Tractable Convex Relaxations of Nonconvex Quadratic Optimization Problems

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Doctor of Philosophy
University of Edinburgh
2024

Abstract

A quadratically constrained quadratic program (QCQP) is an optimization problem in which a (possibly nonconvex) quadratic function is minimized subject to a combination of linear and quadratic constraints. QCQPs cover a very broad family of optimization problems including linear programming and quadratic programming. In addition, since binary variables can be formulated as quadratic constraints, QCQPs also include mixed binary linear or quadratic optimization problems as special cases.

While QCQPs are, in general, NP-hard to solve to global optimality, they can be approximated by various computationally tractable optimization problems. In particular, convex relaxations can be employed to find an approximate solution as well as a guaranteed lower bound on the optimal value of a QCQP in minimization form. A convex relaxation of a QCQP is said to be exact if the lower bound arising from the relaxation agrees with the optimal value of the QCQP, and inexact otherwise. Convex relaxations also play a crucial role for globally solving QCQPs in a branch-and-bound framework.

In this thesis, we consider various convex relaxations of QCQPs, namely, a linear programming relaxation arising from the reformulation-linearization technique (RLT), referred to as the RLT relaxations, and semidefinite programming (SDP) relaxations such as the standard SDP relaxation and the SDP-RLT relaxation given by combining the RLT and SDP relaxations.

We investigate these relaxations for three subclasses of QCQPs. The first subclass is comprised of general quadratic programming problems that involve minimizing a quadratic function subject to linear constraints. The second subclass is given by quadratic programs with box constraints, which is a special class of quadratic programs with constraints given by finite lower and upper bounds on each variable. Finally, the third subclass consists of standard quadratic optimization problems with a hard cardinality constraint, referred to as sparse StQPs, which involve minimizing a quadratic function over the unit simplex with an additional cardinality constraint and admit a formulation as a mixed binary quadratic optimization problem.

First, we consider RLT relaxations of general nonconvex quadratic programs. We investigate the relations between the polyhedral properties of the feasible region of a quadratic program and that of its RLT relaxation. We establish various connections between recession directions, boundedness, and vertices of the two feasible regions. Using these properties, we present a complete description of the set of instances that admit an exact RLT relaxation. We then give a thorough discussion of how our results can be converted into simple algorithmic procedures to construct instances of quadratic programs with exact, inexact, or unbounded RLT relaxations.

Next, for quadratic programs with box constraints, we investigate RLT and SDP-RLT relaxations. We present complete algebraic descriptions of the set of instances that admit

exact RLT relaxations as well as those that admit exact SDP-RLT relaxations. We show that our descriptions can be converted into algorithms for efficiently constructing instances with (i) exact RLT relaxations, (ii) inexact RLT relaxations, (iii) exact SDP-RLT relaxations, and (iv) exact SDP-RLT but inexact RLT relaxations.

Finally, for sparse StQPs, we consider RLT, SDP, and SDP-RLT relaxations. We establish several structural properties of these relaxations in relation to the corresponding relaxations of StQPs without any cardinality constraints and pay particular attention to the rank-one feasible solutions retained by these relaxations. We then utilize these relations to establish several results about the quality of the lower bounds arising from these relaxations. We also present several conditions that ensure the exactness of each relaxation.

Lay Summary

An optimization problem concerns finding the best solution among multiple possible choices. Optimization problems arise in many real-life situations. For example, navigation tools are designed to find the shortest or fastest route. Traders are interested in investment decisions that optimize various risk and return measures.

Many factors can affect the difficulty of solving optimization problems. Based on those factors, optimization problems can be categorized into different classes. An important factor is the notion of convexity that gives rise to convex and nonconvex optimization problems based on the properties of the constraints and the objective function. Convex optimization problems enjoy several desirable properties that can be used to develop efficient solution algorithms. In contrast, nonconvex optimization problems do not possess such properties and are generally hard to solve.

On the other hand, nonconvex optimization problems have more expressive power and arise in a wide range of applications. Nonconvex optimization problems can be approximated by various convex optimization problems. Optimal values of such approximations yield useful bounds on the optimal value of the original nonconvex optimization problem. If the bound arising from a convex approximation agrees with the optimal value of the original problem, then the convex approximation is said to be exact. Otherwise, the approximation is inexact.

In this thesis, we consider several different nonconvex optimization problems and different approaches for constructing approximate convex optimization problems. We compare such approximations in terms of the strengths of the corresponding bounds. We focus on conditions under which a convex approximation is exact or inexact. Based on these conditions, we develop algorithms for constructing instances of a nonconvex optimization problem with known exactness guarantees. Such instances can be used as testbeds for testing exact and approximate solution algorithms.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Yuzhou Qiu*)

To my mother Peiying Zhou
and to loving memory of my grandmother Rongqin Ni

Acknowledgements

My greatest gratitude first and foremost goes to my supervisor Dr E. Alper Yıldırım, who constantly provided me with support, and taught me about optimization and how to do proper research. I could not have completed my PhD without his support and inspiration. I'm particularly grateful to my second supervisor Professor Jacek Gondzio who gave me encouragement and support during my PhD, and also great lectures on interior point methods.

I want to thank the School of Mathematics and Dr E. Alper Yıldırım for funding my PhD scholarship and Conference Travel Funds.

I want to thank Dr. Merve Bodur, Prof Adam Letchford, and Dr. Burak Buke, for being the examiners of my viva and for providing helpful comments and suggestions to improve my thesis.

I had the pleasure of working with my friend Bo Peng and his supervisor, Professor Immanuel Bomze, on the sparse standard quadratic optimization problem. I gratefully acknowledge this collaboration opportunity, which significantly enriched my research experience.

I would like to thank my mother Peiying Zhou, my father Haitao Qiu, my aunt Haihong Qiu, my uncle Dihui Xu, my grandmother Chengbao Chen, Rongqin Ni, my grandfather Daosheng Qiu, Shunhua Zhou, and my brother Zunyue Xu for supporting me during all these years.

Contents

Chapter 1

Introduction

An optimization problem is concerned with finding the best solution among multiple possible choices and has the following expression:

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & f(x) \\ \text{subject to} & x \in F \end{array} \tag{1.1}$$

The function $f : \mathbb{R}^n \to \mathbb{R}$ is called the objective function, where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The vector $x \in \mathbb{R}^n$ are called the decision variables. $F \subseteq \mathbb{R}^n$ is the feasible space that represents the set of all possible choices of x. It is worth emphasizing that '$\inf$' will be more precise than '$\min$' in the objective function, as the minimum might not be attainable. By convention, we will use '$\min$' instead of '$\inf$' for simplicity in this thesis.

Applications of optimization appear everywhere in real life. For instance, machine learning models need to minimize loss functions based on the training set. Such problems can be formulated and solved as optimization problems (see e.g. [84]). In finance, optimization problems can be utilized for selecting portfolios. Such problems can be formulated as mixed integer quadratic optimization problems with minimizing risk as an objective function and acceptable return as constraints [79]. There are many more applications of optimization, for example, in operational research, statistics, computer science, engineering, and physical sciences (see, e.g. [29, 34]).

Many factors influence the difficulty of optimization problems. One can classify optimization problems based on the notion of convexity. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if it satisfies the following condition:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for $x_1, x_2 \in \mathbb{R}^n$ in the domain of f and $\lambda \in [0,1]$. A set $C \subseteq \mathbb{R}^n$ is called a convex set if, for $y_1, y_2 \in C$, the convex combination $\lambda y_1 + (1-\lambda)y_2 \in C$ for $\lambda \in [0,1]$. An optimization problem is referred to as a convex optimization problem if it consists of a convex objective function and a convex feasible region.

A local optimal solution of an optimization problem is a feasible solution that achieves the smallest objective function value among all feasible solutions in a small neighbourhood around it. A global optimal solution is a feasible solution that has the smallest objective function value among all feasible solutions. Any local optimal solution is a global optimal solution for convex optimization problems. In other words, if a solution satisfies local optimality conditions, then it is guaranteed to be an optimal solution. Efficient algorithms such as the ellipsoid method [64, 60, 119] and interior-point methods [115, 117] can be applied to solve convex optimization problems.

A nonconvex optimization problem has either a nonconvex objective function, a nonconvex feasible region, or both. Nonconvex optimization problems cover a wide range of classes of optimization problems. For example, mixed integer optimization problems belong to nonconvex optimization problems. In general, nonconvex optimization problems arise more frequently in real-life applications (see, e.g. [87, 110]). However, they are more challenging and cannot, in general, be solved efficiently compared to convex optimization problems.

In fact, convex optimization problems can be utilized for approximating nonconvex optimization problems. A convex relaxation of (1.1) is a convex optimization problem taking the following expression:

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & g(x) \\ \text{subject to} & x \in \hat{F} \end{array} \tag{1.2}$$

The feasible region $\hat{F} \subseteq \mathbb{R}^n$ of the convex relaxation is a convex set with the property that $F \subseteq \hat{F}$. The objective function $g : \mathbb{R}^n \to \mathbb{R}$ underestimates the original objective function $g(x) \leq f(x)$ for all feasible points in the original feasible region $x \in F$. By construction, the optimal value of the convex relaxation (1.2) is a lower bound on the optimal value of the original problem (1.1).

Convex relaxations play a fundamental role in the design of global solution methods for nonconvex optimization problems. In particular, one of the most prominent algorithmic approaches for globally solving nonconvex optimization problems is based on a branch-and-bound framework [13, 63, 73], in which the feasible region is systematically divided into smaller subregions and a sequence of subproblems is solved to obtain non-decreasing tighter lower and upper bounds on the optimal value in each subregion. The lower bounds in such a scheme are typically obtained by solving a convex relaxation. For instance, several well-known optimization solvers such as `ANTIGONE` [82], `BARON` [109], `CPLEX` [59], and `GUROBI` [55] utilize convex relaxations for globally solving nonconvex quadratic optimization problems.

In this thesis, we investigate various convex relaxations of several classes of nonconvex optimization problems. In the remainder of this chapter, we give the definitions and formulations of these problems. We also introduce several convex relaxations that we focus on.

1.1 Quadratically Constrained Quadratic Optimization Problems

A quadratically constrained quadratic optimization problem (QCQP) is an optimization problem consisting of a quadratic objective function, a finite number of quadratic constraints, and possibly linear constraints.

$$
\begin{array}{llrcll}
\ell^* = \min\limits_{x\in\mathbb{R}^n} & & \frac{1}{2}x^TQx + c^Tx & & & \\
\text{s.t.} & & & & & \\
& & \frac{1}{2}x^TA^ix + (a^i)^Tx + \delta^i & \leq & 0, & i = 1,\dots,p \\
& & \frac{1}{2}x^TB^jx + (b^j)^Tx + \mu^j & = & 0, & j = 1,\dots,s \\
& & G^Tx & \leq & g & \\
& & H^Tx & = & h &
\end{array}
\qquad \text{(QCQP1)}
$$

(QCQP1) consists of quadratic and linear constraints, where matrices $Q, A^i, B^j \in \mathcal{S}^n$ without loss of generality, where $\mathcal{S}^n$ denotes the cone of $n \times n$ real symmetric matrices, $c, a^i, b^j \in \mathbb{R}^n$ are vectors, and $\delta^i, \mu^j \in \mathbb{R}$ are real numbers for $i = 1, \dots, p$ and $j = 1, \dots, s$, where $p, s \in \mathbb{N}$ are integers that define the numbers of quadratic inequality and equality constraints, respectively. In linear constraints, the parameters are $G \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times p}$, $g \in \mathbb{R}^m$, and $h \in \mathbb{R}^p$. We denote the optimal value of (QCQP1) by $\ell^* \in \mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$. We define $\ell^* = -\infty$ if the problem is unbounded. On the other hand, if the problem is infeasible, then we use $\ell^* = +\infty$. If there are no quadratic equations present and all of the matrices Q, A^i involved in (QCQP1) are positive semidefinite, then QCQP1 is convex and can be solved in polynomial time (to arbitrary fixed precision).

QCQP covers a wide range of problems. For instance, if $p = s = 0$ in (QCQP1), the problem turns into a general quadratic optimization problem. If we further assume $Q = 0$, then (QCQP1) becomes a linear optimization problem. Also, linear optimization problems and quadratic optimization problems with binary variables can be formulated as quadratic constraints in (QCQP1), as $x_i \in \{0, 1\}$ if and only if $x_i(1 - x_i) = 0$.

A polynomial optimization problem can be reformulated as a QCQP by a recursive reformulation method called dimension reduction which is a “folklore” result. Recall that the objective function and constraints of a polynomial optimization problem are polynomial functions. For example, an equality constraint $x^3+x = 0$ can be reformulated as $ux+x = 0$ and $u = x^2$. This approach can also be applied to the polynomial objective function. By recursively executing this method, any polynomial optimization problem can be reduced to QCQP with additional constraints and variables.

QCQPs arise in many applications. They capture many optimization problems such as the knapsack problem, the max-cut, and various pseudo-Boolean optimization problems (see [47]). Therefore, the optimization problem given by (QCQP1) encompasses a wide range of problems. Most of these problems are NP-hard, and QCQPs allow us to study these problems in a unifying framework.

1.2 Relaxations of Quadratically Constrained Quadratic Optimization Problems

In this section, we will first give an equivalent formulation, referred to as the lifted formulation, of (QCQP1) and its relation to different [46, 67, 93].

The lifted formulation of (QCQP1) is as follows:

$$
\begin{array}{lrcll}
\ell^* = \min\limits_{x\in\mathbb{R}^n, X\in\mathcal{S}^n} & \frac{1}{2}\langle Q, X\rangle + c^T x & & & \\
\text{s.t.} & & & & \\
& \frac{1}{2}\langle A^i, X\rangle + (a^i)^T x + \delta^i & \leq & 0, & i = 1,\ldots,p \\
& \frac{1}{2}\langle B^j, X\rangle + (b^j)^T x + \mu^j & = & 0, & j = 1,\ldots,s \\
& G^T x & \leq & g & \\
& H^T x & = & h & \\
& X & = & xx^T, &
\end{array}
\qquad \text{(QCQP1-X)}
$$

where we introduce another matrix variable $X \in \mathcal{S}^n$ in this formulation. In the objective function, $\langle U, V\rangle$ is the trace inner product where

$$
\langle U, V\rangle = \mathrm{trace}(U^T V) = \sum_{i=1}^{m}\sum_{j=1}^{n} U_{ij}V_{ij}
$$

for any $U \in \mathbb{R}^{m\times n}$ and $V \in \mathbb{R}^{m\times n}$.

This formulation has a linear objective function, and all the original quadratic constraints become linear but with additional variables. The only nonlinear constraint is $X = xx^T$, which is not convex. A solution x is feasible for (QCQP1) if and only if $(x, X) = (x, xx^T)$ is feasible for (QCQP1-X) with the same objective value. Therefore, (QCQP1) and (QCQP1-X) have the same optimal value.

If we denote the feasible region of (QCQP1-X) by $\mathcal{F}$, a convex relaxation of (QCQP1-X) is obtained by replacing the $\mathcal{F}$ by $\mathrm{conv}(\mathcal{F})$, where $\mathrm{conv}(\cdot)$ denotes the convex hull. The optimal value of this formulation is the same as (QCQP1-X) [93]. In general, it will be difficult to find the convex hull of $\mathcal{F}$ and solve this reformulation (see, e.g. [30, 42]). Many convex relaxations of (QCQP1-X) arise from this reformulation by employing outer approximations of $\mathcal{F}$ using tractable convex sets.

By removing the quadratic constraint $X = xx^T$ in (QCQP1-X), the resulting relaxation is a linear optimization problem, as the objective function and constraints are all linear. This relaxation can be solved efficiently. However, this linear relaxation is weak in general since the off-diagonal elements of X become unconstrained after removing $X = xx^T$. This basic relaxation can be tightened by exploiting the nonconvex constraint $X = xx^T$ in different ways. We discuss convex relaxations with different tightening approaches in the following sections.

1.2.1 RLT Relaxation

Adams and Sherali [1] introduce the reformulation-linearization technique (RLT) for constructing relaxations of zero-one quadratic programming problems. This technique is adapted to nonconvex quadratic optimization problem in [104] and is extended to QCQP in [99]. The RLT relaxation of (QCQP1-X) is obtained by ignoring $X = xx^T$ and enhancing the feasible region $\mathcal{F}$ by generating quadratic constraints implied by linear constraints. Such quadratic constraints are obtained by multiplying each pair of linear inequality constraints and by multiplying each linear equality constraint by a variable [104]. This approach is similar to the McCormick or Al-Khayyal and Falk inequalities based on bilinear functions with bounded constraints [5, 80]. Note that it is not necessary to add the quadratic constraints obtained from multiplying each pair of equality constraints since they are already implied by the aforementioned procedure (see, e.g., [104, Remark 1]). The resulting quadratic constraints and the objective function are then linearized by substituting each quadratic term $x_i x_j$ by a new variable $X_{ij},\ i = 1, \dots, n; j = 1, \dots, n$. The following is the RLT relaxation of (QCQP1-X).

$$
\begin{array}{lrcll}
\ell_R^* = \min\limits_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} & \frac{1}{2}\langle Q, X\rangle + c^T x & & & \\
\text{s.t.} & & & & \\
& \frac{1}{2}\langle A^i, X\rangle + (a^i)^T x + \delta^i & \leq & 0, & i = 1, \dots, p \\
& \frac{1}{2}\langle B^j, X\rangle + (b^j)^T x + \mu^j & = & 0, & j = 1, \dots, s \\
& G^T x & \leq & g & \\
& H^T x & = & h & \\
& G^T X G - G^T x g^T - g x^T G - g g^T & \geq & 0, & \\
& H^T X - h x^T & = & 0, &
\end{array}
\qquad \text{(QCQP1-RLT)}
$$

ℓ_R^* is a lower bound of ℓ^* ($\ell_R^* \leq \ell^*$), since, for any feasible solution x of (QCQP1), (x, xx^T) is feasible for (QCQP1-RLT) with the same objective value. (QCQP1-RLT) is a linear optimization problem which can be solved by interior-point methods in polynomial time [115, 117].

The RLT constraints strengthen the relation between variables x and X. This relaxation can be further tightened by adding semidefinite constraints, which will be discussed in the following section.

1.2.2 SDP Relaxation and SDP-RLT Relaxation

Another way of constructing a convex relaxation is relaxing the quadratic constraint $X = xx^T$ using semidefinite constraints [93]. A matrix $A \in \mathcal{S}^n$ is positive semidefinite, denoted by $A \succeq 0$, if $d^T A d \geq 0$ for any $d \in \mathbb{R}^n$. $X \succeq xx^T$ denotes that $X - xx^T$ is positive semidefinite. $X = xx^T$ is equivalent to two constraints $X \succeq xx^T$ and $xx^T \succeq X$. Removing the nonconvex constraint $xx^T \succeq X$ gives rise to the following optimization problem:

$$
\begin{array}{lrcll}
\ell_S^* = \min\limits_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} & \frac{1}{2}\langle Q, X\rangle + c^T x & & & \\
\text{s.t.} & & & & \\
& \frac{1}{2}\langle A^i, X\rangle + (a^i)^T x + \delta^i & \leq & 0, & i = 1, \dots, p \\
& \frac{1}{2}\langle B^j, X\rangle + (b^j)^T x + \mu^j & = & 0, & j = 1, \dots, s \\
& G^T x & \leq & g & \\
& H^T x & = & h & \\
& \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} & \succeq & 0, &
\end{array}
\qquad \text{(QCQP1-SDP)}
$$

The condition $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$ is equivalent to $X \succeq xx^T$ by the Schur complement. In this way, the relation between X and x can be further strengthened by the matrix inequality.

Similar to the RLT relaxation, for any feasible solution x of (QCQP1), (x, xx^T) is feasible for (QCQP1-SDP) with the same objective value. Therefore, the optimal value of (QCQP1-SDP) provides a lower bound on the optimal value of (QCQP-1) (i.e. $\ell_S^* \leq \ell^*$). (QCQP1-SDP) is a semidefinite program which can be solved efficiently by interior-point methods

[115, 117].

Sherali and Fraticelli [102] introduce the combination of the RLT and SDP relaxation methods. The RLT constraints may further enhance the relation between x and X. We refer to it as the SDP-RLT relaxation. The following is the SDP-RLT relaxation of (QCQP1-X):

$$\begin{array}{lrcll}
\ell^*_{RS} = \min\limits_{x\in\mathbb{R}^n, X\in\mathcal{S}^n} & \frac{1}{2}\langle Q, X\rangle + c^T x & & & \\
\text{s.t.} & & & & \\
& \frac{1}{2}\langle A^i, X\rangle + (a^i)^T x + \delta^i & \le & 0, & i = 1,\ldots,p \\
& \frac{1}{2}\langle B^j, X\rangle + (b^j)^T x + \mu^j & = & 0, & j = 1,\ldots,s \\
& G^T x & \le & g & \\
& H^T x & = & h & \\
& G^T X G - G^T x g^T - g x^T G - g g^T & \ge & 0, & \\
& H^T X - h x^T & = & 0, & \\
& \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} & \succeq & 0, &
\end{array} \tag{QCQP1-SDP-RLT}$$

(QCQP1-SDP-RLT) is the combination of SDP and RLT relaxations that is generally tighter than either of RLT or SDP relaxations and provides a lower bound for the optimal value of the original QCQP:

$$\max\{\ell^*_R, \ell^*_S\} \le \ell^*_{RS} \le \ell^*.$$

Anstreicher [7] compares the SDP and RLT relaxations for nonconvex QCQPs and shows that lower bounds of SDP-RLT relaxation are substantially better than either technique used alone. However, the trade-off is that the SDP-RLT relaxation is computationally more expensive than RLT or SDP relaxation via algorithms for solving semidefinite optimization problems. Therefore, iterative algorithms like cutting plane methods (see, e.g. [101]) can be applied for solving the SDP-RLT relaxation.

1.2.3 Exact and Inexact Relaxations

If the optimal value of a relaxation is the same as the original optimal value, i.e.,

$$\ell^*_U = \ell^*,$$

where ℓ^*_U is the optimal value with relaxation method $U \in \{R, S, RS\}$, we say this relaxation is exact. Otherwise, the relaxation is inexact. Checking whether a relaxation is exact could be a difficult problem. It has been shown that testing whether the SDP relaxation of the max-cut problem is exact is an NP-hard problem [74]. In addition, an exact relaxation only reveals the optimal value (but not necessarily the optimal solution) of the original problem. The optimal solution is recovered if an optimal solution of an exact relaxation is also feasible for the lifted version of the original problem.

QCQP encompasses a wide class of problems. We investigate three subclasses of problems with particular structures and aim to derive stronger conclusions. We hope it will inspire some results for more general QCQP. The three subclasses of problems are, namely, quadratic optimization problems, quadratic optimization problems with box constraints, and standard quadratic optimization problems under cardinality constraints. We will discuss them in the following three sections.

1.3 Quadratic Optimization Problems

A quadratic optimization problem involves minimizing a (possibly nonconvex) quadratic function over a polyhedron. We remark that quadratic optimization is a special case of (QCQP1) with $q = s = 0$:

$$\ell^* = \min_{x \in \mathbb{R}^n} \{q(x) : x \in F\}, \tag{QP}$$

where $q : \mathbb{R}^n \to \mathbb{R}$ and $F \subseteq \mathbb{R}^n$ are given by

$$\begin{aligned} q(x) &= \tfrac{1}{2} x^T Q x + c^T x, && (1.3) \\ F &= \left\{ x \in \mathbb{R}^n : G^T x \leq g, \quad H^T x = h \right\}. && (1.4) \end{aligned}$$

Quadratic optimization problems constitute an important class of problems in global optimization and they arise in a wide variety of applications, ranging from support vector machines and portfolio optimization to various combinatorial optimization problems such as the maximum stable set problem and the MAX-CUT problem. We refer the reader to [47] and the references therein. In addition, quadratic optimization problems also appear as subproblems in sequential quadratic programming algorithms for solving more general classes of nonlinear optimization problems (see, e.g., [85]).

It is well-known that quadratic optimization problems, in general, are NP-hard (see, e.g., [89, 95]). As such, global optimization algorithms for quadratic optimization problems are generally based on spatial branch-and-bound methods. Convex relaxations play the crucial role of generating lower bounds in this framework.

In the next section, we will discuss a further special subclass of (QP) called the quadratic optimization problem with box constraints.

1.4 Quadratic Optimization Problems with Box Constraints

A quadratic optimization problem with box constraints is an optimization problem in which a possibly nonconvex quadratic function is minimized subject to lower and upper bounds on each variable:

$$\ell^* = \min_{x \in \mathbb{R}^n} \{q(x) : x \in F\}, \tag{BoxQP}$$

where $q : \mathbb{R}^n \to \mathbb{R}$ and $F \subseteq \mathbb{R}^n$ are respectively given by

$$q(x) = \tfrac{1}{2} x^T Q x + c^T x, \quad F = \{x \in \mathbb{R}^n : 0 \leq x \leq e\}.$$

where $0 \in \mathbb{R}^n$ denotes the vector of all zeros and $e \in \mathbb{R}^n$ denotes the vector of all ones.

(BoxQP) is a special class of (QCQP1) with the following choices of G and g,

$$G^T = \begin{pmatrix} I \\ -I \end{pmatrix}, \quad g = \begin{pmatrix} e \\ 0 \end{pmatrix},$$

and no equality constraints. Here, $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

(BoxQP) is regarded as a "fundamental problem" in global optimization that appears in a multitude of applications (see, e.g., [38, 58]). If Q is a positive semidefinite matrix, (BoxQP) can be solved in polynomial time [72]. However, if Q is an indefinite or negative semidefinite matrix, then (BoxQP) is an NP-hard problem [89, 95]. It is even NP-hard to approximate a local minimizer of (BoxQP) [2].

1.5 Standard Quadratic Optimization Problems under Cardinality Constraints

The Standard Quadratic Optimization Problem (StQP) consists of minimizing a quadratic form over the standard unit simplex (all vectors with no negative coordinates that sum up to one) and can be expressed as follows:

$$\ell(Q) := \min_{x\in\mathbb{R}^n} \left\{ x^T Q x : x \in F \right\}, \tag{StQP}$$

where $F \subset \mathbb{R}^n$ denotes the standard simplex given by

$$F := \left\{ x \in \mathbb{R}^n : e^T x = 1, \quad x \geq 0 \right\}, \tag{1.5}$$

where $e \in \mathbb{R}^n$ denotes the vector of all ones.

We introduce the following variant under a cardinality constraint, referred to as the *sparse StQP*:

$$\ell_\rho(Q) := \min_{x\in\mathbb{R}^n} \left\{ x^T Q x : x \in F_\rho \right\},$$

where

$$F_\rho := \{ x \in F : \quad \|x\|_0 \leq \rho \} . \tag{1.6}$$

Here, $\|x\|_0$ denotes the number of nonzero components of a vector x and $\rho \in \{1, \dots, n\}$ is the sparsity parameter.

By introducing binary variables, the sparse StQP can be reformulated as a mixed-binary QP:

$$\begin{array}{ll} \ell_\rho(Q) = \min\limits_{x\in\mathbb{R}^n} & x^T Q x \\ \text{s.t.} & \\ & \begin{array}{rcl} e^T x &=& 1 \\ e^T u &=& \rho \\ x &\leq& u \\ u &\in& \{0,1\}^n \\ x &\geq& 0. \end{array} \end{array} \tag{StQP(ρ)}$$

We remark that StQP(ρ) is a special case of (QCQP1) since binary variables can be formulated as quadratic equality constraints.

(StQP) is NP-hard since the maximum clique problem admits a formulation as an instance of (StQP) [50, 83]. Based on the quantitative framework set up by Markowitz [79], a generalization of portfolio selection problems can also be formulated as (StQP). (StQP(ρ)) arises in portfolio section problems with an upper bound on the number of investment options. The cardinality constraint is essential because monitoring costs are charged for each non-zero weight, and an active management fee is charged when performing index-tracking without controlling the number of assets (see, e.g. [14, 15]). We will show that (StQP(ρ))

is, in general, NP-hard in Chapter 4.

1.6 Motivations and Contributions

In this section, we will first briefly discuss our motivations. Recall that convex relaxations can be utilised to solve or approximate difficult nonconvex optimization problems. Exact convex relaxations can recover the optimal value of nonconvex optimization problems. We study different convex relaxations of (QP), (BoxQP), and (StQP(ρ)) and identify exactness conditions for each relaxation. Such conditions shed light on easier subclasses of a difficult optimization problem. These conditions can be used to develop instance generation procedures with prespecified exactness guarantees, which can be of independent interest for testing the performance of exact or approximate algorithms.

Convex relaxations can be tightened by adding further constraints. However, the computational cost of a relaxation also increases with the number of constraints. Therefore, the trade-off between the computational cost of a relaxation and the quality of the lower bound is an important concern. By comparing the strengths and weaknesses of different convex relaxations, we aim to shed light on this trade-off.

Convex relaxations are vital in algorithmic approaches for globally solving nonconvex optimization problems using the branch-and-bound method and are used in several well-known optimization solvers such as `ANTIGONE` [82], `BARON` [109], `CPLEX` [59], and `GUROBI` [55]. In branch-and-bound methods, the feasible region is subdivided into smaller regions, giving rise to a sequence of subproblems. Convex relaxations of these subproblems yield increasingly tight lower bounds. Therefore, tight and cheap convex relaxations significantly enhance the effectiveness of the branch-and-bound method by identifying strong lower bounds early on.

We next discuss our contributions in this thesis.

1.6.1 Quadratic Optimization Problems

We study the RLT relaxation of (QP). We discuss the relations between the polyhedral properties of the feasible region of (QP) and that of its RLT relaxation. In particular, we focus on the relations between recession directions, boundedness, and vertices of the two feasible regions. As a byproduct of our analysis, we obtain a complete algebraic description of the set of instances of (QP) that admit an exact RLT relaxation.

Our contributions are as follows:

1. We show that various properties of the feasible region of (QP) such as boundedness and existence of vertices directly translate to that of its RLT relaxation.

2. We present simple procedures for constructing recession directions and vertices of the feasible region of the RLT relaxation from their counterparts in the original feasible region.

3. For a certain subclass of quadratic optimization problems, we obtain a complete description of the set of all vertices. By observing that every quadratic optimization

problem can be equivalently reformulated in this form, we discuss the implications of this observation on RLT relaxations of general quadratic optimization problems.

4. We identify a necessary and sufficient condition for instances of (QP) that admit an exact RLT relaxation.

5. By using the aforementioned exactness characterization together with the optimality conditions of the RLT relaxation, we present simple algorithmic procedures to construct a family of instances of (QP) with an unbounded, inexact, or exact RLT relaxation.

1.6.2 Quadratic Optimization Problems with Box Constraints

We study the RLT and SDP-RLT relaxations of (BoxQP). Similar to (QP), we describe the set of instances of (BoxQP) that admit exact RLT relaxations and exact SDP-RLT relaxations. We give stronger conclusions on vertices of the RLT relaxation of (BoxQP) than their counterparts for general quadratic optimization problems. We further develop instance generation procedures of (BoxQP). In particular, we propose efficient algorithms for constructing instances with (i) exact RLT relaxations, (ii) inexact RLT relaxations, (iii) exact SDP-RLT relaxations, and (iv) exact SDP-RLT but inexact RLT relaxations.

Our contributions are as follows:

1. By utilizing the recently proposed perspective on convex underestimators induced by convex relaxations [118], we establish several useful properties of each of the two convex underestimators associated with the RLT relaxation and the SDP-RLT relaxation.

2. We present two equivalent algebraic descriptions of the set of instances of (BoxQP) that admit exact RLT relaxations. The first description arises from the analysis of the convex underestimator induced by the RLT relaxation, whereas the second description is obtained by using linear programming duality. In particular, we further show that each component of each vertex of the RLT relaxation of (BoxQP) given by (3.2) lies in the set $\{0, \frac{1}{2}, 1\}$.

3. By relying on the second description of the set of instances with an exact RLT relaxation, we propose an algorithm for efficiently constructing an instance of (BoxQP) that admits an exact RLT relaxation and another algorithm for constructing an instance with an inexact RLT relaxation.

4. We establish that strong duality holds and that primal and dual optimal solutions are attained for the SDP-RLT relaxation and its dual. By relying on this relation, we give an algebraic description of the set of instances of (BoxQP) that admit an exact SDP-RLT relaxation.

5. By utilizing this algebraic description, we propose an algorithm for constructing an instance of (BoxQP) that admits an exact SDP-RLT relaxation and another one for constructing an instance that admits an exact SDP-RLT but an inexact RLT relaxation.

1.6.3 Standard Quadratic Optimization Problems under Cardinality Constraints

We study RLT, SDP, and SDP-RLT relaxations of (StQP(ρ)). In particular, we establish several structural properties of these relaxations and shed light on the relations between each relaxation of (StQP(ρ)) and the corresponding relaxation of (StQP).

Our contributions are as follows:

1. We study the relations between (StQP) and (StQP(ρ)) with the RLT, SDP, and SDP-RLT relaxations, where we compare the feasible region of the relaxations of (StQP) with the projected feasible region of the corresponding relaxations of (StQP(ρ)).

2. We compare the performance of three relaxations of (StQP(ρ)) for different values of ρ. We show that the RLT and SDP relaxations of (StQP(ρ)) are relatively weak and pay more attention to the structural properties of the SDP-RLT relaxation.

3. By explicitly constructing rank-one solutions that violate the original cardinality constraint $\|x\|_0 \leq \rho$ in the projected feasible region of relaxations of (StQP(ρ)), we establish upper bounds on the lower bound arising from (StQP(ρ)).

4. We establish that each rank-one solution in the SDP-RLT relaxation of (StQP(ρ)) is feasible for the SDP-RLT relaxation of (StQP($\rho + 1$)) by proposing a way of constructing such feasible solution in the SDP-RLT relaxation of (StQP($\rho + 1$)).

1.7 Organization

This thesis is organized as follows. In Table 1.1, we summarize our main results for each of the three classes of problems. For each result, we include the main techniques used, the relevant section number, and the complexity result if applicable. We present properties of the RLT relaxation of quadratic optimization problems in Chapter 2. Chapter 3 focuses on the results of RLT and SDP-RLT relaxation of quadratic optimization problems with box constraints. In Chapter 4, we compare the RLT, SDP and SDP-RLT relaxations of (StQP) and (StQP(ρ)). Finally, Chapter 5 gives an overall conclusion of the thesis.

Problem class	Summary of results	Main techniques	Link to the result	Complexity
(QP)	Vertices and recession directions of the RLT relaxation	Polyhedral theory	Section 2.3.2 Section 2.3.3	—
(QP)	Description of instances with exact RLT relaxations	Optimality conditions	Section 2.5	—
(QP)	Instance generation algorithms with unbounded, exact, and inexact RLT relaxations	Optimality conditions, vertex description and recession directions	Section 2.6	$\mathcal{O}(n^2m + nm^2 + n^2p)$ for a given input
(BoxQP)	Vertices in the RLT relaxation	Convex underestimator	Section 3.5.1	—
(BoxQP)	Two descriptions of instances with exact RLT relaxations	Optimality conditions and vertex description	Section 3.5.2 Section 3.5.3	—
(BoxQP)	Description of instances with exact SDP-RLT relaxations	Optimality conditions	Section 3.6.1	—
(BoxQP)	Instance generation algorithms with exact RLT relaxations	Optimality conditions and vertex description	Algorithm 1	$\mathcal{O}(n^2)$
(BoxQP)	Instance generation algorithms with inexact RLT relaxations	Optimality conditions and vertex description	Algorithm 2	$\mathcal{O}(n^2)$
(BoxQP)	Instance generation algorithms with exact SDP-RLT relaxations	Optimality conditions	Algorithm 3	$\mathcal{O}(n^2)$
(BoxQP)	Instance generation algorithms with exact SDP-RLT relaxations and inexact RLT relaxation	Optimality conditions and vertex description	Algorithm 4	$\mathcal{O}(n^2)$
(SpStQP)	Results on RLT relaxation	Polyhedral theory	Section 4.2.1	—
(SpStQP)	Results on SDP relaxation	Matrix analysis	Section 4.2.2	—
(SpStQP)	Results on SDP-RLT relaxation	Matrix analysis	Section 4.2.3	—

Table 1.1: A Summary of Main Results

Chapter 2

Quadratic Optimization Problems

In this chapter, we study the RLT relaxation of general quadratic optimization problems. This chapter is based on [91].

Recall that a quadratic optimization problem takes the following formulation:

$$\ell^* = \min_{x \in \mathbb{R}^n} \left\{ q(x) : x \in F \right\}, \tag{QP}$$

where $q : \mathbb{R}^n \to \mathbb{R}$ and $F \subseteq \mathbb{R}^n$ are given by

$$\begin{aligned} q(x) &= \tfrac{1}{2} x^T Q x + c^T x, \\ F &= \left\{ x \in \mathbb{R}^n : G^T x \leq g, \quad H^T x = h \right\}. \end{aligned}$$

Quadratic optimization problems have a wide range of applications. For example in data mining, they appear in image and signal processing models. In finance, portfolio selection problems can be formulated as quadratic optimization problems. Quadratic optimization problems also appear as subproblems for finding the best search direction of nonlinear optimization problems [51].

When $Q \in \mathcal{S}^n$ is positive semidefinite, one can apply interior-point methods [40] or the ellipsoid method [60, 119] to solve the problem in polynomial time. However, this class of problems is generally NP-hard (see, e.g., [89, 95]).

We consider the polyhedron F in the general form given by (1.4) as opposed to a more convenient form such as the LP standard form for the following reasons. First, many classes of problems such as quadratic optimization problems with box constraints and standard quadratic optimization problems have an associated natural formulation, and converting them to another form generally requires the introduction of additional variables and/or constraints. Such a conversion increases the dimension of the corresponding RLT relaxation, which, in turn, increases the computational cost of solving the relaxation. Second, such a conversion may change the polyhedral structure of the feasible region of (QP). For instance, a nonempty polyhedron F may not have any vertices but any nonempty polyhedron in standard form necessarily has at least one vertex. On the other hand, our goal in this thesis is to identify the relations between the polyhedral properties of the original feasible region F and those of the feasible region of the relaxation $\mathcal{F}$. Third, we illustrate the RLT relaxations arising from two equivalent formulations may not necessarily be equivalent. As such, we adopt the general description.

This chapter is organized as follows. We review the literature in Section 2.1. We present basic results about polyhedra in Section 2.2. We introduce the RLT relaxation and discuss its polyhedral properties in Section 2.3. Section 2.4 is devoted to the discussion of duality and optimality conditions of the RLT relaxation. We introduce the convex underestimators induced by the RLT relaxation and present necessary and sufficient conditions for an exact RLT relaxation in Section 2.5. We discuss how our results can be used to efficiently construct instances of (QP) with exact, inexact, or unbounded RLT relaxations in Section 2.6. Finally, Section 2.7 concludes this chapter.

2.1 Literature Review

For general quadratic optimization problems, the ideas that led to the RLT relaxation were developed by several authors in a series of papers. To the best of our knowledge, the terminology first appears in [100], where the authors develop a branch-and-bound method based on RLT relaxations for solving bilinear quadratic optimization problems, i.e., instances of (QP) for which all the diagonal entries of Q are equal to zero. In [104], this approach is extended to general quadratic optimization problems and several properties such as the effect of redundant constraints on the RLT relaxation are established. The RLT relaxation has been extended to more general classes of discrete and nonconvex optimization problems (see, e.g., [99]).

RLT relaxations of quadratic optimization problems can be further strengthened by adding a set of convex quadratic constraints [104] or by adding semidefinite constraints [6], referred to as the SDP-RLT relaxation. The latter relaxation usually provides much tighter bounds than the RLT relaxation at the expense of significantly higher computational effort. Furthermore, a continuum of linear relaxations between the RLT relaxation and the SDP-RLT relaxation can be obtained by viewing the semidefinite constraint as an infinite number of linear constraints and adding these linear cuts in a cutting plane framework [102]. Alternatively, by using another representation as an infinite number of second-order conic constraints, one can obtain a sequence of second-order conic relaxations that are provably tighter than their linear counterparts [65]. In terms of computational cost, second-order conic relaxations roughly lie between cheaper linear relaxations and more expensive semidefinite relaxations. Alternative convex relaxations can be obtained by relying on the observation that every quadratic optimization problem can be equivalently formulated as an instance of a copositive optimization problem [30], which is a convex but NP-hard problem. Nevertheless, the copositive cone can be approximated by various sequences of tractable convex cones, each of which gives rise to relaxation hierarchies that are exact in the limit. We refer the reader to [118] for a unified treatment of a rather large family of convex relaxations arising from the copositive formulation, and to [7, 12, 20] for comparisons of various convex relaxations.

Despite the fact that there exist many convex relaxations that are provably at least as tight as the RLT relaxation, the latter is used extensively in global optimization algorithms (see, e.g., [9, 99, 109]) due to the fact that state-of-the-art linear optimization solvers can usually scale very well with the size of the problem. Furthermore, they are generally much numerically more stable than second-order conic optimization and semidefinite optimization solvers that are required for solving tighter relaxations. As such, RLT relaxations play a central role in global solution algorithms for nonconvex optimization problems, which motivates our focus on their polyhedral properties.

2.2 Preliminaries

In this section, we review basic facts about polyhedra. We refer the reader to [97] for proofs and further results.

Let $F \subseteq \mathbb{R}^n$ be a nonempty polyhedron given by (1.4). The recession cone of F, denoted by $F_\infty \subseteq \mathbb{R}^n$, is given by

$$F_\infty = \left\{ d \in \mathbb{R}^n : G^T d \leq 0, \quad H^T d = 0 \right\}. \tag{2.1}$$

Note that F_∞ is a polyhedral cone. By the Minkowski-Weyl Theorem, it is finitely generated, i.e., there exists $d^j \in \mathbb{R}^n,\ j = 1, \ldots, t$ such that

$$F_\infty = \operatorname{cone}\left(\left\{d^1, \ldots, d^t\right\}\right) = \left\{ \sum_{j=1}^{t} \lambda_j d^j : \lambda_j \geq 0,\ j = 1, \ldots, t \right\}, \tag{2.2}$$

where $\operatorname{cone}(\cdot)$ denotes the conic hull.

Recall that a hyperplane $\{x \in \mathbb{R}^n : a^T x = \alpha\}$, where $a \in \mathbb{R}^n \backslash \{0\}$ and $\alpha \in \mathbb{R}$, is a *supporting hyperplane* of F if $\alpha = \min\{a^T x : x \in F\}$. A set $F_0 \subseteq F$ is a nonempty *face* of F if $F_0 = F$ or F_0 is given by the intersection of F with a supporting hyperplane. In particular, $F_0 \subseteq F$ is a face of F if and only if there exists a (possibly empty) submatrix $G^0 \in \mathbb{R}^{n \times m_0}$ of G, where $m_0 \leq m$, such that

$$F_0 = \{x \in F : (G^0)^T x = g^0\}, \tag{2.3}$$

where $g^0 \in \mathbb{R}^{m_0}$ denotes the corresponding subvector of g. In particular, F_0 is a *minimal face* of F if and only if it is an affine subspace, i.e., if and only if there exists a submatrix $G^0 \in \mathbb{R}^{n \times m_0}$ of G, where $m_0 \leq m$, and a corresponding subvector $g^0 \in \mathbb{R}^{m_0}$ of g such that

$$F_0 = \{x \in \mathbb{R}^n : (G^0)^T x = g^0, \quad H^T x = h\}. \tag{2.4}$$

Let

$$\phi = \operatorname{rank}\left(\begin{bmatrix} G & H \end{bmatrix}\right) \leq n. \tag{2.5}$$

denotes the rank of the concatenated matrix $\begin{bmatrix} G & H \end{bmatrix}$.

The dimension of each minimal face $F_0 \subseteq F$ is equal to $n - \phi$. Note that every polyhedron has a finite number of minimal faces. In particular, if $\phi = n$, then each minimal face $F_0 \subseteq F$ consists of a single point called a *vertex*. This gives rise to the following useful characterizations of vertices.

Lemma 2.2.1. *Let $F \subseteq \mathbb{R}^n$ be a nonempty polyhedron given by* (1.4) *and let $\hat{x} \in F$. Then, the following statements are equivalent:*

(i) $\hat{x}$ is a vertex of F.

(ii) $\hat{x} - \hat{d} \in F$ and $\hat{x} + \hat{d} \in F$ if and only if $\hat{d} = 0$.

(iii) There exists a partition of $G = \begin{bmatrix} G^0 & G^1 \end{bmatrix}$ *and a corresponding partition of* $g^T = \begin{bmatrix} (g^0)^T & (g^1)^T \end{bmatrix}$ *such that* $(G^0)^T \hat{x} = g^0$, $(G^1)^T \hat{x} < g^1$, *and the matrix* $\begin{bmatrix} G^0 & H \end{bmatrix}$ *has full row rank.*

(iv) There exists $a \in \mathbb{R}^n$ *such that* $\hat{x}$ *is the unique optimal solution of* $\min\{a^T x : x \in F\}$.

Next, we collect several results concerning the recession cone F_∞ given by (2.1).

Lemma 2.2.2. *Let* $F \subseteq \mathbb{R}^n$ *be a nonempty polyhedron given by* (1.4)*. Then, the following statements are equivalent:*

(i) F *has no vertices.*

(ii) F *contains a line.*

(iii) $\phi < n$, *where* ϕ *is defined as in* (2.5).

(iv) There exists $\hat{d} \in \mathbb{R}^n \backslash \{0\}$ *such that* $\hat{d} \in F_\infty$ *and* $-\hat{d} \in F_\infty$ *(i.e.,* F_∞ *contains a line).*

(v) F_∞ *has no vertices.*

Recall that F is a polytope if it is bounded. In this case, $F_\infty = \{0\}$. We next state a useful characterization of polytopes.

Lemma 2.2.3. *Let* $F \subseteq \mathbb{R}^n$ *be a nonempty polyhedron given by* (1.4)*. Then,* F *is bounded if and only if, for every* $z \in \mathbb{R}^n$*, there exists* $(u, w) \in \mathbb{R}^m \times \mathbb{R}^p$ *such that*

$$Gu + Hw = z, \quad u \geq 0. \tag{2.6}$$

Proof. Since F is nonempty, the boundedness of F is equivalent to

$$F_\infty = \left\{ d \in \mathbb{R}^n : G^T d \leq 0, \quad H^T d = 0 \right\} = \{0\}.$$

Therefore, F is bounded if and only if, for every $z \in \mathbb{R}^n$, the optimal value of the linear optimization problem

$$\max\{z^T d : G^T d \leq 0, \quad H^T d = 0\}$$

is equal to zero. The assertion follows from linear optimization duality. □

Finally, we close this section with a useful decomposition result.

Lemma 2.2.4. *Let* $F \subseteq \mathbb{R}^n$ *be a nonempty polyhedron given by* (1.4)*, and let* $F_i \subseteq F,\ i = 1, \ldots, s$ *denote the set of minimal faces of* F*. Then,*

$$F = conv\left(\left\{v^1, \ldots, v^s\right\}\right) + F_\infty, \tag{2.7}$$

where F *is the Minkowski sum of two sets, and* $v^i \in F_i,\ i = 1, \ldots, s$*, and* F_∞ *is given by* (2.1).

