Student Information

Full Name : Yarkın Özcan
Id Number : 2580835

Answer 1

a-)

Since the graph is connected and $|V_0| = 0$ (V_0 : The set of odd degree vertices), the graph has a Eulerian circuit. For example, $g-b-c-a-e-f-j-m-l-k-g-c-d-e-j-i-l-h-d-i-h-g$ is a Eulerian circuit.

b-)

Let's assume the graph G has a Eulerian path that is not a circuit, and let's assume the path is $V_0 - V_1 - V_2 - \cdots - V_n$. Since this is not a circuit, $V_0 \neq V_n$. From this path, we can conclude that V_0 and V_n are odd degree vertices and remaining vertices are even degree. However, in the graph there are no odd degree vertices start or end the Eulerian path. So there is no Eulerian path.

c-)

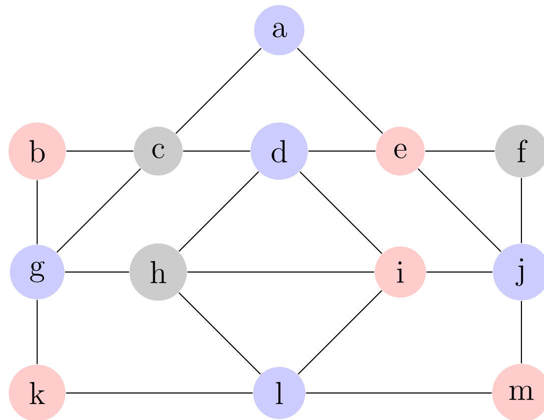
We can check if there is a Hamilton circuit by using the theorem. If for every different pair of vertices in G, their sum of degrees is bigger than n-1 (n=$|V|$) , then there is a Hamilton circuit. However, when we take the vertices b and g as a pair, their sum of degrees is 6 (2 from b, 4 from g), which is not greater or equal to 12 (13-1). However, it is not enough to say there is no Hamilton circuit from this thm. But we can see that the graph has a interior circle in it, which makes it impossible to form a Hamilton Circle because we will be visiting the same vertex 2 times. Hence there is no Hamilton circle.

d-)

There are multiple Hamilton paths. One example is $b-c-a-e-f-j-m-l-k-g-h-i-d$.

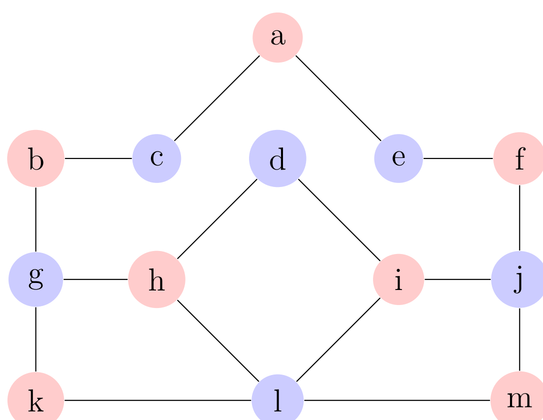
e-)

The chromatic number of G is 3, as shown in the graph



f-)

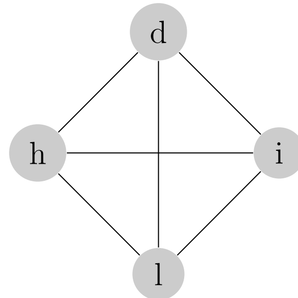
G is not bipartite since its chromatic number is 3. In order for G to be bipartite its chromatic number must be at most 2. So we need to delete min number of edges to make its chromatic number 2. To make that we need to get rid of cycles that consist of odd number of vertices. Firstly, we can delete the edge $g-c$, so that we get rid of $b-c-g$ cycle. Also, if we delete $e-j$, we get rid of $e-f-j$ cycle. And if we delete $h-i$, we get rid of $d-h-i$ and $h-i-l$ (2 different cycles). However there are still 2 odd cycles, namely $b-g-h-d-c$ and $f-j-i-d-e$. To get rid of them, we have to delete the edges $c-d$ and $d-e$. And the resulting graph will be;



Now that its chromatic number is 2, we can say that its bipartite. And the vertices that has the same color will be in the same group.

g-)

A complete graph with 4 nodes must have $\binom{4}{2} = 6$ edges in it and edge node's degree must be 3. In the graph G, we can not form a complete subgraph with 4 vertices. However, if we introduce an edge between d and l, we could have complete subgraph with four vertices.



Answer 2

For these to be isomorphic, we need to check if there is a bijection $f : V_1 --> V_2$ such that $a, b \in E_1$ if and only if $f(a), f(b) \in E_2$. For this to work their;

- Number of nodes,
- Number of edges,
- Degrees of matrices should match.

Both graphs have 8 vertices and 16 edges and all of the vertices in the both graph is degree 4. However, it is not enough to conclude that they are isomorphic. We need to show 1-to-1 correlation for each node. To do this we can define a 1-to-1 and onto function such as;

$$f(a) = a'$$
$$f(b) = g'$$
$$f(e) = h'$$
$$f(f) = b'$$
$$f(d) = c'$$
$$f(h) = d'$$
$$f(g) = f'$$
$$f(c) = e'$$

To see if this function preserves edges we should look at the adjacency of G.

