MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 1 Introduction to Operational Research and Optimization

Lecturer: E. Alper Yıldırım *Week: 1*

1.1 Outline

- What is Operational Research (OR)?
- What is Optimization?
- Mathematical Model and Terminology
- Review Problems

1.2 Operational Research and Optimization

Definition 1.1 Operational Research (OR) *is a branch of applied mathematics that employs analytical models and tools for solving decision problems arising from many different fields such as energy, healthcare, engineering, finance, machine learning, and transportation.*

Definition 1.2 Optimization *is concerned with finding the best solution of a given decision problem among a set of candidate solutions.*

Remark 1.1 *Optimization is one of the fundamental analytical tools in operational research.*

Other tools of OR include dynamic programming, stochastic modelling, simulation, queueing theory, and inventory control models.

1.2.1 Ingredients of an Optimization Problem

An optimization problem consists of the following four ingredients:

- A set of controllable inputs: Set of quantities whose best values we wish to compute (Decision Variables)
- A set of uncontrollable inputs: Set of quantities whose values cannot be changed (Parameters)
- A description of the set of candidate solutions: Usually expressed by some functional relations (Feasible Region)
- A measure of the *goodness* of a candidate solution: Usually measured by a real-valued function (Objective Function)

1.2.2 An Example

Problem 1.3 *What is the shortest (fastest) route from the King's Buildings to the Edinburgh Airport?*

- Decision variables: Which route to choose
- Parameters: Length (travel time) of each route
- Candidate solutions: Set of all possible routes from the King's Buildings to the Edinburgh Airport
- Objective function: Length (travel time) of the route from the King's Buildings to the Edinburgh Airport

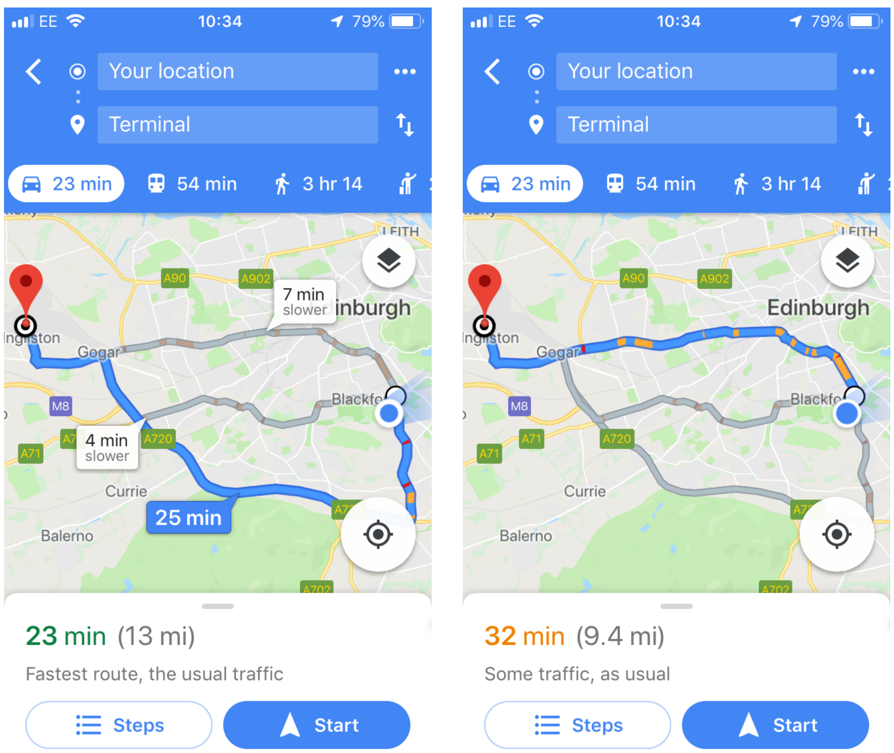

Figure 1.1: The fastest (on the left) and the shortest route (on the right) from King's Buildings to the Edinburgh Airport

As illustrated by Figure 1.1, the best solution depends on the objective function, and one may get different best solutions for different objective functions.

Optimization problems arise in numerous applications, some of which are listed below:

- Manufacturing (cost minimization/profit maximization)
- Wireless networks (throughput maximization/interference minimization)
- Finance (portfolio optimization)

- Physical systems (energy minimization/entropy maximization)
- Machine learning (classification error minimization)
- Transportation and logistics (cost/emission minimization)
- Medicine (treatment optimization/drug design/personalized medicine)
- Many other applications

1.2.3 Mathematical Model and Representation

The mathematical model of an optimization problem consists of the following four ingredients:

- **Decision variables:** Set of quantities whose best values we wish to compute (usually denoted by $x \in \mathbb{R}^n$)
- **Parameters:** Needed to identify the feasible region and the objective function
- **Feasible Region:** $x \in \mathcal{S}$, where $\mathcal{S} \subseteq \mathbb{R}^n$
- **Objective function:** $f : \mathbb{R}^n \to \mathbb{R}$

An optimization problem can therefore be expressed as

$$(\text{P}) \quad \min \{f(x) : x \in \mathcal{S}\}$$

or alternatively as

$$\begin{array}{lll} (\text{P}) & \min & f(x) \\ & \text{s.t. (subject to)} & \\ & & x \in \mathcal{S} \end{array}$$

The first notation is more compact and is, in fact, a set notation. The latter one is more explicit. However, both representations are equivalent and will be used interchangeably throughout the course.

It is customary to use minimization (as opposed to maximization) in optimization problems. However, from a mathematical point of view, one can be converted into the other one easily since

$$f(x^1) \geq f(x^2) \quad \text{if and only if} - f(x^1) \leq -f(x^2),$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $x^1 \in \mathbb{R}^n$, and $x^2 \in \mathbb{R}^n$. Therefore, by negating a function, the ranking of all objective function values is reversed. It follows that we may always consider a minimization problem without loss of generality. This is stated more formally in the following remark.

Remark 1.2 *Maximizing $g(x)$ is equivalent to minimizing $-g(x)$. Therefore, every maximization problem can be converted to an equivalent minimization problem by simply negating the objective function.*

1.2.4 Terminology

Consider an optimization problem given by

$$(\text{P}) \quad \min \{f(x) : x \in \mathcal{S}\}$$

- The set $\mathcal{S}$ of candidate solutions is called the *feasible region* or the *feasible set*.
- Any solution $\hat{x} \in \mathcal{S}$ is called a *feasible solution*.
- A feasible solution $x^* \in \mathcal{S}$ is called an *optimal solution* of (P) if

$$f(x^*) \leq f(\hat{x}), \quad \forall \hat{x} \in \mathcal{S}.$$

  We replace $\leq$ by $\geq$ for a maximization problem.

- The set of all optimal solutions of (P) is denoted by $\mathcal{S}^*$, i.e.,

$$\mathcal{S}^* = \{x^* \in \mathcal{S} : f(x^*) \leq f(\hat{x}), \quad \forall \hat{x} \in \mathcal{S}\}$$

- $z^* \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is the *optimal value* of (P) if $z^* \leq f(\hat{x})$ for all $\hat{x} \in \mathcal{S}$ and there exists a sequence $x^k \in \mathcal{S},\ k = 1, 2, \ldots$ such that $f(x^k) \to z^*$ as $k \to \infty$ (i.e., for a minimization problem, the optimal value is the largest lower bound on the objective function values of all feasible solutions, known as the *infimum* of the objective function values of all feasible solutions).
- If $\mathcal{S}^* \neq \emptyset$, then $z^* = f(x^*)$ for any $x^* \in \mathcal{S}^*$. In this case, we say that the optimal value is *attained*, i.e., there is at least one feasible solution whose objective function value is the same as the optimal value.
- (P) is said to be an *unbounded problem* if there exists a sequence $x^k \in \mathcal{S},\ k = 1, 2, \ldots$ such that $f(x^k) \to -\infty$. In this case, $\mathcal{S}^* = \emptyset$, and we define $z^* = -\infty$. (For a maximization problem, we simply replace $-\infty$ by $+\infty$.)
- If $\mathcal{S} = \emptyset$, then (P) is said to be an *infeasible problem*. In this case, $\mathcal{S}^* = \emptyset$, and we define $z^* = +\infty$. (For a maximization problem, we simply replace $+\infty$ by $-\infty$.)
- If (P) is neither infeasible nor unbounded, then $-\infty < z^* < +\infty$. In this case, $\mathcal{S}^*$ may still be the empty set if there is no feasible solution $\hat{x} \in \mathcal{S}$ such that $f(\hat{x}) = z^*$.

Remarks

1. Every optimization problem has an optimal value $z^* \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, even if it is infeasible or unbounded. Note that $-\infty$ and $+\infty$ are **not** real numbers. The set $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is referred to as the *extended real numbers*.
2. For some optimization problems, the optimal value may be finite (i.e., a real number), but it may not be attained. Here is a simple example:

$$\min \left\{ \frac{1}{x} : x \geq 1 \right\}$$

   Note that $1/x \geq 0$ for each $x \geq 1$ and if we define $x^k = k,\ k = 1, 2, \ldots$, then $f(x^k) = 1/k \to 0$ as $k \to \infty$. Therefore, the optimal value is given by $z^* = 0$. However, no feasible solution achieves the optimal value. In this example, the optimal value is finite but is not attained.
3. By the previous example, the feasible region of an optimization problem may be unbounded and yet the optimal value may be finite (i.e., the optimization problem is not unbounded). Please pay attention to the difference between the unboundedness of an optimization problem and the unboundedness of its feasible region.

Exercises

Question 1.1 *What are the four main ingredients in an optimization problem?*

Question 1.2 *Consider the following optimization problem:*

$$\min\{x : x \geq 1\}$$

1. *What are the decision variables?*
2. *What is the objective function?*
3. *What is the feasible region?*
4. *What is the optimal value?*
5. *Find the set of all optimal solutions.*

Question 1.3 *Consider the following optimization problem:*

$$\min\{0 : x \geq 1\}$$

1. *What are the decision variables?*
2. *What is the objective function?*
3. *What is the feasible region?*
4. *What is the optimal value?*
5. *Find the set of all optimal solutions.*

Question 1.4 *Consider the following optimization problem:*

$$\min\{-x : x \geq 1\}$$

1. *What are the decision variables?*
2. *What is the objective function?*
3. *What is the feasible region?*
4. *What is the optimal value?*
5. *Find the set of all optimal solutions.*

Question 1.5 *Consider the following optimization problem:*

$$\min\left\{\frac{1}{x} : x \geq 1\right\}$$

1. *What are the decision variables?*
2. *What is the objective function?*
3. *What is the feasible region?*
4. *What is the optimal value?*
5. *Find the set of all optimal solutions.*

Question 1.6 *Consider the following optimization problem:*

$$\min \left\{x : x^2 \leq -1\right\}$$

1. *What are the decision variables?*
2. *What is the objective function?*
3. *What is the feasible region?*
4. *What is the optimal value?*
5. *Find the set of all optimal solutions.*

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Lecture 2 — Constrained and Unconstrained Optimization and Introduction to Linear Programming

Lecturer: E. Alper Yıldırım | *Week: 1*

2.1 Outline

- Constrained vs Unconstrained Optimization
- Linear Functions
- Linear Programming
- Review Problems

2.2 Constrained vs Unconstrained Optimization Problem

Recall our generic optimization problem:

$$\text{(P)} \quad \min \{f(x) : x \in \mathcal{S}\},$$

where $x \in \mathbb{R}^n$ denotes the decision variables, $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function and $\mathcal{S} \subseteq \mathbb{R}^n$ is the feasible region.

If $\mathcal{S} = \mathbb{R}^n$, then (P) is called an *unconstrained optimization problem*. In this case, any $x \in \mathbb{R}^n$ is a feasible solution.

Otherwise, $\mathcal{S} \subset \mathbb{R}^n$, i.e., $\mathcal{S}$ is a proper subset of $\mathbb{R}^n$ and (P) is called a *constrained optimization problem*. In this case, the decision variable x is *constrained* to belong to a smaller set $\mathcal{S}$.

2.3 Constrained Optimization

In constrained optimization, the feasible region $\mathcal{S}$ is usually expressed by functional relations:

$$\mathcal{S} = \{x \in \mathbb{R}^n : g_i(x) \geq b_i,\ i \in M_1; \quad \ell_i(x) \leq b_i,\ i \in M_2; \quad h_i(x) = b_i,\ i \in M_3\},$$

where M_1, M_2, and M_3 are finite index sets, some of which may possibly be empty; $g_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_1$; $\ell_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_2$; $h_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_3$; and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$.

Each of the functional relations $g_i(x) \geq b_i,\ i \in M_1$; $\ell_i(x) \leq b_i,\ i \in M_2$; $h_i(x) = b_i,\ i \in M_3$ is called a *constraint*. Note that, for a given $\hat{x} \in \mathbb{R}^n$, $\hat{x} \in \mathcal{S}$ if and only if $\hat{x}$ satisfies *all* of the above constraints *simultaneously*. On the other hand, $\hat{x} \notin \mathcal{S}$ if and only if $\hat{x}$ violates at least one of these constraints.

In the constraints, we generally prefer to have weak inequalities (i.e., $\leq$ and $\geq$) as opposed to strict inequalities (i.e., $<$ and $>$) since the feasible region $\mathcal{S}$ may otherwise not contain its boundary points (i.e., may not be a closed set).

Example 2.1. *If $\mathcal{S} = \{x \in \mathbb{R} : x > 1\}$, then $1 \notin \mathcal{S}$. If we minimize x over $\mathcal{S}$, then the optimal value is given by $z^* = 1$ but $\mathcal{S}^* = \emptyset$ since it is not attained. Therefore, the optimal value is finite but not attained.*

2.3.1 Representation of Constrained Optimization Problems

A constrained optimization problem can be represented using the more compact set notation:

$$(P) \quad \min\left\{f(x) : g_i(x) \geq b_i,\ i \in M_1; \quad \ell_i(x) \leq b_i,\ i \in M_2; \quad h_i(x) = b_i,\ i \in M_3\right\},$$

where $M_1, M_2,$ and M_3 are finite index sets and each of $g_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_1$, $\ell_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_2$, and $h_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_3$ is a constraint, and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$.

Alternatively, we may use a more explicit representation:

$$\begin{array}{lllccl} \text{(P)} & \min & f(x) & & & \\ & \text{s.t.} & & & & \\ & & g_i(x) & \geq & b_i, & i \in M_1, \\ & & \ell_i(x) & \leq & b_i, & i \in M_2, \\ & & h_i(x) & = & b_i, & i \in M_3, \end{array}$$

Note that $\hat{x} \in \mathcal{S}$ (i.e., a feasible solution of (P)) if and only if $g_i(\hat{x}) \geq b_i$ **for each** $i \in M_1$ **and** $\ell_i(\hat{x}) \leq b_i$ **for each** $i \in M_2$ **and** $h_i(x) = b_i$ **for each** $i \in M_3$ (i.e., $\hat{x}$ should satisfy each and every constraint). Therefore, by adding more constraints to (P), the feasible region $\mathcal{S}$ cannot get larger (i.e., either remains unchanged or shrinks).

2.4 Linear Functions

Definition 2.1. *A function $\rho : \mathbb{R}^n \to \mathbb{R}$ is a linear function if*

1. *for all $x \in \mathbb{R}^n$ and for all $y \in \mathbb{R}^n$, $\rho(x + y) = \rho(x) + \rho(y)$;*
2. *for all $x \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{R}$, $\rho(\alpha x) = \alpha\rho(x)$.*

Remark 2.1. *For every linear function $\rho : \mathbb{R}^n \to \mathbb{R}$, we have $\rho(\mathbf{0}) = 0$ by Property 2, where $\mathbf{0} \in \mathbb{R}^n$ denotes the n-dimensional vector of all zeroes.*

2.4.1 Characterisation of Linear Functions

The next proposition gives a complete characterisation of linear functions.

Proposition 2.1. *A function $\rho : \mathbb{R}^n \to \mathbb{R}$ is a linear function if and only if there exists a vector $a \in \mathbb{R}^n$ such that*

$$\rho(x) = a^T x = \sum_{j=1}^{n} a_j x_j.$$

Proof. $\Leftarrow$: If $\rho(x) = a^T x$, then $\rho(x + y) = a^T(x + y) = a^T x + a^T y = \rho(x) + \rho(y)$ for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Similarly, $\rho(\alpha x) = a^T(\alpha x) = \alpha a^T x = \alpha\rho(x)$ for any $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

$\Rightarrow$: Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a linear function and let $x \in \mathbb{R}^n$. Then, $x = \sum_{j=1}^{n} x_j e^j$. Therefore, $\rho(x) = \rho\left(\sum_{j=1}^{n} x_j e^j\right) = \sum_{j=1}^{n} x_j \rho(e^j)$. Therefore, define $a_j = \rho(e^j),\ j = 1, \ldots, n$ and we are done. ☐

2.5 Linear Programming Problem

Recall our generic constrained optimization problem:

$$\begin{array}{lllcll} \text{(P)} & \min & f(x) & & & \\ & \text{subject to (s.t.)} & & & & \\ & & g_i(x) & \geq & b_i, & i \in M_1, \\ & & \ell_i(x) & \leq & b_i, & i \in M_2, \\ & & h_i(x) & = & b_i, & i \in M_3. \end{array}$$

Definition 2.2. *(P) is called a* linear programming problem *if each of* $f : \mathbb{R}^n \to \mathbb{R},\ g_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_1,$ $\ell_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_2$, *and* $h_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_3$ *is a linear function.*

By Proposition 2.1, $\rho : \mathbb{R}^n \to \mathbb{R}$ is a linear function if and only if there exists a vector $a \in \mathbb{R}^n$ such that $\rho(x) = a^T x = \sum_{j=1}^{n} a_j x_j$.

A linear programming problem can therefore be represented as follows:

$$\begin{array}{lllcll} \text{(P)} & \min & c^T x & & & \\ & \text{subject to (s.t.)} & & & & \\ & & (a^i)^T x & \geq & b_i, & i \in M_1, \\ & & (a^i)^T x & \leq & b_i, & i \in M_2, \\ & & (a^i)^T x & = & b_i, & i \in M_3, \end{array}$$

where $c \in \mathbb{R}^n$ and $a^i \in \mathbb{R}^n$ for each $i \in M_1 \cup M_2 \cup M_3$, and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$. Note that some of these index sets may possibly be empty.

Exercises

Question 2.1. *What is the difference between constrained and unconstrained optimization?*

Question 2.2. *Consider the constrained optimization problem* $\min\{f(x) : x \in \mathcal{S}\}$, *where* $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, *and* $\mathcal{S} \subset \mathbb{R}^n$ *is given by*

$$\mathcal{S} = \{x \in \mathbb{R}^n : g_i(x) \geq b_i,\ i \in M_1; \quad \ell_i(x) \leq b_i,\ i \in M_2; \quad h_i(x) = b_i,\ i \in M_3\}.$$

(i) How do the feasible region $\mathcal{S}$ *and the optimal value* z^* *change if we add a new constraint?*

(ii) How do the feasible region $\mathcal{S}$ *and the optimal value* z^* *change if we remove an existing constraint?*

Question 2.3. *For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$, determine whether f is a linear function or not.*

- $f(x) = 3x_1 - 4x_2$
- $f(x) = 3|x_1| - 4x_2$
- $f(x) = 3x_1^2 - 4x_2$
- $f(x) = \max\{3x_1, -4x_2\}$

Question 2.4. *For each of the following optimization problems, determine whether it is a linear programming problem or not.*

- $\min\{x_1^2 + x_2 : x_1 - x_2 \geq 5, \quad x_1 + 2x_2 = 3\}$
- $\min\{|x_1| - x_2 : 3x_1 + 2x_2 \leq -1, \quad x_1 \geq 0\}$
- $\min\{x_1 - x_2 : 3x_1 + 2x_2 \leq -1, \quad x_1^2 + x_2^2 \geq 1\}$
- $\min\{x_1 - 3x_2 : x_1 = 3, \quad x_2 = -2\}$

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 3	Convex Sets and Convex Functions
Lecturer: E. Alper Yıldırım	*Week: 1*

3.1 Outline

- Convex Sets
- Convex Functions
- Concave Functions
- Review Problems

3.2 Convex Sets

Definition 3.1. *Let* $\mathcal{C} \subseteq \mathbb{R}^n$*. We say that* $\mathcal{C}$ *is a* convex set *if* $\forall x \in \mathcal{C}$, $\forall y \in \mathcal{C}$, $\forall \lambda \in [0,1]$, *we have* $\lambda x + (1-\lambda) y \in \mathcal{C}$*. A set that is not convex is said to be* nonconvex.

Geometrically, a set $\mathcal{C} \subseteq \mathbb{R}^n$ is a convex set if and only if, for every two points $x \in \mathcal{C}$ and $y \in \mathcal{C}$, the line segment that joins these two points is entirely contained in $\mathcal{C}$. It is nonconvex if and only if there exist two points $x \in \mathcal{C}$ and $y \in \mathcal{C}$ such that the line segment that joins these two points is not entirely contained in $\mathcal{C}$. Figure 3.1 depicts an example of a convex set and a nonconvex set.

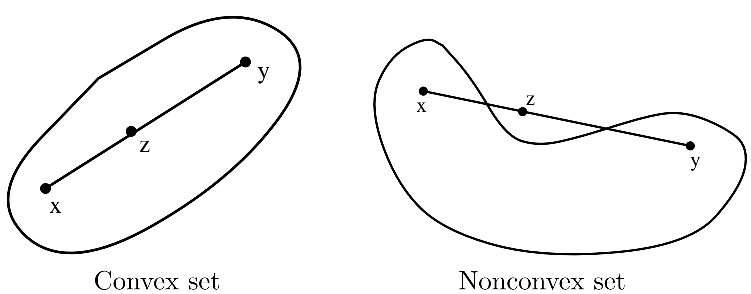

Figure 3.1: An example of a convex set (on the left) and a nonconvex set (on the right)

Remarks

1. Note that $\emptyset$ and $\mathbb{R}^n$ are both convex sets as the definition is trivially satisfied in both cases.
2. Any set $\mathcal{C} \subseteq \mathbb{R}^n$ that consists of a single element (i.e., $\mathcal{C} = \{\hat{x}\}$, where $\hat{x} \in \mathbb{R}^n$) is a convex set since the definition is trivially satisfied.

3. The intersection of any collection of convex sets is a convex set, i.e., convexity is preserved under taking intersections.
4. However, convexity is not necessarily preserved under taking unions.

3.3 Convex Functions

Definition 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is a* convex function *if $\forall x \in \mathbb{R}^n$, $\forall y \in \mathbb{R}^n$, $\forall \lambda \in [0, 1]$, we have*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

If $n = 1$, a function $f : \mathbb{R} \to \mathbb{R}$ is a convex function if and only if, for every two points $x \in \mathbb{R}$ and $y \in \mathbb{R}$, the line segment that joins the two points $(x, f(x)) \in \mathbb{R} \times \mathbb{R}$ and $(y, f(y)) \in \mathbb{R} \times \mathbb{R}$ lies on or above the graph of the function between x and y. It is a nonconvex function if and only if there exist two points $x \in \mathbb{R}$ and $y \in \mathbb{R}$ that violate this property. Figure 3.2 illustrates an example of a convex function and a nonconvex function.

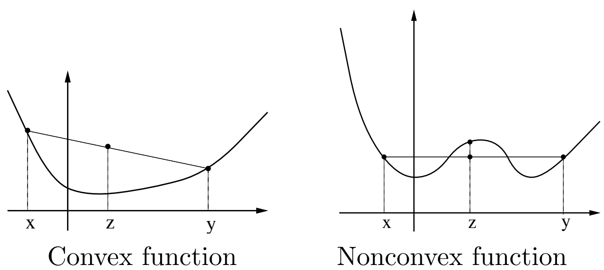

Figure 3.2: An example of a convex function (on the left) and a nonconvex function (on the right)

For a function $f : \mathbb{R}^n \to \mathbb{R}$, where $\cancel{n > 2}$ $n \geq 2$, one can define another function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(\lambda) = f(\lambda x + (1 - \lambda)y),$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Then, you can easily verify that the definition of convexity of the function $f : \mathbb{R}^n \to \mathbb{R}$ is equivalent to the convexity of the one-dimensional function $g : \mathbb{R} \to \mathbb{R}$ between 0 and 1 for every two points $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Therefore, Figure 3.3 essentially captures the definition of convexity for any value of n.

3.4 Epigraphs

Definition 3.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The* epigraph *of f, denoted by $epi(f) \subseteq \mathbb{R}^{n+1}$, is the region above the graph of the function f, i.e.,*

$$epi(f) = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \mathbb{R}^n, \quad z \geq f(x)\}.$$

Note that the epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of $\mathbb{R}^{n+1}$. For $n = 1$, this is easy to visualise as illustrated by Figure 3.3.

