0. \tag{23}$$

It follows that

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho g_0. \tag{24}$$

The first two equations express that the pressure does not depend on x and y but only on z. It also shows that, independently of the shape of a fluid container, the pressure will always be the same at a given depth (in constant gravity).

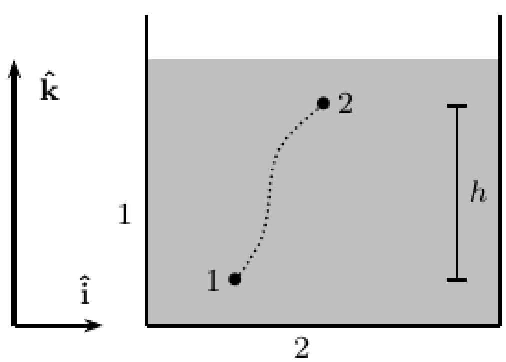

Figure 4: The pressure difference between points 1 and 2 inside a fluid in the Earth's gravity field depends only on their vertical separation. Figure taken from Ref. [1].

For the special case of constant density, $\rho(z) = \rho_0$, the last equation may immediately be integrated. The line integral between points 1 and 2 in Fig. 4, with ρ and $\vec{g}$ constant, is:

$$p_2 - p_1 = \int_1^2 \mathbf{f}_e \cdot d\mathbf{r} = \int_1^2 \rho\mathbf{g} \cdot d\mathbf{r} = -\rho g \int_1^2 dz = -\rho g(z_2 - z_1) = -\rho g h. \tag{25}$$

Thus, the pressure inside a liquid increases with depth:

$$p_1 = p_2 + \rho g h, \tag{26}$$

an expression identical to $(p_1 + C) = (p_2 + C) + \rho g h$. This equation shows that if a force is applied to the free surface of a liquid (for example, with a piston), the pressure increases equally at every point of the fluid. Therefore, the *difference in pressure* between two points depends only on their vertical separation.

3.5.1 Communicating vessels

From the last expression in Eq.4 (24), it follows that $p = \rho g z + C$. At the free surface of the liquid $(z = h)$ the pressure is atmospheric p_0 , so $C = p_0 + \rho g h$. Therefore

$$p = p_0 + \rho\, g(h - z). \tag{27}$$

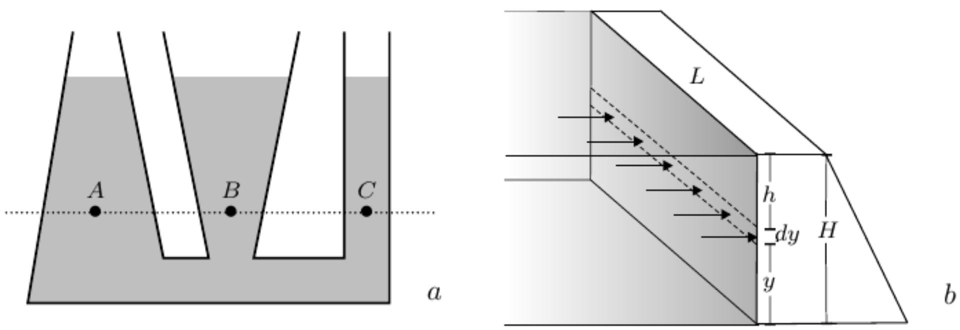

Figure 5: *a*) In communicating vessels, the pressure at points A, B, and C, located at the same height, is the same. *b*) The horizontal hydrostatic force exerted by the water in a dam on a vertical wall differential element of height dy and width L. Figure taken from Ref. [1].

It is therefore true that points at the same depth have the same pressure, independently of the shape of the container (see left panel of Fig. 5). This gives validity to the so-called *principle of communicating vessels* and resolves the so-called *hydrostatic paradox*, according to which—erroneously— the pressure inside a fluid would depend on the shape of the container. As seen here, the principle of communicating vessels is another result derived from the above expression for gravitational external forces $\vec{f}_e$.

According to the erroneous interpretation, the pressure at point B would be greater than at A, since more liquid lies above B than above A. The pressure at C would then be intermediate between those at A and B. The supposed paradox is that in fact

$$P_A = P_B = P_C.$$

This Equation shows that pressure is a scalar quantity, which ensures that *at every point inside the fluid the pressure is the same in all directions.*

3.5.2 The Force on a Dam

Water is filled to a height H behind a dam of width L (see right panel of Fig. 5). Let's determine the resultant force exerted by the water on the dam.

Because pressure increases with depth, the total force on a submerged surface cannot be calculated by simply multiplying the area by a single pressure value. Instead, we must account for the variation of pressure with depth, which requires the use of integration.

Consider a vertical y-axis with $y = 0$ at the bottom of the dam. We divide the face of the dam into narrow horizontal strips located at a height y above the bottom, as illustrated by the dashed strip in the figure.