2.3 Polyhedral Properties of RLT Relaxations

In this section, we look at the corresponding RLT relaxation of (QP). Then, we focus on the relations between the polyhedral properties of the feasible region of (QP) and of its RLT relaxation. We establish several connections between recession directions, boundedness, and vertices of the two feasible regions. For a specific class of quadratic optimization problems, we give a complete characterization of the set of vertices of the feasible region of the RLT relaxation. We finally discuss the implications of this observation on RLT relaxations of general quadratic optimization problems.

2.3.1 RLT Relaxations

In this section, we discuss RLT relaxation, literature review, and optimality conditions of (QP). We also give the notation that will be used in this thesis.

Recall that an instance of (QP) is completely specified by the objective function $q(x)$ and the feasible region F given by (1.3) and (1.4), respectively. The RLT relaxation of (QP) is obtained by generating quadratic constraints implied by linear constraints. Such quadratic constraints are obtained by multiplying each pair of linear inequality constraints and by multiplying each linear equality constraint by a variable. Note that it is not necessary to add the quadratic constraints obtained from multiplying each pair of equality constraints since they are already implied by the aforementioned procedure (see, e.g., [104, Remark 1]). The resulting quadratic constraints and the objective function are then linearized by substituting each quadratic term $x_i x_j$ by a new variable $X_{ij},\ i = 1, \ldots, n; j = 1, \ldots, n.$

For a given instance of (QP), the RLT relaxation of (QP) is therefore given by

$$\text{(RLT)} \quad \ell_R^* = \min_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} \left\{ \tfrac{1}{2}\langle Q, X\rangle + c^T x : (x, X) \in \mathcal{F} \right\},$$

where

$$\mathcal{F} = \left\{ (x, X) \in \mathbb{R}^n \times \mathcal{S}^n : \begin{array}{rcl} G^T x & \leq & g \\ H^T x & = & h \\ H^T X & = & h x^T \\ G^T X G - G^T x g^T - g x^T G + g g^T & \geq & 0 \end{array} \right\}. \tag{2.8}$$

Note that (RLT) is a linear relaxation of (QP) since the objective function and all constraints are linear functions of $(x, X) \in \mathbb{R}^n \times \mathcal{S}^n$ and, for each $\hat{x} \in F$, we have $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}$ with the same objective function value. Therefore,

$$\ell_R^* \leq \ell^*. \tag{2.9}$$

We remark that every other quadratic constraint that can be generated by the pairwise multiplication of linear constraints in F is already implied by $\mathcal{F}$ (see also [104, Remark 1]). Indeed, consider the RLT constraints obtained by multiplying each pair of equality constraints given by $H^T X H - H^T x h^T - h x^T H + h h^T = 0$ as well as those arising from the multiplication of each inequality and equality constraint given by $h g^T - h x^T G - H^T x g^T + H^T X G = 0$. It is easy to see that both are implied by the constraints $H^T x = h$ and $H^T X = h x^T$.

An interesting question is whether all constraints in (2.8) are, in fact, necessary for the RLT relaxation. Our next result identifies a family of instances of (QP) for which $\mathcal{F}$ can be simplified without affecting the lower bound ℓ_R^*.

Proposition 2.3.1. *Suppose that F given by* (1.4) *is nonempty and that $G \in \mathbb{R}^{n\times m}$ and $H \in \mathbb{R}^{n\times p}$ can be permuted into the following block diagonal form:*

$$G = \begin{bmatrix} G^1 & 0 & \cdots & 0 \\ 0 & G^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G^k \end{bmatrix}, \quad H = \begin{bmatrix} H^1 & 0 & \cdots & 0 \\ 0 & H^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H^k \end{bmatrix}, \tag{2.10}$$

where $G^i \in \mathbb{R}^{n_i\times m_i}$ and $H^i \in \mathbb{R}^{n_i\times p_i},\ i = 1,\ldots,k$. Suppose that $Q \in \mathcal{S}^n$, $g \in \mathbb{R}^m$, $h \in \mathbb{R}^p$, and $x \in \mathbb{R}^n$ are permuted accordingly so that

$$Q = \begin{bmatrix} Q^{11} & Q^{12} & \cdots & Q^{1k} \\ Q^{21} & Q^{22} & \cdots & Q^{2k} \\ \vdots & \vdots & \ddots & \vdots \\ Q^{k1} & Q^{k2} & \cdots & Q^{kk} \end{bmatrix}, \quad g = \begin{bmatrix} g^1 \\ g^2 \\ \vdots \\ g^k \end{bmatrix}, \quad h = \begin{bmatrix} h^1 \\ h^2 \\ \vdots \\ h^k \end{bmatrix}, \quad x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{bmatrix},$$

where $Q^{ii} \in \mathcal{S}^{n_i}$, $g^i \in \mathbb{R}^{m_i}$, $h^i \in \mathbb{R}^{p_i}$, and $x^i \in \mathbb{R}^{n_i}$ for each $i = 1,\ldots,k$, and $Q^{ij} = (Q^{ji})^T \in \mathbb{R}^{n_i\times n_j}$ for each $1 \le i < j \le k$. For $(x, X) \in \mathcal{F}$, suppose also that x^i denotes the i-th subvector of x, and X^{ij} denotes the submatrix of X corresponding to Q^{ij}, $i = 1,\ldots,k;\ j = 1,\ldots,k$.

(i) If $Q^{ii} = 0$ for some $i = 1,\ldots,k$, then the RLT lower bound ℓ_R^* remains unchanged if the following RLT constraints are removed from $\mathcal{F}$:

$$\begin{aligned} (H^i)^T X^{ii} &= h^i(x^i)^T \\ (G^i)^T X^{ii} G^i - (G^i)^T x^i (g^i)^T - g^i (x^i)^T G^i + g^i (g^i)^T &\ge 0. \end{aligned}$$

(ii) If $Q^{ij} = 0$ for some $1 \le i < j \le k$, then the RLT lower bound ℓ_R^* remains unchanged if the following RLT constraints are removed from $\mathcal{F}$:

$$\begin{aligned} (H^i)^T X^{ij} &= h^i(x^j)^T \\ (H^j)^T X^{ji} &= h^j(x^i)^T \\ (G^i)^T X^{ij} G^i - (G^i)^T x^i (g^j)^T - g^i (x^j)^T G^j + g^i (g^j)^T &\ge 0. \end{aligned}$$

Proof. We provide the proof of only (i) as the proof of (ii) is very similar. Let $\bar{\mathcal{F}} \subseteq \mathbb{R}^n \times \mathcal{S}^n$ denote the feasible region obtained from $\mathcal{F}$ by removing the two sets of constraints in (i) and let $\bar{\ell}$ denote the optimal value of the RLT relaxation over $\bar{\mathcal{F}}$. Clearly, $\bar{\ell} \le \ell_R^*$ since $\mathcal{F} \subseteq \bar{\mathcal{F}}$. By (2.10) and the structure of the RLT constraints, note that X^{ii} is unrestricted in $\bar{\mathcal{F}}$. Therefore, for any $(\bar{x}, \bar{X}) \in \bar{\mathcal{F}}$, let us define $x = \bar{x}$, $X = \bar{X}$, and redefine $X^{ii} = \bar{x}^i(\bar{x}^i)^T$. It is easy to verify that $(x, X) \in \mathcal{F}$ as it satisfies the two sets of constraints in (i). Furthermore, $\frac{1}{2}\langle Q, \bar{X}\rangle + c^T\bar{x} = \frac{1}{2}\langle Q, X\rangle + c^T x$ since $Q^{ii} = 0$. Therefore, for each feasible solution in $\bar{\mathcal{F}}$, there exists a corresponding feasible solution in $\mathcal{F}$ with the same objective function value. It follows that $\bar{\ell} = \ell_R^*$, which completes the proof. $\square$

Under the assumptions of Proposition 2.3.1, one can compute the same RLT lower bound ℓ_R^* by instead solving a linear optimization problem of a smaller dimension, which may considerably reduce the computational cost of solving (RLT). For instance, if $F = \{x \in \mathbb{R}^n : 0 \leq x_j \leq 1, \quad j = 1, \ldots, n\}$, it follows from Proposition 2.3.1 that one only needs to introduce the RLT constraints $X_{ii} - x_i \leq 0$, $X_{ii} \geq 0$, and $X_{ii} - 2x_i + 1 \geq 0$ whenever $Q_{ii} \neq 0, \ i = 1, \ldots, n$; and the RLT constraints $X_{ij} - x_i \leq 0$, $X_{ij} - x_j \leq 0$, $X_{ij} \geq 0$, and $X_{ij} - x_i - x_j + 1 \geq 0$ whenever $Q_{ij} \neq 0, \ 1 \leq i < j \leq n$.

Henceforth, we assume that $\mathcal{F}$ contains all the RLT constraints as given by (2.8) since Proposition 2.3.1 implies that the RLT lower bound is independent of the exclusion of unnecessary RLT constraints from $\mathcal{F}$.

2.3.2 Recession Cones and Boundedness

In this section, we present several relations between the recession cones associated with the polyhedral feasible regions F and $\mathcal{F}$ of (QP) and (RLT), respectively. We also discuss the boundedness relation between F and $\mathcal{F}$.

Recall that the recession cone of F, denoted by F_∞, is given by (2.1). Similarly, we use $\mathcal{F}_\infty$ to denote the recession cone of $\mathcal{F}$, which is given by

$$\mathcal{F}_\infty = \left\{ (d, D) \in \mathbb{R}^n \times \mathcal{S}^n : \begin{array}{rcl} G^T d & \leq & 0 \\ H^T d & = & 0 \\ H^T D - h d^T & = & 0 \\ G^T D G - G^T d g^T - g d^T G & \geq & 0 \end{array} \right\}. \tag{2.11}$$

Note that

$$(\hat{d}, \hat{D}) \in \mathcal{F}_\infty \Longrightarrow \hat{d} \in F_\infty. \tag{2.12}$$

Our next result gives a recipe for constructing recession directions of $\mathcal{F}$ from recession directions and elements of F.

Proposition 2.3.2. *Let $F \subseteq \mathbb{R}^n$ be a nonempty polyhedron given by* (1.4) *and let $P = \begin{bmatrix} d^1 & \cdots & d^t \end{bmatrix} \in \mathbb{R}^{n \times t}$, where $d^1, \ldots, d^t$ are defined as in* (2.2)*. Then, for each $\hat{d} \in F_\infty$, each $\hat{x} \in F$, and each $\hat{K} \in \mathcal{N}^t$, where $\mathcal{N}^t$ denotes the cone of componentwise nonnegative $t \times t$ real symmetric matrices, we have $(\hat{d}, \hat{D}) \in \mathcal{F}_\infty$, where $\hat{D} = \hat{x}\hat{d}^T + \hat{d}\hat{x}^T + P\hat{K}P^T \in \mathcal{S}^n$.*

Proof. Since $\hat{d} \in F_\infty$, we have $G^T \hat{d} \leq 0$ and $H^T \hat{d} = 0$ by (2.1). Furthermore, we have

$$H^T \hat{D} - h\hat{d}^T = H^T \hat{x}\hat{d}^T + H^T \hat{d}\hat{x}^T + H^T P \hat{K} P^T - h\hat{d}^T = h\hat{d}^T - h\hat{d}^T = 0,$$

where we used $H^T \hat{x} = h$, $H^T \hat{d} = 0$, and $H^T P = 0$. Finally,

$$\begin{array}{rcl} G^T \hat{D} G - G^T \hat{d} g^T - g\hat{d}^T G & = & (G^T \hat{x})(G^T \hat{d})^T + (G^T \hat{d})(G^T \hat{x})^T + G^T P \hat{K} P^T G - (G^T \hat{d}) g^T - g (G^T \hat{d})^T \\ & = & (G^T \hat{x} - g)(G^T \hat{d})^T + (G^T \hat{d})(G^T \hat{x} - g)^T + (G^T P)\hat{K}(G^T P)^T \\ & \geq & 0, \end{array}$$

where the last inequality follows from $G^T \hat{x} \leq g$, $G^T \hat{d} \leq 0$, $G^T P \leq 0$, and $\hat{K} \geq 0$. Therefore, $(\hat{d}, \hat{D}) \in \mathcal{F}_\infty$ by (2.11). $\square$

An interesting question is whether, for each $(\hat{d}, \hat{D}) \in \mathcal{F}_\infty$, there exist some $\hat{x} \in F$ and some $\hat{K} \in \mathcal{N}^t$ such that $\hat{D}$ can be expressed in the form given in Proposition 2.3.2. The following example illustrates that this is not the case.

Example 2.3.1. *Let*

$$F = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1, \quad x_1 + x_2 \geq -1\},$$

i.e., $n = 2$, $m = 2$, $p = 0$, *and*

$$G = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Clearly,

$$F_\infty = \left\{ \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\} = cone \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Let $\hat{d} = 0 \in \mathbb{R}^2$ *and*

$$\hat{D} = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}.$$

By (2.11)*, we have* $(\hat{d}, \hat{D}) \in \mathcal{F}_\infty$ *since* $G^T \hat{D} G = 0 \in \mathcal{S}^2$*. On the other hand, for any* $\hat{x} \in F$ *and any* $\hat{K} \in \mathcal{N}^2$*, we obtain*

$$\begin{aligned} \hat{x}\hat{d}^T + \hat{d}\hat{x}^T + P\hat{K}P^T &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{12} & \hat{K}_{22} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{K}_{11} - 2\hat{K}_{12} + \hat{K}_{22} & -\hat{K}_{11} + 2\hat{K}_{12} - \hat{K}_{22} \\ -\hat{K}_{11} + 2\hat{K}_{12} - \hat{K}_{22} & \hat{K}_{11} - 2\hat{K}_{12} + \hat{K}_{22} \end{bmatrix}, \end{aligned}$$

which implies that $\hat{D}$ *cannot be expressed in this form for any* $\hat{K} \in \mathcal{N}^2$*.*

Next, we discuss the boundedness relation between F and $\mathcal{F}$.

Lemma 2.3.1. F *is nonempty and bounded if and only if* $\mathcal{F}$ *is nonempty and bounded.*

Proof. Suppose that F is nonempty and bounded. Then, $F_\infty = \{0\}$. Clearly, $\mathcal{F}$ is nonempty since, for each $\hat{x} \in F$, we have $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}$. Let $(\hat{d}, \hat{D}) \in \mathcal{F}_\infty$. By (2.12), we obtain $\hat{d} \in F_\infty$, which implies that $\hat{d} = 0$. By (2.11),

$$H^T \hat{D} = 0, \quad G^T \hat{D} G \geq 0.$$

By Lemma 2.2.3, for every $z^1 \in \mathbb{R}^n$ and $z^2 \in \mathbb{R}^n$, there exist $u^1 \in \mathbb{R}^m_+$, $w^1 \in \mathbb{R}^p$, $u^2 \in \mathbb{R}^m_+$, and $w^2 \in \mathbb{R}^p$ such that $Gu^1 + Hw^1 = z^1$ and $Gu^2 + Hw^2 = z^2$. Therefore, for every $z^1 \in \mathbb{R}^n$ and $z^2 \in \mathbb{R}^n$,

$$(z^1)^T \hat{D} z^2 = (Gu^1 + Hw^1)^T \hat{D} (Gu^2 + Hw^2) = (u^1)^T G^T \hat{D} G u^2 \geq 0,$$

where we used $H^T \hat{D} = 0$, $G^T \hat{D} G \geq 0$, $u^1 \geq 0$, and $u^2 \geq 0$. Since the inequality above holds for every $z^1 \in \mathbb{R}^n$ and $z^2 \in \mathbb{R}^n$, we obtain $\hat{D} = 0$. Therefore, $(\hat{d}, \hat{D}) = (0, 0)$, which implies that $\mathcal{F}_\infty = \{(0, 0)\}$. It follows that $\mathcal{F}$ is bounded.

Conversely, if $\mathcal{F}$ is nonempty and bounded, then F is nonempty and bounded since F is the projection of $\mathcal{F}$ onto the x-space. $\square$

2.3.3 Vertices

In this section, we focus on the relations between the vertices of F and those of $\mathcal{F}$. First, we consider the case in which F has no vertices.

Lemma 2.3.2. *Suppose that F is nonempty. If F has no vertices, then $\mathcal{F}$ has no vertices.*

Proof. By Lemma 2.2.2 (iv), there exists a nonzero $\hat{d} \in \mathbb{R}^n \backslash \{0\}$ such that $\hat{d} \in F_\infty$ and $-\hat{d} \in F_\infty$. Let $\hat{x} \in F$ and define $\hat{D} = \hat{x}\hat{d}^T + \hat{d}\hat{x}^T$. By Proposition 2.3.2, we obtain $(\hat{d}, \hat{D}) \in \mathcal{F}_\infty$ and $-(\hat{d}, \hat{D}) \in \mathcal{F}_\infty$, which implies that $\mathcal{F}$ contains a line. By Lemma 2.2.2, $\mathcal{F}$ has no vertices. $\square$

Before we present the relations between the set of vertices of F and that of $\mathcal{F}$, we state a useful technical lemma that will be helpful in the remainder of this section.

Lemma 2.3.3. *Let $A \in \mathbb{R}^{n\times k}$ and $Z \in \mathcal{S}^k$. Then, the system $A^TWA = Z$ has a solution $W \in \mathcal{S}^n$ if and only if the range space of Z is contained in the range space of A^T. Furthermore, if A has full row rank, then the solution is unique.*

Proof. If $A^TWA = Z$ has a solution $W \in \mathcal{S}^n$, then, for any $y \in \mathbb{R}^k$, we have $Zy = A^T(WAy)$, which implies that the range space of Z is contained in the range space of A^T.

Conversely, let $Z = \sum\limits_{j=1}^{\kappa} \lambda_j z^j (z^j)^T$ denote the eigenvalue decomposition of Z, where $\kappa \leq k$ denotes the rank of Z and $z^j \in \mathbb{R}^k,\ j = 1, \ldots, \kappa$. By the hypothesis, the range space of Z, given by $\operatorname{span}\{z^1, \ldots, z^\kappa\}$, is contained in the range space of A^T. Therefore, for each $j = 1, \ldots, \kappa$, there exists $u^j \in \mathbb{R}^n$ such that $z^j = A^T u^j$. It follows that $Z = A^TU\Lambda U^TA$, where $U = \begin{bmatrix} u^1 & \cdots & u^\kappa \end{bmatrix} \in \mathbb{R}^{n\times\kappa}$ and $\Lambda \in \mathcal{S}^\kappa$ is a diagonal matrix whose entries are given by $\lambda_1, \ldots, \lambda_\kappa$. Therefore, $W = U\Lambda U^T$ is a solution of $A^TWA = Z$.

If A has full row rank, then the uniqueness of the solution $W \in \mathcal{S}^n$ follows from the observation that the matrix U is uniquely determined. $\square$

We are now in a position to present the first relation between the set of vertices of F and that of $\mathcal{F}$.

Proposition 2.3.3. *Suppose that F is nonempty. Let $\hat{x} \in F$ and $\hat{X} = \hat{x}\hat{x}^T \in \mathcal{S}^n$. Then, $(\hat{x}, \hat{X})$ is a vertex of $\mathcal{F}$ if and only if $\hat{x}$ is a vertex of F.*

Proof. For each $\hat{x} \in F$, we clearly have $(\hat{x}, \hat{X}) \in \mathcal{F}$, where $\hat{X} = \hat{x}\hat{x}^T \in \mathcal{S}^n$.

First, suppose that $\hat{x} \in \mathbb{R}^n$ is a vertex of F. Let $G^0 \in \mathbb{R}^{n\times m_0}$ and $G^1 \in \mathbb{R}^{n\times m_1}$ denote the submatrices of G, where $m_0 + m_1 = m$, and let $g^0 \in \mathbb{R}^{m_0}$ and $g^1 \in \mathbb{R}^{m_1}$ denote the corresponding subvectors of g such that

$$(G^0)^T\hat{x} = g^0, \quad (G^1)^T\hat{x} < g^1. \tag{2.13}$$

First, we identify the set of active constraints of (RLT) at $(\hat{x}, \hat{X})$:

$$\begin{aligned}
(G^0)^T \hat{x} &= g^0 \\
(G^1)^T \hat{x} &< g^1 \\
H^T \hat{x} &= h \\
H^T \hat{x}\hat{x}^T &= h\hat{x}^T \\
\begin{bmatrix} (G^0)^T \\ (G^1)^T \end{bmatrix} \hat{x}\hat{x}^T \begin{bmatrix} (G^0)^T \\ (G^1)^T \end{bmatrix}^T - \begin{bmatrix} (G^0)^T \\ (G^1)^T \end{bmatrix} \hat{x} \begin{bmatrix} g^0 \\ g^1 \end{bmatrix}^T - \begin{bmatrix} g^0 \\ g^1 \end{bmatrix} \hat{x}^T \begin{bmatrix} (G^0)^T \\ (G^1)^T \end{bmatrix}^T + \begin{bmatrix} g^0 \\ g^1 \end{bmatrix} \begin{bmatrix} g^0 \\ g^1 \end{bmatrix}^T &= \begin{bmatrix} 0 & 0 \\ 0 & r^1(r^1)^T \end{bmatrix},
\end{aligned}$$

where $r^1 = g^1 - (G^1)^T \hat{x} > 0$. Therefore, $r^1(r^1)^T$ is componentwise strictly positive.

By Lemma 2.2.1 (ii), it suffices to show that the system

$$\begin{aligned}
(G^0)^T \hat{d} &= 0 \\
H^T \hat{d} &= 0 \\
H^T \hat{D} &= h\hat{d}^T \\
(G^0)^T \hat{D} G^0 - (G^0)^T \hat{d} (g^0)^T - g^0 \hat{d}^T G^0 &= 0 \\
(G^0)^T \hat{D} G^1 - (G^0)^T \hat{d} (g^1)^T - g^0 \hat{d}^T G^1 &= 0,
\end{aligned}$$

where $(\hat{d}, \hat{D}) \in \mathbb{R}^n \times \mathcal{S}^n$, has a unique solution $(\hat{d}, \hat{D}) = (0, 0)$.

Since $\hat{x}$ is a vertex of F, the matrix $\begin{bmatrix} G^0 & H \end{bmatrix}$ has full row rank by Lemma 2.2.1 (iii). Therefore, we obtain $\hat{d} = 0$ from the first two equations. Substituting $\hat{d} = 0$ into the third and fourth equations, we obtain

$$\begin{bmatrix} (G^0)^T \\ H^T \end{bmatrix} \hat{D} \begin{bmatrix} (G^0)^T \\ H^T \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where we used $H^T \hat{D} = 0$ by the third equation. By Lemma 2.3.3, we obtain $\hat{D} = 0$, which implies that $(\hat{x}, \hat{X})$ is a vertex of $\mathcal{F}$.

Conversely, suppose that $\hat{x}$ is not a vertex of F. By Lemma 2.2.1 (ii), there exists a nonzero $\hat{d} \in \mathbb{R}^n$ such that each of $\hat{d}$ and $-\hat{d}$ is a feasible direction at $\hat{x} \in F$. Using the same partition as in (2.13), we obtain

$$(G^0)^T \hat{d} = 0, \quad H^T \hat{d} = 0.$$

Let $\hat{D} = \hat{d}\hat{x}^T + \hat{x}\hat{d}^T \in \mathcal{S}^n$. We claim that each of $(\hat{d}, \hat{D})$ and $-(\hat{d}, \hat{D})$ is a feasible direction at $(\hat{x}, \hat{X})$. Indeed,

$$\begin{aligned}
H^T(\hat{d}\hat{x}^T + \hat{x}\hat{d}^T) &= h\hat{d}^T \\
(G^0)^T(\hat{d}\hat{x}^T + \hat{x}\hat{d}^T)G^0 - (G^0)^T \hat{d}(g^0)^T - g^0 \hat{d}^T G^0 &= 0 \\
(G^0)^T(\hat{d}\hat{x}^T + \hat{x}\hat{d}^T)G^1 - (G^0)^T \hat{d}(g^1)^T - g^0 \hat{d}^T G^1 &= 0
\end{aligned}$$

Furthermore, since $(G^1)^T \hat{x} < g^1$ and $(G^1)^T \hat{X} G^1 - (G^1)^T \hat{x}(g^1)^T - g^1 \hat{x}^T G^1 + g^1(g^1)^T = r^1(r^1)^T > 0$, where $r^1 = g^1 - (G^1)^T \hat{x} > 0$, it follows that there exists a real number $\epsilon > 0$ such that $(\hat{x}, \hat{X}) + \epsilon(\hat{d}, \hat{D}) \in \mathcal{F}$ and $(\hat{x}, \hat{X}) - \epsilon(\hat{d}, \hat{D}) \in \mathcal{F}$, which implies that $(\hat{x}, \hat{X})$ is not a vertex of $\mathcal{F}$ by Lemma 2.2.1 (ii). □

By Proposition 2.3.3, for each vertex $\hat{x} \in F$, there is a corresponding vertex $(\hat{x}, \hat{X}) \in \mathcal{F}$, where $\hat{X} = \hat{x}\hat{x}^T$. We therefore obtain the following result.

Corollary 2.3.1. *Suppose that F is nonempty. The set of vertices of F is nonempty if and only if the set of vertices of $\mathcal{F}$ is nonempty.*

Proof. The result immediately follows from Lemma 2.3.2 and Proposition 2.3.3. □

Next, we identify another connection between the set of vertices of $\mathcal{F}$ and the set of vertices of F.

Proposition 2.3.4. *Let $v^1 \in F$ and $v^2 \in F$ be two vertices such that $v^1 \neq v^2$. Let $\hat{x} = \frac{1}{2}(v^1 + v^2)$ and $\hat{X} = \frac{1}{2}\left(v^1(v^2)^T + v^2(v^1)^T\right)$. Then, $(\hat{x}, \hat{X})$ is a vertex of $\mathcal{F}$.*

Proof. Let $v^1 \in F$ and $v^2 \in F$ be two vertices. Let $\hat{x} = \frac{1}{2}(v^1+v^2)$ and $\hat{X} = \frac{1}{2}\left(v^1(v^2)^T + v^2(v^1)^T\right)$. First, we verify that $(\hat{x}, \hat{X}) \in \mathcal{F}$. Clearly, we have

$$\begin{aligned} G^T\hat{x} = G^T\left(\tfrac{1}{2}(v^1+v^2)\right) &\leq g \\ H^T\hat{x} = H^T\left(\tfrac{1}{2}(v^1+v^2)\right) &= h \\ H^T\hat{X} = H^T\left(\tfrac{1}{2}\left(v^1(v^2)^T + v^2(v^1)^T\right)\right) = h\left(\tfrac{1}{2}(v^1+v^2)\right)^T &= h\hat{x}^T. \end{aligned}$$

Let us define

$$r^{(1)} = g - G^T v^1 \geq 0, \quad r^{(2)} = g - G^T v^2 \geq 0. \tag{2.14}$$

Then, we obtain

$$G^T\hat{X}G - G^T\hat{x}g^T - g\hat{x}^TG + gg^T = \tfrac{1}{2}\left(r^{(1)}(r^{(2)})^T + r^{(2)}(r^{(1)})^T\right) \geq 0, \tag{2.15}$$

where we used (2.14). Therefore, $(\hat{x}, \hat{X}) \in \mathcal{F}$.

Next, we show that $(\hat{x}, \hat{X})$ is a vertex of $\mathcal{F}$. We define the following submatrices of G and the corresponding subvectors of g:

$$\begin{array}{rcl} (G^0)^T\hat{x} \;=\; (G^0)^Tv^1 & = & (G^0)^Tv^2 = g^0 \\ (G^1)^Tv^1 = g^1 & , & (G^1)^Tv^2 < g^1 \\ (G^2)^Tv^1 < g^2 & , & (G^2)^Tv^2 = g^2 \\ (G^3)^Tv^1 < g^3 & , & (G^3)^Tv^2 < g^3. \end{array}$$

We remark that G^1 and G^2 are nonempty submatrices of G since $v^1 \neq v^2$. By (2.14) and (2.15), we can identify the set of active constraints of (RLT) at $(\hat{x}, \hat{X})$:

$$\begin{aligned} (G^0)^T\hat{x} &= g^0 \\ (G^j)^T\hat{x} &< g^j, \quad j = 1,2,3 \\ H^T\hat{x} &= h \\ H^T\hat{X} &= h\hat{x}^T \\ G^T\hat{X}G - G^T\hat{x}g^T - g\hat{x}^TG + gg^T &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & + & + \\ 0 & + & 0 & + \\ 0 & + & + & + \end{bmatrix}, \end{aligned}$$

where, in the last equation, we assume without loss of generality that $G = \begin{bmatrix} G^0 & G^1 & G^2 & G^3 \end{bmatrix}$ and g is partitioned accordingly, and $+$ denotes a submatrix with strictly positive entries.

Therefore, by Lemma 2.2.1 (iii), it suffices to show that the system

$$\begin{aligned} (G^0)^T \hat{d} &= 0 \\ H^T \hat{d} &= 0 \\ H^T \hat{D} &= h\hat{d}^T \\ (G^i)^T \hat{D} G^j - (G^i)^T \hat{d} (g^j)^T - g^i \hat{d}^T G^j &= 0, \quad (i,j) \in \{(0,0),(0,1),(0,2),(0,3),(1,1),(2,2)\}, \end{aligned}$$

where $(\hat{d}, \hat{D}) \in \mathbb{R}^n \times \mathcal{S}^n$, has a unique solution $(\hat{d}, \hat{D}) = (0,0)$.

Therefore, $(\hat{d}, \hat{D})$ should simultaneously solve the following two systems:

$$\begin{aligned} \begin{bmatrix} (G^0)^T \\ (G^1)^T \\ H^T \end{bmatrix} \hat{D} \begin{bmatrix} (G^0)^T \\ (G^1)^T \\ H^T \end{bmatrix}^T &= \begin{bmatrix} 0 & g^0 \hat{d}^T G^1 & 0 \\ (G^1)^T \hat{d} (g^0)^T & (G^1)^T \hat{d} (g^1)^T + g^1 \hat{d}^T G^1 & (G^1)^T \hat{d} h^T \\ 0 & h\hat{d}^T G^1 & 0 \end{bmatrix}, \\ \begin{bmatrix} (G^0)^T \\ (G^2)^T \\ H^T \end{bmatrix} \hat{D} \begin{bmatrix} (G^0)^T \\ (G^2)^T \\ H^T \end{bmatrix}^T &= \begin{bmatrix} 0 & g^0 \hat{d}^T G^2 & 0 \\ (G^2)^T \hat{d} (g^0)^T & (G^2)^T \hat{d} (g^2)^T + g^2 \hat{d}^T G^2 & (G^2)^T \hat{d} h^T \\ 0 & h\hat{d}^T G^2 & 0 \end{bmatrix}. \end{aligned}$$

Substituting $(G^0)^T v^1 = g^0$ and $H^T v^1 = h$ into the first equation, and $(G^0)^T v^2 = g^0$ and $H^T v^2 = h$ into the second one, we obtain

$$\begin{aligned} \begin{bmatrix} (G^0)^T \\ (G^1)^T \\ H^T \end{bmatrix} \hat{D} \begin{bmatrix} (G^0)^T \\ (G^1)^T \\ H^T \end{bmatrix}^T &= \begin{bmatrix} (G^0)^T \\ (G^1)^T \\ H^T \end{bmatrix} \left(v^1 \hat{d}^T + \hat{d} (v^1)^T \right) \begin{bmatrix} (G^0)^T \\ (G^1)^T \\ H^T \end{bmatrix}^T \\ \begin{bmatrix} (G^0)^T \\ (G^2)^T \\ H^T \end{bmatrix} \hat{D} \begin{bmatrix} (G^0)^T \\ (G^2)^T \\ H^T \end{bmatrix}^T &= \begin{bmatrix} (G^0)^T \\ (G^2)^T \\ H^T \end{bmatrix} \left(v^2 \hat{d}^T + \hat{d} (v^2)^T \right) \begin{bmatrix} (G^0)^T \\ (G^2)^T \\ H^T \end{bmatrix}^T. \end{aligned}$$

Since v^1 and v^2 are vertices of F, Lemma 2.2.1 (iii) implies that each of the two matrices $\begin{bmatrix} G^0 & G^1 & H \end{bmatrix}$ and $\begin{bmatrix} G^0 & G^2 & H \end{bmatrix}$ has full row rank. By Lemma 2.3.3, we obtain

$$\hat{D} = v^1 \hat{d}^T + \hat{d} (v^1)^T = v^2 \hat{d}^T + \hat{d} (v^2)^T,$$

which implies that $(v^1 - v^2)\hat{d}^T + \hat{d}(v^1 - v^2)^T = 0$. Since $v^1 \neq v^2$, we obtain $\hat{d} = 0$. Substituting this into the matrix equations above, we obtain $\hat{D} = 0$ by Lemma 2.3.3, which proves the assertion. $\square$

Propositions 2.3.3 and 2.3.4 identify two sets of vertices of $\mathcal{F}$ under the assumption that F contains at least one vertex. An interesting question is whether every vertex of $\mathcal{F}$ belongs to one of these two sets. The following example based on (BoxQP) illustrates that this is not necessarily true even when F is a polytope.

Example 2.3.2. *Let $n = 2$ and*

$$F = \left\{ x \in \mathbb{R}^2 : 0 \leq x_j \leq 1, \quad j = 1,2 \right\},$$

i.e., we have $m = 4$, $G = \begin{bmatrix} I & -I \end{bmatrix}$, *and* $g^T = \begin{bmatrix} e^T & 0^T \end{bmatrix}$ *without any equality constraints, i.e.,* $p = 0$, *where* $e \in \mathbb{R}^2$, $0 \in \mathbb{R}^2$ *and* $I \in \mathcal{S}^2$ *denotes the identity matrix.* F *has four vertices given by*

$$v^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v^3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v^4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.16}$$

The feasible region of the RLT relaxation is given by

$$\mathcal{F} = \left\{ (x, X) \in \mathbb{R}^2 \times \mathcal{S}^2 : \begin{array}{rcl} x & \leq & e \\ x & \geq & 0 \\ \begin{bmatrix} X - xe^T - ex^T + ee^T & ex^T - X \\ xe^T - X & X \end{bmatrix} & \geq & 0 \end{array} \right\}.$$

By Proposition 2.3.3, there are four vertices of $\mathcal{F}$ *in the form of* $(v^j, v^j(v^j)^T), \ j = 1, \ldots, 4$. *Similarly, by Proposition 2.3.4,* $\mathcal{F}$ *has another set of six vertices in the form of*

$$\left(\tfrac{1}{2}(v^i + v^j), \tfrac{1}{2}(v^i(v^j)^T + v^j(v^i)^T)\right), \ 1 \leq i < j \leq 4.$$

We now claim that $\mathcal{F}$ *has at least one other vertex that does not belong to these two sets. Consider* $(\hat{x}, \hat{X}) \in \mathbb{R}^2 \times \mathcal{S}^2$ *given by*

$$\hat{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \hat{X} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.17}$$

It is easy to verify that $(\hat{x}, \hat{X}) \in \mathcal{F}$ *and that* $(\hat{x}, \hat{X})$ *does not belong to either of the two sets of vertices identified by Proposition 2.3.3 and Proposition 2.3.4. It is an easy exercise to show that* $(\hat{x}, \hat{X}) \pm (\hat{d}, \hat{D}) \in \mathcal{F}$ *if and only if* $(\hat{d}, \hat{D}) = (0, 0)$. *Therefore, we conclude that* $(\hat{x}, \hat{X})$ *is a vertex of* $\mathcal{F}$ *by Lemma 2.2.1 (ii).*

By Example 2.3.2, the two sets of vertices identified in Propositions 2.3.3 and 2.3.4 do not necessarily encompass all vertices of $\mathcal{F}$ in general even if F is a polytope. In the next section, we identify a subclass of quadratic optimization problems for which all vertices of $\mathcal{F}$ are completely characterized by Propositions 2.3.3 and 2.3.4. We then discuss the implications of this observation on general quadratic optimization problems.

2.3.4 A Specific Class of Quadratic Optimization Problems

In this section, we present a specific class of quadratic optimization problems with the property that all vertices of the feasible region $\mathcal{F}$ of the RLT relaxation are precisely given by the union of the two sets identified in Propositions 2.3.3 and 2.3.4.

Consider the class of instances of (QP), where $Q \in \mathcal{S}^n$, $c \in \mathbb{R}^n$, and

$$H = a \in \mathbb{R}^n_+ \backslash \{0\}, \quad h = 1, \quad G = -I \in \mathcal{S}^n, \quad g = 0 \in \mathbb{R}^n, \tag{2.18}$$

where $\mathbb{R}^n_+$ denotes the nonnegative orthant of n-dimensional Euclidean space.

Therefore, the feasible region is given by

$$F = \left\{ x \in \mathbb{R}^n : a^T x = 1, \quad x \geq 0 \right\}. \tag{2.19}$$

Let us define the following index sets:

$$\begin{aligned} \mathbf{P} &= \{ j \in \{1, \ldots, n\} : a_j > 0 \}, & (2.20) \\ \mathbf{Z} &= \{ j \in \{1, \ldots, n\} : a_j = 0 \}. & (2.21) \end{aligned}$$

It is straightforward to verify that the set of vertices of F is

$$V = \left\{ \left(\tfrac{1}{a_j} \right) e^j : a_j > 0 \right\} = \left\{ \left(\tfrac{1}{a_j} \right) e^j : j \in \mathbf{P} \right\}. \tag{2.22}$$

The feasible region $\mathcal{F}$ of the corresponding RLT relaxation is given by

$$\mathcal{F} = \left\{ (x, X) \in \mathbb{R}^n \times \mathcal{S}^n : Xa = x, \quad a^T x = 1, \quad x \geq 0, \quad X \geq 0 \right\}. \tag{2.23}$$

We next present our main result in this section.

Proposition 2.3.5. *Suppose that $\mathcal{F}$ is given by* (2.23)*, where $a \in \mathbb{R}^n_+ \backslash \{0\}$. Then, $(\hat{x}, \hat{X})$ is a vertex of $\mathcal{F}$ if and only if $(\hat{x}, \hat{X}) = (v, vv^T)$ for some $v \in V$, where V is given by* (2.22)*, or $(\hat{x}, \hat{X}) = \left(\frac{1}{2}(v^1 + v^2), \frac{1}{2}(v^1 (v^2)^T + v^2 (v^1)^T) \right)$ for some $v^1 \in V$, $v^2 \in V$, and $v^1 \neq v^2$.*

Proof. By Propositions 2.3.3 and 2.3.4, it suffices to prove the forward implication.

Let us first define

$$\begin{aligned} w^k &= \left(\tfrac{1}{a_k} \right) e^k, \quad k \in \mathbf{P}, & (2.24) \\ W^k &= w^k (w^k)^T = \left(\tfrac{1}{a_k^2} \right) e^k (e^k)^T, \quad k \in \mathbf{P}, & (2.25) \\ z^{ij} &= \tfrac{1}{2} \left(w^i + w^j \right) = \left(\tfrac{1}{2a_i} \right) e^i + \left(\tfrac{1}{2a_j} \right) e^j, \; i \in \mathbf{P}, j \in \mathbf{P}, i \neq j, & (2.26) \\ Z^{ij} &= \tfrac{1}{2} \left(w^i (w^j)^T + w^j (w^i)^T \right) = \left(\tfrac{1}{2a_i a_j} \right) \left(e^i (e^j)^T + e^j (e^i)^T \right), \; i \in \mathbf{P}, j \in \mathbf{P}, i \neq j, & (2.27) \end{aligned}$$

where $\mathbf{P}$ is given by (2.20). By (2.22) and Propositions 2.3.3 and 2.3.4, it follows that each of $(w^k, W^k), \; k \in \mathbf{P}$, and $(z^{ij}, Z^{ij}), \; i \in \mathbf{P}, \; j \in \mathbf{P}, \; i \neq j$, is a vertex of $\mathcal{F}$.

Let $(\hat{x}, \hat{X})$ be a vertex of $\mathcal{F}$. By (2.23) and (2.20), we obtain

$$\hat{X}_{\mathbf{PP}} \, a_{\mathbf{P}} = \hat{x}_{\mathbf{P}}, \quad a_{\mathbf{P}}^T \hat{x}_{\mathbf{P}} = 1, \quad \hat{x} \geq 0, \quad \hat{X} \geq 0. \tag{2.28}$$

First, we claim that $\hat{x}_{\mathbf{Z}} = 0$, $\hat{X}_{\mathbf{PZ}} = 0$, $\hat{X}_{\mathbf{ZP}} = 0$, and $\hat{X}_{\mathbf{ZZ}} = 0$. Indeed, by (2.28), if any of these conditions is not satisfied, it is easy to construct a nonzero $(\hat{d}, \hat{D}) \in \mathbb{R}^n \times \mathcal{S}^n$ such that $(\hat{x}, \hat{X}) \pm (\hat{d}, \hat{D}) \in \mathcal{F}$, which would contradict that $(\hat{x}, \hat{X})$ is a vertex of $\mathcal{F}$ by Lemma 2.2.1 (ii). Therefore,

$$\begin{aligned} \hat{X} &= \sum_{k \in \mathbf{P}} \hat{X}_{kk} e^k (e^k)^T + \tfrac{1}{2} \sum_{i \in \mathbf{P}} \sum_{j \in \mathbf{P} : j \neq i} \hat{X}_{ij} \left(e^i (e^j)^T + e^j (e^i)^T \right) \\ &= \sum_{k \in \mathbf{P}} \mu_k W^k + \sum_{i \in \mathbf{P}} \sum_{j \in \mathbf{P} : j \neq i} \lambda_{ij} Z^{ij}, \end{aligned}$$

where $\mu_k = \hat{X}_{kk}\, a_k^2 \geq 0,\ k \in \mathbf{P}$; $\lambda_{ij} = \hat{X}_{ij}\, a_i a_j \geq 0,\ i \in \mathbf{P},\ j \in \mathbf{P},\ i \neq j$; W^k and Z^{ij} are defined as in (2.25) and (2.27), respectively. By using $W^k a = w^k,\ k \in \mathbf{P}$; $Z^{ij} a = z^{ij},\ i \in \mathbf{P},\ j \in \mathbf{P},\ i \neq j$; and (2.23), the previous equality implies that

$$\hat{x} = \hat{X} a = \sum_{k \in \mathbf{P}} \mu_k w^k + \sum_{i \in \mathbf{P}} \sum_{j \in \mathbf{P}: j \neq i} \lambda_{ij} z^{ij}.$$

By (2.28), we obtain

$$a_{\mathbf{P}}^T \hat{X}_{\mathbf{PP}}\, a_{\mathbf{P}} = \sum_{k \in \mathbf{P}} \hat{X}_{kk}\, a_k^2 + \sum_{i \in \mathbf{P}} \sum_{j \in \mathbf{P}: j \neq i} \hat{X}_{ij}\, a_i a_j = \sum_{k \in \mathbf{P}} \mu_k + \sum_{i \in \mathbf{P}} \sum_{j \in \mathbf{P}: j \neq i} \lambda_{ij} = 1.$$

Therefore, $(\hat{x}, \hat{X})$ is given by a convex combination of $(w^k, W^k),\ k \in \mathbf{P}$, and $(z^{ij}, Z^{ij}),\ i \in \mathbf{P},\ j \in \mathbf{P},\ i \neq j$. By Propositions 2.3.3 and 2.3.4, we conclude that either $(\hat{x}, \hat{X}) = (w^k, W^k)$ for some $k \in \mathbf{P}$, or $(\hat{x}, \hat{X}) = (z^{ij}, Z^{ij})$ for some $i \in \mathbf{P},\ j \in \mathbf{P},\ i \neq j$. This completes the proof. □

Our next result gives a closed-form expression of the lower bound ℓ_R^* for an instance of (QP) in this specific class.

Corollary 2.3.2. *Consider an instance of (QP), where F is given by* (2.19) *and $a \in \mathbb{R}^n_+ \backslash \{0\}$. If ℓ_R^* is finite, then*

$$\ell_R^* = \min \left\{ \min_{v \in V} \left\{ \tfrac{1}{2} v^T Q v + c^T v \right\},\ \min_{v^1 \in V,\ v^2 \in V,\ v^1 \neq v^2} \tfrac{1}{2} \left((v^1)^T Q v^2 + c^T (v^1 + v^2) \right) \right\}, \quad (2.29)$$

where V and $\mathbf{P}$ are given by (2.22) *and* (2.20)*, respectively.*

Proof. If ℓ_R^* is finite, the relation (2.29) follows from Proposition 2.3.5 since (RLT) is a linear optimization problem and the optimal value is attained at a vertex. □

We close this section with a discussion of a well-studied class of quadratic optimization problems that belong to the specific class of quadratic optimization problems identified in this section. An instance of (QP) is referred to as a *standard quadratic optimization problem* (see, e.g., [19]) if

$$H = e \in \mathbb{R}^n, \quad h = 1, \quad G = -I \in \mathcal{S}^n, \quad g = 0 \in \mathbb{R}^n. \quad (2.30)$$

Therefore, the feasible region of a standard quadratic optimization problem is the unit simplex given by

$$F = \left\{ x \in \mathbb{R}^n : e^T x = 1, \quad x \geq 0 \right\}. \quad (2.31)$$

Similarly, the feasible region of the RLT relaxation of a standard quadratic optimization problem is given by

$$\mathcal{F} = \left\{ (x, X) \in \mathbb{R}^n \times \mathcal{S}^n : Xe = x, \quad e^T x = 1, \quad x \geq 0, \quad X \geq 0 \right\}. \quad (2.32)$$

Proposition 2.3.5 gives rise to the following result on standard quadratic optimization problems.

Corollary 2.3.3. *Consider an instance of a standard quadratic optimization problem and let $\mathcal{F}$ denote the feasible region of the RLT relaxation given by* (2.32)*. Then, $(\hat{x}, \hat{X})$ is a vertex of $\mathcal{F}$ if and only if $(\hat{x}, \hat{X}) = (e^j, e^j(e^j)^T)$ for some $j = 1, \ldots, n$, or $(\hat{x}, \hat{X}) = \left(\frac{1}{2}(e^i + e^j), \frac{1}{2}\left(e^i(e^j)^T + e^j(e^i)^T\right)\right)$ for some $1 \leq i < j \leq n$. Furthermore,*

$$\ell_R^* = \min\left\{\min_{k=1,\ldots,n}\left\{\tfrac{1}{2}Q_{kk} + c_k\right\}, \min_{i=1,\ldots,n;\ j=1,\ldots,n;\ i \neq j} \tfrac{1}{2}\left(Q_{ij} + c_i + c_j\right)\right\}.$$

Proof. The first assertion follows from Proposition 2.3.5 and (2.22) by using $a = e \in \mathbb{R}^n_+ \backslash \{0\}$, and the second one from Lemma 2.3.1 and Corollary 2.3.2 since F is bounded. □

In [96], using an alternative copositive formulation of standard quadratic optimization problems in [22], a hierarchy of linear relaxations arising from the sequence of polyhedral approximations of the copositive cone proposed by [39] was considered and the same lower bound given by Corollary 2.3.3 was established for the first level of the hierarchy. Therefore, it is worth noting that the relaxation arising from the copositive formulation turns out to be equivalent to the RLT relaxation arising from the usual formulation of standard quadratic optimization problems as an instance of (QP).

