Copositive Relaxation for General Quadratic Programming

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Etienne De Klerk
Tilburg University

Cornelis Roos
Delft University of Technology

Tamás Terlaky
Lehigh University

Copositive Relaxation for General Quadratic Programming

A.J. Quist, E. de Klerk, C. Roos, T. Terlaky

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Abstract

We consider general, typically nonconvex, Quadratic Programming Problems. The Semi-definite relaxation proposed by Shor provides bounds on the optimal solution, but it does not always provide sufficiently strong bounds if linear constraints are also involved. To get rid of the linear side-constraints, another, stronger convex relaxation is derived. This relaxation uses copositive matrices. Special cases are discussed for which both relaxations are equal. At the end of the paper, the complexity and solvability of the relaxations are discussed.

1 Introduction

A very important problem in optimization is the General Quadratic Programming Problem (GQP) with quadratic (possibly nonconvex) constraints and objective function. This is a very general NP-hard problem [18], including, for example, integer programming and optimization with general polynomial constraints (see, for example [23]). Since the problem is NP-hard, many relaxations of the problem have been studied.

One convex relaxation of the problem is the semi-definite relaxation proposed by Shor [26] (see also [30]). Much progress was made in the last years in developing polynomial-time interior point methods for semi-definite programming (see e.g. the reviews [25] and [30]). Results pertaining to the quality guarantees given by the Shor relaxation for some specific classes of QP problems will be reviewed in Section 3.1. As already noted in the cited article of Shor, linear side-constraints can cause the semidefinite relaxation to work badly; he presented a modification to overcome this problem, by quadratizing the linear constraints. In [22], such a quadratization is worked out for linear equality constraints.

In this paper, we use a convex relaxation for the case with linear inequality constraints. This relaxation uses cone-positive matrices. Using some theoretical results about copositive and cone-positive matrices, we show that this cone-positive relaxation is stronger than the modified Shor-relaxation when linear constraints are involved, but that its complexity is worse (it is NP-hard), although it is a convex optimization problem.

The paper is organized as follows. First, we state the optimization problem, introduce some matrix cones and show some duality results on these cones. In Section 3, both the standard Shor-relaxation and a modified Shor-relaxation are explained. Our cone-positive relaxation is derived using conic duality theory in Section 4. In Section 5, the solution qualities of the Shor-relaxation and the cone-positive relaxation are compared. We derive some conditions on matrix sizes under which both types of relaxation are the same, and show an example where both types of Shor-relaxations give different relaxed solutions, while the cone-positive relaxation gives an exact solution. In Section 6, we discuss the computational complexity of algorithms to solve the relaxations.

2 Problem statement, Notation and Definitions

We consider the General Quadratic Programming problem

$$\begin{array}{rrcl} \max & x^T A_0 x, & & \\ \text{s.t.} & x^T A_i x & = & b_i, \quad i = 1, ..., m, \\ & Dx & \geq & 0. \end{array} \tag{1}$$

Here $A_i,\ i = 0, ..., m$ are symmetric $n \times n$ matrices, D is a $p \times n$ matrix and x is the variable vector of dimension n. Let us point out that this formulation includes quadratic constraints with linear terms; consider a constraint of the form

$$\tilde{x}^T \tilde{A}_i \tilde{x} \ + \ 2\tilde{b}_i^T \tilde{x} \ + \ \tilde{c}_i \ = \ 0 \tag{2}$$

for some i, then we can introduce

$$x = \begin{pmatrix} \tilde{x} \\ 1 \end{pmatrix}, \quad A_i = \begin{pmatrix} \tilde{A}_i & \tilde{b}_i \\ \tilde{b}_i^T & \tilde{c}_i \end{pmatrix}, \quad b_i = 0 \tag{3}$$

to transform (2) to a quadratic constraint of the form appearing in (1).
Problem (1) can be modified to put all nonlinearities into one set of constraints. This is achieved by introducing the variable matrix X, which is defined as the dyadic product

$$X = xx^T.$$

Recall that the trace of a matrix, $\mathbf{Tr}\,(X)$, is the sum of its diagonal elements, and that for two symmetric matrices X and Y:

$$\mathbf{Tr}\,(XY) \ = \ \sum_i \sum_j x_{ij} y_{ij}.$$

The quadratic terms $x^T A_i x,\ i = 1, ..., m$ now can be reduced to linear terms, using the variable X, as follows:

$$x^T A_i x \ = \ \mathbf{Tr}\left(x^T A_i x\right) \ = \ \mathbf{Tr}\left(A_i x x^T\right) \ = \ \mathbf{Tr}\,(A_i X).$$

In this way, the following reformulation of (1) is obtained.

$$\begin{array}{rl} \max & \mathbf{Tr}\,(A_0 X), \\ \text{s.t.} & \mathbf{Tr}\,(A_i X) \ = \ b_i, \quad i = 1, ..., m, \\ & X = xx^T, \\ & Dx \ \geq \ 0. \end{array} \tag{4}$$

The only nonlinearity occurs in the relation between x and X. In the rest of the paper, we will refer to (4) as the standard form. It is not convex in general. To find upper bounds on the optimal value, we will relax (4) by replacing the constraints $X = xx^T,\ Dx \geq 0$ by constraints of the form $X \in \mathcal{A}$, where $\mathcal{A}$ is a convex matrix cone.