The atmospheric pressure acting on both sides of the wall does not exert a net force, so it will not be considered. At a height y measured from the bottom of the dam, the pressure has the value

$$p = \rho g h = \rho g(H - y).$$

The horizontal force that the water exerts on the differential portion of wall of area $L\,dy$ is

$$dF = p\,dA = pL\,dy = \rho g L(H - y)\,dy,$$

so that the total horizontal net force, directed to the right, due to the water is

$$F = \rho g L \int_0^H (H - y)\,dy = \tfrac{1}{2}\rho g L H^2.$$

3.5.3 Pressure gauges

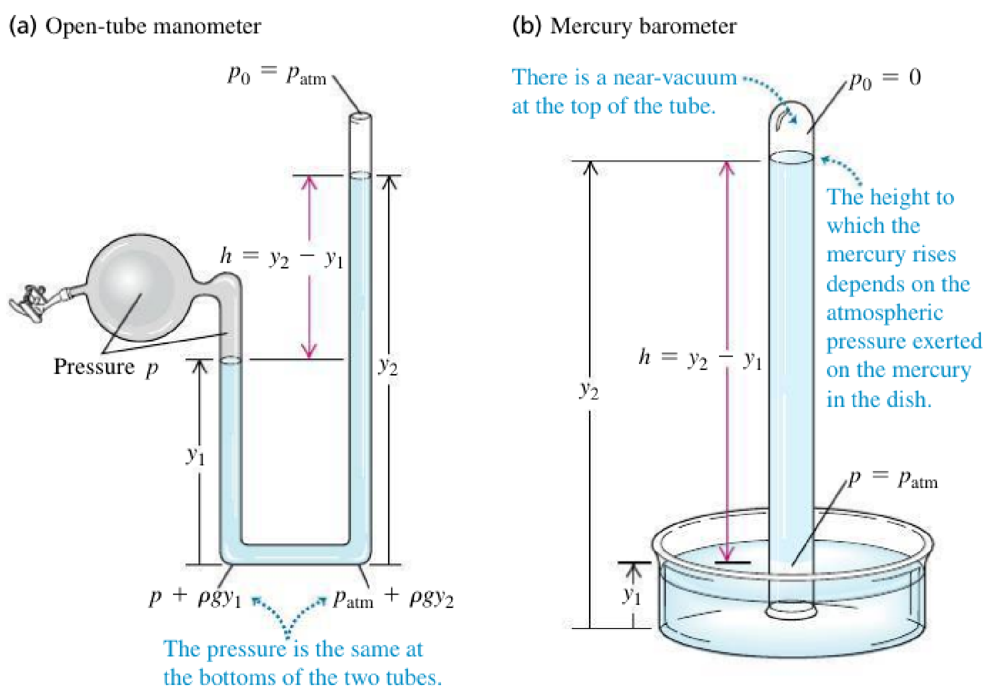

Figure 6: Two types of pressure gauge. Figure taken from Ref. [4]

The simplest pressure gauge is the open-tube *manometer* (left panel of Fig. 6). The U-shaped tube contains a liquid of density ρ, often mercury or water. The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\rm atm}$. The pressure at the bottom of the tube due to the fluid in the left column is $p + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\rm atm} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

$$p + \rho g y_1 = p_{\rm atm} + \rho g y_2,$$

$$p - p_{\rm atm} = \rho g (y_2 - y_1) = \rho g h.$$

Here, p is the *absolute pressure*, and the difference $p - p_{\rm atm}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

Another common pressure gauge is the *mercury barometer*. It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (right panel of Fig. 6). . The space above the mercury column contains only mercury vapor; its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. It follows that

$$p_{\rm atm} = p = 0 + \rho g (y_2 - y_1) = \rho g h.$$

So the height h of the mercury column indicates the atmospheric pressure $p_{\rm atm}$.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "inches of mercury" or "millimeters of mercury" (abbreviated mm Hg). A pressure of 1 mm Hg is called *1 torr*, after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of g, which varies with location, so the pascal is the preferred unit of pressure.

Open Question A manometer tube is partially filled with water. Oil (which does not mix with water) is poured into the left arm of the tube until the oil-water interface is at the midpoint of the tube as shown in Fig. 7. Both arms of the tube are open to the air. Find a relationship between the heights h_{oil} and h_{water}.

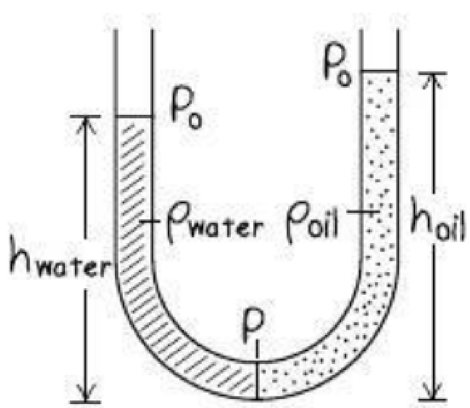

Figure 7: A manometer tube partially filled with water. Figure taken from Ref. [4].

