SURROGATE CONSTRAINTS

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A surrogate constraint is an inequality implied by the constraints of an integer program, and designed to capture useful information that cannot be extracted from the parent constraints individually but is nevertheless a consequence of their conjunction. The use of such constraints as originally proposed by the author has recently been extended in an important way by EGON BALAS and ARTHUR GEOFFRION. Motivated to take advantage of information disregarded in previous definitions of surrogate constraint strength, we build upon the results of Balas and Geoffrion to show how to obtain surrogate constraints that are strongest according to more general criteria. We also propose definitions of surrogate constraint strength that further extend the ideas of the author in 1965 by means of 'normalizations,' and show how to obtain strongest surrogate constraints by reference to these definitions also.

A SURROGATE constraint is an inequality implied by the constraints of an integer program, and designed to capture useful information that cannot be extracted from the parent constraints individually but is nevertheless a consequence of their conjunction. The use of such constraints has recently been extended in an important way by EGON BALAS[2] and ARTHUR GEOFFRION.[5] By modifying the definition of surrogate constraint strength given in reference 6, they have shown how a strongest surrogate constraint can be obtained (according to their respective criteria) by solving a linear program.

A limiting feature of the definition of surrogate constraint strength proposed in reference 6, and also of the modified definitions suggested by Balas and Geoffrion, is an implicit presupposition that information is to be extracted from these constraints only by reference to the lower and upper bounds on the problem variables (assumed to be 0 and 1). However, other restrictions on the problem variables have been shown to yield useful information via the approach of the Multiphase Dual Algorithm.[6, 7] Thus we are motivated to build upon the results of Balas and Geoffrion to prescribe strongest surrogate constraints that accommodate additional restrictions. We also propose definitions of surrogate constraint strength that further extend the ideas of reference 6 by means of 'normalizations,' and show how to obtain strongest surrogate constraints by reference to these definitions also.

In this note we represent the 0–1 integer programming problem as

Maximize cx subject to $Ax \leqq b$, $x \leqq e$, and x integer,

$$x \geqq 0, \tag{1}$$

where $x=(x_1, x_2, \cdots, x_n)^T$, $e=(1, 1, \cdots, 1)^T$, $c=(c_1, c_2, \cdots, c_n)$, $b=(b_1, b_2, \cdots, b_m)$, and $A=(a_{ij})$ for $i=1, \cdots, m$ and $j=1, \cdots, n$. A surrogate constraint is defined to be any inequality $\sum a_j x_j \leqq b_0$ implied by the constraints of (1) (including $x \geqq 0$ and integer), where $Ax \leqq b$ is permitted to include additional inequalities that may be introduced as part of a strategy for solving (1).† In particular, $Ax \leqq b$ is usually assumed to include $-cx \leqq -c_0-1$, where c_0 is the value of cx for the best known feasible discrete solution (we assume the c_j are all integers). Throughout this paper, however, we will be concerned with surrogate constraints obtained as a nonnegative linear combination of the given problem constraints, that is, $ax \leqq b_0$ will be defined by $a=wA$ and $b_0=wb$, where w is a nonnegative m vector.

In reference 6, the strength of a surrogate constraint is defined as follows.

Definition 1. The surrogate constraint $a^*x \leqq b_0^*$ (obtained from $w=w^*$) is stronger than the surrogate constraint $a'x \leqq b_0'$ (obtained from $w=w'$) if $\max_{x\geq 0}\{cx$ subject to $a^*x \leqq b_0^*$, $x \leqq e$ and x integer$\}$ is smaller than $\max_{x\geq 0}\{cx$ subject to $a'x \leqq b_0'$, $x \leqq e$ and x integer$\}$.

A theorem of reference 6 shows how to obtain the strongest surrogate constraints according to Definition 1 when restricting attention to nonnegative linear combinations of two inequalities, and the procedure embodied in this theorem is extended to obtain surrogate constraints as linear combinations of more than two inequalities, although the resulting constraint is not necessarily strongest.

