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Semidefinite relaxations for quadratically constrained quadratic programming: A review and comparisons

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Abstract At the intersection of nonlinear and combinatorial optimization, quadratic programming has attracted significant interest over the past several decades. A variety of relaxations for quadratically constrained quadratic programming (QCQP) can be formulated as semidefinite programs (SDPs). The primary purpose of this paper is to present a systematic comparison of SDP relaxations for QCQP. Using theoretical analysis, it is shown that the recently developed doubly nonnegative relaxation is equivalent to the Shor relaxation, when the latter is enhanced with a partial first-order relaxation-linearization technique. These two relaxations are shown to theoretically dominate six other SDP relaxations. A computational comparison reveals that the two dominant relaxations require three orders of magnitude more computational time than the weaker relaxations, while providing relaxation gaps averaging 3% as opposed to gaps of up to 19% for weaker relaxations, on 700 randomly generated problems with up to 60 variables. An SDP relaxation derived from Lagrangian relaxation, after the addition of redundant nonlinear constraints to the primal, achieves gaps averaging 13% in a few CPU seconds.

Keywords Quadratic programming · Semidefinite programming · Lagrangian relaxation · Nonconvex optimization

Mathematics Subject Classification (2000) 49M29 · 65K05 · 90C22 · 90C26 · 90C30

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1 Introduction

The general quadratically constrained quadratic programming problem can be formulated as:

$$
\begin{aligned}
\text{(QCQP)} \quad f^* = \min_x \ & x^T Q^0 x + (c^0)^T x \\
\text{s.t.} \ & x^T Q^k x + (c^k)^T x \leq b^k, \quad k = 1, \ldots, m \\
& (a^p)^T x = d^p, \quad p = 1, \ldots, q \\
& l \leq x \leq u,
\end{aligned}
$$

where m denotes the number of quadratic constraints and is at least one, Q^k ($k = 0, \ldots, m$) are generally indefinite real $n \times n$ matrices, c^k ($k = 0, \ldots, m$) and a^p ($p = 1, \ldots, q$) are vectors in $\mathbb{R}^n$, b^k ($k = 1, \ldots, m$) and d^p ($p = 1, \ldots, q$) are real numbers. Without loss of generality, the matrices Q^k ($k = 0, \ldots, m$) are assumed to be symmetric. The set $\prod_{i=1}^{n} [l_i, u_i]$ is assumed to be nonempty and bounded, *i.e.*, for all i, $-\infty < l_i \leq u_i < +\infty$. In this case, any QCQP with $x \in [l, u]^n$ can be transformed to an equivalent QCQP with $x' \in [0, 1]^n$ by defining $x_i = (u_i - l_i)x_i' + l_i$. For convenience, we assume $l_i = 0$ and $u_i = 1$ for $i = 1, \ldots, n$.

QCQPs arise in many applications, including facility location, production planning, multiperiod tankage quality problems in refinery processes, circle packing problems, and the max-cut problem [1,21,23–27,41,49]. As general quadratic programming problems are NP-hard [44], the development of suitable relaxations is required for exact solution algorithms. Semidefinite programming (SDP) techniques have received a great deal of attention in the optimization literature recently [48] and many SDP relaxations have been proposed for QCQP [6,7,13,14,20,22,30,38].

For global optimization of general QCQP problems, an immediate question is which relaxation is most appropriate in practice in terms of tightness and computational tractability. In this paper, we present a collection of SDP-based relaxations for QCQP problems, including known relaxations from the literature as well as new relaxations, and investigate the theoretical strength and computational efficiency of these relaxations. The main theoretical comparisons presented in this paper show that, under mild technical conditions, the following hold (see Theorem 1 and Proposition 10):

$$v^{LG} = v^{Shor} = v^{DShor} \leq v^{SD} \leq v^{SC} \leq v^{SRLT} = v^{DNN} \leq f^*,$$

and

$$v^{DShor} \leq v^{DLG1} \leq v^{SRLT},$$

where

- v^{LG} is the value of a standard Lagrangian relaxation of QCQP [45],
- v^{Shor} is the value of the Shor relaxation [39],
- v^{DShor} is the value of the dual of the Shor relaxation [39],
- v^{SD} is the value of the Shor relaxation [39] enhanced with convex/concave envelopes for diagonal elements of the matrix variables,

- v^{SC} is the value of Shor relaxation [39] enhanced with convex/concave envelopes for all matrix variables,
- v^{SRLT} is the value of the Shor relaxation [39] enhanced with a partial first-order RLT [37],
- v^{DNN} is the value of the doubly nonnegative relaxation [18],
- f^* is the optimal solution value of the nonconvex QCQP, and
- v^{DLG1} is an SDP relaxation of QCQP derived from Lagrangian relaxation, after the addition of redundant nonlinear constraints to the primal [32,40,45].

In our computations, we find many instances where v^{DLG1} yields a tighter bound than v^{SC}. Further, it is easy to construct instances where v^{SD} dominates v^{DLG1}. One such example is $\min_x\{-3x^2 + 2x \mid 0 \le x \le 1\}$. Therefore, v^{DLG1} and v^{SD} do not dominate each other.

In Sect. 2, we discuss fundamental polyhedral and Lagrangian relaxations that can be used to construct more complex relaxations. In Sect. 3, we present and analyze the doubly nonnegative relaxation and then use optimality conditions to derive additional relaxations in Sect. 4. Section 5 is devoted to theoretical comparisons of relaxations, including the proofs of Theorem 1 and Proposition 10. Computational comparisons of relaxations for general QCQPs are presented in Sect. 6. Finally, the special case of bilinear programs is addressed in Sect. 7, where a new SDP relaxation is proposed.

1.1 Notation

The following notation is used throughout the paper. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. We use $e \in \mathbb{R}^n$ to represent a vector of ones and e_i to represent the i-th unit vector. $\mathbb{R}^{n\times n}$ refers to the set of real, $n \times n$ matrices. The inner product of two matrices A and B is defined as $A \bullet B = \sum_i \sum_j A_{ij} B_{ij}$. For a set S, $\operatorname{conv} S$ denotes the convex hull of S. For a vector x, $\operatorname{diag}(x)$ denotes a diagonal matrix whose i-th diagonal element equals x_i. For a matrix M, $\operatorname{diag}(M)$ denotes the vector composed of the diagonal of M, while $M_{i\bullet}$ and $M_{\bullet i}$, respectively, denote the i-th row and the i-th column of M. Finally, we use $\mathscr{P}$ and $\mathscr{N}$ to represent the cones of completely positive and doubly nonnegative matrices, respectively, that are defined as:

$$\mathscr{P} = \left\{ A \in \mathbb{R}^{n\times n} \mid A = \sum_{k=1}^{m} x^k (x^k)^T, m \ge 1, x^k \in \mathbb{R}^n_+, k = 1, \ldots, m \right\},$$
$$\mathscr{N} = \{ A \in \mathbb{R}^{n\times n} \mid A \succeq 0, A \ge 0 \}.$$

2 Fundamental polyhedral and Lagrangian relaxations

2.1 Reformulation-linearization technique and McCormick inequalities

Two commonly used approaches to linearize quadratic programming problems are the Reformulation Linearization Technique (RLT) and the convex and concave envelopes of the bilinear terms. The resulting linear relaxations can be used to tighten SDP relaxations.

A recent overview of RLT can be found in [37]. RLT can generate both linear and nonlinear reformulations and relaxations for quadratic programming problems. A complete application of first-level RLT would involve the pairwise multiplication of all, including the quadratic, constraints of (QCQP). We limit attention to a partial first-level RLT that involves the generation of quadratic constraints by pairwise multiplication of only bound constraints and linear constraints of QCQP, followed by linearization of the quadratic constraints by substitution of all quadratic terms with additional variables.

The convex and concave envelopes of a bilinear function over a box, also known as McCormick or Al-Khayyal and Falk inequalities [3,28], have been used to derive linear relaxations for general bilinear programming problems [2], and later used in nonconvex quadratically constrained quadratic programs [4]. In both settings, the envelopes were used to bound individual bilinear terms. This approach has become commonplace for relaxing bilinear terms in global optimization practice [42].

Let $X \in \mathbb{R}^{n\times n}$ be a symmetric matrix. The following linear relaxations can be obtained by both RLT for bound factors ($x \geq 0,\ \ e - x \geq 0$) and McCormick inequalities:

$$\begin{cases} X_{ij} \geq 0, & i, j = 1, \ldots, n \\ X_{ij} - x_j - x_i \geq -1, & i, j = 1, \ldots, n \\ X_{ij} - x_i \leq 0, & i, j = 1, \ldots, n \end{cases}$$

For quadratic programming problems with no linear constraints, *i.e.*, $q = 0$, such as box-constrained quadratic programs, the relaxation obtained by partial first-level RLT is equivalent to that obtained by the McCormick inequalities. However, for general QCQP, the partial first-level RLT invoked here contains additional constraints obtained by products of linear constraints other than bound factors:

$$\begin{aligned} \text{(RLT1)} \quad v^{RLT1} = \min\nolimits_{x,X} \ & Q^0 X + (c^0)^T x \\ \text{s.t.} \quad & Q^k X + (c^k)^T x \leq b^k, \quad k = 1, \ldots, m \\ & (a^p)^T x = d^p, \quad p = 1, \ldots, q \\ & X a^p = d^p x, \quad p = 1, \ldots, q \\ & X \geq 0 \\ & ee^T - ex^T - xe^T + X \geq 0 \\ & xe^T - X \geq 0. \end{aligned}$$

2.2 Shor relaxation and Lagrangian relaxations

Lagrangian duality is an important technique for constrained optimization problems and has been shown to be powerful for obtaining and analyzing semidefinite relaxations for quadratic programming problems [32,38,39,45–47]. For integer quadratic programming with linear constraints, [32] provides a comprehensive review for constructing SDP relaxations by Lagrangian duality. Some of the results remain valid for general QCQP. The Shor relaxation [38,39] is one of the most popular relaxations for quadratic problems. In the sequel, we describe the Shor and Lagrangian relaxations for the general QCQP, and review some important theoretical results.