2.3.5 Implications

In Section 2.3.4, we identified a specific class of quadratic optimization problems with the property that Propositions 2.3.3 and 2.3.4 completely characterize the set of all vertices of the feasible region of the corresponding RLT relaxation. In this section, we first observe that every quadratic optimization problem can be equivalently formulated as an instance of (QP) in this class. We then discuss the implications of this observation on RLT relaxations of general quadratic optimization problems.

Consider a general quadratic optimization problem, where F given by (1.4) is nonempty. By Lemma 2.2.4 and (2.2),

$$F = \operatorname{conv}\left(\left\{v^1, \ldots, v^s\right\}\right) + \operatorname{cone}\left(\left\{d^1, \ldots, d^t\right\}\right), \tag{2.33}$$

where $v^i \in F_i,\ i = 1, \ldots, s$, and each $F_i \subseteq F,\ i = 1, \ldots, s$, denotes a minimal face of F, and $d^1, \ldots, d^t$ are the generators of F_∞. Let us define

$$M = \begin{bmatrix} v^1 & \cdots & v^s \end{bmatrix} \in \mathbb{R}^{n \times s}, \quad P = \begin{bmatrix} d^1 & \cdots & d^t \end{bmatrix} \in \mathbb{R}^{n \times t}. \tag{2.34}$$

By (2.33) and (2.34), $\hat{x} \in F$ if and only if there exists $y \in \mathbb{R}^s_+$ and $z \in \mathbb{R}^t_+$ such that $e^T y = 1$ and $\hat{x} = My + Pz$. Therefore, (QP) admits the following alternative formulation:

$$\text{(QPA)} \quad \min_{y \in \mathbb{R}^s, z \in \mathbb{R}^t} \left\{\tfrac{1}{2}\left((My + Pz)^T Q \,(My + Pz)\right) + c^T(My + Pz) : e^T y = 1, \quad y \geq 0, \quad z \geq 0\right\}.$$

We conclude that every quadratic optimization problem admits an equivalent reformulation as an instance in the specific class identified in Section 2.3.4. However, we remark that this equivalence is mainly of theoretical interest since such a reformulation requires the enumeration of all minimal faces of F and all generators of F_∞, each of which may have an exponential size.

Nevertheless, in this section, we will discuss the relations between the RLT relaxation of (QP) and that of the alternative formulation (QPA) and draw some conclusions.

Let us introduce the following notations:

$$\begin{aligned}
n_A &= s + t && (2.35)\\
Q_A &= \begin{bmatrix} M^TQM & M^TQP \\ P^TQM & P^TQP \end{bmatrix} \in \mathcal{S}^{n_A} && (2.36)\\
c_A &= \begin{bmatrix} M^Tc \\ P^Tc \end{bmatrix} \in \mathbb{R}^{n_A} && (2.37)\\
a_A &= \begin{bmatrix} e \\ 0 \end{bmatrix} \in \mathbb{R}^{n_A} && (2.38)\\
x_A &= \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^{n_A} && (2.39)
\end{aligned}$$

Therefore, (QPA) can be expressed by

$$(\text{QPA}) \qquad \min_{x_A \in \mathbb{R}^{n_A}} \left\{ \tfrac{1}{2}(x_A)^T Q_A x_A + (c_A)^T x_A : x_A \in F_A \right\},$$

where

$$F_A = \left\{ x_A \in \mathbb{R}^{n_A} : (a_A)^T x_A = 1, \quad x_A \geq 0 \right\}. \tag{2.40}$$

Similarly, the RLT relaxation of (QPA) is given by

$$(\text{RLTA}) \quad \ell^*_{RA} = \min_{x_A \in \mathbb{R}^{n_A}, X_A \in \mathcal{S}^{n_A}} \left\{ \frac{1}{2}\langle Q_A, X_A \rangle + (c_A)^T x_A : (x_A, X_A) \in \mathcal{F}_A \right\},$$

where

$$\mathcal{F}_A = \left\{ (x_A, X_A) \in \mathbb{R}^{n_A} \times \mathcal{S}^{n_A} : X_A\, a_A = x_A, \quad (a_A)^T x_A = 1, \quad x_A \geq 0, \quad X_A \geq 0 \right\}. \tag{2.41}$$

We now present the first relation between the RLT relaxations of (QP) and (QPA) given by (RLT) and (RLTA), respectively.

Proposition 2.3.6. *Consider a general quadratic optimization problem, where F given by* (1.4) *is nonempty. Then, $\ell^*_R \leq \ell^*_{RA} \leq \ell^*$.*

Proof. Let $(\hat{x}_A, \hat{X}_A) \in \mathcal{F}_A$ be an arbitrary feasible solution of (RLTA). We will construct a corresponding feasible solution $(\hat{x}, \hat{X}) \in \mathcal{F}$ of (RLT) with the same objective function value. Let

$$\hat{x} = \begin{bmatrix} M & P \end{bmatrix} \hat{x}_A \in \mathbb{R}^n, \quad \hat{X} = \begin{bmatrix} M & P \end{bmatrix} \hat{X}_A \begin{bmatrix} M & P \end{bmatrix}^T \in \mathcal{S}^n, \tag{2.42}$$

where M and P are defined as in (2.34). By (2.33) and (2.38), we conclude that $\hat{x} \in F$, i.e., $G^T\hat{x} \leq g$ and $H^T\hat{x} = h$.

Since $G^Tv^i \leq g$ for each $i = 1, \ldots, s$, and $G^Td^j \leq 0$ for each $j = 1, \ldots, t$, we obtain

$$G^T \begin{bmatrix} M & P \end{bmatrix} - g(a_A)^T \leq 0,$$

where we used (2.34) and (2.38). Since $\hat{X}_A \geq 0$, we have

$$\begin{aligned} 0 &\leq \left(G^T \begin{bmatrix} M & P \end{bmatrix} - g(a_A)^T\right) \hat{X}_A \left(G^T \begin{bmatrix} M & P \end{bmatrix} - g(a_A)^T\right)^T \\ &= G^T \hat{X} G - G^T \hat{x} g^T - g \hat{x}^T G + g g^T, \end{aligned}$$

where we used (2.41) and (2.42) in the second line.

Since $H^T v^i = h$ for each $i = 1, \ldots, s$, and $H^T d^j = 0$ for each $j = 1, \ldots, t$, we obtain

$$H^T \begin{bmatrix} M & P \end{bmatrix} = h(a_A)^T,$$

where we used (2.34) and (2.38). Therefore,

$$\begin{aligned} H^T \hat{X} &= H^T \begin{bmatrix} M & P \end{bmatrix} \hat{X}_A \begin{bmatrix} M & P \end{bmatrix}^T \\ &= h(a_A)^T \hat{X}_A \begin{bmatrix} M & P \end{bmatrix}^T \\ &= h(\hat{x}_A)^T \begin{bmatrix} M & P \end{bmatrix}^T \\ &= h \hat{x}^T, \end{aligned}$$

where we used (2.41) and (2.42) in the third line. Therefore, $(\hat{x}, \hat{X}) \in \mathcal{F}$. Furthermore,

$$\begin{aligned} \tfrac{1}{2}\langle Q, \hat{X} \rangle + c^T \hat{x} &= \tfrac{1}{2} \left\langle Q, \begin{bmatrix} M & P \end{bmatrix} \hat{X}_A \begin{bmatrix} M & P \end{bmatrix}^T \right\rangle + c^T \begin{bmatrix} M & P \end{bmatrix} \hat{x}_A \\ &= \tfrac{1}{2} \left\langle Q_A, \hat{X}_A \right\rangle + (c_A)^T \hat{x}_A, \end{aligned}$$

where we used (2.42) in the first line, and (2.25) and (2.26) in the second line. Therefore, for each $(\hat{x}_A, \hat{X}_A) \in \mathcal{F}_A$, there exists a corresponding solution $(\hat{x}, \hat{X}) \in \mathcal{F}$ with the same objective function value. We conclude that $\ell_R^* \leq \ell_{RA}^* \leq \ell^*$. $\square$

By Proposition 2.3.6, the RLT relaxation (RLTA) of the alternative formulation (QPA) is at least as tight as the RLT relaxation (RLT) of the original formulation (QP). An interesting question is whether the lower bounds arising from the two relaxations are in fact equal (i.e., $\ell_R^* = \ell_{RA}^*$). Our next example illustrates that this is, in general, not true.

Example 2.3.3. *Consider the following instance of (BoxQP) for $n = 2$, where*

$$Q = \begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ -2 \end{bmatrix},$$

and

$$F = \left\{ x \in \mathbb{R}^2 : 0 \leq x_j \leq 1, \quad j = 1, 2 \right\},$$

i.e., we have $p = 0$, $m = 4$, $G = \begin{bmatrix} I & -I \end{bmatrix}$, and $g^T = \begin{bmatrix} e^T & 0^T \end{bmatrix}$, where $e \in \mathbb{R}^2$, $0 \in \mathbb{R}^2$ and $I \in \mathcal{S}^2$ denotes the identity matrix. For the RLT relaxation of the original formulation, we obtain $\ell_R^* = -\frac{3}{2}$, and an optimal solution is given by (2.17). *By* (2.16)*, we have $F = conv(\{v^1, \ldots, v^4\})$. Defining $M = \begin{bmatrix} v^1 \cdots v^4 \end{bmatrix} \in \mathbb{R}^{2 \times 4}$ and using* (2.36)*,* (2.37)*, and* (2.38)*,* *we obtain*

$$Q_A = M^TQM = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & -2 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix}, \quad c_A = M^Tc = \begin{bmatrix} 0 \\ -2 \\ 0 \\ -2 \end{bmatrix}, \quad a_A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, (QP) can be equivalently formulated as (QPA), which is an instance of a standard quadratic optimization problem. By Corollary 2.3.3, we have $\ell^*_{RA} = -1$ and an optimal solution is

$$\hat{x}_A = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \hat{X}_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Therefore, we obtain $\ell^*_R = -\frac{3}{2} < -1 = \ell^*_{RA}$. In fact, for this instance of (QP), we have $\ell^* = -1$, which is attained at $x^* = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Therefore, the RLT relaxation (RLTA) of the alternative formulation (QPA) is not only tighter than that of the original formulation but is, in fact, an exact relaxation.

We next present another example for which $\ell^*_R < \ell^*_{RA} < \ell^*$:

Example 2.3.4. *Consider the following instance of (BoxQP) for $n = 3$, where*

$$Q = \begin{bmatrix} 6 & 9 & -3 \\ 9 & 10 & 13 \\ -3 & 13 & 5 \end{bmatrix}, \quad c = \begin{bmatrix} -6 \\ -18 \\ -10 \end{bmatrix}.$$

The optimal solution of (BoxQP) is attained at $x^* = [1, 0, 1]^T$ with $\ell^* = -13.5$. We obtain $\ell^*_R = -18.5$ and $\ell^*_{RA} = -17$. Therefore, while the RLT relaxation (RLTA) of the alternative formulation (QPA) is tighter than that of the original formulation, it is still inexact.

As illustrated by Examples 2.3.3 and 2.3.4, despite the fact that (QP) and (QPA) are equivalent formulations, (RLTA) may lead to a strictly tighter relaxation of (QP) than (RLT). Therefore, we conclude that the quality of the RLT relaxation may depend on the particular formulation. Recall, however, that the alternative formulation (QPA) may, in general, have an exponential size. We close this section with the following result on the RLT relaxation of the original formulation.

Corollary 2.3.4. *Consider a general quadratic optimization problem, where F given by* (1.4) *is nonempty. Let $v^i \in F_i,\ i = 1, \ldots, s$, where each $F_i \subseteq F,\ i = 1, \ldots, s$, denotes a minimal face of F. Then,*

$$\ell^*_R \le \min \left\{ \min_{k=1,\ldots,s} \left\{ \tfrac{1}{2}(v^k)^TQv^k + c^Tv^k \right\}, \min_{1 \le i < j \le s} \tfrac{1}{2}\left((v^i)^TQv_j + c^T(v^i + v^j) \right) \right\}.$$

Proof. If the RLT relaxation is unbounded, then $\ell^*_R = -\infty$ and the claim is trivially satisfied. If F contains at least one vertex, then the assertion follows from Propositions 2.3.3 and 2.3.4 since each $v^i,\ i = 1, \ldots, s$, is a vertex. Otherwise, by (2.22), the vertices of F_A given by (2.40) are $e^j \in \mathbb{R}^{n_A},\ j = 1, \ldots, s$. The assertion follows directly from Proposition 2.3.6

and Corollary 2.3.2 by observing that $(e^i)^T Q_A e^j = (v^i)^T Q v^j$ for each $i = 1, \ldots, s$ and each $j = 1, \ldots, s$ by (2.34) and (2.36). $\square$

2.4 Duality and Optimality Conditions

In this section, we focus on the dual problem of (RLT) and discuss optimality conditions.

By defining the dual variables $(u, w, R, S) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{p\times n} \times \mathcal{S}^m$ corresponding to the four sets of constraints in (2.8), respectively, the dual of (RLT) is given by

$$
\begin{array}{llrcl}
\text{(RLT-D)} & \displaystyle\max_{u\in\mathbb{R}^m, w\in\mathbb{R}^p, R\in\mathbb{R}^{p\times n}, S\in\mathcal{S}^m} & \multicolumn{3}{l}{-u^T g + w^T h - \frac{1}{2} g^T S g} \\
& \text{s.t.} & & & \\
& & -Gu + Hw - R^T h - GSg & = & c \\
& & R^T H^T + HR + GSG^T & = & Q \\
& & S & \geq & 0 \\
& & u & \geq & 0.
\end{array}
$$

Note that the variable $S \in \mathcal{S}^m$ is scaled by a factor of $\frac{1}{2}$ in (RLT-D). We first review the optimality conditions.

Lemma 2.4.1. *Suppose that (QP) has a nonempty feasible region. Then, $(\hat{x}, \hat{X}) \in \mathcal{F}$ is an optimal solution of (RLT) if and only if there exists $(\hat{u}, \hat{w}, \hat{R}, \hat{S}) \in \mathbb{R}^m_+ \times \mathbb{R}^p \times \mathbb{R}^{p\times n} \times \mathcal{N}^m$ such that*

$$
\begin{aligned}
c &= -G\hat{u} + H\hat{w} - \hat{R}^T h - G\hat{S}g, && (2.43)\\
Q &= \hat{R}^T H^T + H\hat{R} + G\hat{S}G^T, && (2.44)\\
\hat{u}^T \hat{r} &= 0, && (2.45)\\
\left\langle \hat{S}, G^T \hat{X} G + \hat{r} g^T + g \hat{r}^T - g g^T \right\rangle &= 0, && (2.46)
\end{aligned}
$$

where $\hat{r} = g - G^T \hat{x} \in \mathbb{R}^m_+$.

Proof. Since $(\hat{x}, \hat{X}) \in \mathcal{F}$, we have

$$
G^T \hat{X} G - G^T \hat{x} g^T - g \hat{x}^T G + g g^T = G^T \hat{X} G + \hat{r} g^T + g \hat{r}^T - g g^T \geq 0,
$$

where we used $\hat{r} = g - G^T \hat{x}$. The claim now follows from the optimality conditions for (RLT) and (RLT-D). $\square$

We remark that Lemma 2.4.1 gives a recipe for constructing instances of (QP) with a known optimal solution of (RLT) and a finite RLT lower bound on the optimal value. We will discuss this further in Section 2.6.

By Lemma 2.3.1, if F is nonempty and bounded, then $\mathcal{F}$ is nonempty and bounded, which implies that (RLT) has a finite optimal value. By Lemma 2.4.1, we conclude that the (RLT-D) always has a nonempty feasible region under this assumption.

For the first set of vertices of $\mathcal{F}$ given by Proposition 2.3.3, we next establish necessary and sufficient optimality conditions.

Proposition 2.4.1. *Suppose that $v \in F$ is a vertex. Suppose that $G = [G^0 \quad G^1]$ so that $(G^0)^T v = g^0$ and $(G^1)^T v < g^1$, where $G^0 \in \mathbb{R}^{n \times m_0}$, $G^1 \in \mathbb{R}^{n \times m_1}$, $g^0 \in \mathbb{R}^{m_0}$, and $g^1 \in \mathbb{R}^{m_1}$. Then, $(v, vv^T) \in \mathcal{F}$ is an optimal solution of (RLT) if and only if there exists $(\hat{u}, \hat{w}, \hat{R}, \hat{S}) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{p \times n} \times \mathcal{S}^m$, where $\hat{u} \in \mathbb{R}^m$ and $\hat{S} \in \mathcal{S}^m$ can be accordingly partitioned as*

$$\hat{u} = \begin{bmatrix} \hat{u}^0 \\ 0 \end{bmatrix} \in \mathbb{R}^m_+, \quad \hat{S} = \begin{bmatrix} \hat{S}^{00} & \hat{S}^{01} \\ (\hat{S}^{01})^T & 0 \end{bmatrix} \in \mathcal{N}^m, \tag{2.47}$$

where $\hat{u}^0 \in \mathbb{R}^{m_0}$, $\hat{S}^{00} \in \mathcal{S}^{m_0}$, and $\hat{S}^{01} \in \mathbb{R}^{m_0 \times m_1}$, such that (2.43) *and* (2.44) *are satisfied. Furthermore, if $\hat{S}^{00} \in \mathcal{S}^{m_0}$ is strictly positive and $\hat{u}_0 \in \mathbb{R}^{m_0}$ is strictly positive, then $(v, vv^T) \in \mathcal{F}$ is the unique optimal solution of (RLT).*

Proof. Suppose that $v \in \mathbb{R}^n$ is a vertex of F. By Proposition 2.3.3, (v, vv^T) is a vertex of $\mathcal{F}$. Following a similar argument as in the proof of Proposition 2.3.3, we obtain

$$\begin{bmatrix} (G^0)^T \\ (G^1)^T \end{bmatrix} vv^T \begin{bmatrix} (G^0)^T \\ (G^1)^T \end{bmatrix}^T - \begin{bmatrix} (G^0)^T \\ (G^1)^T \end{bmatrix} v \begin{bmatrix} g^0 \\ g^1 \end{bmatrix}^T - \begin{bmatrix} g^0 \\ g^1 \end{bmatrix} v^T \begin{bmatrix} (G^0)^T \\ (G^1)^T \end{bmatrix}^T + gg^T = \begin{bmatrix} 0 & 0 \\ 0 & + \end{bmatrix}.$$

where $+$ denotes a submatrix with strictly positive entries. The first assertion now follows from Lemma 2.4.1.

For the second part, suppose further that $\hat{S}_{00} \in \mathcal{S}^{m_0}$ is strictly positive and $\hat{u}_0 \in \mathbb{R}^{m_0}$ is strictly positive. Let $(\hat{x}, \hat{X}) \in \mathcal{F}$ be an arbitrary feasible solution of (RLT). Then, using the same partitions of G and g as before, we have

$$\begin{aligned} (G^0)^T \hat{x} &= g^0 - r^0 \\ (G^1)^T \hat{x} &= g^1 - r^1 \\ H^T \hat{x} &= h \\ H^T \hat{X} &= h \hat{x}^T \\ (G^i)^T \hat{X} G^j &\geq -r^i (g^j)^T - g^i (r^j)^T + g^i (g^j)^T, \quad (i,j) \in \{(0,0),(0,1),(1,1)\}, \end{aligned}$$

where $r^0 \in \mathbb{R}^{m_0}_+$ and $r^1 \in \mathbb{R}^{m_1}_+$. By Lemma 2.4.1, $(\hat{x}, \hat{X}) \in \mathcal{F}$ is an optimal solution if and only if $r^0 = 0$ and $(G^0)^T \hat{X} G^0 - g^0 (g^0)^T = 0$. Note that any $(\hat{x}, \hat{X}) \in \mathcal{F}$ with this property should satisfy the following equation:

$$\begin{bmatrix} (G^0)^T \\ H^T \end{bmatrix} \hat{X} \begin{bmatrix} (G^0)^T \\ H^T \end{bmatrix}^T = \begin{bmatrix} g^0 \\ h \end{bmatrix} \begin{bmatrix} g^0 \\ h \end{bmatrix}^T.$$

By Lemma 2.3.3, it follows that $\hat{X} = vv^T$ is the only solution to this system since $\begin{bmatrix} G^0 & H \end{bmatrix}$ has full row rank by Lemma 2.2.1 (iii). By Lemma 2.4.1, $(v, vv^T) \in \mathcal{F}$ is the unique optimal solution of (RLT). $\square$

Next, we present necessary and sufficient optimality conditions for the second set of vertices of $\mathcal{F}$ given by Proposition 2.3.4.

Proposition 2.4.2. *Let $v^1 \in F$ and $v^2 \in F$ be two vertices such that $v^1 \neq v^2$. Suppose that $G = [G^0 \quad G^1 \quad G^2 \quad G^3]$ so that $(G^0)^T v^1 = (G^0)^T v^2 = g^0$; $(G^1)^T v^1 = g^1$ and $(G^1)^T v^2 < g^2$; $(G^2)^T v^1 < g^1$ and $(G^2)^T v^2 = g^2$; $(G^3)^T v^1 < g^3$ and $(G^3)^T v^2 < g^3$, where $G^0 \in \mathbb{R}^{n \times m_0}$,*

$G^1 \in \mathbb{R}^{n\times m_1}$, $G^2 \in \mathbb{R}^{n\times m_2}$, $G^3 \in \mathbb{R}^{n\times m_3}$, $g^0 \in \mathbb{R}^{m_0}$, $g^1 \in \mathbb{R}^{m_1}$, $g^2 \in \mathbb{R}^{m_2}$, and $g^3 \in \mathbb{R}^{m_3}$. Then, $(\frac{1}{2}(v^1+v^2), \frac{1}{2}(v^1(v^2)^T + v^2(v^1)^T)) \in \mathcal{F}$ is an optimal solution of (RLT) if and only if there exists $(\hat{u}, \hat{w}, \hat{R}, \hat{S}) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{p\times n} \times \mathcal{S}^m$, where $\hat{u} \in \mathbb{R}^m$ and $\hat{S} \in \mathcal{S}^m$ can be accordingly partitioned as

$$\hat{u} = \begin{bmatrix} \hat{u}^0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^m_+, \quad \hat{S} = \begin{bmatrix} \hat{S}^{00} & \hat{S}^{01} & \hat{S}^{02} & \hat{S}^{03} \\ (\hat{S}^{01})^T & \hat{S}^{11} & 0 & 0 \\ (\hat{S}^{02})^T & 0 & \hat{S}^{22} & 0 \\ (\hat{S}^{03})^T & 0 & 0 & 0 \end{bmatrix} \in \mathcal{N}^m, \tag{2.48}$$

where $\hat{u}^0 \in \mathbb{R}^{m_0}$, $\hat{S}^{kk} \in \mathcal{S}^{m_k},\ k = 0,1,2$, $\hat{S}^{0j} \in \mathbb{R}^{m_0\times m_j},\ j = 1,2,3$, such that (2.43) *and* (2.44) *are satisfied. Furthermore, if each of $\hat{S}^{00} \in \mathcal{S}^{m_0}$, $\hat{S}^{01} \in \mathbb{R}^{m_0\times m_1}$, $\hat{S}^{02} \in \mathbb{R}^{m_0\times m_2}$, $\hat{S}^{11} \in \mathcal{S}^{m_1}$, $\hat{S}^{22} \in \mathcal{S}^{m_2}$, and $\hat{u}^0 \in \mathbb{R}^{m_0}$ is strictly positive, then $(\frac{1}{2}(v^1+v^2), \frac{1}{2}(v^1(v^2)^T + v^2(v^1)^T)) \in \mathcal{F}$ is the unique optimal solution of (RLT).*

Proof. By Proposition 2.3.4, $(\frac{1}{2}(v^1+v^2), \frac{1}{2}(v^1(v^2)^T + v^2(v^1)^T))$ is a vertex of $\mathcal{F}$. The proof is similar to the proof of Proposition 2.4.1. By a similar argument as in the proof of Proposition 2.3.4, we have

$$G^T\hat{X}G - G^T\hat{x}g^T - g\hat{x}^TG + gg^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & + & + \\ 0 & + & 0 & + \\ 0 & + & + & + \end{bmatrix},$$

where we assume that $G = \begin{bmatrix} G^0 & G^1 & G^2 & G^3 \end{bmatrix}$ and g is partitioned accordingly, and $+$ denotes a submatrix with strictly positive entries. The first claim follows from Lemma 2.4.1.

For the second assertion, suppose further that each of $\hat{S}^{00} \in \mathcal{S}^{m_0}$, $\hat{S}^{01} \in \mathbb{R}^{m_0\times m_1}$, $\hat{S}^{02} \in \mathbb{R}^{m_0\times m_2}$, $\hat{S}^{11} \in \mathcal{S}^{m_1}$, $\hat{S}^{22} \in \mathcal{S}^{m_2}$, and $\hat{u}_0 \in \mathbb{R}^{m_0}$ is strictly positive. Let $(\hat{x}, \hat{X}) \in \mathcal{F}$ be an arbitrary solution. Then, using the same partition of G and g, we have

$$\begin{aligned} (G^i)^T\hat{x} &= g^i - r^i, \quad i = 0,1,2,3 \\ H^T\hat{x} &= h \\ H^T\hat{X} &= h\hat{x}^T \\ (G^i)^T\hat{X}G^j &\geq -r^i(g^j)^T - g^i(r^j)^T + g^i(g^j)^T, \quad 0 \leq i \leq j \leq 3, \end{aligned}$$

where $r^i \in \mathbb{R}^{m_i}_+,\ i = 0,1,2,3$, and the last set of inequalities is componentwise. By Lemma 2.4.1, $(\hat{x}, \hat{X}) \in \mathcal{F}$ is an optimal solution if and only if

$$\begin{aligned} r^0 &= 0 \\ (G^0)^T\hat{X}G^0 &= g^0(g^0)^T \\ (G^0)^T\hat{X}G^1 &= -g^0(r^1)^T + g^0(g^1)^T \\ (G^0)^T\hat{X}G^2 &= -g^0(r^2)^T + g^0(g^2)^T \\ (G^1)^T\hat{X}G^1 &= -r^1(g^1)^T - g^1(r^1)^T + g^1(g^1)^T \\ (G^2)^T\hat{X}G^2 &= -r^2(g^2)^T - g^2(r^2)^T + g^2(g^2)^T. \end{aligned}$$

Note that $(\frac{1}{2}(v^1+v^2), \frac{1}{2}(v^1(v^2)^T + v^2(v^1)^T)) \in \mathcal{F}$ with $r^1 = g^1 - \frac{1}{2}(G^1)^T(v^1+v^2)$ and

$r^2 = g^2 - \frac{1}{2}(G^2)^T(v^1 + v^2)$ satisfies this system. Using a similar argument as in the proof of Proposition 2.3.4, one can show that this solution is unique. By Lemma 2.4.1, we conclude that $(\frac{1}{2}(v^1 + v^2), \frac{1}{2}(v^1(v^2)^T + v^2(v^1)^T)) \in \mathcal{F}$ is the unique optimal solution of (RLT). $\square$

2.5 Exact RLT Relaxations

In this section, we present necessary and sufficient conditions for an instance of (QP) to admit an exact RLT relaxation.

First, following from [118], we define the convex underestimator arising from RLT relaxations. To that end, let

$$\mathcal{F}(\hat{x}) = \{(x, X) \in \mathcal{F} : x = \hat{x}\}, \quad \hat{x} \in F. \tag{2.49}$$

We next define the following function:

$$\ell_R(\hat{x}) = \min_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} \left\{ \tfrac{1}{2}\langle Q, X\rangle + c^T x : (x, X) \in \mathcal{F}(\hat{x}) \right\}, \quad \hat{x} \in F. \tag{2.50}$$

By [118], $\ell_R(\cdot)$ is a convex underestimator of $q(\cdot)$ over F, i.e., $\ell_R(\hat{x}) \leq q(\hat{x})$ for each $\hat{x} \in F$, and

$$\ell_R^* = \min_{x \in F} \ell_R(x). \tag{2.51}$$

By (2.50),

$$\ell_R(\hat{x}) = c^T\hat{x} + \ell_R^0(\hat{x}), \tag{2.52}$$

where

$$\begin{array}{llrcl} (\mathrm{RLT})(\hat{x}) & \ell_R^0(\hat{x}) = \displaystyle\min_{X \in \mathcal{S}^n} & & \tfrac{1}{2}\langle Q, X\rangle & \\ & \quad\text{s.t.} & & & \\ & & H^T X & = & h\hat{x}^T \\ & & G^T X G - G^T \hat{x} g^T - g\hat{x}^T G + gg^T & \geq & 0. \end{array}$$

Note that (RLT)$(\hat{x})$ has a nonempty feasible region for each $\hat{x} \in F$ since $\hat{X} = \hat{x}\hat{x}^T$ is a feasible solution. By defining the dual variables $(R, S) \in \mathbb{R}^{p \times n} \times \mathcal{S}^m$ corresponding to the first and second sets of constraints in (RLT)$(\hat{x})$, respectively, the dual of (RLT)$(\hat{x})$ is given by

$$\begin{array}{llrcl} (\mathrm{RLT\text{-}D})(\hat{x}) & \displaystyle\max_{S \in \mathcal{S}^m, R \in \mathbb{R}^{p \times n}} & h^T R\hat{x} + g^T S G^T \hat{x} - \tfrac{1}{2} g^T S g & & \\ & \text{s.t.} & & & \\ & & R^T H^T + HR + GSG^T & = & Q \\ & & S & \geq & 0. \end{array}$$

We start with a useful result on the convex underestimator $\ell_R(\cdot)$.

Lemma 2.5.1. *Suppose that F is nonempty. If there exists $\hat{x} \in F$ such that $\ell_R(\hat{x}) = -\infty$, then $\ell_R(\tilde{x}) = -\infty$ for each $\tilde{x} \in F$. Therefore, $\ell_R^* = -\infty$.*

Proof. Suppose that there exists $\hat{x} \in F$ such that $\ell_R(\hat{x}) = -\infty$. By linear optimization duality, (RLT-D)$(\hat{x})$ is infeasible. Therefore, (RLT-D)$(\tilde{x})$ is infeasible for each $\tilde{x} \in F$ since the feasible region of (RLT-D)$(\hat{x})$ does not depend on $\hat{x} \in F$. Since (RLT)$(\tilde{x})$ has a nonempty feasible region for each $\tilde{x} \in F$, (RLT)$(\tilde{x})$ is unbounded below. By (2.52), $\ell_R(\tilde{x}) = -\infty$ for each $\tilde{x} \in F$. The last assertion simply follows from (2.51). $\square$

We next establish another property of $\ell_R(\cdot)$.

Lemma 2.5.2. *Suppose that F is nonempty and there exists $\hat{x} \in F$ such that $\ell_R(\hat{x}) > -\infty$. Then, $\ell_R(\cdot)$ is a piecewise linear convex function.*

Proof. Suppose that F is nonempty and there exists $\hat{x} \in F$ such that $\ell_R(\hat{x}) > -\infty$. By using a similar argument as in the proof of Lemma 2.5.1, we conclude that $\ell_R(\tilde{x}) > -\infty$ for each $\tilde{x} \in F$. By linear optimization duality, for each $\hat{x} \in F$ the optimal value of (RLT-D)$(\hat{x})$ equals $\ell_R^0(\hat{x})$. By Lemma 2.2.4, there exist feasible solutions $(R^i, S^i) \in \mathbb{R}^{p \times n} \times \mathcal{S}^m,\ i = 1, \ldots, s$, of (RLT-D)$(\hat{x})$ and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^{p \times n} \times \mathcal{S}^m$ such that the feasible region of (RLT-D)$(\hat{x})$ is given by $\operatorname{conv}\{(R^i, S^i) : i = 1, \ldots, s\} + \mathcal{C}$. Since the optimal value of (RLT-D)$(\hat{x})$ is finite, it follows that

$$\ell_R^0(\hat{x}) = \max_{i=1,\ldots,s} \left\{ h^T R^i \hat{x} + g^T S^i G^T \hat{x} - \tfrac{1}{2} g^T S^i g \right\}.$$

The assertion follows from (2.52). $\square$

We remark that a similar result was established in [90] for the convex underestimator $\ell_R(\cdot)$ arising from the RLT relaxation of quadratic optimization problems with box constraints. For this class of problems, the hypothesis of Lemma 2.5.2 is vacuous. Therefore, Lemma 2.5.2 extends this result to RLT relaxations of all quadratic optimization problems under a mild assumption.

We next focus on the description of the set of instances of (QP) that admit an exact RLT relaxation. To that end, let us start with a simple observation.

Lemma 2.5.3. *Suppose that F is nonempty and ℓ^* is finite. Then, the RLT relaxation given by (RLT) is exact, i.e., $\ell_R^* = \ell^*$, if and only if there exists an optimal solution $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}$ of (RLT). Furthermore, in this case, $\hat{x} \in F$ is an optimal solution of (QP).*

Proof. Suppose that the RLT relaxation given by (RLT) is exact, i.e., $\ell_R^* = \ell^* > -\infty$. Then, the set of optimal solutions of (QP) is nonempty by the Frank-Wolfe theorem [45]. Therefore, for any optimal solution $\hat{x} \in F$ of (QP), $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}$ is an optimal solution of (RLT) since $\ell^* = q(\hat{x}) = \frac{1}{2}\langle Q, \hat{x}\hat{x}^T\rangle + c^T\hat{x} = \ell_R^*$.

Conversely, if there exists an optimal solution $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}$ of (RLT), then $\ell_R^* = \frac{1}{2}\langle Q, \hat{x}\hat{x}^T\rangle + c^T\hat{x} = q(\hat{x}) \geq \ell^*$ since $\hat{x} \in F$. Then, $\ell_R^* = \ell^*$ by (2.9). The last assertion follows from these arguments. This completes the proof. $\square$

Let F be nonempty and let $\hat{x} \in F$. Let us define the submatrices G^0, G^1, and the subvectors g^0, g^1 such that (2.13) holds. Consider $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}$ and assume without loss of generality that $G = \begin{bmatrix} G^0 & G^1 \end{bmatrix}$. Therefore,

$$G^T\hat{x}\hat{x}^T G - G^T\hat{x}g^T - g\hat{x}^T G + gg^T = \begin{bmatrix} 0 & 0 \\ 0 & r^1(r^1)^T \end{bmatrix},$$

where $r^1 = g^1 - (G^1)^T \hat{x} > 0$. By Lemma 2.4.1, $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}$ is an optimal solution of (RLT) if and only if $(Q, c) \in \mathcal{E}(\hat{x})$, where

$$\mathcal{E}(\hat{x}) = \left\{ (Q,c) \in \mathcal{S}^n \times \mathbb{R}^n : \begin{array}{rcl} \exists & & (\hat{u}, \hat{w}, \hat{R}, \hat{S}) \in \mathbb{R}^m_+ \times \mathbb{R}^p \times \mathbb{R}^{p \times n} \times \mathcal{N}^m \text{ such that} \\ \hat{u} & = & \begin{bmatrix} \hat{u}^0 \\ 0 \end{bmatrix} \in \mathbb{R}^m_+ \\ \hat{S} & = & \begin{bmatrix} \hat{S}^{00} & \hat{S}^{01} \\ (\hat{S}^{01})^T & 0 \end{bmatrix} \in \mathcal{N}^m \\ c & = & -G\hat{u} + H\hat{w} - \hat{R}^T h - G\hat{S}g \\ Q & = & \hat{R}^T H^T + H\hat{R} + G\hat{S}G^T \end{array} \right\}, \tag{2.53}$$

where $\hat{u}^0 \in \mathbb{R}^{m_0}$, $\hat{S}^{00} \in \mathcal{S}^{m_0}$, and $\hat{S}^{01} \in \mathbb{R}^{m_0 \times m_1}$.

For each $\hat{x} \in F$, it is easy to see that $\mathcal{E}(\hat{x})$ is a polyhedral cone in $\mathcal{S}^n \times \mathbb{R}^n$. By Lemma 2.5.3,

$$-\infty < \ell_R^* = \ell_R(\hat{x}) < +\infty \Longleftrightarrow (Q, c) \in \bigcup_{\hat{x} \in F} \mathcal{E}(\hat{x}). \tag{2.54}$$

We next show that the description of instances of (QP) that admit an exact relaxation given by (2.54) can be considerably simplified.

Proposition 2.5.1. *Suppose that F is nonempty and ℓ^* is finite. Let $F_i \subseteq F,\ i = 1, \ldots, s$ denote the minimal faces of F and let $v^i \in F_i,\ i = 1, \ldots, s$ be an arbitrary point on each minimal face. Then, the RLT relaxation given by (RLT) is exact, i.e., $\ell_R^* = \ell^*$, if and only if*

$$(Q, c) \in \bigcup_{i \in \{1,\ldots,s\}} \mathcal{E}(v^i). \tag{2.55}$$

Furthermore, if $(Q, c) \in \mathcal{E}(v^i)$ for some $i = 1, \ldots, s$, then any $\hat{x} \in F_i$ is an optimal solution of (QP).

Proof. By (2.54), it suffices to show that

$$\bigcup_{i \in \{1,\ldots,s\}} \mathcal{E}(v^i) = \bigcup_{x \in F} \mathcal{E}(x).$$

Clearly, the set on the left-hand side is a subset of the one on the right-hand side. For the reverse inclusion, let $(Q, c) \in \mathcal{E}(\hat{x})$, where $\hat{x} \in F \backslash \left(\bigcup_{i \in \{1,\ldots,s\}} F_i \right)$. Let us define the submatrices G^0, G^1 and the subvectors g^0, g^1 such that (2.13) holds and let $F_0 \subseteq F$ denote the smallest face of F that contains $\hat{x}$. Then, there exists a minimal face $F_i \subseteq F_0$. Let $v = v^i \in F_i$. Assuming that $G = \begin{bmatrix} G^0 & G^1 \end{bmatrix}$, we therefore obtain

$$\begin{array}{rcl} (G^0)^T v & = & g^0, \\ (G^1)^T v & \leq & g^1, \\ G^T v v^T G - G^T v g^T - g v^T G + g g^T & = & \begin{bmatrix} 0 & 0 \\ 0 & r_v^1 (r_v^1)^T \end{bmatrix}, \end{array}$$

where $r_v^1 = g^1 - (G^1)^T v \geq 0$ so that $r_v^1 (r_v^1)^T \in \mathcal{N}^{m_1}$. By (2.53) and Lemma 2.4.1, it follows that $(v, vv^T) = (v^i, v^i (v^i)^T) \in \mathcal{F}$ is an optimal solution of (RLT). Therefore, $(Q, c) \in \mathcal{E}(v^i)$.

The last assertion follows from the observation that the argument is independent of the choice of $v \in F_i$ and Lemma 2.5.3. $\square$

Proposition 2.5.1 presents a complete description of the set of instances of (QP) that admit an exact RLT relaxation and reveals that this property holds if and only if (Q, c) lies in the union of a finite number of polyhedral cones. This result is a generalization of the corresponding result established for RLT relaxations of quadratic optimization problems with box constraints [90]. If ℓ^* is finite, it is worth noticing that the exactness of the RLT relaxation implies that the set of optimal solutions of (QP) either contains a vertex of F or an entire minimal face of F if F has no vertices.

We close this section by establishing a necessary condition for having a finite lower bound from the RLT relaxation whenever F has no vertices.

Proposition 2.5.2. *Suppose that F given by* (1.4) *is nonempty but contains no vertices. Let*

$$L = F_\infty \cap -F_\infty = \left\{d \in \mathbb{R}^n : G^T d = 0, \quad H^T d = 0\right\}$$

denote the lineality space of F, where F_∞ is defined as in (2.1)*. Let $B^1 \in \mathbb{R}^{n \times (n-\phi)}$ be a matrix whose columns form a basis for L (or, equivalently, the null space of $\begin{bmatrix} G & H \end{bmatrix}^T$), where ϕ is defined as in* (2.5)*, and let $B^2 \in \mathbb{R}^{n \times \tau}$ be a matrix whose columns are the extreme directions of $F_\infty \cap L^\perp$, where $L^\perp \subset \mathbb{R}^n$ denotes the orthogonal complement of L. Let $F_i \subseteq F,\ i = 1, \ldots, s$, denote the minimal faces of F and let $v^i \in F_i,\ i = 1, \ldots, s$, be an arbitrary point on each minimal face. If $\ell_R^* > -\infty$, then*

$$\begin{aligned} (B^1)^T Q B^1 &= 0, && (2.56)\\ (B^1)^T Q B^2 &= 0, && (2.57)\\ (B^1)^T (Q v^i + c) &= 0, \quad i = 1, \ldots, s. && (2.58) \end{aligned}$$

Furthermore,

$$q(x + \alpha d) = q(x), \quad \text{for all } x \in F,\ d \in L,\ \alpha \in \mathbb{R}, \tag{2.59}$$

where $q(x)$ is defined as in (1.3)*.*

Proof. Note that L is a nontrivial subspace since the dimension of L given by $n - \phi > 0$ by Lemma 2.2.2 (iii). Furthermore, $F_\infty \cap L^\perp \subseteq \mathbb{R}^n$ is a pointed polyhedral cone (possibly equal to $\{0\}$). Suppose that $\ell_R^* > -\infty$ and let $(\hat{x}, \hat{X}) \in \mathcal{F}$ be an optimal solution of (RLT). By Lemma 2.4.1, there exists $(\hat{u}, \hat{w}, \hat{R}, \hat{S}) \in \mathbb{R}^m_+ \times \mathbb{R}^p \times \mathbb{R}^{p \times n} \times \mathcal{N}^m$ such that (2.43)–(2.46) are satisfied. Let $d^1 \in L$ and $d^2 \in F_\infty$. Since $G^T d^1 = 0$ and $H^T d^1 = H^T d^2 = 0$, it follows from (2.44) that $(d^1)^T Q d^2 = 0$. Since $F_\infty = L + (F_\infty \cap L^\perp)$, we conclude that $(B^1 b)^T Q (B^1 b^1 + B^2 b^2) = b^T (B^1)^T Q B^1 b^1 + b^T (B^1)^T Q B^2 b^2 = 0$ for each $b \in \mathbb{R}^{n-\phi}, b^1 \in \mathbb{R}^{n-\phi}$, and $b^2 \in \mathbb{R}^\tau_+$. We therefore obtain (2.56) and (2.57). Next, for each $v^i,\ i = 1, \ldots, s$, and each $d \in L$, we obtain $d^T (Q v^i + c) = d^T \hat{R}^T h - d^T \hat{R}^T h = 0$ by (2.43) and (2.44). Therefore, $Q v^i + c \in L^\perp$ for each $i = 1, \ldots, s$, which yields (2.58). Finally, let $x \in F$ and $d \in L$. By Lemma 2.2.4, there exist $\lambda_i \geq 0,\ i = 1, \ldots, s$, and $d^2 \in F_\infty$ such that $\sum\limits_{i=1}^s \lambda_i = 1$ and $x = \sum\limits_{i=1}^s \lambda_i v^i + d^2$. Therefore, $d^T (Q x + c) = \sum\limits_{i=1}^s \lambda_i d^T (Q v^i + c) + d^T Q d^2 = 0$ since $Q v^i + c \in L^\perp$ for each $i = 1, \ldots, s$ by (2.58) and $d^T Q d^2 = 0$ by the first part of the proof.

Therefore, by (2.56), we obtain $q(x+\alpha d) = q(x) + \alpha d^T(Qx+c) + \frac{1}{2}\alpha^2 d^T Q d = q(x)$, which establishes the last assertion. This completes the proof. $\square$

Under the hypotheses of Proposition 2.5.2, the objective function of (QP) is constant along each line in F. Note, however, that the conditions (2.56), (2.57), and (2.58) are not sufficient for a finite RLT lower bound. For instance, if there exists $\hat{d} \in F_\infty \cap L^\perp$ such that $\hat{d}^T Q \hat{d} < 0$, then (QP) is unbounded along the ray $x + \lambda \hat{d}$ for any $x \in F$, where $\lambda \geq 0$, which would imply that $\ell^* = \ell_R^* = -\infty$.

In the next subsection, we discuss the implications of our results on the algorithmic construction of instances of (QP) with exact, inexact, or unbounded RLT relaxations.

2.6 Implications on Algorithmic Constructions of Instances

In this section, we discuss how our results can be utilized to design algorithms for constructing an instance of (QP) such that the lower bound from the RLT relaxation and the optimal value of (QP) will have a predetermined relation. In particular, our discussions on instances with exact and inexact RLT relaxations in this section can be viewed as generalizations of the algorithmic constructions which will be discussed in Section 3.5.4 and Section 3.5.5 for quadratic optimization problems with box constraints.

To that end, we will assume that the nonempty feasible region F is fixed and given by (1.4). We will discuss how to construct an objective function in such a way that the resulting instance of (QP) will have an exact, inexact, or unbounded RLT relaxation.

2.6.1 Instances with an Unbounded RLT Relaxation

By Lemma 2.3.1, if F is nonempty and bounded, then the RLT relaxation cannot be unbounded. Therefore, a necessary condition to have an unbounded RLT relaxation is that F is unbounded. In this case, the recession cone $\mathcal{F}_\infty$ given by (2.11) contains a nonzero $(\hat{d}, \hat{D}) \in \mathbb{R}^n \times \mathcal{S}^n$ by Proposition 2.3.2. By linear optimization duality, the RLT relaxation (RLT) is unbounded if and only if

$$\tfrac{1}{2}\langle Q, \hat{D}\rangle + c^T \hat{d} < 0, \quad \text{for some } (\hat{d}, \hat{D}) \in \mathcal{F}_\infty. \tag{2.60}$$

Let $\hat{x} \in F$ and $\hat{d} \in F_\infty \backslash \{0\}$ be arbitrary. Let $\hat{D} = \hat{x}\hat{d}^T + \hat{d}\hat{x}^T \in \mathcal{S}^n$. By Proposition 2.3.2, $(\hat{d}, \hat{D}) \in \mathcal{F}_\infty$. By (2.60), it suffices to choose $(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n$ such that

$$\tfrac{1}{2}\langle Q, \hat{D}\rangle + c^T \hat{d} = \hat{d}^T(Q\hat{x} + c) < 0,$$

which would ensure that the RLT relaxation is unbounded.

While this simple procedure can be used to construct an instance of (QP) with an unbounded RLT relaxation, we remark that the resulting instance of (QP) itself may also be unbounded. In particular, if $\hat{d}^T Q \hat{d} \leq 0$ in the aforementioned procedure, then (QP) will be unbounded along the ray $\hat{x} + \lambda \hat{d}$, where $\lambda \geq 0$. One possible approach to construct an instance of (QP) with a finite optimal value but an unbounded RLT relaxation is to generate $(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n$ in such a way that (2.60) holds while a tighter relaxation of (QP) such as the RLT relaxation strengthened by semidefinite constraints has a finite lower bound. This property can be satisfied by ensuring the feasibility of (Q, c) with respect to the dual problem of tighter relaxation. Such an approach would require the solution of a semidefinite feasibility problem.