Because of the difficulty of determining strongest surrogate constraints according to Definition 1 when there are more than two parent constraints, Egon Balas and Arthur Geoffrion have proposed other similar but more useful definitions. Balas' definition is

Definition 2. The surrogate constraint $a^*x \leqq b_0^*$ is stronger than the surrogate constraint $a'x \leqq b_0'$ if it is stronger by Definition 1 when the requirement "x is integer" is dropped.

Geoffrion's definition of surrogate constraint strength insists that $-cx \leqq -c_0-1$ be given a unit weight and excluded from $Ax \leqq b$. Observing this convention about the form of $Ax \leqq b$, Geoffrion's definition is

Definition 3. The surrogate constraint $(a^*-c)x \leqq b_0^*-c_0-1$ is stronger than the surrogate constraint $(a'-c)x \leqq b_0'-c_0-1$ if $\max_{x\geq 0}\{b_0^*-(a^*-c)x$

† We also allow for (1) to represent an updated form of some original problem in which some of the variables have been assigned specific values by an algorithm such as 1, 2, 3, 6, and 7.

subject to $x \leqq e$ (and x integer)} is smaller than $\max_{x \geqq 0}\{b_0' - (a' - c)x$ subject to $x \leqq e$ (and x integer)}.

We have placed the words "and x integer" in quotations since the definition is unchanged with this stipulation removed. A correspondence between Definition 1 and Definition 3 may be observed as follows. A strongest surrogate constraint by Definition 1 is determined by finding a w to $\text{minimize}_{x \geqq 0}\{\max_{x_j=0 \text{ or } 1} cx$ subject to $b_0 - ax \geqq 0\}$, and by Definition 3 by finding a w to $\text{minimize}_{w \geqq 0}\{\max_{x_j=0 \text{ or } 1} cx + (b_0 - ax)\}$ where Geoffrion adds the inequality $-cx \leqq -c_0 - 1$ to $ax \leqq b_0$. The second minimax problem may clearly be viewed as an approximation to the first in the context of the Lagrange multiplier technique. The approximation becomes precise if the integer requirement of Definition 1 is dropped as in Balas' Definition 2. In fact, if $ax \leqq b_0$ is a strongest surrogate constraint by Definition 2, then $(a-c)x \leqq b_0 - c_0 - 1$ is a strongest surrogate constraint by Definition 3, and conversely. (That is, Balas' relaxed minimax problem and Geoffrion's relaxed minimax problem are equivalent.) Nevertheless, Balas' and Geoffrion's definitions of surrogate constraint strengths are different, and were motivated by different considerations (*see* below).

To state the results due to Balas and Geoffrion we write the dual of (1) (interpreted as a linear program) in the form

$$\text{minimize}_{u,w \geqq 0}\, wb + ue \text{ subject to } wA + u \geqq c. \tag{2}$$

Let w^2, u^2 denote an optimal solution to (2).

THEOREM D2. (Balas). *A surrogate constraint that is strongest in the sense of Definition 2 is obtained by letting $w = w^2$. Moreover, the value of cx in an optimal* continuous *solution to* (1) *is unchanged if $Ax \leqq b$ is replaced by $w^2Ax \leqq w^2b$.*

THEOREM D3. (Geoffrion). *A surrogate constraint that is strongest in the sense of Definition 3 is obtained by letting $w = w^2$ and adding $-cx \leqq -c_0 - 1$ to $w^2Ax \leqq w^2b$.*

The significance of these theorems† lies in the fact that $\max_{x \geqq 0}\{b_0 - ax$ subject to $x \leqq e\}$ and $\max_{x \geqq 0}\{cx$ subject to $ax \leqq b_0$ and $x \leqq e\}$ can readily be determined, and used to expedite the progress of a branch-and-bound algorithm.‡ The use of the former information was proposed by Balas in reference 1. Geoffrion[5] has found that using his surrogate constraint improves the efficiency of a branch-and-bound algorithm that is chiefly organized to exploit such information by a factor of 3 to 20. The use of the

† In reference 9 LINUS SCHRAGE is also credited with the observation that a strongest surrogate constraint can be obtained by solving (2).