The general form of the Lagrangian function is:

$$\left.\begin{aligned} L(x,\lambda,\upsilon,\omega,\mu) &= x^T Q^0 x + {c^0}^T x + \sum_{k=1}^{m} \lambda_k \left(x^T Q^k x + (c^k)^T x - b^k\right) \\ &\quad + \sum_{p=1}^{q} \upsilon_p \left((a^p)^T x - d^p\right) + \omega^T x + \mu^T (x-e) \\ &= x^T \left(Q^0 + \sum_{k=1}^{m} \lambda_k Q^k\right) x + \left(c^0 + \sum_{k=1}^{m} \lambda_k c^k + \sum_{p=1}^{q} \upsilon_p a^p \right. \\ &\quad \left. + \omega + \mu\right)^T x - \left(\sum_{k=1}^{m} \lambda_k b^k + \sum_{p=1}^{q} \upsilon_p d^p + \mu^T e\right), \end{aligned}\right\} \tag{1}$$

where $\lambda \in \mathbb{R}^m_+,\ \ \upsilon \in \mathbb{R}^q,\ \ \omega \in \mathbb{R}^n_-$, and $\mu \in \mathbb{R}^n_+$ are Lagrange multipliers. The Lagrangian dual problem of QCQP is:

$$\begin{aligned} v^{LG} = \max\nolimits_{\lambda,\upsilon,\omega,\mu} &\ \min_{x \in \mathbb{R}^n} L(x,\lambda,\upsilon,\omega,\mu) \\ \text{s.t.} &\quad \lambda \geq 0,\ \omega \leq 0,\ \mu \geq 0. \end{aligned}$$

By weak duality, we always have $v^{LG} \leq f^*$. Furthermore, any feasible solution $(\bar{\lambda},\bar{\upsilon},\bar{\omega},\bar{\mu})$ yields a lower bound for f^*. In addition, the $\min_{x\in\mathbb{R}^n} L(x,\lambda,\upsilon,\omega,\mu)$ goes to negative infinity unless $Q^0+\sum_{k=1}^{m} \lambda_k Q^k \succeq 0$. The latter, is therefore a "hidden constraint."

Strong duality holds if the problem is convex (*i.e.*, if $Q_i \succeq 0,\ \ i = 0,1,\ldots,m$) or there is only one quadratic constraint which means $m = 1,\ \ q = 0$ and no explicit bounds. The former condition is a special case of the classical Lagrangian duality theorem for convex programming, cf. [35]. The latter condition is known as the S-Lemma, which was recently surveyed in [31].

A reformulation of the Lagrangian dual is

$$\begin{aligned} v^{LG} = \max\nolimits_{\xi,\lambda,\upsilon,\omega,\mu} &\ \xi \\ \text{s.t.} &\quad L(x,\lambda,\upsilon,\omega,\mu) - \xi \geq 0, \quad \forall x \in \mathbb{R}^n \\ &\quad \lambda \geq 0,\ \omega \leq 0,\ \mu \geq 0. \end{aligned}$$

After defining a symmetric matrix $A(\xi,\lambda,\upsilon,\omega,\mu) \in \mathbb{R}^{(n+1)\times(n+1)}$ to be

$$\begin{pmatrix} -\left(\sum_{k=1}^{m} \lambda_k b^k + \sum_{p=1}^{q} \upsilon_p d^p + \mu^T e + \xi\right) & \frac{1}{2}\left(c^0 + \sum_{k=1}^{m} \lambda_k c^k + \sum_{p=1}^{q} \upsilon_p a^p + \omega + \mu\right)^T \\ \ldots & Q^0 + \sum_{k=1}^{m} \lambda_k Q^k \end{pmatrix},$$

we obtain

$$L(x, \lambda, \upsilon, \omega, \mu) - \xi = \begin{pmatrix} 1 \\ x \end{pmatrix}^T A(\xi, \lambda, \upsilon, \omega, \mu) \begin{pmatrix} 1 \\ x \end{pmatrix}.$$

The Lagrangian dual can then be rewritten as the SDP problem

$$\begin{aligned} v^{DShor} = \max_{\xi,\lambda,\upsilon,\omega,\mu} \ & \xi \\ \text{s.t.} \quad & A(\xi, \lambda, \upsilon, \omega, \mu) \succeq 0 \\ & \lambda \geq 0, \ \omega \leq 0, \ \mu \geq 0, \end{aligned}$$

whose dual is equivalent to the Shor relaxation [39]:

$$\begin{aligned} \text{(Shor)} \quad v^{Shor} = \min_{x,X} \ & Q^0 \bullet X + (c^0)^T x \\ \text{s.t.} \quad & Q^k \bullet X + (c^k)^T x \leq b^k, \quad k = 1, \ldots, m \\ & (a^p)^T x = d^p, \quad p = 1, \ldots, q \\ & 0 \leq x \leq e \\ & X \succeq xx^T. \end{aligned}$$

This relaxation can also be obtained directly by lifting the QCQP problem into the matrix space (x, X), and relaxing $X = xx^T$ to a semidefinite constraint by ignoring the rank one restriction on X. Based on the above derivation and weak duality, we have:

Proposition 1

$$v^{DShor} = v^{LG} \leq v^{Shor} \leq f^*.$$

Consider now the following condition:

Condition 1 *The SDPs (Shor) and (Dshor) are feasible, and the set*

$$\Big\{(x, X) \mid Q^k \bullet X + (c^k)^T x < b^k, \ k = 1, \ldots, m, \\ (a^p)^T x = d^p, \ p = 1, \ldots, q, \ 0 < x < e, \ X \succ xx^T\Big\}$$

is nonempty.

Then, from conic duality (Theorem 2.4.1 in [9]), we have:

Proposition 2

$$v^{DShor} = v^{LG} = v^{Shor} \leq f^*.$$

From now on, we assume Condition 1 is satisfied.

An alternative approach to relate the Shor relaxation to the Lagrangian relaxation is presented in [22].

Since the Lagrangian dual essentially ignores linear constraints, applying Lagrangian duality to primal problems that contain explicit linear equality constraints and bound constraints usually leads to weak relaxations. An alternative approach is to dualize only the nonlinear constraints as in [43]. The resulting Lagrangian dual problem of QCQP is:

$$\begin{aligned} v_0^{LG} = \max_\lambda \ & \min_{x \in \mathscr{F}} L(x, \lambda, 0, 0, 0) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

where $\mathscr{F} = \{x \in \mathbb{R}^n \mid (a^p)^T x = d^p,\ \ p = 1, \ldots, q,\ \ 0 \leq x \leq e\}$. It is clear that:

Proposition 3

$$v^{LG} \leq v_0^{LG} \leq f^*.$$

In addition, any $\bar{\lambda} \geq 0$ yields a lower bound for f^*. This Lagrangian dual problem may provide tighter bounds for QCQP problems than LG. However, the dual subproblem

$$\min_{x \in \mathscr{F}} L(x, \lambda, 0, 0, 0),$$

which is a linearly constrained quadratic programming problem, while more tractable compared to general QCQP is still not easily solvable.

