Solids at Rest

Contents

Disclaimer

Discussions taken from Alonso Sepúlveda [1], Landau & Lifschits [2], Lautrup [3] books.

1 Objectives

1.1 Objetivos específicos conceptuales:

- OC8. Definir el concepto de tensión superficial, tanto desde la perspectiva de una densidad de energía superficial como desde las fuerzas por unidad de longitud.
- OC9. Enumerar los distintos regímenes de deformación a los que está sometido un sólido o un fluido viscoelástico.
- OC10. Definir el campo de desplazamientos, el tensor de deformaciones y el tensor de elasticidad.
- OC11. Enunciar la ley de Hooke en toda su generalidad.

1.2 Objetivos específicos procedimentales:

- OP8. Escribir el tensor de esfuerzos y la ley de Cauchy.
- OP9. Determinar la relación entre las fuerzas experimentadas por un elemento de volumen en un medio continuo, incluyendo la presión, y el tensor de esfuerzos.

1.3 Objetivos específicos actitudinales:

- OA3. Reconocer la importancia de la notación, del álgebra y del cálculo tensorial en la representación de las cantidades físicas asociadas con los medios continuos.

2 Stress

2.1 Friction

Empirically, such *static friction* can take any magnitude up to a certain maximum, which is proportional to the normal load,

$$T < \mu_0 N. \tag{1}$$

The dimensionless constant of proportionality μ_0 is called the coefficient of static friction, which in our daily doings may take a quite sizable value, say 0.5 or greater. Its value depends on what materials are in contact and on the roughness of the contact surfaces.

If you are able to pull with a sufficient strength, the crate suddenly starts to move, but friction will still be present and you will have to do real work to move the crate any distance. Empirically, the *dynamic* (kinetic or sliding) friction is proportional to the normal load,

$$T = \mu N, \tag{2}$$

with a coefficient of dynamic friction, μ, that is always smaller than the corresponding coefficient of static friction, $\mu < \mu_0$. This is why you have to heave strongly to get the crate set into motion, whereas afterwards a smaller force suffices to keep it going at constant speed.

It is at first sight rather surprising that friction is independent of the size of the contact area. A crate on legs is as hard to drag as a box without, provided they weigh the same. Since larger weight generates larger friction, a car's braking distance will be independent of how heavily it is loaded. In braking a car it is also best to avoid skidding because the static (rolling) friction is larger than sliding friction. Anti-skid brake systems automatically adjust braking pressure to avoid skidding and thus minimize braking distance.

The law of sliding friction goes back to Coulomb (1779) (and also Amontons (1699)). The full story of friction is complicated, and in spite of our everyday familiarity with friction, there is still no universally accepted microscopic explanation of the phenomenon.[1]

2.2 The concept of stress

Shear stress is, just like pressure, defined as force per unit of area, and the standard unit of stress is the same as the unit for pressure, namely the pascal (Pa = N m^{-2}). If a crate on the floor has a contact area A, we may speak both about the average normal stress $\sigma_n = N/A$ and the average tangential (or shear) stress $\sigma_t = T/A$ that the crate exerts on the floor. Depending on the mass distribution of the contents of the crate and the stiffness of its bottom, the local stresses may vary across the contact area A.

2.3 External and internal stress

The stresses acting between the crate and the floor are *external* and are found in the true interface between a body and its environment. In analogy with pressure, we shall also speak about *internal* stresses, even if we may be unable to define a practical way to measure them. Internal stresses abound in the macroscopic world around us. Whenever we come into contact with the environment (and when do we not?) stresses are set up in the materials we touch, and in our own bodies. The precise distribution of stress in a body depends not only on the external forces applied to the body,

[1] See modern tribology references.

but also on the type of material the body is made from and on other macroscopic quantities such as temperature. In the absence of external forces there is usually no stress in a material, although fast cooling may freeze stresses permanently into certain materials, for example glass, and provoke an almost explosive release of stored energy when triggered by a sudden impact.

2.4 Nine components of stress

Shear stress is more complicated than normal stress because there is more than one tangential direction on a surface. In a coordinate system where a force dF_x is applied along the x-direction to a material surface dS_y with its normal in the y-direction, the shear stress will be denoted $\sigma_{xy} = dF_x/dS_y$, instead of just σ. Similarly, if the shear force is applied in the z-direction, the stress would be denoted $\sigma_{zy} = dF_z/dS_y$, and if a normal force had been applied along the y-direction, it would be consistent to denote the normal stress $\sigma_{yy} = dF_y/dS_y$. By convention, the sign is chosen such that a positive value of σ_{yy} corresponds to a pull or *tension*.

