Solids at Rest

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3] books.

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC9. Enumerar los distintos regímenes de deformación a los que está sometido un sólido o un fluido viscoelástico.
- OC10. Definir el campo de desplazamientos, el tensor de deformaciones y el tensor de elasticidad.
- OC11. Enunciar la ley de Hooke en toda su generalidad.

1.2 Objetivos específicos procedimentales:

- OP8. Escribir el tensor de esfuerzos y la ley de Cauchy.
- OP9. Determinar la relación entre las fuerzas experimentadas por un elemento de volumen en un medio continuo, incluyendo la presión, y el tensor de esfuerzos.

1.3 Objetivos específicos actitudinales:

- OA3. Reconocer la importancia de la notación, del álgebra y del cálculo tensorial en la representación de las cantidades físicas asociadas con los medios continuos.

2 Stress

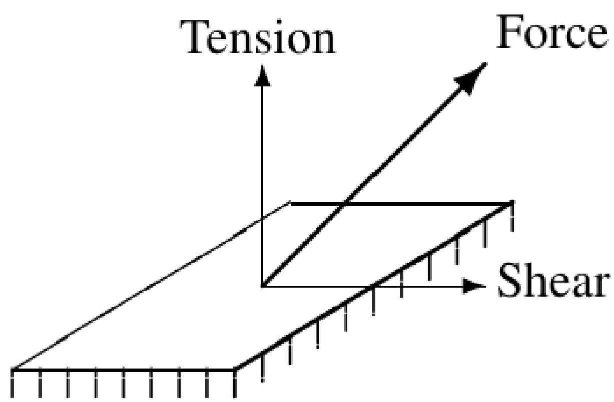
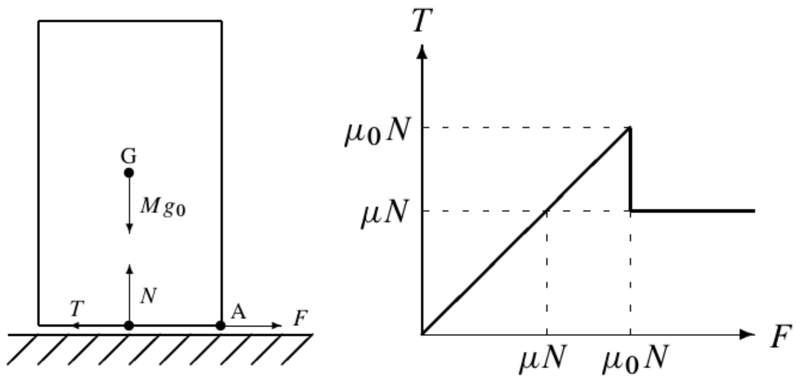


Figure 1: The force on a small piece of a surface can be resolved into a nor- mal pressure force and a tangential shear force. Figure taken from Ref. [3].

- In fluids at rest, the only contact force is pressure, but in solids and viscous fluids in motion, additional *shear stresses* arise—forces acting tangentially to contact surfaces.
- Shear stress is the shear force per unit area and plays a crucial role in maintaining the structural integrity of solids.
- Friction in everyday life is a manifestation of shear stress at contact surfaces between materials.
- Materials are broadly classified as:
  - **Fluids**, which respond to stress by *flowing*.
  - **Solids**, which respond by *deforming*.
- Elastic deformation grows linearly with stress but can lead to permanent plastic deformation or rupture at high stress levels.
- The study of stress applies to all materials—solids, fluids, and intermediate forms—including artificial and exotic substances with special technological uses.
- Contact forces depend not only on spatial position but also on the orientation of the surface; therefore, a more general description using the *stress tensor* is required.
- The stress tensor, consisting of nine components, describes the complete range of contact forces acting within a material.
- Understanding the concept and properties of stress is fundamental to continuum mechanics and central to this chapter.