Another alternative is to reformulate the primal problem before constructing the Lagrangian dual. For instance, one can add redundant constraints or replace constraints with equivalent ones [32,45,46]. To illustrate one of the many possible reformulations, we replace the linear constraints by squared norm constraints as suggested by [32,40,45]. This approach is straightforward to apply to general QCQP problems and leads to medium size SDP relaxations. In particular, we replace the linear constraints $(a^p)^T x = d^p$ by their squared version $x^T a^p (a^p)^T x - 2d^p (a^p)^T x + (d^p)^2 = 0$, $p = 1, \ldots, q$ and add redundant quadratic constraints $x^T e_i e_i^T x \leq 1$ obtained from the box constraints $x \leq e$. Following the same steps as before, the dual of the Lagrangian dual is an SDP relaxation:

$$\begin{aligned} \text{(DLG1)} \quad v^{DLG1} = \min_{x,X} \ & Q^0 \bullet X + (c^0)^T x \\ \text{s.t.} \quad & Q^k \bullet X + (c^k)^T x \leq b^k, \quad k = 1, \ldots, m \\ & a^p (a^p)^T \bullet X - 2d^p (a^p)^T x + (d^p)^2 = 0, \quad p = 1, \ldots, q \\ & \operatorname{diag} X \leq e \\ & 0 \leq x \leq e \\ & X \succeq xx^T. \end{aligned}$$

Proposition 4

$$v^{Shor} \leq v^{DLG1} \leq f^*.$$

Proof The second inequality is obvious. To show the first inequality, assume (x, X) is a feasible solution to (DLG1). We need to show (x, X) is also feasible to the Shor relaxation. Since $X \succeq xx^T$, we have $0 = a^p(a^p)^T \bullet X - 2d^p(a^p)^T x + (d^p)^2 \geq ((a^p)^T x - d^p)^2 \geq 0$. Thus $(a^p)^T x = d^p$ and (x, X) is feasible to (Shor). □

As long as the resulting Lagrangian dual problem can be formulated as an SDP problem, the dual of the Lagrangian dual provides an SDP relaxation of the original problem. On the other hand, all semidefinite relaxations can be regarded as Lagrangian relaxations obtained from certain reformulations of the primal problem. In this sense, Lagrangian relaxation can be thought of as the best possible SDP relaxation of QCQP as argued in [45]. However, it is not clear which particular reformulation should be used to reduce or even close the duality gap, in general.

2.3 Strengthened Shor relaxations

Even though all the original variables have finite bounds, the Shor relaxation may be unbounded since X may not be sufficiently bounded. In practice, additional constraints are often combined with SDP relaxations in order to avoid unboundedness. In [6], it is shown that the combination of SDP and RLT relaxations leads to significantly better bounds than using either technique alone. However, since SDPs with a large number of linear constraints may be computationally expensive, a compromise may be necessary between tightness and tractability of the relaxation. In this section, we present several SDP relaxations for QCQP; these relaxations involve different numbers of constraints.

The following is the Shor relaxation with the addition of convex/concave envelopes for the diagonal components of X:

$$\begin{aligned}
\text{(SD)} \quad v^{SD} = \min\nolimits_{x,X} \ & Q^0 \bullet X + (c^0)^T x \\
\text{s.t.} \quad & Q^k \bullet X + (c^k)^T x \leq b^k, \quad k = 1, \ldots, m \\
& (a^p)^T x = d^p, \quad p = 1, \ldots, q \\
& x \leq e \\
& x - \operatorname{diag} X \geq 0 \\
& e - 2x + \operatorname{diag} X \geq 0 \\
& X \succeq xx^T.
\end{aligned}$$

Constraints $x \leq e$ and $e - 2x + \operatorname{diag} X \geq 0$ can be easily shown to be redundant in this formulation because of $X \succeq xx^T$.

The following is the Shor relaxation after the addition of convex/concave envelopes for the complete matrix X:

$$
\begin{aligned}
(\mathrm{SC}) \quad v^{SC} = \min_{x,X}\ & Q^0 \bullet X + (c^0)^T x \\
\text{s.t.}\quad & Q^k \bullet X + (c^k)^T x \le b^k, \quad k = 1, \dots, m \\
& (a^p)^T x = d^p, \quad p = 1, \dots, q \\
& ee^T - xe^T - ex^T + X \ge 0 \\
& X \ge 0 \\
& xe^T - X \ge 0 \\
& X \succeq xx^T.
\end{aligned}
$$

Finally, the following is the Shor relaxation with partial first-level RLT:

$$
\begin{aligned}
(\mathrm{SRLT}) \quad v^{SRLT} = \min_{x,X}\ & Q^0 \bullet X + (c^0)^T x \\
\text{s.t.}\quad & Q^k \bullet X + (c^k)^T x \le b^k, \quad k = 1, \dots, m \\
& (a^p)^T x = d^p, \quad p = 1, \dots, q \\
& Xa^p = d^p x, \quad p = 1, \dots, q \\
& X \ge 0 \\
& ee^T - xe^T - ex^T + X \ge 0 \\
& xe^T - X \ge 0 \\
& X \succeq xx^T.
\end{aligned}
$$

3 Copositive and doubly nonnegative relaxations

Copositive programming relaxations of QCQP have been obtained by replacing the cone of semidefinite matrices by the cone of completely positive matrices, the dual of which is the cone of copositive matrices [11,12,33]. This approach can provide significantly stronger convex relaxations than linear and semidefinite relaxations for general quadratic programming problems [33], and serve as an exact reformulation of the standard quadratic programming problem [12]. Although optimization over the copositive cone is still an NP-hard problem, the approximation of the copositive cone by the cone of doubly nonnegative matrices can yield a more tractable SDP bound for QCQP. In this section, we describe copositive and doubly nonnegative relaxations for QCQP derived from copositive representations by slightly generalizing results from [18] and [5].

Lifting the QCQP into matrix space, we obtain:

$$
\begin{aligned}
(\mathrm{QCQP}) \quad \min_{x,X}\ & Q^0 \bullet X + (c^0)^T x \\
\text{s.t.}\quad & Q^k \bullet X + (c^k)^T x \le b^k, \quad k = 1, \dots, m \\
& X = xx^T, x \in \mathscr{F},
\end{aligned}
$$

where, as before, $\mathscr{F} = \{x \in \mathbb{R}^n | (a^p)^T x = d^p,\ p = 1, \dots, q,\ x \in [0, 1]^n\}$. Let

$$
\mathscr{L} = \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \mid Q^k \bullet X + (c^k)^T x \le b^k, \quad k = 1, \dots, m \right\}
$$

and

$$\mathscr{C} = \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T | x \in \mathscr{F} \right\}.$$

The feasible region of (QCQP) is $\mathscr{L} \cap \mathscr{C}$. A convex relaxation for the QCQP problem is:

$$\begin{aligned} \text{(CV)} \quad v^{CV} = \min_{x,X} \ & Q^0 \bullet X + (c^0)^T x \\ \text{s.t.} \quad & Q^k \bullet X + (c^k)^T x \leq b^k, \quad k = 1, \ldots, m \\ & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \operatorname{conv} \mathscr{C}. \end{aligned}$$

Since $\mathscr{L} \cap \mathscr{C} \subseteq \operatorname{conv}(\mathscr{L} \cap \mathscr{C}) \subseteq \operatorname{conv} \mathscr{L} \cap \operatorname{conv} \mathscr{C} = \mathscr{L} \cap \operatorname{conv} \mathscr{C}$, we have:

Proposition 5

$$v^{CV} \leq f^*.$$

When QCQP has only linear constraints (*i.e.*, $m = 0$), we have $\mathscr{L} = \mathbb{R}^{(n+1)\times(n+1)}$ and $v^{CV} = f^*$ because $\operatorname{conv}(\mathscr{L} \cap \mathscr{C}) = \operatorname{conv}(\mathscr{C}) = \mathscr{L} \cap \operatorname{conv} \mathscr{C}$ and the objective function is linear in the matrix space $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$, as pointed out in [18]. For problems with general quadratic constraints, this equality holds for some special cases. In [18], this is shown for a single quadratic equality constraint. The following addresses the general QCQP problem:

Proposition 6 *The convex relaxation CV for the QCQP problem is exact, in other words*

$$v^{CV} = f^*$$

if

$$x^T Q^k x + (c^k)^T x \leq (\ or\ \geq) b^k, \quad \forall k = 1, \ldots, m, \ \forall x \in \mathscr{F}.$$

Proof If $x^T Q^k x + (c^k)^T x \leq b^k$ for all $k = 1, \ldots, m$ and $x \in \mathscr{F}$, the quadratic constraints in (QCQP) are redundant. The constraints $Q^k \bullet X + (c^k)^T x \leq b^k$ are also redundant in (CV) because each point in $\mathscr{C}$ satisfies these constraints. Therefore, the constraints being linear, are valid for $\operatorname{conv} \mathscr{C}$. The problem then reduces to the linearly constrained case addressed in [18].

If $x^T Q^k x + (c^k)^T x \geq b^k$ for all $k = 1, \ldots, m$, and $x \in \mathscr{F}$, any point $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$ in $\mathscr{L} \cap \operatorname{conv} \mathscr{C}$ can be expressed as a convex combination of points in $\mathscr{C}$ that satisfy the quadratic constraints as equalities and are, therefore, in $\mathscr{C} \cap \mathscr{L}$. Then, $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in$

$\operatorname{conv}(\mathscr{L} \cap \mathscr{C})$. In other words, $\operatorname{conv}(\mathscr{L} \cap \mathscr{C}) \supseteq \mathscr{L} \cap \operatorname{conv} \mathscr{C}$. Since the objective function of (CV) and (QCQP) is the same linear function in matrix space and $\operatorname{conv}(\mathscr{L} \cap \mathscr{C}) \subseteq \mathscr{L} \cap \operatorname{conv} \mathscr{C}$, we have $v^{CV} = f^*$. □

Now, we consider the convex hull of $\mathscr{C}$. Let $s \in \mathbb{R}^n_+$ be the slacks for the inequality $x \leq 1$. Rewrite all constraints in $\mathscr{F}$ as equalities and lift the set $\mathscr{C}$ to a higher space:

$$\mathscr{C}^+ = \left\{ \begin{pmatrix} 1 \\ x \\ s \end{pmatrix} \begin{pmatrix} 1 \\ x \\ s \end{pmatrix}^T \mid (a^p)^T x = d^p,\ p = 1, \ldots, q, x + s = e, \begin{pmatrix} x \\ s \end{pmatrix} \geq 0 \right\}.$$