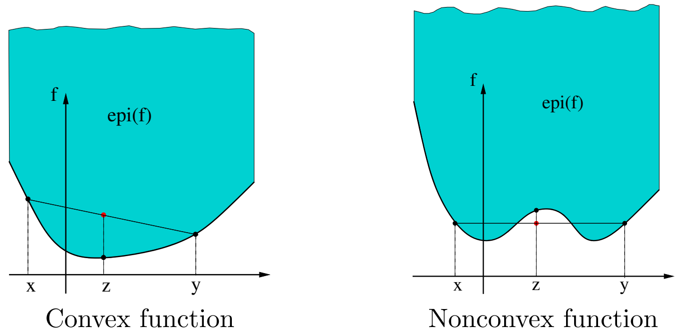

Figure 3.3: An example of the epigraph of a convex function (on the left) and the epigraph of a nonconvex function (on the right)

3.5 Convex Functions and Convex Sets

Proposition 3.1. *$f : \mathbb{R}^n \to \mathbb{R}$ is a convex function if and only if epi$(f) \in \mathbb{R}^{n+1}$ is a convex set.*

Proof. $\Rightarrow$: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $(x, z_1) \in \text{epi}(f)$ and $(y, z_2) \in \text{epi}(f)$. Therefore, $f(x) \leq z_1$ and $f(y) \leq z_2$. Then, for any $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \underbrace{\lambda}_{\geq 0} f(x) + \underbrace{(1-\lambda)}_{\geq 0} f(y) \leq \lambda z_1 + (1-\lambda)z_2.$$

Therefore, $(\lambda x + (1-\lambda)y, \lambda z_1 + (1-\lambda)z_2) = \lambda(x, z_1) + (1-\lambda)(y, z_2) \in \text{epi}(f)$, i.e., $\text{epi}(f) \in \mathbb{R}^{n+1}$ is a convex set.

$\Leftarrow$: Let $\text{epi}(f) \in \mathbb{R}^{n+1}$ be a convex set. For any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we have $(x, f(x)) \in \text{epi}(f)$ and $(y, f(y)) \in \text{epi}(f)$. Since $\text{epi}(f) \in \mathbb{R}^{n+1}$ is a convex set, for any $\lambda \in [0, 1]$, we have $\lambda(x, f(x)) + (1 - \lambda)(y, f(y)) = (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in \text{epi}(f)$. Therefore,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

i.e., $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. □

3.6 Concave Functions

Definition 3.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is a* concave function *if $\forall x \in \mathbb{R}^n$, $\forall y \in \mathbb{R}^n$, $\forall \lambda \in [0, 1]$, we have*

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

As opposed to convexity, note that concavity is only defined for functions. There is no analogous definition for sets. A set that is not convex is called a *nonconvex set.*

Remark 3.1. *$f : \mathbb{R}^n \to \mathbb{R}$ is a concave function if and only if $-f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.*

Exercises

Question 3.1. *For each of the following subsets of* $\mathbb{R}^2$*, determine whether it is a convex set or a nonconvex set.*

- $\mathcal{C} = \{x \in \mathbb{R}^2 : x_2 \leq |x_1|\}$
- $\mathcal{C} = \{x \in \mathbb{R}^2 : x_2 \leq |x_1|, \quad x_1 \geq 0\}$

Question 3.2. *For each of the following functions* $f : \mathbb{R}^2 \to \mathbb{R}$*, determine whether it is a convex function, concave function, both, or neither.*

- $f(x) = 3x_1 - 2x_2$
- $f(x) = 3|x_1| + 2x_2$
- $f(x) = -3|x_1| + 2|x_2|$

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 4 Level Sets, Sublevel and Superlevel Sets

Lecturer: E. Alper Yıldırım *Week: 1*

4.1 Outline

- Level Sets
- Hyperplanes
- Sublevel and Superlevel Sets
- Halfspaces
- Review Problems

4.2 Level Sets

Definition 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. For each $\alpha \in \mathbb{R}$, the level set of f is given by*

$$\mathcal{L}_\alpha(f) = \{x \in \mathbb{R}^n : f(x) = \alpha\}$$

Remarks

1. For every function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\alpha \in \mathbb{R}$, the level set of f given by $\mathcal{L}_\alpha(f)$ is a subset of the domain of the function, i.e., $\mathcal{L}_\alpha(f) \subseteq \mathbb{R}^n$. It is, in fact, the inverse image of α. The union of the sets $\mathcal{L}_\alpha(f)$ over all values of $\alpha \in \mathbb{R}$ is equal to $\mathbb{R}^n$, i.e.,
$$\bigcup_{\alpha \in \mathbb{R}} \mathcal{L}_\alpha(f) = \mathbb{R}^n.$$
2. For a given function $f : \mathbb{R}^n \to \mathbb{R}$, the *range* of f is given by the set of possible output values, i.e., the range of f is given by
$$\bigcup_{x \in \mathbb{R}^n} \{f(x)\} \subseteq \mathbb{R}.$$
3. For a given function $f : \mathbb{R}^n \to \mathbb{R}$ and a given $\alpha \in \mathbb{R}$, the level set $\mathcal{L}_\alpha(f)$ can be equal to the empty set. This is true if and only if α is not in the range of the function f.

4.2.1 Level Sets of Linear Functions

Definition 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function given by $f(x) = a^T x$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. For each $\alpha \in \mathbb{R}$, the level set of f given by*

$$\mathcal{L}_\alpha(f) = \left\{x \in \mathbb{R}^n : a^T x = \alpha\right\}$$

is called a hyperplane.

Remarks

1. If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = ax$, where $a \in \mathbb{R}\backslash\{0\}$, then $\mathcal{L}_\alpha(f)$ is a point on the real line for any $\alpha \in \mathbb{R}$ (zero degrees of freedom).
2. If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x) = a^T x$, where $a \in \mathbb{R}^2\backslash\{\mathbf{0}\}$, then $\mathcal{L}_\alpha(f)$ is a line in $\mathbb{R}^2$ for any $\alpha \in \mathbb{R}$ (one degree of freedom).
3. If $f : \mathbb{R}^3 \to \mathbb{R}$ is given by $f(x) = a^T x$, where $a \in \mathbb{R}^3\backslash\{\mathbf{0}\}$, then $\mathcal{L}_\alpha(f)$ is a plane in $\mathbb{R}^3$ for any $\alpha \in \mathbb{R}$ (two degrees of freedom).
4. More generally, for $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = a^T x$, where $a \in \mathbb{R}^n\backslash\{\mathbf{0}\}$, then $\mathcal{L}_\alpha(f)$ is a hyperplane with $n-1$ degrees of freedom.

4.2.2 Hyperplanes and Convexity

Proposition 4.1. *Every level set of a linear function is a convex set. Therefore, every hyperplane is a convex set.*

Proof. Let $\mathcal{H} \subset \mathbb{R}^n$ be a hyperplane. Then, there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = a^T x$, where $a \in \mathbb{R}^n\backslash\{\mathbf{0}\}$, and an $\alpha \in \mathbb{R}$ such that $\mathcal{H}$ is given by the level set of f:

$$\mathcal{H} = L_\alpha(f) = \left\{x \in \mathbb{R}^n : a^T x = \alpha\right\}.$$

For any $x^1 \in \mathcal{H}$, $x^2 \in \mathcal{H}$, and any $\lambda \in [0,1]$, $a^T(\lambda x^1 + (1-\lambda)x^2) = \lambda a^T x^1 + (1-\lambda)a^T x^2 = \lambda\alpha + (1-\lambda)\alpha = \alpha$. Therefore, $\lambda x^1 + (1-\lambda)x^2 \in \mathcal{H}$, which implies that $\mathcal{H}$ is a convex set. □

4.3 Sublevel Sets and Superlevel Sets

Definition 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. For each $\alpha \in \mathbb{R}$, the sublevel set of f is given by*

$$\mathcal{L}_\alpha^-(f) = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}.$$

Definition 4.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. For each $\alpha \in \mathbb{R}$, the superlevel set of f is given by*

$$\mathcal{L}_\alpha^+(f) = \{x \in \mathbb{R}^n : f(x) \geq \alpha\}.$$

Remarks

1. For a given $\alpha \in \mathbb{R}$, similarly to the level set of f given by $\mathcal{L}_\alpha(f)$, each of the sublevel set $\mathcal{L}_\alpha^-(f)$ and the superlevel set $\mathcal{L}_\alpha^+(f)$ is a subset of $\mathbb{R}^n$. For a given $\alpha \in \mathbb{R}$, the sublevel set $\mathcal{L}_\alpha^-(f)$ or the superlevel set $\mathcal{L}_\alpha^+(f)$ can be the empty set, but not both of them can be empty at the same time for the same value of $\alpha \in \mathbb{R}$.

2. For every function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\alpha \in \mathbb{R}$,

$$\mathcal{L}_\alpha^-(f) \cup \mathcal{L}_\alpha^+(f) = \mathbb{R}^n, \qquad \mathcal{L}_\alpha^-(f) \cap \mathcal{L}_\alpha^+(f) = \mathcal{L}_\alpha(f).$$

4.3.1 Sublevel Sets of Convex Functions

Proposition 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. For each $\alpha \in \mathbb{R}$, the sublevel set of f given by*

$$\mathcal{L}_\alpha^-(f) = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$$

is a convex set.

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\alpha \in \mathbb{R}$. Let $x^1 \in \mathcal{L}_\alpha^-(f)$ and $x^2 \in \mathcal{L}_\alpha^-(f)$. Then, for any $\lambda \in [0,1]$,

$$f\left(\lambda x^1 + (1-\lambda)x^2\right) \leq \underbrace{\lambda}_{\geq 0} \underbrace{f(x^1)}_{\leq \alpha} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{f(x^2)}_{\leq \alpha} \leq \lambda\alpha + (1-\lambda)\alpha = \alpha.$$

Therefore, $\lambda x^1 + (1-\lambda)x^2 \in \mathcal{L}_\alpha^-(f)$, which implies that $\mathcal{L}_\alpha^-(f)$ is a convex set. □

4.3.2 Superlevel Sets of Concave Functions

Corollary 4.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function. For each $\alpha \in \mathbb{R}$, the superlevel set of f given by*

$$\mathcal{L}_\alpha^+(f) = \{x \in \mathbb{R}^n : f(x) \geq \alpha\}$$

is a convex set.

Proof. Since f is a concave function, $-f$ is a convex function and $\mathcal{L}_\alpha^+(f) = \mathcal{L}_{-\alpha}^-(-f)$. The result follows from Proposition 4.2. □

4.4 Convexity and Concavity of Linear Functions

Proposition 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function given by $f(x) = a^T x$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. Then, f is both a convex and a concave function.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function given by $f(x) = a^T x$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. Then, $x^1 \in \mathbb{R}^n$, $x^2 \in \mathbb{R}^n$, and any $\lambda \in [0,1]$, $f(\lambda x^1 + (1-\lambda)x^2) = a^T(\lambda x^1 + (1-\lambda)x^2) = \lambda a^T x^1 + (1-\lambda)a^T x^2 = \lambda f(x^1) + (1-\lambda) f(x^2)$. Therefore, f is both convex and concave. □

4.4.1 Halfspaces

Definition 4.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function given by $f(x) = a^T x$, where $a \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. For each $\alpha \in \mathbb{R}$, each of the sublevel set of f given by*

$$\mathcal{L}^-_\alpha(f) = \{x \in \mathbb{R}^n : a^T x \leq \alpha\}$$

and the superlevel set of f given by

$$\mathcal{L}^+_\alpha(f) = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$$

is called a halfspace.

4.4.2 Convexity of Halfspaces

Corollary 4.7. *Every halfspace is a convex set.*

Proof. Since a halfspace is given by the sublevel set of superlevel set of a linear function and since every linear function is convex and concave by Proposition 4.3, the result follows from Proposition 4.2 and Corollary 4.5. □

4.4.3 Properties of Halfspaces

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function given by $f(x) = a^T x$, where $a \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and let $\mathcal{H} = \mathcal{L}_\alpha(f) = \{x \in \mathbb{R}^n : a^T x = \alpha\}$ be a hyperplane, where $\alpha \in \mathbb{R}$. Let

$$\begin{aligned}\mathcal{H}^- &= \mathcal{L}^-_\alpha(f) = \{x \in \mathbb{R}^n : a^T x \leq \alpha\} \\ \mathcal{H}^+ &= \mathcal{L}^+_\alpha(f) = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}\end{aligned}$$

be the corresponding halfspaces.

1. $\mathcal{H}$ is the boundary of each of $\mathcal{H}^-$ and $\mathcal{H}^+$. Furthermore,

$$\mathcal{H} = \mathcal{H}^- \cap \mathcal{H}^+.$$

2. The vector $a \in \mathbb{R}^n$ is perpendicular to $\mathcal{H}$, i.e., for any $x^1 \in \mathcal{H}$ and $x^2 \in \mathcal{H}$, $a^T(x^1 - x^2) = \alpha - \alpha = 0$.

Exercises

Question 4.1. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x) = x_1^2 + x_2^2$ and $\alpha = 1$, then give a description of $\mathcal{L}_\alpha(f)$.*

Question 4.2. *If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = |x|$ and $\alpha = 1$, then give a description of $\mathcal{L}^+_\alpha(f)$.*

Question 4.3. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x) = -x_1 + x_2$ and $\alpha = 1$, then give a description of $\mathcal{L}^-_\alpha(f)$.*

Question 4.4. *Show that the intersection of any number of halfspaces and hyperplanes in $\mathbb{R}^n$ is a convex set.*

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Lecture 5	Introduction to Convex Optimization
Lecturer: E. Alper Yıldırım	*Week: 2*

5.1 Outline

- Convex Optimization
- Connection with Linear Programming
- Properties of Convex Optimization
- Review Problems

5.2 Quick Review of Lecture 4

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $\alpha \in \mathbb{R}$.

- $\mathcal{L}_\alpha(f) = \{x \in \mathbb{R}^n : f(x) = \alpha\}$ is the level set of f.
- $\mathcal{L}_\alpha^-(f) = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is the sublevel set of f.
- $\mathcal{L}_\alpha^+(f) = \{x \in \mathbb{R}^n : f(x) \geq \alpha\}$ is the superlevel set of f.
- Sublevel sets of convex functions and superlevel sets of concave functions are convex sets.
- A linear function is both convex and concave.
- Level sets of linear functions (hyperplanes) and sublevel and superlevel sets of linear functions (halfspaces) are convex sets.

5.3 Relation with Constrained Optimization

Recall our generic constrained optimization problem:

$$\begin{array}{llrcll} \text{(P)} & \min & f(x) & & & \\ & \text{s.t.} & & & & \\ & & g_i(x) & \geq & b_i, & i \in M_1, \\ & & \ell_i(x) & \leq & b_i, & i \in M_2, \\ & & h_i(x) & = & b_i, & i \in M_3, \end{array}$$

where M_1, M_2, and M_3 are finite index sets; $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function; $g_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_1$; $\ell_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_2$; and $h_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_3$. Each of the functional relations $g_i(x) \geq b_i,\ i \in M_1$; $\ell_i(x) \leq b_i,\ i \in M_2$; $h_i(x) = b_i,\ i \in M_3$ is a constraint.

Remark 5.1. *The feasible region $\mathcal{S} \subseteq \mathbb{R}^n$ of (P) is given by*

$$\mathcal{S} = \{x \in \mathbb{R}^n : g_i(x) \geq b_i,\ i \in M_1; \quad \ell_i(x) \leq b_i,\ i \in M_2; \quad h_i(x) = b_i,\ i \in M_3\}.$$

Therefore, $\mathcal{S} \subseteq \mathbb{R}^n$ is given by the intersection of each of (i) the superlevel set $\mathcal{L}^+_{b_i}(g_i)$ for each $i \in M_1$; (ii) the sublevel set $\mathcal{L}^-_{b_i}(\ell_i)$ for each $i \in M_2$; and (iii) the level set $\mathcal{L}_{b_i}(h_i)$ for each $i \in M_3$, i.e.,

$$\mathcal{S} = \left(\bigcap_{i \in M_1} \mathcal{L}^+_{b_i}(g_i)\right) \bigcap \left(\bigcap_{i \in M_2} \mathcal{L}^-_{b_i}(g_i)\right) \bigcap \left(\bigcap_{i \in M_3} \mathcal{L}_{b_i}(h_i)\right).$$

Therefore, understanding of superlevel, sublevel, and level sets of real-valued functions is fundamental in understanding the geometry of the feasible region $\mathcal{S} \subseteq \mathbb{R}^n$.

5.4 Convex Optimization

Recall our generic constrained optimization problem:

$$\begin{array}{lllll} \text{(P)} & \min & f(x) & & \\ & \text{s.t.} & & & \\ & & g_i(x) & \geq & b_i, \quad i \in M_1, \\ & & \ell_i(x) & \leq & b_i, \quad i \in M_2, \\ & & h_i(x) & = & b_i, \quad i \in M_3, \end{array}$$

where M_1, M_2, and M_3 are finite index sets; $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function; $g_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_1$; $\ell_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_2$; and $h_i : \mathbb{R}^n \to \mathbb{R},\ i \in M_3$.

Definition 5.1. *(P) is called a* convex optimization problem *if (i) f is a convex function; (ii) g_i is a concave function for each $i \in M_1$; (iii) ℓ_i is a convex function for each $i \in M_2$; and (iv) h_i is a linear function for each $i \in M_3$. Otherwise, (P) is called a* nonconvex optimization problem.

Remark 5.2. *Recall that (P) is a linear programming problem if each of f; $g_i,\ i \in M_1$; $\ell_i,\ i \in M_2$; and $h_i,\ i \in M_3$ is a linear function. Since every linear function is both convex and concave by Proposition 4.3, it follows that a linear programming problem is a convex optimization problem.*

5.4.1 Properties of Convex Optimization Problems

Proposition 5.1. *Let (P) be a convex optimization problem. Then, each of the feasible region $\mathcal{S} \subseteq \mathbb{R}^n$ and the set of optimal solutions $\mathcal{S}^* \subseteq \mathbb{R}^n$ is a convex set.*

Proof. If $\mathcal{S} = \emptyset$, then it is convex. Otherwise, each of the level sets of linear functions, sublevel sets of convex functions, and superlevel sets of concave functions is a convex set by Proposition 4.1, Proposition 4.2, and Corollary 4.5, respectively. Since convexity is preserved under taking intersections (see Remark 3 in Section 3.2), $\mathcal{S}$ is a convex set.

If $\mathcal{S}^* = \emptyset$, then it is convex. Otherwise, let $z^* \in \mathbb{R}$ denote the optimal value. For any $x^1 \in \mathcal{S}^*$, $x^2 \in \mathcal{S}^*$, and any $\lambda \in [0, 1]$, note that $\lambda x^1 + (1 - \lambda)x^2 \in \mathcal{S}$ since $\mathcal{S}$ is a convex set. Furthermore, by definition of the optimal value and the convexity of f,

$$z^* \leq f\left(\lambda x^1 + (1 - \lambda)x^2\right) \leq \lambda f(x^1) + (1 - \lambda)f(x^2) = \lambda z^* + (1 - \lambda)z^* = z^*.$$

Therefore, $\lambda x^1 + (1 - \lambda)x^2 \in \mathcal{S}^*$ and $\mathcal{S}^*$ is a convex set. □

Remarks

1. Convex optimization problems possess very nice geometric and theoretical properties.
2. A very large class of convex optimization problems can be efficiently solved by powerful algorithms.
3. For every convex optimization problem, each of the feasible region $\mathcal{S}$ and the set of optimal solutions $\mathcal{S}^*$ is a convex set.
4. Linear programming is a very special class of convex optimization with further additional desirable properties.
5. Henceforth, we will mostly focus on linear programming in this course.

Exercises

Question 5.1. *Determine whether the following optimization problem is a convex optimization problem:*

$$\min\{|x| : x \geq -3, \quad x^2 \leq 4\}$$

Question 5.2. *Determine whether the following optimization problem is a convex optimization problem:*

$$\min\{|x| : x \geq -3, \quad x^2 \leq 4, \quad x^3 = 1\}$$

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Lecture 6 Vertices of Convex Sets and Introduction to Polyhedra

Lecturer: E. Alper Yıldırım *Week: 2*

6.1 Outline

- Vertices of Convex Sets
- Polyhedra and Polytopes
- Review Problems

6.2 Vertices of Convex Sets

Recall that a set $\mathcal{C} \subseteq \mathbb{R}^n$ is a convex set if, for every $x \in \mathcal{C}$ and for every $y \in \mathcal{C}$, all the vectors on the line segment that joins x and y also belong to $\mathcal{C}$.

Definition 6.1. *Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set. A vector $\hat{x} \in \mathcal{C}$ is called a* vertex *of $\mathcal{C}$ if there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, and a corresponding halfspace $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$ such that*

(i) $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$*; and*

(ii) $\mathcal{C} \subseteq \mathcal{H}^+$.

Such a hyperplane $\mathcal{H}$ is called a supporting hyperplane *of $\mathcal{C}$.*

Remarks

1. If $n = 1$, then a hyperplane is a point on the real line and a halfspace is a half line. Therefore, if $\mathcal{C} \subseteq \mathbb{R}$ is a convex set, then $\hat{x} \in \mathcal{C}$ is a vertex of $\mathcal{C}$ if and only if it is an end-point of $\mathcal{C}$. For instance, the convex set $\mathcal{C} = [0, 1]$ has two vertices given by 0 and 1. The convex set $\mathcal{C} = [0, \infty)$ has only one vertex given by 0. The convex set $\mathcal{C} = (0, \infty)$ has no vertices (why not?).

2. If $n = 2$, then a hyperplane is a line in two dimensions and a halfspace is either side of such a line. Therefore, if $\mathcal{C} \subseteq \mathbb{R}$ is a convex set, then $\hat{x} \in \mathcal{C}$ is a vertex of $\mathcal{C}$ if and only if there is a line that intersects $\mathcal{C}$ exactly at the point $\hat{x} \in \mathcal{C}$.

   (a) Let
   $$\mathcal{C} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, \quad x_2 \geq 0\},$$
   i.e., it is the set of all points in the upper semicircle of the unit circle centred at the origin including the boundary points. Then, $\mathcal{C}$ has an infinite number of vertices and the set of all vertices of $\mathcal{C}$ is given by
   $$\mathcal{V} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1, \quad x_2 \geq 0\},$$
   i.e., set of all points on the boundary of the upper semicircle.

(b) Let

$$\mathcal{C} = \{x \in \mathbb{R}^2 : |x_1| \leq 1, \quad |x_2| \leq 1\},$$

i.e., it is the set of all points in the square centred at the origin and four corner points at $[\pm 1, \pm 1]$. There are only four vertices which are precisely given by the four corner points.

(c) Let

$$\mathcal{C} = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1\},$$

i.e., it is a halfspace since it is given by the sublevel set of a linear function (see Definition 4.6). You can easily verify that this convex set has no vertices.

6.3 Vertices and Optimization of Linear Objective Functions

Proposition 6.1. *Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set. A vector $\hat{x} \in \mathcal{C}$ is a vertex of $\mathcal{C}$ if and only if there exists a linear function $\ell(x) = a^T x$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, such that $\hat{x}$ is the* **unique** *optimal solution of the optimization problem*

$$\begin{array}{lll} (P) & \min & a^T x \\ & s.t. & \\ & & x \in \mathcal{C}. \end{array}$$

Proof. $\Rightarrow$: Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set and let $\hat{x} \in \mathcal{C}$ be a vertex of $\mathcal{C}$. Then, there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, and the corresponding halfspace $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$ such that $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$ and $\mathcal{C} \subseteq \mathcal{H}^+$. Since $\mathcal{C} \subseteq \mathcal{H}^+$, we have $a^T x \geq \alpha$ for each $x \in \mathcal{C}$. Since $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$, it follows that $a^T x > \alpha$ for each $x \in \mathcal{C} \backslash \{\hat{x}\}$. Therefore, $\hat{x}$ is the unique optimal solution of (P).

$\Leftarrow$: Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set and let $\hat{x} \in \mathcal{C}$ be the unique optimal solution of (P). Let $\alpha \in \mathbb{R}$ denote the optimal value of (P). Then, $a^T \hat{x} = \alpha$. Since $\hat{x} \in \mathcal{C}$ is the unique optimal solution of (P), we have $a^T x > \alpha$ for each $x \in \mathcal{C} \backslash \{\hat{x}\}$. Therefore, if we define the hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, and the corresponding halfspace $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$, we obtain $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$ and $\mathcal{C} \subseteq \mathcal{H}^+$. Therefore, $\hat{x}$ is a vertex of $\mathcal{C}$. □

Remark 6.1. *Vertices of a convex set play an important role in the minimization of a linear function over that convex set, i.e., each vertex of a convex set is the unique optimal solution for the optimization problem of minimizing some linear function over that set.*

6.4 Polyhedra

Recall that each level set of a linear function is a hyperplane (see Definition 4.2) and that each sublevel or superlevel set of a linear function is a halfspace (see Definition 4.6).

Definition 6.2. *A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a* polyhedron *if it is given by the intersection of a finite number of hyperplanes and a finite number of halfspaces.*

Remark 6.2. *Every polyhedron is a convex set since every hyperplane and every halfspace is a convex set and convexity is preserved under taking intersections (see Remark 2 in Section 3.2).*

6.4.1 Linear Programming and Polyhedra

Recall our generic constrained optimization problem:

$$
\begin{array}{llrcll}
\text{(P)} & \text{min} & f(x) & & & \\
& \text{subject to (s.t.)} & & & & \\
& & g_i(x) & \geq & b_i, & i \in M_1, \\
& & \ell_i(x) & \leq & b_i, & i \in M_2, \\
& & h_i(x) & = & b_i, & i \in M_3,
\end{array}
$$

Remark 6.3. *Recall that (P) is a linear programming problem if each of f; $g_i,\ i \in M_1$; $\ell_i,\ i \in M_2$; and $h_i,\ i \in M_3$ is a linear function. The feasible region of every linear programming problem is a polyhedron since it is given by the intersection of a finite number of hyperplanes and a finite number of halfspaces.*

6.4.2 Bounded Sets and Polytopes

Definition 6.3. *A set $\mathcal{S} \subseteq \mathbb{R}^n$ is* bounded *if there exists a real number $K \in \mathbb{R}$ such that*

$$x \in \mathcal{S} \Rightarrow |x_j| \leq K, \quad j = 1, \ldots, n.$$

Definition 6.4. *A bounded polyhedron is called a* polytope*.*

Exercises

Question 6.1. *In $\mathbb{R}$, does there exist a convex set with no vertices? One vertex? Two vertices? Three vertices?*

Question 6.2. *In $\mathbb{R}^2$, for any $k = 0, 1, \ldots$, show that you can construct a convex set with exactly k vertices.*

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 7	Characterisation of Vertices of Polyhedra
Lecturer: E. Alper Yıldırım	*Week: 2*

7.1 Outline

- Active (Binding) Constraints
- Basic Solutions and Basic Feasible Solutions
- Connection with Vertices
- Review Problems

7.2 Quick Review

- Given a convex set $\mathcal{C} \subseteq \mathbb{R}^n$, $\hat{x} \in \mathcal{C}$ is called a *vertex* of $\mathcal{C}$ if there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, and a corresponding halfspace $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$ such that
  - $\mathcal{C} \cap \mathcal{H} = \{\hat{x}\}$;
  - $\mathcal{C} \subseteq \mathcal{H}^+$.
- A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a *polyhedron* if it is given by the intersection of a finite number of hyperplanes and a finite number of halfspaces.
- Every polyhedron is a convex set.
- The feasible region of every linear programming problem is a polyhedron.