2.6.2 Instances with an Exact RLT Relaxation

By Proposition 2.5.1, the RLT relaxation is exact if and only if there exists $v \in F$ that lies on a minimal face of F such that $(Q, c) \in \mathcal{E}(v)$, where $\mathcal{E}(v)$ is defined as in (2.53). This result can be used to easily construct an instance of (QP) with an exact RLT relaxation.

The first step requires the computation of a point $v \in F$ that lies on a minimal face of F. Then, it suffices to choose $\hat{u} \in \mathbb{R}^m_+$ and $\hat{S} \in \mathcal{N}^n$ as in (2.53). Finally, choosing an arbitrary $(\hat{w}, \hat{R}) \in \mathbb{R}^p \times \mathbb{R}^{p \times n}$ and defining Q and c using (2.53), we ensure that $(Q, c) \in \mathcal{E}(v)$. It follows from Proposition 2.5.1 that $(v, vv^T) \in \mathcal{F}$ and $(\hat{u}, \hat{w}, \hat{R}, \hat{S}) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{p \times n} \times \mathcal{S}^m$ are optimal solutions of (RLT) and (RLT-D), respectively. By Proposition 2.5.1, we conclude that the RLT relaxation is exact and that $v \in F$ is an optimal solution of (QP). We remark that this procedure not only ensures an exact RLT relaxation but also yields an instance of (QP) with a predetermined optimal solution $v \in F$.

2.6.3 Instances with an Inexact and Finite RLT Relaxation

First, we assume that F has at least two distinct vertices $v^1 \in F$ and $v^2 \in F$. In this case, one can choose $\hat{u} \in \mathbb{R}^m_+$ and $\hat{S} \in \mathcal{N}^m$ such that the assumptions of the second part of Proposition 2.4.2 are satisfied. Then, by choosing an arbitrary $(\hat{w}, \hat{R}) \in \mathbb{R}^p \times \mathbb{R}^{p \times n}$ and defining Q and c using (2.44) and (2.43), respectively, we obtain that $(\frac{1}{2}(v^1 + v^2), \frac{1}{2}(v^1(v^2)^T + v^2(v^1)^T)) \in \mathcal{F}$ is the unique optimal solution of (RLT). Therefore, the RLT relaxation has a finite lower bound ℓ^*_R. By Lemma 2.5.3, we conclude that the RLT relaxation is inexact, i.e., $-\infty < \ell^*_R < \ell^*$.

We next consider the case in which F has no vertices. In this case, $\mathcal{F}_\infty$ also has no vertices by Lemma 2.3.2. Therefore, the RLT relaxation can never have a unique optimal solution. However, our next result shows that we can extend the procedure above to construct an instance of (QP) with an inexact but finite RLT lower bound under a certain assumption on F.

Lemma 2.6.1. *Suppose that F given by* (1.4) *has no vertices but has two distinct minimal faces $F_1 \subseteq F$ and $F_2 \subseteq F$. Let $v^1 \in F_1$ and $v^2 \in F_2$. Suppose that $G = [G^0 \quad G^1 \quad G^2 \quad G^3]$ is defined as in Proposition 2.4.2. Let $\hat{u} \in \mathbb{R}^m_+$ and $\hat{S} \in \mathcal{N}^m$ be such that the assumptions of the second part of Proposition 2.4.2 are satisfied. Let $(\hat{w}, \hat{R}) \in \mathbb{R}^p \times \mathbb{R}^{p \times n}$ be arbitrary. If Q and c are defined by* (2.45) *and* (2.46)*, respectively, then the RLT relaxation of the resulting instance of (QP) is inexact and satisfies $-\infty < \ell^*_R < \ell^*$.*

Proof. Arguing similarly to the first part of the proof of Proposition 2.4.2, we conclude that $(\frac{1}{2}(v^1 + v^2), \frac{1}{2}(v^1(v^2)^T + v^2(v^1)^T)) \in \mathcal{F}$ is an optimal solution of the RLT relaxation of the resulting instance of (QP). Therefore, $-\infty < \ell^*_R$.

Next, we argue that the RLT relaxation is inexact. First, since each of F_1 and F_2 are minimal faces, they are affine subspaces given by

$$\begin{array}{rcl} F_1 & = & \{x \in \mathbb{R}^n : (G^0)^T x = g^0, \quad (G^1)^T x = g^1, \quad H^T x = h\}, \\ F_2 & = & \{x \in \mathbb{R}^n : (G^0)^T x = g^0, \quad (G^2)^T x = g^2, \quad H^T x = h\}. \end{array}$$

Suppose, for a contradiction, that the RLT relaxation is exact. Then, by Proposition 2.5.1, there exists $v \in F_0$, where $F_0 \subseteq F$ is a minimal face of F, such that (v, vv^T) is an optimal solution of (RLT). Let us define $r = g - G^T v \geq 0$ and partition r accordingly as

$$(G^i)^T v = g^i - r^i, \quad i = 0, 1, 2, 3,$$

where $r^i \in \mathbb{R}^{m_i}_+,\ i = 0, 1, 2, 3$. By Lemma 2.4.1, (v, vv^T) and $(\hat{u}, \hat{w}, \hat{R}, \hat{S}) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{p \times n} \times \mathcal{S}^m$ satisfy the optimality conditions (2.43)–(2.46). By (2.45) and $\hat{u}^0 > 0$, we conclude that $r^0 = 0$. On the other hand,

$$G^T vv^T G - G^T v g^T - g v^T G + g g^T = G^T vv^T G + r g^T + g \hat{r}^T - g g^T = r r^T.$$

By (2.46), we obtain $r^1 = 0$ and $r^2 = 0$ since $\hat{S}^{11} \in \mathcal{S}^{m_1}$ and $\hat{S}^{22} \in \mathcal{S}^{m_2}$ are strictly positive. It follows that $v \in F_0$ satisfies

$$(G^0)^T v = g^0, \quad (G^1)^T v = g^1, \quad (G^2)^T v = g^2, \quad H^T v = h,$$

i.e., v lies on a face whose dimension is strictly smaller than each of F_1 or F_2. This contradicts our assumption that each of F_1 and F_2 is a minimal face of F. We therefore conclude that (RLT) cannot have an optimal solution of the form (v, vv^T), where v lies on a minimal face of F, or equivalently $(Q, c) \notin \mathcal{E}(v)$ for any v that lies on a minimal face of F, where $\mathcal{E}(v)$ is defined as in (2.53). By Proposition 2.5.1, we conclude that the RLT relaxation is inexact, i.e., $-\infty < \ell_R^* < \ell^*$. $\square$

The next example illustrates the algorithmic construction of Lemma 2.6.1.

Example 2.6.1. *Suppose that F is given as in Example 2.3.1. Note that F has no vertices since it contains the line $\{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$. F has two minimal faces given by*

$$\begin{array}{rcl} F_1 & = & \{x \in \mathbb{R}^2 : x_1 + x_2 = 1\}, \\ F_2 & = & \{x \in \mathbb{R}^2 : x_1 + x_2 = -1\}. \end{array}$$

Therefore, F satisfies the assumptions of Lemma 2.6.1. Let us choose $v^1 = [1 \quad 0]^T \in F_1$ and $v_2 = [0 \quad -1]^T \in F_2$. We obtain

$$\hat{x} = \tfrac{1}{2}(v^1 + v^2) = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \quad \hat{X} = \tfrac{1}{2}(v^1 (v^2)^T + v^2 (v^1)^T) = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}.$$

Since $g - G^T \hat{x} > 0$, we choose $\hat{u} = 0 \in \mathbb{R}^2$. Using the partition given in Proposition 2.4.2, we obtain that G^0 is an empty matrix, $G^1 = [1 \quad 1]^T$, $G^2 = [-1 \quad -1]^T$, and G^3 is an empty matrix. By the second part of Proposition 2.4.2, we choose

$$\hat{S} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, by (2.45) *and* (2.46)*, we obtain*

$$Q = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By Proposition 2.4.2, $\ell_R^* = \frac{1}{2}\langle Q, \hat{X}\rangle + c^T\hat{x} = -\frac{3}{2}$. Note that $L = F_\infty$ as given in Example 2.3.1. In view of Proposition 2.5.2, it is worth noticing that (2.56) *and* (2.58) *are satisfied whereas* (2.57) *is vacuous in this example. Furthermore, for each $x \in F$ such that $x_1 + x_2 = \beta$, where $\beta \in [-1, 1]$, we have $q(x) = \frac{3}{2}\beta^2 + \beta$. Therefore, the minimum value is attained at $\beta^* = -\frac{1}{3}$. It follows that $\ell^* = -\frac{1}{6}$ and the set of optimal solutions of (QP) is given by $\left\{x \in \mathbb{R}^2 : x_1 + x_2 = -\frac{1}{3}\right\}$. Therefore, $-\infty < \ell_R^* < \ell^*$.*

2.6.4 Implications of One Minimal Face

We finally consider the case in which F has exactly one minimal face. For any instance of (QP) with this property, we show that the RLT relaxation is either exact or unbounded below.

Lemma 2.6.2. *Consider an instance of (QP), where F given by* (1.4) *is nonempty and has exactly one minimal face. Then, the RLT relaxation is either exact or unbounded below.*

Proof. Suppose that F has one minimal face $F_0 \subseteq F$ and let $v \in F_0$. Suppose that $G = [G^0 \quad G^1]$ so that $(G^0)^T v = g^0$ and $(G^1)^T v < g^1$, where $G^0 \in \mathbb{R}^{n \times m_0}$, $G^1 \in \mathbb{R}^{n \times m_1}$, $g^0 \in \mathbb{R}^{m_0}$, and $g^1 \in \mathbb{R}^{m_1}$. First, we claim that the set of inequalities $(G^1)^T x \leq g^1$ is redundant for F. Let $\hat{x} \in F$ be arbitrary. By Lemma 2.2.4,

$$F = \{v\} + F_\infty,$$

where F_∞ is given by (2.1). Therefore, there exists $\hat{d} \in F_\infty$ such that $\hat{x} = v + \hat{d}$. Therefore, $(G^1)^T\hat{x} = (G^1)^T v + (G^1)^T\hat{d} < g^1$ since $(G^1)^T v < g^1$ and $(G^1)^T\hat{d} \leq 0$ by (2.1). Therefore, $(G^1)^T x \leq g^1$ is implied by $(G^0)^T x \leq g^0$ and $H^T x = h$. By [104, Proposition 2], all of the RLT constraints obtained from $(G^1)^T x \leq g^1$ are implied by the RLT constraints obtained from $(G^0)^T x \leq g^0$ and $H^T x = h$. Therefore, we have $(\hat{x}, \hat{X}) \in \mathcal{F}$ if and only if $(G^0)^T\hat{x} \leq g^0$, $H^T\hat{x} = h$, $H^T\hat{X} = h\hat{x}^T$, and

$$(G^0)^T\hat{X}G^0 - (G^0)^T\hat{x}(g^0)^T - g^0\hat{x}^T G^0 + g^0(g^0)^T \geq 0.$$

Note that all of the inequality constraints of $\mathcal{F}$ are active at (v, vv^T) (or at any $(v', v'(v')^T)$, where $v' \in F_0$, if v is not a vertex of F). If the feasible region of the dual problem given by (RLT-D) is nonempty, then any $(v, vv^T) \in \mathcal{F}$, where $v \in F_0$, satisfies the optimality conditions of Lemma 2.4.1 together with any feasible solution of (RLT-D). It follows that any such $(v, vv^T) \in \mathcal{F}$ is an optimal solution of (RLT). By Lemma 2.5.3, we conclude that the RLT relaxation is exact. On the other hand, if (RLT-D) is infeasible, then (RLT) is unbounded below by linear optimization duality. The assertion follows. □

We conclude this section with the following corollary.

Corollary 2.6.1. *Suppose that* F *given by* (1.4) *has exactly one vertex* $v \in F$*. Then,* $\mathcal{F}$ *given by* (2.8) *has exactly one vertex* (v, vv^T)*. Furthermore, if* $F = \{v\}$*, then* $\mathcal{F} = \{(v, vv^T)\}$*.*

Proof. Suppose that F has one vertex $v \in F$. By Proposition 2.3.3, (v, vv^T) is a vertex of $\mathcal{F}$. By Proposition 2.5.1 and Lemma 2.6.2, we either have $(Q, c) \in \mathcal{E}(v)$, where $\mathcal{E}(v)$ is defined as in (2.53), in which case, the RLT relaxation is exact and (v, vv^T) is an optimal solution of (RLT), or $(Q, c) \notin \mathcal{E}(v)$ and the RLT relaxation is unbounded below. By Lemma 2.2.1 (iv), we conclude that (v, vv^T) is the unique vertex of $\mathcal{F}$. If $F = \{v\}$, then $\mathcal{F}$ is bounded by Lemma 2.3.1 and contains a unique vertex (v, vv^T) by the first part. Therefore, by Lemma 2.2.4, we conclude that $\mathcal{F} = \{(v, vv^T)\}$. $\square$

Corollary 2.6.1 reveals that the description of $\mathcal{F}$ is independent of the particular representation of F if F consists of a single point. We remark that, in general, this is not the case. For instance, let $F_1 = \{x \in \mathbb{R}^n : e^T x = 1\}$, where $e \in \mathbb{R}^n$ denotes the vector of all ones. Then, the RLT procedure yields $\mathcal{F}_1 = \{(x, X) \in \mathbb{R}^n \times \mathcal{S}^n : e^T x = 1, \quad Xe = x\}$. On the other hand, consider $F_2 = \{x \in \mathbb{R}^n : e^T x \leq 1, \quad -e^T x \leq -1\}$. Clearly, $F_1 = F_2$. However, the feasible region of the RLT relaxation is now given by $\mathcal{F}_2 = \{(x, X) \in \mathbb{R}^n \times \mathcal{S}^n : e^T x = 1, \quad 1 - 2e^T x + e^T X e = 0\} = \{(x, X) \in \mathbb{R}^n \times \mathcal{S}^n : e^T x = 1, \quad e^T X e = 1\}$. It is easy to see that $\mathcal{F}_1 \subset \mathcal{F}_2$ for each $n \geq 2$.

2.7 Summary

In this chapter, we studied various relations between the polyhedral properties of the feasible region of a (QP) and its RLT relaxation. We presented a complete description of the set of instances of quadratic optimization problems that admit exact RLT relaxations. We then discussed how our results can be used to construct (QPs) with an exact, inexact, and unbounded RLT relaxation.

For RLT relaxations of (QP), we are able to establish a partial characterization of the set of vertices of the feasible region of the RLT relaxation. We intend to work on a complete characterization of this set in the near future. Such a characterization may have further algorithmic implications for constructing a larger set of instances with inexact but finite RLT relaxations.

In the next chapter, we will investigate the RLT and SDP-RLT relaxations of (BoxQP). We draw stronger conclusions on the RLT relaxation of (BoxQP), in particular, on vertices of the feasible region of the RLT relaxation. We further give descriptions and generation algorithms of the set of instances of (BoxQP) that admit exact and inexact RLT relaxations and exact SDP-RLT relaxations.

Chapter 3

Quadratic Optimization Problems with Box Constraints

In this chapter, we consider the RLT and SDP-RLT relaxations of quadratic optimization problems with box constraints. This chapter is based on [90].

Recall that a quadratic optimization problem with box constraints is a nonconvex quadratic optimization problem minimized subject to lower and upper bounds on each variable:

$$\ell^* = \min_{x \in \mathbb{R}^n} \{q(x) : x \in F\}, \qquad \text{(BoxQP)}$$

where $q : \mathbb{R}^n \to \mathbb{R}$ and $F \subseteq \mathbb{R}^n$ are respectively given by

$$q(x) = \tfrac{1}{2} x^T Q x + c^T x, \quad F = \{x \in \mathbb{R}^n : 0 \leq x \leq e\}.$$

where $0 \in \mathbb{R}^n$ denotes the vector of all zeros and $e \in \mathbb{R}^n$ denotes the vector of all ones.

If $Q \in \mathcal{S}^n$ is positive semidefinite in the objective function, (BoxQP) can be solved in polynomial time [72]. However, if Q is an indefinite or negative semidefinite matrix, then (BoxQP) has been shown to be an NP-hard problem [89]. Furthermore, when all diagonal elements of Q are non-positive, there exists an optimal solution which is an extreme point [56].

There exist numerous algorithms for solving (BoxQP). Under the assumption that Q is negative semidefinite, Konno [68] proposes a cutting plane approach by reformulating the problem into a bilinear programming problem. Gould and Toint [53] provide local methods for solving (BoxQP) and more general quadratic programming problems. There are also methods for solving (BoxQP) in a global manner [3, 4, 37, 38, 43, 44, 54, 56, 88, 103, 105]. Vandenbussche and Nemhauser [112, 113] apply a branch and bound method on the first-order KKT conditions and solve linear programming relaxations at each node. Based on this algorithm, Burer and Vandenbussche [32] solve semidefinite programming relaxations at each node instead of linear programs. Sherali and Tuncbilek [104] develop the Reformulation Linearization Technique (RLT) relaxation for solving (BoxQP) and further nonconvex quadratic programming problems.

Our main goal is to describe the set of instances of (BoxQP) that admit exact RLT relaxations (i.e., $\ell^*_R = \ell^*$) as well as those that admit exact SDP-RLT relaxations (i.e., $\ell^*_{RS} = \ell^*$). Such descriptions shed light on easier subclasses of a difficult optimization problem. Compared to a general quadratic program given by (QP), we establish a stronger result about the set of vertices in the RLT relaxation of (BoxQP). Furthermore, this stronger result enables us to present an alternative description of instances of (BoxQP) that admit exact RLT relaxations. In addition, relying on these descriptions, we develop efficient algorithms for constructing an instance of (BoxQP) that admits an exact or inexact RLT relaxation and exact SDP-RLT relaxation.

This chapter is organized as follows. We discuss the RLT and SDP-RLT relaxation of (BoxQP) and define our notation in Section 3.1. We briefly review the literature in Section 3.2. We review the optimality conditions in Section 3.3. We present several properties of the convex underestimators arising from the RLT and SDP-RLT relaxations in Section 3.4. Section 3.5 focuses on the description of instances with exact RLT relaxations and presents two algorithms for constructing instances, one with exact RLT relaxations and another one with inexact RLT relaxations. SDP-RLT relaxations are treated in Section 3.6, which includes an algebraic description of instances with exact SDP-RLT relaxations and two algorithms for constructing instances, one with exact SDP-RLT relaxations and another one with exact SDP-RLT but inexact RLT relaxations. We present several numerical examples, preliminary computational results, and a brief discussion in Section 3.7. Finally, Section 3.8 concludes this chapter.

3.1 RLT and SDP-RLT Relaxations

By using a simple "lifting" idea, (BoxQP) can be equivalently reformulated as

$$\text{(L-BoxQP)} \quad \ell^* = \min_{(x,X)\in\mathcal{F}} \tfrac{1}{2}\langle Q, X\rangle + c^T x,$$

where $\langle A, B\rangle = \operatorname{trace}(A^T B) = \sum\limits_{i=1}^{p}\sum\limits_{j=1}^{q} A_{ij}B_{ij}$ for any $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{p\times q}$, and

$$\mathcal{F} = \left\{(x, X) \in \mathbb{R}^n \times \mathcal{S}^n : 0 \leq x \leq e, \quad X_{ij} = x_i x_j, \quad 1 \leq i \leq j \leq n\right\}. \tag{3.1}$$

Since (L-BoxQP) is an optimization problem with a linear objective function over a nonconvex feasible region, one can replace $\mathcal{F}$ by $\operatorname{conv}(\mathcal{F})$, where $\operatorname{conv}(\cdot)$ denotes the convex hull, without affecting the optimal value. Many convex relaxations of (BoxQP) arise from this reformulation by employing outer approximations of $\operatorname{conv}(\mathcal{F})$ using tractable convex sets.

A well-known relaxation of $\operatorname{conv}(\mathcal{F})$ is obtained by replacing the nonlinear equalities $X_{ij} = x_i x_j$ by the so-called McCormick inequalities [80], which gives rise to the RLT relaxation of (BoxQP) (see, e.g., [103]):

$$\text{(R)} \quad \ell^*_R = \min_{(x,X)\in\mathbb{R}^n\times\mathcal{S}^n} \left\{\frac{1}{2}\langle Q, X\rangle + c^T x : (x, X) \in \mathcal{F}_R\right\},$$

where

$$\mathcal{F}_R = \left\{(x, X) \in \mathbb{R}^n \times \mathcal{S}^n : \begin{array}{rcccl} 0 & \leq & x & \leq & e \\ \max\{x_i + x_j - 1, 0\} & \leq & X_{ij} & \leq & \min\{x_i, x_j\}, \quad 1 \leq i \leq j \leq n \end{array}\right\}. \tag{3.2}$$

The RLT relaxation (R) of (BoxQP) can be further strengthened by adding tighter semidefinite constraints [106], giving rise to the SDP-RLT relaxation:

$$\text{(RS)} \quad \ell^*_{RS} = \min_{(x,X)\in\mathbb{R}^n\times\mathcal{S}^n} \left\{ \frac{1}{2}\langle Q, X\rangle + c^T x : (x,X) \in \mathcal{F}_{RS} \right\},$$

where

$$\mathcal{F}_{RS} = \left\{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n : (x,X) \in \mathcal{F}_R, \quad X - xx^T \succeq 0\right\}. \tag{3.3}$$

The optimal value of each of the RLT and SDP-RLT relaxations, denoted by ℓ^*_R and ℓ^*_{RS}, respectively, yields a lower bound on the optimal value of (BoxQP). The SDP-RLT relaxation is clearly at least as tight as the RLT relaxation, i.e.,

$$\ell^*_R \le \ell^*_{RS} \le \ell^*. \tag{3.4}$$

Every convex relaxation of a nonconvex quadratic optimization problem obtained through lifting induces a convex underestimator on the objective function over the feasible region [118]. In the following sections, we introduce duality and optimality conditions of (BoxQP), and the convex underestimators induced by RLT and SDP-RLT relaxations and establish several properties of these underestimators.

For an instance of (BoxQP) given by $(Q,c) \in \mathcal{S}^n \times \mathbb{R}^n$, the set of vertices of (BoxQP) is denoted by

$$V \;=\; \{x \in F : x_j \in \{0,1\}, \quad j = 1,\ldots,n\}. \tag{3.5}$$

For $\hat{x} \in F$, we define the following index sets:

$$\mathbf{L} \;=\; \mathbf{L}(\hat{x}) = \{j \in \{1,\ldots,n\} : \hat{x}_j = 0\}, \tag{3.6}$$

$$\mathbf{B} \;=\; \mathbf{B}(\hat{x}) = \{j \in \{1,\ldots,n\} : 0 < \hat{x}_j < 1\}, \tag{3.7}$$

$$\mathbf{U} \;=\; \mathbf{U}(\hat{x}) = \{j \in \{1,\ldots,n\} : \hat{x}_j = 1\}. \tag{3.8}$$

For simplicity, we refer to $\mathbf{L}(\hat{x})$, $\mathbf{B}(\hat{x})$ and $\mathbf{U}(\hat{x})$ as $\mathbf{L}$, $\mathbf{B}$ and $\mathbf{U}$, respectively, which will be clear in the content.

3.2 Literature Review

Quadratic optimization problems with box constraints have been extensively studied in the literature. Since our focus is on convex relaxations in this thesis, we will mainly restrict our review accordingly.

Recall that the lifted feasible region of (BoxQP) to be

$$\mathcal{F} = \{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n : 0 \le x \le e, \quad X_{ij} = x_i x_j, \quad 1 \le i \le j \le n\}, \tag{3.9}$$

and the set $\text{conv}(\mathcal{F})$, where $\mathcal{F}$ is given by (3.9), has been investigated in several papers (see, e.g., [6, 8, 28, 31, 41, 116]). This is a nonpolyhedral convex set even for $n = 1$. However, it turns out that $\text{conv}(\mathcal{F})$ is closely related to the so-called Boolean quadric polytope [86] that arises in unconstrained binary quadratic optimization problems, which can be formulated as an instance of (BoxQP) [92], and is given by $\mathrm{conv}(\mathcal{F}^-)$, where

$$\mathcal{F}^- = \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}} : x_i \in \{0,1\}, \quad z_{ij} = x_i x_j, \quad 1 \leq i < j \leq n\right\}. \tag{3.10}$$

The RLT relaxation of $\mathrm{conv}(\mathcal{F}^-)$, denoted by $\mathcal{F}_R^-$, is given by

$$\mathcal{F}_R^- = \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}} : \begin{array}{rcccl} 0 & \leq & x & \leq & e \\ \max\{x_i + x_j - 1, 0\} & \leq & z_{ij} & \leq & \min\{x_i, x_j\}, \quad 1 \leq i < j \leq n \end{array}\right\}, \tag{3.11}$$

which is very similar to the feasible region of RLT relaxation of (BoxQP), except that McCormick inequalities are only applied to $1 \leq i < j \leq n$. In particular, $\mathcal{F}_R^- = \mathrm{conv}(\mathcal{F}^-)$ for $n = 2$ [80]. Padberg [86] identifies several facets of $\mathrm{conv}(\mathcal{F}^-)$ and shows that the components of each vertex of $\mathcal{F}_R^-$ are in the set $\{0, \frac{1}{2}, 1\}$. Yajima and Fujie [116] show how to extend the valid inequalities for $\mathcal{F}^-$ in [86] to $\mathcal{F}$. Burer and Letchford [31] extend this result further by observing that $\mathrm{conv}(\mathcal{F}^-)$ is the projection of $\mathrm{conv}(\mathcal{F})$ onto the "common variables". They also give a description of the set of extreme points of $\mathrm{conv}(\mathcal{F})$. We refer the reader to [28, 41, 48] for further refinements, a computational procedure based on such valid inequalities, and generalizations to domain-constrained optimization problems, respectively.

Anstreicher [6] reports computational results illustrating that the SDP-RLT relaxation significantly improves the RLT relaxation and gives a theoretical justification of the improvement by comparing the feasible region of RLT and SDP-RLT relaxation of (BoxQP) for $n = 2$. Anstreicher and Burer [8] show that the feasible region of SDP-RLT relaxation of (BoxQP) is equivalent to $\mathrm{conv}(\mathcal{F})$ if and only if $n \leq 2$. In particular, this implies that the SDP-RLT relaxation of (BoxQP) is always exact for $n \leq 2$.

We next briefly review the literature on exact convex relaxations. Several papers have identified conditions under which a particular convex relaxation of a class of optimization problems is exact. For quadratically constrained quadratic optimization problems, we refer the reader to [10, 33, 57, 61, 65, 75, 107, 114] for various exactness conditions for second-order cone or semidefinite relaxations. Recently, a large family of convex relaxations of general quadratic optimization problems was considered in a unified manner through induced convex underestimators and a general algorithmic procedure was proposed for constructing instances with inexact relaxations for various convex relaxations [118].

In this work, our focus is on algebraic descriptions and algorithmic constructions of instances of (BoxQP) that admit exact and inexact RLT relaxations as well as those that admit exact SDP-RLT relaxations and exact SDP-RLT but inexact RLT relaxations. Therefore, our focus is similar to presenting descriptions of such instances of standard quadratic optimization problems for the RLT relaxation in [96] and the SDP-RLT relaxation in [52].

3.3 Optimality Conditions

In this section, we first review first-order and second-order optimality conditions for (BoxQP).

Let $\hat{x} \in F$ be a local minimizer of (BoxQP). By the first-order optimality conditions, there exists $(\hat{r}, \hat{s}) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$
\begin{aligned}
Q\hat{x} + c + \hat{r} - \hat{s} &= 0, &(3.12)\\
\hat{r}_{\mathbf{L}\cup\mathbf{B}} &= 0, &(3.13)\\
\hat{s}_{\mathbf{B}\cup\mathbf{U}} &= 0, &(3.14)\\
\hat{r} &\geq 0, &(3.15)\\
\hat{s} &\geq 0. &(3.16)
\end{aligned}
$$

Note that $\hat{r} \in \mathbb{R}^n$ and $\hat{s} \in \mathbb{R}^n$ are the Lagrange multipliers corresponding to the constraints $x \leq e$ and $x \geq 0$ in (BoxQP), respectively.

For a local minimizer $\hat{x} \in F$ of (BoxQP), the second-order optimality conditions are given by

$$d^T Q d \geq 0, \quad \forall\, d \in D(\hat{x}), \tag{3.17}$$

where $D(\hat{x})$ is the set of feasible directions at $\hat{x}$ at which the directional derivative of the objective function vanishes, i.e.,

$$D(\hat{x}) := \{d \in \mathbb{R}^n : (Q\hat{x} + c)^T d = 0, \quad d_{\mathbf{L}} \geq 0, \quad d_{\mathbf{U}} \leq 0\}. \tag{3.18}$$

Note, in particular, that

$$\hat{x} \in F \text{ is a local minimizer} \Rightarrow Q_{\mathbf{BB}} \succeq 0. \tag{3.19}$$

In fact, $\hat{x} \in F$ is a local minimizer of (BoxQP) if and only if the first-order and second-order optimality conditions given by (3.12)–(3.16) and (3.17), respectively, are satisfied (see, e.g., [62, 78]).

3.4 Properties of RLT and SDP-RLT Relaxations

Given an instance of (BoxQP), recall that the RLT relaxation is given by

$$\text{(R)} \quad \ell_R^* = \min_{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n} \left\{ \frac{1}{2}\langle Q, X \rangle + c^T x : (x, X) \in \mathcal{F}_R \right\},$$

where $\mathcal{F}_R$ is given by (3.2), and the SDP-RLT relaxation by

$$\text{(RS)} \quad \ell_{RS}^* = \min_{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n} \left\{ \frac{1}{2}\langle Q, X \rangle + c^T x : (x, X) \in \mathcal{F}_{RS} \right\},$$

where $\mathcal{F}_{RS}$ is given by (3.3).

Every convex relaxation of a nonconvex quadratic optimization problem obtained through lifting induces a convex underestimator on the objective function over the feasible region [118]. In the following sections, we introduce duality and optimality conditions of Box QP, and the convex underestimators induced by RLT and SDP-RLT relaxations and establish several properties of these underestimators.

3.4.1 Convex Underestimators

In this section, we introduce the convex underestimators induced by the RLT and SDP-RLT relaxations. Let us first define the following sets parametrized by $\hat{x} \in F$:

$$
\begin{aligned}
\mathcal{F}_R(\hat{x}) &= \{(x, X) \in \mathcal{F}_R : x = \hat{x}\}, \quad \hat{x} \in F, &(3.20)\\
\mathcal{F}_{RS}(\hat{x}) &= \{(x, X) \in \mathcal{F}_{RS} : x = \hat{x}\}, \quad \hat{x} \in F. &(3.21)
\end{aligned}
$$

For each $\hat{x} \in F$, we clearly have $\{(\hat{x}, \hat{x}\hat{x}^T)\} \subseteq \mathcal{F}_{RS}(\hat{x}) \subseteq \mathcal{F}_R(\hat{x})$ and

$$\bigcup_{\hat{x} \in F} \mathcal{F}_R(\hat{x}) = \mathcal{F}_R, \quad \bigcup_{\hat{x} \in F} \mathcal{F}_{RS}(\hat{x}) = \mathcal{F}_{RS}.$$

Next, we define the following functions:

$$
\begin{aligned}
\ell_R(\hat{x}) &= \min_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} \left\{\tfrac{1}{2}\langle Q, X\rangle + c^T x : (x, X) \in \mathcal{F}_R(\hat{x})\right\}, \quad \hat{x} \in F, &(3.22)\\
\ell_{RS}(\hat{x}) &= \min_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} \left\{\tfrac{1}{2}\langle Q, X\rangle + c^T x : (x, X) \in \mathcal{F}_{RS}(\hat{x})\right\}, \quad \hat{x} \in F. &(3.23)
\end{aligned}
$$

Note that the functions $\ell_R(\cdot)$ and $\ell_{RS}(\cdot)$ return the best objective function value of the corresponding relaxation subject to the additional constraint that $x = \hat{x}$. By [118], each of $\ell_R(\cdot)$ and $\ell_{RS}(\cdot)$ is a convex function over F satisfying the relations

$$\ell_R(\hat{x}) \leq \ell_{RS}(\hat{x}) \leq q(\hat{x}), \quad \hat{x} \in F, \qquad (3.24)$$

and

$$
\begin{aligned}
(\mathrm{R1}) \quad \ell_R^* &= \min_{x \in F} \ell_R(x), &(3.25)\\
(\mathrm{RS1}) \quad \ell_{RS}^* &= \min_{x \in F} \ell_{RS}(x). &(3.26)
\end{aligned}
$$

The convex underestimators $\ell_R(\cdot)$ and $\ell_{RS}(\cdot)$ allow us to view the RLT and SDP-RLT relaxations in the original space $\mathbb{R}^n$ of (BoxQP) by appropriately projecting out the lifted variables $X \in \mathcal{S}^n$ that appear in each of (R) and (RS). As such, (R1) and (RS1) can be viewed as "reduced" formulations of the RLT relaxation and the SDP-RLT relaxation, respectively. In the remainder of this chapter, we will alternate between the two equivalent formulations (R) and (R1) for the RLT relaxation as well as (RS) and (RS1) for the SDP-RLT relaxation.

3.4.2 Properties of Convex Underestimators

We now present several properties of the convex underestimators $\ell_R(\cdot)$ and $\ell_{RS}(\cdot)$ given by (3.22) and (3.23), respectively.

First, we start with the observation that $\ell_R(\cdot)$ has a very specific structure with a simple closed-form expression.

Lemma 3.4.1. *$\ell_R(\cdot)$ is a piecewise linear convex function on F given by*

$$\ell_R(\hat{x}) = \frac{1}{2}\left(\sum_{(i,j):Q_{ij}>0} Q_{ij}\max\{0, \hat{x}_i + \hat{x}_j - 1\} + \sum_{(i,j):Q_{ij}<0} Q_{ij}\min\{\hat{x}_i, \hat{x}_j\}\right) + c^T\hat{x}, \quad \hat{x} \in F. \tag{3.27}$$

Proof. For each $\hat{x} \in F$, the relation (3.27) follows from (3.22) and (3.2). It follows that $\ell_R(\cdot)$ is a piecewise linear convex function on F since it is given by the sum of a finite number of piecewise linear convex functions. □

In contrast with $\ell_R(\cdot)$ given by the optimal value of a simple linear optimization problem with bound constraints, $\ell_{RS}(\cdot)$ does not, in general, have a simple closed-form expression as it is given by the optimal value of a semidefinite optimization problem.

The next result states a useful decomposition property regarding the sets $\mathcal{F}_R(\hat{x})$ and $\mathcal{F}_{RS}(\hat{x})$.

Lemma 3.4.2. *For any $\hat{x} \in F$, $(\hat{x}, \hat{X}) \in \mathcal{F}_R(\hat{x})$ if and only if there exists $\hat{M} \in \mathcal{M}_R(\hat{x})$ such that $\hat{X} = \hat{x}\hat{x}^T + \hat{M}$, where*

$$\mathcal{M}_R(\hat{x}) = \left\{ M \in \mathcal{S}^n : \begin{array}{rcl} M_{ij} & \leq & \min\{\hat{x}_i - \hat{x}_i\hat{x}_j, \hat{x}_j - \hat{x}_i\hat{x}_j\}, \quad i \in \mathbf{B}, j \in \mathbf{B}, \\ M_{ij} & \geq & \max\{-\hat{x}_i\hat{x}_j, \hat{x}_i + \hat{x}_j - 1 - \hat{x}_i\hat{x}_j\}, \quad i \in \mathbf{B}, j \in \mathbf{B}, \\ M_{ij} & = & 0, \quad \textit{otherwise.} \end{array} \right\}, \tag{3.28}$$

where $\mathbf{B}$ is given by (3.7)*. Furthermore, $(\hat{x}, \hat{X}) \in \mathcal{F}_{RS}(\hat{x})$ if and only if $\hat{M} \in \mathcal{M}_{RS}(\hat{x})$, where*

$$\mathcal{M}_{RS}(\hat{x}) = \{M \in \mathcal{S}^n : M \in \mathcal{M}_R(\hat{x}), \quad M \succeq 0\}. \tag{3.29}$$

Proof. Both assertions follow from (3.20), (3.2), (3.21), (3.3), and the decomposition $\hat{X} = \hat{x}\hat{x}^T + \hat{M}$. □

By Lemma 3.4.2, we remark that M_{ij} has a negative lower bound and a positive upper bound in (3.28) if and only if $i \in \mathbf{B}$ and $j \in \mathbf{B}$. Therefore, for any $\hat{x} \in \mathcal{F}$ and any $(\hat{x}, \hat{X}) \in \mathcal{F}_R$ (and hence any $(\hat{x}, \hat{X}) \in \mathcal{F}_{RS}$), we obtain

$$\hat{X}_{ij} = \hat{x}_i\hat{x}_j, \quad i \notin \mathbf{B}, \text{ or } j \notin \mathbf{B}. \tag{3.30}$$

This observation yields the following result.

Corollary 3.4.1. *For any vertex $v \in F$, $\mathcal{F}_R(v) = \mathcal{F}_{RS}(v) = \{(v, vv^T)\}$.*

Proof. The claim directly follows from (3.30) since $\mathbf{B} = \emptyset$. □

The decomposition in Lemma 3.4.2 can be translated into the functions $\ell_R(\cdot)$ and $\ell_{RS}(\cdot)$.

Lemma 3.4.3. *For each $\hat{x} \in F$,*

$$\ell_R(\hat{x}) = q(\hat{x}) + \tfrac{1}{2}\min_{M \in \mathcal{M}_R(\hat{x})} \langle Q, M\rangle, \tag{3.31}$$

$$\ell_{RS}(\hat{x}) = q(\hat{x}) + \tfrac{1}{2}\min_{M \in \mathcal{M}_{RS}(\hat{x})} \langle Q, M\rangle, \tag{3.32}$$

where $\mathcal{M}_R(\hat{x})$ and $\mathcal{M}_{RS}(\hat{x})$ are given by (3.28) *and* (3.29)*, respectively.*

Proof. The assertions directly follow from (3.22), (3.23), and Lemma 3.4.2. □

By Lemma 3.4.3, we can easily establish the following properties.

Lemma 3.4.4. *Let $\hat{x} \in F$ and let $\mathbf{B} = \mathbf{B}(\hat{x})$, where $\mathbf{B}(\hat{x})$ is given by* (3.7)*.*

(i) $\ell_R(\hat{x}) = q(\hat{x})$ if and only if $\hat{x}$ is a vertex of F or $Q_{\mathbf{BB}} = 0$.

(ii) $\ell_{RS}(\hat{x}) = q(\hat{x})$ if and only if $\hat{x}$ is a vertex of F or $Q_{\mathbf{BB}} \succeq 0$.

Proof. By Lemma 3.4.3, $\ell_R(\hat{x}) = q(\hat{x})$ (resp., $\ell_{RS}(\hat{x}) = q(\hat{x})$) if and only if $\min\limits_{M \in \mathcal{M}_R(\hat{x})} \langle Q, M \rangle = 0$ (resp., $\min\limits_{M \in \mathcal{M}_{RS}(\hat{x})} \langle Q, M \rangle = 0$). The assertions now follow from Lemma 3.4.2. □

Lemma 3.4.4 immediately gives rise to the following results about the underestimator $\ell_{RS}(\cdot)$.

Corollary 3.4.2. *(i) If $Q \succeq 0$, then $\ell_{RS}(\hat{x}) = q(\hat{x})$ for each $\hat{x} \in F$.*

(ii) For any local or global minimizer $\hat{x} \in F$ of (BoxQP), we have $\ell_{RS}(\hat{x}) = q(\hat{x})$.

Proof. If $Q \succeq 0$, then $Q_{\mathbf{BB}} \succeq 0$ for each $\mathbf{B} \subseteq \{1, \ldots, n\}$. Therefore, both assertions follow from Lemma 3.4.4(ii) since $Q_{\mathbf{BB}} \succeq 0$ at any local or global minimizer of (BoxQP) by (3.19). □

Corollary 3.4.2(i) in fact holds for SDP relaxations of general quadratic optimization problems and a result similar to Corollary 3.4.2(ii) was established for general quadratic optimization problems with a bounded feasible region [118]. We remark that Corollary 3.4.2(ii) presents a desirable property of the SDP-RLT relaxation, which is a necessary condition for its exactness by (3.24) and (3.26). However, this necessary condition, in general, is not sufficient for exactness.

3.5 Exact and Inexact RLT Relaxations

In this section, we focus on instances of (BoxQP) that admit exact and inexact RLT relaxations. We first establish a useful property of the set of optimal solutions of RLT relaxations. Using this property, we present two equivalent but different algebraic descriptions of instances with exact RLT relaxations. Then, we present an algorithm for constructing instances of (BoxQP) with an exact RLT relaxation and another algorithm for constructing instances with an inexact RLT relaxation.

3.5.1 Optimal Solutions of RLT Relaxation

In this section, we extend polyhedral properties to optimal solutions of (BoxQP). Our first result establishes the existence of a minimizer of (R1) with a very specific structure. Recall that, (R1) is

$$\text{(R1)} \quad \ell_R^* = \min_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} \left\{ \tfrac{1}{2} \langle Q, X \rangle + c^T x : (x, X) \in \mathcal{F} \right\}$$

where

$$\mathcal{F} = \{(x, X) \in \mathbb{R}^n \times \mathcal{S}^n : 0 \leq x \leq e, \quad X_{ij} = x_i x_j, \quad 1 \leq i \leq j \leq n\}.$$

Proposition 3.5.1. *For the RLT relaxation of any instance of (BoxQP), there exists an optimal solution $\hat{x} \in F$ of (R1), where (R1) is given by* (3.25)*, such that $\hat{x}_j \in \{0, \frac{1}{2}, 1\}$ for each $j = 1, \ldots, n$.*

Proof. Let $\hat{x} \in F$ be an optimal solution of (R1), i.e., $\ell_R^* = \ell_R(\hat{x})$. Suppose that there exists $k \in \{1, \ldots, n\}$ such that $\hat{x}_k \notin \{0, \frac{1}{2}, 1\}$. By appropriately perturbing $\hat{x}$, we will show that one can construct another optimal solution $\tilde{x} \in F$ such that $\ell_R(\tilde{x}) = \ell_R(\hat{x}) = \ell_R^*$ and $\tilde{x}_j \in \{0, \frac{1}{2}, 1\}$ for each $j = 1, \ldots, n$.

Let

$$\alpha = \min\{\hat{x}_k, 1 - \hat{x}_k\} \in (0, \tfrac{1}{2})$$

and let

$$\begin{aligned} \alpha_l &= \max\left\{\left(\max_{j:\min\{\hat{x}_j, 1-\hat{x}_j\} < \alpha} \min\{\hat{x}_j, 1 - \hat{x}_j\}\right), 0\right\}, \\ \alpha_u &= \min\left\{\left(\min_{j:\min\{\hat{x}_j, 1-\hat{x}_j\} > \alpha} \min\{\hat{x}_j, 1 - \hat{x}_j\}\right), \tfrac{1}{2}\right\}, \end{aligned}$$

with the usual conventions that the minimum and the maximum over the empty set are defined to be $+\infty$ and $-\infty$, respectively. Note that $0 \le \alpha_l < \alpha < \alpha_u \le \frac{1}{2}$ and

$$\min\{\hat{x}_j, 1 - \hat{x}_j\} \in [0, \alpha_l] \cup [\alpha_u, 1 - \alpha_u] \cup [1 - \alpha_l, 1] \tag{3.33}$$

Let us define the following index sets:

$$\begin{aligned} \mathbb{I}_1 &= \{j \in \{1, \ldots, n\} : \hat{x}_j = \alpha\}, \\ \mathbb{I}_2 &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 1 - \alpha\}, \\ \mathbb{I}_3 &= \{j \in \{1, \ldots, n\} : \hat{x}_j \in [0, \alpha_l] \cup [\alpha_u, 1 - \alpha_u] \cup [1 - \alpha_l, 1]\}. \end{aligned}$$

Note that $\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3$ is a partition of the index set by the definitions of α_l and α_u, and we have $k \in \mathbb{I}_1 \cup \mathbb{I}_2$. Let us define a direction $\hat{d} \in \mathbb{R}^n$ by

$$\hat{d}_j = \begin{cases} 1, & j \in \mathbb{I}_1, \\ -1, & j \in \mathbb{I}_2, \\ 0, & j \in \mathbb{I}_3. \end{cases}$$

Consider $x^\beta = \hat{x} + \beta\hat{d}$. It is easy to verify that $x^\beta \in F$ for each $\beta \in [\alpha_l - \alpha, \alpha_u - \alpha]$. We claim that $\ell_R(x^\beta)$ is a linear function of β on this interval. By (3.27), it suffices to show that each term is a linear function.

First, let us focus on the term given by $\max\{0, x_i^\beta + x_j^\beta - 1\} = \max\{0, \hat{x}_i + \hat{x}_j - 1 + \beta\hat{d}_i + \beta\hat{d}_j\}$, where $i = 1, \ldots, n;\ j = 1, \ldots, n$. It suffices to show that the sign of $\hat{x}_i + \hat{x}_j - 1 + \beta\hat{d}_i + \beta\hat{d}_j$ does not change for each $\beta \in [\alpha_l - \alpha, \alpha_u - \alpha]$ and for each $i = 1, \ldots, n;\ j = 1, \ldots, n$. Clearly, $\hat{x}_i + \hat{x}_j - 1 + \beta\hat{d}_i + \beta\hat{d}_j = \hat{x}_i + \hat{x}_j - 1$ if $\{i, j\} \subseteq \mathbb{I}_3$; or $i \in \mathbb{I}_1, j \in \mathbb{I}_2$; or $i \in \mathbb{I}_2, j \in \mathbb{I}_1$. For the remaining cases, it follows from the definitions of

$\mathbb{I}_1, \mathbb{I}_2$, and $\mathbb{I}_3$ that

$$\hat{x}_i+\hat{x}_j-1+\beta\hat{d}_i+\beta\hat{d}_j \in \begin{cases} [2\alpha_l-1, 2\alpha_u-1], & \{i,j\} \subseteq \mathbb{I}_1, \\ [1-2\alpha_u, 1-2\alpha_l], & \{i,j\} \subseteq \mathbb{I}_2, \\ [\alpha_l-1, \alpha_l+\alpha_u-1] \cup [\alpha_l+\alpha_u-1, 0] \cup [0, \alpha_u], & i \in \mathbb{I}_1, j \in \mathbb{I}_3;\ \text{or } i \in \mathbb{I}_3, j \in \mathbb{I}_1, \\ [-\alpha_u, 0] \cup [0, 1-\alpha_l-\alpha_u] \cup [1-\alpha_l-\alpha_u, 1-\alpha_l], & i \in \mathbb{I}_2, j \in \mathbb{I}_3;\ \text{or } i \in \mathbb{I}_3, j \in \mathbb{I}_2. \end{cases}$$

Our claim now follows from $0 \leq \alpha_l < \alpha_u \leq \frac{1}{2}$. Therefore, $\max\{0, \hat{x}_i + \hat{x}_j - 1 + \beta\hat{d}_i + \beta\hat{d}_j\}$ is a linear function of β on the interval $[\alpha_l - \alpha, \alpha_u - \alpha]$ for each $i = 1, \ldots, n;\ j = 1, \ldots, n$.