	μ_0	μ
Glass/glass	0.9	0.4
Rubber/asphalt	0.9	0.7
Steel/steel	0.7	0.6
Metal/metal	0.6	0.4
Wood/wood	0.4	0.3
Steel/ice	0.1	0.05
Steel/teflon	0.05	0.05

Figure 2: Left: Balance of forces on a crate at rest on a horizontal floor. Center: Sketch of tangential reaction (traction) T as a function of applied force F. Up to $F = \mu_0 N$, the traction adjusts itself to the applied force, $T = F$. At $F = \mu_0 N$, the tangential reaction drops abruptly to a lower value, $T = \mu N$, and stays there regardless of the applied force. Right: Typical friction coefficients for various combinations of materials. Figures taken from Ref. [3].

2.1 Friction

- **Friction as a shear force:**
  - Friction between solid bodies is a type of shear force.
  - It enables us to hold, grab, drag, and rub objects without them slipping.
  - Everyday tasks, from stirring coffee to lighting a fire, involve working against friction.
- **Static friction:**
  - Occurs when an object is at rest and resists being set into motion.
  - The static frictional force T balances the applied force F up to a maximum value (see Fig. 2):
    $$T < \mu_0 N,$$
    where μ_0 is the *coefficient of static friction* and N is the normal load.
  - The coefficient μ_0 is dimensionless and typically around 0.5 or higher in everyday materials.
  - Its magnitude depends on the materials in contact and the roughness of their surfaces.
- **Dynamic (kinetic) friction:**
  - Once motion starts, friction still acts but with a smaller magnitude than static friction.
  - The frictional force is proportional to the normal load:
    $$T = \mu N,$$
    where μ is the *coefficient of dynamic friction*, satisfying $\mu < \mu_0$.
  - After motion begins, a smaller force $F = \mu N$ maintains constant speed.

- Work done against dynamic friction is converted into heat:

$$P = FU,$$

where P is the power and U is the velocity.

- Static and dynamic friction are both independent of the size of the contact area, so that a crate on legs is as hard to drag as one without. The best way to diminish the force necessary to drag the crate is to place it on wheels.

- **Comparison:**
  - Static friction prevents motion; dynamic friction resists motion once it starts.
  - Static friction does not produce work, while dynamic friction continuously dissipates energy as heat.

2.2 Stress fields

- Stress fields describe the local distribution of normal and tangential forces acting on a surface.
- Measured in pascals (Pa = N m^{-2}), analogous to pressure.
- Types of stresses:
  - *Tension stress:* acts along the normal and pulls outward.
  - *Pressure stress:* acts along the normal and pushes inward.
  - *Shear stress:* acts tangentially along the surface.

2.2.1 External and internal stresses:

- *External stresses* act between a body and its environment.
- *Internal stresses* act across imaginary surfaces within the body and exist even in the absence of external forces.
- Internal stresses depend on material composition, geometry, temperature, and external loading.
- Rapid temperature changes (e.g., fast cooling) can permanently "freeze" internal stresses, especially in brittle materials like glass.

2.2.2 Tensile strength and yield stress:

- When external forces grow large, materials may deform plastically or fracture.
- *Tensile strength:* the maximum stress a material can withstand before failure.
- *Yield stress:* the stress at which plastic deformation begins.
- *Ductile materials* (e.g., copper) can stretch without breaking; *brittle materials* (e.g., cast iron) fail suddenly.

- Typical tensile strengths:
  - Metals: several hundred megapascals.
  - Carbon fibers: several gigapascals.
  - Carbon nanotubes: up to 50 GPa.

2.3 The stress tensor:

- Stresses at a point are described by nine components σ_{ij} in Cartesian coordinates:

$$\sigma_{ij} = \frac{dF_i}{dS_j}. \tag{1}$$

- These include three normal stresses (σ_{xx}, σ_{yy}, σ_{zz}) and six shear stresses.
- *Cauchy's stress hypothesis* asserts that these nine numbers are in fact all that is needed to determine the force

$$d\vec{F} = (dF_x, dF_y, dF_z)$$

on an arbitrary surface element,

$$d\vec{S} = (dS_x, dS_y, dS_z),$$

according to the rule

$$\begin{aligned} dF_x &= \sigma_{xx} dS_x + \sigma_{xy} dS_y + \sigma_{xz} dS_z, \\ dF_y &= \sigma_{yx} dS_x + \sigma_{yy} dS_y + \sigma_{yz} dS_z, \\ dF_z &= \sigma_{zx} dS_x + \sigma_{zy} dS_y + \sigma_{zz} dS_z. \end{aligned} \tag{2}$$