2.1 Matrix cones, definitions and properties

It is assumed in the rest of this paper that all matrices, except D, are symmetric. By $A \geq 0$ we mean that each entry of A is nonnegative. With $x^i,\ i = 1, ..., k$ we mean a set of vectors. The subscript notiation x_i refers to an element of the vector x. The set $\mathcal{M}$ is the set of symmetric square matrices of size depending on the context. We will consider some special classes of matrices that form cones in $\mathcal{M}$. The reader can easily verify that all these cones are convex. For some of these cone definitions, we need the polyhedral set

$$\mathcal{D} \ = \ \{x : Dx \geq 0\}$$

that defines the feasible area of (4) with respect to the linear constraints. The matrix cones under consideration are:

- Nonnegative matrices:
  $\mathcal{N} \;=\; \{N \in \mathcal{M} : N \geq 0\}.$
  $\mathcal{N}^D = \;\{X \in \mathcal{M} :\; X = D^T ND,\; N \in \mathcal{N}\} \;=\; D^T \mathcal{N} D.$
  $\mathcal{N}_D = \;\{X \in \mathcal{M} :\; DXD^T \in \mathcal{N}\}.$
- Positive-semidefinite (PSD-) matrices:
  $\mathcal{S} \;=\; \{S \in \mathcal{M} : x^T Sx \;\geq\; 0 \;\; \forall\, x\} \;=\; \{S \in \mathcal{M} : S = \sum_{i=1}^k s^i {s^i}^T, k \;\geq\; 1\},$
  where the s^i's are vectors of appropriate dimensions.
- Cone-positive matrices (matrices that are semidefinite on the polyhedral cone $\mathcal{D}$):
  $\mathcal{C}_D = \;\{C \in \mathcal{M} : x^T Cx \geq 0 \; \forall x \in \mathcal{D}\}.$
- Completely cone-positive matrices:
  $\mathcal{B}_D = \;\{B \in \mathcal{M} : B \;=\; \sum_{i=1}^k s^i {s^i}^T,\; k \;\geq\; 1,\; s^i \in \mathcal{D},\;\; i = 1, ..., k\}.$
- Copositive matrices (cone-positive on the nonnegative orthant):
  $\mathcal{C} \;=\; \{C \in \mathcal{M} : x^T Cx \geq 0 \; \forall x \geq 0\}.$
- Completely positive matrices (completely cone-positive on the nonnegative orthant):
  $\mathcal{B} \;=\; \{B \in \mathcal{M} : B \;=\; \sum_{i=1}^k s^i {s^i}^T,\; k \;\geq\; 1,\; s^i \geq 0,\;\; i = 1, ..., k\}.$

There are several relations between these matrix cones. Let I be the identity matrix and O the zero matrix of appropriate size, then

$$\mathcal{N} = \mathcal{N}^I = \mathcal{N}_I, \quad \mathcal{B} = \mathcal{B}_I, \; \mathcal{C} = \mathcal{C}_I, \quad \mathcal{S} = \mathcal{B}_O = \mathcal{C}_O.$$

Definition 1 *For a given closed convex matrix cone $\mathcal{A} \subseteq \mathcal{M}$, its dual cone $\mathcal{A}^*$, is defined as follows:*

$$\mathcal{A}^* \;=\; \{X \in \mathcal{M} : \mathbf{Tr}\,(XY) \geq 0 \;\forall\, Y \in \mathcal{A}\}.$$

Note that $\mathcal{A}^{**} \;=\; \mathcal{A}$, which is a well-known result that can be found in e.g. [1].

Lemma 1 $(\mathcal{N}^D)^* \;=\; \mathcal{N}_D.$

Proof: Given any $Y \in \mathcal{N}_D$, it holds for all $Z \;=\; D^T ND \in \mathcal{N}^D$ that

$$\mathbf{Tr}\,(ZY) \;=\; \mathbf{Tr}\left(D^T NDY\right) \;=\; \mathbf{Tr}\left(NDYD^T\right) \;\geq\; 0,$$

since $N \;\geq\; 0$ and $DYD^T \;\geq\; 0$. Thus $\mathcal{N}_D \;\subseteq\; (\mathcal{N}^D)^*$. To prove the other implication, let us consider a $Y \;\notin\; \mathcal{N}_D$. Then $(DYD^T)_{ij} \;<\; 0$ for some pair (i,j). Let E_{ij} be the matrix with ones at positions (i,j) and (j,i), and zeros elsewhere. Then $E_{ij} \in \mathcal{N}$, $D^T E_{ij} D \in \mathcal{N}^D$ and

$$\mathbf{Tr}\left(YD^T E_{ij} D\right) \;=\; \mathbf{Tr}\left(DYD^T E_{ij}\right) \;<\; 0,$$

so $Y \notin (\mathcal{N}^D)^*$, which proves the lemma. $\square$

Taking $D = I$ we obtain as a consequence, that the cone $\mathcal{N}$ is self-dual, which is a well-known fact.

Lemma 2 $\mathcal{B}_D^* \;=\; \mathcal{C}_D.$

Proof: Given any $Y \in \mathcal{C}_D$, it holds for all $Z = \sum_{i=1}^k z^i {z^i}^T \;\in\; \mathcal{B}_D$ that

$$\mathbf{Tr}\,(YZ) \;=\; \sum_{i=1}^k \mathbf{Tr}\left(Yz^i {z^i}^T\right) \;=\; \sum_{i=1}^k {z^i}^T Yz^i \;\geq\; 0,$$

so $\mathcal{C}_D \subseteq \mathcal{B}_D^*$.
To prove the other implication, let us consider a $Y \notin \mathcal{C}_D$. According to (2.1) a vector $z \in \mathcal{D}$ exists such that

$$z^T Y z < 0.$$

Now define $Z = zz^T$, then $Z \in \mathcal{B}_D$ and

$$\mathbf{Tr}\,(YZ) = \mathbf{Tr}\left(z^T Y z\right) = z^T Y z < 0,$$

so $Y \notin \mathcal{B}_D^*$. It follows that $\mathcal{B}_D^* \subseteq \mathcal{C}_D$, hence $\mathcal{C}_D = \mathcal{B}_D^*$. □

Taking $D = I$, we get as a consequence that $\mathcal{B}^* = \mathcal{C}$, which is also proved in [12]. Taking $D = O$ we get that $\mathcal{S}$ is self-dual, which is also a well-known result.

