# Student Information

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# Answer 1

For this question, we can use the Handshaking Theorem.

$2|E| = \sum_{v \in V} deg(v)$, where $v's$ are the nodes in the cube graph $Q_n$ and $E$ is the set of edges. Now we need to know the degree of each node in a $Q_n$. In a $Q_n$, there are $2^n$ nodes and each node is labelled with n length binary strings. And there is an edge between 2 nodes if their labels differ in exactly one digit. For instance, for a $Q_n$, let's take a node with the label 000...0(n length). This node will be connected to the nodes that have the labels as follows;

100....0
010....0
001....0
.
.
.
000....1
Here we have n different nodes. So since this node will be connected to n different nodes, its degree will be n. This is valid for the every node in the $Q_n$. So the sum becomes $2^n * n = 2|E|$. So $a_n = n * 2^{n-1}$. We can rewrite this statement as;
$a_{n+1} = (n+1) * 2^n$
$a_{n+1} = n * 2^n + 2^n$
$a_{n+1} = 2 * (n * 2^{n-1}) + 2^n$
Here we can replace $(n * 2^{n-1})$ with $a_n$
$a_{n+1} = 2 * a_n + 2^n$
$a_n = 2 * a_{n-1} + 2^{n-1}$ where $n \geq 1$ and $a_0 = 0$

# Answer 2

We know that $\frac{1}{1-x} =< 1, 1, 1, ... > <=> 1 + x + x^2 + x^3 + ...$
If we take the derivative, we get;
$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... <=> < 1, 2, 3, 4, .. >$
Now we can shift right once by multiplying the closed form with x;
$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + ... <=> < 0, 1, 2, 3, 4, .. >$
Now, we can multiply the closed form with 3.
$\frac{3x}{(1-x)^2} <=> < 0, 3, 6, 9, ... >$
From here, all we need to do is to add 1 to all coefficients.

$\frac{3x}{(1-x)^2} + \frac{1}{1-x} <=> < 0, 3, 6, 9, ... > + < 1, 1, 1, 1, 1, ... >$
The final result is $\frac{1+2x}{(1-x)^2}$ where $|x| < 1$.

## Answer 3

$$\sum_{n=1}^{\infty} a_n * x^n = \sum_{n=1}^{\infty} a_{n-1} * x^n + \sum_{n=1}^{\infty} 2^n * x^n$$

$$A(x) = \sum_{n=0}^{\infty} a_n * x^n$$

$$A(x) - a_0 = x * \sum_{n=1}^{\infty} a_{n-1} * x^{n-1} + \sum_{n=1}^{\infty} 2^n * x^n$$

$$\sum_{n=1}^{\infty} 2^n * x^n = 2x + (2x)^2 + (2x)^3 + \cdots = \frac{1}{1-2x} - 1$$

$$A(x) - 1 = x * A(x) + \frac{1}{1-2x} - 1$$

$$A(x) = \frac{1}{(1-2x)*(1-x)}$$

$$\frac{1}{(1-2x)*(1-x)} = \frac{B}{1-2x} + \frac{C}{1-x}$$

$$B * (1-x) + C * (1-2x) = 1$$

$$B + C = 1$$

$$B + 2C = 0$$

$$B = 2, C = -1$$

$$\frac{2}{1-2x} <=> 2* < 1, 2, 4, \cdots >$$

$$\frac{-1}{1-x} <=> -1* < 1, 1, 1, \cdots >$$

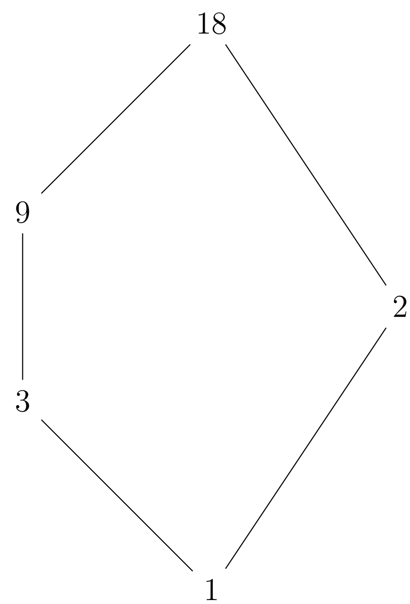
$$\frac{2}{1-2x} + \frac{-1}{1-x} <=>< 1, 3, 7, 10, \cdots >$$

$$A(x) = \sum_{n=1}^{\infty} (2^{n+1} - 1) * x^n$$

$$a_n = 2^{n+1} - 1$$

# Answer 4

**a-)**




**b-)**

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**c-)**

R is a lattice since it satisfies the conditions of partial ordering;

- Reflexive ( $\forall x \in A \ x|x$ )
- Antisymmetric ( $\forall x \forall y \in A \ x|y \land y|x \implies x = y$)
- Transitive ( $\forall x \forall y \forall z \in A \ x|y \land y|z \implies x|z$)

And also in the Hasse diagram of R every different pair or vertices has a unique least upper bound and greatest lower bound.

**d-)**

$$
\begin{bmatrix}1&1&1&1&1\\0&1&0&0&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix} \vee \begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\1&0&1&1&0\\1&1&1&1&1\end{bmatrix} = \begin{bmatrix}1&1&1&1&1\\1&1&0&0&1\\1&0&1&1&1\\1&0&1&1&1\\1&1&1&1&1\end{bmatrix}
$$

$R$ $\qquad$ $R^{-1}$ $\qquad$ The symmetric closure

**e-)**

2 and 9 are not comparable since 2 does not divide 9 whereas 3 and 18 are comparable since 3 dividies 18.

# Answer 5

**a-)**

We can represent this binary relation as n x n matrix. Since we want this relation to be reflexive all the diagonal entries will be 1. And since we want the relation to be symmetric, the upper triangle and the lower triangle in the matrix will be same. So we only need to calculate how many different lower (or upper since they will be same) triangle we can get. To calculate that, we need to find how many nodes are there in the lower (or upper) triangle in a n x n matrix.

There are total $n^2$ number of nodes, we can exclude n diagonal entries since they are all 1 (no need to calculate), the remaining $n^2 - n$ will be distributed to lower and upper triangles evenly. So we need to calculate how many different $\frac{n^2-n}{2}$ nodes we can get. Each node can either be 1 or 0, so there are 2 options. By product rule, we get $2^{\frac{n^2-n}{2}}$.

**b-)**

Again we can represent this binary relation with a n x n matrix. Again since due reflexive property n diagonal entries will be 1 (no need to calculate them). Now for the remaining nodes, due to antisymmetric property, for each pair of nodes that are symmetric to each other with respect diagonal entries, there will be 3 possibilities;

- 1 0
- 0 0
- 0 1

When we exclude the diagonal entries and divide the remaining nodes by 2, we will get the number of symmetric nodes, which is $\frac{n^2-n}{2}$. And since there are 3 possibilities for each pair, by the product rule, we will get $3^{\frac{n^2-n}{2}}$.

# Answer 6

We can disprove this by giving a counterexample. Let's take an antisymmetrix as follows;

$$M_R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_{R^2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^3} = M_{R^2 R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

At this point, we can take the logical or operator with the 3 matrices we get;

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

From this point, we don't need to find $M_{R^4}$ since we already get a matrix that has all entries 1 and the resulting matrix will be same hence the transitive closure is symmetric and we get a counterexample.