The components of the force are expressed compactly as

$$dF_i = \sum_j \sigma_{ij}\, dS_j. \tag{3}$$

- The nine stress components form the *stress tensor field* $\sigma_{ij}(x, t)$.
- The stress tensor is written as

$$\vec{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{4}$$

- The stress acting on a surface element $d\vec{S}$ is given by

$$d\vec{F} = \vec{\sigma} \cdot d\vec{S}. \tag{5}$$

- Writing $d\vec{S} = \hat{n}\, dS$, the stress per unit area becomes

$$\frac{d\vec{F}}{dS} = \vec{\sigma} \cdot \hat{n}, \tag{6}$$

called the *stress vector*.

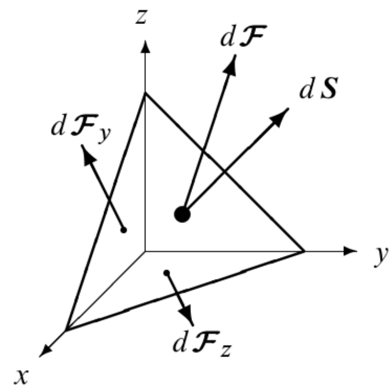


Figure 3: The tiny triangle and its projections form a tetrahedron. The (vector) force acting on the triangle in the xy-plane is called $d\vec{F}_z$, and the force acting on the triangle in the zx-plane is dF_y. The force $d\vec{F}_x$ acting on the triangle in the yz-plane is hidden from view. The force $d\vec{F}$ acts on the skew triangle. Figure taken from Ref. [3].

- Although $\vec{\sigma} \cdot \hat{n}$ behaves like a vector, it is not a true vector field since it depends on the surface orientation.

Proof of Cauchy's stress hypothesis: we take again a surface element in the shape of a tiny triangle (see Fig. 3) with area vector

$$d\vec{S} = (dS_x, dS_y, dS_z).$$

This triangle and its projections on the coordinate planes form together a little body in the shape of a right tetrahedron. Let the external (vector) forces acting on the three triangular faces of the tetrahedron be denoted, respectively,

$$d\vec{F}_x,\ d\vec{F}_y,\ \text{and}\ d\vec{F}_z.$$

Adding the external force $d\vec{F}$ acting on the skew face and a possible volume force $f\, dV$, Newton's Second Law for the small tetrahedron becomes

$$\vec{w}\, dM = \vec{f}\, dV + d\vec{F}_x + d\vec{F}_y + d\vec{F}_z + d\vec{F}, \tag{7}$$

where $\vec{w}$ is the center-of-mass acceleration of the tetrahedron and $dM = \rho\, dV$ its mass.

The volume of the tetrahedron scales like the third power of its linear size, whereas the surface area only scales like the second power. Making the tetrahedron progressively smaller, the body force term and the acceleration term will vanish faster than the surface terms. In the limit of a truly infinitesimal tetrahedron, only the surface terms survive, so that we must have

$$d\vec{F}_x + d\vec{F}_y + d\vec{F}_z + d\vec{F} = 0.$$

Taking into account that the area projections dS_x, dS_y, and dS_z point into the tetrahedron, we define the stress vectors along the coordinate directions as

$$\vec{\sigma}_x = -\frac{d\vec{F}_x}{dS_x}, \quad \vec{\sigma}_y = -\frac{d\vec{F}_y}{dS_y}, \quad \vec{\sigma}_z = -\frac{d\vec{F}_z}{dS_z}.$$

Consequently,

$$d\vec{F} = \vec{\sigma}_x \, dS_x + \vec{\sigma}_y \, dS_y + \vec{\sigma}_z \, dS_z. \tag{8}$$

This shows that the force on an arbitrary surface element may be written as a linear combination of three basic stress vectors along the coordinate axes. Introducing the nine coordinates of the three stress vectors,

$$\vec{\sigma}_x = (\sigma_{xx}, \sigma_{yx}, \sigma_{zx}), \quad \vec{\sigma}_y = (\sigma_{xy}, \sigma_{yy}, \sigma_{zy}), \quad \vec{\sigma}_z = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz}),$$

we arrive at Cauchy's hypothesis.