Let us now consider the term $\min\{x_i^\beta, x_j^\beta\} = \min\{\hat{x}_i + \beta\hat{d}_i, \hat{x}_j + \beta\hat{d}_j\}$. By the choice of $\hat{d}$, it is easy to see that the order of the components of x^β remains unchanged for each $\beta \in [\alpha_l - \alpha, \alpha_u - \alpha]$, i.e., if $\hat{x}_i \leq \hat{x}_j$, then $\hat{x}_i + \beta\hat{d}_i \leq \hat{x}_j + \beta\hat{d}_j$ for each $i = 1, \ldots, n;\ j = 1, \ldots, n$. It follows that $\min\{\hat{x}_i + \beta\hat{d}_i, \hat{x}_j + \beta\hat{d}_j\}$ is a linear function of β on the interval $[\alpha_l - \alpha, \alpha_u - \alpha]$.

Since the third term in (3.27) is also a linear function of β on the interval $[\alpha_l - \alpha, \alpha_u - \alpha]$, it follows that $\ell_R(x^\beta)$ is a linear function on $[\alpha_l - \alpha, \alpha_u - \alpha]$. Therefore, by the optimality of $\hat{x}$ in (R1), $\ell_R(x^\beta)$ is a constant function on this interval. If $\alpha_l = 0$ and $\alpha_u = \frac{1}{2}$, then the alternate optimal solution $x^\beta = \hat{x} + \beta\hat{d}$ obtained by setting $\beta = \alpha_l - \alpha = -\alpha$ or by setting $\beta = \alpha_u - \alpha = \frac{1}{2} - \alpha$ satisfies the desired property by 3.33. Otherwise, we set (i) $\beta = \alpha_l - \alpha$ if $\alpha_l > 0$, or (ii) $\beta = \alpha_u - \alpha$ if $\alpha_u > \frac{1}{2}$. In each case, we have $\min\{x_j^\beta, 1 - x_j^\beta\} \in [0, \alpha_l] \cup [\alpha_u, \frac{1}{2}]$ for each $j = 1, \ldots, n$. Starting now with the new optimal solution x_β, one can repeat the same procedure in an iterative manner to arrive at an alternate optimal solution with the desired property. Note that this procedure is finite since either, at each iteration, α_l strictly decreases in case (i) or α_u strictly increases in case (ii) at each iteration. $\square$

We can utilize Proposition 3.5.1 to obtain the following result about the set of optimal solutions of (R).

Corollary 3.5.1. *There exists an optimal solution $(\hat{x}, \hat{X}) \in \mathbb{R}^n \times \mathcal{S}^n$ of the RLT relaxation (R) such that $\hat{x}_j \in \{0, \frac{1}{2}, 1\}$ for each $j = 1, \ldots, n$ and $\hat{X}_{ij} \in \{0, \frac{1}{2}, 1\}$ for each $i = 1, \ldots, n;\ j = 1, \ldots, n$ such that*

$$\begin{bmatrix} \hat{X}_{\mathbf{LL}} & \hat{X}_{\mathbf{LB}} & \hat{X}_{\mathbf{LU}} \\ \hat{X}_{\mathbf{BL}} & \hat{X}_{\mathbf{BB}} & \hat{X}_{\mathbf{BU}} \\ \hat{X}_{\mathbf{UL}} & \hat{X}_{\mathbf{UB}} & \hat{X}_{\mathbf{UU}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{X}_{\mathbf{BB}} & \frac{1}{2} e_{\mathbf{B}} e_{\mathbf{U}}^T \\ 0 & \frac{1}{2} e_{\mathbf{U}} e_{\mathbf{B}}^T & e_{\mathbf{U}} e_{\mathbf{U}}^T \end{bmatrix}, \quad \hat{X}_{ij} \in \{0, \tfrac{1}{2}\}, \quad i \in \mathbf{B},\ j \in \mathbf{B}.$$

Proof. By Proposition 3.5.1, there exists $\hat{x} \in F$ such that $\ell_R^* = \ell_R(\hat{x})$ and $\hat{x}_j \in \{0, \frac{1}{2}, 1\}$ for each $j = 1, \ldots, n$. Define $\hat{X} \in \mathcal{S}^n$ such that

$$\hat{X}_{ij} = \begin{cases} \max\{0, \hat{x}_i + \hat{x}_j - 1\}, & \text{if } Q_{ij} > 0, \\ \min\{\hat{x}_i, \hat{x}_j\}, & \text{if } Q_{ij} < 0, \\ 0, & \text{otherwise.} \end{cases} \quad i = 1, \ldots, n;\ j = 1, \ldots, n.$$

Note that $(\hat{x}, \hat{X}) \in \mathcal{F}_R$ by (3.2) and $\ell_R(\hat{x}) = \frac{1}{2}\langle Q, \hat{X} \rangle + c^T\hat{x}$ by Lemma 3.4.1. Therefore, $(\hat{x}, \hat{X})$ is an optimal solution of (R) with the desired property. $\square$

The next result follows from Corollary 3.5.1.

Corollary 3.5.2. *For each vertex $(\hat{x}, \hat{X}) \in \mathcal{F}_R$, $\hat{x}_j \in \{0, \frac{1}{2}, 1\}$ for each $j = 1, \ldots, n$ and $\hat{X}_{ij} \in \{0, \frac{1}{2}, 1\}$ for each $i = 1, \ldots, n;\ j = 1, \ldots, n$.*

Proof. Since $\mathcal{F}_R$ is a polytope, $(\hat{x}, \hat{X}) \in \mathcal{F}_R$ is a vertex if and only if there exists a $(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n$ such that $(\hat{x}, \hat{X}) \in \mathcal{F}_R$ is the unique optimal solution of (R). The assertion follows from Corollary 3.5.1. □

We remark that Padberg [86] established a similar result for the set $\mathcal{F}_R^-$ given by (3.11), which is the projection of $\mathcal{F}_R$ on the onto the "common variables". In fact, Padberg's result is implied by Corollary 3.5.2, which extends the same result to $\mathcal{F}_R$ since each vertex of $\mathcal{F}_R^-$ is a projection of some vertex in $\mathcal{F}_R$. In contrast with the proof of [86], which relies on linearly independent active constraints, our proof uses a specific property of the set of optimal solutions of the reduced formulation (R1).

3.5.2 First Description of Instances with Exact RLT Relaxations

In this section, we present our first description of the set of instances of (BoxQP) with an exact RLT relaxation. We start with a useful property of such instances.

Proposition 3.5.2. *For any instance of (BoxQP), the RLT relaxation is exact, i.e., $\ell_R^* = \ell^*$, if and only if there exists a vertex $v \in F$ such that v is an optimal solution of (R1), where (R1) is given by* (3.25).

Proof. Suppose that $\ell_R^* = \ell^*$. Then, by (3.24) and (3.25), for any optimal solution $x^* \in F$ of (BoxQP), we have $q(x^*) = \ell^* = \ell_R^* \leq \ell_R(x^*) \leq q(x^*) = \ell^*$, which implies that $\ell_R^* = \ell_R(x^*) = q(x^*)$. By Lemma 3.4.4(i), either x^* is a vertex of F, in which case, we are done, or $Q_{\mathbf{BB}} = 0$, where $\mathbf{B}$ is given by (3.7). In the latter case, since $\hat{x}_{\mathbf{L}}^* = 0$ and $\hat{x}_{\mathbf{U}}^* = e_{\mathbf{U}}$ by (3.6) and (3.8), respectively, we obtain

$$\ell_R^* = \ell^* = q(x^*) = \frac{1}{2} e_{\mathbf{U}}^T Q_{\mathbf{UU}} e_{\mathbf{U}} + (x_{\mathbf{B}}^*)^T Q_{\mathbf{BU}} e_{\mathbf{U}} + c_{\mathbf{U}}^T e_{\mathbf{U}} + c_{\mathbf{B}}^T x_{\mathbf{B}}^* = \frac{1}{2} e_{\mathbf{U}}^T Q_{\mathbf{UU}} e_{\mathbf{U}} + c_{\mathbf{U}}^T e_{\mathbf{U}},$$

where the last equality follows from the identity $Q_{\mathbf{BU}} e_{\mathbf{U}} + c_{\mathbf{B}} = 0$ by (3.12)–(3.14). Therefore, for the vertex $v \in F$ given by $v_j = 1$ for each $j \in \mathbf{U}$ and $v_j = 0$ for each $j \in \mathbf{L} \cup \mathbf{B} = 0$, we obtain $q(v) = \ell^* = \ell_R(v)$ by Lemma 3.4.4(i).

Conversely, if there exists a vertex $v \in F$ such that $\ell_R(v) = \ell_R^*$, then we have $\ell^* \leq q(v) = \ell_R(v) = \ell_R^*$ by Lemma 3.4.4(i). The assertion follows from (3.4). □

Proposition 3.5.2 presents an important property of the set of instances of (BoxQP) with exact RLT relaxations in terms of the set of optimal solutions and gives rise to the following corollary.

Corollary 3.5.3. *For any instance of (BoxQP), the RLT relaxation is exact if and only if there exists a vertex $v \in F$ such that (v, vv^T) is an optimal solution of (R). Furthermore, in this case, v is an optimal solution of (BoxQP).*

Proof. The assertion directly follows from Proposition 2.5.1, Proposition 3.5.2, Lemma 3.4.4(i), and Corollary 3.4.1. □

By Corollary 3.5.3, if the set of optimal solutions of (BoxQP) does not contain a vertex, then the RLT relaxation is inexact. Note that if $q(\cdot)$ is a concave function, then the set of optimal solutions of (BoxQP) contains at least one vertex. However, this is an NP-hard problem (see, e.g., [95]), which implies that the RLT relaxation can be inexact even if the set of optimal solutions of (BoxQP) contains at least one vertex. The next example illustrates that the RLT relaxation can be inexact even if every optimal solution of (BoxQP) is a vertex.

Example 3.5.1. *Consider an instance of (BoxQP) with*

$$Q = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note that

$$q(x) = \frac{1}{2}\left(-x_1^2 + x_2^2 - 4x_1x_2\right) + x_1 + x_2 = \frac{1}{2}\left(x_2 - x_1\right)^2 + (x_1 + x_2)(1 - x_1),$$

which implies that $q(x) \geq 0$ *for each* $x \in F$ *and* $q(x) = 0$ *if and only if* $x \in \{0, e\}$*. Therefore,* $\ell^* = 0$ *and the set of optimal solution is given by* $\{0, e\}$*, which consists of two vertices of* F*. However, for* $\hat{x} = \frac{1}{2}e \in \mathbb{R}^2$*, it is easy to verify that* $\ell_R(\hat{x}) = -1/4 < \ell^*$*. In fact,* $\ell_R^* = \ell_R(\hat{x}) = -1/4$ *and* $\ell_R(x) > \ell_R(\hat{x})$ *for each* $x \in F \backslash \{\hat{x}\}$*. Figure 3.1 illustrates the functions* $q(\cdot)$ *and* $\ell_R(\cdot)$*.*

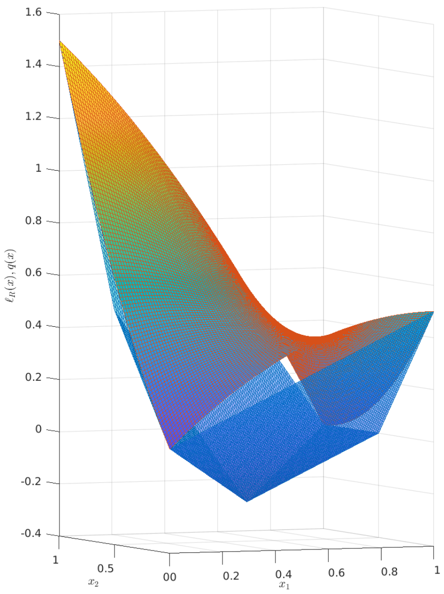


Figure 3.1: Graphs of $q(\cdot)$ (above) and $\ell_R(\cdot)$ (below) for Example 3.5.1

We next present our first description of the set of instances that admit an exact RLT relaxation. To that end, let us define

$$\mathcal{E}_R = \{(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell^* = \ell_R^*\}. \tag{3.34}$$

i.e., $\mathcal{E}_R$ denotes the set of all instances of (BoxQP) that admit an exact RLT relaxation. For a given $\hat{x} \in F$, let us define

$$\mathcal{O}_R(\hat{x}) = \{(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell_R(\hat{x}) = \ell_R^*\} = \{(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell_R(\hat{x}) \leq \ell_R(x), \quad x \in F\}, \tag{3.35}$$

i.e., $\mathcal{O}_R(\hat{x})$ denotes the set of instances of (BoxQP) such that $\hat{x}$ is a minimizer of (R1), where (R1) is given by (3.25). By (3.27), it is easy to see that $\mathcal{O}_R(\hat{x})$ is a convex cone for each $\hat{x} \in F$. Our next result provides an algebraic description of the set $\mathcal{E}_R$.

Proposition 3.5.3. *Let $V \subset F$ denote the set of vertices of F given by* (3.5) *and let $V^+ = \{x \in F : x_j \in \{0, \frac{1}{2}, 1\}, \quad j = 1, \ldots, n\}$. Then, $\mathcal{E}_R$ defined as in* (3.34) *is given by the union of a finite number of polyhedral cones and admits the following description:*

$$\mathcal{E}_R = \bigcup_{v \in V} \mathcal{O}_R(v) = \bigcup_{v \in V} \left(\bigcap_{\hat{x} \in V^+} \{(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell_R(v) \leq \ell_R(\hat{x})\} \right). \tag{3.36}$$

Proof. By Proposition 3.5.2, the RLT relaxation is exact if and only if there exists a vertex $v \in F$ such that $\ell_R(v) = \ell_R^*$, which, together with (3.35), leads to the first equality in (3.36). By Proposition 3.5.1, the set V^+ contains at least one minimizer of $\ell_R(\cdot)$, which implies the second equality in (3.36). $\mathcal{E}_R$ is the union of a finite number of polyhedral cones since $\ell_R(x)$ is a linear function of (Q, c) for each fixed $x \in F$ by Lemma 3.4.1 and V^+ is a finite set. □

For each $v \in V$, we remark that Proposition 3.5.3 gives a description of $\mathcal{O}_R(v)$ using 3^n linear inequalities since $|V^+| = 3^n$. In fact, for each $v \in V$, the convexity of $\ell_R(\cdot)$ on F implies that it suffices to consider only those $\hat{x} \in V_+$ such that $\hat{x}_j \in \{0, \frac{1}{2}\}$ for $j \in \mathbf{L}(v)$ and $\hat{x}_j \in \{\frac{1}{2}, 1\}$ for $j \in \mathbf{U}(v)$, where $\mathbf{L}(v)$ and $\mathbf{U}(v)$ are given by (3.6) and (3.8), respectively, which implies a simpler description of $\mathcal{E}_R$ with 2^n inequalities. Due to the exponential number of such inequalities, this description is not very useful for efficiently checking if a particular instance of (BoxQP) admits an exact RLT relaxation. Similarly, this description cannot be used easily for constructing such an instance of (BoxQP). In the next section, we present an alternative description of $\mathcal{E}_R$ using linear optimization duality, which gives rise to algorithms for efficiently constructing instances of (BoxQP) with exact or inexact RLT relaxations.

3.5.3 An Alternative Description of Instances with Exact RLT Relaxations

In this section, our main goal is to present an alternative description of the set $\mathcal{E}_R$ defined as in (3.34) using duality.

Recall that the RLT relaxation is given by

$$\text{(R)} \quad \ell_R^* = \min_{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n} \left\{ \frac{1}{2}\langle Q, X \rangle + c^T x : (x, X) \in \mathcal{F}_R \right\},$$

where, $\mathcal{F}_R$ given by (3.2), can be expressed in the following form:

$$\mathcal{F}_R = \left\{ (x, X) \in \mathbb{R}^n \times \mathcal{S}^n : \begin{array}{rcl} x & \leq & e \\ x & \geq & 0 \\ X - xe^T - ex^T + ee^T & \geq & 0 \\ -X + ex^T & \geq & 0 \\ X & \geq & 0 \end{array} \right\}, \tag{3.37}$$

and the dual problem of (R) is given by

$$
\begin{array}{llrcl}
\text{(R-D)} & \displaystyle\max_{(r,s,W,Y,Z)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n\times\mathbb{R}^{n\times n}\times\mathcal{S}^n} & \multicolumn{3}{c}{-e^Tr - \frac{1}{2}e^TWe} \\
& \text{s.t.} & & & \\
& & -r + s - We + Y^Te & = & c \\
& & W - Y - Y^T + Z & = & Q \\
& & r & \geq & 0 \\
& & s & \geq & 0 \\
& & W & \geq & 0 \\
& & Y & \geq & 0 \\
& & Z & \geq & 0.
\end{array}
$$

Note that the variables $(W, Y, Z) \in \mathcal{S}^n \times \mathbb{R}^{n\times n} \times \mathcal{S}^n$ in (R-D) are scaled by a factor of $\frac{1}{2}$. First, we start with optimality conditions for (R) and (R-D).

Lemma 3.5.1. $(\hat{x}, \hat{X}) \in \mathcal{F}_R$ *is an optimal solution of (R) if and only if there exists* $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n} \times \mathcal{S}^n$ *such that*

$$
\begin{array}{rcll}
Q & = & \hat{W} - \hat{Y} - \hat{Y}^T + \hat{Z} & (3.38) \\
c & = & -\hat{r} + \hat{s} - \hat{W}e + \hat{Y}^Te & (3.39) \\
\hat{r}^T(e - \hat{x}) & = & 0 & (3.40) \\
\hat{s}^T\hat{x} & = & 0 & (3.41) \\
\langle \hat{W}, \hat{X} - \hat{x}e^T - e\hat{x}^T + ee^T \rangle & = & 0 & (3.42) \\
\langle \hat{Y}, e\hat{x}^T - \hat{X} \rangle & = & 0 & (3.43) \\
\langle \hat{Z}, \hat{X} \rangle & = & 0 & (3.44) \\
\hat{r} & \geq & 0 & (3.45) \\
\hat{s} & \geq & 0 & (3.46) \\
\hat{W} & \geq & 0 & (3.47) \\
\hat{Y} & \geq & 0 & (3.48) \\
\hat{Z} & \geq & 0. & (3.49)
\end{array}
$$

Proof. The assertion follows from strong duality since each of (R) and (R-D) is a linear optimization problem. □

Lemma 3.5.1 gives rise to an alternative description of the set of instances of (BoxQP) that admits an exact RLT relaxation.

Corollary 3.5.4. $(Q, c) \in \mathcal{E}_R$*, where* $\mathcal{E}_R$ *is defined as in* (3.34)*, if and only if there exists a vertex* $v \in F$ *and there exists* $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n} \times \mathcal{S}^n$ *such that the relations* (3.38)–(3.49) *hold, where* $(\hat{x}, \hat{X}) = (v, vv^T)$.

Proof. Note that $(Q, c) \in \mathcal{E}_R$ if and only if there exists a vertex $v \in F$ such that (v, vv^T) is an optimal solution of (R) by Corollary 3.5.3. The assertion now follows from Lemma 3.5.1. □

In the next section, we discuss how Corollary 3.5.4 can be utilized to construct instances of (BoxQP) with exact and inexact RLT relaxations.

3.5.4 Construction of Instances with Exact RLT Relaxations

In this section, we describe an algorithm for constructing instances of (BoxQP) with an exact RLT relaxation. Algorithm 1 is based on designating a vertex $v \in F$ and constructing an appropriate dual feasible solution that satisfies optimality conditions together with $(v, vv^T) \in \mathcal{F}_R$.

Algorithm 1 (BoxQP) Instance with an Exact RLT Relaxation

Require: $n; \mathbf{L} \subseteq \{1, \ldots, n\}$
Ensure: $(Q, c) \in \mathcal{E}_R$
1: $\mathbf{U} \leftarrow \{1, \ldots, n\} \backslash \mathbf{L}$
2: Choose an arbitrary $\hat{r} \in \mathbb{R}^n$ such that $\hat{r} \geq 0$ and $\hat{r}_{\mathbf{L}} = 0$.
3: Choose an arbitrary $\hat{s} \in \mathbb{R}^n$ such that $\hat{s} \geq 0$ and $\hat{s}_{\mathbf{U}} = 0$.
4: Choose an arbitrary $\hat{W} \in \mathcal{S}^n$ such that $\hat{W} \geq 0$ and $\hat{W}_{\mathbf{LL}} = 0$.
5: Choose an arbitrary $\hat{Y} \in \mathbb{R}^{n \times n}$ such that $\hat{Y} \geq 0$ and $\hat{Y}_{\mathbf{LU}} = 0$.
6: Choose an arbitrary $\hat{Z} \in \mathcal{S}^n$ such that $\hat{Z} \geq 0$ and $\hat{Z}_{\mathbf{UU}} = 0$.
7: $Q \leftarrow \hat{W} - \hat{Y} - \hat{Y}^T + \hat{Z}$
8: $c \leftarrow -\hat{r} + \hat{s} - \hat{W}e + \hat{Y}^T e$

The following result establishes the correctness of Algorithm 1.

Proposition 3.5.4. *Algorithm 1 returns $(Q, c) \in \mathcal{E}_R$, where $\mathcal{E}_R$ is defined as in* (3.34)*. Conversely, any $(Q, c) \in \mathcal{E}_R$ can be generated by Algorithm 1 with appropriate choices of $\mathbf{L} \subseteq \{1, \ldots, n\}$ and $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n} \times \mathcal{S}^n$.*

Proof. Let $\mathbf{L} \subseteq \{1, \ldots, n\}$ and define $\mathbf{U} = \{1, \ldots, n\} \backslash \mathbf{L}$. Let $v \in F$ be the vertex given by $v_j = 0,\ j \in \mathbf{L}$ and $v_j = 1,\ j \in \mathbf{U}$. It is easy to verify that $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n} \times \mathcal{S}^n$ and $(\hat{x}, \hat{X}) = (v, vv^T)$ satisfy the hypotheses of Corollary 3.5.4, which establishes the first assertion. The second assertion also follows from Corollary 3.5.4. □

By considering all possible subsets $\mathbf{L} \subseteq \{1, \ldots, n\}$, Proposition 3.5.4 yields an alternative characterization of $\mathcal{E}_R$ given by the union of 2^n polyhedral cones (cf. Proposition 3.5.3). In contrast with the description in Proposition 3.5.3, this alternative description enables us to easily construct an instance of (BoxQP) with a known optimal vertex and an exact RLT relaxation (cf. Corollary 3.5.3). Note, however, that even the alternative description is not very useful for effectively checking if $(Q, c) \in \mathcal{E}_R$ due to the exponential number of polyhedral cones.

3.5.5 Construction of Instances with Inexact RLT Relaxations

In this section, we propose an algorithm for constructing instances of (BoxQP) with an inexact RLT relaxation. Algorithm 2 is based on constructing a dual optimal solution of (R-D) in such a way that no feasible solution of the form $(v, vv^T) \in \mathcal{F}_R$ can be an optimal solution of (R), where $v \in F$ is a vertex.

The next result establishes that the output from Algorithm 2 is an instance of (BoxQP) with an inexact RLT relaxation.

Proposition 3.5.5. *Algorithm 2 returns $(Q, c) \notin \mathcal{E}_R$, where $\mathcal{E}_R$ is defined as in* (3.34)*.*

Algorithm 2 (BoxQP) Instance with an Inexact RLT Relaxation

Require: $n; \mathbf{B} \subseteq \{1, \ldots, n\}; \mathbf{B} \neq \emptyset$

Ensure: $(Q, c) \notin \mathcal{E}_R$

1: Choose an arbitrary $\mathbf{L} \subseteq \{1, \ldots, n\} \backslash \mathbf{B}$

2: $\mathbf{U} \leftarrow \{1, \ldots, n\} \backslash (\mathbf{B} \cup \mathbf{L})$

3: Choose an arbitrary $k \in \mathbf{B}$.

4: Choose an arbitrary $\hat{r} \in \mathbb{R}^n$ such that $\hat{r}_{\mathbf{U}} \geq 0$ and $\hat{r}_{\mathbf{L} \cup \mathbf{B}} = 0$.

5: Choose an arbitrary $\hat{s} \in \mathbb{R}^n$ such that $\hat{s}_{\mathbf{L}} \geq 0$ and $\hat{s}_{\mathbf{B} \cup \mathbf{U}} = 0$.

6: Choose an arbitrary $\hat{W} \in \mathcal{S}^n$ such that $\hat{W}_{\mathbf{LL}} = 0,\ \hat{W}_{\mathbf{LB}} = 0,\ \hat{W}_{\mathbf{BL}} = 0,\ \hat{W}_{kk} > 0$, and $\hat{W}_{ij} \geq 0$ otherwise.

7: Choose an arbitrary $\hat{Y} \in \mathbb{R}^{n \times n}$ such that $\hat{Y}_{\mathbf{LB}} = 0,\ \hat{Y}_{\mathbf{LU}} = 0,\ \hat{Y}_{\mathbf{BB}} = 0,\ \hat{Y}_{\mathbf{BU}} = 0$ and $\hat{Y}_{ij} \geq 0$ otherwise.

8: Choose an arbitrary $\hat{Z} \in \mathcal{S}^n$ such that $\hat{Z}_{\mathbf{BU}} = 0,\ \hat{Z}_{\mathbf{UB}} = 0,\ \hat{Z}_{\mathbf{UU}} = 0,\ \hat{Z}_{kk} > 0$, and $\hat{Z}_{ij} \geq 0$ otherwise.

9: $Q \leftarrow \hat{W} - \hat{Y} - \hat{Y}^T + \hat{Z}$

10: $c \leftarrow -\hat{r} + \hat{s} - \hat{W} e + \hat{Y}^T e$

Proof. Consider the partition $(\mathbf{L}, \mathbf{B}, \mathbf{U})$ of the index set $\{1, \ldots, n\}$ as defined in Algorithm 2, where $\mathbf{B} \neq \emptyset$. Clearly, $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n} \times \mathcal{S}^n$ is a feasible solution of (R-D). We will construct a feasible solution $(\hat{x}, \hat{X}) \in \mathbb{R}^n \times \mathcal{S}^n$ of (R) that satisfies the optimality conditions of Lemma 3.5.1.

Consider the following solution $(\hat{x}, \hat{X}) \in \mathbb{R}^n \times \mathcal{S}^n$:

$$\hat{x}_{\mathbf{L}} = 0, \quad \hat{x}_{\mathbf{B}} = \tfrac{1}{2} e_{\mathbf{B}} \quad \hat{x}_{\mathbf{U}} = e_{\mathbf{U}},$$

and

$$\begin{bmatrix} \hat{X}_{\mathbf{LL}} & \hat{X}_{\mathbf{LB}} & \hat{X}_{\mathbf{LU}} \\ \hat{X}_{\mathbf{BL}} & \hat{X}_{\mathbf{BB}} & \hat{X}_{\mathbf{BU}} \\ \hat{X}_{\mathbf{UL}} & \hat{X}_{\mathbf{UB}} & \hat{X}_{\mathbf{UU}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} e_{\mathbf{B}} e_{\mathbf{U}}^T \\ 0 & \frac{1}{2} e_{\mathbf{U}} e_{\mathbf{B}}^T & e_{\mathbf{U}} e_{\mathbf{U}}^T \end{bmatrix},$$

By Lemma 3.4.2, $(\hat{x}, \hat{X}) \in \mathcal{F}_R$. By Steps 4, 5, 6, 7, and 8 of Algorithm 2, it is easy to verify that (3.40), (3.41), (3.42), (3.43), and (3.44) are respectively satisfied. Therefore, by Lemma 3.5.1, we conclude that $(\hat{x}, \hat{X})$ is an optimal solution of (R) and $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z})$ is an optimal solution of (R-D).

We next argue that the RLT relaxation is inexact. Let $(\tilde{x}, \tilde{X}) \in \mathcal{F}_R$ be an arbitrary optimal solution of (R). By Lemma 3.4.2, $(\tilde{x}, \tilde{X})$ and $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z})$ satisfy the conditions (3.40), (3.41), (3.42), (3.43), and (3.44). By (3.44) and Step 8 of Algorithm 2, we obtain $\tilde{X}_{kk} = 0$ since $\hat{Z}_{kk} > 0$. Since $\hat{W}_{kk} > 0$ by Step 6 of Algorithm 2, the relation (3.42) implies that $\tilde{X}_{kk} - 2\tilde{x}_k + 1 = 0$, i.e., $\tilde{x}_k = \frac{1}{2}$ since $\tilde{X}_{kk} = 0$. By Lemma 3.5.1, we conclude that $\tilde{x}_k = \frac{1}{2}$ for each optimal solution $(\tilde{x}, \tilde{X}) \in \mathcal{F}_R$ of (R). By Corollary 3.5.3, we conclude that $(Q, c) \notin \mathcal{E}_R$. □

Algorithm 2 can be used to generate an instance of (BoxQP) with an inexact RLT relaxation. Note that Algorithm 2 constructs an instance (Q, c) with the property that at least one component $\hat{x}_k$ is fractional at every optimal solution $(\hat{x}, \hat{X})$ of (R), which is sufficient for having an inexact RLT relaxation by Corollary 3.5.3 since each optimal solution $(\hat{x}, \hat{X})$

of (R) of an instance $(Q, c) \notin \mathcal{E}_R$ should have a fractional component $\hat{x}_k$. However, an instance with an inexact RLT relaxation may not necessarily have the property that every optimal solution $(\hat{x}, \hat{X})$ of (R) has the same fractional component $\hat{x}_k$. In particular, note that an instance generated by Algorithm 2 cannot have a concave objective function since $Q_{kk} = \hat{W}_{kk} + \hat{Z}_{kk} > 0$. On the other hand, for the specific instance $(Q, c) \in \mathcal{S}^3 \times \mathbb{R}^3$ in [8] given by $Q = \frac{1}{3}ee^T - I$, where $I \in \mathcal{S}^3$ denotes the identity matrix, and $c = 0$, the objective function is concave and the optimal value is given by $\ell^* = -\frac{1}{3}$, which is attained at any vertex that has exactly one component equal to 1. For $\hat{x} = \frac{1}{2}e \in \mathbb{R}^3$, we have $\ell_R(\hat{x}) = -\frac{1}{2} < -\frac{1}{3} = \ell^*$ by Lemma 3.4.1, which implies that the RLT relaxation is inexact on this instance. Therefore, in contrast with Algorithm 1, we conclude that Algorithm 2 may not necessarily generate all possible instances $(Q, c) \notin \mathcal{E}_R$.

3.6 Exact and Inexact SDP-RLT Relaxations

In this section, we focus on the set of instances of (BoxQP) that admit exact and inexact SDP-RLT relaxations. We give a complete algebraic description of the set of instances of (BoxQP) that admit an exact SDP-RLT relaxation. In addition, we develop an algorithm for constructing such an instance of (BoxQP) as well as for constructing an instance of (BoxQP) with an exact SDP-RLT relaxation but an inexact RLT relaxation.

Similar to the RLT relaxation, let us define

$$\mathcal{E}_{RS} = \{(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell^* = \ell_{RS}\}, \tag{3.50}$$

i.e., $\mathcal{E}_{RS}$ denotes the set of all instances of (BoxQP) that admit an exact SDP-RLT relaxation. By (3.4), the SDP-RLT relaxation of any instance of (BoxQP) is at least as tight as the RLT relaxation. It follows that

$$\mathcal{E}_R \subseteq \mathcal{E}_{RS}, \tag{3.51}$$

where $\mathcal{E}_R$ is given by (3.34).

By Corollary 3.4.2 and (3.26), we clearly have $(Q, c) \in \mathcal{E}_{RS}$ whenever $Q \succeq 0$. Furthermore, the SDP-RLT relaxation is always exact (i.e., $\mathcal{E}_{RS} = \mathcal{S}^n \times \mathbb{R}^n$) if and only if $n \leq 2$ [8].

For the RLT relaxation, Proposition 3.5.1 established the existence of an optimal solution of the RLT relaxation with a particularly simple structure. This observation enabled us to characterize the set of instances of (BoxQP) with an exact RLT relaxation as the union of a finite number polyhedral cones (see Proposition 3.5.3). In contrast, the next result shows that the set of optimal solutions of the SDP-RLT relaxation cannot have such a simple structure, as every feasible solution can be a unique optimal solution for (RS1).

Lemma 3.6.1. *For any $\hat{x} \in F$, there exists an instance (Q, c) of (BoxQP) such that $\hat{x}$ is the unique optimal solution of (RS1), where (RS1) is given by* (3.26)*, and $(Q, c) \in \mathcal{E}_{RS}$.*

Proof. For any $\hat{x} \in F$, consider an instance of (BoxQP) with $(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n$, where $Q \succ 0$ and $c = -Q\hat{x}$. We obtain $q(x) = \frac{1}{2}\left((x - \hat{x})^T Q (x - \hat{x}) - \hat{x}^T Q \hat{x}\right)$. Since $Q \succ 0$, $\hat{x}$ is the unique unconstrained minimizer of $q(x)$. By Lemma 3.4.4, since $Q \succ 0$, we have $\ell_{RS}(x) = q(x)$ for each $x \in F$, which implies that $\ell^* = q(x^*) = \ell_{RS}(x^*) = \ell^*_{RS}$. It follows that $(Q, c) \in \mathcal{E}_{RS}$. The uniqueness follows from the strict convexity of $\ell_{RS}(\cdot)$ since $\ell_{RS}(x) = q(x)$ for each $x \in F$. $\square$

In the next section, we rely on duality theory to obtain a description of the set $\mathcal{E}_{RS}$.

3.6.1 The Dual Problem

In this section, we present the dual of the SDP-RLT relaxation given by (RS) and establish several useful properties.

Recall that the SDP-RLT relaxation is given by

$$\text{(RS)} \quad \ell_{RS}^* = \min_{(x,X)\in\mathbb{R}^n\times\mathcal{S}^n} \left\{ \frac{1}{2}\langle Q, X\rangle + c^T x : (x,X) \in \mathcal{F}_{RS} \right\},$$

where $\mathcal{F}_{RS}$ is given by (3.3).

By using the same set of dual variables $(r, s, W, Y, Z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n} \times \mathcal{S}^n$ as in (R-D) corresponding to the common constraints in $\mathcal{F}_R$ and $\mathcal{F}_{RS}$ (see (3.37)), and defining the dual variable

$$\begin{bmatrix} \beta & h^T \\ h & H \end{bmatrix} \in \mathcal{S}^{n+1},$$

where $\beta \in \mathbb{R}$, $h \in \mathbb{R}^n$, and $H \in \mathcal{S}^n$, corresponding to the additional semidefinite constraint, the dual problem of (RS) is given by

$$\begin{array}{llrcl}
\text{(RS-D)} & \displaystyle\max_{(r,s,W,Y,Z,\beta,h,H)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n\times\mathbb{R}^{n\times n}\times\mathcal{S}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n} & \multicolumn{3}{l}{-e^T r - \frac{1}{2}e^T We - \frac{1}{2}\beta} \\
& \text{s.t.} & & & \\
& & -r + s - We + Y^T e + h & = & c \\
& & W - Y - Y^T + Z + H & = & Q \\
& & r & \geq & 0 \\
& & s & \geq & 0 \\
& & W & \geq & 0 \\
& & Y & \geq & 0 \\
& & Z & \geq & 0 \\
& & \begin{bmatrix} \beta & h^T \\ h & H \end{bmatrix} & \succeq & 0.
\end{array}$$

In contrast with linear optimization problem, strong duality and attainment of optimal solutions may fail in semidefinite optimization problem (see, e.g., [111]). We first establish well-known sufficient conditions for strong duality and attainment.

Lemma 3.6.2. *Strong duality holds between (RS) and (RS-D), and optimal solutions are attained in both (RS) and (RS-D).*

Proof. Note that $\mathcal{F}_{RS}$ is a nonempty and bounded set since $0 \leq x_j \leq 1$ and $X_{jj} \leq 1$ for each $j = 1, \ldots, n$. Therefore, the set of optimal solutions of (RS) is nonempty. Let $\hat{x} = \frac{1}{2}e \in \mathbb{R}^n$ and let $\hat{X} = \hat{x}\hat{x}^T + \epsilon I \in \mathcal{S}^n$, where $\epsilon \in (0, \frac{1}{4})$. By Lemma 3.4.2, $(\hat{x}, \hat{X}) \in \mathcal{F}_{RS}$. Furthermore, it is a strictly feasible solution of (RS) since $(\hat{x}, \hat{X})$ satisfies all the constraints strictly. Strong duality and attainment in (RS-D) follow from conic duality (see, e.g., [94]). $\square$

Lemma 3.6.2 allows us to give a complete characterization of optimality conditions for the pair (RS) and (RS-D).

Lemma 3.6.3. $(\hat{x}, \hat{X}) \in \mathcal{F}_{RS}$ *is an optimal solution of (RS) if and only if there exists*

$$(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}, \hat{\beta}, \hat{h}, \hat{H}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n} \times \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$$

such that

$$\begin{aligned} Q &= \hat{W} - \hat{Y} - \hat{Y}^T + \hat{Z} + \hat{H}, & (3.52)\\ c &= -\hat{r} + \hat{s} - \hat{W}e + \hat{Y}^T e + \hat{h}, & (3.53)\\ \begin{bmatrix} \hat{\beta} & \hat{h}^T \\ \hat{h} & \hat{H} \end{bmatrix} &\succeq 0 & (3.54)\\ \left\langle \begin{bmatrix} 1 & \hat{x}^T \\ \hat{x} & \hat{X} \end{bmatrix}, \begin{bmatrix} \hat{\beta} & \hat{h}^T \\ \hat{h} & \hat{H} \end{bmatrix} \right\rangle &= 0, & (3.55) \end{aligned}$$

and (3.40)–(3.49) *are satisfied.*

Proof. The claim follows from strong duality between (RS) and (RS-D), which holds by Lemma 3.6.2. □

Using Lemma 3.6.3, we obtain the following description of the set of instances of (BoxQP) with an exact SDP-RLT relaxation.

Proposition 3.6.1. $(Q, c) \in \mathcal{E}_{RS}$*, where* $\mathcal{E}_{RS}$ *is defined as in* (3.50)*, if and only if there exists* $\hat{x} \in F$ *and there exists* $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}, \hat{\beta}, \hat{h}, \hat{H}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n} \times \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ *such that the conditions of Lemma 3.6.3 are satisfied, where* $(\hat{x}, \hat{X}) = (\hat{x}, \hat{x}\hat{x}^T)$*. Furthermore, in this case,* $\hat{x}$ *is an optimal solution of (BoxQP).*

Proof. Suppose that $(Q, c) \in \mathcal{E}_{RS}$. Let $\hat{x} \in F$ be an optimal solution of (BoxQP). By Corollary 3.4.2(ii), we obtain $\ell^*_{RS} = \ell^* = q(\hat{x}) = \ell_{RS}(\hat{x})$. Therefore, $\hat{x}$ is an optimal solution of (RS1) given by (3.26). Let $(\hat{x}, \hat{X}) = (\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}_{RS}$. We obtain $\frac{1}{2}\langle Q, \hat{X} \rangle + c^T \hat{x} = q(\hat{x}) = \ell^*_{RS}$, which implies that $(\hat{x}, \hat{x}\hat{x}^T)$ is an optimal solution of (RS). The claim follows from Lemma 3.6.3.

For the reverse implication, note that $(\hat{x}, \hat{X}) = (\hat{x}, \hat{x}\hat{x}^T)$ is an optimal solution of (RS) by Lemma 3.6.3. By a similar argument and using (3.4), we obtain $\ell^* \leq q(\hat{x}) = \ell^*_{RS} \leq \ell^*$, which implies that $\ell^*_{RS} = \ell^*$, or equivalently, that $(Q, c) \in \mathcal{E}_{RS}$.

The second assertion follows directly from the previous arguments. □

In the next section, by relying on Proposition 3.6.1, we propose two algorithms to construct instances of (BoxQP) with different exactness guarantees.

3.6.2 Construction of Instances with Exact SDP-RLT Relaxations

In this section, we present an algorithm for constructing instances of (BoxQP) with an exact SDP-RLT relaxation. Similar to Algorithm 1, Algorithm 3 is based on designating $\hat{x} \in F$ and constructing an appropriate dual feasible solution that satisfies optimality conditions together with $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}_{RS}$.

The next proposition establishes the correctness of Algorithm 3.

Algorithm 3 (BoxQP) Instance with an Exact SDP-RLT Relaxation

Require: n, $\hat{x} \in F$

Ensure: $(Q, c) \in \mathcal{E}_{RS}$

1: $\mathbf{L} \leftarrow \mathbf{L}(\hat{x})$, $\mathbf{B} \leftarrow \mathbf{B}(\hat{x})$, $\mathbf{U} \leftarrow \mathbf{U}(\hat{x})$

2: Choose an arbitrary $\hat{r} \in \mathbb{R}^n$ such that $\hat{r}_{\mathbf{U}} \geq 0$ and $\hat{r}_{\mathbf{L} \cup \mathbf{B}} = 0$.

3: Choose an arbitrary $\hat{s} \in \mathbb{R}^n$ such that $\hat{s}_{\mathbf{L}} \geq 0$ and $\hat{s}_{\mathbf{B} \cup \mathbf{U}} = 0$.

4: Choose an arbitrary $\hat{W} \in \mathcal{S}^n$ such that $\hat{W}_{\mathbf{B} \cup \mathbf{L}, \mathbf{B} \cup \mathbf{L}} = 0$ and $\hat{W}_{ij} \geq 0$ otherwise.

5: Choose an arbitrary $\hat{Y} \in \mathbb{R}^{n \times n}$ such that $\hat{Y}_{\mathbf{LB}} = 0,\ \hat{Y}_{\mathbf{LU}} = 0,\ \hat{Y}_{\mathbf{BB}} = 0,\ \hat{Y}_{\mathbf{BU}} = 0$ and $\hat{Y}_{ij} \geq 0$ otherwise.

6: Choose an arbitrary $\hat{Z} \in \mathcal{S}^n$ such that $\hat{Z}_{\mathbf{B} \cup \mathbf{U}, \mathbf{B} \cup \mathbf{U}} = 0$ and $\hat{Z}_{ij} \geq 0$ otherwise.

7: Choose an arbitrary $\hat{H} \in \mathcal{S}^n$ such that $\hat{H} \succeq 0$.

8: $\hat{h} \leftarrow -\hat{H}\hat{x}$, $\hat{\beta} \leftarrow -\hat{h}^T \hat{x}$

9: $Q \leftarrow \hat{W} - \hat{Y} - \hat{Y}^T + \hat{Z} + \hat{H}$

10: $c \leftarrow -\hat{r} + \hat{s} - \hat{W}e + \hat{Y}^T e + \hat{h}$

Proposition 3.6.2. *Algorithm 3 returns* $(Q, c) \in \mathcal{E}_{RS}$*, where* $\mathcal{E}_{RS}$ *is defined as in* (3.50)*. Conversely, any* $(Q, c) \in \mathcal{E}_{RS}$ *can be generated by Algorithm 3 with appropriate choices of* $\hat{x} \in F$ *and* $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}, \hat{\beta}, \hat{h}, \hat{H}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n} \times \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$.

Proof. Since $\hat{H} \succeq 0$, it follows from Steps 8 and 9 of Algorithm 3 that

$$\begin{bmatrix} \hat{\beta} & \hat{h}^T \\ \hat{h} & \hat{H} \end{bmatrix} = \begin{bmatrix} \hat{x}^T \\ -I \end{bmatrix} \hat{H} \begin{bmatrix} \hat{x}^T \\ -I \end{bmatrix}^T \succeq 0. \tag{3.56}$$

Therefore, $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}, \hat{\beta}, \hat{h}, \hat{H}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n} \times \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ is a feasible solution of (RS-D). Furthermore, the identity in (3.56) also implies that

$$\left\langle \begin{bmatrix} 1 & \hat{x}^T \\ \hat{x} & \hat{x}\hat{x}^T \end{bmatrix}, \begin{bmatrix} \hat{\beta} & \hat{h}^T \\ \hat{h} & \hat{H} \end{bmatrix} \right\rangle = \begin{bmatrix} 1 \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} \hat{\beta} & \hat{h}^T \\ \hat{h} & \hat{H} \end{bmatrix} \begin{bmatrix} 1 \\ \hat{x} \end{bmatrix} = 0.$$

It is easy to verify that the conditions of Lemma 3.6.3 are satisfied with $(\hat{x}, \hat{X}) = (\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{F}_{RS}$. Both assertions follow from Proposition 3.6.1. $\square$

By Proposition 3.6.2, we conclude that $\mathcal{E}_{RS}$ is given by the union of infinitely many convex cones each of which can be represented by semidefinite and linear constraints.

Similar to Algorithm 1, we remark that Algorithm 3 can be utilized to generate an instance of (BoxQP) with an exact SDP-RLT relaxation such that any designated feasible solution $\hat{x} \in F$ is an optimal solution of (BoxQP).

3.6.3 Construction of Instances with Exact SDP-RLT and Inexact RLT Relaxations

Recall that the SDP-RLT relaxation of any instance of (BoxQP) is at least as tight as the RLT relaxation. In this section, we present another algorithm for constructing instances of (BoxQP) that admit an exact SDP-RLT relaxation but an inexact RLT relaxation, i.e., an instance in $\mathcal{E}_{RS} \backslash \mathcal{E}_R$ (cf. (3.51)). In particular, this algorithm can be used to construct instances of (BoxQP) such that the SDP-RLT relaxation not only strengthens the RLT relaxation, but also yields an exact lower bound.

Note that Algorithm 3 is capable of constructing all instances of (BoxQP) in the set $\mathcal{E}_{RS}$. On the other hand, if one chooses $\hat{x} \in V$ and $\hat{H} = 0$ in Algorithm 3, which, in turn, would imply that $\hat{h} = 0$ and $\hat{\beta} = 0$, it is easy to verify that the choices of the remaining parameters satisfy the conditions of Algorithm 1, which implies that the resulting instance would already have an exact RLT relaxation, i.e., $(Q, c) \in \mathcal{E}_R$.

In this section, we present Algorithm 4, where we use a similar idea as in Algorithm 2, i.e., we aim to construct an instance of (BoxQP) such that $(\hat{x}, \hat{x}\hat{x}^T)$ is the unique optimal solution of (RS), where $\hat{x} \in F \backslash V$.