‡ Geoffrion's Definition 3 is aimed at getting a good surrogate constraint for exploiting the former information (when $a = wA - c$, $b_0 = wb - c_0 - 1$), and Balas' Definition 2 is aimed at getting a good surrogate constraint for exploiting the latter. Also, Balas proposes using a single such constraint whereas Geoffrion uses several (as suggested in reference 6).

latter information was proposed in connection with the Multiphase Dual Algorithm and also suggested by BERTIER, NGHIEM, AND ROY.[3] Together with Balas' surrogate constraint theorem, it forms a key part of Balas' promising new Filter Algorithm.[2]

In seeking other definitions of surrogate constraint strength, we are motivated by the fact that surrogate constraints can be exploited efficiently not only by reference to $x \leqq e$, but also by reference to other inequalities, as, in particular, $U_0 \geqq e^T x \geqq L_0$[6] and $U_k \geqq \sum_{j \epsilon S_k} x_j \geqq L_k$, where the S_k are nested sets of indices.[7] We shall represent a set of 'exploiting' inequalities in matrix form by $Qx \leqq d$. Then we are in general interested in solving

$$\text{maximize}_{x \geqq 0}\, cx \text{ subject to } Ax \leqq b \quad \text{and} \quad Qx \leqq d, \tag{3}$$

where some or all of the components of x are constrained to integer values.

The dual of (3) interpreted as a linear program may be written

$$\text{minimize}_{w,\,u \geqq 0}\, wb + ud \text{ subject to } wA + uQ \geqq c. \tag{4}$$

An optimal solution to (3) (as a linear program) will be denoted by x^3 and an optimal solution to (4) by w^4, u^4.

We will first propose an immediate generalization of Balas' definition of surrogate constraint strength and then give a corresponding generalization of his theorem that provides a strongest constraint in the sense of the new definition. For the purpose of the discussion to follow, we assume that $a = wA + uQ$ and $b_0 = wb + ud$.

Definition 4. The surrogate constraint $a^* x \leqq b_0{}^*$ (for w, $u = w^*$, u^*) is stronger than $a'x \leqq b_0{}'$ (for w, $u = w'$, u') if $\max_{x \geqq 0} \{cx$ subject to $a^* x \leqq b_0{}^*$ and $Qx \leqq d\}$ is smaller than $\max_{x \geqq 0} \{cx$ subject to $a'x \leqq b_0{}'$ and $Qx \leqq d\}$.

THEOREM D4. *A strongest surrogate constraint in the sense of Definition 4 is obtained by setting $w = w^4$ and $u_i = u_i^4$ or 0 (as desired) for each component u_i of u. Moreover, the value of cx in an optimal continuous solution to* (3) *is unchanged by replacing $Ax \leqq b$ with the resulting strongest $ax \leqq b_0$.*

We need only prove D4 for the case $u = 0$, since we may assume that any subset of the constraints of $Qx \leqq d$ (with index set T, say) is also included in $Ax \leqq b$. Hence by setting $w = w^4$ and $u = 0$ relative to such an augmented A matrix we accomplish the same thing as by setting $w = w^4$ and $u_i = \delta_i u_i^4$ for a nonaugmented A matrix, where $\delta_i = 1$ if $i \epsilon T$ and 0 otherwise.

The theorem is proved most easily (for $u = 0$) by stating it in a slightly different form. Consider the dual problems

$$\text{maximize}_{x \geqq 0}\, cx \text{ subject to } a^4 x \leqq b^4 \quad \text{and} \quad Qx \leqq d; \tag{5}$$

$$\text{minimize}_{w_0,\,u_0 \geqq 0}\, w_0 b^4 + ud \text{ subject to } w_0 a^4 + uQ \geqq c, \tag{6}$$

where $a^4 = w^4 A$ and $b^4 = w^4 b$. Let x^5 denote an optimal solution to (5) and

w_0^6, u^6 denote an optimal solution to (6). Then an equivalent statement of Theorem D4 is

THEOREM D4′. $cx^3 = cx$.

Proof. Note that $w_0=1$, $u=u^4$ is feasible for (6). Thus $w_0^6b^4+u^6d \leqq w^4b+u^4d$. On the other hand, $w=w_0^6w^4$, $u=u^4$ is clearly feasible for (4), and hence the foregoing inequality also holds in the opposite direction, implying $cx^3=cx^5$ by the dual theorem of linear programming.†

We remark that an application of Theorem D4 that is particularly useful in the context of references 6 and 7 occurs by replacing cx with e^Tx.