2.4 Total force

Including a volume force density f_i, the total force on a body of volume V with surface S becomes

$$F_i = \int_V f_i \, dV + \oint_S \sum_j \sigma_{ij} \, dS_j.s \tag{9}$$

Using Gauss' theorem, this may be written as a single volume integral,

$$F_i = \int_V f_i^* \, dV, \tag{10}$$

where

$$f_i^* = f_i + \sum_j \nabla_j \sigma_{ij}, \tag{11}$$

is the *effective force density.* The effective force is not just a formal quantity, because the total force on a material particle is $d\vec{F} = \vec{f}^* \, dV$. In matrix notation this reads

$$\vec{f}^* = \vec{f} + \vec{\nabla} \cdot \vec{\sigma}^{\mathrm{T}}, \tag{12}$$

where $\vec{\sigma}^{\mathrm{T}}$ is the transposed matrix, defined as $\sigma^{\mathrm{T}}_{ji} = \sigma_{ij}$.

2.4.1 Mechanical Pressure

- **Hydrostatic equilibrium:** When the only contact force is pressure, the stress tensor is isotropic:

  $$\boldsymbol{\sigma} = -p\,\mathbf{1}, \qquad \text{or equivalently,} \qquad \sigma_{ij} = -p\,\delta_{ij},$$

  where $\mathbf{1}$ is the unit matrix and δ_{ij} is the Kronecker delta.

- **General case:** In real materials, the stress tensor usually has off-diagonal (shear) components. The diagonal components (σ_{xx}, σ_{yy}, σ_{zz}) differ and correspond to the normal stresses, often interpreted as "pressures":

  $$p_x = -\sigma_{xx}, \qquad p_y = -\sigma_{yy}, \qquad p_z = -\sigma_{zz}.$$

  These components do *not* form a true vector, since they do not transform properly under rotation.

- **Mechanical pressure definition:** The *mechanical pressure* is defined as the average of the normal stresses:

$$p = \frac{1}{3}(p_x + p_y + p_z) = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}).$$

  This definition ensures that pressure is a scalar field, invariant under Cartesian coordinate transformations, because it depends on the *trace* of the stress tensor:

$$\mathrm{Tr}\,\boldsymbol{\sigma} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}.$$

- **Physical meaning:** The mechanical pressure corresponds to the isotropic part of the stress tensor. It represents the mean normal stress acting equally in all directions and is the only scalar combination of stress components that remains invariant under coordinate changes.

2.5 Mechanical equilibrium

In mechanical equilibrium, the total force on any body must vanish, for if it does not the body will begin to move. So the general condition is that $\vec{F} = 0$ for all volumes V. In particular, requiring that the force on each and every material particle must vanish, we arrive at *Cauchy's equilibrium equation(s)*:

$$f_i + \sum_j \nabla_j \sigma_{ij} = 0. \tag{13}$$

In spite of their simplicity, these partial differential equations govern mechanical equilibrium in all kinds of continuous matter, be it solid, fluid, or otherwise. For a fluid at rest, where pressure is the only stress component, we have $\sigma_{ij} = -p\,\delta_{ij}$, and recover the equation of hydrostatic equilibrium, $f_i - \nabla_i p = 0$.

It is instructive to explicitly write out the three individual equations contained in Cauchy's equilibrium equation:

$$\begin{aligned} f_x + \nabla_x\sigma_{xx} + \nabla_y\sigma_{xy} + \nabla_z\sigma_{xz} &= 0, \\ f_y + \nabla_x\sigma_{yx} + \nabla_y\sigma_{yy} + \nabla_z\sigma_{yz} &= 0, \\ f_z + \nabla_x\sigma_{zx} + \nabla_y\sigma_{zy} + \nabla_z\sigma_{zz} &= 0. \end{aligned} \tag{14}$$

These equations are in themselves not sufficient to determine the state of continuous matter, but must be supplemented by suitable *constitutive equations* connecting stress and state. For fluids at rest, the equation of state serves this purpose by relating hydrostatic pressure to mass density and temperature. In elastic solids, the constitutive equations are more complicated and relate stress to displacement.