2.5 Cauchy's stress hypothesis

Altogether, it therefore appears to be necessary to use at least nine numbers to indicate the state of stress in a given point of a material in a Cartesian coordinate system. *Cauchy's stress hypothesis* asserts that the force $d\mathbf{F} = (dF_x, dF_y, dF_z)$ on an arbitrary surface element, $dS = (dS_x, dS_y, dS_z)$, is of the form

$$\begin{aligned}
dF_x &= \sigma_{xx} dS_x + \sigma_{xy} dS_y + \sigma_{xz} dS_z, \\
dF_y &= \sigma_{yx} dS_x + \sigma_{yy} dS_y + \sigma_{yz} dS_z, \\
dF_z &= \sigma_{zx} dS_x + \sigma_{zy} dS_y + \sigma_{zz} dS_z,
\end{aligned} \tag{3}$$

where each coefficient $\sigma_{ij} = \sigma_{ij}(x, t)$ depends on position and time, and thus is a field in the normal sense of the word. Collecting them in a matrix,

$$\sigma = \{\sigma_{ij}\} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{4}$$

The force may be written compactly as a matrix equation,

$$d\mathbf{F} = \sigma \cdot d\mathbf{S}. \tag{5}$$

The force per unit of area is $d\mathbf{F}/dS = \sigma \cdot \mathbf{n}$, where $\mathbf{n}$ is the normal to the surface. It is sometimes called the *stress vector*, although it is not a vector field in the usual sense because it depends on the normal.

2.6 The stress tensor

Together the nine fields $\{\sigma_{ij}\}$ make up a single geometric object, called the *stress tensor*, first introduced by Cauchy in 1822. Using index notation, we may write

$$\boxed{dF_i = \sum_j \sigma_{ij} dS_j.} \tag{6}$$

Since the force dF_i as well as the surface element dS_j are vectors, it follows that σ_{ij} is indeed a tensor. This collection of nine fields cannot be viewed geometrically as consisting of nine scalars or three vector fields, but must be considered together as one geometrical object, a *tensor field* $\sigma_{ij}(x,t)$ which is neither scalar nor vector.

A stress tensor field of the form

$$\{\sigma_{ij}\} = \{x_i x_j\} = \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix}$$

is a tensor product and thus by construction a true tensor. The stress 'vector' acting on a surface with normal in the direction of the x-axis is

$$\sigma_x = \sigma \cdot e_x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} x.$$

2.7 Hydrostatic pressure

For the special case of hydrostatic equilibrium, where the only contact force is pressure, comparison shows that the stress tensor must be

$$\sigma = -p\,\mathbf{1}, \tag{7}$$

where $\mathbf{1}$ is the (3×3) unit matrix. In tensor notation this becomes

$$\sigma_{ij} = -p\,\delta_{ij}, \tag{8}$$

where δ_{ij} is the Kronecker delta.

3 Mechanical pressure

Generally, however, the stress tensor will have both diagonal and off-diagonal non-vanishing components. A diagonal component behaves like a (negative) pressure, and one often defines the pressures along different coordinate axes to be

$$p_x = -\sigma_{xx}, \qquad p_y = -\sigma_{yy}, \qquad p_z = -\sigma_{zz}. \tag{9}$$

Since they may be different, it is not clear what the meaning of *the* pressure in a point should be. The *mechanical pressure* or simply the pressure is defined to be the average of the three pressures along the axes,

$$p = \frac{1}{3}(p_x + p_y + p_z) = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}). \tag{10}$$

This makes sense because the sum over the diagonal elements of a matrix, the trace $\mathrm{Tr}\,\sigma = \sum_i \sigma_{ii}$, is invariant under Cartesian coordinate transformations. Defining pressure this way ensures it behaves as a scalar field, taking the same value in all coordinate systems.

4 9.4 Mechanical equilibrium

Including a volume force density f_i, the total force on a body of volume V with surface S becomes according to (9.6)

$$\boxed{F_i = \int_V f_i \, dV + \oint_S \sum_j \sigma_{ij} \, dS_j.} \tag{11}$$

Using Gauss' theorem, this may be written as a single volume integral,

$$F_i = \int_V f_i^* \, dV, \tag{12}$$

where

$$\boxed{f_i^* = f_i + \sum_j \nabla_j \sigma_{ij},} \tag{13}$$

is the *effective force density*. The effective force is not just a formal quantity, because the total force on a material particle is $d\mathbf{F} = \mathbf{f}^* \, dV$. As in hydrostatics, this may be demonstrated by considering a small box-shaped particle.