Algorithm 4 (BoxQP) Instance with an Exact SDP-RLT Relaxation and an Inexact RLT Relaxation

Require: n, $\hat{x} \in F \backslash V$
Ensure: $(Q, c) \in \mathcal{E}_{RS} \backslash \mathcal{E}_R$
1: $\mathbf{L} \leftarrow \mathbf{L}(\hat{x})$, $\mathbf{B} \leftarrow \mathbf{B}(\hat{x})$, $\mathbf{U} \leftarrow \mathbf{U}(\hat{x})$
2: Choose an arbitrary $\hat{r} \in \mathbb{R}^n$ such that $\hat{r}_{\mathbf{U}} \geq 0$ and $\hat{r}_{\mathbf{L} \cup \mathbf{B}} = 0$.
3: Choose an arbitrary $\hat{s} \in \mathbb{R}^n$ such that $\hat{s}_{\mathbf{L}} \geq 0$ and $\hat{s}_{\mathbf{B} \cup \mathbf{U}} = 0$.
4: Choose an arbitrary $\hat{W} \in \mathcal{S}^n$ such that $\hat{W}_{\mathbf{B} \cup \mathbf{L}, \mathbf{B} \cup \mathbf{L}} = 0$ and $\hat{W}_{ij} \geq 0$ otherwise.
5: Choose an arbitrary $\hat{Y} \in \mathbb{R}^{n \times n}$ such that $\hat{Y}_{\mathbf{LB}} = 0,\ \hat{Y}_{\mathbf{LU}} = 0,\ \hat{Y}_{\mathbf{BB}} = 0,\ \hat{Y}_{\mathbf{BU}} = 0$ and $\hat{Y}_{ij} \geq 0$ otherwise.
6: Choose an arbitrary $\hat{Z} \in \mathcal{S}^n$ such that $\hat{Z}_{\mathbf{B} \cup \mathbf{U}, \mathbf{B} \cup \mathbf{U}} = 0$ and $\hat{Z}_{ij} \geq 0$ otherwise.
7: Choose an arbitrary $\hat{H} \in \mathcal{S}^n$ such that $\hat{H} \succ 0$.
8: $\hat{h} \leftarrow -\hat{H}\hat{x}$, $\hat{\beta} \leftarrow -\hat{h}^T\hat{x}$
9: $Q \leftarrow \hat{W} - \hat{Y} - \hat{Y}^T + \hat{Z} + \hat{H}$
10: $c \leftarrow -\hat{r} + \hat{s} - \hat{W}e + \hat{Y}^T e + \hat{h}$

Note that Algorithm 3 and Algorithm 4 are almost identical, except that, in Step 7, we require that $\hat{H} \succ 0$ in Algorithm 4 as opposed to $\hat{H} \succeq 0$ in Algorithm 3. The next result establishes that the output from Algorithm 4 is an instance of (BoxQP) with an exact SDP-RLT but inexact RLT relaxation.

Proposition 3.6.3. *Algorithm 4 returns $(Q, c) \in \mathcal{E}_{RS} \backslash \mathcal{E}_R$, where $\mathcal{E}_R$ and $\mathcal{E}_{RS}$ are defined as in* (3.34) *and* (3.50)*, respectively.*

Proof. By the observation preceding the statement, it follows from Propositions 3.6.1 and 3.6.2 that $(Q, c) \in \mathcal{E}_{RS}$ and that $(\hat{x}, \hat{x}\hat{x}^T)$ is an optimal solution of (RS). First, we show that this is the unique optimal solution of (RS). Suppose, for a contradiction, that there exists another optimal solution $(\tilde{x}, \tilde{X}) \in \mathcal{F}_{RS}$. Note that, for any $A \succeq 0$ and $B \succeq 0$, $\langle A, B \rangle = 0$ holds if and only if $AB = 0$. Therefore, it follows from (3.55) that $\hat{h} - \hat{H}\tilde{x} = 0$. Since $\hat{H} \succ 0$, we obtain $\tilde{x} = \hat{x}$ by Step 8. By (3.56), we obtain

$$\left\langle \begin{bmatrix} 1 & \hat{x}^T \\ \hat{x} & \tilde{X} \end{bmatrix}, \begin{bmatrix} \hat{\beta} & \hat{h}^T \\ \hat{h} & \hat{H} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 & \hat{x}^T \\ \hat{x} & \tilde{X} \end{bmatrix}, \begin{bmatrix} \hat{x}^T \\ -I \end{bmatrix} \hat{H} \begin{bmatrix} \hat{x}^T \\ -I \end{bmatrix}^T \right\rangle = \langle \hat{H}, \tilde{X} - \hat{x}\hat{x}^T \rangle = 0.$$

Since $\hat{H} \succ 0$ by Step 7 and $\tilde{X} - \hat{x}\hat{x}^T \succeq 0$, it follows that $\tilde{X} = \hat{x}\hat{x}^T$, which contradicts our assumption. It follows that $(\hat{x}, \hat{x}\hat{x}^T)$ is the unique optimal solution of (RS), or equivalently, that $\hat{x}$ is the unique optimal solution of (RS1) given by (3.26). By Proposition 3.6.1 and (3.24), we conclude that $(Q, c) \in \mathcal{E}_{RS}$ and that $\hat{x} \in F \backslash V$ is the unique optimal solution of (BoxQP). By Corollary 3.5.3, $(Q, c) \notin \mathcal{E}_R$, which completes the proof. $\square$

Algorithm 4 can be used to construct an instance in the set $\mathcal{E}_R \backslash \mathcal{E}_{RS}$. In particular, we remark that the family of instances used in the proof of Lemma 3.6.1 can be constructed by Algorithm 4 by simply choosing $(\hat{r}, \hat{s}, \hat{W}, \hat{Y}, \hat{Z}) = (0, 0, 0, 0, 0)$ and any $\hat{x} \in F$. In particular, similar to Algorithm 2, it is worth noting that any instance constructed by Algorithm 4 necessarily satisfies $Q_{kk} > 0$ for each $k \in \mathbf{B}$. On the other hand, recall that the SDP-RLT relaxation is always exact for $n \leq 2$. It follows that the instance in Example 3.5.1 belongs to $\mathcal{E}_{RS} \backslash \mathcal{E}_R$. However, it cannot be constructed by Algorithm 4 since (RS) does not have a unique optimal solution. Therefore, similar to our discussion about Algorithm 2, we conclude that the set of instances that can be constructed by Algorithm 4 may not necessarily encompass all instances in $\mathcal{E}_R \backslash \mathcal{E}_{RS}$.

3.6.4 A Stronger Relaxation

In this section, we present a stronger version of the SDP-RLT relaxation and briefly discuss the implications of our results on this relaxation.

The SDP-RLT relaxation (RS) can be further strengthened by adding the so-called triangle inequalities (see [86, 31]):

$$\mathcal{F}_{RST} = \left\{ (x, X) \in \mathcal{F}_{RS} : \begin{array}{rcll} x_i + x_j + x_k - X_{ij} - X_{jk} - X_{ik} & \leq & 1, & 1 \leq i < j < k \leq n \\ X_{ij} + X_{ik} - x_i - X_{jk} & \leq & 0, & 1 \leq i < j < k \leq n \\ X_{ij} + X_{jk} - x_j - X_{ik} & \leq & 0, & 1 \leq i < j < k \leq n \\ X_{ik} + X_{jk} - x_k - X_{ij} & \leq & 0, & 1 \leq i < j < k \leq n \end{array} \right\}. \tag{3.57}$$

We can now define the following relaxation, referred to as the SDP-RLT-TRI relaxation:

$$(\text{RST}) \quad \ell^*_{RST} = \min_{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n} \left\{ \frac{1}{2} \langle Q, X \rangle + c^T x : (x, X) \in \mathcal{F}_{RST} \right\}.$$

Clearly, $\ell^*_{RS} \leq \ell^*_{RST} \leq \ell^*$ since $\mathcal{F}_{RST} \subseteq \mathcal{F}_{RS}$. Let us similarly define

$$\mathcal{E}_{RST} = \{(Q, c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell^* = \ell_{RST}\}. \tag{3.58}$$

We readily obtain $\mathcal{E}_R \subseteq \mathcal{E}_{RS} \subseteq \mathcal{E}_{RST}$. In fact, computational results indicate that this stronger relaxation is usually exact on small- to medium-scale instances (see, e.g., [28, 76]).

Denoting the Lagrangian dual of (RST) by (RST-D), it is easy to show that Lemma 3.6.2 also holds for this primal-dual pair. Therefore, by relying on the corresponding versions of Lemma 3.6.3 and Proposition 3.6.1 for this relaxation, one can easily extend Algorithm 3 and Algorithm 4 to construct an instance of (BoxQP) with an exact SDP-RLT-TRI relaxation and an instance with an exact SDP-RLT-TRI but inexact RLT relaxation.

3.7 Examples and Discussion

In this section, we present numerical examples generated by each of the four algorithms given by Algorithms 1–4 with computational experiments. We then report computational results on randomly generated instances using Algorithms 1–4 and close the section with a brief discussion.

3.7.1 Examples Generated by Algorithms 1–4

In this section, we present instances of (BoxQP) generated by each of the four algorithms given by Algorithms 1–4. Our main goal is to demonstrate that our algorithms are capable of generating nontrivial instances of (BoxQP) with predetermined exactness or inexactness guarantees.

Example 3.7.1. *Let $n = 2$, $\mathbf{L} = \{1\}$, and $\mathbf{U} = \{2\}$ in Algorithm 1. Then, by Steps 2–6, we have*

$$\hat{r} = \begin{bmatrix} 0 \\ \hat{r}_2 \end{bmatrix}, \quad \hat{s} = \begin{bmatrix} \hat{s}_1 \\ 0 \end{bmatrix}, \quad \hat{W} = \begin{bmatrix} 0 & \hat{W}_{12} \\ \hat{W}_{12} & \hat{W}_{22} \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} \hat{Y}_{11} & 0 \\ \hat{Y}_{21} & \hat{Y}_{22} \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} \\ \hat{Z}_{12} & 0 \end{bmatrix},$$

where each of $\hat{r}_2, \hat{s}_1, \hat{W}_{12}, \hat{W}_{22}, \hat{Y}_{11}, \hat{Y}_{21}, \hat{Y}_{22}, \hat{Z}_{11}, \hat{Z}_{12}$ is a nonnegative real number. By Steps 7 and 8, we obtain

$$Q = \begin{bmatrix} \hat{Z}_{11} - 2\hat{Y}_{11} & \hat{W}_{12} + \hat{Z}_{12} - \hat{Y}_{21} \\ \hat{W}_{12} + \hat{Z}_{12} - \hat{Y}_{21} & \hat{W}_{22} - 2\hat{Y}_{22} \end{bmatrix}, \quad c = \begin{bmatrix} \hat{s}_1 + \hat{Y}_{11} + \hat{Y}_{21} - \hat{W}_{12} \\ \hat{Y}_{22} - \hat{r}_2 - \hat{W}_{12} - \hat{W}_{22} \end{bmatrix}.$$

For instance, if we choose $\hat{Y}_{11} = \hat{Y}_{22} = \hat{W}_{12} = \hat{Z}_{12} = \hat{Y}_{21} = 0$, $\hat{Z}_{11} + \hat{W}_{22} > 0$, $\hat{r}_2 \geq 0$, and $\hat{s}_1 \geq 0$, then $Q \succeq 0$, which implies that $q(x)$ is a convex function. If we choose $\hat{Z}_{11} = \hat{W}_{22} = \hat{W}_{12} = \hat{Z}_{12} = \hat{Y}_{21} = 0$, $\hat{Y}_{11} + \hat{Y}_{22} > 0$, $\hat{r}_2 \geq 0$, and $\hat{s}_1 \geq 0$, then $-Q \succeq 0$, which implies that $q(x)$ is a concave function. Finally, if we choose $\hat{Y}_{11} = \hat{W}_{22} = \hat{W}_{12} = \hat{Z}_{12} = \hat{Y}_{21} = 0$, $\hat{Z}_{11} > 0$, $\hat{Y}_{22} > 0$, $\hat{r}_2 \geq 0$, and $\hat{s}_1 \geq 0$, then Q is indefinite, which implies that $q(x)$ is an indefinite quadratic function. For each of the three choices, the RLT relaxation is exact and $\hat{x} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ is an optimal solution of the resulting instance of (BoxQP). Note that by choosing $\hat{x} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and setting $\hat{H} = 0$ and $\hat{h} = 0$ in Algorithm 3, the same observations carry over.

Example 3.7.2. *Let $n = 3$, $\mathbf{L} = \{1\}$, $\mathbf{B} = \{2\}$, and $\mathbf{U} = \{3\}$ in Algorithm 2. Then, by Steps 3–8, we have $k = 2$ and*

$$\hat{r} = \begin{bmatrix} 0 \\ 0 \\ \hat{r}_3 \end{bmatrix}, \quad \hat{s} = \begin{bmatrix} \hat{s}_1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{W} = \begin{bmatrix} 0 & 0 & \hat{W}_{13} \\ 0 & \hat{W}_{22} & \hat{W}_{23} \\ \hat{W}_{13} & \hat{W}_{23} & \hat{W}_{33} \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} \hat{Y}_{11} & 0 & 0 \\ \hat{Y}_{21} & 0 & 0 \\ \hat{Y}_{31} & \hat{Y}_{32} & \hat{Y}_{33} \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} & \hat{Z}_{13} \\ \hat{Z}_{12} & \hat{Z}_{22} & 0 \\ \hat{Z}_{13} & 0 & 0 \end{bmatrix},$$

where each of $\hat{r}_3, \hat{s}_1, \hat{W}_{13}, \hat{W}_{23}, \hat{W}_{33}, \hat{Y}_{11}, \hat{Y}_{21}, \hat{Y}_{31}, \hat{Y}_{32}, \hat{Y}_{33}, \hat{Z}_{11}, \hat{Z}_{12}, \hat{Z}_{13}$ is a nonnegative real number, $\hat{W}_{22} > 0$ and $\hat{Z}_{22} > 0$. By Steps 9 and 10, we obtain

$$Q = \begin{bmatrix} \hat{Z}_{11} - 2\hat{Y}_{11} & \hat{Z}_{12} - \hat{Y}_{21} & \hat{W}_{13} + \hat{Z}_{13} - \hat{Y}_{31} \\ \hat{Z}_{12} - \hat{Y}_{21} & \hat{W}_{22} + \hat{Z}_{22} & \hat{W}_{23} - \hat{Y}_{32} \\ \hat{W}_{13} + \hat{Z}_{13} - \hat{Y}_{31} & \hat{W}_{23} - \hat{Y}_{32} & \hat{W}_{33} - 2\hat{Y}_{33} \end{bmatrix}, \quad c = \begin{bmatrix} \hat{s}_1 + \hat{Y}_{11} + \hat{Y}_{21} + \hat{Y}_{31} - \hat{W}_{13} \\ \hat{Y}_{32} - \hat{W}_{22} - \hat{W}_{23} \\ \hat{Y}_{33} - \hat{W}_{13} - \hat{W}_{23} - \hat{W}_{33} - \hat{r}_3 \end{bmatrix}.$$

If we set each of the parameters $\hat{r}_3, \hat{s}_1, \hat{W}_{13}, \hat{W}_{23}, \hat{W}_{33}, \hat{Y}_{11}, \hat{Y}_{21}, \hat{Y}_{31}, \hat{Y}_{32}, \hat{Y}_{33}, \hat{Z}_{11}, \hat{Z}_{12}, \hat{Z}_{13}$ to zero, and choose any $\hat{W}_{22} > 0$ and $\hat{Z}_{22} > 0$, then $Q \succeq 0$, which implies that $q(x)$ is a convex function. On the other hand, if we set each of the parameters $\hat{r}_3, \hat{s}_1, \hat{W}_{13}, \hat{W}_{23}, \hat{W}_{33}, \hat{Y}_{21}, \hat{Y}_{31}$

, $\hat{Y}_{32}, \hat{Z}_{11}, \hat{Z}_{12}, \hat{Z}_{13}$ *to zero, and choose any* $\hat{W}_{22} > 0$*,* $\hat{Z}_{22} > 0$ *and* $\hat{Y}_{11} + \hat{Y}_{33} > 0$*, then* Q *is indefinite, which implies that* $q(x)$ *is an indefinite quadratic function. For each of the two choices, the RLT relaxation is inexact. Recall that an instance generated by Algorithm 2 cannot have a concave objective function since* $Q_{kk} = \hat{W}_{kk} + \hat{Z}_{kk} > 0$.

Example 3.7.3. *Let* $n = 3$ *and let* $\hat{x} \in V \backslash F$ *be such that* $\mathbf{L} = \{1\}$*,* $\mathbf{B} = \{2\}$*, and* $\mathbf{U} = \{3\}$ *in Algorithm 4. Then, by Steps 2–6, we have*

$$\hat{r} = \begin{bmatrix} 0 \\ 0 \\ \hat{r}_3 \end{bmatrix}, \quad \hat{s} = \begin{bmatrix} \hat{s}_1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{W} = \begin{bmatrix} 0 & 0 & \hat{W}_{13} \\ 0 & 0 & \hat{W}_{23} \\ \hat{W}_{13} & \hat{W}_{23} & \hat{W}_{33} \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} \hat{Y}_{11} & 0 & 0 \\ \hat{Y}_{21} & 0 & 0 \\ \hat{Y}_{31} & \hat{Y}_{32} & \hat{Y}_{33} \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} & \hat{Z}_{13} \\ \hat{Z}_{12} & 0 & 0 \\ \hat{Z}_{13} & 0 & 0 \end{bmatrix},$$

where each of $\hat{r}_3, \hat{s}_1, \hat{W}_{13}, \hat{W}_{23}, \hat{W}_{33}, \hat{Y}_{11}, \hat{Y}_{21}, \hat{Y}_{31}, \hat{Y}_{32}, \hat{Y}_{33}, \hat{Z}_{11}, \hat{Z}_{12}, \hat{Z}_{13}$ *is a nonnegative real number. By Step 7,* $\hat{H} \succ 0$ *is arbitrarily chosen. By Step 8, we have* $\hat{h} = -\hat{H}\hat{x}$ *and* $\hat{\beta} = -\hat{h}^T\hat{x}$*. By Steps 9 and 10, we therefore obtain*

$$\begin{aligned} Q &= \begin{bmatrix} \hat{Z}_{11} - 2\hat{Y}_{11} & \hat{Z}_{12} - \hat{Y}_{21} & \hat{W}_{13} + \hat{Z}_{13} - \hat{Y}_{31} \\ \hat{Z}_{12} - \hat{Y}_{21} & 0 & \hat{W}_{23} - \hat{Y}_{32} \\ \hat{W}_{13} + \hat{Z}_{13} - \hat{Y}_{31} & \hat{W}_{23} - \hat{Y}_{32} & \hat{W}_{33} - 2\hat{Y}_{33} \end{bmatrix} + \hat{H}, \\ c &= \begin{bmatrix} \hat{s}_1 + \hat{Y}_{11} + \hat{Y}_{21} + \hat{Y}_{31} - \hat{W}_{13} \\ \hat{Y}_{32} - \hat{W}_{23} \\ \hat{Y}_{33} - \hat{W}_{13} - \hat{W}_{23} - \hat{W}_{33} - \hat{r}_3 \end{bmatrix} + \hat{h}. \end{aligned}$$

If we set each of the parameters $\hat{r}_3, \hat{s}_1, \hat{W}_{13}, \hat{W}_{23}, \hat{W}_{33}, \hat{Y}_{11}, \hat{Y}_{21}, \hat{Y}_{31}, \hat{Y}_{32}, \hat{Y}_{33}, \hat{Z}_{11}, \hat{Z}_{12}, \hat{Z}_{13}$ *to zero, then* $Q = \hat{H} \succ 0$*, which implies that* $q(x)$ *is a strictly convex function. On the other hand, if we set each of the parameters* $\hat{r}_3, \hat{s}_1, \hat{W}_{13}, \hat{W}_{23}, \hat{W}_{33}, \hat{Y}_{21}, \hat{Y}_{31}, \hat{Y}_{32}, \hat{Z}_{11}, \hat{Z}_{12}, \hat{Z}_{13}$ *to zero, and choose* $\hat{Y}_{11} + \hat{Y}_{33} = \frac{1}{2}(\hat{H}_{11} + \hat{H}_{22} + \hat{H}_{33}) > 0$*, then* Q *is indefinite since* $trace(Q) = 0$ *and* $Q \neq 0$*, which implies that* $q(x)$ *is an indefinite quadratic function. For each of the two choices,* $\hat{x}$ *is the unique optimal solution of the resulting instance of (BoxQP) and the SDP-RLT relaxation is exact whereas the RLT relaxation is inexact. Recall that an instance generated by Algorithm 4 cannot have a concave objective function since* $Q_{kk} = \hat{H}_{kk} > 0$*. Indeed, such an instance of (BoxQP) necessarily has an optimal solution at a vertex whereas Algorithm 4 ensures that the resulting instance of (BoxQP) has a unique solution* $\hat{x} \in F \backslash V$.

3.7.2 Computational Experiments

In this section, in an attempt to shed light on the computational cost of globally solving instances of (BoxQP) generated by Algorithms 1–4 using a state-of-the-art solver, we report preliminary results.

In our experiments, we chose $n \in \{25, 50, 75, 100\}$. For each choice of n and each of the four algorithms given by Algorithms 1–4, we generated 100 random instances of (BoxQP), giving rise to a total of 1600 instances. For each algorithm, all nonnegative (resp., positive) parameters were generated uniformly from the set of integers $\{0, 1, \ldots, 10\}$ (resp., $\{1, 2, \ldots, 10\}$). For Algorithm 1 (resp., Algorithms 2–4), each component was assigned to index sets $\mathbf{L}$ and $\mathbf{U}$ (resp, $\mathbf{L}$, $\mathbf{B}$, and $\mathbf{U}$) with equal probabilities while ensuring that $\mathbf{B} \neq \emptyset$ in Algorithms 2–4. For Algorithms 3 and 4, each $\hat{x}_j,\ j \in \mathbf{B}$ was chosen uniformly from the finite set $\{0.01, 0.02, \ldots, 0.99\}$. Regarding the matrices $\hat{H} \succeq 0$ in Algorithm 3 and $\hat{H} \succ 0$

in Algorithm 4, we generated a random matrix $\hat{A} \in \mathbb{R}^{n \times n}$ whose entries are uniformly chosen from the set $\{-5, -4, \ldots, 4, 5\}$ and computed its QR-factorization $\hat{A} = \hat{Q}\hat{R}$. We then generated a diagonal matrix $\hat{\Lambda} \in \mathcal{S}^n$ with entries uniformly chosen from the sets $\{0, 1, \ldots, 10\}$ and $\{1, 2, \ldots, 10\}$ for Algorithm 3 and Algorithm 4, respectively, and set $\hat{H} = \hat{Q}\hat{\Lambda}\hat{Q}^T$.

We implemented our algorithms in Julia version 1.8.5 [16] and solved the instances of (BoxQP) calling CPLEX version 22.1.1.0 [59] via the modeling language JuMP [77]. We imposed a time limit of 600 seconds on each instance. Our computational experiments were carried out on a 64-bit HP workstation with 24 threads (2 sockets, 6 cores per socket, 2 threads per core) running Ubuntu Linux with 96 GB of RAM and Intel Xeon CPU E5-2667 processors with a clock speed of 2.90 GHz. Note, however, that we set the number of threads to one by setting `CPX_PARAM_THREADS = 1`. The default settings were used for all of the other parameters of CPLEX[1].

	Optimal								Time Limit							
	Number of Instances				Average Time				Number of Instances				Average Gap (%)			
n	Alg1	Alg2	Alg3	Alg4	Alg1	Alg2	Alg3	Alg4	Alg1	Alg2	Alg3	Alg4	Alg1	Alg2	Alg3	Alg4
25	100	100	100	100	0.00	0.22	0.87	0.30	0	0	0	0	-	-	-	-
50	100	91	82	90	0.00	51.22	65.22	48.80	0	9	18	10	-	0.88	0.05	0.05
75	100	10	14	29	0.00	180.58	220.43	237.01	0	90	86	71	-	0.89	0.05	0.04
100	100	1	2	2	0.00	14.01	115.90	52.91	0	99	98	98	-	1.40	0.06	0.05

Table 3.1: Summary statistics

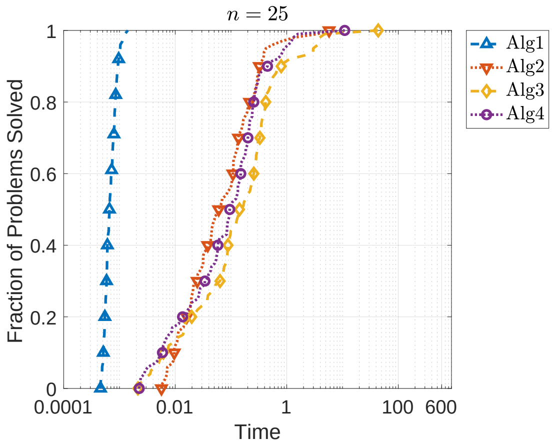
We present a summary of our results in Table 3.1, which is organized as follows. The first column denotes the dimension n. The second set of columns reports our results corresponding to the instances that were solved to global optimality by CPLEX within the time limit whereas the third set is devoted to the instances that were terminated due to the time limit. Each of the second and third sets of columns is further subdivided into two sets of columns, the first of which reports the number of instances in each category. The second sets of columns report the average solution time (in seconds) and the average gap (in percentages) reported by the solver, respectively. Finally, the columns Alg1, Alg2, Alg3, and Alg4 report the results corresponding to each of Algorithms 1–4, respectively.

Table 3.1 illustrates that all of the instances generated by Algorithm 1 can be solved very quickly regardless of the value of n. Recall that each of these instances admits an exact RLT relaxation and the solver seems to be capable of exploiting this property. On the other hand, the instances generated by Algorithms 2–4 seem to be computationally more demanding as n increases. In fact, while all instances with $n = 25$ are solved to global optimality fairly quickly, almost all instances hit the time limit for $n = 100$. Concerning the average gap, while Algorithms 3 and 4 seem to be somewhat less sensitive to n, we notice a stronger correlation for Algorithm 2.

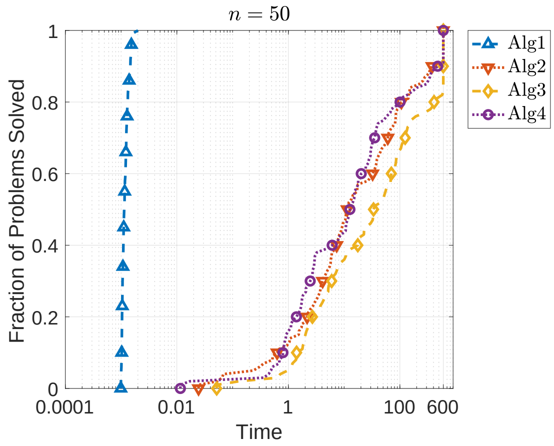
In an attempt to provide more insight into the distribution of the solution times of instances, we report empirical cumulative distribution functions in Figure 3.2. Each of the four plots in Figure 3.2 corresponds to each choice of n. The horizontal axis denotes the solution time in logarithmic scale and the vertical axis represents the fraction of instances solved. Markers indicate the solution time of every 10^{th} instance. We used identical axis

[1] All of the instances, our detailed results, and our codes for generating and solving the instances are publicly available at `https://github.com/eayildirim/BoxQPInstanceGeneration`

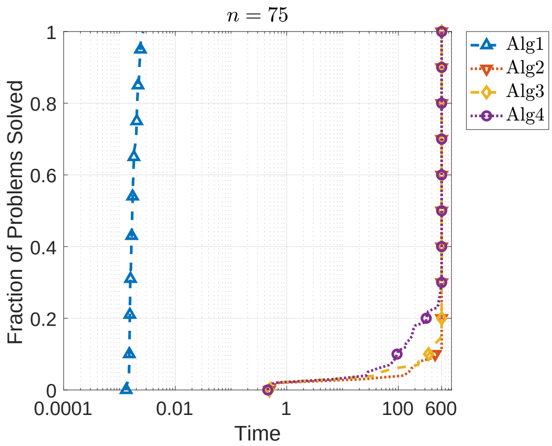
limits in all figures to facilitate an easier comparison. Note that we include the solution times of all instances, including the ones that were terminated due to the time limit.



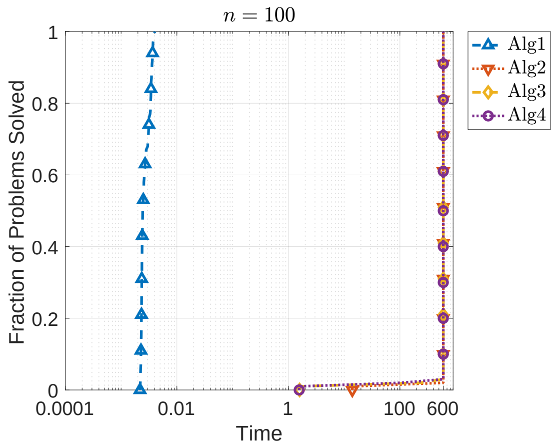
(a) $n = 25$



(b) $n = 50$



(c) $n = 75$



(d) $n = 100$

Figure 3.2: Cumulative distribution functions of solution times

Figure 3.2 clearly illustrates that the instances generated by Algorithm 1 can be solved much faster than the instances generated by each of the remaining three algorithms for each choice of n. The instances generated by each of Algorithms 2–4 exhibit a somewhat similar distribution in terms of solution times. Interestingly, the sets of instances generated by Algorithm 3 seem to be slightly more challenging than those generated by Algorithm 4. As n increases, a larger fraction of instances hit the time limit.

We therefore conclude that Algorithms 2–4 are capable of generating computationally more demanding instances, especially as n increases. It is particularly worth noticing that Algorithms 3 and 4 can generate computationally challenging instances even though they admit an exact SDP-RLT relaxation. Therefore, such instances could be useful for testing the performance of algorithms in the future since an optimal solution is already known.

3.7.3 Discussion

We close this section with a discussion of the four algorithms given by Algorithms 1–4. Note that all instances of (BoxQP) can be divided into the following four sets:

$$\begin{array}{rcl}\mathcal{E}_1 & = & \{(Q,c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell_R^* = \ell_{RS}^* = \ell^*\}, \\ \mathcal{E}_2 & = & \{(Q,c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell_R^* < \ell_{RS}^* = \ell^*\}, \\ \mathcal{E}_3 & = & \{(Q,c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell_R^* = \ell_{RS}^* < \ell^*\}, \\ \mathcal{E}_4 & = & \{(Q,c) \in \mathcal{S}^n \times \mathbb{R}^n : \ell_R^* < \ell_{RS}^* < \ell^*\}.\end{array}$$

We clearly have $\mathcal{E}_1 = \mathcal{E}_R$, and any such instance can be constructed by Algorithm 1. On the other hand, Algorithm 2 returns an instance in $\mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$. Any instance in $\mathcal{E}_1 \cup \mathcal{E}_2$ can be constructed by Algorithm 3. Finally, Algorithm 4 outputs an instance in the set $\mathcal{E}_2 = \mathcal{E}_{RS} \backslash \mathcal{E}_R$.

Note that one can generate a specific instance of (BoxQP) with an inexact SDP-RLT relaxation by extending the example in Section 3.5.5 [8]. Let $n = 2k+1 \geq 3$ and consider the instance $(Q,c) \in \mathcal{S}^n \times \mathbb{R}^n$ given by $Q = \frac{1}{n}ee^T - I$, where $I \in \mathcal{S}^n$ denotes the identity matrix, and $c = 0$. Since Q is negative semidefinite, the optimal solution of (BoxQP) is attained at one of the vertices. It is easy to verify that any vertex $v \in F$ with k (or $k+1$) components equal to 1 and the remaining ones equal to zero is an optimal solution, which implies that $\ell^* = \frac{1}{2}\left(\frac{k^2}{n} - k\right)$. Let $\hat{x} = \frac{1}{2}e$ and

$$\begin{array}{rcl}\hat{X} & = & \hat{x}\hat{x}^T + \hat{M} \\ & = & \dfrac{1}{4}ee^T + \dfrac{1}{4(n-1)}\left(nI - ee^T\right) \\ & = & \dfrac{1}{4}\left(1 + \dfrac{1}{n-1}\right)I + \dfrac{1}{4}\left(1 - \dfrac{1}{n-1}\right)ee^T.\end{array}$$

It is easy to verify that $(\hat{x}, \hat{X}) \in \mathcal{F}_{RS}$. Therefore,

$$\ell_{RS}^* \leq \frac{1}{2}\langle Q, \hat{X}\rangle + c^T\hat{x} = -\frac{n}{8}.$$

Using $n = 2k+1$, we conclude that $\ell_{RS}^* < \ell^*$, i.e., the SDP-RLT relaxation is inexact. Finally, this example can be extended to an even dimension $n = 2k \geq 4$ by simply constructing the same example corresponding to $n = 2k-1$ and then adding a component of zero to each of $\hat{x}$ and c, and adding a column and row of zeros to each of Q and $\hat{X}$.

An alternative way of constructing instances of (BoxQP) with an inexact SDP-RLT relaxation is via using clique inequalities, a family of facet-definining inequalities for the boolean quadric polytope [86], which are not implied by the constraints of the SDP-RLT relaxation. For instance, one can consider the clique inequality $x_1 + x_2 + x_3 - X_{12} - X_{13} - X_{23} \leq 1$ with $n = 3$ and maximize the left-hand side without lifting, i.e., maximize $x_1 + x_2 + x_3 - x_1x_2 - x_1x_3 - x_2x_3$. This yields an instance of (QP) with

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

For this instance, the SDP-RLT relaxation is inexact since $\ell^*_{RS} = -1.125 < \ell^* = -1$.

An interesting question is whether an algorithm can be developed for generating more general instances with inexact SDP-RLT relaxations, i.e., the set of instances given by $\mathcal{E}_3 \cup \mathcal{E}_4$. One possible approach is to use a similar idea as in Algorithms 2 and 4, i.e., designate an optimal solution $(\hat{x}, \hat{X}) \in \mathcal{F}_{RS}$, which is not in the form of (v, vv^T) for any vertex $v \in F$, and identify the conditions on the other parameters so as to guarantee that $(\hat{x}, \hat{X})$ is the unique optimal solution of the SDP-RLT relaxation (RS). Note that Lemma 3.6.3 can be used to easily construct an instance of (BoxQP) such that any feasible solution $(\hat{x}, \hat{X}) \in \mathcal{F}_{RS}$ is an optimal solution of (RS). In particular, the condition (3.55) can be satisfied by simply choosing an arbitrary matrix $B \in \mathcal{S}^k$ such that $B \succeq 0$, and by defining

$$\begin{bmatrix} \hat{\beta} & \hat{h}^T \\ \hat{h} & \hat{H} \end{bmatrix} = PBP^T,$$

where $P \in \mathbb{R}^{(n+1)\times k}$ is a matrix whose columns form a basis for the nullspace of the matrix

$$\begin{bmatrix} 1 & \hat{x}^T \\ \hat{x} & \hat{X} \end{bmatrix}.$$

For instance, the columns of P can be chosen to be the set of eigenvectors corresponding to zero eigenvalues. However, this procedure does not necessarily guarantee that $(\hat{x}, \hat{X}) \in \mathcal{F}_{RS}$ is the unique optimal solution of (RS). Therefore, a characterization of the extreme points and the facial structure of $\mathcal{F}_{RS}$ may shed light on the algorithmic construction of such instances. We intend to investigate this direction in the near future for the SDP-RLT as well as the stronger SDP-RLT-TRI relaxation.

3.8 Summary

In this chapter, we considered RLT and SDP-RLT relaxations of quadratic optimization problems with box constraints. We presented algebraic descriptions of instances of (BoxQP) that admit exact RLT relaxations as well as those that admit exact SDP-RLT relaxations. Using these descriptions, we proposed four algorithms for efficiently constructing an instance of (BoxQP) with predetermined exactness or inexactness guarantees. Our preliminary computational experiments revealed that Algorithms 2–4 are capable of generating computationally challenging instances. In particular, we remark that Algorithms 1, 3, and 4 can be used to construct an instance of (BoxQP) with a known optimal solution, which may be of independent interest for computational purposes.

In the next chapter, we study convex relaxations of (StQP) and (StQP(ρ)). We compare the RLT, SDP, and SDP-RLT relaxations of (StQP) and (StQP(ρ)) and focus on the projected feasible regions of those relaxations of (StQP(ρ)).

Chapter 4

Standard Quadratic Optimization Problems under Cardinality Constraints

In this chapter, we will discuss the results of the RLT, SDP, and SDP-RLT relaxations of the sparse constrained standard quadratic optimization problem. This chapter is based on the technical report [18].

Recall that the Standard Quadratic optimization Problem (StQP) is given by

$$\ell(Q) := \min_{x\in\mathbb{R}^n} \left\{ x^T Q x : x \in F \right\}, \qquad \text{(StQP)}$$

$$F := \left\{ x \in \mathbb{R}^n : e^T x = 1, \quad x \geq 0 \right\}$$

And the sparse StQP take the following expression:

$$\begin{array}{lrcl} \ell_\rho(Q) = \min\limits_{x\in\mathbb{R}^n} & x^T Q x & & \\ \text{s.t.} & & & \\ & e^T x & = & 1 \\ & e^T u & = & \rho \\ & x & \leq & u \\ & u & \in & \{0,1\}^n \\ & x & \geq & 0. \end{array} \qquad \text{(StQP}(\rho))$$

The class of StQPs provides a quite versatile modelling tool (see, e.g., [19]). Applications are numerous, ranging from the famous Markowitz portfolio problem in finance, evolutionary game theory in economics and quadratic resource allocation problems, through machine learning (background–foreground clustering in image analysis), to the life sciences —– e.g., in population genetics (selection models) and ecology (replicator dynamics).

StQPs appear also quite naturally as subproblems in copositive-conic relaxations of mixed-integer or combinatorial optimization problems of all sorts [11]. Finally, using barycentric coordinates, every quadratic optimization problem over a polytope with known (and not too many) vertices can be rephrased as an StQP.

The aforementioned structural simplicity does not preclude coexistence of an exponential number of (local or global) solutions to some StQPs. Some of these solutions may be sparse (and will be so with high probability in the average case, see below), others may have many positive coordinates. However, in important applications like some variants of sparse portfolio optimization problems where one is interested in investments with a limited number of assets (see, e.g., [81] and the references therein), sparsity of a solution must be enforced by an additional, explicit hard constraint on the number of positive coordinates. Introducing this cardinality constraint can render StQPs NP-hard even if the Hessian is positive-definite.

This chapter is organized as follows. We discuss some preliminary results in Section 4.1. In Section 4.2, we consider several convex relaxations of (StQP(ρ)). Section 4.2.1 focuses on the RLT relaxation of (StQP(ρ)) and presents several results in comparison with the RLT relaxation of (StQP). The SDP relaxation of (StQP(ρ)) is treated in Section 4.2.2 and compared with that of (StQP). In Section 4.2.3, we then study the convex relaxation of (StQP(ρ)) given by combining the RLT and SDP relaxations and compare it with that of (StQP). We conclude this chapter in Section 4.3.

4.1 Preliminary Results

Recall that the Standard Quadratic optimization Problem (StQP) is given by

$$\ell(Q) := \min_{x \in \mathbb{R}^n} \left\{ x^T Q x : x \in F \right\}, \tag{StQP}$$

$$F := \left\{ x \in \mathbb{R}^n : e^T x = 1, \quad x \geq 0 \right\}$$

There is an exponential number, namely $2^n - 1$, of faces of F, which form the "combinatorial" reason for NP-hardness. Indeed, if the active set $\{i : x_i^* = 0\}$ at the global solution x^* is known exactly, locating the solution (i.e., determining x^* or a value-equivalent alternative with the same set of zero coordinates) reduces to solving an $n \times n$ linear equation system. The same holds true for locating local solutions and even first-order critical (KKT) points. This phenomenon may be the reason why recently iterative first-order methods were proposed, which can achieve identification of the correct active set in finite time [26].

For any instance of (StQP), not all faces of F can contain an isolated (local or global) solution in their relative interior, as there is an upper bound on their cardinality given by Sperner's theorem on the maximal antichain (and Stirling's asymptotics) [108], namely

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \sim \sqrt{\frac{2}{\pi n}}\, 2^n \quad \text{as } n \to \infty \,. \tag{4.1}$$

Scozzari and Tardella [98] note that solutions can occur only in the relative interior of a face restricted to which the objective function is strictly convex. Nevertheless, recent research [27] has shown an exponential behavior regarding the number of local (or global) solutions: in the worst case, an instance of (StQP) of order n can have at least

$$(15120)^{n/24} \approx (1.4933)^n \tag{4.2}$$

coexisting optimal solutions, a lower bound that currently seems to be the largest one known. The other bad news is that rounding on the standard simplex is, from the asymptotic point of view, also not always successful [23]. In spite of all this, (StQP) admits a polynomial-time approximation scheme (PTAS) [21].

All of the above observations refer to the worst case, of course. Several researchers turned to the average case, modelled by randomly chosen instances. Already in 1988, Kingman [66] observed that very large polymorphisms (i.e., solutions x^* with more than $C\sqrt{n}$ positive coordinates where C is a positive constant independent of n) are atypical. More recently, in a series of papers Kontogiannis and Spirakis [69, 70, 71] looked at models with several independent and identically distributed (e.g., Gaussian or uniform) entries of $Q \in \mathcal{S}^n$ and proved, among other results, that the expected number of (local) solutions does not grow faster than $\exp(0.138n) \approx (1.148)^n$, way smaller than the worst-case lower bound in (4.2). Based upon more recent research by Chen and coauthors [35, 36], under quite reasonable distributional assumptions modeling the random average case, the probability that the global solution has more than two positive coordinates (i.e., that it does not lie on an edge of F) is asymptotically vanishing faster than

$$K\frac{(\log n)^2}{n} \quad \text{with } n \to \infty\,,$$

where $K > 0$ is a universal constant [27, Proposition 1].

When we handle the StQP with a hard cardinality constraint

$$\ell_\rho(Q) := \min_{x\in\mathbb{R}^n} \left\{ x^T Q x : x \in F_\rho \right\},$$

where

$$F_\rho := \{x \in F : \quad \|x\|_0 \le \rho\}\ ,$$

the elements of F_ρ will be referred to as ρ-sparse. When ρ is fixed independently of n, F_ρ is the union of $\binom{n}{\rho} = \mathcal{O}(n^\rho)$ faces of F, a number polynomial in n. In each of these faces, due to (4.1), at most $\binom{\rho}{\lfloor\frac{\rho}{2}\rfloor}$ local solutions to (1.6) can coexist, so we end up with a polynomial set of candidates which makes problem (1.6) solvable in polynomial time, again for universally fixed ρ. We will prove sparse StQP is NP-hard even when Q is positive-definite due to the combinatorial nature of the sparsity term $\|x\|_0$ in the following.

Proposition 4.1.1. *If (n, ρ) are considered as input, the sparse StQP is NP-hard even if Q is positive-definite.*

Proof. Our arguments are similar to those in [49] who dealt with (sign-unconstrained) sparse portfolio selection. In the online supplement to [49], this problem is reduced to the k-subset sum problem by the following technique: consider an instance of the latter problem of the form to decide, given n integers $\{a_1, \ldots, a_n\}$ and ρ, whether or not there is a ρ-element subset $S \subset \{1, \ldots, n\}$ with $\sum_{i\in S} a_i = 0$. This amounts to find a binary vector $v \in \{0, 1\}^n$ such that $\|v\|_0 = \rho$ and $a^T v = 0$ which in turn is equivalent to $v^T H v = 0$ where $H = aa^T$, or else certify that no such v exists. To this end [49], consider the minimization of the sum of a squared Euclidean distance and $v^T H v$ over $\mathbb{R}^n$ subject to the constraint $\|v\|_0 \le \rho$. We do basically the same but scale it, and additionally pose sign constraints: consider the quadratic objective $q(x) = \|x - \frac{1}{\rho}\, e\|_2^2 + x^T H x = x^T(H + I)x - \frac{2}{\rho} e^T x + n$ and minimize it over $x \in F_\rho$. Over this set, we have $\|x - \frac{1}{\rho}\, e\|_2^2 \ge \frac{n-\rho}{\rho^2}$ with equality if and only if $v := \rho x$ is a binary vector. Therefore we obtain $\ell_\rho(Q) \ge \frac{n-\rho}{\rho^2} + \frac{2}{\rho} - n$ for this problem (where $Q = \frac{1}{2}\, \nabla^2 q = H + I \succ 0$). If this inequality is strict, we have a certificate of unsolvability of the k-subset sum problem. Otherwise, we have $\ell_\rho(Q) = \frac{n-\rho}{\rho^2} + \frac{2}{\rho} - n$, then necessarily v is binary and satisfies $a^T v = 0$. Hence solving for $\ell_\rho(Q)$ cannot be easier than solving the k-subset sum problem, which is NP-complete. This proves the claim. $\square$

Evidently, any (feasible or optimal) solution of the sparse StQP is a feasible solution to (StQP) with guaranteed ρ-sparsity, which can be crucial. Even if ρ is fixed to a moderate number, say to 6, and for medium-scale dimensions, say $n = 100$, polynomial worst-case behavior would not help much in practical optimization since $n^\rho = 10^{12}$. This emphasizes the need for tractable relaxations of the sparse StQP.

We start with some simple observations.

Lemma 4.1.1. *The following relations hold:*

$$\ell(Q) = \ell_n(Q) \leq \ell_{n-1}(Q) \leq \ldots \leq \ell_2(Q) \leq \ell_1(Q)\,, \tag{4.3}$$

with

$$\ell_1(Q) = \min_{1\leq k\leq n} Q_{kk}\,, \tag{4.4}$$

and

$$\ell_2(Q) = \min\left\{\min\left\{\tfrac{Q_{ii}Q_{jj}-Q_{ij}^2}{Q_{ii}+Q_{jj}-2Q_{ij}} : Q_{ij} < \min\{Q_{ii}, Q_{jj}\}, 1 \leq i < j \leq n\right\}, \ell_1(Q)\right\}\,. \tag{4.5}$$

Furthermore, we have $\ell(Q) = \ell_\rho(Q)$ if and only if (StQP) has a ρ-sparse optimal solution.

Proof. The relations (4.3) and (4.4) follow from $F_1 = \{e^1, e^2, \ldots, e^n\} \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = F$, where F_ρ and F are given by (1.4) and (1.6), respectively, and $e^i \in \mathbb{R}^n$ denotes the ith unit vector, $i = 1, \ldots, n$. For $\rho = 2$, a straightforward discussion of univariate quadratics over the edges $\operatorname{conv}(\{e^i, e^j\})\,,\ 1 \leq i < j \leq n$ (in case these are strictly convex functions yielding a minimizer in the relative interior of the edge) is sufficient to establish (4.5). The last assertion is trivial. $\square$

The condition $\ell(Q) = \ell_2(Q)$ is related to edge-convexity of the instance of (StQP) as discussed in [98, Theorem 1] but we will not dive into details here. Rather observe that the effort to calculate $\ell_2(Q)$, obviously an upper bound of $\ell(Q)$, is the same as for the closed-form lower bound $\ell^{\text{ref}}(Q) \leq \ell(Q)$ proposed in [24]. The bracket

$$\ell^{\text{ref}}(Q) \leq \ell(Q) \leq \ell_2(Q)$$

shrinks to a singleton (i.e. the discussed bounds are exact) if and only if all off-diagonal entries of Q are equal, in which case, an optimal solution x^* to (StQP) must satisfy $\|x^*\|_0 \leq 2$ (see [24, Theorem 2] and (4.5)).