Given two cones of matrices $\mathcal{A}_1$ and $\mathcal{A}_2$, we define

$$\mathcal{A}_1 + \mathcal{A}_2 = \{X : X = A_1 + A_2,\ A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2\}.$$

We will use in this paper the well known relation $(\mathcal{A}_1 + \mathcal{A}_2)^* = \mathcal{A}_1^* \cap \mathcal{A}_2^*$.
We end this section with stating a primal-dual pair of conic optimization problems, which will be used extensively in this paper. This pair can be found in e.g. [31].

Lemma 3 *Given a convex cone $\mathcal{A} \in \mathcal{M}$, the following two problems form a convex primal-dual pair of optimization problems.*

$$\begin{array}{llll} (P) & \max & \mathbf{Tr}\,(A_0 X) \\ & s.t. & \mathbf{Tr}\,(A_i X) = b_i,\ i = 1, \ldots, m \\ & & X \in \mathcal{A} \end{array} \qquad \begin{array}{lll} (D) & \min & b^T y \\ & s.t. & \sum_{i=1}^m y_i A_i - A_0 \in \mathcal{A}^* \\ & & \end{array}$$

The optimal values of the primal and dual problem are equal if the generalized Slater constraint qualification holds, i.e. if an X exists such that

$$X \in \mathrm{int}\{\mathcal{A}\},\ \mathbf{Tr}\,(A_i X) = b_i,\ i = 1, \ldots, m$$

and an y exists such that

$$\sum_{i=1}^m y_i A_i - A_0 \in \mathrm{int}\{\mathcal{A}^*\}.$$

3 Shor-relaxation

In this section, we present the Shor-relaxation and review known results on its quality guarantees for QP. Initially, linear constraints are ignored. In the second subsection, we consider the case with linear inequality constraints.

3.1 Quadratic constraints only

In this section we deal with the problem (4) without the linear inequality constraints. The Shor-relaxation relaxes the constraint $X = xx^T$ to $X \in \mathcal{S}$, to obtain

$$\begin{array}{rl} \max & \mathbf{Tr}\,(A_0 X), \\ \text{s.t.} & \mathbf{Tr}\,(A_i X) = b_i, \quad i = 1, ..., m, \\ & X \in \mathcal{S}. \end{array} \tag{5}$$

Using the second definition of $\mathcal{S}$, we note that the Shor-relaxation in fact relaxes the rank-one constraint on X.
The corresponding dual problem is (according to Lemma 3)

$$\begin{array}{ll} \min & b^T y, \\ \text{s.t.} & \sum_{i=1}^m y_i A_i \; - \; A_0 \; \in \mathcal{S}. \end{array} \tag{6}$$

Several new results concerning quality guarantees of the Shor relaxation have been published recently. Lovász and Schrijver [14] first considered the following Boolean QP problem which arises in combinatorial optimization, namely

$$q^{\max} = \max\left\{x^T Q x \; : \; x_i \in \{-1,1\} \; (\forall i)\right\}, \tag{7}$$

which has as Shor relaxation[1]

$$\bar{q} = \max\left\{\mathbf{Tr}\,(QX) \; : \; \mathrm{diag}(X) = e, \; X \in \mathcal{S}\right\}, \tag{8}$$

with e the all-one vector. For this general relaxation Nesterov [20] recently proved that

$$\bar{q} - \underline{q} \geq q^{\max} - q^{\min} \geq \frac{4-\pi}{\pi}\left(\bar{q} - \underline{q}\right)$$

where $(q^{\min}, q^{\max})$ is the range of feasible objective values in (7), and $(\underline{q}, \bar{q})$ is the range of feasible values in the relaxation problem (8). Moreover, a random feasible solution x to (7) can be computed from the solution to the relaxation (8). The expected objective value of x, say $E(x)$, satisfies

$$\frac{q^{\max} - E(x)}{q^{\max} - q^{\min}} < \frac{4}{7}.$$

Such an algorithm is called a (randomized) μ-approximation algorithm, with $\mu = 4/7$ in this case.

The same bounds were subsequently obtained by Ye [32] for the 'box-constrained' problem where $x_i \in \{-1,1\}$ is replaced by $-1 \leq x_i \leq 1$ in problem (7); these results were further extended in [33] to include *simple quadratic constraints* of the form: $\sum_{i=1}^n a_i x_i^2 = b$.

These recent results are extensions of work by Goemans and Williamson [9] on the *maximal cut problem*, i.e. the problem of finding a cut of maximal weight through a graph with weighted edges. The special structure of this problem actually allows the bound $\bar{q} \leq 1.14 q^{\max}$. The randomized algorithm moreover produces a cut with expected value greater than $0.878 q^{\max}$ in this case.

These positive results unfortunately do not include QP problems involving general linear inequality constraints. In fact, Bellare and Rogaway [2] have showed that the generic QP problem

$$\max\left\{x^T Q x \; : \; Ax \leq b, \; x \in [0,1]^n\right\}$$

has no polynomial time μ-approximation algorithm for some $\mu \in (0,1)$, unless P=NP. This latter result motivates the alternative convex relaxations studied in this paper, which cannot be solved in polynomial time in general, but are tighter than the Shor relaxation.