Problems

1. The gate CD in the left panel of Fig. 8 can rotate about C. What horizontal force F, applied at a height h, is required to prevent the gate from rotating under the action of the water on its left, if its width is L?

2. What is the force per unit length required to prevent the cylinder in the right panel of Fig. 8 from rolling under the action of the water located to its left?

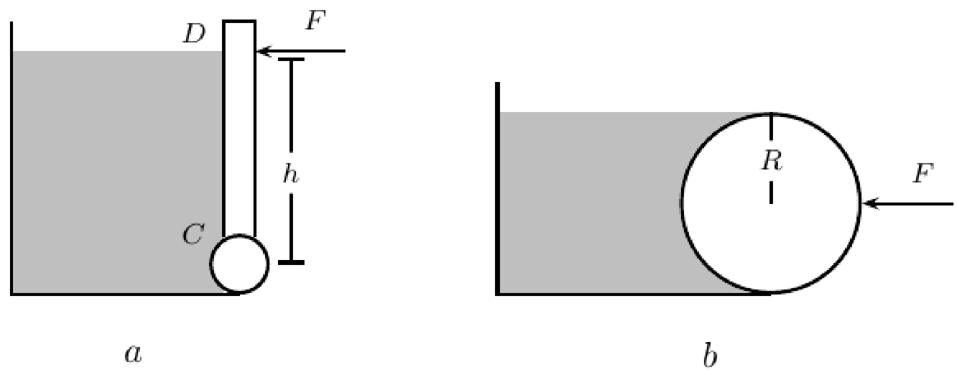

Figure 8: Hydrostatic force on a flat gate in a dam (a) and on the surface of a horizontal cylinder (b). Figure taken from Ref. [1].

3.6 Equation of state

The local equation of hydrostatic equilibrium with an externally given gravitational field is not sufficient by itself; we need a relation between pressure and density. Thermodynamics provides this relationship through the *equation of state*, which relates density ρ, pressure p, and absolute temperature T:

$$F(\rho, p, T) = 0. \tag{28}$$

This relation is valid for any macroscopic amount of homogeneous isotropic fluid in thermodynamic equilibrium. In continuum physics, where conditions vary from point to point, it is assumed that each material particle is in thermodynamic equilibrium with its surroundings, so the equation of state holds locally:

$$F(\rho(x), p(x), T(x)) = 0. \tag{29}$$

3.6.1 The Ideal Gas Law

The *ideal gas law* is the most famous equation of state, often attributed to Clapeyron (1834). In terms of the volume V of a mass M of gas and the number of moles $n = M/M_{\text{mol}}$, it is written as

$$pV = nR_{\text{mol}}T, \tag{30}$$

where $R_{\text{mol}} = 8.31447\,\text{J}\,\text{K}^{-1}\,\text{mol}^{-1}$ is the *universal molar gas constant*. Since $\rho = M/V = nM_{\text{mol}}/V$, the ideal gas law can be expressed in the local form:

$$p = R\rho T, \quad R = \frac{R_{\text{mol}}}{M_{\text{mol}}}. \tag{31}$$

Remarks. The specific gas constant R varies for different gases (see Table 3). Extensions of the ideal gas law include corrections for excluded molecular volume and intermolecular forces. The ideal gas law also applies to mixtures of gases if the average molar mass is used.

	$10^3 M_{\text{mol}}$ (kg mol^{-1})	$10^{-3} R$ (J K^{-1} kg^{-1})
H_2	2.0	4.157
He	4.0	2.079
Ne	20.2	0.412
N_2	28.0	0.297
O_2	32.0	0.260
Ar	39.9	0.208
CO_2	44.0	0.189
Air	29.0	0.287

Table 3: The molar mass, M_{mol}, and the specific gas constant, $R = R_{\text{mol}}/M_{\text{mol}}$, for a few gases.

The microscopic origin of the ideal gas law is as follows. The equipartition theorem tells us that the average kinetic energy of a molecule of mass m is

$$\frac{1}{2}m\langle v^2\rangle = \frac{1}{2}m\langle v_x^2\rangle + \frac{1}{2}m\langle v_y^2\rangle + \frac{1}{2}m\langle v_z^2\rangle = \frac{3}{2}k_B T, \tag{32}$$

from which

$$p = \frac{1}{3}\rho\langle v^2\rangle = \rho\frac{k_B T}{m} = \rho\frac{N_A k_B T}{N_A m} = \rho\frac{R_{\text{mol}} T}{M_{\text{mol}}}. \tag{33}$$

3.6.2 Case: Isothermal atmosphere

Let us assume a constant temperature, $T(x) = T_0$, and combine the equation of hydrostatic equilibrium with the ideal gas law. Then

$$\frac{dp}{dz} = -\rho g_0 = -\frac{g_0}{RT_0}p. \tag{34}$$

This is an ordinary differential equation for the pressure, and using the initial condition $p = p_0$ for $z = 0$, we find the solution

$$p = p_0 e^{-z/h_0}, \qquad h_0 = \frac{RT_0}{g_0} = \frac{p_0}{\rho_0 g_0}. \tag{35}$$

In the last step we have again used the ideal gas law at $z = 0$ to show that the expression for h_0 is identical to the incompressible atmospheric scale height.