Based on Proposition 2.1 in [18], we have:

$$\operatorname{conv} \mathscr{C}^+ = \left\{ \begin{pmatrix} 1 & x^T & s^T \\ x & X & Z \\ s & Z^T & S \end{pmatrix} \in \mathscr{P} \mid (a^p)^T x = d^p,\ p = 1, \ldots, q;\ x + s = e; \right.$$
$$\left. a^p (a^p)^T \bullet X = (d^p)^2;\ \operatorname{diag}(X + 2Z + S) = e \right\},$$

where $s = e - x,\ Z = xe^T - X$, and $S = ee^T - xe^T - ex^T + X$. After relaxing $\operatorname{conv} \mathscr{C}^+$ in (x, X) space, we get:

$$\operatorname{conv} \mathscr{C} \subseteq \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathscr{P} \mid (a^p)^T x = d^p,\ p = 1, \ldots, q; \right.$$
$$\left. a^p (a^p)^T \bullet X = (d^p)^2;\ ee^T - xe^T - ex^T + X \geq 0;\ xe^T - X \geq 0 \right\}.$$

Note that the bound constraint $x \leq e$ is dominated by the last two inequalities. Then, the convex hull of $\mathscr{C}$ is outer-approximated by a copositive constraint, which makes the convex relaxation (CV) a convex copositive programming problem. However, the problem is still NP-hard. A natural approach is to further outer-approximate the cone of completely positive matrices by computable convex cones.

It is known that $\mathscr{P} \subseteq \mathscr{N}$. A relaxation of $\operatorname{conv} \mathscr{C}$ therefore is:

$$\mathscr{C}_R = \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathscr{N} \mid (a^p)^T x = d^p,\ p = 1, \ldots, q; \right.$$
$$\left. a^p (a^p)^T \bullet X = (d^p)^2;\ ee^T - xe^T - ex^T + X \geq 0; xe^T - X \geq 0 \right\}.$$

The constraint $Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathscr{N}$ is a linear and semidefinite constraint $Y \succeq 0$, $Y \geq 0$. Since $\operatorname{conv} \mathscr{C} \subseteq \mathscr{C}_R$, we have an SDP relaxation, known as the doubly nonnegative relaxation:

$$
\begin{aligned}
\text{(DNN)}\quad v^{DNN} = \min_{x,X}\ & Q^0 \bullet X + (c^0)^T x \\
\text{s.t.}\ & Q^k \bullet X + (c^k)^T x \le b^k, \quad k = 1, \dots, m \\
& (a^p)^T x = d^p, \quad p = 1, \dots, q \\
& a^p (a^p)^T \bullet X = (d^p)^2 \\
& ee^T - xe^T - ex^T + X \ge 0 \\
& xe^T - X \ge 0 \\
& X \ge 0 \\
& X \succeq xx^T.
\end{aligned}
$$

The inequalities $ee^T - xe^T - ex^T + X \ge 0, \quad xe^T - X \ge 0$ and $X \ge 0$ are the convex/concave envelopes of $X_{ij} = x_i x_j$ over $0 \le x \le e$. It is then clear that:

Proposition 7

$$v^{DNN} \le v^{CV} \le f^*.$$

For low-dimensional problems, [5] proposed an exact doubly nonnegative representation for $\mathscr{C}$ when $\mathscr{F}$ is a simplex and an exact positive semidefinite representation for $\mathscr{C}$ when $\mathscr{F}$ is a two-dimensional box, based on the fact that $\mathscr{P} = \mathscr{N}$ for sets of dimension less than or equal to 4. Although $\operatorname{conv}\mathscr{C} \subset \mathscr{C}_R$ for higher dimensions, the result of [5] demonstrates the potential strength of approximating $\operatorname{conv}\mathscr{C}$ by combining polyhedral and semidefinite programming relaxations to construct relaxations for QCQP problems.

The doubly nonnegative relaxation (DNN) can also be obtained by Lagrangian duality for QCQP with the following redundant constraints:

$$
\begin{aligned}
& x^T a^p (a^p)^T x = (d^p)^2 \\
& xx^T - xe^T - ex^T + ee^T \ge 0 \\
& xe^T - xx^T \ge 0 \\
& xx^T \ge 0.
\end{aligned}
$$

4 Relaxations of KKT conditions

In this section, we discuss SDP relaxations for QCQP that incorporate the first order Karush–Kuhn–Tucker (KKT) conditions. Previous studies that enforce KKT conditions in SDP relaxations for quadratic programming problems can be found in [20] for linearly constrained QP, and in [19] for box QP.

Using the Lagrangian function (1), any locally optimal solution x of QCQP must satisfy the following two sets of first-order conditions for some $\lambda \in \mathbb{R}^m_+,\ \upsilon \in \mathbb{R}^q,\ \omega \in \mathbb{R}^n_-$, and $\mu \in \mathbb{R}^n_+$; stationarity:

$$2\left(Q^0 + \sum_{k=1}^{m} \lambda_k Q^k\right)x + c^0 + \sum_{k=1}^{m} \lambda_k c^k + \sum_{p=1}^{q} \upsilon_p a^p + \omega + \mu = 0, \tag{2}$$

and complementary:

$$\lambda_k(x^T Q^k x + {c^k}^T x - b^k) = 0, \quad k = 1, \ldots, m \tag{3}$$

$$\omega_i x_i = 0, \quad i = 1, \ldots, n \tag{4}$$

$$\mu_i(x_i - 1) = 0, \quad i = 1, \ldots, n. \tag{5}$$

Proposition 8 *Let $(x, \lambda, \upsilon, \mu)$ satisfy the stationarity and complementarity conditions. Then, we have*

$$\begin{aligned} x^T Q^0 x + (c^0)^T x = \frac{1}{2}(c^0)^T x + \frac{1}{2}\sum_{k=1}^{m} \lambda_k (c^k)^T x - \sum_{k=1}^{m} \lambda_k b^k \\ -\frac{1}{2}\sum_{p=1}^{q} \upsilon_p d^p - \frac{1}{2}\sum_{i=1}^{n} \mu_i. \end{aligned} \tag{6}$$

This result follows by pre-multiplying (2) by x^T and substituting (3), (4) and (5) in the resulting equation.

The optimality conditions can be added to the original problem, thereby reducing the feasible region while lifting the problem into a higher space that includes the Lagrange multipliers as variables. After dropping the non-quadratic Eq. (3), Eqs. (2), (4), (5), and (6) can be relaxed and formulated as semidefinite constraints and combined with other SDP relaxations for QCQP. We lift the problem into matrix space by considering

$$\begin{pmatrix} 1 \\ x \\ \lambda \\ \upsilon \\ \omega \\ \mu \end{pmatrix} \begin{pmatrix} 1 \\ x \\ \lambda \\ \upsilon \\ \omega \\ \mu \end{pmatrix}^T \succeq 0$$

and substituting all quadratic terms by new variables

$$Y = \begin{pmatrix} 1 & x^T & \lambda^T & \upsilon^T & \omega^T & \mu^T \\ & X & X^{\lambda T} & X^{\upsilon T} & X^{\omega T} & X^{\mu T} \\ & & S^1 & S^2 & S^3 & S^4 \\ & & & S^5 & S^6 & S^7 \\ & & & & S^8 & S^9 \\ & & & & & S^{10} \end{pmatrix} \succeq 0.$$

For example, Eq. (5) can be represented by

$$\mathrm{diag}(X^{\mu}) - \mu = 0.$$

After defining matrices

$$C = \begin{pmatrix} (c^1)^T \\ \vdots \\ (c^m)^T \end{pmatrix} \in \mathbb{R}^{m \times n}$$

and

$$Q'^i = \begin{pmatrix} Q^1_{i\bullet} \\ \vdots \\ Q^m_{i\bullet} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

for $i = 1, \ldots, n$, we obtain the following set of semidefinite constraints according to the KKT conditions of QCQP:

$$\left.\begin{array}{l} 2(Q^0_{i\bullet})^T x + 2Q'^i \bullet X^\lambda + c^0{}_i + \sum_{k=1}^m \lambda_k c^k{}_i + \sum_{p=1}^q \upsilon_p a_i^p \\ \qquad +\omega_i + \mu_i = 0, \quad i = 1, \ldots, n \\ Q^0 \bullet X + \frac{1}{2}(c^0)^T x - \frac{1}{2} C \bullet X^\lambda + \sum_{k=1}^m \lambda_k b^k \\ \qquad + \frac{1}{2}\sum_{p=1}^q \upsilon_p d^p + \frac{1}{2}\sum_{i=1}^n \mu_i = 0 \\ \mathrm{diag}(X^\omega) = 0 \\ \mathrm{diag}(X^\mu) - \mu = 0 \\ \lambda \geq 0, \omega \leq 0, \mu \geq 0 \\ Y_{11} = 1, \ Y \succeq 0. \end{array}\right\} \tag{7}$$