3.2 Linear inequality constraints

It was observed in [26] that the linear inequality constraints cannot be handled the same way as the quadratic equality constraints. In this section, we will show, that the homogeneous linear constraints of (4),

$$d^{j^T} x \; \geq \; 0, \quad j = 1, ..., p,$$

[1]Note that if $x_i \in \{-1,1\}$ then $\mathrm{diag}(xx^T) = e$.

will completely vanish in the Shor-relaxation, when handled as general quadratic constraints of the form (2). When treating the constraints in this way, we have to introduce

$$\overline{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad \overline{A}_{m+j} = \begin{pmatrix} \mathbf{0} & d^j/2 \\ {d^j}^T/2 & 0 \end{pmatrix}, \quad j = 1, ..., p.$$

We also extend $A_i,\ i = 0, ..., m$ and X with one row and column containing zeros, and define

$$\overline{X} := \begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix} (x^T\ 1).$$

Problem (4) then becomes

$$\begin{array}{ll} \max & \mathbf{Tr}\left(\overline{A}_0 \overline{X}\right), \\ \text{s.t.} & \mathbf{Tr}\left(\overline{A}_i \overline{X}\right) = b_i, \quad i = 1, ..., m, \\ & \mathbf{Tr}\left(\overline{A}_j \overline{X}\right) \geq 0, \quad j = m+1, ..., m+p, \\ & \overline{X} = \overline{x}\overline{x}^T, \\ & \overline{x}_{n+1} = 1. \end{array} \tag{9}$$

The Shor-relaxation replaces the rank-one constraint by the constraint $\overline{X} \in \mathcal{S}$. According to the second definition of $\mathcal{S}$ in Section 2.1, this means that the Shor-relaxation requires that

$$\overline{X} \equiv \begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix} = \sum_{i=1}^{k} \begin{pmatrix} \xi^i \\ z_i \end{pmatrix} ({\xi^i}^T\ z_i),$$

for some positive number k. This is equivalent to the system

$$\begin{array}{rcl} X & = & \sum_{i=1}^k \xi^i {\xi^i}^T \\ x & = & \sum_{i=1}^k z_i \xi^i \\ \sum_{i=1}^k z_i^2 & = & 1. \end{array}$$

Using (9), this leads to the following formulation of the Shor-relaxation, expressed using the original-size matrices $A_i,\ i = 0, ..., m$:

$$\begin{array}{ll} \max & \mathbf{Tr}\left(A_0 X\right), \\ \text{s.t.} & \mathbf{Tr}\left(A_i X\right) = b_i, \quad i = 1, ..., m, \\ & X = \sum_{i=1}^k \xi^i {\xi^i}^T \\ & D(\sum_{i=1}^k z_i \xi^i) \geq 0 \\ & \sum_{i=1}^k z_i^2 = 1. \end{array} \tag{10}$$

Now, given a k, and vectors $\xi^i,\ i = 1, ..., k$ such that the first two sets of constraints are satisfied, add the vector $\xi^{k+1} = 0$, let $z_i = 0,\ i = 1, ..., k$ and let $z_{k+1} = 1$, to see that we can always satisfy the last two sets of constraints. Thus, the linear constraints are in fact ignored in the Shor-relaxation, and we conclude that the Shor-relaxation can be written as (5), regardless of the matrix D.

3.3 Improved Shor-relaxation

Shor presents an alternative way to treat the linear constraints. In the primal problem (4), the linear constraints

$${d^j}^T x \geq 0, \quad j = 1, ..., p$$

are replaced by the constraints

$$x^T d^i {d^j}^T x \geq 0, \quad i, j = 1, ..., p.$$

Using the transformation

$$x^T d^i {d^j}^T x \;=\; \mathbf{Tr}\left(d^i {d^j}^T X\right) \;=\; {d^j}^T X d^i \;=\; (DXD^T)_{ji},$$

we can put these constraints together in the matrix inequality

$$DXD^T \geq 0 \quad (\Leftrightarrow \quad X \in \mathcal{N}_D). \tag{11}$$

We obtain the following relaxation of (4):

$$\begin{array}{rl} \max & \mathbf{Tr}\,(A_0 X) \\ \text{s.t.} & \mathbf{Tr}\,(A_i X) \;=\; b_i, \quad i = 1, ..., m \\ & X \;\in\; \mathcal{S} \,\cap\, \mathcal{N}_D \end{array} \tag{12}$$

with its dual

$$\begin{array}{rl} \min & b^T y \\ \text{s.t.} & -(A_0 \;+\; \sum_{i=1}^m y_i A_i) \;\in \mathcal{S} \;+\; \mathcal{N}^D. \end{array} \tag{13}$$

Compared to the original problem (4), in (12) the rank-one constraint on X is relaxed, but the matrix X should in a sense be positive with respect to D. More precisely, X should be of the form

$$\sum_{\ell=1}^{k} x^\ell {x^\ell}^T,$$

where $k \in \mathbb{Z}_+$ and

$$\sum_{\ell=1}^{k} {d^i}^T x^\ell {x^\ell}^T d^j \;\geq\; 0, \quad i, j = 1, ..., p.$$

By a simple example, we show that this relaxation works better than the standard Shor-relaxation.

Example 1 *Consider the example (with optimum value 0)*

$$\begin{array}{rl} \max & x^T \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} x \\ s.t. & Ix \;\geq\; 0. \end{array} \tag{14}$$

The original Shor relaxation reads

$$\begin{array}{rl} \max & \mathbf{Tr}\left(\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} X\right) \\ s.t. & X \;\in\; \mathcal{S}. \end{array}$$

The feasible set and the optimum value of this relaxation are unbounded, as can be seen by choosing

$$X \;=\; \lim_{\alpha\to\infty} \begin{pmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix}.$$

This corresponds to the fact that the dual problem

$$\begin{array}{rl} \min & 0 \\ s.t. & \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \;\in\; \mathcal{S} \end{array}$$

is infeasible. With the constraints (11) the modified Shor-relaxation becomes

$$\begin{array}{rl} \max & \mathbf{Tr}\left(\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} X\right) \\ s.t. & X \;\in\; \mathcal{S} \,\cap\, \mathcal{N}_I \;=\; \mathcal{S} \,\cap\, \mathcal{N}. \end{array}$$

This problem has optimum value 0. The dual problem

$$\begin{array}{ll} \min & 0 \\ s.t. & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in \mathcal{S} + \mathcal{N}^I = \mathcal{S} + \mathcal{N}, \end{array} \tag{15}$$

also has optimum value 0, as can be seen by observing that

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Although in the above example, the improved Shor-relaxation was tight, this is in general not true. The third type of relaxation, that we state in the next section, is still tighter than the improved Shor-relaxation. When we refer to the Shor-relaxation in the rest of the paper, we mean the improved Shor-relaxation, unless we refer to the original Shor-relaxation explicitly.