In the isothermal atmosphere the pressure thus decreases exponentially with height on a characteristic length scale again set by $h_0 \approx 8728\,\text{m}$ calculated for 1 atm and 25°C. The pressure at the top of Mount Everest ($z = 8848\,\text{m}$) is now finite and predicted to be 368 hPa.

3.6.3 Barotropic equation of state

Sometimes there exists a so-called *barotropic* relationship between density and pressure,

$$F(\rho(x), p(x)) = 0, \tag{36}$$

which does not depend on the local temperature $T(x)$.

Polytropic relation

The first law of thermodynamics in differential form reads

$$T\,ds = de + p\,d\left(\frac{1}{\rho}\right), \tag{37}$$

where s is the entropy, e the specific internal energy, p the pressure, and ρ the mass density. For an isentropic process we have $ds = 0$, so

$$de = -p\,d\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\,d\rho. \tag{38}$$

For an ideal gas, $e = c_v T$ and $p = \rho R T$. Thus

$$c_v\,dT = \frac{p}{\rho^2}\,d\rho, \qquad T = \frac{p}{\rho R}. \tag{39}$$

Eliminating T we obtain

$$c_v\,d\left(\frac{p}{\rho R}\right) = \frac{p}{\rho^2}\,d\rho, \tag{40}$$

or equivalently

$$\frac{c_v}{R}\left(\frac{dp}{\rho} - \frac{p}{\rho^2}\,d\rho\right) = \frac{p}{\rho^2}\,d\rho. \tag{41}$$

Now, recalling that

$$\gamma \equiv \frac{c_p}{c_v} = 1 + \frac{R}{c_v},$$

we can simplify the relation to

$$\frac{dp}{p} = \gamma\,\frac{d\rho}{\rho}. \tag{42}$$

Integrating both sides gives

$$\ln p = \gamma \ln \rho + \text{const}, \tag{43}$$

which leads to the **polytropic relation**

$$p = C\,\rho^{\gamma}, \tag{44}$$

where C is a constant (the *polytropic constant*), and γ is the *polytropic index* (see Table 4).

3.7 Isobars, Isoclines, and Potentials

For a fluid of density ρ —in general dependent on position— in a gravitational field $\vec{g}$:

$$-\rho\vec{g} + \vec{\nabla}p = 0 \qquad \text{or:} \qquad \rho\vec{\nabla}\Phi + \vec{\nabla}p = 0. \tag{45}$$

Taking the curl of the second equation,

$$\rho\vec{\nabla}\times\vec{\nabla}\Phi + \vec{\nabla}\rho\times\vec{\nabla}\Phi + \vec{\nabla}\times\vec{\nabla}p = 0. \tag{46}$$

Since $\vec{\nabla}\times\vec{\nabla}\Phi \equiv 0$ and $\vec{\nabla}\times\vec{\nabla}p \equiv 0$, it follows that, in a fluid of variable density, this condition holds:

$$\vec{\nabla}\rho\times\vec{\nabla}\Phi = 0 \qquad \text{or:} \qquad \vec{\nabla}\rho\times\vec{g} = 0. \tag{47}$$

	γ	$10^{-3}c_p$
H_2	1.41	14.30
He	1.63	5.38
Ne	1.64	1.05
N_2	1.40	1.04
O_2	1.40	0.91
Ar	1.67	0.52
CO_2	1.30	0.82
Air	1.40	1.00

Table 4: Table of the adiabatic index and the specific heat at constant pressure ($c_p = \gamma R/(\gamma-1)$) for a few nearly ideal gases. Units: $\mathrm{J\,K^{-1}\,kg^{-1}}$.

This means that, point by point, the direction of the gravitational field lines coincides with that of the density gradient, or, equivalently, that the surfaces of equal density (isoclines) and the gravitational equipotentials are coincident.

Considering

$$\vec{\nabla}\Phi + \frac{\vec{\nabla}p}{\rho} = 0,$$

and its curl is taken, it follows that for a compressible fluid

$$\vec{\nabla}\rho \times \vec{\nabla}p = 0, \tag{48}$$

from which it is concluded that the surfaces of constant pressure (isobars) and those of constant density coincide. From this equation it follows that the pressure is an *exclusive functional* of density: $p = p(\rho)$. Indeed,

$$\vec{\nabla}\rho \times \vec{\nabla}p = \vec{\nabla}\rho \times \left(\frac{dp}{d\rho}\vec{\nabla}\rho\right) = \frac{dp}{d\rho}\vec{\nabla}\rho \times \vec{\nabla}\rho = 0. \tag{49}$$

In an analogous way, it can be shown that Φ is a functional of ρ: $\Phi = \Phi(\rho)$. Thus, the surfaces of Φ, ρ and p constant are the same. This result is valid regardless of the form of the gravitational field. In particular, if $\vec{g} = -\hat{k}g$, the surfaces Φ, ρ and p will be horizontal planes. Outside the Earth's mass, assumed spherical, it is true that

$$\mathbf{g} = -\frac{GM\hat{\mathbf{r}}}{r^2}, \qquad \Phi = -\frac{GM}{r},$$

so the three surfaces are spherical and concentric.