2.5.1 Symmetry

There is one very general condition which may normally be imposed, namely the symmetry of the stress tensor:

$$\sigma_{ij} = \sigma_{ji}. \tag{15}$$

Symmetry only affects the shear stress components, requiring

$$\sigma_{xy} = \sigma_{yx}, \qquad \sigma_{yz} = \sigma_{zy}, \qquad \sigma_{zx} = \sigma_{xz}, \tag{16}$$

and thus reduces the number of independent stress components from nine to six.

Being thus a symmetric matrix, the stress tensor may be diagonalized. The eigenvectors define the principal directions of stress and the eigenvalues the principal tensions or stresses. In the principal basis, there are no off-diagonal elements, i.e., shear stresses, only pressures. The principal basis is generally different from point to point in space.

2.5.2 Boundary Conditions

- **Cauchy's equation of mechanical equilibrium** requires appropriate boundary conditions on the surface of a body, since it is valid only within the body's volume.
- The **stress tensor** $\vec{\sigma}$ is a local physical quantity that can be assumed continuous in regions where the material properties vary smoothly.
- Across real boundaries or interfaces, where material properties may change abruptly, Newton's Third Law implies that the two sides exert equal and opposite forces on each other (in the absence of surface tension).
- The **stress vector**
$$\vec{\sigma} \cdot \hat{\hat{n}} = \left\{ \sum_j \sigma_{ij} n_j \right\}$$
must therefore be continuous across a surface with normal $\hat{n}$.
- This condition is expressed as the vanishing of the surface discontinuity of the stress vector:
$$[\vec{\sigma} \cdot \hat{n}] = 0.$$
where the brackets [ ] denote the difference between the two sides of the surface.
- This does *not* imply that all stress tensor components are continuous—only three linear combinations (those corresponding to the stress vector) are constrained to be continuous.
- The **mechanical pressure** is, therefore, not necessarily continuous across interfaces. In general continuum mechanics, the mechanical pressure loses the simple, intuitive interpretation it had in hydrostatics.

3 Strain

3.1 Displacement field

$$\vec{u} = \vec{r}' - \vec{r}. \tag{17}$$
$$d\vec{u} = d\vec{r}' - d\vec{r}. \tag{18}$$

$$dl'^2 = dl^2 + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l} dx_k dx_l \tag{19}$$
$$= dl^2 + 2u_{ik} dx_i dx_k, \tag{20}$$

with the strain tensor

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right). \tag{21}$$

4 Hooke's law

4.1 Deformation regimes

In solid continuum mechanics, deformation regimes are classified according to the magnitude of the deformation and the material response. Broadly, three regimes are distinguished:

1. **Infinitesimal (small) deformations.**
   In this regime, displacements and strains are very small compared to unity. The linearized strain tensor is valid, geometric nonlinearities such as large rotations or nonlinear displacement gradients are neglected, and the principle of superposition applies. This approximation underlies most problems in classical elasticity, such as stress analysis in beams and plates.

2. **Finite (large) deformations.**
   Here, displacements or strains are not negligible, and nonlinear kinematics must be employed. Quantities such as the deformation gradient tensor and the Green–Lagrange strain are used. Large rotations, shear, and finite strains are fully accounted for, and the governing equations are nonlinear. Applications include rubber elasticity, biomechanics of soft tissues, and geomechanics.

3. **Extreme (highly nonlinear) deformations.**
   This regime involves very large strains, often accompanied by material nonlinearities or instabilities. The material response may be anisotropic or rate-dependent, and phenomena such as plasticity, viscoelasticity, viscoplasticity, damage, or fracture must be considered. Examples include crash simulations, metal forming at large strains, and failure analysis.

In summary: the *small deformation regime* corresponds to linear elasticity, the *finite deformation regime* requires nonlinear kinematics but can still be elastic or elastic–plastic, while the *extreme regime* involves very large strains typically coupled with inelastic effects such as plasticity, fracture, or damage.

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