7.3 Active (Binding) Constraints

Consider a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\},$$

where M_1, M_2, and M_3 are finite sets, and $a^i \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ for each $i \in M_1 \cup M_2 \cup M_3$.

Definition 7.1. *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron and let $\hat{x} \in \mathbb{R}^n$. The set of indices of* active *(or* binding*) constraints at $\hat{x}$ is given by*

$$I(\hat{x}) = \left\{ i \in M_1 \cup M_2 \cup M_3 : (a^i)^T \hat{x} = b_i \right\}.$$

Note that $M_3 \subseteq I(\hat{x})$ for each $\hat{x} \in \mathcal{P}$.

7.4 Basic Solutions and Basic Feasible Solutions

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\}.$$

Definition 7.2. *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron and let $\hat{x} \in \mathbb{R}^n$.*

(i) $\hat{x}$ is a basic solution *if all of the equality constraints are active at $\hat{x}$ (i.e., $M_3 \subseteq I(\hat{x})$) and the set $\{a^i : i \in I(\hat{x})\} \subset \mathbb{R}^n$ contains n linearly independent vectors (i.e., the set $\{a^i : i \in I(\hat{x})\}$ spans $\mathbb{R}^n$).*

(ii) $\hat{x}$ is a basic feasible solution *if $\hat{x}$ is a* basic solution *and $\hat{x}$ is feasible (i.e., $\hat{x} \in \mathcal{P}$).*

7.4.1 Basic Feasible Solutions and Vertices

Proposition 7.1. *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron and let $\hat{x} \in \mathcal{P}$. Then, $\hat{x}$ is a basic feasible solution of $\mathcal{P}$ if and only if $\hat{x}$ is a vertex of $\mathcal{P}$.*

Proof. $\Rightarrow$: Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron and let $\hat{x} \in \mathcal{P}$ be a basic feasible solution of $\mathcal{P}$. Let $I(\hat{x}) = \big\{i \in M_1 \cup M_2 \cup M_3 : (a^i)^T \hat{x} = b_i\big\}$. We need to construct a vector $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and a real number $\alpha \in \mathbb{R}$ such that $a^T \hat{x} = \alpha$ and $a^T x > \alpha$ for each $x \in \mathcal{P} \backslash \{\hat{x}\}$. Let $I \subseteq I(\hat{x})$ be such that the set $\{a^i : i \in I\}$ is linearly independent and spans $\mathbb{R}^n$ (i.e., it is a basis for $\mathbb{R}^n$). Let $a = \sum\limits_{i \in I \cap M_1} (a^i) + \sum\limits_{i \in I \cap M_2} (-a^i) + \sum\limits_{i \in I \cap M_3} (a^i)$. Note that $a \neq \mathbf{0}$ since the set $\{a^i : i \in I\}$ is linearly independent. Let $\alpha = a^T \hat{x} = \sum\limits_{i \in I \cap M_1} (a^i)^T \hat{x} + \sum\limits_{i \in I \cap M_2} (-a^i)^T \hat{x} + \sum\limits_{i \in I \cap M_3} (a^i)^T \hat{x} = \sum\limits_{i \in I \cap M_1} b_i + \sum\limits_{i \in I \cap M_2} (-b_i) + \sum\limits_{i \in I \cap M_3} b_i$. Let $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$ and $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$. Then, $\hat{x} \in \mathcal{P} \cap \mathcal{H}$. For any $x \in \mathcal{P}$, we have $a^T x = \sum\limits_{i \in I \cap M_1} (a^i)^T x + \sum\limits_{i \in I \cap M_2} (-a^i)^T x + \sum\limits_{i \in I \cap M_3} (a^i)^T x \geq \sum\limits_{i \in I \cap M_1} b_i + \sum\limits_{i \in I \cap M_2} (-b_i) + \sum\limits_{i \in I \cap M_3} b_i = \alpha$. Therefore, $\mathcal{P} \subseteq \mathcal{H}^+$. Finally, for any $x \in \mathcal{P} \cap \mathcal{H}$, we have $(a^i)^T x = b_i$ for each $i \in I$, which implies that $(a^i)^T (x - \hat{x}) = 0$ for each $i \in I$. Since the set $\{a^i : i \in I\}$ is a basis for $\mathbb{R}^n$, it follows that $x - \hat{x} = \mathbf{0}$, which implies that $\mathcal{P} \cap \mathcal{H} = \{\hat{x}\}$. Therefore, $\hat{x}$ is a vertex of $\mathcal{P}$.

$\Leftarrow$: Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron and let $\hat{x} \in \mathcal{P}$ be a vertex of $\mathcal{P}$. Then, there exists a vector $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, a real number $\alpha \in \mathbb{R}$, a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : a^T x = \alpha\}$ and a corresponding halfspace $\mathcal{H}^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$ such that $\mathcal{P} \subseteq \mathcal{H}^+$ and $\mathcal{P} \cap \mathcal{H} = \{\hat{x}\}$. Let $I(\hat{x}) = \big\{i \in M_1 \cup M_2 \cup M_3 : (a^i)^T \hat{x} = b_i\big\}$. Suppose, for a contradiction, that $\hat{x}$ is not a basic feasible solution. Since $\hat{x}$ is feasible, it is then not a basic solution. Therefore, the set $\{a^i : i \in I(\hat{x})\}$ does not contain n linearly independent vectors. Then, there exists a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $(a^i)^T d = 0$ for each $i \in I(\hat{x})$. Let $\epsilon > 0$ be a real number. Consider $\hat{x} + \epsilon d$ and $\hat{x} - \epsilon d$. Note that $M_3 \subseteq I(\hat{x})$ since $\hat{x} \in \mathcal{P}$. Since $(a^i)^T d = 0$ for each $i \in I(\hat{x})$, we have $(a^i)^T (\hat{x} + \epsilon d) = b_i$ and $(a^i)^T (\hat{x} - \epsilon d) = b_i$ for each $\epsilon > 0$ and each $i \in I(\hat{x})$. For each $i \in M_1 \backslash I(\hat{x})$, we have $(a^i)^T \hat{x} > b_i$, which implies that $(a^i)^T (\hat{x} + \epsilon d) \geq b_i$ and $(a^i)^T (\hat{x} - \epsilon d) \geq b_i$ if ϵ is sufficiently small but positive. Similarly, for each $i \in M_2 \backslash I(\hat{x})$, we have $(a^i)^T \hat{x} < b_i$, which implies that $(a^i)^T (\hat{x} + \epsilon d) \leq b_i$ and $(a^i)^T (\hat{x} - \epsilon d) \leq b_i$ if ϵ is sufficiently small but positive. Therefore, there exists $\epsilon^* > 0$ such that $\hat{x} - \epsilon^* d \in \mathcal{P}$ and $\hat{x} + \epsilon^* d \in \mathcal{P}$. Since $a^T x > \alpha$ for each $x \in \mathcal{P} \backslash \{\hat{x}\}$, we have $a^T (\hat{x} - \epsilon^* d) = \alpha - \epsilon^* a^T d > \alpha$ and $a^T (\hat{x} + \epsilon^*) = \alpha + \epsilon^* a^T d > \alpha$. Hence, $a^T d < 0$ and $a^T d > 0$, which is a contradiction. Therefore, $\hat{x}$ is a basic feasible solution.

$\square$

Remarks

1. For a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$, there is a one-to-one correspondence between vertices and basic feasible solutions.
2. The definition of a vertex is geometric and is therefore not very useful in an algorithmic framework.
3. On the other hand, the definition of a basic feasible solution is algebraic, i.e., for a given polyhedron $\mathcal{P}$ and a given vector $\hat{x}$, one can check if $\hat{x}$ is a basic feasible solution by simply using tools from linear algebra.

Exercises

Question 7.1. *Consider the following polyhedron:*

$$\mathcal{P} = \{x \in \mathbb{R}^2 : x_1 \geq 1,\ x_1 - x_2 \geq 0,\ x_1 \geq 0,\ x_2 \leq 1,\ x_1 + x_2 = 2\}$$

For each of the following vectors in $\mathbb{R}^2$, determine whether it is a basic solution, basic feasible solution, both, or neither.

(i) $x^1 = [1, 1]^T$

(ii) $x^2 = [3/2, 1/2]^T$

(iii) $x^3 = [2, 0]^T$

(iv) $x^4 = [0, 2]^T$

(v) $x^5 = [2, 2]^T$

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 8 — Existence and Finiteness of Basic Feasible Solutions

Lecturer: E. Alper Yıldırım *Week: 2*

8.1 Outline

- Existence of Basic Feasible Solutions
- Finiteness of Basic Feasible Solutions
- Review Problems

8.2 Quick Review

Let

$$\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1;\ (a^i)^T x \leq b_i,\ i \in M_2; (a^i)^T x = b_i,\ i \in M_3\}$$

be a polyhedron and let $\hat{x} \in \mathbb{R}^n$.

- A constraint is active (or binding) at $\hat{x}$ if it is satisfied with equality.
- Let $I(\hat{x})$ denote the set of indices of all active constraints at $\hat{x}$.
- $\hat{x}$ is a *basic solution* if all of the equality constraints are active at $\hat{x}$ (i.e., $M_3 \subseteq I(\hat{x})$) and the set $\{a^i : i \in I(\hat{x})\} \subset \mathbb{R}^n$ contains n linearly independent vectors (i.e., the set $\{a^i : i \in I(\hat{x})\}$ spans $\mathbb{R}^n$).
- $\hat{x}$ is a *basic feasible solution* if $\hat{x}$ is a *basic solution* and $\hat{x}$ is feasible (i.e., $\hat{x} \in \mathcal{P}$).
- $\hat{x}$ is a basic feasible solution of $\mathcal{P}$ if and only if $\hat{x}$ is a vertex of $\mathcal{P}$.

8.3 Existence of Vertices

Question 1. *Does every nonempty polyhedron necessarily have at least one vertex?*

Definition 8.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1;\ (a^i)^T x \leq b_i,\ i \in M_2;\ (a^i)^T x = b_i,\ i \in M_3\}$ be a polyhedron. $\mathcal{P}$* contains a line *if there exists a vector $\tilde{x} \in \mathcal{P}$ and a nonzero vector $d \in \mathbb{R}^n$ such that $\tilde{x} + \lambda d \in \mathcal{P}$ for every real number λ.*

Consider the line in $\mathbb{R}^2$ given by $x_1 + x_2 = 2$. Alternatively, the same line can be represented by a point on the line and the direction of the line. For instance, $\tilde{x} = [1, 1]^T$ is a vector on this line and starting from this point, one can move in the direction $d = [1, -1]^T$ or in its opposite direction and will always remain on this line. Therefore,

$$\{x \in \mathbb{R}^n : x_1 + x_2 = 2\} = \{\tilde{x} + \lambda d : \lambda \in \mathbb{R}\}.$$

The latter representation holds for any line in $\mathbb{R}^n$. Note that Definition 8.1 uses the second representation.

The next proposition gives a complete characterisation of polyhedra that contain at least one vertex.

Proposition 8.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1;\ (a^i)^T x \leq b_i,\ i \in M_2;\ (a^i)^T x = b_i,\ i \in M_3\}$ be a nonempty polyhedron. $\mathcal{P}$ has at least one vertex if and only if it does not contain a line.*

Proof. $\Rightarrow$: We will use proof by contrapositive, i.e., we will show that if a nonempty polyhedron contains a line, then it does not have any vertices. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron that contains a line. Then, there exists a vector $\tilde{x} \in \mathcal{P}$ and a nonzero vector $d \in \mathbb{R}^n$ such that $\tilde{x} + \lambda d \in \mathcal{P}$ for every real number λ. Since $\tilde{x} \in \mathcal{P}$, we have $(a^i)^T \tilde{x} \geq b_i$ for each $i \in M_1$, $(a^i)^T \tilde{x} \leq b_i$ for each $i \in M_2$, and $(a^i)^T \tilde{x} = b_i$ for each $i \in M_3$. Since $\tilde{x} + \lambda d \in \mathcal{P}$ for every real number λ, we have $(a^i)^T(\tilde{x} + \lambda d) \geq b_i$ for each $i \in M_1$, $(a^i)^T(\tilde{x} + \lambda d) \leq b_i$ for each $i \in M_2$, and $(a^i)^T(\tilde{x} + \lambda d) = b_i$ for each $i \in M_3$. Therefore, we have $(a^i)^T d = 0$ for each $i \in M_1 \cup M_2 \cup M_3$. Therefore, for any $x \in \mathcal{P}$, since $I(x) \subseteq M_1 \cup M_2 \cup M_3$, it follows that $(a^i)^T d = 0$ for each $i \in I(x)$. Therefore, the set $\{a^i : i \in I(x)\}$ cannot contain n linearly independent vectors for any $x \in \mathcal{P}$. It follows that $\mathcal{P}$ does not have any vertices.

$\Leftarrow$: Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron that does not contain a line. We need to show that $\mathcal{P}$ has at least one vertex. By the argument in the previous part of the proof, there does not exist a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $(a^i)^T d = 0$ for each $i \in M_1 \cup M_2 \cup M_3$. Let $\hat{x} \in \mathcal{P}$ be an arbitrary vector and let $I(\hat{x}) = \left\{i \in M_1 \cup M_2 \cup M_3 : (a^i)^T \hat{x} = b_i\right\}$. If the set $\{a^i : i \in I(\hat{x})\}$ contains n linearly independent vectors, then we are done since $\hat{x}$ is a vertex. Otherwise, there exists a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $(a^i)^T d = 0$ for each $i \in I(\hat{x})$. Since $M_3 \subseteq I(\hat{x})$, we have $(a^i)^T d = 0$ for each $i \in M_3$. Consider the line $\hat{x} + \lambda d$, where $\lambda \in \mathbb{R}$. Since $\mathcal{P}$ does not contain a line, there exist a nonzero $\lambda^* \in \mathbb{R}$ such that $x^* = \hat{x} + \lambda^* d \in \mathcal{P}$ and an index $i^* \in (M_1 \cup M_2) \backslash I(\hat{x})$ such that $(a^{i^*})^T x^* = b_{i^*}$. Therefore, $(a^{i^*})^T d \neq 0$ and $I(x^*) \supseteq I(\hat{x}) \cup \{i^*\}$. We claim that a^{i^*} is not a linear combination of the vectors in the set $\{a^i : i \in I(\hat{x})\}$. Otherwise, there would exist real numbers $\alpha_i,\ i \in I(\hat{x})$ such that $a^{i^*} = \sum\limits_{i \in I(\hat{x})} \alpha_i a^i$. Then, $(a^{i^*})^T d = \sum\limits_{i \in I(\hat{x})} \alpha_i (a^i)^T d = 0$, which contradicts with $(a^{i^*})^T d \neq 0$. Therefore, the number of linearly independent vectors indexed by $I(x^*)$ is at least one larger than that indexed by $I(\hat{x})$. By repeating this procedure as many times as needed, we obtain a feasible solution whose set of active constraints contains a subset of n linearly independent vectors. Therefore, $\mathcal{P}$ contains a vertex. $\square$

8.3.1 Implications on Polytopes

Remark 8.1. *Recall that a bounded polyhedron is called a polytope.*

Corollary 8.2. *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polytope. Then, $\mathcal{P}$ has at least one vertex.*

Proof. Since $\mathcal{P}$ is nonempty and bounded, it cannot contain a line. By Proposition 8.1, $\mathcal{P}$ has at least one vertex. $\square$

8.4 Number of Vertices of a Polyhedron

Proposition 8.2. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1;\ (a^i)^T x \leq b_i,\ i \in M_2;\ (a^i)^T x = b_i,\ i \in M_3\}$ be a nonempty polyhedron. Then, $\mathcal{P}$ contains at most a finite number of vertices.*

Proof. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron. If $\mathcal{P}$ contains a line, then it has no vertices. Otherwise, for any $x \in \mathcal{P}$, let $I(x) = \left\{i \in M_1 \cup M_2 \cup M_3 : (a^i)^T x = b_i\right\}$. Note that $M_3 \subseteq I(x)$. Therefore, $I(x) = M_3 \cup J$, where $J \subseteq M_1 \cup M_2$. The number of different subsets of $M_1 \cup M_2$ is given by $2^{|M_1| + |M_2|}$, which is a finite number. Therefore, for any $x \in \mathcal{P}$, $I(x)$ can be equal to a finite number of different sets. Among those different sets, only a subset of them will satisfy the condition that the set $\{a^i : i \in I(x)\}$ contains n linearly independent vectors. Finally, if $x^1 \in \mathcal{P}$ and $x^2 \in \mathcal{P}$ are such that $I(x^1) = I(x^2) = I$ and the set $\{a^i : i \in I\}$ contains n linearly independent vectors, then $x^1 = x^2$ since $(a^i)^T(x^1 - x^2) = 0$ for each $i \in I$. Therefore, $\mathcal{P}$ contains at most a finite number of vertices. □

8.4.1 Number of Vertices of a General Convex Set

Consider the following convex set $\mathcal{C} \subseteq \mathbb{R}^2$:

$$\mathcal{C} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}.$$

Note that $\mathcal{C}$ is the circle centred at the origin with radius 1. It is easy to show that $\mathcal{C} \subseteq \mathbb{R}^2$ is a convex set. However, $\mathcal{C}$ is not a polyhedron since it cannot be written as the intersection of a finite number of halfspaces and hyperplanes. Note that $\mathcal{C}$ has an **infinite** number of vertices and the set of all vertices of $\mathcal{C}$ is given by

$$\mathcal{V} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\},$$

i.e., set of all points on the boundary of the circle. As illustrated by this example, a polyhedron is a very special kind of convex set.

Exercises

Question 8.1. *Let $\mathcal{P}^1 \subset \mathbb{R}^n$ be a polyhedron that contains at least one vertex and let $\mathcal{P}^2 \subset \mathbb{R}^n$ be an arbitrary nonempty polyhedron. Let $\mathcal{P} = \mathcal{P}^1 \cap \mathcal{P}^2$. Suppose that $\mathcal{P}$ is nonempty.*

(i) Show that $\mathcal{P}$ is a polyhedron.

(ii) Show that $\mathcal{P}$ contains at least one vertex.

Question 8.2. *Consider the following polyhedron:*

$$\mathcal{P} = \{x \in \mathbb{R}^n : x_j \geq 0,\ j = 1, \ldots, n;\ x_j \leq 1,\ j = 1, \ldots, n\}$$

How many vertices does $\mathcal{P}$ have?

MATH11111: Fundamentals of Optimization	Fall 2023
Lecture 9	Graphical Solution Method
Lecturer: E. Alper Yıldırım	*Week: 3*

9.1 Outline

- Linear Programming with One Variable
- Linear Programming with Two Variables
- Review Problems

9.2 Introduction

In this lecture, we will try to start addressing the following question.

Question 1. *How can we solve a linear programming problem?*

Remark 9.1. *Solving an optimization problem means*

- *finding the optimal value and an optimal solution (if any); or*
- *detecting that the problem is unbounded; or*
- *verifying that the problem is infeasible.*

Recall the general linear programming problem:

$$\begin{array}{llrcll} \text{(P)} & \min & c^T x & & & \\ & \text{s.t.} & & & & \\ & & (a^i)^T x & \geq & b_i, & i \in M_1, \\ & & (a^i)^T x & \leq & b_i, & i \in M_2, \\ & & (a^i)^T x & = & b_i, & i \in M_3, \end{array}$$

where $c \in \mathbb{R}^n$; $a^i \in \mathbb{R}^n$ for each $i \in M_1 \cup M_2 \cup M_3$; $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$; and M_1, M_2, and M_3 are finite index sets.

9.2.1 One Decision Variable ($n = 1$)

1. If $n = 1$ (i.e., there is only one decision variable $x_1 \in \mathbb{R}$), then each constraint is in the form of $(a_1^i)x_1 \geq b_i$, or $(a_1^i)x_1 \leq b_i$, or $(a_1^i)x_1 = b_i$, where $a_1^i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ for each $i \in M_1 \cup M_2 \cup M_3$.
2. The feasible region is either the empty set, or a single point, or a line segment, or a half line (assuming $a_1^i \neq 0,\ i \in M_1 \cup M_2 \cup M_3$).
3. The objective function is given by $c_1 x_1$, where $c_1 \in \mathbb{R}$.

4. If $c_1 > 0$, the optimal solution is the smallest feasible solution (if any); or the problem is unbounded and the optimal value is given by $z^* = -\infty$; or the problem is infeasible and the optimal value is given by $z^* = +\infty$.
5. If $c_1 < 0$, the optimal solution is the largest feasible solution (if any); or the problem is unbounded and the optimal value is given by $z^* = -\infty$; or the problem is infeasible and the optimal value is given by $z^* = +\infty$.
6. If $c_1 = 0$, then any feasible solution is an optimal solution and $z^* = 0$; or the problem is infeasible and the optimal value is given by $z^* = +\infty$. (Such an optimization problem is called a *feasibility problem.*)

Observations

1. If $c_1 < 0$, we keep moving in the feasible region towards $+\infty$ (i.e., in the direction of $-c_1 > 0$).
2. If $c_1 > 0$, we keep moving in the feasible region towards $-\infty$ (i.e., in the direction of $-c_1 < 0$).
3. If $n = 1$, the set of optimal solutions, denoted by $\mathcal{P}^*$, is either the empty set, a single point (i.e., a vertex or an end point), or equal to the feasible region.
4. Therefore, if $\mathcal{P}^*$ is nonempty and $\mathcal{P}$ contains at least one vertex, then $\mathcal{P}^*$ contains at least one vertex.

9.2.2 Two Decision Variables ($n = 2$)

Let us now assume $n = 2$ (i.e., there are two decision variables x_1 and x_2).

Example 9.1. *Consider the following two-variable linear programming problem:*

$$
\begin{array}{lrcrcl}
\min & c_1x_1 & + & c_2x_2 & & \\
s.t. & & & & & \\
 & x_1 & + & 2x_2 & \leq & 4 \\
 & 3x_1 & + & x_2 & \leq & 6 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

We will consider three different objective functions given by (i) $c = [1, -2]^T$, (ii) $c = [-2, -4]^T$, and (iii) $c = [0, 0]^T$.

Let us first ignore the objective function and draw the feasible region. Note that the feasible region does not depend on the objective function. In this example, the feasible region, which is depicted in Figure 9.1, is a nonempty polytope (i.e., bounded polyhedron).

(i) Let $c = [1, -2]^T$. Since this is a minimization problem, the improving direction is given by $-c = [-1, 2]^T$, which points in the northwest direction (see Figure 9.2). Therefore, the unique optimal solution is given by the intersections of the boundaries of the two constraints $x_1 + 2x_2 \leq 4$ and $x_1 \geq 0$. Therefore, we can solve for the following system simultaneously:

$$
\begin{aligned}
x_1 + 2x_2 &= 4 \\
x_1 \qquad &= 0
\end{aligned}
$$

We obtain $x_1 = 0$ and $x_2 = 2$. Therefore, $\mathcal{P}^* = \{[0, 2]^T\}$, i.e., it is given by a single vertex. Substituting this solution into the objective function, we obtain $z^* = 1 \cdot (0) - 2 \cdot 2 = -4$.

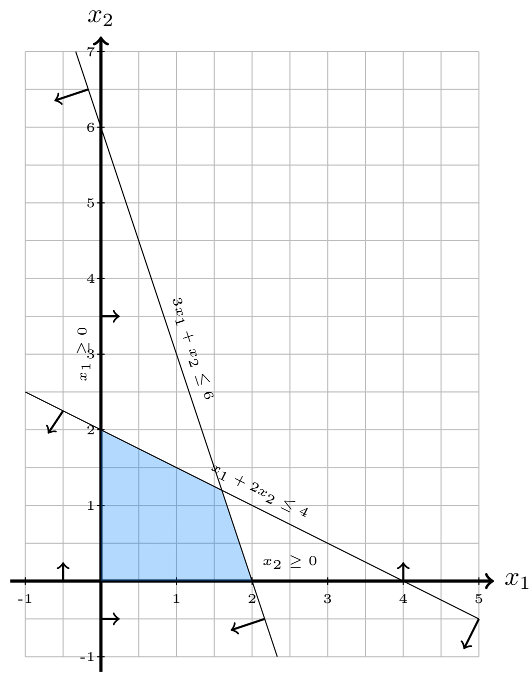


Figure 9.1: Feasible region of Example 9.1 (blue region)

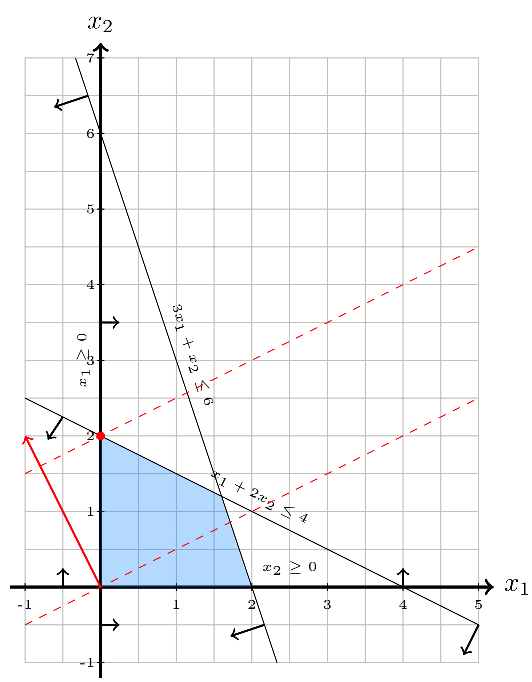


Figure 9.2: Feasible region of Example 9.1(i) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the optimal solution (red circle)

(ii) *Let* $c = [-2, -4]^T$. *Since this is a minimization problem, the improving direction is given by* $-c = [2, 4]^T$, *which points in the northeast direction (see Figure 9.3). Therefore, any feasible solution on the boundary of the constraint* $x_1 + 2x_2 \leq 4$ *is an optimal solution.*

To compute the coordinates of the end points of this line segment, we first consider the end point at the top left. Note that this point is given by the intersections of the boundaries of the two constraints $x_1 + 2x_2 \leq 4$ *and* $x_1 \geq 0$. *By part (i), we know that* $x^1 = [0, 2]^T$.