3.8 Pressure potential

Since the gravitational field is conservative and can be obtained from the gradient of the gravitational potential $\Phi(x)$, it follows that

$$\vec{g}(x) = -\vec{\nabla}\Phi(x). \tag{50}$$

In terms of the potential, the hydrostatics equilibrium equation ($\vec{\nabla}p = \rho\vec{g}$) may now be written as

$$\nabla\Phi + \frac{\nabla p}{\rho} = 0. \tag{51}$$

In the case of a constant density, $\rho(x) = \rho_0$, we obtain that

$$\Phi^* \equiv \Phi + \frac{p}{\rho_0} \tag{52}$$

is also constant. In flat-Earth gravity, $\Phi = g_0 z$.

It is always possible to integrate the hydrostatic equation for any barotropic fluid with $\rho = \rho(p)$. Introducing the *pressure potential*,

$$w(p) = \int \frac{dp}{\rho(p)}, \tag{53}$$

it turns out that hydrostatic equilibrium may be written as

$$\Phi^* = \Phi + w(p). \tag{54}$$

Φ^* is dubbed the *effective potential*.

Isothermal gas

Under isothermal conditions, the pressure potential of an ideal gas is calculated by means of the ideal gas law:

$$w = \int \frac{RT_0}{p}\, dp = RT_0 \log p. \tag{55}$$

Polytropic fluid

For fluids obeying a polytropic relation, the pressure potential becomes

$$w = \int C\gamma\rho^{\gamma-1}\frac{d\rho}{\rho} = C\frac{\gamma}{\gamma-1}\rho^{\gamma-1} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}. \tag{56}$$

When the fluid is an ideal gas with $p = R\rho T$, this takes the simpler form

$$w = \frac{\gamma}{\gamma-1}RT = c_p T, \qquad c_p = \frac{\gamma}{\gamma-1}R, \tag{57}$$

where c_p is the specific heat at constant pressure.

3.9 Bulk modulus

All fluids compress if the pressure increases, resulting in a decrease in volume or an increase in density. A common way to describe the compressibility of a fluid is by the bulk modulus of elasticity. It is defined the *bulk modulus* as the pressure increase dp per *fractional decrease* in volume, $-dV/V$, or

$$K = \frac{dp}{-dV/V} = \frac{dp}{d\rho/\rho} = \rho\frac{dp}{d\rho}. \tag{58}$$

In the second step we have used the constancy of the mass $M = \rho V$ of the fluid in the volume to derive that $dM = \rho dV + V d\rho = 0$, from which we get $-dV/V = d\rho/\rho$.

The above definition makes immediate sense for a barotropic fluid, where $p = p(\rho)$ is a function of density. For general fluid states it is necessary to specify the conditions under which the bulk modulus is defined, for example whether the temperature is held constant (isothermal) or whether there is no heat transfer (adiabatic or isentropic). Thus, the equation of state for an ideal gas implies that the isothermal bulk modulus is

$$K_T = \left(\rho \frac{\partial p}{\partial \rho}\right)_T = p, \tag{59}$$

where the index—as commonly done in thermodynamics—indicates that the temperature T is held constant. Similarly, for an isentropic ideal gas obeying the polytropic relation that the *isentropic bulk modulus* becomes

$$K_S = \left(\rho \frac{dp}{d\rho}\right)_S = \gamma p, \tag{60}$$

where the index indicates that the entropy S is held constant. It is larger than the isothermal bulk modulus by a factor of $\gamma > 1$, because adiabatic compression also increases the temperature of the gas, which further increases the pressure.

The definition of the bulk modulus (and the above equation) shows that it is measured in the same units as pressure, for example pascals, bars, or atmospheres. The bulk modulus is actually a measure of *incompressibility*, because the larger it is, the greater is the pressure increase that is needed to obtain a given fractional increase in density. The inverse bulk modulus $\beta = 1/K$ may be taken as a measure of *compressibility.*

The bulk modulus for water at standard conditions is approximately 2100 MPa (310,000 psi), or 21 000 times the atmospheric pressure. For air at standard conditions, B is equal to 1 atm. In general, B for a gas is equal to the pressure of the gas. To cause a 1% change in the density of water a pressure of 21 MPa (210 atm) is required. This is an extremely large pressure needed to cause such a small change; thus liquids are often assumed to be incompressible. For gases, if significant changes in density occur, say 4%, they should be considered as compressible; for small density changes under 3% they may also be treated as incompressible. This occurs for atmospheric airspeeds under about 100 m/s (220 mph), which includes many airflows of engineering interest: air flow around automobiles, landing and take-off of aircraft, and air flow in and around buildings.