Recall that, by introducing binary variables, the sparse StQP can be reformulated as a mixed-binary QP:

$$\begin{array}{llrcl}
\ell_\rho(Q) = \min\limits_{x\in\mathbb{R}^n} & x^TQx & & & \\
\text{s.t.} & & & & \\
& & e^Tx & = & 1 \\
& & e^Tu & = & \rho \\
& & x & \le & u \\
& & u & \in & \{0,1\}^n \\
& & x & \ge & 0,
\end{array} \tag{StQP(ρ)}$$

While it turns out that all relaxations behave as expected for the case of $\rho = n$, already for the cases $\rho = 1$ and $\rho = 2$ (which cannot be excluded with a high probability in the random average case models) and other moderate sparsity values, there is a sharp contrast on the feasible region between the relaxations, which contributes to the motivation of this study. Typically, applications would require models with sparsity (significantly) less than half of the dimension, for which we obtain more interesting results.

We focus on RLT, SDP, and SDP-RLT relaxations of (StQP(ρ)), all more tractable than the conic ones presented in [25, Section 3.2] for general quadratic optimization problems. In particular, we establish several structural properties of these relaxations and shed light on the relations between each relaxation of (StQP(ρ)) and the corresponding relaxation of (StQP). We then draw several conclusions about the relations between different relaxations as well as the strength of each relaxation.

We will also pay particular attention to the case of rank-one solutions to the relaxations (all of them use matrix variables by lifting), in particular, because they certify optimality if optimal to the relaxed problems, and also because in algorithmic frameworks, we may (warm-)start with some (good) feasible solutions to the original problem of larger sparsity than desired.

4.2 Convex Relaxations: RLT and SDP

In this section, we consider several well-known convex relaxations of (StQP(ρ)), which use LP and SDP methods. We study their properties and establish relations between each relaxation of (StQP(ρ)) and the corresponding relaxation of (StQP).

4.2.1 RLT Relaxation

In this section, we consider the RLT relaxation.

We first start with the RLT relaxation of (StQP):

$$\ell^{R1}(Q) := \min_{x\in\mathbb{R}^n, X\in\mathcal{S}^n} \left\{\langle Q, X\rangle : (x,X) \in \mathcal{F}^{R1}\right\}, \tag{R1}$$

where

$$\mathcal{F}^{R1} := \left\{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n : e^Tx = 1, \quad Xe = x, \quad x \ge 0, \quad X \ge 0\right\}. \tag{4.6}$$

Note that $x \ge 0$ is a redundant constraint in $\mathcal{F}^{R1}$ since it is implied by $Xe = x$ and $X \ge 0$. Furthermore, it is easy to see that $\mathcal{F}^{R1}$ is a polytope. We first recall the following result about (R1).

According to the proposition 2.3.5, the set of vertices of $\mathcal{F}^{R1}$ is given by

$$\left\{(e^i, e^i(e^i)^T) : i = 1, \ldots, n\right\} \cup \left\{\left(\frac{1}{2}(e^i + e^j), \frac{1}{2}(e^i(e^j)^T + e^j(e^i)^T)\right) : 1 \leq i < j \leq n\right\}. \tag{4.7}$$

Therefore,

$$\ell^{R1}(Q) = \min_{1 \leq i \leq j \leq n} Q_{ij} \leq \ell(Q).$$

Furthermore, (R1) is exact (i.e., $\ell^{R1}(Q) = \ell(Q)$) if and only if

$$\min_{1 \leq i \leq j \leq n} Q_{ij} = \min_{1 \leq k \leq n} Q_{kk}.$$

Proposition 2.3.5 implies that (R1) is exact if and only if the minimum entry of Q is on the diagonal. In this case, (StQP) has a 1-sparse optimal solution, i.e., the optimal solution of (StQP) without any cardinality constraint is already the sparsest possible solution. Furthermore, by Lemma 4.1.1, we immediately obtain

$$\ell(Q) = \ell_n(Q) = \ell_{n-1}(Q) = \ldots = \ell_1(Q) = \min_{1 \leq k \leq n} Q_{kk}. \tag{4.8}$$

By reformulating the binarity constraint $u_j \in \{0, 1\}$ with $u_j^2 = u_j,\ j = 1, \ldots, n$ in (StQP(ρ)) and linearizing the quadratic terms xx^T, xu^T, and uu^T by X, R, and U, respectively, we obtain the following RLT relaxation:

$$\begin{array}{lrcl}
\ell_\rho^{R1}(Q) := \displaystyle\min_{x\in\mathbb{R}^n, u\in\mathbb{R}^n, X\in\mathcal{S}^n, U\in\mathcal{S}^n, R\in\mathbb{R}^{n\times n}} & \langle Q, X\rangle & & \\
\text{s.t.} & & & \\
& e^T x & = & 1 \\
& e^T u & = & \rho \\
& x & \leq & u \\
& x & \geq & 0 \\
& \operatorname{diag}(U) & = & u \\
& Xe & = & x \\
& R^T e & = & u \\
& Re & = & \rho x \\
& Ue & = & \rho u \\
& X - R^T - R + U & \geq & 0 \\
& X - R^T & \leq & 0 \\
& R - U & \leq & 0 \\
& X, R, U & \geq & 0,
\end{array} \tag{R1(ρ)}$$

where $\operatorname{diag}(U)$ denotes the diagonal component of U.

Before we continue, let us remark that the constraints $x \leq u$ and $x \geq 0$ are redundant in (R1(ρ)) since they are implied by the constraints $Xe = x$, $X \geq 0$, $R^T e = u$, and $X - R^T \leq 0$. Likewise, they imply $u \geq 0$ and $R \geq 0$. Furthermore, it is easy to verify that $R - U \leq 0$ and $U \geq 0$ are implied by the constraints $X - R^T \leq 0$ and $X - R^T - R + U \geq 0$. Note that

$u \le e$ is not implied in this formulation.

Let us denote the projection of the feasible region of (R1(ρ)) onto (x, X) by

$$\mathcal{F}_\rho^{R1} := \left\{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n : (x,u,X,U,R) \text{ is (R1}(\rho)\text{)-feasible for some } (u,U,R) \in \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n}\right\}. \tag{4.9}$$

Note that

$$\ell_\rho^{R1}(Q) = \min_{(x,X)\in\mathbb{R}^n\times\mathcal{S}^n} \left\{\langle Q, X\rangle : (x,X) \in \mathcal{F}_\rho^{R1}\right\}. \tag{4.10}$$

Clearly, we have $\mathcal{F}_\rho^{R1} \subseteq \mathcal{F}^{R1}$ for $1 \le \rho \le n$, where $\mathcal{F}^{R1}$ is given by (4.6). Our next result gives a description of $\mathcal{F}_\rho^{R1}$ in closed form for each $\rho \in \{1,\dots,n\}$.

Lemma 4.2.1. *For $\mathcal{F}_\rho^{R1}$,*

(i) For $\rho = 1$, we have $\mathcal{F}_1^{R1} = \left\{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n : e^T x = 1, \quad X = Diag(x), \quad x \ge 0\right\}$, where $Diag(x)$ is a diagonal matrix whose diagonal entries are given by x.

(ii) For each $\rho \in \{2,3,\dots,n\}$, we have $\mathcal{F}_\rho^{R1} = \mathcal{F}^{R1}$, where $\mathcal{F}^{R1}$ is given by (4.6).

Proof. (i) Let $(x,X) \in \mathcal{F}_1^{R1}$. Then $e^T x = 1$ and $x \ge 0$. Moreover, there exists $(u,U,R) \in \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n}$ such that (x,u,X,U,R) is (R1(ρ))-feasible with $\rho = 1$. Since $U \ge 0$, $\operatorname{diag}(U) = u$, and $Ue = u$, we obtain $U = \operatorname{Diag}(u)$. Since $R - U \le 0$ and $R \ge 0$, we obtain that R is a diagonal matrix. Similarly, using $X - R^T \le 0$, we conclude that X is a diagonal matrix. Since $Xe = x$, we obtain that $X = \operatorname{Diag}(x)$. Conversely, if $e^T x = 1$, $X = \operatorname{Diag}(x)$, and $x \ge 0$, then it is easy to verify that $(x,u,X,U,R) = (x,x,X,X,X)$ is (R1(ρ))-feasible. It follows that $(x,X) \in \mathcal{F}_1^{R1}$.
(ii) Let $\rho \in \{2,3,\dots,n\}$. We clearly have $\mathcal{F}_\rho^{R1} \subseteq \mathcal{F}^{R1}$. Evidently, $\mathcal{F}^{R1}$ is a polytope, so for the reverse inclusion, it suffices to show that each vertex of $\mathcal{F}^{R1}$ belongs to $\mathcal{F}_\rho^{R1}$. By Proposition 2.3.5, the set of vertices of $\mathcal{F}^{R1}$ is given by (4.7). If $(x,X) = (e^i, e^i(e^i)^T)$ for some $i = 1,\dots,n$, then choose an arbitrary $u \in \{0,1\}^n$ such that $u_i = 1$ and $e^T u = \rho$. If, on the other hand, $(x,X) = (\frac{1}{2}(e^i + e^j), \frac{1}{2}(e^i(e^j)^T + e^j(e^i)^T))$ for some $1 \le i < j \le n$, then choose an arbitrary $u \in \{0,1\}^n$ such that $u_i = 1$, $u_j = 1$, and $e^T u = \rho$. In both cases then, define $R = xu^T$ and $U = uu^T$. It is easy to verify that $(x,u,X,U,R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n}$ is (R1(ρ))-feasible, which implies that each vertex of $\mathcal{F}^{R1}$ belongs to $\mathcal{F}_\rho^{R1}$. We conclude that $\mathcal{F}_\rho^{R1} = \mathcal{F}^{R1}$. □

Lemma 4.2.1 immediately gives rise to the following results.

Corollary 4.2.1. *(i) For $\rho = 1$,* (R1(ρ)) *is exact, i.e., $\ell_1^{R1}(Q) = \ell_1(Q)$.*

(ii) For each $\rho \in \{2,3,\dots,n\}$, we have $\ell^{R1}(Q) = \ell_\rho^{R1}(Q) = \min\limits_{1\le i\le j\le n} Q_{ij}$.

Proof. Both assertions follow from Lemma 4.2.1, Lemma 4.1.1, and (4.10). □

We arrive at the following exactness result for the classical RLT relaxation of sparse StQPs:

Theorem 4.2.1. (R1(ρ)) *is exact (i.e., $\ell_\rho^{R1}(Q) = \ell_\rho(Q)$) if and only if $\rho = 1$ or $\min\limits_{1\le i\le j\le n} Q_{ij} = \min\limits_{1\le k\le n} Q_{kk}$.*

Proof. By Corollary 4.2.1(i), (R1(ρ)) is exact for $\rho = 1$. Let $\rho \in \{2, 3, \dots, n\}$. If (R1($\rho$)) is exact, then Lemma 4.1.1 and Corollary 4.2.1(ii) imply that $\ell_\rho^{R1}(Q) = \min_{1 \le i \le j \le n} Q_{ij} = \ell^{R1}(Q) \le \ell(Q) \le \ell_\rho(Q) = \ell_\rho^{R1}(Q) = \ell^{R1}(Q)$. The claim follows from Proposition 2.3.5. Conversely, if $\min_{1 \le i \le j \le n} Q_{ij} = \min_{1 \le k \le n} Q_{kk}$, then $\ell_\rho^{R1}(Q) = \ell^{R1}(Q) = \min_{1 \le i \le j \le n} Q_{ij} = \min_{1 \le k \le n} Q_{kk} = \ell(Q) = \ell_\rho(Q)$ by Lemma 4.1.1, Corollary 4.2.1(ii), and Proposition 2.3.5. Therefore, (R1(ρ)) is exact. □

By Theorem 4.2.1, (R1(ρ)) is exact if and only if $\rho = 1$ or (StQP) itself already has a 1-sparse optimal solution. Otherwise, in view of Lemma 4.1.1 and the relation $\ell^{R1}(Q) \le \ell(Q)$, it follows from Corollary 4.2.1 that, for each $\rho \ge 2$, the lower bound $\ell_\rho^{R1}(Q)$ arising from (R1(ρ)) is, in general, quite weak as it already agrees with the lower bound $\ell^{R1}(Q)$ obtained from the RLT relaxation (R1) of (StQP).

4.2.2 SDP Relaxation

Next, we consider the SDP relaxation of (StQP) following (QCQP1-SDP) which is given by

$$\ell^{R2}(Q) := \inf_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} \left\{ \langle Q, X \rangle : (x, X) \in \mathcal{F}^{R2} \right\}, \tag{R2}$$

where

$$\mathcal{F}^{R2} := \left\{ (x, X) \in \mathbb{R}^n \times \mathcal{S}^n : e^T x = 1, \quad x \ge 0, \quad X \succeq x x^T \right\}, \tag{4.11}$$

is a closed convex set not necessarily bounded, which necessitates the use of '$\inf$' in (R2). Indeed, we have the following well-known result about (R2); we include a short proof for the sake of completeness.

Lemma 4.2.2. *If $Q \succeq 0$, then (R2) is exact (i.e., $\ell^{R2}(Q) = \ell(Q)$). If $Q \not\succeq 0$, then $\ell^{R2}(Q) = -\infty$.*

Proof. If $Q \succeq 0$, then, for any (R2)-feasible solution $(x, X) \in \mathbb{R}^n \times \mathcal{S}^n$, we have $\langle Q, X \rangle \ge x^T Q x$ since $X \succeq x x^T$, which implies that $\ell(Q) \ge \ell^{R2}(Q) \ge \ell(Q)$. If $Q \not\succeq 0$, then there exists $d \in \mathbb{R}^n$ such that $d^T Q d < 0$. Let $x \in \mathbb{R}^n$ be any feasible solution of (StQP) and let $X(\lambda) = x x^T + \lambda d d^T$, where $\lambda \ge 0$. The assertion follows by observing that $(x, X(\lambda)) \in \mathcal{F}^{R2}$ for each $\lambda \ge 0$ and that the objective function of (R2) evaluated at $(x, X(\lambda))$ tends to $-\infty$ as $\lambda \to \infty$. □

The SDP relaxation of (StQP(ρ)) following (QCQP1-SDP) is given by

$$\begin{aligned}
\ell_\rho^{R2}(Q) := \min_{x \in \mathbb{R}^n, u \in \mathbb{R}^n, X \in \mathcal{S}^n, U \in \mathcal{S}^n, R \in \mathbb{R}^{n \times n}} \quad & \langle Q, X \rangle \\
\text{s.t.} \quad & \\
e^T x &= 1 \\
e^T u &= \rho \\
\operatorname{diag}(U) &= u \\
x &\le u \\
x &\ge 0 \\
\begin{bmatrix} 1 & x^T & u^T \\ x & X & R \\ u & R^T & U \end{bmatrix} &\succeq 0.
\end{aligned} \tag{R2(ρ)}$$

Note that the constraint $u \leq e$ is implied by $\mathrm{diag}(U) = u$ and the semidefiniteness constraint.

Similar to the RLT relaxation of (StQP(ρ)), let us introduce the following projection of the feasible region of (R2(ρ)) onto (x, X):

$$\mathcal{F}_\rho^{R2} := \left\{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n : (x,u,X,U,R) \text{ is (R2}(\rho)\text{)-feasible for some } (u,U,R) \in \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n}\right\}. \tag{4.12}$$

We again observe that

$$\ell_\rho^{R2}(Q) = \min_{(x,X)\in\mathbb{R}^n\times\mathcal{S}^n} \left\{\langle Q, X\rangle : (x,X) \in \mathcal{F}_\rho^{R2}\right\}. \tag{4.13}$$

The following result gives a complete description of $\mathcal{F}_\rho^{R2}$ for each $\rho = 1, 2, \ldots, n$.

Lemma 4.2.3. *For each $\rho \in \{1, 2, \ldots, n\}$, we have $\mathcal{F}_\rho^{R2} = \mathcal{F}^{R2}$, where $\mathcal{F}^{R2}$ is given by* (4.11)*.*

Proof. We clearly have $\mathcal{F}_\rho^{R2} \subseteq \mathcal{F}^{R2}$. For the reverse inclusion, let $(x, X) \in \mathcal{F}^{R2}$ so that $e^T x = 1$ and $x \geq 0$, so also $x \leq e$. Furthermore $X = xx^T + M$ for some $M \succeq 0$. Define $u = x + \left(\frac{\rho-1}{n-1}\right)(e - x)$ so that $e^T u = \rho$ and $0 \leq x \leq u \leq e$. Let $R = xu^T$ and $U = uu^T + D$, where $D \in \mathcal{S}^n$ is a diagonal matrix such that $D_{jj} = u_j - (u_j)^2 \geq 0,\ j = 1, \ldots, n$. Note that $\mathrm{diag}(U) = u$ and

$$\begin{bmatrix} X & R \\ R^T & U \end{bmatrix} - \begin{bmatrix} x \\ u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}^T = \begin{bmatrix} M & 0 \\ 0 & D \end{bmatrix} \succeq 0.$$

By the Schur complement lemma, it follows that $(x, u, X, U, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n}$ is (R2(ρ))-feasible. Therefore $(x, X) \in \mathcal{F}_\rho^{R2}$. $\square$

Lemma 4.2.3 reveals that none of the feasible solutions of $\mathcal{F}^{R2}$ is cut off in the projection of the feasible region of (R2(ρ)) for any choice of $\rho \in \{1, 2, \ldots, n\}$. In view of (4.13), we obtain the following corollary.

Corollary 4.2.2. *For any $\rho \in \{1, \ldots, n\}$, we have $\ell_\rho^{R2}(Q) = \ell^{R2}(Q)$.*

Proof. The assertion follows from (R2), (4.13), and Lemma 4.2.3. $\square$

Now we obtain the following exactness result for the SDP relaxation of the sparse StQP:

Theorem 4.2.2. (R2(ρ)) *is exact (i.e., $\ell_\rho^{R2}(Q) = \ell_\rho(Q)$) if and only if $Q \succeq 0$ and (StQP) has a ρ-sparse optimal solution.*

Proof. The assertion follows from Lemma 4.2.2, Corollary 4.2.2, and Lemma 4.1.1. $\square$

Theorem 4.2.2 shows that (R2(ρ)) provides a finite lower bound if and only if $Q \succeq 0$. Furthermore, in this case, the bound is tight if and only if the problem (StQP) without any cardinality constraint already has a ρ-sparse optimal solution. It follows that (R2(ρ)), in general, is a weak relaxation. We close this section by specializing Theorem 4.2.2 to the particular case with a rank-one $Q \in \mathcal{S}^n$.

Corollary 4.2.3. *Let $Q = vv^T$, where $v \in \mathbb{R}^n$. If $v \in \mathbb{R}^n_+$ or $-v \in \mathbb{R}^n_+$ or $v_i = 0$ for some $i \in \{1, \ldots, n\}$, then $\ell_\rho^{R2}(Q) = \ell_\rho(Q)$ for each $\rho \in \{1, \ldots, n\}$. Otherwise, $\ell_1^{R2}(Q) < \ell_1(Q)$ and $\ell_\rho^{R2}(Q) = \ell_\rho(Q)$ for each $\rho \in \{2, \ldots, n\}$.*

Proof. Let $Q = vv^T$, where $v \in \mathbb{R}^n$. Note that $x^TQx = (v^Tx)^2 \geq 0$ for each $x \in \mathbb{R}^n$. If $v \in \mathbb{R}^n_+$ (resp., $-v \in \mathbb{R}^n_+$ or $v_i = 0$ for some $i \in \{1, \ldots, n\}$), then (StQP) has a 1-sparse optimal solution given by $e^j \in \mathbb{R}^n$, where $j = \arg \min\limits_{1 \leq i \leq n} v_i$ (resp., $j = \arg \min\limits_{1 \leq i \leq n} (-v_i)$ or $i = j$). The assertion follows from Theorem 4.2.2. Otherwise, there exist $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$ such that $v_i < 0 < v_j$. Therefore, setting $x = \frac{v_j}{v_j - v_i} e^i - \frac{v_i}{v_j - v_i} e^j$, we obtain $x \in F$ and $x^TQx = (v^Tx)^2 = 0 = \ell(Q)$. On the other hand $\ell_1(Q) = \min\limits_{1 \leq k \leq n} Q_{kk} = \min\limits_{1 \leq k \leq n} v_k^2 > 0$ by Lemma 4.1.1. $\square$

A comparison of Corollary 4.2.3 and Theorem 4.2.1 reveals that the SDP relaxation (R2(ρ)) can be strictly weaker than the RLT relaxation (R1(ρ)) for $\rho = 1$, even when $Q \succeq 0$.

4.2.3 SDP-RLT Relaxation

We will consider the SDP-RLT relaxations of (StQP(ρ)) and (StQP) obtained by combining the corresponding RLT relaxations and SDP relaxations. In particular, our objective is to shed light on the properties of the combined relaxation in relation to those of the two individual relaxations.

The SDP-RLT relaxation of (StQP) is given by

$$\ell^{R3}(Q) := \min_{x \in \mathbb{R}^n, X \in \mathcal{S}^n} \left\{ \langle Q, X \rangle : (x, X) \in \mathcal{F}^{R3} \right\}, \tag{R3}$$

where

$$\mathcal{F}^{R3} := \left\{ (x, X) \in \mathbb{R}^n \times \mathcal{S}^n : e^Tx = 1, \quad Xe = x, \quad x \geq 0, \quad X \geq 0, \quad X \succeq xx^T \right\}. \tag{4.14}$$

A complete description of instances of (StQP) that admit exact SDP-RLT relaxations is given below.

Theorem 4.2.3 (Gökmen and Yıldırım [52]). (R3) *is exact (i.e., $\ell^{R3}(Q) = \ell(Q)$) if and only if (i) $n \leq 4$; or (ii) $n \geq 5$ and there exist $x \in F$, $P \succeq 0$, $N \in \mathcal{S}^n$, $N \geq 0$, $\lambda \in \mathbb{R}$ such that $Px = 0$, $x^TNx = 0$, and $Q = P + N + \lambda E$. Furthermore, for any such decomposition, $x \in F$ is an optimal solution of (StQP) and $\ell^{R3}(Q) = \ell(Q) = \lambda$.*

We next consider the SDP-RLT relaxation of (StQP(ρ)):

$$
\begin{array}{rrcl}
\ell_\rho^{R3}(Q) := & \min\limits_{x\in\mathbb{R}^n, u\in\mathbb{R}^n, X\in\mathcal{S}^n, U\in\mathcal{S}^n, R\in\mathbb{R}^{n\times n}} & & \langle Q, X\rangle \\
\text{s.t.} & & & \\
 & e^T x & = & 1 \\
 & e^T u & = & \rho \\
 & x & \leq & u \\
 & x & \geq & 0 \\
 & \operatorname{diag}(U) & = & u \\
 & Xe & = & x \\
 & R^T e & = & u \\
 & Re & = & \rho\, x \\
 & Ue & = & \rho\, u \\
 & X - R^T - R + U & \geq & 0 \\
 & X - R^T & \leq & 0 \\
 & R - U & \leq & 0 \\
 & X, R, U & \geq & 0 \\
 & \begin{bmatrix} 1 & x^T & u^T \\ x & X & R \\ u & R^T & U \end{bmatrix} & \succeq & 0.
\end{array}
\tag{R3(ρ)}
$$

Similar to the RLT and SDP relaxations, consider the projection of the feasible region of (R3(ρ)) onto (x, X) given by

$$
\mathcal{F}_\rho^{R3} = \left\{(x, X) \in \mathbb{R}^n \times \mathcal{S}^n : (x, u, X, U, R) \text{ is (R3(}\rho\text{))-feasible for some } (u, U, R) \in \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n}\right\}. \tag{4.15}
$$

In a similar manner as above, we have

$$
\ell_\rho^{R3}(Q) = \min_{(x,X)\in\mathbb{R}^n\times\mathcal{S}^n} \left\{\langle Q, X\rangle : (x, X) \in \mathcal{F}_\rho^{R3}\right\}. \tag{4.16}
$$

It is also easy to see that

$$
\mathcal{F}_\rho^{R3} \subseteq \mathcal{F}_\rho^{R1} \cap \mathcal{F}_\rho^{R2}, \tag{4.17}
$$

where $\mathcal{F}_\rho^{R1}$ and $\mathcal{F}_\rho^{R2}$ are given by (4.9) and (4.12), respectively. Therefore,

$$
\max\{\ell_\rho^{R1}(Q), \ell_\rho^{R2}(Q)\} \leq \ell_\rho^{R3}(Q) \leq \ell_\rho(Q) \quad \text{for all } \rho \in \{1, \ldots, n\}, \tag{4.18}
$$

which implies that (R3(ρ)) is at least as tight as each of (R1(ρ)) and (R2(ρ)).

Our first result follows from the previous results on weaker relaxations.

Corollary 4.2.4. *If (i) $\rho = 1$, or (ii) $\min\limits_{1\leq i\leq j\leq n} Q_{ij} = \min\limits_{1\leq k\leq n} Q_{kk}$, or (iii) $Q \succeq 0$ and (StQP) has a ρ-sparse optimal solution, then* (R3(ρ)) *is exact.*

Proof. The assertion follows from Theorem 4.2.1, Theorem 4.2.2, and (4.18). □

Projected Feasible Sets and Their Inner Approximations

We now focus on the sets $\mathcal{F}_\rho^{R3}$, $\rho \in \{1,\ldots,n\}$. By Lemma 4.2.1, Lemma 4.2.3, (4.14), and (4.17),

$$\mathcal{F}_1^{R3} \subseteq \left\{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n : e^T x = 1,\ X = \mathrm{Diag}(x),\ X \succeq xx^T,\ x \geq 0\right\} \subseteq \mathcal{F}^{R3}, \quad (4.19)$$

$$\mathcal{F}_\rho^{R3} \subseteq \left\{(x,X) \in \mathbb{R}^n \times \mathcal{S}^n : e^T x = 1,\ Xe = x,\ X \succeq xx^T,\ X \geq 0,\ x \geq 0\right\} = \mathcal{F}^{R3},\ \ \rho \geq 2. \quad (4.20)$$

Next, we consider inner approximations of the sets $\mathcal{F}_\rho^{R3}$, where $\rho \in \{1,2,\ldots,n\}$.

Proposition 4.2.1. *For any fixed $\rho \in \{1,\ldots,n\}$, consider the corresponding formulation* (StQP(ρ))*. Then, we have*

$$conv\left\{(x, xx^T) : x \in F_\rho\right\} \subseteq \mathcal{F}_\rho^{R3}, \quad (4.21)$$

where F_ρ and $\mathcal{F}_\rho^{R3}$ are given by (1.6) *and* (4.15)*, respectively.*

Proof. For any (StQP(ρ))-feasible solution $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$, we define $X = xx^T$, $R = xu^T$, and $U = uu^T$. Then obviously $(x,u,X,U,R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n}$ is (R3(ρ))-feasible. The claim now follows by (4.15) and the convexity of $\mathcal{F}_\rho^{R3}$. □

In the remainder of this section, we identify further properties of the sets $\mathcal{F}_\rho^{R3}$, where $\rho \in \{1,2,\ldots,n\}$, and their implications on the tightness of the lower bound $\ell_\rho^{R3}(Q)$.

The Extremely Sparse Case $\rho = 1$

In this section, we give an exact description of the set $\mathcal{F}_1^{R3}$ and discuss its implications. We start with a technical lemma.

Lemma 4.2.4. *Let $a \in \mathbb{R}^n$ and define $A := Diag(a) - aa^T$. Then, the following statements are equivalent:*

(a) $a \in \mathbb{R}^n_+$ and $e^T a \leq 1$.

(b) A is positive-semidefinite.

Proof. (a) implies (b) by diagonal dominance: if $a \in \mathbb{R}^n_+$ and $e^T a \leq 1$, then $\mathrm{Diag}(a) - aa^T$ is diagonally dominant since

$$a_i - a_i^2 - \sum_{j\in\{1,\ldots,n\}\setminus\{i\}} |-a_i a_j| = a_i - a_i^2 - a_i \sum_{j\in\{1,\ldots,n\}\setminus\{i\}} a_j = a_i\left(1 - e^T a\right) \geq 0, \quad i = 1,\ldots,n\,.$$

Hence A must be positive-semidefinite. It remains to show that (b) implies (a). Now, if A is positive-semidefinite, then its diagonal entries satisfy $a_i - a_i^2 \geq 0$, $i = 1,\ldots,n$, which implies that $0 \leq a \leq e$. Furthermore, $e^T\left(\mathrm{Diag}(a) - aa^T\right)e = e^T a - (e^T a)^2 \geq 0$, which implies that $e^T a \leq 1$. □

By Lemma 4.2.4, it is easy to see that the constraint $X - xx^T \succeq 0$ on the right-hand side of (4.19) is redundant. Therefore, by Lemma 4.2.1, we obtain

$$\mathcal{F}_1^{R3} \subseteq \left\{(x, X) \in \mathbb{R}^n \times \mathcal{S}^n : e^T x = 1, \quad X = \mathrm{Diag}(x), \quad x \geq 0\right\} = \mathcal{F}_1^{R1}. \tag{4.22}$$

Our next result shows that the inclusion in (4.22) actually holds with equality, thereby yielding an exact description of $\mathcal{F}_1^{R3}$.

Lemma 4.2.5. *We have*

$$\mathcal{F}_1^{R3} = \mathcal{F}_1^{R1} = conv\left\{(x, xx^T) : x \in F_1\right\} = conv\left\{(e^j, e^j(e^j)^T) : j \in \{1, \ldots, n\}\right\}, \tag{4.23}$$

where $\mathcal{F}_1^{R3}$ and $\mathcal{F}_1^{R1}$ are defined as in (4.15) *and* (4.9)*, respectively.*

Proof. The assertion follows from the observation that $\mathcal{F}_1^{R1} = \mathrm{conv}\left\{(e^j, e^j(e^j)^T) : 1 \leq j \leq n\right\}$ in conjunction with Proposition 4.2.1 and (4.22). □

Lemma 4.2.5 reveals that the SDP-RLT relaxation (R3(ρ)) is identical to the RLT relaxation (R1(ρ)) for $\rho = 1$: semidefinite constraints in (R3(ρ)) are redundant.

Case of Larger Cardinality $\rho \geq 2$

In this section, we focus on the sets $\mathcal{F}_\rho^{R3}$, where $\rho \in \{2, 3, \ldots, n\}$, and establish several properties and relations. Our first result strengthens the inner approximation of $\mathcal{F}_\rho^{R3}$ given by Proposition 4.2.1.

Lemma 4.2.6. *We have*

$$\left\{(x, X) \in \mathcal{F}^{R3} : x \in F_\rho\right\} \subseteq \mathcal{F}_\rho^{R3}, \quad \textit{for all } \rho \in \{2, \ldots, n\},$$

where F_ρ, $\mathcal{F}^{R3}$, and $\mathcal{F}_\rho^{R3}$ are given by (1.6)*,* (4.14) *and* (4.15)*, respectively.*

Proof. Fix $\rho \in \{2, \ldots, n\}$ and let $(x, X) \in \mathcal{F}^{R3}$ with $\|x\|_0 \leq \rho$. Choose $u \in \{0, 1\}^n$ such that $x \leq u$ and $e^T u = \rho$. Define $R = xu^T$ and $U = uu^T$. Clearly, $\mathrm{diag}(U) = u$, $R^T e = u, Re = \rho x$, $Ue = \rho u$, $R - U \leq 0$, $R \geq 0$, and $U \geq 0$. Since $X = xx^T + M$ for some $M \succeq 0$, we obtain

$$\begin{bmatrix} X & R \\ R^T & U \end{bmatrix} - \begin{bmatrix} x \\ u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}^T = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \succeq 0.$$

Next, we consider the constraint $X - R^T \leq 0$. Since $X \geq 0$ and $Xe = x$, we obtain $0 \leq X_{ij} \leq \min\{x_i, x_j\}$ for each $1 \leq i \leq j \leq n$. Therefore, if $\min\{x_i, x_j\} = 0$, then $X_{ij} - u_i x_j = -u_i x_j \leq 0$. On the other hand, if $\min\{x_i, x_j\} > 0$, then $u_i = 1$, which implies that $X_{ij} - u_i x_j = X_{ij} - x_j \leq 0$. It follows that $X - R^T \leq 0$.

Finally, we need to show that $X - R - R^T + U \geq 0$. For each $1 \leq i \leq j \leq n$, if $\min\{x_i, x_j\} = 0$, then $X_{ij} = 0$ and $\min\{R_{ij}, R_{ji}\} = \min\{x_i u_j, x_j u_i\} = 0$. Therefore,

$$X_{ij} - R_{ij} - R_{ji} + U_{ij} = 0 - \max\{R_{ij}, R_{ji}\} - 0 + u_i u_j = -\max\{x_i u_j, x_j u_i\} + u_i u_j \geq 0,$$

since $x \leq u$. Here, we used the lattice identity $v + w = \min\{v, w\} + \max\{v, w\}$. On the other hand, if $\min\{x_i, x_j\} > 0$, then $u_i = u_j = 1$, which implies that $X_{ij} - R_{ij} - R_{ji} + U_{ij} = X_{ij} - x_i - x_j + 1$. For any $1 \leq i < j \leq n$, since $x_i + x_j \leq 1$, we clearly have $X_{ij} - x_i - x_j + 1 \geq 0$ since $X \geq 0$. Finally, if $i = j$, since $X \succeq 0$ and $Xe = x$, we obtain

$$X_{ii} - 2x_i + 1 = (e^i - e)^T X (e^i - e) \geq 0, \quad i = 1, \ldots, n,$$

which completes the proof. □

By Lemma 4.2.6, none of the solutions in $(x, X) \in \mathcal{F}^{R3}$ with $x \in F_\rho$ is cut off by the projection $\mathcal{F}_\rho^{R3}$. This observation gives rise to the following corollary.

Corollary 4.2.5. *(i) For each $\rho \in \{2, \ldots, n\}$, if there exists an optimal solution $(x, X) \in \mathbb{R}^n \times \mathcal{S}^n$ of* (R3) *such that $\|x\|_0 \leq \rho$, then $\ell^{R3}(Q) = \ell_\rho^{R3}(Q)$.*

(ii) We have $\mathcal{F}_n^{R3} = \mathcal{F}^{R3}$ and $\ell^{R3}(Q) = \ell_n^{R3}(Q)$.

Proof. (i) We clearly have $\ell^{R3}(Q) \leq \ell_\rho^{R3}(Q)$ by (R3), (4.16), and (4.20). The reverse inequality follows from Lemma 4.2.6.

(ii) As $F_n = F$, the first equality follows from (4.20) and Lemma 4.2.6, and the second one from the first assertion (i).

□

By Corollary 4.2.5, we can identify a particular set of instances of (StQP(ρ)) that admit an exact SDP-RLT relaxation.

Corollary 4.2.6. *Let $\rho \in \{2, \ldots, n\}$, $x \in F_\rho$, and $\lambda \in \mathbb{R}$. Let $P \succeq 0$ be such that $Px = 0$, and let $N \in \mathcal{S}^n$ be such that $N \geq 0$ and $x^T N x = 0$. If $Q = P + N + \lambda E$, then the SDP-RLT relaxation* (R3(ρ)) *is exact, i.e., $\ell_\rho^{R3}(Q) = \ell_\rho(Q)$.*

Proof. Under the hypotheses, Theorem 4.2.3 implies that $x \in F$ is an optimal solution of (StQP) and $\ell^{R3}(Q) = \ell(Q) = \lambda$. The assertion follows from Corollary 4.2.5(i) and Lemma 4.1.1. □

Rank-One Elements of $\mathcal{F}_\rho^{R3}$

Recall that each solution $(x, X) \in \mathcal{F}^{R3}$, where $x \in F_\rho$, is retained in the projection $\mathcal{F}_\rho^{R3}$, $\rho = 1, \ldots, n$ by Lemma 4.2.6. In this section, our goal is to shed light on the relations between $\mathcal{F}_\rho^{R3}$ and the set of solutions $(x, X) \in \mathcal{F}^{R3}$, where $\|x\|_0 > \rho$.

First, it follows from Proposition 4.2.1 and Lemma 4.2.5 that

$$\mathcal{F}_1^{R3} \subseteq \mathcal{F}_\rho^{R3} \quad \text{for all } \rho \in \{2, \ldots, n\}, \tag{4.24}$$

which, in turn, implies that $(x, X) = (\frac{1}{n}e, \frac{1}{n}I) \in \mathcal{F}_\rho^{R3}$ for each $\rho \in \{1, \ldots, n\}$ by Lemma 4.2.5. Therefore, for each $\rho \in \{1, \ldots, n-1\}$, there exists $(x, X) \in \mathcal{F}_\rho^{R3}$ such that $\|x\|_0 > \rho$.

Let us restrict our attention to the subset of "rank-one solutions" $(x, X) \in \mathcal{F}^{R3}$, i.e., those with $\|x\|_0 = \nu > \rho$ and $X = xx^T$. Note that $\langle Q, X \rangle = x^T Q x$ for each rank-one solution. This, in turn, enables us to compare $\ell_\rho^{R3}(Q)$ and $\ell_\nu(Q)$ for some $\nu > \rho$.

We start with the following result for $\rho = 1$.

Corollary 4.2.7. $(x, xx^T) \in \mathcal{F}_1^{R3}$ *if and only if* $x \in F_1$.

Proof. The claim follows from Lemma 4.2.5. $\square$

By Corollary 4.2.7, each rank-one solution $(x, xx^T) \in \mathcal{F}^{R3}$, where $\|x\|_0 > 1$, is cut off by $\mathcal{F}_1^{R3}$. We next focus on $\mathcal{F}_\rho^{R3}$ for $\rho \geq 2$. To that end, we first state a technical result about the feasible region of (R3(ρ)).

Lemma 4.2.7. *Let* $\rho \in \{1, \dots, n\}$ *and let* $(x, u, X, U, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n}$ *be (R3(ρ))-feasible. Then,*

$$(\rho - 2)\, u_i + 2R_{ii} + (1 - \rho)x_i - X_{ii} \geq 0, \quad \textit{for all } i \in \{1, \dots, n\}. \tag{4.25}$$

Proof. Suppose that $(x, u, X, U, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n}$ is (R3(ρ))-feasible. Let us fix $i \in \{1, \dots, n\}$. For each $j \in \{1, \dots, n\}$ such that $j \neq i$, we have

$$U_{ij} - R_{ij} - R_{ji} + X_{ij} \geq 0.$$

Therefore,

$$\begin{aligned} 0 &\leq \sum_{j \in \{1,\dots,n\} \setminus \{i\}} (U_{ij} - R_{ij} - R_{ji} + X_{ij}) \\ &= (\rho u_i - u_i) - (\rho x_i - R_{ii}) - (u_i - R_{ii}) + (x_i - X_{ii}) \\ &= (\rho - 2)\, u_i + 2R_{ii} + (1 - \rho)x_i - X_{ii}, \end{aligned}$$

where we used $\operatorname{diag}(U) = u$, $Xe = x$, $R^T e = u$, $Re = \rho\, x$, and $Ue = \rho\, u$ in the second line. The assertion follows. $\square$

Using this technical result, we can establish the following result about rank-one solutions for $\rho = 2$.

Corollary 4.2.8. *For each* $x \in F$ *such that* $\|x\|_0 \geq 4$*, we have* $(x, xx^T) \notin \mathcal{F}_2^{R3}$.

Proof. We prove the contrapositive. Let $\rho = 2$ and let $(x, xx^T) \in \mathcal{F}_\rho^{R3}$. Then, there exists $(u, U, R) \in \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n}$ such that $(x, u, X, U, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n}$ is (R3(ρ))-feasible, where $X = xx^T$. Since $X = xx^T$, it follows from the positive semidefiniteness constraint and the Schur complement lemma that $R = xu^T$. By Lemma 4.2.7, we obtain

$$(\rho - 2)\, u_i + 2x_i u_i + (1 - \rho)x_i - x_i^2 \geq 0 \quad \text{for all } i \in \{1, \dots, n\}.$$

Using $\rho = 2$, for each $i \in \{1, \dots, n\}$ such that $x_i > 0$, we obtain

$$u_i \geq \frac{1 + x_i}{2}.$$

Summing over all $i \in \{1, \dots, n\}$ with $x_i > 0$, and observing $\sum\limits_{i: x_i > 0} x_i = e^T x = 1$, we arrive at

$$2 = \sum_i u_i \geq \sum_{i: x_i > 0} u_i \geq \frac{\|x\|_0 + 1}{2},$$

which implies that $\|x\|_0 \leq 3$. The assertion follows. $\square$

By Corollary 4.2.8, each rank-one solution $(x, xx^T) \in \mathcal{F}^{R3}$, where $\|x\|_0 > 3$, is cut off by $\mathcal{F}_2^{R3}$. Furthermore, for each $x \in F$ such that $\|x\|_0 = 3$, if $(x, xx^T) \in \mathcal{F}^{R3}$, then the proof of Corollary 4.2.8 implies that there exists a unique $u \in \mathbb{R}^n$ given by $u = \frac{1}{2}(x+e) = x+\frac{1}{2}(e-x)$ such that $(x, u, xx^T, U, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^{n\times n}$ is (R3(ρ))-feasible. Our next result establishes that such a (R3(ρ))-feasible solution can always be constructed and that the choice of u can be generalized to larger values of ρ.

Theorem 4.2.4. *We have*

$$\begin{aligned}
\left\{(x, xx^T) \in \mathcal{F}^{R3} : x \in F_{2\rho-1}\right\} &\subseteq \mathcal{F}_\rho^{R3} \quad \text{for all } \rho \in \left\{2, \ldots, \left\lfloor \tfrac{n+1}{2} \right\rfloor\right\}, && (4.26)\\
\left\{(x, xx^T) \in \mathcal{F}^{R3} : x \in F\right\} &\subseteq \mathcal{F}_\rho^{R3} \quad \text{for all } \rho \in \left\{\left\lfloor \tfrac{n+1}{2} \right\rfloor + 1, \ldots, n\right\}, && (4.27)\\
\left\{(x, xx^T) \in \mathcal{F}^{R3} : x \in G_\rho\right\} &\subseteq \mathcal{F}_\rho^{R3} \quad \text{for all } \rho \in \left\{2, \ldots, \left\lfloor \tfrac{n}{2} \right\rfloor\right\}, && (4.28)
\end{aligned}$$

where we define for $\rho \in \left\{2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$

$$G_\rho := \left\{x \in F : \|x\|_0 > 2\rho - 1, \quad \max_{1\le i<j\le n: x_i x_j > 0} \frac{x_i x_j}{1 - x_i - x_j} \le \frac{(\rho-1)(\rho-2)}{(\|x\|_0 - 2)(\|x\|_0 - 2\rho + 1)}\right\}. \tag{4.29}$$

Proof. By Corollary 4.2.5(ii), we have $\mathcal{F}_n^{R3} = \mathcal{F}^{R3}$, which implies (4.27) for $\rho = n$. Therefore, let $\rho \in \{2, \ldots, n-1\}$. By Lemma 4.2.6, it suffices to focus on rank-one solutions (x, xx^T), where $x \in F$ with $\|x\|_0 \ge \rho + 1$. We abbreviate $\nu := \|x\|_0$ to ease notation. Our proof is constructive. Let us define $u \in \mathbb{R}^n$ as follows:

$$u_i = \begin{cases} x_i + \lambda(1 - x_i), & \text{if } x_i > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\lambda := \frac{\rho - 1}{\nu - 1} \in (0, 1). \tag{4.30}$$

Note that $0 \le x \le u \le e$ and $e^T u = \rho$. Let us define $X = xx^T$, $R = xu^T$, and

$$U = uu^T + U^1 + U^2,$$

where

$$\begin{aligned}
U^1 &:= \alpha\left(\mathrm{Diag}(x) - xx^T\right),\\
U^2 &:= \beta\left(\mathrm{Diag}(a) - aa^T\right),
\end{aligned}$$

and α, β, and $a \in \mathbb{R}^n$ are given by

$$\alpha := \frac{(\nu-\rho)(\nu-\rho-1)}{(\nu-1)(\nu-2)} \ge 0, \tag{4.31}$$

$$\beta := \frac{(\nu-\rho)(\rho-1)}{\nu-2} > 0, \tag{4.32}$$

$$a_i := \begin{cases} \frac{1-x_i}{\nu-1}, & \text{if } x_i > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.33}$$

Note that $a \in \mathbb{R}^n_+$, $e^T a = 1$, and $Ue = \rho u$, because $U^1 e = U^2 e = 0$. Furthermore,

$$\begin{bmatrix} X & R \\ R^T & U \end{bmatrix} - \begin{bmatrix} x \\ u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & U^1 + U^2 \end{bmatrix} \succeq 0,$$

where we used Lemma 4.2.4. In addition, if $x_i = 0$, then $U_{ii} = 0 = u_i$. If $x_i > 0$, then it follows as well that $U_{ii} = u_i$, along the following lines:

$$\begin{array}{rcl} U_{ii} & = & u_i^2 + \alpha(x_i - x_i^2) + \beta(a_i - a_i^2) \\ & = & ((1-\lambda)x_i + \lambda)^2 + \alpha(x_i - x_i^2) + \frac{\beta}{\nu-1}(1-x_i) - \frac{\beta}{(\nu-1)^2}(1-x_i)^2 \\ & = & \left((1-\lambda)^2 - \alpha - \frac{\beta}{(\nu-1)^2}\right) x_i^2 + \left(2\lambda(1-\lambda) + \alpha - \frac{\beta}{\nu-1} + \frac{2\beta}{(\nu-1)^2}\right) x_i \\ & & +\lambda^2 + \frac{\beta}{\nu-1} - \frac{\beta}{(\nu-1)^2} \, . \end{array} \tag{4.34}$$

We claim the last expression of (4.34) equals $(1-\lambda)x_i + \lambda = u_i$, which follows by equating the coefficients of x_i^2, x_i, and 1, in above expression and re-arranging all terms with λ to the right-hand side:

$$\left\{ . \begin{array}{rcccc} \alpha + \frac{\beta}{(\nu-1)^2} & = & (1-\lambda)^2 & = & \frac{(\nu-\rho)^2}{(\nu-1)^2} \\ \alpha + \frac{\beta(3-\nu)}{(\nu-1)^2} & = & (1-\lambda)(1-2\lambda) & = & \frac{(\nu-\rho)(\nu-2\rho+1)}{(\nu-1)^2} \\ 0 + \frac{\beta(\nu-2)}{(\nu-1)^2} & = & \lambda(1-\lambda) & = & \frac{(\nu-\rho)(\rho-1)}{(\nu-1)^2} \end{array} \right. . \tag{4.35}$$

Observe that the system (4.35) has a unique solution given by (4.31) and (4.32) since subtracting the second equation from the first one yields the third equation. Therefore, we obtain that $\operatorname{diag}(U) = u$. We clearly have $X \geq 0$, $R \geq 0$, and $X - R^T = (x-u)x^T \leq 0$. Finally, we focus on $X - R - R^T + U \geq 0$ since each of $R - U \leq 0$ and $U \geq 0$ is implied by these constraints. If $x_i = 0$, then $U_{ii} - 2R_{ii} + X_{ii} = 0 \geq 0$. On the other hand, if $x_i > 0$, we have

$$U_{ii} - 2R_{ii} + X_{ii} = u_i^2 + U_{ii}^1 + U_{ii}^2 - 2x_i u_i + x_i^2 = (u_i - x_i)^2 + U_{ii}^1 + U_{ii}^2 \geq 0,$$

where we used $U^1 \succeq 0$ and $U^2 \succeq 0$. Similarly, $U_{ij} - R_{ij} - R_{ji} + X_{ij} = 0 \geq 0$ whenever $1 \leq i < j \leq n$ and $x_i x_j = 0$. On the other hand, if $1 \leq i < j \leq n$ and $x_i x_j > 0$, we obtain

$$\begin{array}{rcl} U_{ij} - R_{ij} - R_{ji} + X_{ij} & = & u_i u_j - \alpha x_i x_j - \beta a_i a_j - x_i u_j - x_j u_i + x_i x_j \\ & = & (u_i - x_i)(u_j - x_j) - \alpha x_i x_j - \beta a_i a_j \\ & = & \lambda^2 (1-x_i)(1-x_j) - \alpha x_i x_j - \frac{\beta}{(\nu-1)^2}\,(1-x_i)(1-x_j) \\ & = & \left(\lambda^2 - \frac{\beta}{(\nu-1)^2}\right)(1 - x_i - x_j) + \left(\lambda^2 - \frac{\beta}{(\nu-1)^2} - \alpha\right) x_i x_j \\ & = & \frac{(\rho-1)(\rho-2)}{(\nu-1)(\nu-2)}\,(1 - x_i - x_j) + \frac{2\rho-1-\nu}{\nu-1}\, x_i x_j \, , \end{array}$$

where we used (4.30), (4.31), and (4.32) to derive the last equation. Since $\rho \geq 2$, $\nu \geq 3$, $1 - x_i - x_j \geq 0$ and $x_i x_j > 0$, it follows that $U_{ij} - R_{ij} - R_{ji} + X_{ij} \geq 0$ if $\nu \leq 2\rho - 1$, which establishes (4.26) and (4.27). If, on the other hand, $\nu > 2\rho - 1$, then $U_{ij} - R_{ij} - R_{ji} + X_{ij} \geq 0$ by (4.29), giving rise to (4.28). This completes the proof. □

Before we proceed to the important consequences of the above result, let us motivate the construction in its proof, in particular the choice of λ and the other constants.