Considering now the end point at the bottom right, this point is given by the intersections of the boundaries of the two constraints $x_1 + 2x_2 \leq 4$ *and* $3x_1 + x_2 \leq 6$. *Therefore, we can solve for the following system simultaneously:*

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + x_2 &= 6 \end{aligned}$$

We obtain $x_1^2 = 8/5$ *and* $x_2^2 = 6/5$. *Therefore,* $\mathcal{P}^* = \{\lambda x^1 + (1-\lambda)x^2 : \lambda \in [0, 1]\}$, *i.e., it is given by a line segment. Note that any point on this line segment is an optimal solution. Substituting any point on this line segment into the objective function (say,* $[0, 2]^T$*), we obtain* $z^* = -2 \cdot (0) - 4 \cdot 2 = -8$.

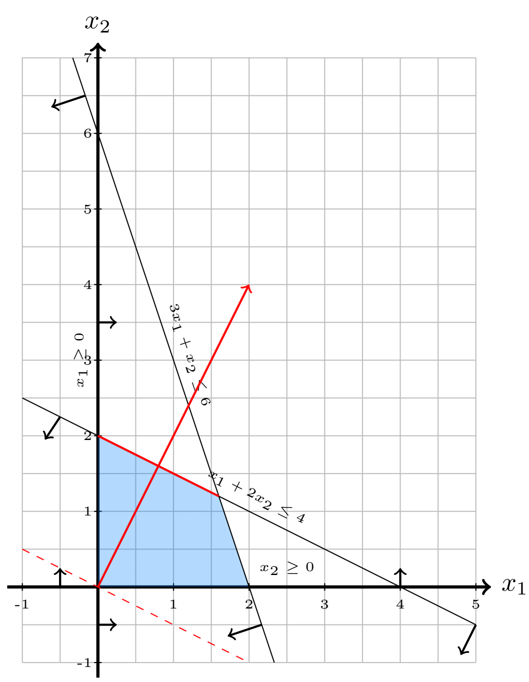


Figure 9.3: Feasible region of Example 9.1(ii) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red line), and the set of optimal solutions (red line segment)

(iii) *Let* $c = [0, 0]^T$. *Note that, in this case, the objective function does not depend on* x_1 *and* x_2, *i.e., the objective function value of* any *feasible solution is equal to* 0. *Therefore, in this case any feasible solution is an optimal solution, i.e.,* $\mathcal{P}^* = \mathcal{P}$, *where* $\mathcal{P}$ *denotes the feasible region. The optimal value is given by* $z^* = 0$ *(see Figure 9.4).*

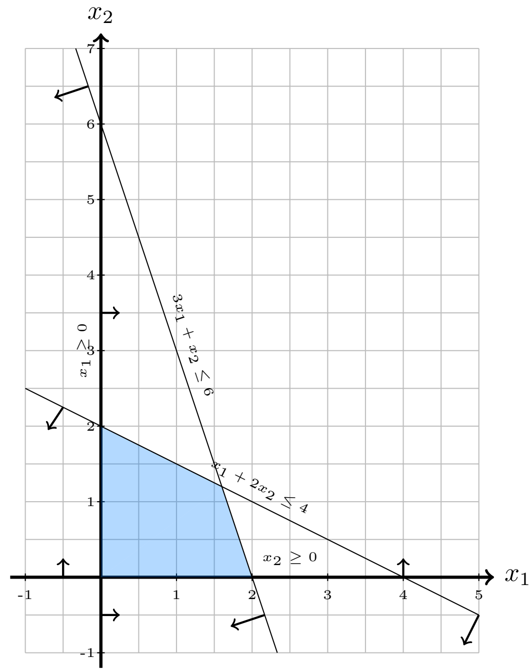


Figure 9.4: Feasible region of Example 9.1(iii) (blue region) and the set of optimal solutions (blue region)

Suppose now that we change the second constraint in Example 9.1 as follows:

$$\begin{array}{lrcrcr} \min & c_1x_1 & + & c_2x_2 & & \\ \text{s.t.} & & & & & \\ & x_1 & + & 2x_2 & \leq & 4 \\ & x_1 & + & x_2 & \leq & -1 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

We will consider three different objective functions given by (i) $c = [1, -2]^T$, (ii) $c = [-2, -4]^T$, and (iii) $c = [0, 0]^T$.

Once again, let us first ignore the objective function and draw the feasible region. Recall that the feasible region does not depend on the objective function. As illustrated by Figure 9.5, the halfspaces corresponding to the four constraints do not have a common point. In fact, it is easy to see that the three constraints $x_1 \geq 0$, $x_2 \geq 0$, and $x_1 + x_2 \leq -1$ cannot be satisfied simultaneously. Therefore, the feasible region of this modified problem is the empty set, i.e., $\mathcal{P} = \emptyset$. As a result, regardless of the objective function, this linear programming problem is infeasible and we define the optimal value $z^* = +\infty$ since it is a minimization problem. Clearly, $\mathcal{P}^* = \emptyset$, i.e., there is no optimal solution since there is no feasible solution.

Observations

1. If $n = 2$ and $\mathcal{P}$ is a nonempty polytope, then $\mathcal{P}^*$ is always nonempty, and $\mathcal{P}^*$ is either a single vertex, or a line segment, or $\mathcal{P}^* = \mathcal{P}$.
2. Note that $\mathcal{P}$ always contains at least one vertex (i.e., a corner point).
3. Note that $\mathcal{P}^*$ always contains at least one vertex of $\mathcal{P}$.

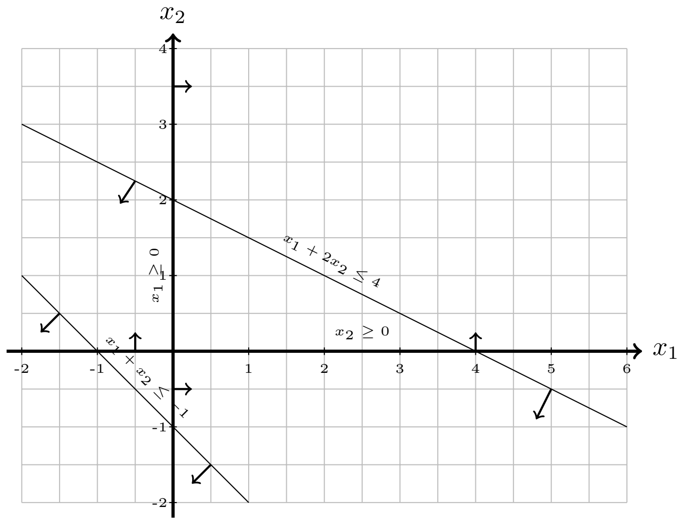


Figure 9.5: Feasible region of the modified Example 9.1

4. Such a linear programming problem cannot be unbounded.
5. If $\mathcal{P} = \emptyset$ (i.e., the problem is infeasible), then $\mathcal{P}^* = \emptyset$. Recall that, in this case, we define $z^* = +\infty$ for a minimization problem.

Example 9.2. *Consider the following linear programming problem:*

$$\begin{array}{llcrcl} \min & c_1x_1 & + & c_2x_2 & & \\ s.t. & & & & & \\ & -x_1 & + & 2x_2 & \le & 2 \\ & x_1 & & & \ge & 0 \\ & & & x_2 & \ge & 0 \end{array}$$

We will consider five different objective functions given by (i) $c = [1, -1]^T$*, (ii)* $c = [1, 0]^T$*, (iii)* $c = [2, -4]^T$*, (iv)* $c = [-1, 1]^T$*, and (v)* $c = [0, 0]^T$*.*

Once again, let us first ignore the objective function and draw the feasible region. Recall that the feasible region does not depend on the objective function. In this example, the feasible region, which is depicted in Figure 9.6, is a nonempty polyhedron. Note that, in contrast with Example 9.1, it is unbounded, i.e., it is a polyhedron but not a polytope.

(i) Let $c = [1, -1]^T$*. Since this is a minimization problem, the improving direction is given by* $-c = [-1, 1]^T$*, which points in the northwest direction (see Figure 9.7). Therefore, the unique optimal solution is given by the intersections of the boundaries of the two constraints* $x_1 + 2x_2 \le 2$ *and* $x_1 \ge 0$*. Therefore, we can solve for the following system simultaneously:*

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ x_1 \qquad &= 0 \end{aligned}$$

We obtain $x_1 = 0$ *and* $x_2 = 1$*. Therefore,* $\mathcal{P}^* = \{[0, 1]^T\}$*, i.e., it is given by a single vertex. Substituting this solution into the objective function, we obtain* $z^* = 1 \cdot (0) - 1 \cdot 1 = -1$*.*

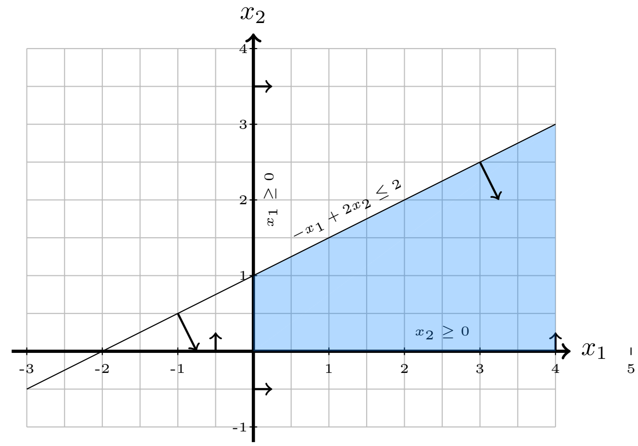


Figure 9.6: Feasible region of Example 9.2 (blue region)

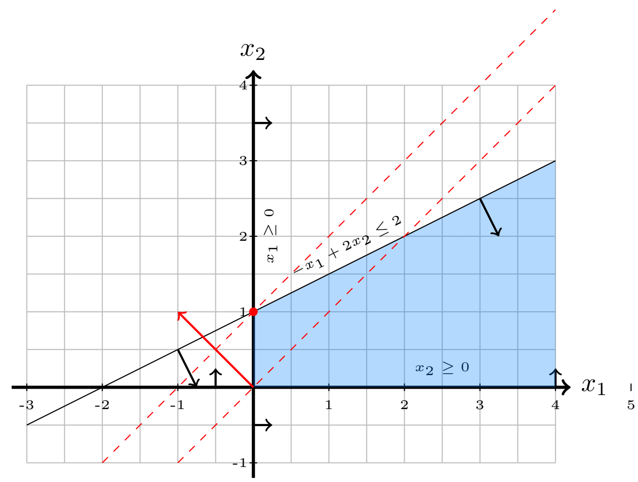


Figure 9.7: Feasible region of Example 9.2(i) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the optimal solution (red circle)

(ii) *Let $c = [1, 0]^T$. Since this is a minimization problem, the improving direction is given by $-c = [-1, 0]^T$, which points to the west parallel to the x_1 axis (see Figure 9.8). Therefore, any feasible solution on the boundary of the constraint $x_1 \ge 0$ is an optimal solution.*

To compute the coordinates of the end points of this line segment, we first consider the end point at the bottom. Note that this point is given by the intersections of the boundaries of the two constraints $x_1 \ge 0$ and $x_2 \ge 0$. Solving the equation system simultaneously, we obtain $x^1 = [0, 0]^T$.

Considering now the end point at the top, this point is given by the intersections of the boundaries of the two constraints $-x_1 + 2x_2 \le 2$ and $x_1 \ge 0$. Solving the equation system simultaneously, we obtain $x^2 = [0, 1]^T$.

Therefore, $\mathcal{P}^* = \{\lambda x^1 + (1-\lambda)x^2 : \lambda \in [0, 1]\}$, i.e., it is given by a line segment. Note that any point on this line segment is an optimal solution. Substituting any point on this line segment into the objective function (say, $[0, 0]^T$), we obtain $z^* = 1 \cdot (0) - 0 \cdot (0) = 0$.

(iii) *Let $c = [2, -4]^T$. Since this is a minimization problem, the improving direction is given by $-c =$*

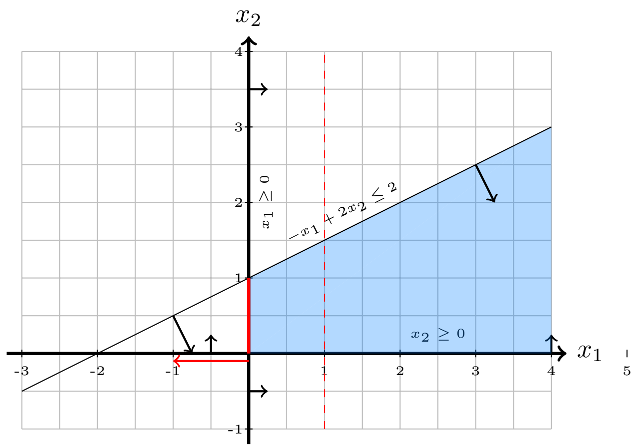


Figure 9.8: Feasible region of Example 9.2(ii) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the set of optimal solutions (red line segment)

$[-2, 4]^T$, which points in the northwest direction (see Figure 9.9). Therefore, any feasible solution on the boundary of the constraint $-x_1 + 2x_2 \leq 2$ is an optimal solution.

To compute the end point of this half line, we note that it is given by the intersection of the boundary of the constraints $-x_1 + 2x_2 \leq 2$ and $x_1 \geq 0$. Solving the equation system simultaneously, we obtain $x^1 = [0, 1]^T$.

A half line is given by a starting point and a direction. To find the direction, simply pick any point on this half line different from x^1, say $x^2 = [2, 2]^T$. The direction can be computed by $d = x^2 - x^1 = [2, 1]^T$. Therefore, $\mathcal{P}^* = \{x^1 + \lambda d : \lambda \geq 0\}$, i.e., it is given by a half line. Note that any point on this half line is an optimal solution. Substituting any point on this line segment into the objective function (say, $[0, 1]^T$), we obtain $z^* = 2 \cdot (0) - 4 \cdot (1) = -4$.

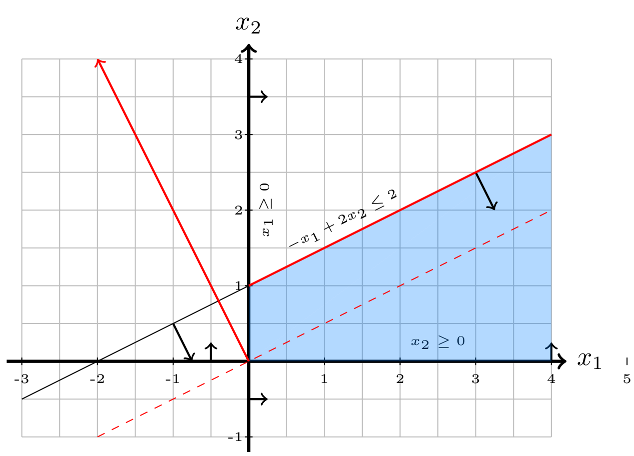


Figure 9.9: Feasible region of Example 9.2(iii) (blue region), improving direction (solid red arrow), contour lines of the objective function (dashed red lines), and the set of optimal solutions (red half line)

(iv) Let $c = [-1, 1]^T$. Since this is a minimization problem, the improving direction is given by $-c = [1, -1]^T$, which points in the southeast direction (see Figure 9.10). Note that the contour lines of the objective function will always have a nonempty intersection with the feasible region in the improving

direction. Therefore, this problem is unbounded, i.e., $z^* = -\infty$ *and* $\mathcal{P}^* = \emptyset$ *since no feasible solution achieves this objective function value (i.e., there is no best feasible solution).*

You can verify that starting at $x^1 = [0,0]^T$ *and moving in the direction* $d = [1,0]^T$*, if you consider the half line* $\{x^1 + \lambda d : \lambda \geq 0\}$*, which is contained in the feasible region, the objective function value along this half line is given by* $z(\lambda) = (-1) \cdot (0 + \lambda) + 1 \cdot (0) = -\lambda$*, which tends to* $-\infty$ *as* λ *tends to* $+\infty$*. Such a direction is called* a direction of unboundedness.

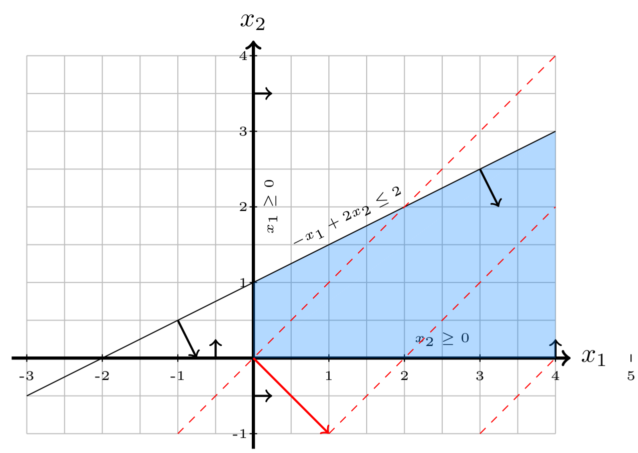


Figure 9.10: Feasible region of Example 9.2(iv) (blue region), improving direction (solid red arrow), and contour lines of the objective function (dashed red lines)

(v) $c = [0,0]^T$*. As in Example 9.1(iii), the objective function does not depend on* x_1 *and* x_2*, i.e., the objective function value of* any *feasible solution is equal to* 0*. Therefore, in this case any feasible solution is an optimal solution, i.e.,* $\mathcal{P}^* = \mathcal{P}$*, where* $\mathcal{P}$ *denotes the feasible region. The optimal value is given by* $z^* = 0$ *(see Figure 9.11).*

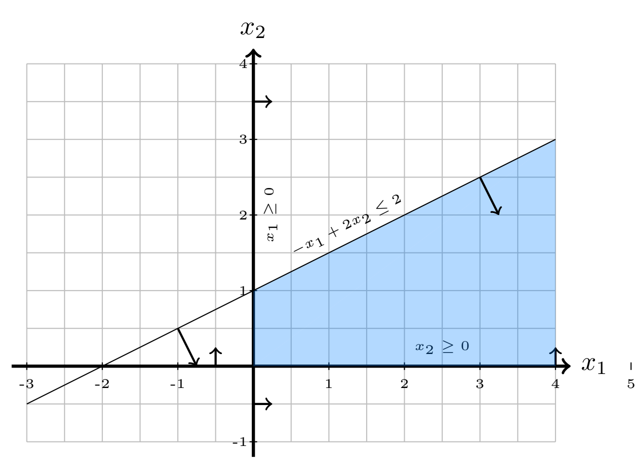


Figure 9.11: Feasible region of Example 9.2(v) (blue region) and the set of optimal solutions (blue region)

Observations

1. If $n = 2$ and $\mathcal{P}$ is a nonempty and unbounded polyhedron that has at least one vertex, then $\mathcal{P}^*$ is either the empty set, or a single vertex, or a line segment, or a half line, or $\mathcal{P}^* = \mathcal{P}$.
2. If $\mathcal{P}$ contains at least one vertex and $\mathcal{P}^* \neq \emptyset$, then $\mathcal{P}^*$ always contains at least one vertex of $\mathcal{P}$.
3. Such a linear programming problem may have a finite optimal value or may be unbounded (even though $\mathcal{P}$ itself is unbounded).
4. If $\mathcal{P} = \emptyset$, then $\mathcal{P}^* = \emptyset$.
5. Recall that a nonempty and unbounded polyhedron may not have any vertices (if it contains a line).

Remarks

1. Linear programming problems with one or two decision variables can be easily solved by using the graphical method.
2. We keep moving in the feasible region towards the improving direction, given by $-c$ for a minimization problem, until the last point of contact (if any).
3. The graphical method can be extended to linear programming problems with three decision variables but drawing three-dimensional objects is much more tricky.
4. For linear programming problems with at least four decision variables, we need to develop a different algorithm.

Exercises

Question 9.1. *Can you construct a linear programming problem with exactly two optimal solutions?*

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 10	Optimality of Vertices and Vertex Enumeration
Lecturer: E. Alper Yıldırım	*Week: 3*

10.1 Outline

- Optimality of Vertices
- Vertex Enumeration
- Review Problems

10.2 Optimality of Vertices

In Lecture 9, we discussed the graphical solution method for solving linear programming problems with one or two decision variables. In each of the examples, if the feasible region $\mathcal{P}$ contains at least one vertex and the set of optimal solutions $\mathcal{P}^*$ is nonempty, then $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$. In this lecture, we will show that this property is satisfied by every polyhedron.

Proposition 10.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1; (a^i)^T x \leq b_i,\ i \in M_2; (a^i)^T x = b_i,\ i \in M_3\}$ be a nonempty polyhedron that contains at least one vertex and let $c \in \mathbb{R}^n$. Consider the linear programming problem given by*

$$\min\{c^T x : x \in \mathcal{P}\}.$$

Suppose that the set of optimal solutions, denoted by $\mathcal{P}^*$, is nonempty. Then, $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$ (i.e., there exists at least one optimal solution which is a vertex of $\mathcal{P}$).

Proof. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron and let $\mathcal{P}^* \subseteq \mathcal{P}$ denote the set of optimal solutions. Let $z^* \in \mathbb{R}$ denote the optimal value of the linear programming problem $\min\{c^T x : x \in \mathcal{P}\}$. Then, $c^T x^* = z^*$ for any $x^* \in \mathcal{P}^*$. Therefore, $\mathcal{P}^* = \mathcal{P} \cap \{x \in \mathbb{R}^n : c^T x = z^*\}$. It follows that $\mathcal{P}^*$ itself is a polyhedron. Since $\mathcal{P}^* \subseteq \mathcal{P}$ and $\mathcal{P}$ does not contain a line, then $\mathcal{P}^*$ does not contain a line. Therefore, $\mathcal{P}^*$ has at least one vertex by Proposition 8.1.

Let $x^* \in \mathcal{P}^*$ denote a vertex of $\mathcal{P}^*$. We claim that x^* is also a vertex of $\mathcal{P}$. Let

$$I(x^*) = \left\{i \in M_1 \cup M_2 \cup M_3 : (a^i)^T x^* = b_i\right\}.$$

If the set $\{a^i : i \in I(x^*)\}$ contains n linearly independent vectors, then we are done since x^* is a vertex of $\mathcal{P}$. Otherwise, there exists a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $(a^i)^T d = 0$ for each $i \in I(x^*)$. By using a similar argument as in the proof of Proposition 7.1, there exists a real number $\epsilon^* > 0$ such that $x^* - \epsilon^* d \in \mathcal{P}$ and $x^* + \epsilon^* d \in \mathcal{P}$. Since $c^T x \geq z^*$ for each $x \in \mathcal{P}$, we obtain $c^T(x^* - \epsilon^* d) = z^* - \epsilon^* c^T d \geq z^*$ and $c^T(x^* + \epsilon^* d) = z^* + \epsilon^* c^T d \geq z^*$. Therefore, $c^T d = 0$. However, x^* is a vertex of $\mathcal{P}^*$, which implies that the set $\{c\} \cup \{a^i : i \in I(x^*)\}$ contains n linearly independent vectors. Therefore, $d = \mathbf{0}$, which is a contradiction. It follows that x^* is also a vertex of $\mathcal{P}$.

□

Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1; (a^i)^T x \leq b_i,\ i \in M_2; (a^i)^T x = b_i,\ i \in M_3\}$ be a nonempty polyhedron that contains at least one vertex. By Proposition 6.1, since $\mathcal{P}$ is a convex set, a vector $\hat{x} \in \mathcal{P}$ is a vertex of $\mathcal{P}$ if and only if **there exists** a vector a, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, such that $\hat{x}$ is the **unique** optimal solution of the linear programming problem $\min\{a^T x : x \in \mathcal{P}\}$. By Proposition 10.1, for **every** vector $c \in \mathbb{R}^n$, there exists at least one vertex of $\mathcal{P}$ which is an optimal solution of $\min\{c^T x : x \in \mathcal{P}\}$, under the assumption that $\mathcal{P}^*$ is nonempty.