Small density changes in liquids can be very significant when large pressure changes are present. For example, they account for "water hammer," which can be heard shortly after the sudden closing of a valve in a pipeline; when the valve is closed an internal pressure wave propagates down the pipe, producing a hammering sound due to pipe motion when the wave reflects from the closed valve or pipe elbows.

The bulk modulus can also be used to calculate the speed of sound in a liquid:

$$c = \sqrt{\left(\frac{dp}{d\rho}\right)_T} = \sqrt{\frac{K_T}{\rho}} \tag{61}$$

This yields approximately 1450 m/s (4800 ft/s) for the speed of sound in water at standard conditions.

3.10 Questions

1. A rubber hose is attached to a funnel, and the free end is bent around to point upward. When water is poured into the funnel, it rises in the hose to the same level as in the funnel, eventhough the funnel has a lot more water in it than the hose does. Why? What supports the extra weight of the water in the funnel?

2. Suppose the door of a room makes an airtight but frictionless fit in its frame. Do you think you could open the door if the air pressure on one side were standard atmospheric pressure and the air pressure on the other side differed from standard by 1%? Explain.

3. At a certain depth in an incompressible liquid, the absolute pressure is p. At twice this depth, will the absolute pressure be equal to $2p$, greater than $2p$, or less than $2p$? Justify your answer.

4. A meteorologist states that the barometric pressure is 28.5 inches of mercury. Convert this pressure to kilopascals.

   (A) 98.6 kPa (B) 97.2 kPa (C) 96.5 kPa (D) 95.6 kPa

5. If the pressure in the air shown in Fig. 9 is increased by 10 kPa, the magnitude of H will be nearest (initially $H = 16$ cm):

   (A) 8.5 cm (B) 10.5 cm (C) 16 cm (D) 24.5 cm

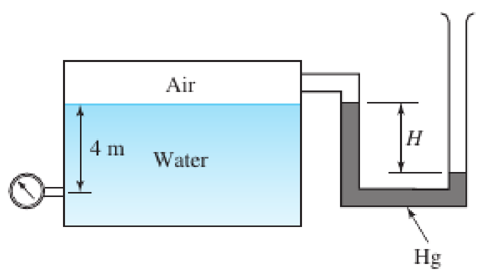

Figure 9: Question 5.

6. The rectangular gate shown in Fig. 10 is 3 m wide. The force P needed to hold the gate in the position shown is nearest:

   (A) 24.5 kN (B) 32.7 kN (C) 98 kN (D) 147 kN

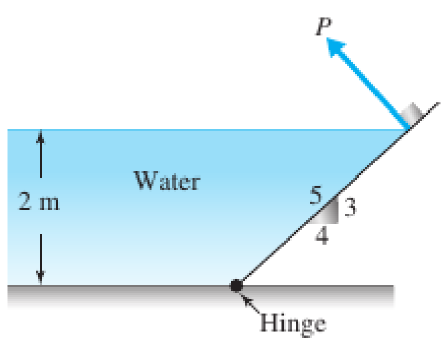

Figure 10: Question 6.

7. A force $P = 300$ kN is needed to just open the gate of Fig. 11 with $R = 1.2$ m and $H = 4$ m. How wide is the gate?

   (A) 2.98 m (B) 3.67 m (C) 4.32 m (D) 5.16 m

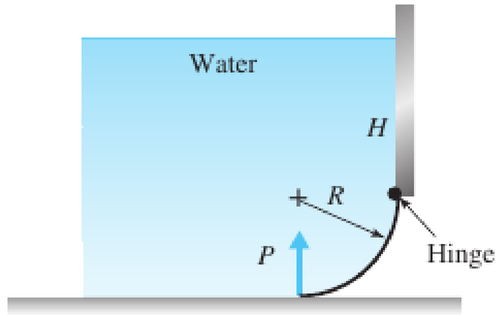

Figure 11: Question 7.

3.11 Problems

1. What is the volume change of $2\,\mathrm{m}^3$ of water at 20°C due to an applied pressure of 10 MPa?

2. Two engineers wish to estimate the distance across a lake. One pounds two rocks together under water on one side of the lake and the other submerges his head and hears a small sound 0.62 s later, as indicated by a very accurate stopwatch. What is the distance between the two engineers?

3. A pressure is applied to 20 L of water. The volume is observed to decrease to 18.7 L. Calculate the applied pressure.

4. **Black Smokers.** Black smokers are hot volcanic vents that emit smoke deep in the ocean floor. Many of them teem with exotic creatures, and some biologists think that life on earth may have begun around such vents. The vents range in depth from about 1500 m to 3200 m below the surface. What is the gauge pressure at a 3200-m deep vent, assuming that the density of water does not vary? Express your answer in pascals and atmospheres.