Observation 4.2.1. *Let $\rho \in \{2, \ldots, n\}$ and let $x \in F$. Assume that $u_i = \tau x_i + b$ if $x_i > 0$ with $0 < \tau < 1$, while $u_i = 0$ if $x_i = 0$. Furthermore, assume that $U_{ij} = cx_i + cx_j + d$ if $x_i x_j > 0$ while $U_{ij} = 0$ if $x_i x_j = 0$ for $1 \le i < j \le n$. It is easy to verify that the choices of u and U in the proof of Theorem 4.2.4 are in this form. Then, the best choice of τ, b, c and d ensuring that $(x, u, X, U, R) = (x, u, xx^T, U, xu^T)$ is R3(ρ)-feasible, is the choice in the proof of Theorem 4.2.4.*

Proof. Let $\rho \in \{2, \ldots, n\}$ and let $x \in F$. Again, abbreviate $\nu = \|x\|_0$. From $e^T(\tau x + b) = e^T u = \rho$, we derive $b = \frac{\rho-\tau}{\nu} \in (0,1)$ as $\rho > 1 > \tau$ and $\rho - \tau < \rho < \nu$. Furthermore, the constraints $x \le u \le e$ become

$$x_i \le \min\left\{\frac{1-b}{\tau}, \frac{b}{1-\tau}\right\} = \min\left\{\frac{\nu-\rho+\tau}{\nu\tau}, \frac{\rho-\tau}{\nu(1-\tau)}\right\} \quad \text{for all } i = 1, \ldots, n\,.$$

Since $g(\tau) := \frac{\nu-\rho+\tau}{\nu\tau}$ decreases and $h(\tau) := \frac{\rho-\tau}{\nu(1-\tau)}$ increases with $\tau \in (0,1)$, the maximum of $\min\{g(\tau), h(\tau)\}$ is attained at τ^* satisfying $g(\tau^*) = h(\tau^*)$, and this value ensures that the formulation covers as many $x \in F$ as possible. Hence the best choice of τ would be the solution τ^* of $g(\tau^*) = h(\tau^*)$, namely $\tau^* = \frac{\nu-\rho}{\nu-1}$, which is exactly our choice in the proof of Theorem 4.2.4 with $\lambda = 1 - \tau^* = \frac{\rho-1}{\nu-\rho}$. Since $U_{ii} = u_i$ and $Ue = \rho u$, we have for $x_i > 0$

$$\tau^* x_i + b + \sum_{j \ne i : x_j > 0} (cx_i + cx_j + d) = \rho(\tau^* x_i + b) \quad \text{or}$$

$$(\nu-1)d + c + \frac{\rho-\tau^*}{\nu} + (\nu-1)cx_i + (\tau^* - c)x_i = \rho\tau^* x_i + \frac{\rho(\rho-\tau^*)}{\nu},$$

which implies, comparing coefficients of x_i and 1, that

$$\begin{cases} (\nu-1)c + \tau^* - c = \rho\tau^* \quad \text{and} \\ (\nu-1)d + c + \frac{\rho-\tau^*}{\nu} = \frac{\rho(\rho-\tau^*)}{\nu}\,, \end{cases}$$

so that $c = \frac{(\rho-1)\tau^*}{\nu-2} = \frac{(\rho-1)(\nu-\rho)}{(\nu-1)(\nu-2)}$ and $d = \frac{\rho-1}{\nu(\nu-1)(\nu-2)}[(\nu-2)\rho - 2\tau^*(\nu-1)] = \frac{(\rho-1)(\rho-2)}{(\nu-1)(\nu-2)}$, substituting $\tau^* = \frac{\nu-\rho}{\nu-1}$. This justifies our choice of c and d in the proof of Theorem 4.2.4. □

Example 4.2.1. *The condition* (4.29) *is sufficient but not necessary. For $n = 6$ and $\rho = 3$, the point*

$$x = [0.6, 0.2, 0.05, 0.05, 0.05, 0.05]^T \in F$$

violates (4.29) *since*

$$0.6 = \frac{(0.6)\,(0.2)}{1-0.6-0.2} = \max_{1 \le i < j \le n : x_i x_j > 0} \frac{x_i x_j}{1 - x_i - x_j} > \frac{(\rho-1)(\rho-2)}{(\|x\|_0 - 2)\,(\|x\|_0 - 2\rho + 1)} = 0.5\,,$$

while there exists $(u, U, R) \in \mathbb{R}^6 \times \mathcal{S}^6 \times \mathbb{R}^{6\times 6}$ such that $(x, u, X, U, R) = (x, u, xx^T, U, xu^T)$ is (SDP-RLT(3))-feasible. One choice of u and U is $u = [0.8866, 0.5512, 0.3905, 0.3906, 0.3906, 0.3905]^T$

and

$$U = \begin{bmatrix} 0.8866 & 0.4674 & 0.3264 & 0.3265 & 0.3265 & 0.3264 \\ 0.4674 & 0.5512 & 0.1588 & 0.1588 & 0.1588 & 0.1588 \\ 0.3264 & 0.1588 & 0.3905 & 0.0986 & 0.0986 & 0.0986 \\ 0.3265 & 0.1588 & 0.0986 & 0.3906 & 0.0986 & 0.0986 \\ 0.3265 & 0.1588 & 0.0986 & 0.0986 & 0.3906 & 0.0986 \\ 0.3264 & 0.1588 & 0.0986 & 0.0986 & 0.0986 & 0.3905 \end{bmatrix}.$$

Note that u is not given by an affine function of x in the sense of Observation 4.2.1.

Theorem 4.2.4 reveals that an increasingly larger and nontrivial set of rank-one solutions is contained in the sets $\mathcal{F}_\rho^{R3}$ as ρ increases. Note that G_ρ given by (4.29) is a nonconvex set. Our next result gives further insight into this set by providing a piecewise convex inner approximation.

Lemma 4.2.8. *We have $G_2 = \emptyset$. Furthermore, for all $\rho \in \left\{3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, define*

$$\delta_{\rho,\nu} := 2\left[\left(\tau_{\rho,\nu}^2 + \tau_{\rho,\nu}\right)^{1/2} - \tau_{\rho,\nu}\right], \tag{4.36}$$

with

$$\tau_{\rho,\nu} := \frac{(\rho-1)(\rho-2)}{(\nu-2)(\nu-2\rho+1)}. \tag{4.37}$$

Then,

$$H_\rho := \bigcup_{\nu=2\rho}^{n} \{x \in F : \|x\|_0 = \nu, \quad x_i + x_j \leq \delta_{\rho,\nu},\ 1 \leq i < j \leq n\} \subseteq G_\rho \quad \textit{if } \rho \in \left\{3, \ldots, \left\lfloor \tfrac{n}{2} \right\rfloor\right\}, \tag{4.38}$$

where G_ρ is defined as in (4.29)*. Moreover, we have*

$$\left\{(x, xx^T) \in \mathcal{F}^{R3} : x \in H_\rho\right\} \subseteq \mathcal{F}_\rho^{R3} \quad \textit{for all } \rho \in \left\{3, \ldots, \left\lfloor \tfrac{n}{2} \right\rfloor\right\}. \tag{4.39}$$

Proof. For $\rho = 2$, the upper bound in (4.29) equals zero, which implies that $G_2 = \emptyset$. Let us fix $\rho \in \left\{3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$ and let $x \in H_\rho$. Then, $x \in F$, $\|x\|_0 = \nu > 2\rho - 1$, and it is easy to verify that

$$\max_{1 \leq i < j \leq n : x_i x_j > 0} \frac{x_i x_j}{1 - x_i - x_j} \leq \frac{\delta_{\rho,\nu}^2}{4(1 - \delta_{\rho,\nu})} = \tau_{\rho,\nu},$$

where the last equality follows from (4.36) and (4.37). Both inclusions (4.38) and (4.39) now follow from Theorem 4.2.4 by observing that $\|x\|_0 = \nu$. □

For fixed $\rho \in \left\{3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, it is worth noticing that $\tau_{\rho,\nu}$ given by (4.37) is a decreasing function of ν, which, in turn, implies that $\delta_{\rho,\nu}$ given by (4.36) is a decreasing function of ν. Therefore, the positive components of the elements of H_ρ given by Lemma 4.2.8 tend to get closer to each other as ν increases. For instance, if $\rho = 3$, then $\delta_{\rho,\nu}$ equals 0.7321, 0.5798, and 0.4805 for $\nu = 6$, $\nu = 7$, and $\nu = 8$, respectively. Note that the point x of Example 4.2.1 satisfies $x \notin G_3$, readily certifying $x \notin H_3$ since $x_1 + x_2 = 0.8 > 0.7321$.

Theorem 4.2.4 gives rise to several results about rank-one solutions of $\mathcal{F}_\rho^{R3}$. Our next result gives a complete description of such solutions for $\rho = 2$.

Corollary 4.2.9. *We have* $(x, xx^T) \in \mathcal{F}_2^{R3}$ *if and only if* $x \in F_3$.

Proof. By Theorem 4.2.4, for any $x \in F_3$, we have $(x, xx^T) \in \mathcal{F}_2^{R3}$ by (4.26). The assertion follows from Corollary 4.2.8. $\square$

For $\rho = 1$ and $\rho = 2$, it follows from Corollary 4.2.7 and Corollary 4.2.9 that $(x, xx^T) \in \mathcal{F}_\rho^{R3}$ if and only if $x \in F_1$ and $x \in F_3$, respectively. On the other hand, for $\rho \geq 3$, Theorem 4.2.4 gives rise to our next result, which reveals that such a nontrivial upper bound on $\|x\|_0$ concerning rank-one solutions of $\mathcal{F}_\rho^{R3}$ does not exist.

Lemma 4.2.9. *Let* $\rho \in \{3, \ldots, n-1\}$. *Then, for any* $\nu \in \{\rho+1, \ldots, n\}$, *there exists* $x \in F_\nu$ *such that* $(x, xx^T) \in \mathcal{F}_\rho^{R3}$.

Proof. Let $\rho \in \{3, \ldots, n-1\}$ and $\nu \in \{\rho+1, \ldots, n\}$. By Theorem 4.2.4, the assertion clearly holds for any $x \in F_\nu$ such that $\|x\|_0 = \nu \leq 2\rho - 1$. Suppose that $\nu > 2\rho - 1$. By Lemma 4.2.8, it suffices to construct an $x \in F_\nu$ such that $\|x\|_0 = \nu$ and $x \in H_\rho$, where H_ρ is defined as in (4.38). Let $x \in F_\nu$ be given by

$$x_i = \begin{cases} \frac{1}{\nu}\,, & \text{if } i \in \{1, \ldots, \nu\}\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

We therefore need to verify that

$$\frac{2}{\nu} \leq 2\left[\left(\tau_{\rho,\nu}^2 + \tau_{\rho,\nu}\right)^{1/2} - \tau_{\rho,\nu}\right]\,,$$

where $\tau_{\rho,\nu}$ is given by (4.37). Rearranging and simplifying the terms, the above inequality reduces to

$$\frac{1}{\nu(\nu-2)} \leq \tau_{\rho,\nu}\,.$$

By (4.37), this inequality holds if

$$\frac{\nu - 2\rho + 1}{\nu} \leq (\rho - 1)(\rho - 2)\,.$$

Since $\rho \geq 3$ (and thus $2\rho - 1 > 0$), we even have

$$\frac{\nu - 2\rho + 1}{\nu} \leq 1 \leq (\rho - 1)(\rho - 2)\,,$$

which establishes the assertion. $\square$

Following our earlier discussion about the positive components of the elements of the set H_ρ, we remark that all such components of the solution constructed in the proof of Lemma 4.2.9 are equal. Our next result establishes another useful property of the rank-one solutions of $\mathcal{F}_\rho^{R3}$.

Theorem 4.2.5. *For each* $\rho \in \{1, \ldots, n-1\}$, *if* $(x, xx^T) \in \mathcal{F}_\rho^{R3}$, *then* $(x, xx^T) \in \mathcal{F}_{\rho+1}^{R3}$.

Proof. If $\rho = 1$, then the claim follows from Corollary 4.2.7 and Corollary 4.2.9. Therefore, let $\rho \in \{2, \ldots, n-1\}$ and let $(x, xx^T) \in \mathcal{F}_\rho^{R3}$. Let us define $\nu = \|x\|_0$. If $\rho \in$

$\{2, \ldots, \lfloor \frac{n+1}{2} \rfloor - 1\}$ and $\nu \leq 2(\rho+1) - 1 = 2\rho + 1$; or if $\rho \in \{\lfloor \frac{n+1}{2} \rfloor, \ldots, n\}$, then the assertion follows from Theorem 4.2.4. Therefore, let us assume that $\rho \in \{2, \ldots, \lfloor \frac{n+1}{2} \rfloor - 1\}$ and $\nu > 2\rho + 1$. For each $\rho \geq 3$, we remark that the set of rank-one solutions with this property is nonempty by Lemma 4.2.9.

Since $(x, xx^T) \in \mathcal{F}_\rho^{R3}$, there exists $(u, U, R) \in \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n}$ such that (x, u, xx^T, U, R) is (R3(ρ))-feasible. Since $X = xx^T$, we have $R = xu^T$ and $U - uu^T \succeq 0$ by the Schur complement lemma. We will construct $(u', U', R') \in \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^{n \times n}$ such that (x, u', xx^T, U', R') is (R3($\rho + 1$))-feasible. To this end, given u with $I_u := \{i : u_i > 0\}$, we construct a convex combination $u' = (1 - \lambda)u + \lambda e(u)$ where $e(u) := \sum_{i \in I_u} e_i$ and $\lambda := \frac{1}{\mu}$ where

$$\mu := \|u\|_0 - \rho > \rho + 1 \geq 3\,,$$

by assumption on $\|u\|_0 \geq \|x\|_0 = \nu > 2\rho + 1$. Therefore, $u' = u + s$, where $s \in \mathbb{R}^n$ is given by

$$s_i = \begin{cases} \frac{1 - u_i}{\|u\|_0 - \rho}, & \text{if } u_i > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The inequality $s_i \geq 0$ follows from $u_i \leq 1$. Therefore, we obtain $0 \leq x \leq u \leq u' \leq e$. Furthermore, $e^T s = 1$, which implies that $e^T u' = \rho + 1$. Since $X = xx^T$, we define $R' = x(u')^T = R + xs^T$. Finally, we define

$$U' = u'(u')^T + \frac{\mu - 2}{\mu}\left(U - uu^T\right) + \mathrm{Diag}(s) - ss^T.$$

By the Schur complement lemma,

$$\begin{bmatrix} X & R' \\ (R')^T & U' \end{bmatrix} - \begin{bmatrix} x \\ u' \end{bmatrix} \begin{bmatrix} x \\ u' \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\mu - 2}{\mu}\left(U - uu^T\right) + \mathrm{Diag}(s) - ss^T \end{bmatrix} \succeq 0,$$

where we used $U - uu^T \succeq 0$, $\mu > 3$, and Lemma 4.2.4 for $\mathrm{Diag}(s) - ss^T \succeq 0$. Therefore, the semidefiniteness constraint is satisfied. We clearly have $Xe = x$, $R'e = (\rho + 1)x$, $(R')^T e = u'$, and $U'e = (\rho + 1)u'$.

We next focus on the constraint $\mathrm{diag}(U') = u'$. If $u_i = 0$, then $u_i' = u_i = U_{ii} = U_{ii}' = 0$ since $s_i = 0$. If $u_i > 0$, then

$$\begin{aligned} U_{ii}' &= (u_i')^2 + \frac{\mu - 2}{\mu}\left(U_{ii} - u_i^2\right) + s_i - s_i^2 \\ &= (u_i + s_i)^2 + \frac{\mu - 2}{\mu}\left(u_i - u_i^2\right) + s_i - s_i^2 \\ &= \frac{2}{\mu}u_i^2 + \frac{\mu - 2}{\mu}u_i + s_i + 2u_i s_i \\ &= \frac{1}{\mu}\left(2u_i^2 + (\mu - 2)u_i + 1 - u_i + 2u_i(1 - u_i)\right) \\ &= \frac{(\mu - 1)u_i + 1}{\mu} \\ &= u_i', \end{aligned}$$

where we used $\mathrm{diag}(U) = u$ in the second line and the definition of s in the fourth line. This establishes $\mathrm{diag}(U') = u'$.

Furthermore, we have $X \geq 0$, $R' = R + xs^T \geq 0$, and $X - (R')^T = X - R^T - sx^T \leq 0$ since $X - R^T \leq 0$, $x \geq 0$, and $s \geq 0$. We next verify $X - (R')^T - R' + U' \geq 0$. Recall again that the remaining inequality constraints are implied. For the diagonal components, we have

$$U'_{ii} - 2R'_{ii} + X_{ii} = u'_i - 2x_i u'_i + x_i^2 \geq (u'_i)^2 - 2x_i u'_i + x_i^2 = (u_i - x_i)^2 \geq 0, \quad i = 1, \ldots, n,$$

where we used $\mathrm{diag}(U') = u'$ and $0 \leq x \leq u \leq e$. If $1 \leq i < j \leq n$, then

$$\begin{aligned}
U'_{ij} - R'_{ij} - R'_{ji} + X_{ij} &= u'_i u'_j + \frac{\mu - 2}{\mu}(U_{ij} - u_i u_j) - s_i s_j - x_i u'_j - x_j u'_i + x_i x_j \\
&= (u_i + s_i)(u_j + s_j) + \frac{\mu - 2}{\mu}(U_{ij} - u_i u_j) - s_i s_j \\
&\quad -x_i(u_j + s_j) - x_j(u_i + s_i) + x_i x_j \\
&= u_i s_j + s_i u_j - x_i u_j - x_i s_j - x_j u_i - x_j s_i + x_i x_j + \frac{2}{\mu} u_i u_j + \frac{\mu - 2}{\mu} U_{ij} \\
&= \frac{\mu - 2}{\mu}(U_{ij} - x_i u_j - x_j u_i + x_i x_j) + \frac{2}{\mu}(u_i u_j - x_i u_j - x_j u_i + x_i x_j) \\
&\quad + u_i s_j + s_i u_j - x_i s_j - x_j s_i \\
&\geq \frac{2}{\mu}((u_i - x_i)(u_j - x_j)) + (s_j(u_i - x_i) + s_i(u_j - x_j)) \\
&\geq 0,
\end{aligned}$$

where we used $\mu > 3$ and $U_{ij} - R_{ij} - R_{ji} + X_{ij} = U_{ij} - x_i u_j - x_j u_i + x_i x_j \geq 0$ to derive the first inequality, and $0 \leq x \leq u$ together with $s \geq 0$ to arrive at the final one. This completes the proof. □

Theorem 4.2.5 establishes the nested behavior of the sets of rank-one solutions of $\mathcal{F}_\rho^{R3}$ with respect to ρ. We can give the following result about the tightness of the lower bound $\ell_\rho^{R3}(Q)$ arising from (R3).

Corollary 4.2.10. *We have*

$$\ell_\rho^{R3}(Q) \leq \ell_{2\rho-1}(Q) \leq \ell_\rho(Q), \qquad \text{for all } \rho \in \left\{2, \ldots, \left\lfloor \tfrac{n+1}{2} \right\rfloor \right\}, \ \text{while} \tag{4.40}$$

$$\ell_\rho^{R3}(Q) \leq \ell(Q) \leq \ell_\rho(Q), \qquad \text{for all } \rho \in \left\{ \left\lfloor \tfrac{n+1}{2} \right\rfloor + 1, \ldots, n \right\}. \tag{4.41}$$

Proof. The first inequalities follow from (4.16) and from (4.26) and (4.27), respectively, whereas the second inequalities follow from Lemma 4.1.1. □

Corollary 4.2.10 reveals that the lower bound $\ell_\rho^{R3}(Q)$ can be potentially quite weak especially for larger values of ρ.

4.3 Summary

A standard quadratic optimization problem with hard cardinality constraints can be exactly reformulated as a mixed-binary QP. Therefore, it is tempting to use tractable LP- or SDP-based relaxations, either in a straightforward way or by suitable combinations. The aim is to achieve tight rigorous bounds with a computational effort that scales well with the problem size. Our analysis on the exactness reveals that some caveats are in place when following this approach. In unfavorable circumstances (e.g., if the cardinality constraints are not stringent enough), the resulting bounds are quite weak.

We also characterized the exactness of the bounds and studied the behavior of rank-one solutions to the relaxations, and several structural properties of these relaxations in relation to the corresponding relaxations of StQPs.

The reader is referred to [17] for a theoretical and computational comparison of other convex relaxations of standard quadratic optimization problems with cardinality constraints.

Chapter 5

Conclusion and Future Research Directions

In this thesis, we investigated various convex relaxations of several classes of nonconvex optimization problems. We compared different convex relaxations according to their strengths and weaknesses. We gave descriptions of the set of instances that admit exact or inexact relaxations. Based on such descriptions, we proposed algorithms for generating instances with known optimal solutions to the problems and prespecified exactness guarantees of the relaxations.

For (QP), we studied various relations between the polyhedral properties of the feasible region of a quadratic optimization problem and its RLT relaxation. We presented necessary and sufficient conditions for the set of instances of quadratic optimization problems that admit exact RLT relaxations. We were able to establish a partial characterization of the set of vertices of the feasible region of the RLT relaxation. We then discussed how our results can be used to construct quadratic optimization problems with an exact, inexact, and unbounded RLT relaxation.

For (BoxQP), we considered RLT and SDP-RLT relaxations of quadratic optimization problems with box constraints. We presented algebraic descriptions of instances of (BoxQP) that admit exact RLT relaxations as well as those that admit exact SDP-RLT relaxations. By using these descriptions, we proposed four algorithms for efficiently constructing an instance of (BoxQP) with predetermined exactness or inexactness guarantees. Our preliminary computational experiments revealed that Algorithms 2–4 (instances with inexact RLT or exact SDP-RLT relaxations) are capable of generating computationally challenging instances. In particular, we remark that Algorithms 1 (exact RLT), 3 (exact SDP-RLT), and 4 (exact SDP-RLT with inexact RLT) can be used to construct an instance of (BoxQP) with a known optimal solution, which may be of independent interest for computational purposes.

For (StQP(ρ)), we compared its RLT, SDP, and SDP-RLT relaxations with their counterparts for (StQP). We found that RLT and SDP relaxation of (StQP(ρ)) are relatively weak. Both relaxations achieve the same lower bound as the relaxations without the cardinality constraint. We investigated the relation between the feasible region of (StQP(ρ)) and its SDP-RLT relaxation. We characterized the exactness of the bounds and studied the behavior of rank-one solutions to the SDP-RLT relaxations.

For (QP), we intend to work on a complete characterization of the set of vertices of the feasible region of the RLT relaxation in the near future. Such a characterization may have further algorithmic implications for constructing a larger set of instances with inexact but finite RLT relaxations. Our results in this thesis establish several properties of RLT relaxations of quadratic optimization problems. Another interesting question is how the structural properties change for higher-level RLT relaxations and SDP-RLT relaxations. Additionally, our preliminary computational results suggest that the lower bounds from the RLT relaxation after the conversion to the standard form and the one from the RLT relaxation of the original formulation always agree. We leave the theoretical investigation of this observation as a future research direction.

For (BoxQP), we intend to investigate the facial structure of the feasible region of the SDP-RLT relaxation and exploit it to develop algorithms for generating instances of (BoxQP) with an inexact SDP-RLT relaxation in the near future. Another interesting direction is the computational complexity of determining whether, for a given instance of (BoxQP), the RLT or the SDP-RLT relaxation is exact. Our algebraic descriptions do not yield an efficient procedure for this problem. An efficient recognition algorithm may have significant implications for extending the reach of global solvers for (BoxQP).

For (StQP(ρ)), an interesting question is whether Theorem 4.2.5 concerning the nested property of rank-one solutions could be extended to general feasible solutions. Another direction could be the study of tighter conic-based relaxations than the SDP-RLT relaxation. While these avenues are beyond the scope of the present work, they remain on our research agenda for the near future.

We also intend to see whether our results can be extended to the general framework or other subclasses of problems. One interesting direction is to study the vertices of the RLT relaxation of (QCQP1) by extending Proposition 2.3.3 and Proposition 2.3.4. For other subclasses of problems, the disjoint bilinear optimization problem admits a reduced version of the RLT relaxation implied by Proposition 2.3.1. We intend to investigate the difference between the polyhedral properties of the standard RLT relaxation and the reduced version of the RLT relaxation.

Bibliography

[1] W. P. Adams and H. D. Sherali. A tight linearization and an algorithm for zero-one quadratic programming problems. *Manag. Sci.*, 32(10):1274–1290, 1986.

[2] A. A. Ahmadi and J. Zhang. On the complexity of finding a local minimizer of a quadratic function over a polytope. *Math. Program.*, 195(1):783–792, 2022.

[3] I. G. Akrotirianakis and C. A. Floudas. Computational experience with a new class of convex underestimators: Box-constrained NLP problems. *J. Glob. Optim.*, 29(3):249–264, 2004.

[4] I. G. Akrotirianakis and C. A. Floudas. A new class of improved convex underestimators for twice continuously differentiable constrained NLPs. *J. Glob. Optim.*, 30(4):367–390, 2004.

[5] F. A. Al-Khayyal and J. E. Falk. Jointly constrained biconvex programming. *Math. Oper. Res.*, 8(2):273–286, 1983.

[6] K. M. Anstreicher. Semidefinite programming versus the reformulation-linearization technique for nonconvex quadratically constrained quadratic programming. *J. Glob. Optim.*, 43:471–484, 2009.

[7] K. M. Anstreicher. On convex relaxations for quadratically constrained quadratic programming. *Math. Program.*, 136(2):233–251, 2012.

[8] K. M. Anstreicher and S. Burer. Computable representations for convex hulls of low-dimensional quadratic forms. *Math. Program.*, 124(1-2):33–43, 2010.

[9] C. Audet, P. Hansen, B. Jaumard, and G. Savard. A branch and cut algorithm for nonconvex quadratically constrained quadratic programming. *Math. Program.*, 87(1):131–152, 2000.

[10] G. Azuma, M. Fukuda, S. Kim, and M. Yamashita. Exact SDP relaxations of quadratically constrained quadratic programs with bipartite graph structures. *J. Glob. Optim.*, 2023. To appear.

[11] R. Badenbroek and E. de Klerk. An analytic center cutting plane method to determine complete positivity of a matrix. *INFORMS J. Comput.*, 34(2):1115–1125, 2022.

[12] X. Bao, N. V. Sahinidis, and M. Tawarmalani. Semidefinite relaxations for quadratically constrained quadratic programming: A review and comparisons. *Math. Program.*, 129(1):129–157, 2011.

[13] E. Beale. *Mathematical Programming in Practice*. Pitman international text. Pitman, 1968.

[14] D. Bertsimas, C. Darnell, and R. Soucy. Portfolio construction through mixed-integer programming at Grantham, Mayo, Van Otterloo and Company. *Interfaces*, 29(1):49–66, 1999.

[15] D. Bertsimas and R. Weismantel. *Optimization over Integers*. Dynamic Ideas, Charlestown, MA., 01 2005.

[16] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah. Julia: A fresh approach to numerical computing. *SIAM Rev.*, 59(1):65–98, 2017.

[17] I. Bomze, B. Peng, Y. Qiu, and E. A. Yildirim. Tighter yet more tractable relaxations and nontrivial instance generation for sparse standard quadratic optimization, 2024.

[18] I. Bomze, B. Peng, Y. Qiu, and E. A. Yıldırım. On tractable convex relaxations of standard quadratic optimization problems under sparsity constraints, 2023.

[19] I. M. Bomze. On standard quadratic optimization problems. *J. Glob. Optim.*, 13(4):369–387, 1998.

[20] I. M. Bomze. Copositive relaxation beats Lagrangian dual bounds in quadratically and linearly constrained quadratic optimization problems. *SIAM J. Optim.*, 25(3):1249–1275, 2015.

[21] I. M. Bomze and E. de Klerk. Solving standard quadratic optimization problems via linear, semidefinite and copositive programming. *J. Glob. Optim.*, 24(2):163–185, 2002.

[22] I. M. Bomze, M. Dür, E. de Klerk, C. Roos, A. J. Quist, and T. Terlaky. On copositive programming and standard quadratic optimization problems. *J. Glob. Optim.*, 18(4):301–320, 2000.

[23] I. M. Bomze, S. Gollowitzer, and E. A. Yıldırım. Rounding on the standard simplex: Regular grids for global optimization. *J. Glob. Optim.*, 59(2–3):243–258, 2014.

[24] I. M. Bomze, M. Locatelli, and F. Tardella. New and old bounds for standard quadratic optimization: Dominance, equivalence and incomparability. *Math. Program.*, 115(1, Ser. A):31–64, 2008.

[25] I. M. Bomze and B. Peng. Conic formulation of QPCCs applied to truly sparse QPs. *Comput. Optim. Appl.*, 84:703–735, 2023.

[26] I. M. Bomze, F. Rinaldi, and S. Rota-Bulò. First-order methods for the impatient: Support identification in finite time with convergent Frank-Wolfe variants. *SIAM J. Optim.*, 29(3):2211–2226, 2019.

[27] I. M. Bomze, W. Schachinger, and R. Ullrich. The complexity of simple models – a study of worst and typical hard cases for the standard quadratic optimization problem. *Math. Oper. Res.*, 43(2):651–674, 2018.

[28] P. Bonami, O. Günlük, and J. T. Linderoth. Globally solving nonconvex quadratic programming problems with box constraints via integer programming methods. *Math. Program. Comput.*, 10(3):333–382, 2018.

[29] S. P. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[30] S. Burer. On the copositive representation of binary and continuous nonconvex quadratic programs. *Math. Program.*, 120(2):479–495, 2009.

[31] S. Burer and A. N. Letchford. On nonconvex quadratic programming with box constraints. *SIAM J. Optim.*, 20(2):1073–1089, 2009.

[32] S. Burer and D. Vandenbussche. A finite branch-and-bound algorithm for nonconvex quadratic programming via semidefinite relaxations. *Math. Program.*, 113(2):259–282, 2008.

[33] S. Burer and Y. Ye. Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs. *Math. Program.*, 181(1):1–17, 2020.

[34] D.-S. Chen, R. G. Batson, and Y. Dang. *Applied integer programming: modeling and solution*. John Wiley & Sons, 2011.

[35] X. Chen and J. Peng. New analysis on sparse solutions to random standard quadratic optimization problems and extensions. *Math. Oper. Res.*, 40(3):725–738, 2015.

[36] X. Chen, J. Peng, and S. Zhang. Sparse solutions to random standard quadratic optimization problems. *Math. Program.*, 141(1):273–293, 2013.

[37] B. Contesse. Une caractérisation compléte des minima locaux en programmation quadratique. *Numer. Math. (Heidelb.)*, 34:315–332, 1980.

[38] P. L. De Angelis, P. M. Pardalos, and G. Toraldo. Quadratic programming with box constraints. In I. M. Bomze, T. Csendes, R. Horst, and P. M. Pardalos, editors, *Developments in Global Optimization*, pages 73–93. Springer, 1997.

[39] E. de Klerk and D. V. Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM J. Optim.*, 12(4):875–892, 2002.

[40] I. I. Dikin. Iterative solution of problems of linear and quadratic programming. *Sov. Math., Dokl.*, 8:674–675, 1967.

[41] H. Dong and J. T. Linderoth. On valid inequalities for quadratic programming with continuous variables and binary indicators. In M. X. Goemans and J. R. Correa, editors, *Integer Programming and Combinatorial Optimization - 16th International Conference, IPCO 2013, Valparaíso, Chile, March 18-20, 2013. Proceedings*, volume 7801 of *Lecture Notes in Computer Science*, pages 169–180, Heidelberg, 2013. Springer Berlin.

[42] G. Eichfelder and J. Povh. On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets. *Optim. Lett.*, 7(6):1373–1386, 2013.

[43] C. A. Floudas. *Deterministic global optimization: theory, methods and applications*. Springer Science & Business Media, 2013.

[44] C. A. Floudas, I. G. Akrotirianakis, S. Caratzoulas, C. A. Meyer, and J. Kallrath. Global optimization in the 21st century: Advances and challenges. *Comput. Chem. Eng.*, 29(6):1185–1202, 2005.

[45] M. Frank and P. Wolfe. An algorithm for quadratic programming. *Nav. Res. Logist. Q.*, 3(1-2):95–110, 1956.

[46] T. Fujita and M. Kojima. Semidefinite relaxation for nonconvex programs. *J. Glob. Optim.*, 10:367–380, 1997.

[47] F. Furini, E. Traversi, P. Belotti, A. Frangioni, A. M. Gleixner, N. Gould, L. Liberti, A. Lodi, R. Misener, H. D. Mittelmann, N. V. Sahinidis, S. Vigerske, and A. Wiegele. QPLIB: A library of quadratic programming instances. *Math. Program. Comput.*, 11(2):237–265, 2019.

[48] L. Galli and A. N. Letchford. Valid inequalities for quadratic optimisation with domain constraints. *Discr. Optim.*, 41:100661, 2021.

[49] J. Gao and D. Li. Optimal cardinality constrained portfolio selection. *Oper. Res.*, 61(3):745–761, 2013.

[50] L. E. Gibbons, D. W. Hearn, P. M. Pardalos, and M. V. Ramana. Continuous characterizations of the maximum clique problem. *Math. Oper. Res.*, 22(3):754–768, 1997.

[51] P. E. Gill, W. Murray, and M. H. Wright. *Practical optimization*. SIAM, 2019.

[52] Y. G. Gökmen and E. A. Yıldırım. On standard quadratic programs with exact and inexact doubly nonnegative relaxations. *Math. Program.*, 193(Ser. A):365–403, 2022.

[53] N. I. Gould and P. L. Toint. Numerical methods for large-scale non-convex quadratic programming. In A. H. Siddiqi and M. Kočvara, editors, *Trends in Industrial and Applied Mathematics*, pages 149–179. Springer, 2002.

[54] L. Grippo and S. Lucidi. A differentiable exact penalty function for bound constrained quadratic programming problems. *Optim.*, 22(4):557–578, 1991.

[55] Gurobi Optimization, LLC. Gurobi Optimizer Reference Manual, 2023.

[56] P. Hansen, B. Jaumard, M. Ruiz, and J. Xiong. Global minimization of indefinite quadratic functions subject to box constraints. *Nav. Res. Logist.*, 40(3):373–392, 1993.

[57] N. Ho-Nguyen and F. Kilinç-Karzan. A second-order cone based approach for solving the trust-region subproblem and its variants. *SIAM J. Optim.*, 27(3):1485–1512, 2017.

[58] R. Horst, P. M. Pardalos, and N. V. Thoai. *Introduction to Global Optimization*, volume 48 of *Nonconvex Optimization and Its Applications*. Springer, New York, NY, 2000.

[59] IBM ILOG CPLEX Optimization Studio. User's Manual for CPLEX, 2023.

[60] D. Iudin and A. S. Nemirovskii. Evaluation of informational complexity of mathematical-programming programs. *Matekon*, 13(2):3–25, 1977.

[61] V. Jeyakumar and G. Li. Trust-region problems with linear inequality constraints: Exact SDP relaxation, global optimality and robust optimization. *Math. Program.*, 147(1-2):171–206, 2014.

[62] L. Jiaquan, S. Tiantai, and D. Dingzhu. On the necessary and sufficient condition of the local optimal solution of quadratic programming. *Chin. Ann. Math. Ser. B*, 3(5):625–630, 1982.

[63] D. Kennedy. Some branch and bound techniques for nonlinear optimization. *Math. Program.*, 42:147–157, 1988.

[64] L. Khachiyan. Polynomial algorithms in linear programming. *USSR Comput. Math. & Math. Phys. (GB)*, 20(1):53–72, 1980.

[65] S. Kim and M. Kojima. Exact solutions of some nonconvex quadratic optimization problems via SDP and SOCP relaxations. *Comput. Optim. Appl.*, 26(2):143–154, 2003.

[66] J. F. C. Kingman. Typical polymorphisms maintained by selection at a single locus. *J. Appl. Probab.*, 25(A):113–125, 1988.

[67] M. Kojima and L. Tunçel. Cones of matrices and successive convex relaxations of nonconvex sets. *SIAM J. Optim.*, 10(3):750–778, 2000.

[68] H. Konno. Maximizing a convex quadratic function over a hypercube. *J. Oper. Res. Soc. Jpn.*, 23(2):171–189, 1980.

[69] S. Kontogiannis and P. Spirakis. Counting stable strategies in random evolutionary games. In X. Deng and D. Du, editors, *Proceedings of the 16th Annual International Symposium on Algorithms and Computation (ISAAC 2005)*, volume 3827 of *LNCS*, pages 839–848. Springer, Heidelberg, 2006.

[70] S. C. Kontogiannis and P. G. Spirakis. On the support size of stable strategies in random games. *Theor. Comput. Sci.*, 410(8–10):933–942, 2009.

[71] S. C. Kontogiannis and P. G. Spirakis. Well supported approximate equilibria in bimatrix games. *Algorithmica*, 57(2):653–667, 2010.

[72] M. Kozlov, S. Tarasov, and L. Khachiyan. The polynomial solvability of convex quadratic programming. *USSR Comput. Math. & Math. Phys. (GB)*, 20(5):223–228, 1980.

[73] A. H. Land and A. G. Doig. An automatic method of solving discrete programming problems. *Econometrica*, 28(3):497–520, 1960.

[74] M. Laurent and S. Poljak. On a positive semidefinite relaxation of the cut polytope. *Linear Algebra Its Appl.*, 223:439–461, 1995.

[75] M. Locatelli. Exactness conditions for an SDP relaxation of the extended trust region problem. *Optim. Lett.*, 10(6):1141–1151, 2016.

[76] M. Locatelli, V. Piccialli, and A. M. Sudoso. Fix and bound: An efficient approach for solving large-scale quadratic programming problems with box constraints. *arXiv preprint arXiv:2211.08911*, 2022.

[77] M. Lubin, O. Dowson, J. D. Garcia, J. Huchette, B. Legat, and J. P. Vielma. Jump 1.0: recent improvements to a modeling language for mathematical optimization. *Math. Program. Comput.*, 15(3):581–589, Sep 2023.

[78] A. Majthay. Optimality conditions for quadratic programming. *Math. Program.*, 1(1):359–365, 1971.

[79] H. Markowitz. Portfolio selection. *J. Finance*, 1:71–91, 1952.

[80] G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part i—convex underestimating problems. *Math. Program.*, 10(1):147–175, 1976.

[81] L. Mencarelli and C. D'Ambrosio. Complex portfolio selection via convex mixed-integer quadratic programming: a survey. *Int. Trans. Oper. Res.*, 26(2):389–414, 2019.

[82] R. Misener and C. A. Floudas. ANTIGONE: Algorithms for continuous/integer global optimization of nonlinear equations. *J. Glob. Optim.*, 59(2-3):503–526, 2014.

[83] T. S. Motzkin and E. G. Straus. Maxima for graphs and a new proof of a theorem of turán. *Can. J. Math.*, 17:533–540, 1965.

[84] C. Ning and F. You. Optimization under uncertainty in the era of big data and deep learning: When machine learning meets mathematical programming. *Comput. Chem. Eng.*, 125:434–448, 2019.

[85] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer, New York, NY, USA, second edition, 2006.

[86] M. Padberg. The Boolean quadric polytope: Some characteristics, facets and relatives. *Math. Program.*, 45(1):139–172, 1989.

[87] P. Pardalos, J. R. Birge, D.-Z. Du, C. Floudas, J. Mockus, H. Sherali, and G. Stavroulakis. *Nonconvex Optimization and Its Applications*. Springer, 1994.

[88] P. M. Pardalos. Global optimization algorithms for linearly constrained indefinite quadratic problems. *Comput. Math. Appl.*, 21(6-7):87–97, 1991.

[89] P. M. Pardalos and S. A. Vavasis. Quadratic programming with one negative eigenvalue is NP-hard. *J. Glob. Optim.*, 1:15–22, 1991.

[90] Y. Qiu and E. A. Yıldırım. On exact and inexact RLT and SDP-RLT relaxations of quadratic programs with box constraints. *J. Glob. Optim.*, May 2024.

[91] Y. Qiu and E. A. Yıldırım. Polyhedral properties of RLT relaxations of nonconvex quadratic programs and their implications on exact relaxations. *Math. Program.*, Mar 2024.

[92] M. Raghavachari. On connections between zero-one integer programming and concave programming under linear constraints. *Oper. Res.*, 17(4):680–684, 1969.

[93] M. Ramana. *An algorithmic analysis of multiquadratic and semidefinite programming problems*. PhD thesis, The Johns Hopkins University, 1994. Copyright - Database copyright ProQuest LLC; ProQuest does not claim copyright in the individual underlying works; Last updated - 2023-02-23.

[94] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.

[95] S. Sahni. Computationally related problems. *SIAM J. Comput.*, 3(4):262–279, 1974.

[96] G. Sağol and E. A. Yıldırım. Analysis of copositive optimization based linear programming bounds on standard quadratic optimization. *J. Glob. Optim.*, 63(1):37–59, 2015.

[97] A. Schrijver. *Theory of Linear and Integer Programming*. Wiley-Interscience series in discrete mathematics and optimization. Wiley, 1999.

[98] A. Scozzari and F. Tardella. A clique algorithm for standard quadratic programming. *Discrete Appl. Math.*, 156(13):2439–2448, 2008.

[99] H. D. Sherali and W. P. Adams. *A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems*. Springer US, Boston, MA, 1999.

[100] H. D. Sherali and A. Alameddine. A new reformulation-linearization technique for bilinear programming problems. *J. Glob. Optim.*, 2(4):379–410, 1992.

[101] H. D. Sherali, E. Dalkiran, and J. Desai. Enhancing RLT-based relaxations for polynomial programming problems via a new class of v-semidefinite cuts. *Comput. Optim. Appl.*, 52(2):483–506, 2012.

[102] H. D. Sherali and B. M. P. Fraticelli. Enhancing RLT relaxations via a new class of semidefinite cuts. *J. Glob. Optim.*, 22(1-4):233–261, 2002.

[103] H. D. Sherali and C. H. Tuncbilek. A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique. *J. Glob. Optim.*, 2(1):101–112, 1992.

[104] H. D. Sherali and C. H. Tunçbilek. A reformulation-convexification approach for solving nonconvex quadratic programming problems. *J. Glob. Optim.*, 7(1):1–31, 1995.

[105] H. D. Sherali and C. H. Tunçbilek. New reformulation linearization/convexification relaxations for univariate and multivariate polynomial programming problems. *Oper. Res. Lett.*, 21(1):1–9, 1997.

[106] N. Z. Shor. Quadratic optimization problems. *Soviet J. Comput. Syst. Sci.*, 25(6):1–11, 1987.

[107] S. Sojoudi and J. Lavaei. Exactness of semidefinite relaxations for nonlinear optimization problems with underlying graph structure. *SIAM J. Optim.*, 24(4):1746–1778, 2014.

[108] E. Sperner. Ein satz über untermengen einer endlichen menge. *Math. Z.*, 27(1):544–548, 1928.

[109] M. Tawarmalani and N. V. Sahinidis. A polyhedral branch-and-cut approach to global optimization. *Math. Program.*, 103:225–249, 2005.

[110] D. Trehan. Non-convex optimization: a review. In *2020 4th International conference on intelligent computing and control systems (ICICCS)*, pages 418–423. IEEE, 2020.

[111] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Rev.*, 38(1):49–95, 1996.

[112] D. Vandenbussche and G. L. Nemhauser. A branch-and-cut algorithm for nonconvex quadratic programs with box constraints. *Math. Program.*, 102(3):559–575, 2005.

[113] D. Vandenbussche and G. L. Nemhauser. A polyhedral study of nonconvex quadratic programs with box constraints. *Math. Program.*, 102(3):531–557, 2005.

[114] A. L. Wang and F. Kilinç-Karzan. On the tightness of SDP relaxations of QCQPs. *Math. Program.*, 193(1):33–73, 2022.

[115] S. J. Wright. *Primal-dual interior-point methods*. SIAM, 1997.

[116] Y. Yajima and T. Fujie. A polyhedral approach for nonconvex quadratic programming problems with box constraints. *J. Glob. Optim.*, 13:151–170, 1998.

[117] Y. Ye. *Interior point algorithms: theory and analysis*. John Wiley & Sons, 2011.

[118] E. A. Yıldırım. An alternative perspective on copositive and convex relaxations of nonconvex quadratic programs. *J. Glob. Optim.*, 82(1):1–20, 2022.

[119] D. B. Yudin and A. S. Nemirovskii. Informational complexity and efficient methods for the solution of convex extremal problems. *Matekon*, 13(2):22–45, 1976.