10.3 A Naive Enumeration Algorithm

By Proposition 10.1, if $\mathcal{P}$ has at least one vertex and $\mathcal{P}^* \neq \emptyset$, then the set of optimal solutions contains at least one vertex. By Proposition 8.2, every polyhedron has a finite number of vertices. Therefore, by combining these two observations, we may consider a naive enumeration idea for solving general linear programming problems by simply computing all the vertices, evaluating the objective function value at each vertex, and returning the vertex with the best objective function value:

Naive Enumeration Algorithm:

1. Compute all vertices of the polyhedron.
2. Compute the objective function at each vertex.
3. Output the vertex with the best objective function value.

10.3.1 Computing Vertices

Consider a general polyhedron given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\}.$$

Below, we give an algorithm for enumerating all vertices by using the equivalence between vertices of a polyhedron and basic feasible solutions (see Proposition 7.1).

Algorithm for Enumeration of Vertices of a Polyhedron:

1. For each subset $J \subseteq M_1 \cup M_2$ do
   (a) Let $I = M_3 \cup J$.
   (b) If the set $\{a^i : i \in I\}$ contains n linearly independent vectors, then let $I^* \subseteq I$ be such that the set $\{a^i : i \in I^*\}$ contains exactly n linearly independent vectors.
      i. Solve the system $(a^i)^T x = b_i,\ i \in I^*$ to obtain the unique solution $\hat{x} \in \mathbb{R}^n$. ($\hat{x}$ is a basic solution.)
      ii. Check if $\hat{x} \in \mathcal{P}$ (i.e., if $\hat{x}$ is a basic feasible solution).

Note that the above algorithm terminates after a finite number of iterations by successfully computing all vertices of $\mathcal{P}$ (if any) since there are only a finite number of choices for the set I. Therefore, this algorithm can be used in Step 1 of the Naive Enumeration Algorithm. However, it has a number of shortcomings outlined below:

1. A polyhedron may have an exponential number of vertices.
2. A polyhedron may not contain a vertex.
3. A polyhedron may not contain any vectors at all (i.e., it can be equal to the empty set).
4. The naive enumeration algorithm cannot detect if a linear programming problem is infeasible or unbounded.

Therefore, we need to develop a more clever algorithm for solving linear programming problems that can also correctly detect unboundedness and infeasibility.

Exercises

Question 10.1. *For each value of $k = 0, 1, \ldots$, show that you can construct a linear programming problem with two decision variables such that the set of optimal solutions $\mathcal{P}^*$ contains exactly k vertices.*

MATH11111: Fundamentals of Optimization	Fall 2023
Lecture 11 Polyhedra and Linear Programming in Standard Form	
Lecturer: E. Alper Yıldırım	*Week: 3*

11.1 Outline

- Polyhedra in Standard Form
- Linear Programming in Standard Form
- Review Problems

11.2 Introduction and Motivation

Recall that we can solve linear programming problems with up to 3 variables geometrically using the graphical solution method. If there is a larger number of variables, we need an alternative method. The naive vertex enumeration algorithm outlined in Lecture 10 has several drawbacks. In this lecture, we will start to discuss building blocks to develop a more effective algorithm for solving a general linear programming problem. To that end, we will identify a particular convenient form for a polyhedron and a linear programming problem.

11.3 Polyhedra in Standard Form

Definition 11.1. *A polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ is said to be* in standard form *if $\mathcal{P}$ is given by*

$$\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x = b_i, \; i = 1, \ldots, m; \; x_j \geq 0, \; j = 1, \ldots, n\}.$$

11.3.1 Relation with General Polyhedra

Consider a general polyhedron given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\}.$$

Therefore, $\mathcal{P}$ is in standard form if

- each constraint $(a^i)^T x \geq b_i, \; i \in M_1$ is given by $(e^j)^T x = x_j \geq 0$ for some $j \in J_1$, where $J_1 \subseteq \{1, \ldots, n\}$; and
- each constraint $(a^i)^T x \leq b_i, \; i \in M_2$ is given by $-(e^j)^T x = -x_j \leq 0$ for some $j \in J_2$, where $J_2 \subseteq \{1, \ldots, n\}$; and

- $J_1 \cup J_2 = \{1, \ldots, n\}$.

It follows that a polyhedron in standard form is a special case of a general polyhedron.

11.4 Linear Programming in Standard Form

Definition 11.2. *A linear programming is said to be* in standard form *if it is given by*

$$
\begin{array}{lllll}
(P) & \min & c^T x & & \\
 & s.t. & & & \\
 & & (a^i)^T x & = & b_i, \quad i = 1, \ldots, m, \\
 & & x_j & \geq & 0, \quad j = 1, \ldots, n.
\end{array}
$$

Remark 11.1. *Similar to the relation between a polyhedron in standard form and a general polyhedron, a linear programming in standard form is a special case of general linear programming. Note that the objective function is minimized in a linear programming problem in standard form.*

11.4.1 Transformation to Standard Form

Proposition 11.1. *Any linear programming problem can be transformed into an equivalent linear programming problem in standard form.*

Remark 11.2. 1. *"Equivalent" means there is a one-to-one correspondence between the feasible solutions of two problems and they have the same optimal value.*

2. *Therefore, a linear programming in standard form is not really a special case.*

Proof. If (P) is already in standard form, we are done. Otherwise, for each $i \in M_1$ such that the corresponding constraint is not given by $(a^i)^T x = (e^j)^T x = x_j \geq 0$ for some $j \in \{1, \ldots, n\}$, we define a new nonnegative variable s_i and replace $(a^i)^T x \geq b_i$ by $(a^i)^T x - s_i = b_i$ and $s_i \geq 0$. For each $i \in M_2$ such that the corresponding constraint is not given by $(a^i)^T x = -(e^j)^T x = -x_j \leq 0$ for some $j \in \{1, \ldots, n\}$, we define a new nonnegative variable s_i and replace $(a^i)^T x \leq b_i$ by $(a^i)^T x + s_i = b_i$ and $s_i \geq 0$. Therefore, we replace each inequality constraint by equality constraints. For each $j = 1, \ldots, n$ such that $x_j \geq 0$ is not included in the inequality constraints, if $x_j \leq 0$ is a constraint, then we can define a new variable by $x'_j = -x_j \geq 0$ and replace each occurrence of x_j by $-x'_j$. Otherwise, we can define two new nonnegative variables $x_j^+ \geq 0$ and $x_j^- \geq 0$ and replace each occurrence of x_j by $x_j^+ - x_j^-$. We therefore obtain an equivalent linear programming problem in standard form. $\square$

11.4.2 Advantages of Linear Programming in Standard Form

1. Linear programming problems in standard form have a simpler structure than general linear programming problems.

2. This transformation allows us to develop an algorithm only for linear programming problems in standard form and use it to solve all linear programming problems.

3. A polyhedron in standard form has certain desirable properties.

11.4.3 Existence of Vertices in Polyhedra in Standard Form

Proposition 11.2. *Let*

$$\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x = b_i,\ i = 1, \ldots, m;\ x_j \geq 0,\ j = 1, \ldots, n\}$$

be a nonempty polyhedron in standard form. Then, $\mathcal{P}$ *has at least one vertex.*

Proof. Since $\mathcal{P}$ is a nonempty polyhedron, it suffices to show that $\mathcal{P}$ does not contain a line and then the claim follows directly from Proposition 8.1. Note that $\mathcal{P} \subseteq \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_j \geq 0,\ j = 1, \ldots, n\}$. Since $\mathbb{R}^n_+$ does not contain a line, $\mathcal{P}$ cannot contain a line. Therefore, $\mathcal{P}$ has at least one vertex. □

Remark 11.3. *Note that, in contrast, a general nonempty polyhedron may not have any vertices.*

Exercises

Question 11.1. *Convert the following linear programming problem into standard form:*

$$\begin{array}{llllllllll}
\max & -x_1 & - & x_2 & + & x_3 & - & 2x_4 & & \\
s.t. & & & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 6 \\
 & & & x_2 & - & 3x_3 & + & 2x_4 & \leq & 3 \\
 & x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & \geq & 9 \\
 & x_1 \geq 0 & , & x_2 \geq 0 & , & x_3 \leq 0 & . & & &
\end{array}$$

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 12 Full Row Rank Assumption in Standard Form Polyhedra

Lecturer: E. Alper Yıldırım *Week: 3*

12.1 Outline

- Polyhedra in Standard Form: Different Cases
- Full Row Rank Assumption
- Review Problems

12.2 A Compact Representation

Recall that a linear programming in standard form is given by

$$\begin{array}{llrcll} \text{(P)} & \min & c^T x & & & \\ & \text{s.t.} & & & & \\ & & (a^i)^T x & = & b_i, & i = 1, \ldots, m, \\ & & x_j & \geq & 0, & j = 1, \ldots, n. \end{array}$$

Let us define $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$ as follows:

$$A = \begin{bmatrix} (a^1)^T \\ (a^2)^T \\ \vdots \\ (a^m)^T \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then, (P) can be represented by

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}$$

12.3 Polyhedra in Standard Form

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form. Then, $\mathcal{P} = \mathcal{L} \cap \mathbb{R}^n_+$, where $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\}$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$. Consider the columns of $A \in \mathbb{R}^{m \times n}$:

$$A = \begin{bmatrix} A^1 & A^2 & \ldots & A^n \end{bmatrix},$$

where $A^1, \ldots, A^n \in \mathbb{R}^m$. Therefore, $Ax = b$ if and only if $b = \sum\limits_{j=1}^{n} A^j x_j$, i.e., if and only if $b \in \text{span}\{A^1, \ldots, A^n\}$.

Let $r = \text{rank(A)}$. Note that r equals the largest number of linearly independent columns $\{A^1, \dots, A^n\}$, or equivalently, the largest number of linearly independent rows $\{(a^1)^T, \dots, (a^m)^T\}$. Therefore, $r \le \min\{m, n\}$.

Case 1: Suppose that $b \notin \text{span}\{A^1, \dots, A^n\}$. Then, $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\} = \emptyset$, and $\mathcal{P} = \mathcal{L} \cap \mathbb{R}^n_+ = \emptyset$. In this case, we have $r < m$, i.e., A does not have full row rank.

Case 2: Suppose now that $b \in \text{span}\{A^1, \dots, A^n\}$.

Case 2a: Suppose that $r = n \le m$. In this case, the set $\{A^1, \dots, A^n\}$ is linearly independent, i.e., A has full column rank. Then, there is a unique solution $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} = b$, i.e., $\mathcal{L} = \{\hat{x}\}$. If $\hat{x} \ge \mathbf{0}$, then $\mathcal{P} = \mathcal{L} = \{\hat{x}\}$, otherwise $\mathcal{P} = \emptyset$.

Case 2b: Suppose that $r = m \le n$. In this case, the set $\{A^1, \dots, A^n\} \subset \mathbb{R}^m$ contains m linearly independent vectors, or A has full row rank, i.e., $\text{span}\{A^1, \dots, A^n\} = \mathbb{R}^m$. For any $b \in \mathbb{R}^m$, the set $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$. If there exists $\hat{x} \in \mathcal{L}$ such that $\hat{x} \ge \mathbf{0}$, then $\mathcal{P} \neq \emptyset$. Otherwise, $\mathcal{P} = \emptyset$.

Case 2c: Suppose now that $r < m$ and $r < n$. In this case, A does not have full row rank and does not have full column rank. Therefore, the set $\{a^1, \dots, a^m\} \subset \mathbb{R}^n$ contains r linearly independent vectors.

Suppose that $\{a^1, \dots, a^r\}$ are linearly independent (otherwise we can rearrange the columns). Then, for each $j = r + 1, \dots, m$, there exist real numbers $\lambda^j_1, \dots, \lambda^j_r$ such that $a^j = \sum\limits_{i=1}^{r} \lambda^j_i a^i$. Since $b \in \text{span}\{A^1, \dots, A^n\}$, there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} = b$, i.e., $(a^i)^T \hat{x} = b_i, \ i = 1, \dots, m$.

For each $j = r + 1, \dots, m$, if we multiply $(a^i)^T \hat{x} = b_i$ by λ^j_i for each $i = 1, \dots, r$ and add them up, we obtain $\sum\limits_{i=1}^{r} \lambda^j_i (a^i)^T \hat{x} = \sum\limits_{i=1}^{r} \left(\lambda^j_i a^i\right)^T \hat{x} = (a^j)^T \hat{x} = \sum\limits_{i=1}^{r} \lambda^j_i b_i$. Similarly, for each $j = r + 1, \dots, m$, since $(a^j)^T \hat{x} = b_j$, we obtain $b_j = \sum\limits_{i=1}^{r} \lambda^j_i b_i$.

Therefore, for each $j = r + 1, \dots, m$, the equation $(a^j)^T \hat{x} = b_j$ is given by a linear combination of the first r equations and is therefore redundant. We can remove the redundant equations $(a^j)^T x = b_j$ for each $j = r + 1, \dots, m$ without changing the set of solutions. Let

$$\hat{A} = \begin{bmatrix} (a^1)^T \\ (a^2)^T \\ \vdots \\ (a^r)^T \end{bmatrix} \in \mathbb{R}^{r \times n}, \quad \hat{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix} \in \mathbb{R}^r.$$

Then, $Ax = b$ if and only if $\hat{A}x = \hat{b}$. Furthermore, $\text{rank}(\hat{A}) = r$, i.e., $\hat{A}$ has full row rank. Therefore, if $b \in \text{span}\{A^1, \dots, A^n\}$ and $r < m$ and $r < n$, we can remove the redundant equations from $Ax = b$ and the reduced system has full row rank. Observe that the reduced system is in the form of Case 2b.

Let us now summarise the different cases for the system $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If A does not have full row rank, then there are two possibilities:

1. The system $Ax = b$ has no solution (see Case 1) and $\mathcal{L} = \mathcal{P} = \emptyset$.
2. The system $Ax = b$ has at least one solution and the system can be reduced to $\hat{A}x = \hat{b}$, where $\hat{A}$ has full row rank (see Case 2c).

Both of these cases can be easily checked using Gaussian elimination.

12.3.1 Full Row Rank Assumption on A

Assumption 12.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be polyhedron in standard form. Suppose that $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$. Then, we can assume that A has full row rank.*

Exercises

Question 12.1. *Consider the system $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.*

(i) Show that the set of solutions to this system of equations is a convex set.

(ii) Show that the set of solutions to this system of equations is either the empty set, or consists of one solution, or an infinite number of solutions.

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 13 Basic Solutions of Polyhedra in Standard Form

Lecturer: E. Alper Yıldırım *Week: 4*

13.1 Outline

- Characterisation of Basic Feasible Solutions of Polyhedra in Standard Form
- Enumeration of Vertices
- Review Problems

13.2 Basic Solutions and Basic Feasible Solutions of Polyhedra in Standard Form

Question 1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form. Can we identify simple conditions for basic solutions and basic feasible solutions of $\mathcal{P}$?*

13.2.1 Review of General Polyhedra

Let us recall the definitions of basic solutions and basic feasible solutions for general polyhedra. Consider a general polyhedron $\mathcal{P} \subset \mathbb{R}^n$ given by

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} (a^i)^T x \geq b_i, & i \in M_1, \\ (a^i)^T x \leq b_i, & i \in M_2, \\ (a^i)^T x = b_i, & i \in M_3 \end{array} \right\}.$$

For $\hat{x} \in \mathbb{R}^n$, let $I(\hat{x}) = \left\{i \in M_1 \cup M_2 \cup M_3 : (a^i)^T \hat{x} = b_i\right\}$.

- $\hat{x}$ is a *basic solution* if all of the equality constraints are active at $\hat{x}$ (i.e., $M_3 \subseteq I(\hat{x})$) and the set $\{a^i : i \in I(\hat{x})\}$ contains n linearly independent vectors (i.e., the set $\{a^i : i \in I(\hat{x})\}$ spans $\mathbb{R}^n$).
- $\hat{x}$ is a *basic feasible solution* if $\hat{x}$ is a *basic solution* and $\hat{x}$ is feasible (i.e., $\hat{x} \in \mathcal{P}$).

In this lecture, we will specialise this definition to a polyhedron in standard form.

13.2.2 Polyhedra in Standard Form

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form.

- If $\hat{x} \in \mathbb{R}^n$ is a basic solution, then $A\hat{x} = b$, i.e., $(a^i)^T \hat{x} = b_i, \; i = 1, \ldots, m$.

- The inequality constraints in $\mathcal{P}$ are given by $x_j \geq 0$, i.e., $(e^j)^T x \geq 0,\ j = 1, \ldots, n$, where $e^j \in \mathbb{R}^n$ is the vector of all zeroes except for 1 in the jth coordinate, $j = 1, \ldots, n$.
- Label the constraints $(a^i)^T x = b_i$ by $i = 1, \ldots, m$ and the constraints $(e^j)^T x \geq 0,\ j = 1, \ldots, n$ using $m + 1, \ldots, m + n$.
- Then, $I(\hat{x}) = \{1, \ldots, m\} \cup \{m + j : (e^j)^T \hat{x} = 0\}$.
- Note that $m + j \in I(\hat{x})$ if and only if $(e^j)^T \hat{x} = \hat{x}_j = 0$, where $j = 1, \ldots, n$.
- Let us define the following index sets:

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}. \end{aligned}$$

- We have $\hat{B} \cup \hat{N} = \{1, \ldots, n\}$ and $\hat{B} \cap \hat{N} = \emptyset$.
- Note that $j \in \hat{N}$ if and only if $m + j \in I(\hat{x})$ (i.e., the constraint $x_j \geq 0$ is active at $\hat{x}$).
- Let $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ denote the submatrix of A consisting only of columns A^j, where $j \in \hat{B}$, and define $A_{\hat{N}} \in \mathbb{R}^{m \times |\hat{N}|}$ similarly.

Proposition 13.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form and let $A\hat{x} = b$. Then, $\hat{x} \in \mathbb{R}^n$ is a basic solution of $\mathcal{P}$ if and only if the submatrix $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank (i.e., the columns of $A_{\hat{B}}$ are linearly independent). Furthermore, $\hat{x} \in \mathbb{R}^n$ is a basic feasible solution (i.e., a vertex) of $\mathcal{P}$ if $\hat{x}$ is a basic solution and $\hat{x} \geq \mathbf{0}$.*

Proof. $\Rightarrow$: Suppose that $\hat{x} \in \mathbb{R}^n$ is a basic solution of $\mathcal{P}$ and let $I(\hat{x}) = \{1, \ldots, m\} \cup \{m + j : (e^j)^T \hat{x} = 0\}$. Then, $\operatorname{span}\left(\{a^i : i = 1, \ldots, m\} \cup \{e^j : \hat{x}_j = 0\}\right) = \mathbb{R}^n$. Suppose, for a contradiction, that $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ does not have full column rank. Therefore, there exists a vector $d_{\hat{B}} \in \mathbb{R}^{|\hat{B}|} \backslash \{\mathbf{0}\}$ such that $A_{\hat{B}} d_{\hat{B}} = \mathbf{0}$. Then, if we define $d_{\hat{N}} = \mathbf{0} \in \mathbb{R}^{|\hat{N}|}$, we obtain $A_{\hat{B}} d_{\hat{B}} + A_{\hat{N}} d_{\hat{N}} = \mathbf{0}$. Since $d_{\hat{B}} \neq \mathbf{0}$, we obtain a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $Ad = \mathbf{0}$. Hence, $(a^i)^T d = 0,\ i = 1, \ldots, m$. Furthermore, for each $j \in \hat{N}$, since $d_j = 0$, we obtain $(e^j)^T d = d_j = 0$. Since $d \neq \mathbf{0} \in \mathbb{R}^n$, it follows that $\operatorname{span}\left(\{a^i : i = 1, \ldots, m\} \cup \{e^j : \hat{x}_j = 0\}\right) \neq \mathbb{R}^n$. We obtain a contradiction.

$\Leftarrow$: Suppose that $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank. Suppose, for a contradiction, that $\hat{x}$ is not a basic solution. Then, $\operatorname{span}\left(\{a^i : i = 1, \ldots, m\} \cup \{e^j : \hat{x}_j = 0\}\right) \neq \mathbb{R}^n$, i.e., there exists a vector $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that $(a^i)^T d = 0,\ i = 1, \ldots, m$, and $(e^j)^T d = d_j = 0$ for each $j \in \{1, \ldots, n\}$ such that $\hat{x}_j = 0$. It follows that $Ad = \mathbf{0}$, or equivalently, $A_{\hat{B}} d_{\hat{B}} + A_{\hat{N}} d_{\hat{N}} = \mathbf{0}$, where $d_{\hat{B}} \in \mathbb{R}^{|\hat{B}|}$ and $d_{\hat{N}} \in \mathbb{R}^{|\hat{N}|}$ are the subvectors of d that contain only the coordinates in $\hat{B}$ and $\hat{N}$, respectively. Since $(e^j)^T d = d_j = 0$ for each $j \in \hat{N}$, we obtain $d_{\hat{N}} = \mathbf{0} \in \mathbb{R}^{|\hat{N}|}$. Therefore, we have $A_{\hat{B}} d_{\hat{B}} + A_{\hat{N}} d_{\hat{N}} = A_{\hat{B}} d_{\hat{B}} = \mathbf{0}$. Since $d \neq \mathbf{0}$ and $d_{\hat{N}} = \mathbf{0} \in \mathbb{R}^{|\hat{N}|}$, it follows that $d_{\hat{B}} \in \mathbb{R}^{|\hat{B}|} \backslash \{\mathbf{0}\}$. This contradicts our hypothesis that $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank. $\square$

13.2.3 A Simpler Characterisation Under the Full Row Rank Assumption

Recall that if $\mathcal{L} = \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$, we can assume that A has full row rank.

Question 2. *Under the full row rank assumption on A, can we obtain a simpler characterisation of basic solutions and basic feasible solutions?*

Proposition 13.2. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form and suppose that A has full row rank. Then, $\hat{x} \in \mathbb{R}^n$ is a basic solution of $\mathcal{P}$ if and only if there exist two index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ such that all of the following conditions hold:*

(i) $B \cup N = \{1, \ldots, n\}$, $|B| = m$ *and* $|N| = n - m$.

(ii) The matrix $A_B \in \mathbb{R}^{m \times m}$ *is invertible (i.e., nonsingular).*

(iii) $\hat{x}_B = (A_B)^{-1} b$ *and* $\hat{x}_N = \mathbf{0}$.

Furthermore, $\hat{x} \in \mathbb{R}^n$ *is a basic feasible solution (i.e., a vertex) of* $\mathcal{P}$ *if* $\hat{x}$ *is a basic solution and* $\hat{x} \geq \mathbf{0}$.

Proof. $\Rightarrow$: Let $\hat{x} \in \mathbb{R}^n$ be a basic solution of $\mathcal{P}$. Then, $A\hat{x} = b$. Recall the sets

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}. \end{aligned}$$

By Proposition 13.1, the submatrix $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank, which implies that $|\hat{B}| \leq m$.

Case 1: If $|\hat{B}| = m$, then $A_{\hat{B}} \in \mathbb{R}^{m \times m}$ and $\text{rank}(A_{\hat{B}}) = m$, which implies that $A_{\hat{B}}$ is an invertible matrix. Define $B = \hat{B}$ and $N = \hat{N}$. Since $A_{\hat{B}} \hat{x}_{\hat{B}} = A_B \hat{x}_B = b$, $\hat{x}_N = \hat{x}_{\hat{N}} = \mathbf{0}$ and A_B is invertible, the claim follows.

Case 2: If $|\hat{B}| < m$, then since the maximum number of linearly independent columns of A is equal to m and the submatrix $A_{\hat{B}} \in \mathbb{R}^{m \times |\hat{B}|}$ has full column rank, we can find $m - |\hat{B}|$ columns in $A_{\hat{N}}$ so that the set of all columns of $A_{\hat{B}}$ and the $m - |\hat{B}|$ columns in $A_{\hat{N}}$ are linearly independent. Let $J \subseteq \hat{N}$ denote the indices of those $m - |\hat{B}|$ columns in $A_{\hat{N}}$. Define $B = \hat{B} \cup J$ and $N = \hat{N} \backslash J$. Note that $|B| = m$ and $A_B \in \mathbb{R}^{m \times m}$ is an invertible matrix. Since $A_{\hat{B}} \hat{x}_{\hat{B}} = A_B \hat{x}_B = b$, $\hat{x}_N = \mathbf{0}$ and A_B is invertible, the claim follows.

$\Leftarrow$: Let $\hat{x} \in \mathbb{R}^n$ be such that there exist two index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ that satisfy conditions (i), (ii), and (iii). By condition (iii), we have $A\hat{x} = A_B \hat{x}_B + A_N \hat{x}_N = A_B \hat{x}_B = A_B (A_B)^{-1} b = b$. Therefore, $A\hat{x} = b$. Recall the sets

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}. \end{aligned}$$

By condition (iii), we readily obtain $\hat{B} \subseteq B$ and $N \subseteq \hat{N}$. By condition (ii), $A_{\hat{B}}$ has full column rank since $\hat{B} \subseteq B$ and the columns of A_B are linearly independent. By Proposition 13.1, $\hat{x}$ is a basic solution of $\mathcal{P}$. The claim follows. $\square$

13.2.4 Terminology

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form. Let $\hat{x} \in \mathbb{R}^n$ be a basic solution of $\mathcal{P}$. Let

$$\begin{aligned} \hat{B} &= \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}, \\ \hat{N} &= \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}. \end{aligned}$$

and let B and N be two index sets such that $\hat{B} \subseteq B \subseteq \{1, \ldots, n\}$ and $N \subseteq \hat{N} \subseteq \{1, \ldots, n\}$, $B \cup N = \{1, \ldots, n\}$, $|B| = m$, and $|N| = n - m$.