Most of the new variables defined for the quadratic terms in the matrix Y do not appear in the equations. In order to reduce the dimension of matrix variables involved in the semidefinite constraints, we drop all complementarity conditions and reformulate the semidefinite constraints. Let

$$Y' = \begin{pmatrix} 1 & x^T & \lambda^T \\ x & X & X^{\lambda T} \\ \lambda & X^\lambda & S \end{pmatrix}.$$

The reduced set of KKT constraints is:

$$\left.\begin{array}{l} 2(Q^0_{i\bullet})^T x + v2Q'^i \bullet X^\lambda + c^0{}_i + \sum_{k=1}^m \lambda_k c^k{}_i + \sum_{p=1}^q \upsilon_p a_i^p + \mu_i \geq 0, \quad i = 1, \ldots, n \\ Q^0 \bullet X + \frac{1}{2}(c^0)^T x - \frac{1}{2} C \bullet X^\lambda + \sum_{k=1}^m \lambda_k b^k + \frac{1}{2}\sum_{p=1}^q \upsilon_p d^p + \frac{1}{2}\sum_{i=1}^n \mu_i = 0 \\ \lambda \geq 0, \mu \geq 0 \\ Y' \succeq 0. \end{array}\right\} \tag{8}$$

By derivation, any optimal solution of QCQP must satisfy the two sets of semidefinite constraints (7) and (8), which can therefore be embedded in other SDP relaxations. In [19], it was shown that, for box QP, incorporating the first and second order optimality conditions in Shor's relaxation with upper bounds for diagonal elements of the matrix variables enhances the performance of branch-and-bound algorithms despite the fact that it has been shown theoretically that adding the optimality conditions does not tighten the relaxation at the root node. [20] also presents empirical evidence showing that addition of these optimality conditions tightens the feasible region of SDP relaxations for linearly constrained QP in the context of branch-and-bound. Consider the following SDP:

$$\begin{aligned} \min_{x,X}\ & Q^0 \bullet X + (c^0)^T x \\ \text{s.t.}\ \ & Q^k \bullet X + (c^k)^T x \le b^k, \quad k = 1, \ldots, m \\ & (a^p)^T x = d^p, \quad p = 1, \ldots, q \\ & x \le e \\ & x - \operatorname{diag} X \ge 0 \\ & X \succeq xx^T. \end{aligned}$$

We can use these KKT constraints to further tighten the strengthened Shor relaxations, such as (SD). One possibility is to add all KKT constraints to (SD):

$$\begin{aligned} \text{(KKT1)}\quad v^{KKT1} = \min_{x,X}\ & Q^0 \bullet X + (c^0)^T x \\ \text{s.t.}\ \ & Q^k \bullet X + (c^k)^T x \le b^k, \quad k = 1, \ldots, m \\ & (a^p)^T x = d^p, \quad p = 1, \ldots, q \\ & x \le e \\ & x - \operatorname{diag} X \ge 0 \\ & X \succeq xx^T \\ & \text{Constraints (7).} \end{aligned}$$

Another possibility is the relaxation (SD) with a reduced set of KKT constraints:

$$\begin{aligned} \text{(KKT2)}\quad v^{KKT2} = \min_{x,X}\ & Q^0 \bullet X + (c^0)^T x \\ \text{s.t.}\ \ & Q^k \bullet X + (c^k)^T x \le b^k, \quad k = 1, \ldots, m \\ & (a^p)^T x = d^p, \quad p = 1, \ldots, q \\ & x \le e \\ & x - \operatorname{diag} X \ge 0 \\ & X \succeq xx^T \\ & \text{Constraints (8).} \end{aligned}$$

5 Equivalence and dominance of different SDP relaxations

Proposition 9 *The Shor relaxation with RLT and the doubly nonnegative relaxation are equivalent. In other words,*

$$v^{SRLT} = v^{DNN}.$$

Proof Assume (x, X) is a feasible solution of the Shor relaxation with RLT. Since $(a^p)^T x = d^p$ and $Xa^p = d^p x$ for any $p = 1, \ldots, q$, we have

$$(a^p)^T X a^p = d^p (a^p)^T x = (d^p)^2.$$

That is $a^p (a^p)^T \bullet X = (d^p)^2$. Thus (x, X) is also feasible to the doubly nonnegative relaxation, and $v^{SRLT} \geq v^{DNN}$.

On the other hand, assume that (x, X) is a feasible solution of the doubly nonnegative relaxation. Since $(a^p)^T x = d^p$ and $a^p (a^p)^T \bullet X = (d^p)^2$ for any $p = 1, \ldots, q$, we have

$$(a^p)^T (X - xx^T) a^p = 0.$$

Since $X - xx^T \succeq 0$, there exists a symmetric matrix B such that $X - xx^T = B^2$. That is $(a^p)^T B^2 a^p = 0$. Then $Ba^p = 0$ and

$$(X - xx^T) a^p = 0 = Xa^p - d^p x.$$

Thus, (x, X) is also feasible to the SRLT relaxation, and $v^{SRLT} \leq v^{DNN}$. □

Proposition 1 in [17] also provides a result analogous to Proposition 9.

Proposition 10

$$v^{Shor} \leq v^{DLG1} \leq v^{SRLT}.$$

Proof The first inequality was proved in Sect. 2.2. The second inequality holds because $(a^p)^T x = d^p$ and $a^p (a^p)^T \bullet X = (d^p)^2$ imply $a^p (a^p)^T \bullet X - 2d^p (a^p)^T x + (d^p)^2 = 0$. □

Theorem 1

$$v^{LG} = v^{Shor} = v^{DShor} \leq v^{SD} \leq v^{SC} \leq v^{SRLT} = v^{DNN} \leq f^*.$$

Proof The first two equalities are from Proposition 2. The next three inequalities are based on the following facts:

1. $x - \operatorname{diag} X \geq 0$ and $\operatorname{diag} X \geq 0$ imply $x \geq 0$.
2. $X \geq 0$ and $xe^T - X \geq 0$ imply $x \geq 0$.
3. $xe^T - X \geq 0$ and $ee^T - xe^T - ex^T + X \geq 0$ imply $x \leq e$.

The last equality in the statement of the theorem comes from Proposition 9. Finally, the last inequality follows from the arguments given in Sect. 3. □

6 Computational comparison of relaxations

In this section, we supplement the above theoretical comparisons of relaxations with computational comparisons. The primary motivation is that, even when a dominance relationship has been established, the practical solvability of tight relaxations still needs to be investigated. Although SDP problems can be solved in polynomial time, the computational effort required for some SDP relaxations, especially those combining linear and semidefinite constraints, can be significantly large. In the computations, we made no effort to identify the best possible way for implementing any particular relaxation. For instance, we did not consider row generation schemes for relaxations with a large number of constraints. Our primary objective has been to study the relative strengths of relaxations and identify relaxations that are easy to solve with general-purpose SDP codes.

The test problems used here were randomly generated. The variables x were bounded in a unit box $[0, 1]^n$, and the number of variables n was varied from 20 to 60. The vectors c^k, a^p, b^k, and d^p, and the matrices Q^k were formed with components randomly generated in the interval $[-1, 1]$ from uniform distributions. The only exception was the values for b^k, which were generated similarly in the range $[0, 100]$.

It is known that QCQPs with many quadratic constraints are much harder to solve than those with only bounds or linear constraints. In addition, some SDP relaxations essentially ignore linear constraints and lead to weak bounds for problems with many linear constraints. For these reasons, we decided to construct test problems in which the numbers of quadratic and linear constraints were varied according to the following combinations:

1. QCQPs with $m = 1, q = n/10$.
2. QCQPs with $m = 1, q = n/5$.
3. QCQPs with $m = n/2, q = n/10$.
4. QCQPs with $m = n, q = n/10$.

We also decided to investigate whether the density and the fraction of negative eigenvalues of matrices Q^k affect the quality of bounds and performance of SDP solvers used to solve the relaxations under study. For this purpose, we constructed matrices Q^k with density 25, 50 and 100%. For the fully dense problems, we generated matrices with 25, 50, 75, and 100% of negative eigenvalues. In all cases, the matrices Q^k were obtained as a product $Z^k D^k (Z^k)^t$. Here, Z^k is a random unitary matrix, while the diagonal matrix D^k has all its negative and positive elements randomly generated in $[-1, 0]$ and $[0, 1]$, respectively. The number of negative elements of D^k was set to the desired number of negative eigenvalues for Q^k. For each specification (the dimension, the density, and the percentage of negative eigenvalues of Q^k), we generated and solved five instances. Overall, we tested 700 instances with each relaxation, with the exception of one relaxation for which additional problems were required to obtain a clear sense of the results. As some of these problems are still beyond the capabilities of exact solvers, in the calculation of relaxation gaps, we used the best feasible solution obtained by BARON, MINOS, CONOPT, and SNOPT, where each solver was executed with a time limit of 10,000 s under GAMS [16]. All computational times are reported on Intel Xeon 1,700 Mhz processors.