4 Cone-positive relaxation

In the original Shor-relaxation, we found that the constraints $X = xx^T, \ \ Dx \ \geq \ 0$, are relaxed to the semidefiniteness constraint

$$X = \sum_{i=1}^{k} x^i {x^i}^T, \quad k \geq 1, \tag{16}$$

thus 'forgetting' the linear constraints. The improved Shor-relaxation adds to (16) the additional constraint

$$DXD^T \equiv \sum_{i=1}^{k} Dx^i {x^i}^T D^T \geq 0. \tag{17}$$

We still can do better. In the following relaxation, we add to (16) the set of constraints

$$Dx^i \geq 0, \quad i = 1, ..., k. \tag{18}$$

Theorem 1 *The GQP as stated in (4) can be relaxed by the following 'completely cone-positive' problem*

$$\begin{array}{ll} \max & \mathbf{Tr}\,(A_0 X) \\ s.t. & \mathbf{Tr}\,(A_i X) = b_i, \quad i = 1, ..., m, \\ & X \in \mathcal{B}_D \end{array} \tag{19}$$

with its 'cone-positive' dual

$$\begin{array}{ll} \min & b^T y \\ s.t. & \sum_{i=1}^m y_i A_i - A_0 \in \mathcal{C}_D. \end{array} \tag{20}$$

Proof: The cone $\mathcal{B}_D$ consists of all matrices

$$X = \sum_{i=1}^{k} x^i {x^i}^T, \quad k \geq 1, \quad Dx^i \geq 0, \quad i = 1, ..., k.$$

The set of feasible solutions of (4) contains of the matrices

$$X = xx^T, \quad Dx \geq 0,$$

so it is contained in $\mathcal{B}_D$. The primal problem (19) is therefore a relaxation of (4). Using Lemma 2 and Lemma 3, the dual follows. □

Since $\mathcal{B}_D$ and $\mathcal{C}_D$ are convex cones, both problems are convex optimization problems. Moreover, since (18) implies (17), the relaxation is at least as tight as the Shor-relaxation. In the following, we will derive the relation between the two relaxations in more detail, both with regard to quality and solvability.

5 Comparison of the quality of relaxations

In this section, we will discuss the relation between the feasible areas of the Shor-relaxation and the cone-positive relaxation, and study under which conditions both relaxations are equal. We do this by considering the dual feasible sets of the Shor-relaxation and the cone-positive relaxation. A relaxation will be called stronger than a second one if the first one gives better bounds on the primal solution. We first consider the special case that $D = I$.

5.1 Important case: $D = I$

Many practical optimization problems deal with nonnegativity constraints. When the set of linear inequalities reduces to the constraint that all variables are nonnegative, D becomes the identity matrix. The cone-positive relaxation then reduces to what we will call the copositive programming problem:

$$\begin{array}{rl} \max & \mathbf{Tr}\,(A_0 X) \\ \text{s.t.} & \mathbf{Tr}\,(A_i X) \;=\; b_i, \quad i = 1, ..., m, \\ & X \;\in\; \mathcal{B} \end{array}$$

with its dual

$$\begin{array}{rl} \min & b^T y \\ \text{s.t.} & \sum_{i=1}^m y_i A_i \;-\; A_0 \;\in\; \mathcal{C}. \end{array}$$

We study its quality compared to the Shor-relaxation, by comparing the dual cones $S + \mathcal{N}^D$ of the Shor-relaxation and $\mathcal{C}$ of the copositive relaxation. Since $D = I$, $\mathcal{N}^D$ reduces to $\mathcal{N}$. Clearly,

$$\mathcal{C} \;\supseteq\; \mathcal{N} \;+\; \mathcal{S},$$

since both a nonnegative matrix and a semidefinite matrix are copositive (it follows also since (18) implies (17)). This implies that the copositive dual has a feasible region that is at least as large as the Shor dual, thus the copositive relaxation is at least as strong as the Shor-relaxation.
For matrix size $n \leq 4$, both relaxations are equivalent. This follows from a result that is proved by Diananda in [8]. We state it as a lemma here since it is also needed for the case $D \neq I$.

Lemma 4 *For matrix size $n \leq 4$,*

$$\mathcal{C} \;=\; \mathcal{N} \;+\; \mathcal{S}.$$

The proof is not difficult but lengthy, and is not repeated here. For $n = 5$, the cited article provides an example matrix, described in more detail in [12], that is in $\mathcal{C}$ but not in $\mathcal{N} + \mathcal{S}$, namely the matrix

$$J \;=\; \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}. \tag{21}$$

We only show here that it is in $\mathcal{C}$. The matrix corresponds to a quadratic function, that may be expressed in different ways:

$$\begin{array}{rcl} x^T J x &=& (x_1 - x_2 + x_3 + x_4 - x_5)^2 + 4x_2x_4 + 4x_3(x_5 - x_4), \\ &=& (x_1 - x_2 + x_3 - x_4 + x_5)^2 + 4x_2x_5 + 4x_1(x_4 - x_5). \end{array}$$

For $x \geq 0$, the first formulation shows nonnegativity of $x^T Jx$ if $x_4 \leq x_5$, the second formulation shows nonnegativity if $x_4 \geq x_5$, thus $x^T Jx \geq 0$ for all $x \geq 0$.