Definition 13.1. 1. *For each $j \in B$, the variable x_j is called a* basic variable. *Note that $\hat{x}_j \geq 0$ for each $j \in B$ if $\hat{x}$ is a basic feasible solution.*

2. *For each $j \in N$, the variable x_j is called a* nonbasic variable. *Note that $\hat{x}_j = 0$ for each $j \in N$ if $\hat{x}$ is a basic solution or a basic feasible solution.*

3. *The invertible matrix $A_B \in \mathbb{R}^{m \times m}$ is called the* basis matrix.

13.2.5 Enumeration of Vertices of Polyhedra in Standard Form

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form and suppose that A has full row rank. Below is an algorithm for enumerating (i.e., computing) all vertices of $\mathcal{P}$.

Enumeration of Vertices:

(i) Choose m linearly independent columns from A and let $B \subseteq \{1, \ldots, n\}$ denote the set of indices of these columns.

(ii) Define $N = \{1, \ldots, n\} \backslash B$.

(iii) Solve the system $A_B \, \hat{x}_B = b$, i.e., compute $\hat{x}_B = (A_B)^{-1} b$ and obtain the values of basic variables.

(iv) Define $\hat{x}_N = \mathbf{0} \in \mathbb{R}^{|N|}$, i.e., set all nonbasic variables to zero.

(v) $\hat{x} \in \mathbb{R}^n$ is a basic solution.

(vi) If $\hat{x}_B \geq \mathbf{0}$ (i.e., values of all basic variables are nonnegative), then $\hat{x} \in \mathbb{R}^n$ is a basic feasible solution.

Remarks

1. Recall that every polyhedron has at most a finite number of vertices.

2. Every nonempty polyhedron in standard form has at least one vertex.

3. Under the full row rank assumption on A, the number of basic solutions of a polyhedron in standard form with m equations and n variables is bounded above by $\binom{n}{m}$.

4. Therefore, if A has full row rank, then the number of basic feasible solutions is bounded above by $\binom{n}{m}$ since the set of basic feasible solutions is a subset of basic solutions.

5. However, if $m = 10$ and $n = 20$ (considered a "tiny" linear programming problem), we have $\binom{n}{m} = 184756$.

Exercises

Question 13.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a polyhedron in standard form and suppose that A has full row rank. Suppose that $\mathcal{P} = \emptyset$.*

(i) Does $\mathcal{P}$ have any basic feasible solutions?

(ii) Does $\mathcal{P}$ have any basic solutions?

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 14 Fundamental Theorem of Linear Programming

Lecturer: E. Alper Yıldırım *Week: 4*

14.1 Outline

- Existence of an Optimal Solution
- Fundamental Theorem of Linear Programming
- Review Problems

14.2 Overview

In this lecture, we will establish the fundamental theorem of linear programming. This theorem will form the theoretical basis to develop a solution method for a general linear programming problem.

Let

$$\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i, \ i \in M_1; (a^i)^T x \leq b_i, \ i \in M_2; (a^i)^T x = b_i, \ i \in M_3\}$$

be a nonempty polyhedron that contains at least one vertex and let $c \in \mathbb{R}^n$.

Consider the general linear programming problem given by

$$\text{(P)} \quad \min\{c^T x : x \in \mathcal{P}\}.$$

Earlier, we established the following result: If $\mathcal{P}$ contains at least one vertex, and the set of optimal solutions of (P), denoted by $\mathcal{P}^*$, is nonempty, then $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$ (i.e., there exists at least one optimal solution which is a vertex of $\mathcal{P}$).

However, note that this is still a partial result since (i) we need to assume that $\mathcal{P}$ contains at least one vertex, and (ii) that $\mathcal{P}^* \neq \emptyset$.

If a linear programming problem is infeasible or unbounded, then it does not have any optimal solutions. In general, an optimization problem may not have any optimal solutions even if it has a finite optimal value:

$$\min\{1/x : x \geq 1\}$$

Note that the optimal value is 0, and even though the problem has a finite optimal value, this value is not attained by any feasible solution.

Question 1. *Does every linear programming problem with a finite optimal value have at least one optimal solution?*

14.3 Existence of an Optimal Solution

The following proposition establishes that every linear programming problem in standard form with a finite optimal value has at least one optimal solution (i.e., the optimal value is attained by at least one feasible solution).

Proposition 14.1. *Let (P) denote a linear programming problem in standard form.*

$$\begin{array}{lll} (P) & \min & c^T x \\ & s.t. & \\ & & \begin{array}{rcl} Ax & = & b, \\ x & \geq & \mathbf{0}. \end{array} \end{array}$$

If (P) has a finite optimal value, then the set of optimal solutions, denoted by $\mathcal{P}^*$, is nonempty. Furthermore, $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$.

Proof. Since the feasible region, denoted by $\mathcal{P}$, is nonempty, we may assume that A has full row rank. We will show the following result. For every feasible solution $\bar{x} \in \mathcal{P}$, there exists a vertex $\hat{x} \in \mathcal{P}$ such that $c^T\hat{x} \leq c^T\bar{x}$. Let $\bar{x} \in \mathcal{P}$ be an arbitrary feasible solution.

Case 1: If $\bar{x}$ is a vertex, then we set $\hat{x} = \bar{x}$ and this proves our claim.

Case 2: Suppose that $\bar{x}$ is not a vertex. Let $\bar{B} = \{j \in \{1,\ldots,n\} : \bar{x}_j \neq 0\} = \{j \in \{1,\ldots,n\} : \bar{x}_j > 0\}$, and $\bar{N} = \{j \in \{1,\ldots,n\} : \bar{x}_j = 0\}$. Then, $A_{\bar{B}} \in \mathbb{R}^{m \times |\bar{B}|}$ does not have full column rank. Therefore, there exists a nonzero vector $d_{\bar{B}} \in \mathbb{R}^{|\bar{B}|}$ such that $A_{\bar{B}} d_{\bar{B}} = \mathbf{0} \in \mathbb{R}^m$. Define $d_{\bar{N}} = \mathbf{0} \in \mathbb{R}^{|\bar{N}|}$. Let $d \in \mathbb{R}^n$ be a vector that has $d_{\bar{B}}$ and $d_{\bar{N}}$ as its subvectors. Therefore, $Ad = A_{\bar{B}} d_{\bar{B}} + A_{\bar{N}} d_{\bar{N}} = \mathbf{0} \in \mathbb{R}^m$. If $c^T d > 0$, then replace d by $-d$. Note that $Ad = \mathbf{0}$ still holds. We can assume that $c^T d \leq 0$. Consider the line $\bar{x} + \lambda d$, where $\lambda \in \mathbb{R}$. Note that, for any real number $\lambda \in \mathbb{R}$, we have

$$A(\bar{x} + \lambda d) = A\bar{x} + \lambda A d = b,$$

and

$$(\bar{x} + \lambda d)_{\bar{B}} = \bar{x}_{\bar{B}} + \lambda d_{\bar{B}}, \quad (\bar{x} + \lambda d)_{\bar{N}} = \bar{x}_{\bar{N}} + \lambda d_{\bar{N}} = \mathbf{0}.$$

Case 2a: Suppose that $c^T d < 0$. We claim that there exists $j \in \bar{B}$ such that $d_j < 0$ (i.e., the vector $d_{\bar{B}}$ has at least one negative component.

Suppose that the claim is false. Then, $d_{\bar{B}} \geq 0$. Therefore, $\bar{x} + \lambda d \geq \mathbf{0}$ for each $\lambda \geq 0$, which implies that $\bar{x} + \lambda d \in \mathcal{P}$ for each $\lambda \geq 0$.

Since $c^T d < 0$, we obtain

$$c^T(\bar{x} + \lambda d) = c^T\bar{x} + \lambda c^T d \to -\infty$$

as $\lambda \to +\infty$. It follows that (P) is an unbounded problem, which contradicts our hypothesis.

Therefore, our claim holds. Since $\bar{x}_{\bar{B}} > 0$, $d_{\bar{B}}$ has at least one negative component, and $\lambda \geq 0$, there exists a real number $\lambda^* > 0$ such that $\bar{x} + \lambda d \geq \mathbf{0}$ if $\lambda \in [0, \lambda^*]$ and at least one component of $(\bar{x} + \lambda^* d)_{\bar{B}} = \bar{x}_{\bar{B}} + \lambda^* d_{\bar{B}}$ is equal to zero. Letting $\lambda = \lambda^*$, we have $(\bar{x} + \lambda^* d)_{\bar{N}} = \bar{x}_{\bar{N}} + \lambda^* d_{\bar{N}} = \mathbf{0}$. Therefore, we obtain a new feasible solution $\hat{x} = \bar{x} + \lambda^* d$ with a better objective function value since

$$c^T\hat{x} = c^T(\bar{x} + \lambda^* d) = c^T\bar{x} + \lambda^* c^T d < c^T\bar{x},$$

and $\hat{B} \subset \bar{B}$, where $\hat{B} = \{j \in \{1,\ldots,n\} : \hat{x}_j \neq 0\} = \{j \in \{1,\ldots,n\} : \hat{x}_j > 0\}$. If $A_{\hat{B}}$ has full column rank, then $\hat{x}$ is a vertex of $\mathcal{P}$ and we are done. Otherwise, we can repeat the same procedure starting with $\hat{x}$ as our new feasible solution.

Case 2b: Suppose that $c^T d = 0$. Since $\mathcal{P}$ is a nonempty polyhedron in standard form, it cannot contain a line. Therefore, the line $\bar{x} + \lambda d$ should eventually exit $\mathcal{P}$. Let $\lambda^* \in \mathbb{R}$ denote the value of λ corresponding to the last point of contact with $\mathcal{P}$ before it exits $\mathcal{P}$. Let $\hat{x} = \bar{x} + \lambda^* d$. By a similar reasoning, there should exist $j \in \bar{B}$ such that $\hat{x}_j = 0$. Therefore,

$$c^T \hat{x} = c^T(\bar{x} + \lambda^* d) = c^T \bar{x} + \lambda^* c^T d = c^T \bar{x},$$

i.e., the objective function value of $\hat{x}$ is no worse than that of $\bar{x}$ and $\hat{B} \subset \bar{B}$, where $\hat{B} = \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\} = \{j \in \{1, \ldots, n\} : \hat{x}_j > 0\}$. If $A_{\hat{B}}$ has full column rank, then $\hat{x}$ is a vertex of $\mathcal{P}$ and we are done. Otherwise, we can repeat the same procedure starting with $\hat{x}$ as our new feasible solution.

Therefore, it follows from both cases that, for any feasible solution $\bar{x} \in \mathcal{P}$, there exists a vertex $\hat{x} \in \mathcal{P}$ such that $c^T \hat{x} \leq c^T \bar{x}$. Since $\mathcal{P}$ contains at most a finite number of vertices denoted by $\{x^1, \ldots, x^k\}$, we obtain that the optimal value, denoted by z^*, satisfies:

$$z^* = \min_{i=1,\ldots,k} c^T x^i.$$

Therefore, $\mathcal{P}^* \neq \emptyset$ and $\mathcal{P}^*$ contains at least one vertex of $\mathcal{P}$.

□

14.4 A Generalisation

In this section, we give a partial extension of Proposition 14.1 to general linear programming problems.

Corollary 14.1. *Every linear programming problem with a finite optimal value has at least one optimal solution.*

Proof. Note that every linear programming problem can be converted into an equivalent linear programming problem in standard form. Therefore, the result follows from Proposition 14.1, and the fact that there is a one-to-one correspondence between the feasible solutions of the two linear programming problems. □

Remark 14.1. *Note that Corollary 14.1 holds even if the feasible region does not contain any vertices. However, we can no longer guarantee that the set optimal solutions contains at least one vertex since the feasible region may not necessarily contain any vertices in the general case.*

14.5 Fundamental Theorem of Linear Programming

By combining all the results thus far, we can state the following fundamental theorem.

Theorem 14.1 (Fundamental Theorem of Linear Programming)**.** *Let (P) denote a linear programming problem in standard form.*

$$\begin{array}{lll} (P) & \min & c^T x \\ & s.t. & \\ & & \begin{array}{rcl} Ax & = & b, \\ x & \geq & 0. \end{array} \end{array}$$

Then, exactly one of the following three statements is true for (P):

1. *(P) is infeasible and $z^* = +\infty$.*

2. *The optimal value is finite* $(-\infty < z^* < +\infty)$ *and there is at least one vertex* $x^* \in \mathcal{P}$ *in the set of optimal solutions (i.e.,* $c^T x^* = z^*$*).*

3. *(P) is unbounded and* $z^* = -\infty$.

Theorem 14.1 has many interesting implications. Recall that every linear programming problem can be converted into an equivalent linear programming problem in standard form. By Theorem 14.1, if the optimal value is finite, it suffices to search only over the finite number of vertices.

However, the following three problems still remain:

1. What if there are too many vertices?
2. How do we detect that (P) is infeasible?
3. How do we detect that (P) is unbounded?

We will attempt to answer these questions in the following lectures.

Exercises

Question 14.1. *Let (P) denote a linear programming problem in standard form.*

$$\begin{array}{llrcl} (P) & \min & c^T x & & \\ & s.t. & & & \\ & & Ax & = & b, \\ & & x & \geq & 0. \end{array}$$

Suppose that the optimal value of (P) is finite, $(-\infty < z^* < +\infty)$. *Show that the set of optimal solutions, denoted by* $\mathcal{P}^*$, *is a nonempty polyhedron that does not contain a line.*

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 15 Optimality Conditions

Lecturer: E. Alper Yıldırım *Week: 4*

15.1 Outline

- Feasible Directions
- Optimality Conditions
- Review Problems

15.2 Overview

In this lecture, we will start to discuss the building blocks of an algorithm, called the *simplex method*, for solving linear programming problems. Without loss of generality, we will assume that the linear programming problem is in standard form and that it satisfies the full row rank assumption.

15.2.1 Setup and Assumptions

Let (P) denote a linear programming problem in standard form.

$$\begin{array}{llrcl} \text{(P)} & \min & c^T x & & \\ & \text{s.t.} & & & \\ & & Ax & = & b, \\ & & x & \geq & \mathbf{0}, \end{array}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$.

Assumptions:

1. A has full row rank. (This assumption can always be ensured by preprocessing.)
2. We assume that the feasible region, denoted by $\mathcal{P}$, is nonempty. (We will revisit this assumption later on.)
3. We assume that we have computed a vertex $\hat{x}$ of $\mathcal{P}$. (We will revisit this assumption later on.)

15.3 Optimality Conditions

Remark 15.1. *By the Fundamental Theorem of Linear Programming, we know that (P) contains a vertex as an optimal solution if the optimal value is finite. Recall that* $\hat{x}$ *is a vertex by our assumption.*

Question 1. *Under what conditions would $\hat{x}$ be an optimal solution of (P)?*

- Note that $\hat{x}$ is an optimal solution of (P) if and only if $\mathcal{P}$ does not contain any other vector $\bar{x} \in \mathbb{R}^n$ such that $c^T\bar{x} < c^T\hat{x}$.
- Let $d = \bar{x} - \hat{x} \in \mathbb{R}^n$. Then, $c^T d = c^T(\bar{x} - \hat{x}) = c^T\bar{x} - c^T\hat{x}$.
- Therefore, $\hat{x}$ is an optimal solution of (P) if and only if $c^T d \geq 0$ for all $d \in \mathbb{R}^n$ such that $\hat{x} + d \in \mathcal{P}$.

Question 2. *How can we characterise such vectors $d \in \mathbb{R}^n$?*

15.3.1 Feasible Directions and Optimality Conditions

In this section, given an arbitrary $\bar{x} \in \mathcal{P}$ (not necessarily a vertex), we focus on the set of all directions $d \in \mathbb{R}^n$ such that $\bar{x} + d \in \mathcal{P}$.

If $\bar{x} + d \in \mathcal{P}$, then $\lambda(\bar{x} + d) + (1 - \lambda)\bar{x} = \bar{x} + \lambda d \in \mathcal{P}$ for each $\lambda \in [0, 1]$ since $\mathcal{P}$ is a convex set.

Definition 15.1. *Let $\bar{x} \in \mathcal{P}$. A vector $d \in \mathbb{R}^n$ is said to be a* feasible direction *at $\bar{x}$ if there exists $\lambda^* > 0$ such that $\bar{x} + \lambda^* d \in \mathcal{P}$.*

Remark 15.2.

1. *A vector $d \in \mathbb{R}^n$ is a feasible direction at $\bar{x}$ if, starting from $\bar{x}$, we can move in the direction of d at least for a while without leaving $\mathcal{P}$.*
2. *Note that $d = \mathbf{0} \in \mathbb{R}^n$ is a feasible direction at any feasible solution.*

The following proposition establishes the significance of feasible directions.

Proposition 15.1. *Let $\bar{x} \in \mathcal{P}$. Then, $\bar{x}$ is an optimal solution of (P) if and only if $c^T d \geq 0$ for all feasible directions $d \in \mathbb{R}^n$ at $\bar{x}$.*

Proof. $\Rightarrow$: Suppose that $\bar{x}$ is an optimal solution of (P). Suppose, for a contradiction, that there exists a feasible direction $\bar{d} \in \mathbb{R}^n$ such that $c^T\bar{d} < 0$. Then, there exists a real number $\lambda^* > 0$ such that $\bar{x} + \lambda^* d \in \mathcal{P}$. We obtain $c^T(\bar{x} + \lambda^* d) = c^T\bar{x} + \lambda^* c^T\bar{d} < c^T\bar{x}$, contradicting the optimality of $\bar{x}$.

$\Leftarrow$: Suppose that $c^T d \geq 0$ for all feasible directions $d \in \mathbb{R}^n$ at $\bar{x}$. Suppose, for a contradiction, that $\bar{x}$ is not optimal. Then, there exists $x^* \in \mathcal{P}$ such that $c^T x^* < c^T\bar{x}$. Then, let $d = x^* - \bar{x}$. Clearly, d is a feasible direction at $\bar{x}$ and $c^T d < 0$, contradicting our hypothesis. $\square$

15.3.2 Feasible Directions and Optimality Conditions at a Vertex

In this section, given a **vertex** $\hat{x} \in \mathcal{P}$, we aim to shed light on the set of feasible directions at $\hat{x}$.

Question 3. *Proposition 15.1 holds for any feasible solution $\bar{x} \in \mathcal{P}$. How can we specialise it to a vertex $\hat{x} \in \mathcal{P}$?*

Recall that $\hat{x}$ is a vertex if and only if there exist disjoint index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ such that $|B| = m$, $|N| = n - m$, $A_B \in \mathbb{R}^{m \times m}$ is invertible, $\hat{x}_B = (A_B)^{-1}b \geq \mathbf{0}$, and $\hat{x}_N = \mathbf{0} \in \mathbb{R}^{|N|}$.

Let $d \in \mathbb{R}^n$ be a feasible direction at $\hat{x}$. Then, $\hat{x} + \lambda^* d \in \mathcal{P}$ for some $\lambda^* > 0$.

- Therefore, $A(\hat{x} + \lambda^* d) = A\hat{x} + \lambda^* Ad = b$, i.e., $Ad = \mathbf{0}$.
- $\hat{x} + \lambda^* d \geq \mathbf{0}$, i.e., $\hat{x}_B + \lambda^* d_B \geq \mathbf{0}$ and $\hat{x}_N + \lambda^* d_N = \lambda^* d_N \geq \mathbf{0}$.

We therefore obtain the following result: If $d \in \mathbb{R}^n$ is a feasible direction at $\hat{x}$, then $Ad = A_B d_B + A_N d_N = \mathbf{0}$, $\hat{x}_B + \lambda^* d_B \geq \mathbf{0}$, and $d_N \geq \mathbf{0}$.

Conversely, pick an arbitrary vector $d_N \in \mathbb{R}^{n-m}$ such that $d_N \geq \mathbf{0}$. In order to extend d_N to $d \in \mathbb{R}^n$ such that $Ad = A_B d_B + A_N d_N = \mathbf{0}$, set $d_B = -(A_B)^{-1} A_N d_N$.

- Then, for any $\lambda \in \mathbb{R}$, $A(\hat{x} + \lambda d) = A\hat{x} + \lambda Ad = b + \mathbf{0} = b$.
- Note that $\hat{x}_N + \lambda d_N = \lambda d_N \geq \mathbf{0}$ for all $\lambda \geq 0$.
- Consider $\hat{x}_B + \lambda d_B$. Note that d_B may not necessarily be a nonnegative vector!
  (i) **Case 1:** If there exists a real number $\lambda^* > 0$ such that $\hat{x} + \lambda^* d \in \mathcal{P}$ (i.e., $\hat{x}_B + \lambda^* d_B \geq \mathbf{0}$), then d is a feasible direction at $\hat{x}$.
  (ii) **Case 2:** Otherwise, d is not a feasible direction at $\hat{x}$.

This observation motivates the following discussion. Given a vertex $\hat{x}$ of $\mathcal{P}$, let

$$\mathcal{D} = \{d \in \mathbb{R}^n : d_N \geq \mathbf{0}, \quad d_B = -(A_B)^{-1} A_N d_N\}.$$

Let $\hat{\mathcal{D}} \subseteq \mathbb{R}^n$ denote the set of all feasible directions at $\hat{x}$. We therefore obtain

$$\hat{\mathcal{D}} \subseteq \mathcal{D},$$

i.e., the set $\hat{\mathcal{D}}$ contains all feasible directions at a vertex $\hat{x}$ of $\mathcal{P}$. However, not every element of $\mathcal{D}$ is necessarily a feasible direction at $\hat{x}$ (i.e., $\mathcal{D}$ can be strictly larger than $\hat{\mathcal{D}}$). We will revisit this issue later on.

Corollary 15.2. *Let $\hat{x} \in \mathbb{R}^n$ be a vertex of $\mathcal{P}$. If $c^T d \geq 0$ for each $d \in \mathcal{D}$, then $\hat{x}$ is an optimal solution of (P).*

Proof. Since $\hat{\mathcal{D}} \subseteq \mathcal{D}$ and $c^T d \geq 0$ for each $d \in \mathcal{D}$, we obtain $c^T d \geq 0$ for all feasible directions $d \in \mathbb{R}^n$ at $\hat{x}$. Therefore, $\hat{x}$ is an optimal solution of (P) by Proposition 15.1. □

15.3.3 Optimality Conditions and Reduced Costs

Let $\hat{x} \in \mathbb{R}^n$ be a vertex of $\mathcal{P}$ and let $d \in \mathcal{D}$. Then, $d_N \geq \mathbf{0}$ and $d_B = -(A_B)^{-1} A_N d_N$. Therefore,

$$\begin{aligned} c^T d &= c_B^T d_B + c_N^T d_N \\ &= -c_B^T (A_B)^{-1} A_N d_N + c_N^T d_N \\ &= -\sum_{j \in N} c_B^T (A_B)^{-1} A^j d_j + \sum_{j \in N} c_j d_j \\ &= \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} d_j. \end{aligned}$$

Definition 15.3. *For each index $j \in N$, the parameter $\bar{c}_j$ is called the* reduced cost *of the variable x_j.*

Using the reduced costs, we can arrive at a simplified set of optimality conditions at a vertex $\hat{x} \in \mathcal{P}$.

Corollary 15.4. *Let $\hat{x} \in \mathbb{R}^n$ be a vertex of $\mathcal{P}$. If $\bar{c}_j \geq 0$ for each $j \in N$, then $\hat{x}$ is an optimal solution of (P).*

Proof. Suppose that $\bar{c}_j \geq 0$ for each $j \in N$. Then, $c^T d = \sum_{j \in N} \bar{c}_j d_j \geq 0$ for each $d \in \mathcal{D}$ since $d_j \geq 0$ for each $j \in N$. The result follows from Corollary 15.2. □

Remark 15.3. *Therefore, by computing $n - m$ reduced costs and checking whether each one is nonnegative, we can verify the optimality of a vertex $\hat{x}$!*

Here are further questions that we will address in the following lectures:

- Corollary 15.4 states that the nonnegativity of all reduced costs at a vertex $\hat{x}$ is **sufficient** to ensure its optimality.
- Is this condition also **necessary** (i.e., is it true that all reduced costs should be nonnegative at an optimal vertex)?
- Related to the previous question, what if there is a negative reduced cost? What can we conclude?

Exercises

Question 15.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ be a nonempty polyhedron in standard form and let $\bar{x} \in \mathcal{P}$ be an arbitrary feasible solution. Show that the set of all feasible directions at $\bar{x}$ is a nonempty convex set.*

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 16	Degeneracy
Lecturer: E. Alper Yıldırım	*Week: 4*

16.1 Outline

- Necessity of Optimality Conditions
- Degeneracy in General Polyhedra
- Degeneracy in Standard Form Polyhedra
- Review Problems

16.2 Overview

In this lecture, we will discuss the concept of *degeneracy* and its effect on optimality conditions.

Let us recall our setup and assumptions. Let (P) denote a linear programming problem in standard form.

$$
\begin{array}{llrcl}
\text{(P)} & \min & c^T x & & \\
& \text{s.t.} & & & \\
& & Ax & = & b, \\
& & x & \geq & \mathbf{0},
\end{array}
$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$.

Assumptions:

1. A has full row rank. (This assumption can always be ensured by preprocessing.)
2. We assume that the feasible region, denoted by $\mathcal{P}$, is nonempty. (We will revisit this assumption later on.)
3. We assume that we have computed a vertex $\hat{x}$ of $\mathcal{P}$. (We will revisit this assumption later on.)

16.2.1 Reduced Costs Revisited

Let $\mathcal{P}$ denote the feasible region of (P) and let $\hat{x}$ be a vertex of $\mathcal{P}$ with disjoint index sets B and N, where $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ such that $|B| = m$, $|N| = n - m$, $A_B \in \mathbb{R}^{m \times m}$ is invertible, $\hat{x}_B = (A_B)^{-1} b \geq \mathbf{0}$, and $\hat{x}_N = \mathbf{0} \in \mathbb{R}^{|N|}$.