There are various software packages available for solving SDP, an overview of which can be found in [29]. Among the state-of-the-art SDP solvers, we considered three packages: SDPA 7.3.0 [50], CSDP 6.0.1 [15] and DSDP 5.8 [10], mainly because they are available in subroutine libraries that can be easily embedded in larger applications, such as branch-and-bound algorithms. All solvers provide a primal as well as dual solution value of the SDP under consideration and terminate when these values are within a predefined tolerance. In all cases, the value of the dual solution provides a valid lower bound for the QCQP and was used to report relaxation gaps. Hence, the relaxation gaps presented in the sequel represent the sum of the actual gap of the relaxation under study and the gap introduced by the SDP solver while solving the SDP relaxation; the latter was no more than 1% unless otherwise stated below. Since the actual gap of the SDP relaxations is much larger than 1% for most cases, our computational results provide a meaningful basis of comparisons of SDP relaxations.

In the first experiment, SDPA, CSDP, and DSDP were used to solve (DNN), one of the most difficult SDP relaxations considered, with default settings for problems with up to 50 variables. Figure 1 shows the relative gap for (DNN) obtained by the solvers and the computational time taken by the solvers as a function of problem size. From the figure, it can be seen that SDPA requires higher computational effort and provides more accurate solutions, while DSDP is faster but does not meet the 1% optimality tolerance. CSDP achieves the same accuracy as SDPA with less computational time. Further comparisons between SDPA and CSDP with a 5% instead of the default 1% tolerance led to average CPU time reductions of 25% for SDPA and 50% for CSDP. After this preliminary experimentation, we decided to use CSDP to carry out detailed computational experiments with the relaxations under study, using a relative termination tolerance of 1%.

The remainder of the experiments are devoted to the computational comparison of seven different relaxations that were presented earlier in the paper. In Fig. 2, we compare the tightness and computational effort required by these SDP relaxations as a function of problem size. As expected, (DNN) and (SRLT) are equivalent and provide the tightest lower bounds for QCQP problems. On average, the relative gap of these relaxations is 3%, while the computational time required averaged about 400 s per problem. The strengthened Shor relaxation (SC), which is a relaxation of

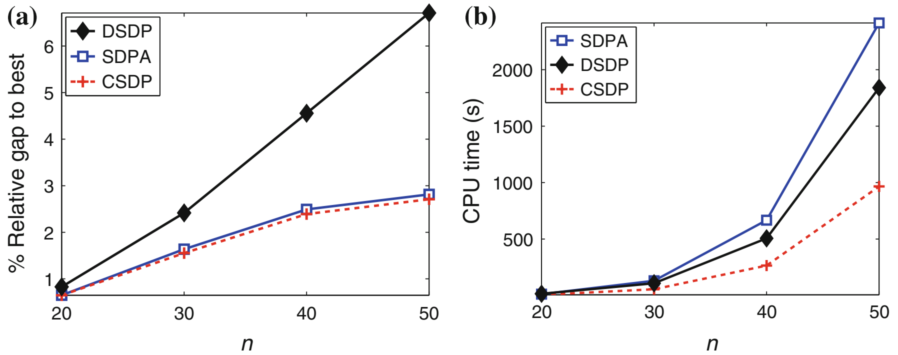

Fig. 1 Comparison of SDP solvers applied to (DNN) relaxation (from results with 480 problems)

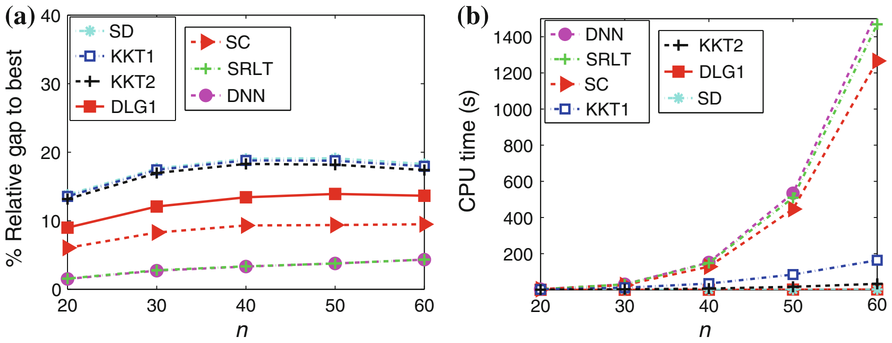

Fig. 2 Comparison of seven different relaxations (from results with 700 problems)

(DNN)/(SRLT), provides an average gap of 9% in an average of 359 s. Since the dimensions of both (DNN) and (SC) grow fast with problem size, solving these relaxations for larger QCQP problems is challenging for current SDP solvers.

The other four relaxations take considerably less computational time. (DLG1) achieves the best lower bounds of these four relaxations, with a relative gap of 13% on average. Relaxation (SD) provides a 19% relative gap on average. In fact, (DLG1) provided at least as good bounds (SD) for all problems we solved. On the other hand, even though (DLG1) was on average 4% tighter than (SC), there were many problems for which (SC) provided stronger bounds than (DLG1). Adding the constraint $\operatorname{diag}(X) \leq x$ to (DLG1) results in an insignificant improvement over (DLG1) as it strengthens this relaxation by an average of only 0.16%.

Neither (KKT1) nor (KKT2) leads to better lower bounds than (SD), despite the fact that they contain additional constraints and require more computational time than (SD). Relaxation (SC) is second only to (DNN)/(SRLT) in terms of relaxation gap. However, since the computational requirements of (SC) follow closely those of (DNN)/(SRLT), the latter relaxations dominate (SC). Clearly, (DNN)/(SRLT) is the best approach for obtaining tight bounds for general QCQP problems. On the other hand, (DLG1) is the best relaxation amongst all others. It provides weaker bounds than (DNN)/(SRLT) but at a small fraction of the computational requirements. Overall, all other relaxations are dominated by (DNN)/(SRLT) and/or (DLG1), in terms of quality and/or efficiency of solution. The remainder of this section presents more detailed results for the dominant relaxations.

In Fig. 3, we show the effect of the number of quadratic constraints m and linear constraints q on the SDP relaxations (DLG1) and (DNN)/(SRLT). It is known that Lagrangian dual relaxations are, in general, weak for QCQP problems with many linear constraints. The Lagrangian relaxation (DLG1) does not suffer from this because it includes a squared form of the linear constraints. From the figure, we see that increasing the number of linear constraints has a more profound effect on (DLG1) than on (DNN)/(SRLT). On the other hand, increasing the number of quadratic constraints has a similar effect on both relaxations.

To investigate the effect of the density of matrices Q^k on the relaxations, we plot the relative gap and the computational time for problems with density of 25, 50, and 100% in Fig. 4. The figure shows that sparse and dense problems tend to be easier than

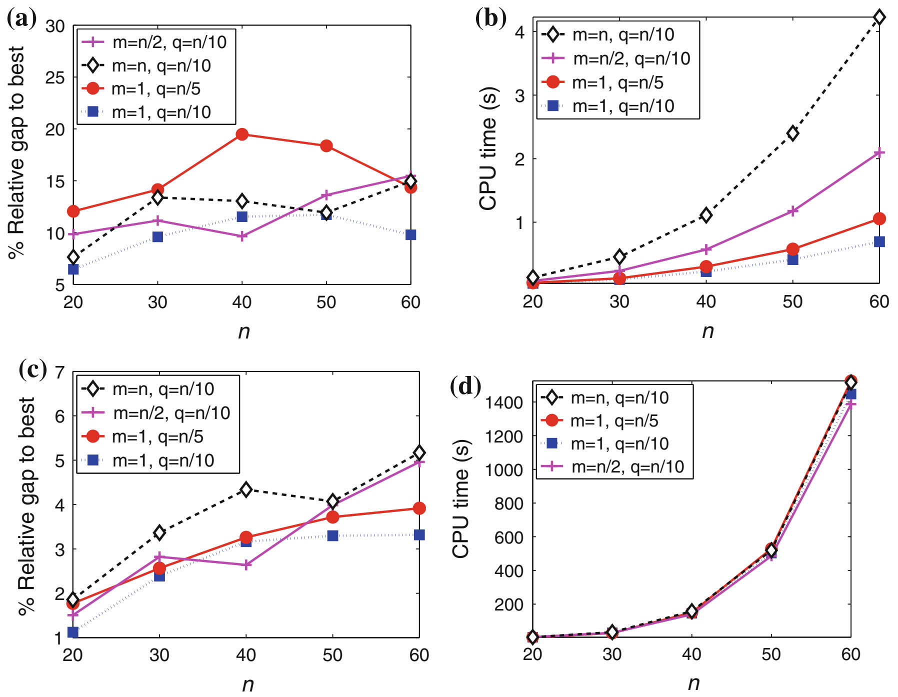

Fig. 3 Effect of the constraint structures on (DLG1) (**a**, **b**) and (DNN)/(SRLT) (**c**, **d**) (from results with 700 problems)

problems with 50% density for smaller problems but no clear trend is observed, in general. The effect of matrix density on the computational effort is also not significant. The main reason could be that, even when the matrices in the SDP relaxations are relatively sparse, the Schur complement, which is computed by SDP solvers implementing interior point methods, is still dense except for extremely sparse problems.