The abovementioned results are summarized in the following theorem.

Theorem 2 *The Shor-relaxation is not stronger than the copositive relaxation. Both relaxations are the same for matrix size $n \leq 4$. For $n > 4$, there exist problems for which the copositive relaxation is stronger than the Shor-relaxation.*

Below, we illustrate this theorem with an example where $n = 5$.

Example 2 *In this example, the original Shor-relaxation, the improved Shor-relaxation and the Copositive relaxation give three different bounds. The problem reads*

$$\begin{array}{ll} \min & x^T Jx \\ s.t. & \sum_{i=1}^{5} x_i^2 \ = \ 1 \\ & x \ \geq \ 0, \end{array}$$

with J as defined in (21). Since J is copositive, the solution is nonnegative. The exact solution is 0, for example obtained by $x = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0)^T$.
The original Shor-relaxation from Section 3.2 reads

$$\begin{array}{lll} (P) & \min & \mathbf{Tr}\,(JX) \\ & s.t. & \mathbf{Tr}\,(X) \ = \ 1, \\ & & X \in \mathcal{S} \end{array} \qquad \begin{array}{lll} (D) & \max & y \\ & s.t. & J - yI \in \mathcal{S} \end{array}$$

The optimum solution value of the dual problem corresponds to the smallest eigenvalue of J, which is equal to $1 - \sqrt{5} \ \approx \ -1.236$.
The improved Shor-relaxation from Section 3.3 reads

$$\begin{array}{lll} (P) & \min & \mathbf{Tr}\,(JX) \\ & s.t. & \mathbf{Tr}\,(X) \ = \ 1, \\ & & X \in \mathcal{S} \cap \mathcal{N} \end{array} \qquad \begin{array}{lll} (D) & \max & y \\ & s.t. & J - yI \in \mathcal{S} + \mathcal{N} \end{array}$$

For this problem, the optimum value $2 - \sqrt{5} \ \approx \ -0.23606$. This is easily checked by taking the primal solution

$$X = \frac{1}{10} \begin{pmatrix} 2 & \sqrt{5}-1 & 0 & 0 & \sqrt{5}-1 \\ \sqrt{5}-1 & 2 & \sqrt{5}-1 & 0 & 0 \\ 0 & \sqrt{5}-1 & 2 & \sqrt{5}-1 & 0 \\ 0 & 0 & \sqrt{5}-1 & 2 & \sqrt{5}-1 \\ \sqrt{5}-1 & 0 & 0 & \sqrt{5}-1 & 2 \end{pmatrix} \ \in \ \mathcal{S} \cap \mathcal{N}.$$

and the dual matrix

$$J - (2 - \sqrt{5})I = \begin{pmatrix} \sqrt{5}-1 & -1 & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & -1 \\ -1 & \sqrt{5}-1 & -1 & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & -1 & \sqrt{5}-1 & -1 & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & -1 & \sqrt{5}-1 & -1 \\ -1 & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & -1 & \sqrt{5}-1 \end{pmatrix} + \frac{\sqrt{5}-1}{2} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{S} + \mathcal{N}.$$

The Copositive relaxation reads

$$\begin{array}{lll} (P) & \min & \mathbf{Tr}\,(JX) \\ & s.t. & \mathbf{Tr}\,(X) \ = \ 1, \\ & & X \in \mathcal{B} \end{array} \qquad \begin{array}{lll} (D) & \max & y \\ & s.t. & J - yI \in \mathcal{C} \end{array}$$

Since $J \in \mathcal{C}$, the optimum value for (D) is greater than or equal to zero. But the solution $X = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0)^T (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0)$ is feasible for (P) with optimum value 0, so this relaxation gives the exact solution 0.

5.2 General case

To compare the quality of the cone-positive relaxation with the quality of the Shor-relaxation for general D, we compare the dual cones $\mathcal{C}_D$ of the cone-positive dual and $\mathcal{S}+\mathcal{N}^D$ of the Shor dual. Firstly, since (18) implies (17), the cone-positive relaxation is at least as strong as the Shor-relaxation, implying that $\mathcal{S}+\mathcal{N}^D \subseteq \mathcal{C}_D$.
To derive conditions under which both cones are the same, we use the intermediate cone

$$\mathcal{CS}_D \;=\; \{Y : Y = D^T CD + S,\ C \in \mathcal{C},\ S \in \mathcal{S}\} \;=\; D^T \mathcal{C} D \;+\; \mathcal{S}$$

and find conditions under which $\mathcal{C}_D = \mathcal{CS}_D$ resp. $\mathcal{CS}_D = \mathcal{S}+\mathcal{N}^D$. For the first equality, we refer without proof to the following lemma from Martin and Jacobson [16].

Lemma 5 (Martin and Jacobson) *Let $D \in \mathbb{R}^{p\times n}$. Then $\mathcal{C}_D \;=\; \mathcal{CS}_D$ if one of the following three conditions is satisfied:*

1. *rank $D \;=\; p$;*
2. *$p \le 4$ and there exists an x such that $Dx > 0$;*
3. *$n = 2$ and there exists an x such that $Dx > 0$.*

Under the first condition, the restriction that a strictly interior point x of $\mathcal{D}$ exists is automatically satisfied. An extensive proof of the lemma can be found in [16], together with a proof that counterexamples exist for the case $n = 3$, $p = 5$. The following lemma deals with the second equality.