We will now extend Geoffrion's result by similarly considering what happens when $Qx \leqq d$ replaces $x \leqq e$. However, we will go beyond this by also prescribing surrogate constraints that are strongest according to other kinds of definitions.

The generalized definition of surrogate constraint strength in Geoffrion's sense is

Definition 5. The surrogate constraint $(a^*-c)x \leqq b_0^*-c_0-1$ is stronger than $(a'-c)x \leqq b_0'-c_0-1$ if $\max_{x \geqq 0}\{b_0^*-(a^*-c)x$ subject to $Qx \leqq d$ (and x integer)$\}$ is smaller than $\max_{x \geqq 0}\{b_0'-(a'-c)x$ subject to $Qx \leqq d$ (and x integer)$\}$.

As before, we have stipulated "and x integer" in quotations since, for the particular inequalities $Qx \leqq d$ relevant for references 6 and 7, the definition is equivalent if x is allowed to be continuous.

It might be guessed from our foregoing remarks that Geoffrion's theorem D3 generalizes by replacing $w=w^2$ with $w=w^4$ and $u_i=u_i^4$ or 0, and this is true. Instead of proving this directly, we turn now to considerations that yield this result as a byproduct.

To motivate our discussion, let us examine Definition 5 from a different perspective. Instead of segregating $-cx \leqq -c_0-1$ from $Ax \leqq b$, assume that it is the first constraint of this matrix inequality. Definition 5 can then be seen to define a strongest surrogate constraint as one that always assigns w_1 the value 1 and then picks the remaining $w_i \geqq 0$ to minimize b_0-ax, where x is selected in turn to maximize this quantity subject to $Qx \leqq d$. However, instead of requiring $w_1=1$, it would sometimes seem more appealing to measure surrogate constraint strength by requiring a normalization such as $b_0=1$ or -1, thus reflecting the notion that $99 \leqq a^*x \leqq 100$ is a tighter inequality than $1 \leqq a'x \leqq 2$ (which is clearly true, for example, if $a^*=50a'$). One might also (or alternatively) require $wAe=k_0$ for some constant k_0,

† Theorem D4′ can also be viewed as a direct consequence of the sufficiency theorem for Lagrange multipliers. By invoking the strong complementary slackness theorem one can observe, in the spirit of Balas,[2] that there exists an optimal pair x^3, x^5 such that $x_j^3=0$ implies $x_j^5=0$ and $Q_ix^3=d_i$ implies $Q_ix^5=d_i$, (where Q_i is the ith row of Q).

which has the interpretation $\sum |a_j| = k_0$ if the coefficients a_{ij} all have the same sign.

To permit ourselves flexibility, we will in general express a 'desirable normalization' by the matrix inequality $wP \geqq h$. We will also allow $Ax \leqq b$ to include some of the constraints of $Qx \leqq d$, and stipulate that the surrogate constraint $ax \leqq b_0$ be given by $a = wA$ and $b_0 = wb$.†

Definition 6. Given the normalization $wP \geqq h$ satisfied by $w = w^*$ and $w = w'$, the surrogate constraint $a^*x \leqq b_0^*$ is stronger than $a'x \leqq b_0'$ if $\max_{x \geqq 0}$ $\{b_0^* - a^*x$ subject to $Qx \leqq d\}$ is smaller than $\max_{x \geqq 0} \{b_0' - a'x$ subject to $Qx \leqq d\}$.

To obtain a strongest surrogate constraint according to Definition 6, note that we seek a vector w to

$$\min_{\substack{w \geqq 0 \\ wP \geqq h}} \max_{\substack{x \geqq 0 \\ Qx \leqq d}} wb - wAx. \tag{7}$$

The expression (7) is closely related to that of a constrained game, and may be expressed as a linear program by an essentially analogous procedure to that given by CHARNES,[4] provided the proper assumptions are accommodated. We give these assumptions and their implications in the next theorem.