Figure 5 presents the relaxation gap and computational time for relaxations (DLG1) and (DNN)/(SRLT) for problems for which the fraction of negative eigenvalues was 0.25, 0.50, 0.75, and 1. In both cases, the relaxation gap increases as we move away from the reverse convex case (100% negative eigenvalues) and then decreases again as we approach the convex case (0% negative eigenvalues). This gap peaks between 50 and 75% negative eigenvalues for (DNN)/(SRLT) and around 10% for (DLG1). The former is clear from Fig. 5c, while the latter is seen in Fig. 6, which provides more detailed results for additional fractions of negative eigenvalues for (DLG1). The performance of (SD), (SC), (KKT1) and (KKT2) was found to be similar to that for (DLG1).

7 SDP relaxations for bilinear programming

Bilinear programming is a special case of general QCQP, where the matrices Q^k, $k = 0, 1, \ldots, m$ have no nonzero diagonal elements. For this case, all quadratic expressions

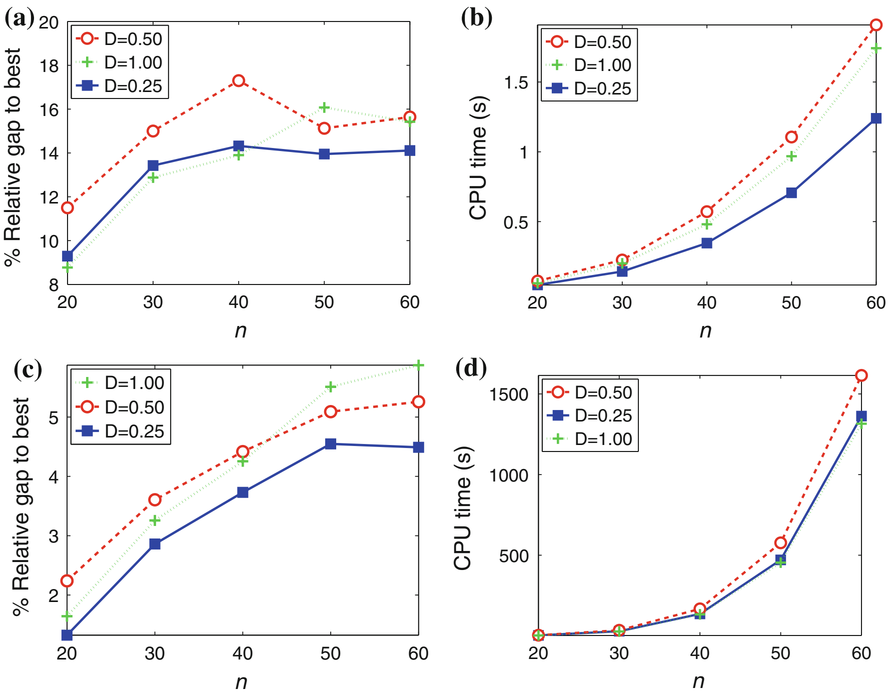

Fig. 4 Effect of matrix density (D) on (DLG1) (**a**, **b**) and (DNN)/(SRLT) (**c**, **d**) (from results with 300 problems)

in the optimization problem are bilinear functions. In this section, we propose an SDP relaxation for this particular problem based on the convex envelope of bilinear functions.

The convex envelope of bilinear and more general multilinear functions has been studied extensively [8,34,36]. We first give the definition of a multilinear function.

Definition 1 A function $M(x_1, \dots, x_n)$ is said to be a general multilinear function if, for each $i \in \{1, \dots, n\}$, the function $M(x_1^0, \dots, x_i, \dots, x_n^0)$ depends linearly on the vector x_i, provided that all the remaining $n-1$ vector arguments are fixed.

Let $M(x)$ be a multilinear function over a hyper-rectangle P and let $\text{vert}(P) = \{\pi^t\}$ denote the set of vertices of P. In [34], it was shown that the convex envelope of the multilinear function over a hyper-rectangle is polyhedral, and is generated by restricting attention to the extreme points of the hyper-rectangle. The convex envelope of $M(x)$ over P can thus be expressed in a higher-dimensional space as follows:

$$\operatorname*{conv}_P M(x) = \min_{\varphi} \left\{ \sum_t \varphi_t M(\pi^t) \mid x = \sum_t \varphi_t \pi^t,\ \sum_t \varphi_t = 1, \varphi_t \geq 0 \right\}. \tag{9}$$

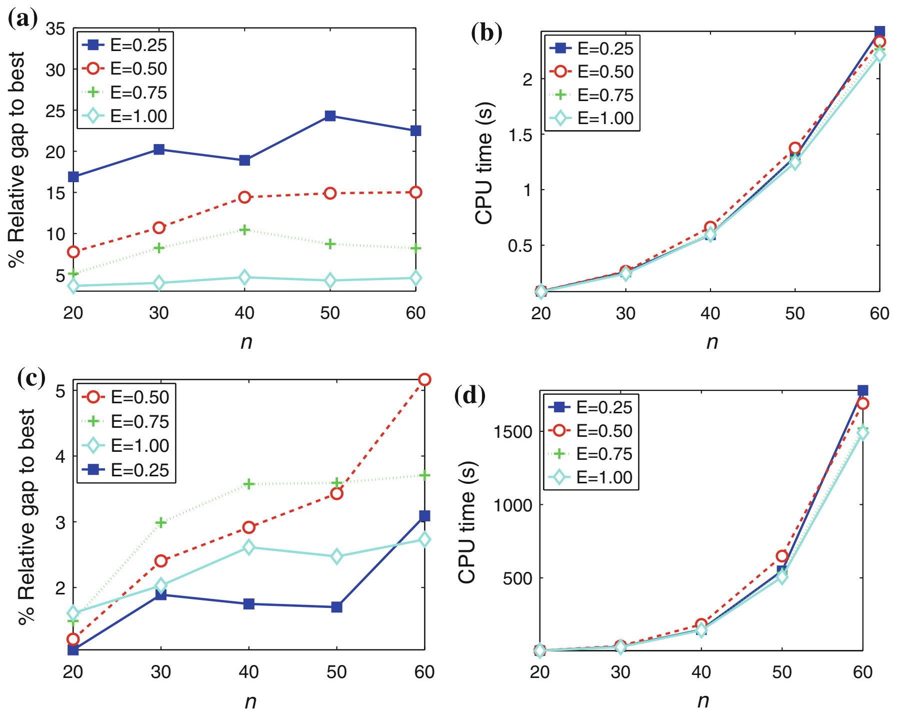

Fig. 5 Effect of the fraction (E) of negative eigenvalues on (DLG1) (**a**, **b**) and (DNN)/(SRLT) (**c**, **d**) (from results with 400 problems)

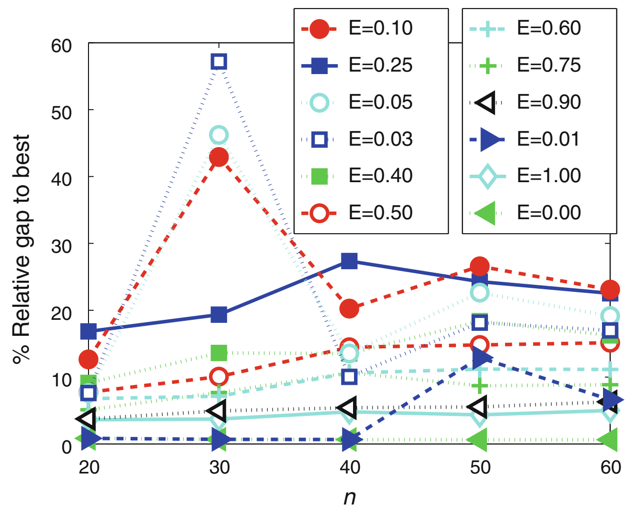

Fig. 6 Effect of the fraction (E) of negative eigenvalues on (DLG1) (detailed results on 1,200 problems)

We will show that the following is an SDP relaxation for bilinear programming problems:

$$
\begin{aligned}
\text{(BL)}\quad v^{BL} = \min_{x,X'}\ & Q^0 \bullet X' + (c^0)^T x \\
\text{s.t.}\quad & Q^k \bullet X' + (c^k)^T x \le b^k, \quad k = 1, \ldots, m \\
& (a^p)^T x = d^p, \quad p = 1, \ldots, q \\
& 0 \le x \le e \\
& \operatorname{diag} X' = x \\
& X' \succeq xx^T.
\end{aligned}
$$

Assume that the relative interior of the following set is nonempty:

$$
\Big\{(x, X) \mid Q^k \bullet X + (c^k)^T x < b^k, \quad k = 1, \ldots, m,\ (a^p)^T x = d^p, \quad p = 1, \ldots, q,\\
0 < x < e,\ \operatorname{diag} X = x,\ X \succeq xx^T\Big\}
$$

Proposition 11 *Consider the QCQP problem with* $\operatorname{diag}(Q^k) = 0, k = 0, 1, \ldots, m$. *The SDP (BL) provides a lower bound for the QCQP problem. In other words,*

$$v^{BL} \le f^*.$$

Thus (BL) is a valid relaxation for the general bilinear programming problem.

To prove Proposition 11, we need the following proposition.