Lemma 6 *The cones $\mathcal{CS}_D$ and $\mathcal{S}+\mathcal{N}^D$ are equal if and only if rank$(D) \le 4$.*

Proof: It is immediately clear that $\mathcal{CS}_D \;\supseteq\; \mathcal{S}+\mathcal{N}^D$ since $\mathcal{C} \supseteq \mathcal{N}$. To prove that $\mathcal{CS}_D \;\subseteq\; \mathcal{S}+\mathcal{N}^D$ for rank$(D) \le 4$, suppose the matrix D has size $p \times n$ and rank $r < p$, and let

$$Y \;=\; D^T CD + S_1 \;\in\; \mathcal{CS}_D,\ C \in \mathcal{C},\ S_1 \in \mathcal{S}$$

be given. The rows of D are ordered in such a way that the first r rows are independent and the last $p-r$ rows can be expressed as linear combination of the first r rows. Define, for all x such that $Dx \ge 0$, the vector

$$y(x) \;=\; Dx.$$

Then the last $p-r$ elements of $y(x)$ are linear functions of $y_1, ..., y_r$. This implies that the quadratic form

$$x^T D^T CDx \;=\; y^T Cy$$

is only a quadratic form in $y_1, ..., y_r$. Therefore, the $p \times p$ matrix C can be replaced by an $r \times r$ copositive matrix $\tilde{C}$. We use Lemma 4 to obtain that for $r \le 4$, the matrix $\tilde{C}$ can be replaced by $\tilde{N} + \tilde{S}$, $\tilde{N} \in \mathcal{N}$, $\tilde{S} \in \mathcal{S}$. Using

$$\overline{C} \;=\; \begin{pmatrix} \tilde{C} & 0 \\ 0 & 0 \end{pmatrix},\quad \overline{N} \;=\; \begin{pmatrix} \tilde{N} & 0 \\ 0 & 0 \end{pmatrix},\quad \overline{S} \;=\; \begin{pmatrix} \tilde{S} & 0 \\ 0 & 0 \end{pmatrix},$$

we obtain

$$Y = D^T CD + S_1 = D^T \overline{C} D + S_1 = D^T \overline{N} D + D^T \overline{S} D + S_1 = D^T ND + S_2 \in \mathcal{S}+\mathcal{N}^D.$$

To show the necessity part of the lemma, a counterexample for the case that rank$(D) \ge 5$ is easily obtained by choosing $C = J$ (J from (21)) and $D = I$. □

Now, combining Lemma 5 and Lemma 6 gives the following theorem.

Theorem 3 *The Shor-relaxation is not stronger than the cone-positive dual. Both relaxations coincide if $\mathcal{D} = \{x : Dx \geq 0\}$ has a strictly interior point, and either $p \leq 4$ or $n = 2$.*

Since it is proven that counterexamples exist to Lemma 5, there exist also counterexamples for the case that $\mathcal{D}$ has no strictly interior point, or none of the conditions of Theorem 3 is satisfied.
We may conclude that the two relaxations are the same only for a very small number of side-constraints, or for very small matrix sizes. For larger problems, the cone-positive dual may provide a stronger relaxation. In the next section, we will study the solvability of the two relaxations.

6 Solvability

All described relaxations are convex programming problems in conic form. In principle, all convex optimization problems can be cast in the conic form. Nesterov and Nemirovskii [19] show that such conic optimization problems can be solved by sequential minimization techniques, where the conic constraint is discarded and a barrier term is added to the objective. Suitable barriers are called *self-concordant*. These are smooth convex functions with second derivatives which are Lipschitz continuous with respect to a local metric (the metric induced by the Hessian of the function itself). More precisely, the definition is as follows.

Definition 2 (Self-concordant barrier) *A convex function $f : C \mapsto \mathbb{R}$ is called self-concordant if*

$$\left| \sum_{i,j,k} \left(\frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k \right) \right| \leq k \langle \nabla^2 f(x) h, h \rangle^{3/2},$$

for all $x \in \operatorname{int}(C)$ and $h \in \mathbb{R}^n$, where $k \geq 0$ is a fixed constant. A strictly convex self-concordant function f is called a ν-self-concordant barrier if $k = 2$ and it also satisfies

$$2\langle \nabla f(x), h \rangle - \langle \nabla^2 f(x) h, h \rangle \leq \nu$$

for all $h \in \mathbb{R}^n$.

Self-concordant barriers go to infinity as the boundary of the cone is approached, and can be minimized efficiently by Newton's method.[2]

Using such a barrier, one can formulate optimization algorithms that require only a polynomially bounded number of iterations using exact arithmetic to compute ϵ-optimal solutions.

Theorem 4 (Nesterov and Nemirovskii) *Given a ν-self-concordant barrier for a closed convex domain G, one can associate with this barrier interior-point methods, for minimizing linear objectives over G, that require at most $\mathcal{O}\left(\sqrt{\nu} \log\left(\frac{1}{\epsilon}\right)\right)$ iterations to compute an ϵ-optimal solution from a suitable starting solution.*

The function

$$B_{\mathcal{S}}(X) \;=\; -\log(\det(X)) \tag{22}$$

is n-self-concordant for $\mathcal{S}$ [19], and the standard Shor-relaxation is therefore solvable in $\mathcal{O}\left(\sqrt{n} \log\left(\frac{1}{\epsilon}\right)\right)$ iterations, provided the Slater condition holds for both primal and dual

[2] A well-written introductory text dealing with self-concordance is [13].

problems.[3] Likewise, the function

$$B_{\mathcal{N}}(X) \;=\; -\sum_{i,j=1}^{n} \log(X_{i,j}) \;=\; -e^T \log[X]e$$

is a $\frac{1}{2}n(n+1)$-self-concordant barrier for $\mathcal{N}$, where $\log[X]$ denotes an element-wise operator. Applying simple combination rules, it follows that a barrier for $\mathcal{N}_D$ is given by

$$B_{\mathcal{P}}(X) \;=\; -e^T \log[DXD^T]e. \tag{23}$$

It is easy to see that a barrier for $\mathcal{S} \cap \mathcal{N}_D$ is obtained by summation of the two barriers (22) and (23):

$$B_{\mathcal{S}\cap\mathcal{P}} \;=\; -\log(\det(X)) - e^T \log[DXD^T]e. \tag{24}$$

It follows that the modified Shor-relaxation is solvable in $\mathcal{O}\left(n \log\left(\frac{1}{\epsilon}\right)\right)$ iterations.