- For each index $j \in N$, the reduced cost of the variable x_j, denoted by $\bar{c}_j$, is given by $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j$.

- If $\bar{c}_j \geq 0$ for each $j \in N$, then $\hat{x}$ is an optimal solution of (P).

Question 1. *If $\hat{x}$ is an optimal solution of (P), is it necessarily true that $\bar{c}_j \geq 0$ for each $j \in N$?*

In particular, we know from Lecture 15 that the nonnegativity of reduced costs at a vertex is sufficient to guarantee its optimality. We are interested in whether this condition is also necessary for optimality. If this is the case, then the nonnegativity of all reduced costs at a vertex would be equivalent to its optimality.

As illustrated by the following example, the answer turns out to be no. We will investigate the reasons. Later, we will see that the necessity of this condition holds under some additional assumptions on the vertex.

16.3 An Example

Consider the following linear programming problem with two variables:

$$\begin{array}{lrcrcl} \min & 3x_1 & - & 2x_2 & & \\ \text{s.t.} & & & & & \\ & -3x_1 & + & 3x_2 & \leq & 6 \\ & -x_1 & + & 2x_2 & \leq & 0 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

Using the graphical method, you can verify that the unique optimal solution is given by $x^* = [0,0]^T$ and the optimal value is $z^* = 0$.

Let us convert this problem into standard form by defining x_3 and x_4 as the nonnegative slack variables for the first and the second constraints, respectively:

$$A = \begin{bmatrix} -3 & 3 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

Since we obtain an equivalent linear programming problem in standard form, the unique optimal solution is $x^* = [0,0,6,0]^T$ and $z^* = 0$. For this optimal solution, we obtain

$$\begin{aligned} \hat{B} &= \{j \in \{1,\ldots,4\} : x_j^* \neq 0\} = \{3\} \\ \hat{N} &= \{j \in \{1,\ldots,4\} : x_j^* = 0\} = \{1,2,4\}. \end{aligned}$$

Note that $m = 2$ and $n = 4$ in this example since there are two equality constraints and four decision variables in standard form. It is easy to verify that A has full row rank as its rows are linearly independent. Therefore, we need to find index sets $B \subseteq \{1,\ldots,4\}$ and $N \subseteq \{1,\ldots,4\}$ such that (i) $\hat{B} \subseteq B$, (ii) $N \subseteq \hat{N}$, (iii) $B \cup N = \{1,\ldots,4\}$, (iv) $B \cap N = \emptyset$, (v) $|B| = m = 2$ (therefore, $|N| = n - m = 4 - 2$), and (vi) $A_B \in \mathbb{R}^{2\times 2}$ is invertible.

Let us pick $B = \{3,4\}$. Note that $\hat{B} \subseteq B$, $|B| = 2$, and $A_B \in \mathbb{R}^{2\times 2}$ is invertible since A_B consisting of the third and fourth columns of A is simply the 2×2 identity matrix. Then, $N = \{1,2\}$.

However, we obtain

$$\bar{c}_2 = c_2 - c_B^T(A_B)^{-1}A^2 = -2 < 0$$

even though x^* is optimal and $2 \in N$.

This example shows that the nonnegativity of reduced costs is not necessary for the optimality of a vertex.

In the previous example, if we had used $B = \{2, 3\}$ instead, we would have still had $\hat{B} \subseteq B$, A_B is invertible, and we would have obtained $N = \{1, 4\}$. The reduced costs would then be given by

$$\begin{aligned} \bar{c}_1 &= c_1 - c_B^T (A_B)^{-1} A^1 = 2 \geq 0, \\ \bar{c}_4 &= c_4 - c_B^T (A_B)^{-1} A^4 = 0 \geq 0. \end{aligned}$$

With these choices of index sets B and N, we would be able to verify the optimality of $\hat{x}$!

In this example, note that $|\hat{B}| = 1$, whereas we need to have two indices in the set B since $m = 2$. Hence, we have some flexibility in choosing B. If we had a basic solution with $|\hat{B}| = 2$, then we would have only one choice given by $B = \hat{B}$.

16.4 Degeneracy

In this section, we introduce *degeneracy*, which is the underlying reason for the behaviour of reduced costs in the example above.

16.4.1 Degeneracy in General Polyhedra

Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x \geq b_i,\ i \in M_1; (a^i)^T x \leq b_i,\ i \in M_2; (a^i)^T x = b_i,\ i \in M_3\}$ be a polyhedron and let $\hat{x} \in \mathbb{R}^n$. Recall that $I(\hat{x}) = \left\{i \in M_1 \cup M_2 \cup M_3 : (a^i)^T \hat{x} = b_i\right\}$. A point $\hat{x} \in \mathbb{R}^n$ is a basic solution of $\mathcal{P}$ if and only if $M_3 \subseteq I(\hat{x})$ and the set $\{a^i : i \in I(\hat{x})\}$ contains n linearly independent vectors.

Definition 16.1. *A basic solution $\hat{x} \in \mathbb{R}^n$ of $\mathcal{P}$ is said to be* degenerate *if $|I(\hat{x})| > n$ (i.e., if more than n constraints are active at $\hat{x}$). Otherwise (i.e., if $|I(\hat{x})| = n$), it is said to be* nondegenerate.

16.4.2 Degeneracy in Standard Form Polyhedra

Let $\mathcal{P} = \{x \in \mathbb{R}^n : (a^i)^T x = b_i,\ i = 1, \ldots, m; x_j \geq 0,\ j = 1, \ldots, n\}$ be a polyhedron and let $\hat{x} \in \mathbb{R}^n$. Recall that $\hat{x}$ is a basic solution if and only if there exist disjoint index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ such that $|B| = m$, $|N| = n - m$, $A_B \in \mathbb{R}^{m \times m}$ is invertible, $\hat{x}_B = (A_B)^{-1} b$ and $\hat{x}_N = \mathbf{0}$.

Recall that $\hat{B} = \{j \in \{1, \ldots, n\} : \hat{x}_j \neq 0\}$ and $\hat{N} = \{j \in \{1, \ldots, n\} : \hat{x}_j = 0\}$. Note that each of the m equality constraints is active at $\hat{x}$ and the inequality constraints $x_j \geq 0$ for each $j \in \hat{N}$ is active at $\hat{x}$.

Therefore,

$$|I(\hat{x})| = m + |\hat{N}| = m + n - |\hat{B}|.$$

Since $\hat{x}$ is a basic solution, we have $\hat{B} \subseteq B$, therefore, $|\hat{B}| \leq |B| = m$.

- **Case 1:** If $|\hat{B}| = |B| = m$ (i.e., $B = \hat{B}$), then $|I(\hat{x})| = m + n - |\hat{B}| = n$. Then, $\hat{x}$ is nondegenerate. Note that all basic variables are different from zero in this case.

- **Case 2:** If $|\hat{B}| < |B| = m$ (i.e., $\hat{B} \subset B$), then $|I(\hat{x})| = m + n - |\hat{B}| > n$. Then, $\hat{x}$ is degenerate. Note that at least one basic variable is equal to zero in this case.

Definition 16.2. *Let $\hat{x} \in \mathbb{R}^n$ be basic solution of $\mathcal{P}$ in standard form. If $|\hat{B}| < m$, then $\hat{x}$ is degenerate. Otherwise (i.e., if $|\hat{B}| = m$), then $\hat{x}$ is nondegenerate.*

Remarks

1. If $\hat{x} \in \mathbb{R}^n$ is a basic **feasible** solution of $\mathcal{P}$ in standard form with index sets B and N, then $\hat{x}_B \geq \mathbf{0}$ and $\hat{x}_N = \mathbf{0}$.
2. Therefore, $\hat{x}$ is a nondegenerate basic feasible solution if and only if $\hat{x}_B > \mathbf{0}$, i.e., each basic variable is strictly positive.
3. $\hat{x}$ is a degenerate basic feasible solution if and only if the value of at least one basic variable is equal to zero.

Exercises

Question 16.1. *Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq 0\}$ be polyhedron in standard form, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$, and let $\hat{x} \in \mathcal{P}$ be a degenerate vertex with k positive components, where $k < m$. How many different possible choices are there to define the index set B?*

MATH11111: Fundamentals of Optimization	Fall 2023
Lecture 17	Optimality Conditions Under Nondegeneracy
Lecturer: E. Alper Yıldırım	*Week: 5*

17.1 Outline

- Optimality Conditions Under Nondegeneracy
- Nondegeneracy vs Degeneracy
- Review Problems

17.2 Review and Setup

Let (P) denote a linear programming problem in standard form.

$$\begin{array}{llrcl} \text{(P)} & \min & c^T x & & \\ & \text{s.t.} & & & \\ & & Ax & = & b, \\ & & x & \geq & \mathbf{0}, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$.

- Let $\hat{x}$ be a vertex of $\mathcal{P}$ with corresponding index sets B and N.
- For each index $j \in N$, the reduced cost of the nonbasic variable x_j, denoted by $\bar{c}_j$, is given by $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j$.
- If $\bar{c}_j \geq 0$ for each $j \in N$, then $\hat{x}$ is an optimal solution of (P).
- However, we may have $\bar{c}_j < 0$ for some $j \in N$ if $\hat{x}$ is a degenerate optimal solution of (P).

In this lecture, we will consider the effect of nondegeneracy on the optimality conditions in linear programming.

17.3 Optimality Conditions Under Nondegeneracy

The next proposition illustrates that we can establish stronger optimality conditions under the nondegeneracy assumption.

Proposition 17.1. *Let (P) be a linear programming problem in standard form and let $\hat{x}$ be a nondegenerate vertex of $\mathcal{P}$ with corresponding index sets B and N. Then, $\hat{x}$ is an optimal solution of (P) if and only if $\bar{c}_j \geq 0$ for each $j \in N$.*

Proof. $\Leftarrow$: Follows from Corollary 15.4.

$\Rightarrow$: Let $\hat{x}$ be a nondegenerate vertex of $\mathcal{P}$ with corresponding index sets B and N such that $\hat{x}$ is optimal. Suppose, for a contradiction, that there exists $j^* \in N$ such that $\bar{c}_{j^*} < 0$. Let us construct a feasible direction $d \in \mathbb{R}^n$ at $\hat{x}$ as follows. We set $d_{j^*} = 1$ and $d_j = 0$ for each $j \in N\backslash\{j^*\}$. Note that $d_N \geq \mathbf{0}$. Let $d_B = -(A_B)^{-1}A_N d_N \in \mathbb{R}^m$ so that $Ad = A_B d_B + A_N d_N = \mathbf{0}$. We claim that d is a feasible direction at $\hat{x}$. We need to show that there exists some real number $\lambda^* > 0$ such that $\hat{x} + \lambda^* d \in \mathcal{P}$. Clearly, $A(\hat{x} + \lambda d) = A\hat{x} + \lambda A d = b + \mathbf{0} = b$ for any $\lambda \in \mathbb{R}$, and $\hat{x}_N + \lambda d_N = \lambda d_N \geq \mathbf{0}$ for any $\lambda \geq 0$. Consider $\hat{x}_B + \lambda d_B$.

Case 1: If d_B has at least one negative component, then define $\lambda^* = \min\limits_{j \in B: d_j < 0} \frac{-\hat{x}_j}{d_j}$. Since $\hat{x}$ is nondegenerate, we obtain $\hat{x}_j > 0$ for each $j \in B$. Therefore, $\lambda^* > 0$ and it is finite. Note that $\hat{x}_B + \lambda d_B \geq \mathbf{0}$ for each $\lambda \in [0, \lambda^*]$. Therefore, $\hat{x} + \lambda d \in \mathcal{P}$ for each $\lambda \in [0, \lambda^*]$. In addition, by setting $\lambda = \lambda^*$, we obtain $c^T(\hat{x} + \lambda^* d) = c^T\hat{x} + \lambda^* c^T d = c^T\hat{x} + \lambda^*(c_B^T d_B + c_N^T d_N) = c^T\hat{x} + \lambda^*(-c_B^T(A_B)^{-1}A_N d_N + c_N^T d_N) = c^T\hat{x} + \lambda^* \sum\limits_{j \in N} \bar{c}_j d_j = c^T\hat{x} + \lambda^* \bar{c}_{j^*} d_{j^*} = c^T\hat{x} + \lambda^* \bar{c}_{j^*} < c^T\hat{x}$, which contradicts the optimality of $\hat{x}$.

Case 2: If $d_B \geq \mathbf{0}$, then $\hat{x}_B + \lambda d_B \geq \mathbf{0}$ for any $\lambda \geq 0$. Therefore, $\hat{x} + \lambda d \in \mathcal{P}$ for each $\lambda \geq 0$. Furthermore, $c^T(\hat{x} + \lambda d) = c^T\hat{x} + \lambda \bar{c}_{j^*} \to -\infty$ as $\lambda \to \infty$, which implies that (P) is unbounded, which contradicts the optimality of $\hat{x}$. Therefore, such an index $j^* \in N$ cannot exist, i.e., we should have $\bar{c}_j \geq 0$ for each $j \in N$. □

Remark 17.1. *This result implies that the nonnegativity of all reduced costs is not only sufficient but also necessary for optimality under the nondegeneracy assumption.*

17.4 Nondegeneracy vs Degeneracy

17.4.1 Nondegenerate Case

Let $\hat{x}$ be a nondegenerate vertex of $\mathcal{P}$ (i.e., $\hat{x}_j > 0$ for each $j \in B$). Let $\hat{\mathcal{D}}$ denote the set of feasible directions at $\hat{x}$ and let $\mathcal{D} = \{d \in \mathbb{R}^n : d_N \geq \mathbf{0}, \quad d_B = -(A_B)^{-1}A_N d_N\}$. We always have

$$\hat{\mathcal{D}} \subseteq \mathcal{D}.$$

.

Conversely, for any $d \in \mathcal{D}$, we have $d_N \geq \mathbf{0}$.

1. **Case 1:** If $d_B = -(A_B)^{-1}A_N d_N \geq \mathbf{0}$, then $\hat{x} + \lambda d \in \mathcal{P}$ for each $\lambda \geq 0$. Therefore, $d \in \hat{\mathcal{D}}$.

2. **Case 2:** If $d_B = -(A_B)^{-1}A_N d_N$ has at least one negative component, then recall that $\lambda^* = \min\limits_{j \in B: d_j < 0} \frac{-\hat{x}_j}{d_j} > 0$. We obtain that $\hat{x} + \lambda d \in \mathcal{P}$ for each $\lambda \in [0, \lambda^*]$. Therefore, $d \in \hat{\mathcal{D}}$.

Therefore, we arrive at the following important observation:

> If $\hat{x}$ is nondegenerate, then $\hat{\mathcal{D}} = \mathcal{D}$.

In particular, this observation is the underlying reason for the result stated in Proposition 17.1.

17.4.2 Degenerate Case

On the other hand, let $\hat{x}$ be a degenerate vertex of $\mathcal{P}$ (i.e., $\hat{x}_j = 0$ for some $j \in B$) and let $d \in \mathcal{D}$. Consider the case in which $d_B = -(A_B)^{-1} A_N d_N$ has at least one negative component and recall that $\lambda^* = \min\limits_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j}$.

Therefore, if $d_j < 0$ for an index $j \in B$ such that $\hat{x}_j = 0$, we may have $\lambda^* = 0$. We can no longer guarantee that d is a feasible direction at $\hat{x}$. Therefore, we may have $\hat{\mathcal{D}} \subset \mathcal{D}$, which implies that we may have have $c^T d \geq 0$ for all $d \in \hat{\mathcal{D}}$ (i.e., $\hat{x}$ is optimal), but $c^T \bar{d} < 0$ for a vector $\bar{d} \in \mathcal{D} \backslash \hat{\mathcal{D}}$ (in particular, see the example in Section 16.3 of Lecture 16). This is why the nonnegativity of reduced costs of nonbasic variables is not necessary for the optimality of a degenerate vertex.

We will revisit the issue of degeneracy later.

Exercises

Question 17.1. *Can you construct a polyhedron $\mathcal{P} \subset \mathbb{R}^2$ that has at least one nondegenerate vertex and at least one degenerate vertex?*

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 18 — Development of the Simplex Method

Lecturer: E. Alper Yıldırım *Week: 5*

18.1 Outline

- Development of the Simplex Method
- Review Problems

18.2 Review and Setup

Let (P) denote a linear programming problem in standard form.

$$
\begin{array}{llrcl}
\text{(P)} & \min & c^T x & & \\
& \text{s.t.} & & & \\
& & Ax & = & b, \\
& & x & \geq & \mathbf{0},
\end{array}
$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$.

- Let $\hat{x}$ be a vertex of $\mathcal{P}$ with corresponding index sets B and N.
- For each index $j \in N$, the reduced cost of the variable x_j, denoted by $\bar{c}_j$, is given by $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j$.
- If $\bar{c}_j \geq 0$ for each $j \in N$, then $\hat{x}$ is an optimal solution of (P).
- If $\hat{x}$ is nondegenerate, then $\bar{c}_j \geq 0$ for each $j \in N$ if and only if $\hat{x}$ is an optimal solution of (P).

18.3 Nondegeneracy Assumption

In this lecture, we will first assume that $\hat{x}$ is a nondegenerate vertex of $\mathcal{P}$ with corresponding index sets B and N (we will revisit this assumption later on). Compute reduced costs $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j$ for each $j \in N$.

Case 1

If $\bar{c}_j \geq 0$ for each $j \in N$, then $\hat{x}$ is an optimal solution of (P) by Proposition 17.1. Stop.

Case 2

Suppose that there exists a $j^* \in N$ such that $\bar{c}_{j^*} < 0$ (i.e., $\hat{x}$ is not an optimal solution by Proposition 17.1).

Let $d \in \mathbb{R}^n$ be such that $d_{j^*} = 1$ and $d_j = 0$ for each $j \in N \backslash \{j^*\}$. Let $d_B = -(A_B)^{-1} A_N d_N \in \mathbb{R}^m$. Note that $d \in \mathbb{R}^n$ is a feasible direction at $\hat{x}$ such that $c^T d = \bar{c}_{j^*} < 0$ (i.e., it is an *improving* feasible direction).

Case 2a

If $d_B \geq \mathbf{0}$ (i.e., $d_j \geq 0$ for each $j \in B$), then $\hat{x} + \lambda d \in \mathcal{P}$ for each $\lambda \geq 0$ and $c^T(\hat{x} + \lambda d) = c^T \hat{x} + \lambda \bar{c}_{j^*} \to -\infty$ as $\lambda \to \infty$. We have discovered a feasible direction d at $\hat{x}$ along which the objective function can be decreased indefinitely. The problem is unbounded. Stop.

Case 2b

If d_B has at least one negative component, then define $\lambda^* = \min\limits_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j}$ (recall that $\hat{x}_j > 0$ for each $j \in B$ due to nondegeneracy and $\lambda^* > 0$ is a finite number).

Let $k^* \in B$ such that $\lambda^* = \frac{-\hat{x}_{k^*}}{d_{k^*}} = \min\limits_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j}$. Note that $d_{k^*} < 0$.

Let $\bar{x} = \hat{x} + \lambda^* d$. Note that $\bar{x} \in \mathcal{P}$ and $c^T \bar{x} = c^T(\hat{x} + \lambda^* d) = c^T \hat{x} + \lambda^* c^T d = c^T \hat{x} + \lambda^* \bar{c}_{j^*} < c^T \hat{x}$ (i.e., $\bar{x}$ is a better feasible solution than $\hat{x}$).

We therefore obtain the following relations:

$$
\begin{aligned}
\lambda^* &= \frac{-\hat{x}_{k^*}}{d_{k^*}} = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} \\
\bar{x}_j &= \hat{x}_j + \lambda^* d_j \geq 0, \quad j \in B \backslash \{k^*\} \\
\bar{x}_{k^*} &= \hat{x}_{k^*} + \lambda^* d_{k^*} = \hat{x}_{k^*} + \left(\frac{-\hat{x}_{k^*}}{d_{k^*}} \right) d_{k^*} = 0 \\
\bar{x}_{j^*} &= \hat{x}_{j^*} + \lambda^* d_{j^*} = \lambda^* > 0 \\
\bar{x}_j &= \hat{x}_j + \lambda^* d_j = 0, \quad j \in N \backslash \{j^*\}
\end{aligned}
$$

Table 18.1 summarises the similarities and differences between $\bar{x}$ and $\hat{x}$.

Index	$\hat{x}$	$\bar{x}$
$j \in B \backslash \{k^*\}$	$\hat{x}_j > 0$	$\bar{x}_j \geq 0$
k^*	$\hat{x}_{k^*} > 0$	$\bar{x}_{k^*} = 0$
j^*	$\hat{x}_{j^*} = 0$	$\bar{x}_{j^*} > 0$
$j \in N \backslash \{j^*\}$	$\hat{x}_j = 0$	$\bar{x}_j = 0$

Table 18.1: Comparison of $\hat{x}$ and $\bar{x}$

As illustrated by Table 18.1, we can make the following observations:

- Only one of the nonbasic variables at $\hat{x}$ becomes positive at $\bar{x}$.

- All the remaining nonbasic variables remain at zero.
- At least one of the basic variables at $\hat{x}$ becomes zero at $\bar{x}$.
- All the remaining basic variables at $\hat{x}$ remain nonnegative at $\bar{x}$.
- Let $\bar{B} = (B\backslash\{k^*\}) \cup \{j^*\}$ and $\bar{N} = (N\backslash\{j^*\}) \cup \{k^*\}$.
- Note that $|\bar{B}| = m$, $|\bar{N}| = n - m$, and $\bar{B} \cup \bar{N} = \{1, \ldots, n\}$.

The following proposition outlines a very important property of the new feasible solution $\bar{x}$ and forms the backbone of the *simplex method*.

Proposition 18.1. *$\bar{x}$ is a vertex of $\mathcal{P}$.*

Proof. Since $\bar{x} \in \mathcal{P}$, $\bar{x}_j = 0$ for each $j \in \bar{N}$, and $|\bar{B}| = m$, it suffices to show that $A_{\bar{B}} \in \mathbb{R}^{m\times m}$ is an invertible matrix, i.e., its columns are linearly independent. Since $A_B \in \mathbb{R}^{m\times m}$ is an invertible matrix and $\bar{B} = (B\backslash\{k^*\}) \cup \{j^*\}$, the columns of $A_{\bar{B}}$ corresponding to indices $j \in B\backslash\{k^*\}$ are clearly linearly independent. Suppose, for a contradiction, that the columns of $A_{\bar{B}}$ are not linearly independent. Then, the column A^{j^*} should be in the span of the columns of A_B corresponding to indices $j \in B\backslash\{k^*\}$. Suppose that A^{k^*} is the ℓth column of A_B, where $\ell \in \{1, \ldots, m\}$. Then, there exist real numbers $\mu_j,\ j = 1, \ldots, m,\ j \neq \ell$, such that $A^{j^*} = \sum\limits_{j\in\{1,\ldots,m\}\backslash\{\ell\}} \mu_j A_B e^j$. Recall that

$$d_B = -(A_B)^{-1} A_N d_N = -(A_B)^{-1}\left(\sum_{j\in N} A^j d_j\right) = -(A_B)^{-1} A^{j^*}.$$

Therefore,

$$d_B = -(A_B)^{-1} A^{j^*} = -\sum_{j\in\{1,\ldots,m\}\backslash\{\ell\}} \mu_j (A_B)^{-1} A_B e^j = -\sum_{j\in\{1,\ldots,m\}\backslash\{\ell\}} \mu_j e^j,$$

which implies that $(d_B)_\ell = d_{k^*} = 0$ since k^* is the ℓth index in the set B. This contradicts $d_{k^*} < 0$. $\square$

The next definition relates the starting vertex $\hat{x}$ and the new vertex $\bar{x}$.

Definition 18.1. *Let $\hat{x} \in \mathcal{P}$ be a vertex and let $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ denote the indices of the basic and nonbasic variables, respectively. Let $\bar{x} \in \mathcal{P}$ be a vertex such that $\bar{x} \neq \hat{x}$ and let $\bar{B} \subseteq \{1, \ldots, n\}$ and $\bar{N} \subseteq \{1, \ldots, n\}$ denote the indices of the basic and nonbasic variables, respectively. Then, $\hat{x}$ and $\bar{x}$ are called* adjacent *if B and $\bar{B}$ have exactly $m - 1$ common indices. The line segment that joins $\hat{x}$ and $\bar{x}$ is called an* edge *of the feasible region $\mathcal{P}$.*

Remark 18.1. *Note that the vertices $\hat{x}$ and $\bar{x}$ in Case 2b are adjacent.*

Proposition 18.1 reveals that the new solution $\bar{x}$ computed in Case 2b is also a vertex of $\mathcal{P}$ that is adjacent to $\hat{x}$. Recall that $\bar{x}$ is a better feasible solution than $\hat{x}$, i.e., $c^T\bar{x} < c^T\hat{x}$.

If we assume that $\bar{x}$ is also a nondegenerate vertex, then we can repeat the same procedure, starting from $\bar{x}$. This procedure can be turned into an algorithm, called the *simplex method*.