THEOREM D6. *If $\{x \geqq 0 : Qx \geqq d\}$ is nonempty and bounded, then w is optimal for (7) (and hence gives a strongest surrogate constraint by Definition 6) if and only if w is optimal for the linear program*

$$\textit{minimize}_{w,\, u \geqq 0}\ wb + ud \textit{ subject to } wA + uQ \geqq 0 \textit{ and } wP \geqq h. \tag{8}$$

Moreover, if there is a finite feasible optimum for (7) [or (8)], then

$$\min_w (\max_x) = \max_x (\min_w).$$

Proof:‡ To prove w is optimal for (7) if and only if u is optimal for (8):

$$\min_{\substack{w \geqq 0 \\ wP \geqq h}} [wb + \max_{\substack{x \geqq 0 \\ Qx \leqq d}} -wAx] = \min_{\substack{w \geqq 0 \\ wP \geqq h}} [wb + \min_{\substack{u \geqq 0 \\ uQ \geqq -wA}} ud]$$

$$= \min_{\substack{w,\, u \geqq 0 \\ wP \geqq h \\ wA + uQ \geqq 0}} [wb + ud].$$

To prove

$$\min_w (\max_x) = \max_x (\min_w):$$

the dual of (8) is

† We nevertheless retain the preference for excluding $Qx \leqq d$ from $Ax \leqq b$ so that the solution set $\{x | ax \leqq b_0, Qx \leqq b \text{ and } x \geqq 0\}$ will be as small as possible for the surrogate constraints defined to be strongest.

‡ I am indebted to EGON BALAS and ARTHUR GEOFFRION for suggesting the present concise form of this proof.

$$\begin{aligned}\max_{\substack{y,x\geqq 0\\ Py+Ax\leqq b\\ Qx\leqq d}} hy &= \max_{\substack{x\geqq 0\\ Qx\leqq d}}\,[\max_{\substack{y\geqq 0\\ Py\leqq b-Ax}} hy]\\ &= \max_{\substack{x\geqq 0\\ Qx\leqq d}}\,[\min_{\substack{w\geqq 0\\ wP\geqq h}} wb-wAx].\end{aligned} \tag{9}$$

It may be remarked that the assumption of the theorem that $x\geqq 0$, $Qx\leqq d$ implies x is finite is consistent with the kinds of inequalities that are generally exploited in reference 7, and, of course, immediately accords with $x\leqq U$ and $e^T x\leqq U_0$.

Given this property of $Qx\leqq d$, $wP\geqq h$ can represent any of the normalizations expressed by $\sum_{i\epsilon S} k_i w_i = k_0$ and $\sum_{i\notin S} h_i w_i = h_0$, where S is any subset (possibly empty) of $\{1, 2, \cdots, m\}$ and $k_0 k_i > 0$ for some $i\epsilon S$ and $h_0 h_i > 0$ for some $i\notin S$. Using such normalizations, (8) will be assured of a finite feasible solution whenever problem (3) has a feasible solution. (Of course, S may also be replaced by several disjoint sets.) Specific instances of the foregoing are Geoffrion's $w_1 = 1$ and the suggested $wb = 1$ and $wb = -1$ (respectively if $b\not\leq 0$ and $b\not\geq 0$). It may also be seen that if $wP\geqq h$ is $w_1 = 1$ then (8) is precisely the dual of (3), and Theorem D6 thus implies a generalized version of Geoffrion's Theorem D3.

It is perhaps worthwhile to point out that Definition 6 can itself be generalized by allowing the normalization $wP\geqq h$ to be replaced by $wP+vM\geqq h$ for w, $v\geqq 0$, and the inequality $Qx\leqq d$ to be replaced by $Qx+Rz\leqq d$ for x, $z\geqq 0$. (The new inequalities can simultaneously include the old inequalities plus others.) Theorem D6 then correspondingly generalizes by replacing (8) and (9) with

$$\begin{aligned}&\text{minimize}_{v,w,u\geqq 0}\, wb+ud \text{ subject to } wA+uQ\geqq 0,\\ &\qquad\qquad wP+vM\geqq h, \quad\text{and}\quad uR\geqq 0,\end{aligned} \tag{8'}$$

$$\begin{aligned}&\text{maximize}_{x,y,z\geqq 0}\, hy \text{ subject to } Ax+Py\leqq b,\\ &\qquad\qquad Qx+Rz\leqq d, \quad\text{and}\quad My\leqq 0,\end{aligned} \tag{9'}$$

where we now assume that $x\geqq 0$, $z\geqq 0$, and $Qx+Rz\leqq d$ implies both x and z are finite.