A self-concordant barrier also exists for the copositive cone $\mathcal{C}$, according to the following theorem.

Theorem 5 (Nesterov and Nemirovskii) *Given a closed convex domain $G \subset \mathbb{R}^n$, there exists a self concordant barrier for G. If G does not contain any one-dimensional subspace of $\mathbb{R}^n$, then we can take the barrier*

$$F(x) : \mathrm{int}(G) \to \mathbb{R} : \;\; F(x) = \mathcal{O}(1) \ln |G^0(x)|,$$

where $\mathcal{O}(1)$ is an appropriately chosen absolute constant,

$$G^0(x) = \{\varphi \in \mathbb{R}^n : \varphi^T(y-x) \leq 1 \;\forall y \in G\}$$

is the polar of G with respect to the point x, and $|\cdot|$ denotes the n-dimensonal volume.

The problem is that this theorem only gives an implicit definition of the barrier, and no explicit expression is known in general.

In fact, there is a reason to suspect that a self-concordant barrier for $\mathcal{C}$ can not be evaluated in a polynomial number of operations. This reason is that detecting copositivity of a matrix is an NP-complete problem [17, 18]; the copositive optimization problem is therefore NP-hard, i.e. the recognition version of the problem is NP-complete[4]. In comparison, the recognition version of a general semidefinite programming problem (without the Slater condition) is not NP-complete, unless NP=co-NP [24]. On the other hand, the convexity of the copositive optimization problem suggests that it is in some sense an 'easier' NP-hard problem than the original GQP problem.

Remark 1: To find a barrier, we should know something about the boundaries of the cone. The semidefinite cone $\mathcal{S}$ is bounded by the surface $\det(X) = 0$, and $\det(X) > 0$ in $\mathrm{int}(\mathcal{S})$. However, the copositive cone is bounded by parts of all $2^n - 1$ surfaces $\det(\overline{Y}_{JJ}) = 0$, where $\overline{Y}_{JJ}$ is a principal submatrix of Y obtained by deleting all rows and columns with indices not in the set $J \subseteq \{1, ..., n\}$. Moreover, these determinants may be zero for some matrices in $\mathrm{int}(\mathcal{C})$.

[3] A review of known complexity results for semidefinite programming is given in [25].

[4] Roughly speaking, the recognition version of an optimization problem is the problem to find whether there exists a rational solution with objective value equal to or better than a fixed constant. See [21] for a detailed treatment.

Remark 2: In a recent paper by Güler [10], a new way of constructing self-concordant barries for so-called *homogeneous cones*[5] was presented. Unfortunately, this kind of construction is impossible for the copositive cone, since it is not homogeneous.

As an alternative to barrier methods, one might use gradient-based methods for optimization. Usually, such methods contain a line-search. For such a line-search, it is important to detect whether or not a matrix is in the cone. Again, detecting whether a matrix is in $\mathcal{C}$ is already an NP-complete problem. There are several tests for detection of copositivity. As an example, we recall the following theorem of Keller, stated in [6]. It makes use of *cofactors:* determinants of submatrices where one row and one column are deleted.

Theorem 6 (Keller) *A matrix Y is copositive if and only if each principal submatrix for which the cofactors of its last row are nonnegative, has a nonnegative determinant.*

This theorem shows that not only the determinants of some principal submatrices are important, even cofactors of the principal submatrices have to be considered. Another feature of this test is that it is stated in a conditional form: there may be principal submatrices with negative determinant, if they have also a negative cofactor in the last row. Other tests for copositivity can be found in e.g. [3, 4, 7, 11, 15, 27, 28, 29]. In [5], the test of [4] is used to develop an algorithm that solves Quadratic Programs with indefinite objective function and linear constraints to global optimality. This algorithm is finite but exponential. We may conclude that, for general copositive optimization problems, an algorithm with polynomial bound on the iteration count is unlikely.
It is topic of further research to see whether one can exploit the structure of certain subclasses, in order to make efficient algorithms. For tridiagonal matrices, the test in Theorem 6 can be simplified. One can show that only principal submatrices with connected index sets have to be considered, since all other principal submatrices are block-diagonal. There are $n(n-1)/2$ such submatrices, and testing one of them requires the computation of at most n determinants, each of order at most $n-1$. Computation of a determinant of order n can be performed using pivoting in $\mathcal{O}(n^3)$ steps, and thus we obtain the following corollary to Theorem 6.

Corollary 1 *The copositivity of a tridiagonal matrix can be checked in at most $\mathcal{O}(n^6)$ operations.*

Practical computations in this case can be much simplified using the sparsity patterns. Efficient algorithms for this case may therefore be possible, and are the subject of further study.

7 Conclusion

The copositive relaxation for general quadratic programming problems is tighter than the Shor-relaxation. Solving the copositive relaxation is less efficient, since even checking whether a point is feasible is an NP-complete problem, despite the fact that the feasible region is convex. However, since examples show that the relaxation is really tighter than the Shor-relaxation, it might be interesting to investigate algorithms to solve special cases, to study their behaviour and see how it compares to the Shor-relaxation.

[5]An open convex cone $\mathcal{K}$ is called homogeneous if for each pair $x, y \in \mathcal{K}$ there exists an isomorphism $g : \mathcal{K} \mapsto \mathcal{K}$ satisfying $g(x) = y$. Examples include the nonnegative orthant and the cone of positive semidefinite matrices; homogeneous cones are not necessarily self-dual.

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