5. **Oceans on Mars.** Scientists have found evidence that Mars may once have had an ocean 0.500 km deep. The acceleration due to gravity on Mars is 3.71 $\mathrm{m/s^2}$. (a) What would be the gauge pressure at the bottom of such an ocean, assuming it was freshwater? (b) To what depth would you need to go in the earth's ocean to experience the same gauge pressure?

6. (a) Calculate the difference in blood pressure between the feet and top of the head for a person who is 1.65 m tall. (b) Consider a cylindrical segment of a blood vessel 2.00 cm long and 1.50 mm in diameter. What additional outward force would such a vessel need to withstand in the person's feet compared to a similar vessel in her head?

7. In intravenous feeding, a needle is inserted in a vein in the patient's arm and a tube leads from the needle to a reservoir of fluid (density $1050\,\mathrm{kg/m^3}$) located at height h above the arm. The top of the reservoir is open to the air. If the gauge pressure inside the vein is 5980 Pa, what is the minimum value of h that allows fluid to enter the vein? Assume the needle diameter is large enough that you can ignore the viscosity of the fluid.

8. You are designing a diving bell to withstand the pressure of seawater at a depth of 250 m. a) What is the gauge pressure at this depth? (You can ignore changes in the density of the water with depth.) b) At this depth, what is the net force due to the water outside and the

air inside the bell on a circular glass window 30.0 cm in diameter if the pressure inside the diving bell equals the pressure at the surface of the water? (Ignore the small variation of pressure over the surface of the window.)

9. There is a maximum depth at which a diver can breathe through a snorkel tube because as the depth increases, so does the pressure difference, which tends to collapse the diver's lungs. Since the snorkel connects the air in the lungs to the atmosphere at the surface, the pressure inside the lungs is atmospheric pressure. What is the external–internal pressure difference when the diver's lungs are at a depth of 6.1 m (about 20 ft)? Assume that the diver is in freshwater. (A scuba diver breathing from compressed air tanks can operate at greater depths than can a snorkeler, since the pressure of the air inside the scuba diver's lungs increases to match the external pressure of the water.)

10. The lower end of a long plastic straw is immersed below the surface of the water in a plastic cup. An average person sucking on the upper end of the straw can pull water into the straw to a vertical height of 1.1 m above the surface of the water in the cup. a) What is the lowest gauge pressure that the average person can achieve inside his lungs? b) Explain why your answer in part (a) is negative.

11. An electrical short cuts off all power to a submersible diving vehicle when it is 30 m below the surface of the ocean. The crew must push out a hatch of area $0.75\,\mathrm{m}^2$ and weight 300 N on the bottom to escape. If the pressure inside is 1.0 atm, what downward force must the crew exert on the hatch to open it?

12. A tall cylinder with a cross-sectional area $12.0\,\mathrm{cm}^2$ is partially filled with mercury; the surface of the mercury is 8.00 cm above the bottom of the cylinder. Water is slowly poured in on top of the mercury, and the two fluids don't mix. What volume of water must be added to double the gauge pressure at the bottom of the cylinder?

13. A closed container is partially filled with water. Initially, the air above the water is at atmospheric pressure ($1.01 \times 10^5\,\mathrm{Pa}$) and the gauge pressure at the bottom of the water is 2500 Pa. Then additional air is pumped in, increasing the pressure of the air above the water by 1500 Pa. (a) What is the gauge pressure at the bottom of the water? (b) By how much must the water level in the container be reduced, by drawing some water out through a valve at the bottom of the container, to return the gauge pressure at the bottom of the water to its original value of 2500 Pa? The pressure of the air above the water is maintained at 1500 Pa above atmospheric pressure.

14. **Exploring Venus.** The surface pressure on Venus is 92 atm, and the acceleration due to gravity there is $0.894g$. In a future exploratory mission, an upright cylindrical tank of benzene is sealed at the top but still pressurized at 92 atm just above the benzene. The tank has a diameter of 1.72 m, and the benzene column is 11.50 m tall. Ignore any effects due to the very high temperature on Venus. (a) What total force is exerted on the inside surface of the bottom of the tank? (b) What force does the Venusian atmosphere exert on the outside surface of the bottom of the tank? (c) What total inward force does the atmosphere exert on the vertical walls of the tank?

15. **The Great Molasses Flood.** On the afternoon of January 15, 1919, an unusually warm day in Boston, a 17.7 m-high, 27.4 m-diameter cylindrical metal tank used for storing molasses

ruptured. Molasses flooded into the streets in a 5 m-deep stream, killing pedestrians and horses and knocking down buildings. The molasses had a density of 1600 kg/m^3. If the tank was full before the accident, what was the total outward force the molasses exerted on its sides? (Hint: Consider the outward force on a circular ring of the tank wall of width dy and at a depth y below the surface. Integrate to find the total outward force. Assume that before the tank ruptured, the pressure at the surface of the molasses was equal to the air pressure outside the tank.)