4.1 Cauchy's equation of equilibrium

In mechanical equilibrium, the total force on any body must vanish, for if it does not the body will begin to move. So the general condition is that $\mathbf{F} = 0$ for all volumes V. In particular, requiring that the force on each and every material particle must vanish, we arrive at *Cauchy's equilibrium equation(s)*:

$$f_i + \sum_j \nabla_j \sigma_{ij} = 0. \tag{14}$$

In spite of their simplicity, these partial differential equations govern mechanical equilibrium in all kinds of continuous matter, be it solid, fluid, or otherwise. For a fluid at rest, where pressure is the only stress component, we have $\sigma_{ij} = -p \, \delta_{ij}$, and recover the equation of hydrostatic equilibrium, $f_i - \nabla_i p = 0$.

It is instructive to explicitly write out the three individual equations contained in Cauchy's equilibrium equation:

$$\begin{aligned} f_x + \nabla_x \sigma_{xx} + \nabla_y \sigma_{xy} + \nabla_z \sigma_{xz} &= 0, \\ f_y + \nabla_x \sigma_{yx} + \nabla_y \sigma_{yy} + \nabla_z \sigma_{yz} &= 0, \\ f_z + \nabla_x \sigma_{zx} + \nabla_y \sigma_{zy} + \nabla_z \sigma_{zz} &= 0. \end{aligned} \tag{15}$$

These equations are in themselves not sufficient to determine the state of continuous matter, but must be supplemented by suitable *constitutive equations* connecting stress and state. For fluids at rest, the equation of state serves this purpose by relating hydrostatic pressure to mass density and temperature. In elastic solids, the constitutive equations are more complicated and relate stress to displacement.

5 Symmetry

There is one very general condition (also going back to Cauchy) which may normally be imposed, namely the symmetry of the stress tensor:

$$\boxed{\sigma_{ij} = \sigma_{ji}.} \tag{16}$$

Symmetry only affects the shear stress components, requiring

$$\sigma_{xy} = \sigma_{yx}, \qquad \sigma_{yz} = \sigma_{zy}, \qquad \sigma_{zx} = \sigma_{xz}, \tag{17}$$

and thus reduces the number of independent stress components from nine to six.

Being thus a symmetric matrix, the stress tensor may be diagonalized. The eigenvectors define the principal directions of stress and the eigenvalues the principal tensions or stresses. In the principal basis, there are no off-diagonal elements, i.e., shear stresses, only pressures. The principal basis is generally different from point to point in space.

6 Strain

6.1 Displacement field

$$\vec{u} = \vec{r}' - \vec{r}. \tag{18}$$

$$d\vec{u} = d\vec{r}' - d\vec{r}. \tag{19}$$

$$dl'^2 = dl^2 + 2\frac{\partial u_i}{\partial x_k}dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}dx_k dx_l \tag{20}$$

$$= dl^2 + 2u_{ik}dx_i dx_k, \tag{21}$$

with the strain tensor

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right). \tag{22}$$

7 Hooke's law

7.1 Deformation regimes

In solid continuum mechanics, deformation regimes are classified according to the magnitude of the deformation and the material response. Broadly, three regimes are distinguished:

1. **Infinitesimal (small) deformations.**
   In this regime, displacements and strains are very small compared to unity. The linearized strain tensor is valid, geometric nonlinearities such as large rotations or nonlinear displacement gradients are neglected, and the principle of superposition applies. This approximation underlies most problems in classical elasticity, such as stress analysis in beams and plates.

2. **Finite (large) deformations.**
   Here, displacements or strains are not negligible, and nonlinear kinematics must be employed. Quantities such as the deformation gradient tensor and the Green–Lagrange strain are used. Large rotations, shear, and finite strains are fully accounted for, and the governing equations are nonlinear. Applications include rubber elasticity, biomechanics of soft tissues, and geomechanics.

3. **Extreme (highly nonlinear) deformations.**
   This regime involves very large strains, often accompanied by material nonlinearities or instabilities. The material response may be anisotropic or rate-dependent, and phenomena such as plasticity, viscoelasticity, viscoplasticity, damage, or fracture must be considered. Examples include crash simulations, metal forming at large strains, and failure analysis.

In summary: the *small deformation regime* corresponds to linear elasticity, the *finite deformation regime* requires nonlinear kinematics but can still be elastic or elastic–plastic, while the *extreme regime* involves very large strains typically coupled with inelastic effects such as plasticity, fracture, or damage.

References

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[2] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, vol. 7 of *Course of Theoretical Physics*. Elsevier Butterworth-Heinemann, New York, 1986.

[3] B. Lautrup, *Physics of Continuous Matter: Exotic and Everyday Phenomena in the Macroscopic World*. CRC Press, Boca Raton, 2 ed., 2011.