Proposition 12 *Consider the following linearly constrained bilinear programming problem:*

$$
\begin{aligned}
\textit{(LQP)}\quad f_l^* = \min_x\ & x^T Qx + (c)^T x \\
\text{s.t.}\quad & (a^p)^T x = d^p, \quad p = 1, \ldots, q \\
& 0 \le x \le e,
\end{aligned}
$$

where $\operatorname{diag}(Q) = 0$. *The following SDP provides a lower bound for (LQP).*

$$
\begin{aligned}
\textit{(LBL)}\quad v_l^{BL} = \min_{x,X'}\ & Q \bullet X' + (c)^T x \\
\text{s.t.}\quad & (a^p)^T x = d^p, \quad p = 1, \ldots, q \\
& 0 \le x \le e \\
& \operatorname{diag} X' = x \\
& X' \succeq xx^T.
\end{aligned}
$$

In other words,

$$v_l^{BL} \le f_l^*.$$

Proof Assume x is a feasible solution of (LQP). Let $P = [0, 1]^n$, $\text{vert}(P) = \{\pi^t\} \in \{0, 1\}^n$. Since $x \in P$, there exist multipliers φ_t such that

$$x = \sum_t \varphi_t \pi^t, \ \sum_t \varphi_t = 1, \ \varphi_t \geq 0. \tag{10}$$

Since $\text{diag}(Q) = 0$, the quadratic expression $x^T Qx$ is a bilinear and, hence, multilinear function. Using (9), there exist φ_t satisfying Eq. (10) and such that

$$x^T Qx \geq \sum_t \varphi_t (\pi^t)^T Q\pi^t.$$

Letting $X' = \sum_t \varphi_t pi^t (\pi^t)^T$, we have

$$x^T Qx \geq Q \bullet X'. \tag{11}$$

Consider the matrix $\begin{pmatrix} 1 & x^T \\ x & X' \end{pmatrix}$. By Eq. (10), we have:

$$\begin{pmatrix} 1 & x^T \\ x & X' \end{pmatrix} = \sum_t \varphi_t \begin{pmatrix} 1 & (\pi^t)^T \\ \pi^t & \pi^t (\pi^t)^T \end{pmatrix} = \sum_t \varphi_t \begin{pmatrix} 1 \\ \pi^t \end{pmatrix} \begin{pmatrix} 1 \\ \pi^t \end{pmatrix}^T \succeq 0.$$

In addition, $\text{diag}\, X' = \sum_t \varphi_t \pi^t = x$. Then, we have

$$\begin{aligned} X' &\succeq xx^T \\ \text{diag}\, X' &= x. \end{aligned}$$

It follows that (x, X') is feasible to the SDP (LBL). Furthermore, because of Eq. (11), we have

$$x^T Qx + (c)^T x \geq Q \bullet X' + (c)^T x.$$

Therefore $v_l^{BL} \leq f_l^*$. □

Then, we show Proposition 11 is valid.

Proof We start from the Lagrangian dual problem of QCQP v_0^{LG} in Sect. 2.2. Since $\text{diag}(Q^k) = 0$ for $k = 0, 1, \ldots, m$, the dual subproblem $v_0^{SLG} = \min_{x \in \chi} L(x, \lambda, 0, 0, 0)$ is a linearly constrained bilinear programming problem. Using

Proposition 12, an SDP relaxation for the dual subproblem is:

$$
\begin{aligned}
\text{(LBL0)} \quad v_{l0}^{BL} = \min_{x,X'} \; & \left(Q^0 + \sum_{k=1}^{m} \lambda_k Q^k\right) \bullet X' + \left(c^0 + \sum_{k=1}^{m} \lambda_k c^k\right)^T x - \sum_{k=1}^{m} \lambda_k b^k \\
\text{s.t.} \quad & (a^p)^T x = d^p, \quad p = 1, \ldots, q \\
& 0 \le x \le e \\
& \operatorname{diag} X' = x \\
& X' \succeq x x^T .
\end{aligned}
$$

The dual of (LBL0) is

$$
\begin{aligned}
\text{(DLBL0)} \quad v_{l0}^{DBL} = \max_{\upsilon,\omega,\mu,\alpha} \; & -\sum_{p=1}^{q} \upsilon_p d^p - \mu^T e \\
\text{s.t.} \quad & A(\lambda, \upsilon, \omega, \mu, \alpha) \succeq 0 \\
& \omega \le 0, \;\; \mu \ge 0,
\end{aligned}
$$

where the symmetric matrix $A(\lambda, \upsilon, \omega, \mu, \alpha)$ equals

$$
\begin{pmatrix}
-\left(\sum_{k=1}^{m} \lambda_k b^k\right) & \frac{1}{2}\left(c^0 + \sum_{k=1}^{m} \lambda_k c^k + \sum_{p=1}^{q} \upsilon_p a^p + \omega + \mu + \alpha\right)^T \\
\ldots & Q^0 + \sum_{k=1}^{m} \lambda_k Q^k - \operatorname{diag}(\alpha)
\end{pmatrix}.
$$

By weak duality, we have

$$
v_{l0}^{DBL} \le v_{l0}^{BL} \le v_0^{SLG}, \quad \forall \lambda .
$$

Thus, $v^{DBL} \le v_0^{LG}$, where v^{DBL} is as follows:

$$
\begin{aligned}
\text{(DBL)} \quad v^{DBL} = \max_{\lambda,\upsilon,\omega,\mu,\alpha} \; & -\sum_{p=1}^{q} \upsilon_p d^p - \mu^T e \\
\text{s.t.} \quad & A(\lambda, \upsilon, \omega, \mu, \alpha) \succeq 0 \\
& \lambda \ge 0, \omega \le 0, \;\; \mu \ge 0.
\end{aligned}
$$

The dual of (DBL) is:

$$
\begin{aligned}
\text{(BL)}\quad v^{BL} = \min\nolimits_{x,X'}\ & Q^0 \bullet X' + (c^0)^T x \\
\text{s.t.}\quad & Q^k \bullet X' + (c^k)^T x \le b^k, \quad k = 1, \dots, m \\
& (a^p)^T x = d^p, \quad p = 1, \dots, q \\
& 0 \le x \le e \\
& \operatorname{diag} X' = x \\
& X' \succeq xx^T.
\end{aligned}
$$

With the Slater Condition, we have $v^{BL} = v^{DBL}$. Therefore:

$$v^{BL} \le v_0^{LG} \le f^*.$$

□

The validation of this proposition can also be shown by the equivalence between (BL) and (SD).

Proposition 13 *For the QCQP problem with* $\operatorname{diag}(Q^k) = 0,\ k = 0, 1, \dots, m$*, the SDP relaxation (BL) and the Shor relaxation with convex/concave envelopes for diagonal elements (SD) are equivalent. In other words,*

$$v^{SD} = v^{BL}.$$

Proof Assume (x, X') is a feasible solution of (BL). Since $\operatorname{diag} X' = x$ and $0 \le x \le e$, we have

$$
\begin{aligned}
&\operatorname{diag} X' \ge 0 \\
&\operatorname{diag} X' \le x \\
&\operatorname{diag} X' \ge 2x - e.
\end{aligned}
$$

Thus (x, X') is also feasible to (SD), and $v^{SD} \le v^{BL}$. On the other hand, assume (x, X) is a feasible solution of (SD). Since $X \succeq xx^T$, we have $\operatorname{diag} X \ge 0$. Therefore:

$$0 \le \operatorname{diag} X \le x \le e.$$

If $\operatorname{diag} X = x$, then (x, X) is also feasible to (BL), and $v^{SD} \ge v^{BL}$. Otherwise, let $z = x - \operatorname{diag} X \ge 0$ and $X' = X + \operatorname{diag} z$. Since $X \succeq xx^T$, we have

$$
\begin{aligned}
&\operatorname{diag} X' = x, \\
&X' - xx^T = X - xx^T + \operatorname{diag} z \succeq 0.
\end{aligned}
$$

Since $\operatorname{diag}(Q^k) = 0,\ k = 0, 1, \dots, m,\ (x, X')$ is feasible to (BL) and $Q^0 \bullet X + (c^0)^T x = Q^0 \bullet X' + (c^0)^T x$. Thus, $v^{SD} \ge v^{BL}$, which completes the proof. □

8 Conclusions

In this paper, we reviewed a number of existing and presented a number of new SDP relaxations for general QCQP problems. It was shown that relaxations with bounds on diagonal elements of the matrix variable X can provide fast approximations, while complete bounding of all the components of X is necessary for tight relaxations. Whereas much sharper than other relaxations, the (DNN)/(SRLT) relaxations are also time consuming for current SDP solvers. In a recent work, Burer [17] is addressing the development of a decomposition algorithm to solve bound-constrained (DNNs). When extended to general constraints, this approach is likely to provide tight bounds for QCQP problems at more realistic computing times compared to current SDP solvers. Towards the same end, future research should also address the development of approximate solution techniques for the dual of (DNN)/(SRLT). At the time of this writing the computational results with (DLG1), an SDP relaxation derived from Lagrangian relaxation after the addition of redundant nonlinear constraints to the primal, suggest that this relaxation may achieve the best balance between gap quality and solvability with current SDP solvers. Comparisons of these relaxations in the context of a branch-and-bound algorithm should be addressed by future research.

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