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad A_H = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

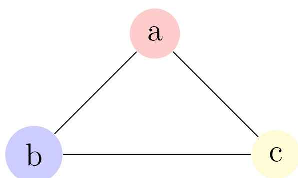
Where in A_G rows and columns are a, b, c, d, e, f, g, h, and for A_H rows and columns are $a', g', e', c', h', b', f', d'$ This matrices holds for both graphs and we can conclude that the function f preserves edges and it is an isomorphism and hence G and H are isomorphic.

Answer 3

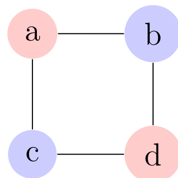
a-)

For a graph to be bipartite its chromatic number must be at most 1. So we need to color these graphs to get their chromatic number.

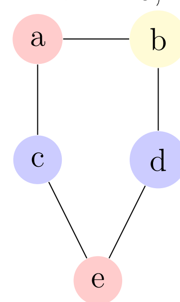
For n=3;



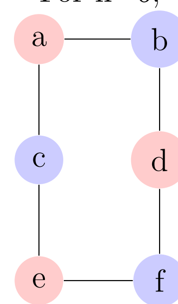
For n=4;



For n=5;



For n=6;



From these graphs, we can conclude that when n is an odd number, the chromatic number of the graph becomes 3 and when n is an even number, the chromatic number of the graph becomes 2. Hence, C_n is bipartite when n is even, and not bipartite when n is odd. Also if we remove an edge from a cycle graph and make it a line graph and start coloring them, if the initial cycle graph had odd number of vertices, the first vertex and final vertex would have the same color which we cannot introduce an edge between them(unless we make 1 of them a different color by adding a new color to graph). However, if the initial cycle graph had even number of vertices, than in the line graph the final vertex and the start vertex would have different colors and we can just introduce an edge between them.

b-)

For cube graph, independent from n, $x(Q_n) = 2$ for $n \geq 1$ because there will be no cycles that will consist of odd number of vertices, this is because if two vertices a,b are connected on a Q_n, there will be no vertex such that a and b are both connected to because from the fundamentals of a cube graph we know that a and b's binary string labels are differed by one character and there are no binary string such that will differ from both a and b by 1 character. Hence there are no 3 vertex cycles to deal with and we can just use 2 colors to color entire graph. So Q_n is bipartite for $n \geq 1$.

Answer 4

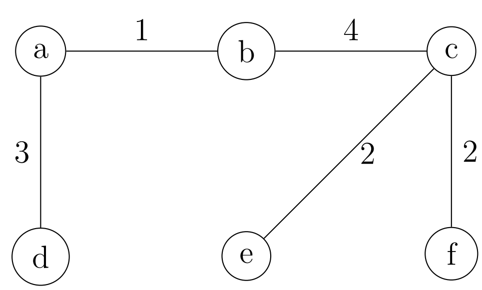
a-)

We can use Kruskal's algorithm to form a MST(minimum spanning tree). Firstly each node forms a single connected component. To form a MST, in each step we need to include minimum weight edge that connects two nodes from different connected component and we need to repeat this process until the number of edges are 1 less than the number of nodes. Firstly, we need to add the edge $a-b$ since it has the minimum weight 1. Secondly we can add any edge from $e-c, c-f, e-f$

since their weights are all 2, for this case I chose $e - f$. Thirdly we can add either $e - c$ or $c - f$, I chose $c - f$ for this problem. After that we should not add the remaining weight 2 edge since it won't be connecting nodes from different connected components. Instead, we need to add the edge $a - d$ since it has the minimum weight from the remaining edges. And lastly we need to add the edge $b - c$. At this point, we need to stop since we have 5 edges which is 1 less than 6(number of vertices).

1. $a - b$
2. $c - e$
3. $c - f$
4. $a - d$
5. $b - c$

b-)



c-)

The MST is not unique since in the second step we could have chosen $e - f$ and we could have a MST as;

1. $a - b$
2. $e - f$
3. $c - f$
4. $a - d$
5. $b - c$

This is also a MST, so we can form it in different ways.

Answer 5

a-)

We can say that $n = b + l$ where b is the number of branch nodes and l is the number of leaf nodes. Also by Handshaking Theorem, we know that $2|E| = \sum_{v \in V} deg(v)$ where E is the set of edges and V is the set of vertices. We also know that leaf nodes' degree is 1 and branch nodes'(except the root node) degree is 3(2 for its children and 1 for its parent). And root node's degree is 2(only from its children). And we can get the equation;

$2|E| = 2 + l * 1 + (b - 1) * 3$

We also know that since this graph is a tree $|E| + 1 = n = b + l$. Hence we get;

$1 + \frac{l}{2} + \frac{3b}{2} - \frac{3}{2} + 1 = n$

$n - \frac{1}{2} = \frac{l+3b}{2}$

$2n - 1 = l + 3b$ if we combine this equation with $n = b + l$, we get;

$l = \frac{n+1}{2}$

b-)

In a tree, there are no cycles and hence we can just use 2 colors to color entire tree. For instance if a parent node has black color all of its children will have white color and their children will have black color. So $X(T) = 2$ for every tree.

c-)

For a m-ary tree with n vertices to have the max height, at most 1 children among m children from the same parent must also have children. Hence, at all the levels except the first level(root) which is h levels, there will be m vertices. So the total number of nodes can be found by;

$h * m + 1 = n$ and we get that the upper bound for h is $\frac{n-1}{m}$.