16. What is the pressure in the water pipe shown in the figure Fig. 12?

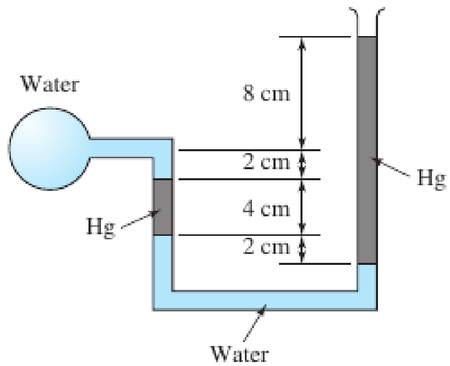

Figure 12: Problem.

17. Determine the force P needed to hold the 4-m wide gate in the position shown in Fig. 13.

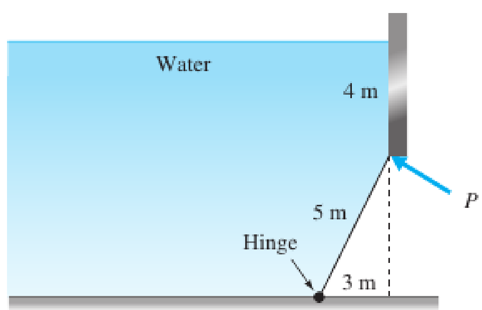

Figure 13: Problem.

18. Find the force P needed to hold the 10-m-long cylindrical object in position as shown in Fig. 14. Here $S \equiv \rho/\rho_{\text{water}}$.

19. Find the force P if the parabolic gate shown in Fig. 15 has a width w.

20. **Hydrostatic atmospheres** For this exercise you will need the following constants: Boltzmann's constant $k_B = 1.380658 \times 10^{-16}$ erg K^{-1}, the atomic mass unit $m = 1.6605 \times 10^{-24}$ g, the solar radius $R_\odot = 6.9599 \times 10^{10}$ cm, the solar mass $M_\odot = 1.98892 \times 10^{33}$ g, the gravitational constant $G = 6.6743 \times 10^{-8}$ cm^3 g^{-1} s^{-2}, the earth radius $R_\oplus = 6.378 \times 10^8$ cm and the earth mass $M_\oplus = 5.974 \times 10^{27}$. You will calculate characterizations of the sun and earth atmospheres, assuming that the ideal gas law to be valid $p = \frac{\rho k_B T}{\mu m}$.

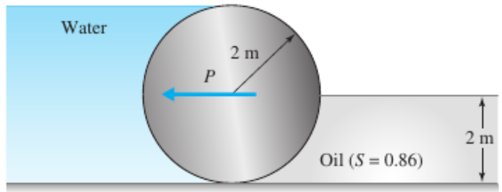

Figure 14: Problem.

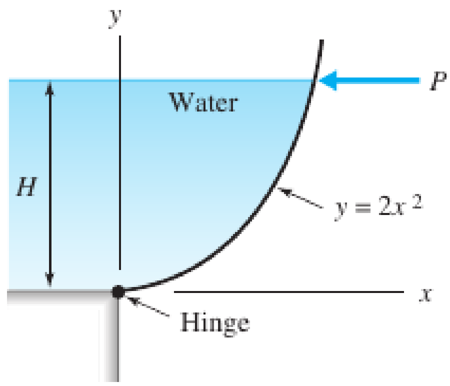

Figure 15: Problem.

i) In addition to the ideal gas assumption, we further assume that pressure and density are related according to

$$p = K\rho^{1+\frac{1}{n}},$$

where K and n are two constants. If $n \to \infty$, what does this imply for the temperature profiles in the atmospheres?

ii) Let's start with the case of the sun's photosphere. You can assume a planar atmosphere and constant T. For the purpose of computing the effect of gravity, you may further assume that the atmosphere has a total mass content that is negligible relative to the mass of the sun and an equally negligible radial extent relative to the radius of the sun (which also justifies the planar assumption). Starting from momentum conservation in the hydrostatic case,

$$\nabla p = f,$$

show that the pressure profile obeys

$$p = p_0 \exp\left[-(z - z_0)\frac{\mu m g}{k_B T}\right],$$

where g is gravitational acceleration.

iii) Identify a measure of "scale height" in this expression for pressure and interpret the meaning of a "scale height" in an atmosphere.

iv) Compute the scale heights for the solar photosphere at temperature $T_\odot = 5770\,\mathrm{K}$ and the earth at $T_\oplus = 300\,\mathrm{K}$. The solar photosphere is highly ionized and has a mean molecular weight $\mu = 0.6$, while the earth atmosphere consists largely of dinitrogen (N_2) and has a mean molecular weight $\mu = 28$. The scale heights for sun and earth should come out as $H = 2.9 \times 10^7$ cm and 9.0×10^5 cm respectively.

v) Mauna Kea in Hawaii hosts a number of the world's most powerful telescopes requiring astronomers to work at an altitude of 4.2 km. What is the implication for the atmospheric *density* at this elevation relative to sea level?

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