An application for the generalized version of Theorem D6 occurs, for example, when it is desired to obtain a surrogate constraint only from some of the rows of A (as when these rows have all nonnegative coefficients and the normalization $\sum |a_j| = k_0$ is desired). Then write A as $\begin{pmatrix} A \\ M\end{pmatrix}$, where the new A consists of the rows from which the surrogate constraint is to be obtained. In order to reflect the influence of the remaining part (M) of the original A matrix inequalities a normalization involving v may be used such as $ve=1$. The generalized version of Theorem D6 then prescribes a strongest surrogate constraint relative to this normalization.

Our final definition of surrogate constraint strength is based on the idea that it may sometimes be useful to replace a min (max) objective with a min (Expected Value) objective.

Definition 7. Given the normalization $wP\geqq h$ satisfied by $w=w^*$ and $w=w'$, the surrogate constraint $a^*x\leqq b_0^*$ is stronger than $a'x\leqq b_0'$ if $E(b_0^*-a^*x)<E(b_0'-a'x)$, where E denotes expected value.

To apply this definition, we assume for each j that $0\leqq x_j\leqq U_j$ and probabilities $p_j^k=\text{pr}\ (x_j=k)$ have been assigned for $k=0, 1, \cdots, U_j$.†

Let g be the column n-vector whose jth component is $g_j=\sum_{k=1}^{U_j} p_j^k$. Then we may write $E(b_0-ax)=b_0-ag$. Thus to obtain a strongest surrogate constraint in the sense of Definition 7, we wish to solve the linear program

$$\text{minimize}_{w\geqq 0}\, w(b-Ag) \text{ subject to } wP\geqq h. \tag{10}$$

Note that if it is desired to assure $E(-cx)\leqq -c_0-1$ it is reasonable to assign the constraint $-cx\leqq -c_0-1$ a relatively large weight. For example, one possibility for $wP\geqq h$ is a set of constraints of the form $w_1=k_1>0$, $w_i\leqq k_i$ for $i>1$, and $wb\leqq k_0$ (or $\geqq K_0$). Then problem (10) is simply a knapsack problem without integer requirements on the variables, and is quickly solved by taking ratios (*see*, e.g., reference 6).

REFERENCES

1. EGON BALAS, "An Additive Algorithm for Solving Linear Programs with Zero-One Variables," *Opns. Res.* **13,** 517–545 (1965).
2. ———, "Discrete Programming by the Filter Method with Extension to Mixed-Integer Programming and Application to Machine-Sequencing," *ICC Report N. 66/10,* International Computation Center, Centre International De Calcul, November, 1966.
3. P. BERTIER, PH. T. NGHIEM, AND BERNARD ROY, "Programmes lineaires en nombres entiers," *Metra.* **4,** No. 3 (1965).
4. A. CHARNES, "Constrained Games and Linear Programming," *Proc. National Academy of Sciences* (July, 1953).
5. ARTHUR M. GEOFFRION, "Implicit Enumeration Using an Imbedded Linear Program," *Working Paper No.* **120**, Western Management Science Institute, May, 1967.
6. FRED GLOVER, "A Multiphase-Dual Algorithm for the Zero-One Integer Programming Problem," *Opns Res.* **13,** 879–919 (1965).
7. ———, "The Multiphase-Dual Algorithm for Pure and Mixed Integer Programming" (in preparation).

† *See* reference 7 for results bearing on the assignment of such probabilities and other connections with strongest surrogate constraints in a nondeterministic context.

8. ———, "A Note on Some Stochastic Approaches to Nonstochastic Discrete Programming," *Management Sci. Res. Memo*, Graduate School of Indus. Admin., Carnegie Mellon University, Aug., 1967.
9. S. Woiler, "Implicit Enumeration Algorithms for Discrete Optimization Problems," Technical Report, Department of Industrial Engineering, Stanford University, May 1967.