18.4 The Simplex Method

Let (P) be a linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}$$

Assume that $\mathcal{P} \neq \emptyset$, A has full row rank, and we have a vertex $\hat{x}$ of $\mathcal{P}$. Let $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ denote the indices corresponding to basic and nonbasic variables, respectively. Here is the outline of the simplex method:

1. Compute the reduced costs $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j, \ j \in N$.
2. **Case 1:** If $\bar{c}_j \geq 0$ for each $j \in N$, then stop. $\hat{x}$ is an optimal solution of (P) and $z^* = c^T \hat{x}$.
3. **Case 2:** If there exists a $j^* \in N$ such that $\bar{c}_{j^*} < 0$, then let $d \in \mathbb{R}^n$ be such that $d_{j^*} = 1$ and $d_j = 0$ for each $j \in N \backslash \{j^*\}$ and $d_B = -(A_B)^{-1} A_N d_N = -(A_B)^{-1} A^{j^*} \in \mathbb{R}^m$.
   (a) **Case 2a:** If $d_B \geq \mathbf{0}$, then stop. (P) is unbounded along the direction $d \in \mathbb{R}^n$.
   (b) **Case 2b:** If d_B has at least one negative component, then define $\lambda^* = \min_{j \in B : d_j < 0} \frac{-\hat{x}_j}{d_j} = \frac{-\hat{x}_{k^*}}{d_{k^*}}$. Set $\bar{x} = \hat{x} + \lambda^* d$, $\bar{B} = (B \backslash \{k^*\}) \cup \{j^*\}$ and $\bar{N} = (N \backslash \{j^*\}) \cup \{k^*\}$. Set $\hat{x} \leftarrow \bar{x}$, $B \leftarrow \bar{B}$, and $N \leftarrow \bar{N}$. Go to Step 1.

Remarks

1. In Case 2, if there is more than one negative reduced cost, we can pick any one arbitrarily.
2. However, for any $j \in N$ such that $\bar{c}_j < 0$, recall that $c^T(\hat{x} + \lambda d) = c^T \hat{x} + \lambda \bar{c}_j$.
3. Therefore, $\bar{c}_j$ gives us the rate of decrease of the objective function and it makes sense to choose the most negative reduced cost.
4. Reduced costs are only computed for nonbasic variables. Suppose that $j \in B$ is the ℓth index in B. We have $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j = c_j - c_B^T (A_B)^{-1} A_B e^\ell = c_j - (c_B)_\ell = c_j - c_j = 0$. **Therefore, the reduced cost of any basic variable is equal to zero.**
5. The computation of λ^* in Case 2b is referred to as the *minimum ratio test.*
6. The computation of reduced costs at a vertex and the case checks is referred to as an *iteration of the simplex method.*

18.5 Finite Convergence of the Simplex Method

An interesting and relevant question is whether the simplex method always terminates after a finite number of iterations. The following proposition addresses this question under the nondegeneracy assumption.

Proposition 18.2. *Let (P) be a linear programming problem in standard form. Suppose that the feasible region $\mathcal{P}$ is nonempty, A has full row rank, and every vertex of $\mathcal{P}$ is nondegenerate. Then, the simplex method terminates after a finite number of iterations with exactly one of the following two possible outcomes:*

1. *An optimal vertex $\hat{x}$ with a finite optimal value, or*

2. *A vertex $\hat{x}$ and a direction d along which the objective function is unbounded below.*

Proof. Recall that every linear programming problem in standard form with a nonempty feasible region has at least one vertex. Therefore, we can always find a starting vertex (for example by using complete enumeration). Since every vertex of $\mathcal{P}$ is nondegenerate, Case 1 is a necessary and sufficient condition of optimality. Furthermore, we have $\lambda^* > 0$ whenever we perform Case 2b, therefore we always move to a different vertex. Since the objective function improves at the next vertex, we cannot go back to an earlier vertex that was already visited. Since every linear programming problem has at most a finite number of vertices, the result follows. $\square$

18.6 Discussion and Questions

Proposition 18.2 shows that the simplex method terminates after a finite number of iterations under the stated assumptions. It can either find an optimal vertex or detect that the problem is unbounded.

This development gives rise to a number of questions:

- **Question 1:** Is there a simple implementation of the simplex method?
- **Question 2:** What happens if there are degenerate vertices?
- **Question 3:** What if there are no feasible solutions (i.e., $\mathcal{P} = \emptyset$)?
- **Question 4:** Is there a simple procedure to obtain a starting vertex (without using complete enumeration)?

We will address each of these questions in the following lectures.

Exercises

Question 18.1. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a nonempty polyhedron in standard form and let $\hat{x}$ be a nondegenerate vertex of $\mathcal{P}$. Suppose that there exists a $j^* \in N$ such that $\bar{c}_{j^*} < 0$ and d_B has at least one negative component. Can the next vertex $\bar{x} = \hat{x} + \lambda^* d$ be a degenerate vertex?*

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 19	Simplex Method in Dictionary Form
Lecturer: E. Alper Yıldırım	*Week: 5*

19.1 Outline

- Simplex Method in Dictionary Form
- Review Problems

19.2 Overview

In this lecture, we will discuss a simple implementation of the simplex method, an algorithm for solving linear programming problems. We will illustrate this idea on two examples.

19.3 Example 1

Consider the following linear programming problem:

$$\begin{array}{lrcrcl}
\min & 3x_1 & - & 2x_2 & & \\
\text{s.t.} & & & & & \\
 & -3x_1 & + & 3x_2 & \leq & 6 \\
 & -4x_1 & + & 2x_2 & \leq & 2 \\
 & x_1 & - & 2x_2 & \leq & 2 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}$$

Since the problem is not in standard form, we introduce nonnegative slack variables x_3, x_4, and x_5 for the first, second, and the third inequality constraints, respectively.

We therefore obtain the following equivalent linear programming problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\min & 3x_1 & - & 2x_2 & & & & & & & & \\
\text{s.t.} & & & & & & & & & & & \\
 & -3x_1 & + & 3x_2 & + & x_3 & & & & & = & 6 \\
 & -4x_1 & + & 2x_2 & & & + & x_4 & & & = & 2 \\
 & x_1 & - & 2x_2 & & & & & + & x_5 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}$$

Note that we have $m = 3$ and $n = 5$, and

$$A = \begin{bmatrix} -3 & 3 & 1 & 0 & 0 \\ -4 & 2 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

In order to solve this problem using the simplex method, we need to have a vertex as a starting solution.

Let $\hat{x} = [0, 0, 6, 2, 2]^T \in \mathbb{R}^5$. Note that $\hat{x}$ is a feasible solution since it satisfies all of the constraints. Furthermore, $\hat{B} = \{j \in \{1, \ldots, 5\} : \hat{x}_j \neq 0\} = \{3, 4, 5\}$ and $\hat{N} = \{j \in \{1, \ldots, 5\} : \hat{x}_j = 0\} = \{1, 2\}$.

Since $|\hat{B}| = m = 3$ and

$$A_{\hat{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is clearly invertible, $\hat{x}$ is a vertex with $B = \hat{B}$ and $N = \hat{N}$.

In this solution, x_3, x_4, and x_5 are the basic variables, and x_1 and x_2 are nonbasic variables. The current objective function value at $\hat{x}$ is equal to 0.

Remark: We use x to denote a general solution and $\hat{x}$ to denote a specific solution. A similar convention will be used in general for other notation as well.

Dictionary 1

Let us define a new variable z to denote the objective function and express each basic variable as a function of nonbasic variables:

$$\begin{array}{rcrcrcr} z & = & 0 & + & 3x_1 & - & 2x_2 \\ x_3 & = & 6 & + & 3x_1 & - & 3x_2 \\ x_4 & = & 2 & + & 4x_1 & - & 2x_2 \\ x_5 & = & 2 & - & x_1 & + & 2x_2 \end{array}$$

This is called a *dictionary*. The first row, referred to as Row 0, corresponds to the objective function. The remaining three rows, denoted by Row 1, Row 2, and Row 3, correspond to each of the three basic variables.

- Since x_1 and x_2 are nonbasic, $\hat{x}_1 = \hat{x}_2 = 0$.
- x_3, x_4, and x_5 are basic variables, and $\hat{x}_3 = 6$, $\hat{x}_4 = 2$, and $\hat{x}_5 = 2$.
- The objective function value is $\hat{z} = 0$.

Looking at Row 0, we see that we can reduce z by increasing x_2 since it has a negative coefficient. We will therefore try to increase the value of x_2 while keeping $x_1 = 0$. By Rows 1, 2, and 3, this change will affect the values of each of x_3, x_4, and x_5.

- By Row 1, $x_3 = 6 - 3x_2 \geq 0$ if and only if $x_2 \leq 2$.

- By Row 2, $x_4 = 2 - 2x_2 \geq 0$ if and only if $x_2 \leq 1$.
- By Row 3, $x_5 = 2 + 2x_2$. We can therefore increase x_2 indefinitely and continue to have $x_5 \geq 0$.

The smallest bound is given by Row 2. This step is called the *minimum ratio test*. If $x_2 = 1$, we obtain $x_4 = 0$. Therefore, we will move x_2 to the left-hand side and x_4 to the left-hand side in Row 2.

Therefore, x_2 is the new basic variable, called the *entering variable*, and x_4 is the new nonbasic variable, called the *leaving variable*.

We now move x_2 to the left-hand side of Row 2 and move x_4 to the right-hand side of Row 2.

By Row 2, we obtain $x_2 = 1 + 2x_1 - (1/2)x_4$.

We now substitute this for x_2 in the right-hand sides of Rows 0, 1, and 3.

Dictionary 2

After the steps above, we arrive at the following dictionary:

$$\begin{array}{rcrcrcr} z & = & -2 & - & x_1 & + & x_4 \\ x_3 & = & 3 & - & 3x_1 & + & \frac{3}{2}x_4 \\ x_2 & = & 1 & + & 2x_1 & - & \frac{1}{2}x_4 \\ x_5 & = & 4 & + & 3x_1 & - & x_4 \end{array}$$

- We have $B = \{3, 2, 5\}$ and $N = \{1, 4\}$.
- The basic variables are given by $\hat{x}_3 = 3$, $\hat{x}_2 = 1$, and $\hat{x}_5 = 4$.
- The nonbasic variables are given by $\hat{x}_1 = 0$ and $\hat{x}_4 = 0$.
- The objective function value is $\hat{z} = -2 < 0$.

Since A_B is invertible, this is a vertex. We have therefore obtained a new vertex with a strictly better objective function value than the previous vertex.

By Row 0, x_1 is the entering variable since it has a negative coefficient on the right-hand side of Row 0.

- Note that Rows 2 and 3 do not give any restrictions (or upper bounds) on x_1.
- Row 1 is the only row that gives a finite upper bound of 1 on the value of x_1.

Therefore, x_1 is the entering variable and x_3 is the leaving variable.

We now move x_1 to the left-hand side of Row 1 and x_3 to the right-hand side of Row 1.

By Row 1, $x_1 = 1 - \frac{1}{3}x_3 + \frac{1}{2}x_4$.

We now substitute this expression for x_1 in Rows 0, 2, and 3.

Dictionary 3

After the steps above, we arrive at the following dictionary:

$$\begin{array}{rcrcrcr} z & = & -3 & + & \frac{1}{3}x_3 & + & \frac{1}{2}x_4 \\ x_1 & = & 1 & - & \frac{1}{3}x_3 & + & \frac{1}{2}x_4 \\ x_2 & = & 3 & - & \frac{2}{3}x_3 & + & \frac{1}{2}x_4 \\ x_5 & = & 7 & - & x_3 & + & \frac{1}{2}x_4 \end{array}$$

- We have $B = \{1, 2, 5\}$ and $N = \{3, 4\}$.
- The basic variables are given by $\hat{x}_1 = 1$, $\hat{x}_2 = 3$, and $\hat{x}_5 = 7$.
- The nonbasic variables are given by $\hat{x}_3 = 0$ and $\hat{x}_4 = 0$.
- The objective function value is $\hat{z} = -3 < -2$.

Since A_B is invertible, this is a vertex.

By Row 0, the coefficients of both nonbasic variables on the right-hand side of Row 0 are nonnegative. Therefore, we can no longer improve the objective function (i.e., there are no eligible entering variables).

Termination and Discussion

- We have found an optimal solution given by $x^* = [1, 3, 0, 0, 7]^T$ and the optimal value is $z^* = c^T x^* = -3$ after performing two iterations.
- Note that we started with a vertex having an objective function value of zero, and each vertex was strictly better than the previous one.
- Each of the three vertices was nondegenerate since the basic variables were always strictly positive.
- In each dictionary, Row 0 expressed the objective function in terms of nonbasic variables in that dictionary, and Rows 1, 2, and 3 expressed each basic variable in terms of nonbasic variables in that dictionary.
- We used Row 0 to determine the entering variable and used Rows 1, 2, and 3 to determine the leaving variable.

19.4 Relation with the Simplex Method

Question 1. *How is the above procedure related to the simplex method?*

Let (P) denote a linear programming problem in standard form.

$$\begin{array}{lllcl} \text{(P)} & \min & c^T x & & \\ & \text{s.t.} & & & \\ & & Ax & = & b, \\ & & x & \geq & \mathbf{0}, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$. Let $\mathcal{P}$ denote the (nonempty) feasible region.

- Let $\hat{x} \in \mathcal{P}$ be a vertex with index sets B and N.
- Since $Ax = b$ for each $x \in \mathcal{P}$, we obtain $A_B x_B + A_N x_N = b$.
- Multiplying both sides by $(A_B)^{-1}$ from the left, we obtain $x_B + (A_B)^{-1} A_N x_N = (A_B)^{-1} b$, i.e.,

$$x_B = (A_B)^{-1} b - (A_B)^{-1} A_N x_N.$$

- Since $z = c^T x = c_B^T x_B + c_N^T x_N$, we can substitute $x_B = (A_B)^{-1} b - (A_B)^{-1} A_N x_N$ and obtain

$$z = c_B^T (A_B)^{-1} b - c_B^T (A_B)^{-1} A_N x_N + c_N^T x_N = c_B^T (A_B)^{-1} b + c_N^T x_N - c_B^T (A_B)^{-1} A_N x_N.$$

- Note that $c_N^T x_N = \sum_{j \in N} c_j x_j$ and $A_N x_N = \sum_{j \in N} A^j x_j$.

We therefore arrive at the following system of equations:

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Since $\hat{x} \in \mathcal{P}$ is a vertex with index sets B and N, we obtain $\hat{x}_N = \mathbf{0}$ and $\hat{x}_B = (A_B)^{-1} b$. Therefore, $\hat{z} = c^T \hat{x} = c_B^T \hat{x}_B + c_N^T \hat{x}_N = c_B^T (A_B)^{-1} b$.
- The expressions $c_B^T (A_B)^{-1} b$ and $(A_B)^{-1} b$ are the current values of the objective function value and the basic variables at $\hat{x}$, respectively.
- The coefficients of the nonbasic variables on the right-hand side of Row 0 are the reduced costs.
- If $\bar{c}_{j^*} < 0$, then the column $-(A_B)^{-1} A^{j^*}$ yields d_B obtained by setting $d_{j^*} = 1$ and $d_j = 0$ for each $j \in N \backslash \{j^*\}$.

19.5 Example 2

Consider the following linear programming problem:

$$\begin{array}{lrcrcl} \min & -x_1 & + & x_2 & & \\ \text{s.t.} & & & & & \\ & -3x_1 & + & 3x_2 & \leq & 6 \\ & -4x_1 & + & 2x_2 & \leq & 2 \\ & x_1 & - & 2x_2 & \leq & 2 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

The equivalent problem in standard form is given by

$$\begin{array}{lrcrcrcrcrcl}
\min & -x_1 & + & x_2 & & & & & & & & \\
\text{s.t.} & & & & & & & & & & & \\
& -3x_1 & + & 3x_2 & + & x_3 & & & & & = & 6 \\
& -4x_1 & + & 2x_2 & & & + & x_4 & & & = & 2 \\
& x_1 & - & 2x_2 & & & & & + & x_5 & = & 2 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0.
\end{array}$$

Let $\hat{x} = [0, 0, 6, 2, 2]^T$.

Dictionary 1

$$\begin{array}{rcrcrcr}
z & = & 0 & - & x_1 & + & x_2 \\
x_3 & = & 6 & + & 3x_1 & - & 3x_2 \\
x_4 & = & 2 & + & 4x_1 & - & 2x_2 \\
x_5 & = & 2 & - & x_1 & + & 2x_2
\end{array}$$

- We have $B = \{3, 4, 5\}$ and $N = \{1, 2\}$.
- The values of basic variables are given by $\hat{x}_3 = 6$, $\hat{x}_4 = 2$, and $\hat{x}_5 = 2$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$ and $\hat{x}_2 = 0$.
- The objective function value is $\hat{z} = 0$.
- Since A_B is invertible, this is a vertex.
- By Row 0, x_1 is the entering variable and x_5 is the leaving variable by Row 3.
- By Row 3, $x_1 = 2 + 2x_2 - x_5$. Substitute this expression for x_1 in Rows 0, 1, and 2.

Dictionary 2

$$\begin{array}{rcrcrcr}
z & = & -2 & - & x_2 & + & x_5 \\
x_3 & = & 12 & + & 3x_2 & - & 3x_5 \\
x_4 & = & 10 & + & 6x_2 & - & 4x_5 \\
x_1 & = & 2 & + & 2x_2 & - & x_5
\end{array}$$

- We have $B = \{3, 4, 1\}$ and $N = \{2, 5\}$.
- The values of basic variables are given by $\hat{x}_3 = 12$, $\hat{x}_4 = 10$, and $\hat{x}_1 = 2$.
- The values of nonbasic variables are given by $\hat{x}_2 = 0$ and $\hat{x}_5 = 0$.
- The objective function value is $\hat{z} = -2 < 0$.
- Since A_B is invertible, this is a vertex.

- By Row 0, x_2 is the entering variable.
- By Rows 1, 2, and 3, there is no leaving variable, i.e., we can increase x_2 indefinitely without ever leaving the feasible region!
- Therefore, the problem is unbounded along the direction $d = [2, 1, 3, 6, 0]^T$.

19.6 Summary

Consider a linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq 0\}$$

19.6.1 Setup and Initialisation

Assume that the feasible region is nonempty, A has full row rank, and we have an initial basic feasible solution $\hat{x}$ with index sets B and N. Define a new variable z corresponding to the objective function. Set up the initial dictionary by

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b - \sum_{j \in N} (A_B)^{-1} A^j x_j \end{aligned}$$

19.6.2 Procedure

1. If $\bar{c}_j \geq 0$ for each $j \in N$ in Row 0, then stop. $\hat{x}$ is optimal and the optimal value is $z^* = c_B^T (A_B)^{-1} b$.
2. If there exists $j^* \in N$ such that $\bar{c}_j < 0$, then
   (a) If the coefficients of x_{j^*} on the right-hand sides of Rows 1 through m are all nonnegative, then stop. The problem is unbounded along the direction d given by $d_{j^*} = 1$, $d_j = 0$ for each $j \in N \backslash \{j^*\}$, and $d_B = -(A_B)^{-1} A^{j^*}$.
   (b) Otherwise, x_{j^*} is the entering variable.
      i. Apply the minimum ratio test to determine the leaving variable x_{k^*}, where $k^* \in B$.
      ii. Use the row corresponding to x_{k^*} to move x_{j^*} to the left-hand side and x_{k^*} to the right-hand side.
      iii. Substitute this expression for x_{j^*} in every other row, including Row 0.
      iv. Update the dictionary, the index sets B and N, and the current vertex $\hat{x}$ accordingly and go to Step 1.

19.7 Dictionary vs Tableau

You may have previously studied the simplex method in tableau form (as opposed to the dictionary form). Both dictionary and tableau forms are in fact equivalent. The tableau form is easier to implement. However, it is somewhat more "cryptic." We adopt dictionaries since they are more explicit and easier to understand.

19.8 Discussion and Questions

We discussed a simple and intuitive implementation of the simplex method. In both examples, all visited vertices were nondegenerate and we had an "easily identifiable" starting vertex (i.e., set all original variables to zero and each slack variables to the corresponding right-hand side value).

- **Question 1:** What happens if there are degenerate vertices?
- **Question 2:** What if there are no feasible solutions (i.e., $\mathcal{P} = \emptyset$)?
- **Question 3:** Is there a simple procedure to obtain a starting vertex if it is not easy to identify one?

We will address these questions in the subsequent lectures.

Exercises

Question 19.1. *Consider Example 1 again:*

$$\begin{array}{lrcrcl}
\min & 3x_1 & - & 2x_2 & & \\
s.t. & & & & & \\
 & -3x_1 & + & 3x_2 & \leq & 6 \\
 & -4x_1 & + & 2x_2 & \leq & 2 \\
 & x_1 & - & 2x_2 & \leq & 2 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}$$

After adding the slack variables x_3, x_4, and x_5 for the first, second, and the third inequality constraints, respectively, here are the basic feasible solutions computed by the simplex method: (i) $\hat{x}^1 = [0, 0, 6, 2, 2]^T$; (ii) $\hat{x}^2 = [0, 1, 3, 0, 4]^T$; (iii) $\hat{x}^3 = [1, 3, 0, 0, 7]^T$. Solve the given problem using the graphical method and identify the feasible solutions of the original problem corresponding to $\hat{x}^1, \hat{x}^2$, and $\hat{x}^3$.

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 20	Simplex Method and Degeneracy
Lecturer: E. Alper Yıldırım	*Week: 7*

20.1 Outline

- Effects of Degeneracy on the Simplex Method
- Bland's Rule
- Review Problems

20.2 Overview

- Recall that a nondegenerate vertex is an optimal solution if and only if all reduced costs of nonbasic variables are nonnegative.
- At a degenerate vertex, if all reduced costs of nonbasic variables are nonnegative, then it is optimal.
- However, a degenerate vertex may be an optimal solution and yet reduced costs of some nonbasic variables may still be negative.
- In this lecture, we will discuss if and how degeneracy affects the simplex method.

20.3 An Example

Consider the following linear programming problem:

$$\begin{array}{llllllllll}
\min & -10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 & & \\
\text{s.t.} & & & & & & & & & \\
 & 0.5x_1 & - & 5.5x_2 & - & 2.5x_3 & + & 9x_4 & \leq & 0 \\
 & 0.5x_1 & - & 1.5x_2 & - & 0.5x_3 & + & x_4 & \leq & 0 \\
 & x_1 & & & & & & & \leq & 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

Since the problem is not in standard form, we introduce nonnegative slack variables x_5, x_6, and x_7 for the first, second, and the third inequality constraints, respectively.

We obtain the following equivalent linear programming problem in standard form:

$$\begin{array}{llllllllllllllll}
\min & -10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 & & & & & & & & \\
\text{s.t.} & & & & & & & & & & & & & & & \\
 & 0.5x_1 & - & 5.5x_2 & - & 2.5x_3 & + & 9x_4 & + & x_5 & & & & & = & 0 \\
 & 0.5x_1 & - & 1.5x_2 & - & 0.5x_3 & + & x_4 & & & + & x_6 & & & = & 0 \\
 & x_1 & & & & & & & & & & & + & x_7 & = & 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & , & x_7 & \geq & 0
\end{array}$$

We therefore have

$$A = \begin{bmatrix} 0.5 & -5.5 & -2.5 & 9 & 1 & 0 & 0 \\ 0.5 & -1.5 & -0.5 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} -10 \\ 57 \\ 9 \\ 24 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}.$$

Note that $m = 3$ and $n = 7$. We use $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ as the starting vertex, with $B = \{5, 6, 7\}$ and $N = \{1, 2, 3, 4\}$.

Dictionary 1

$$\begin{array}{rcrcrcrcrcr} z & = & 0 & - & 10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 \\ x_5 & = & 0 & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\ x_6 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\ x_7 & = & 1 & - & x_1 & & & & & & \end{array}$$

- $B = \{5, 6, 7\}$ and $N = \{1, 2, 3, 4\}$.
- The values of basic variables are given by $\hat{x}_5 = 0$, $\hat{x}_6 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_3 = 0$, and $\hat{x}_4 = 0$.
- The objective function value is $\hat{z} = 0$.
- Note that $\hat{x}$ is degenerate since there is at least one basic variable with a value of zero.
- By Row 0, x_1 is the entering variable. By Rows 1 and 2, x_5 and x_6 both achieve the minimum ratio.
- We will break ties in favour of the basic variable with the smallest index, i.e., x_5 is the leaving variable.
- By Row 1, $x_1 = 0 + 11x_2 + 5x_3 - 18x_4 - 2x_5$. Substitute this expression for x_1 in Rows 0, 2, and 3.

Dictionary 2

$$\begin{array}{rcrcrcrcrcr} z & = & 0 & - & 53x_2 & - & 41x_3 & + & 204x_4 & + & 20x_5 \\ x_1 & = & 0 & + & 11x_2 & + & 5x_3 & - & 18x_4 & - & 2x_5 \\ x_6 & = & 0 & - & 4x_2 & - & 2x_3 & + & 8x_4 & + & x_5 \\ x_7 & = & 1 & - & 11x_2 & - & 5x_3 & + & 18x_4 & + & 2x_5 \end{array}$$

- $B = \{1, 6, 7\}$ and $N = \{2, 3, 4, 5\}$.
- The values of basic variables are given by $\hat{x}_1 = 0$, $\hat{x}_6 = 0$, and $\hat{x}_7 = 1$.

- The values of nonbasic variables are given by $\hat{x}_2 = 0$, $\hat{x}_3 = 0$, $\hat{x}_4 = 0$, and $\hat{x}_5 = 0$.
- The objective function value is $\hat{z} = 0$.
- Note that the objective function value has not improved!
- In fact, we are still at the same vertex given by $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$.
- The only change is the new index sets B and N.
- By Row 0, we can either increase x_2 or x_3.
- Since x_2 has the most negative reduced cost, we will choose x_2 as the entering variable.
- By Row 2, x_6 is the leaving variable.
- Using Row 2, we obtain $x_2 = 0 - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6$.
- We substitute this expression for x_2 in Rows 0, 1, and 3.

Dictionary 3

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 14.5x_3 & + & 98x_4 & + & 6.75x_5 & + & 13.25x_6 \\
x_1 & = & 0 & - & 0.5x_3 & + & 4x_4 & + & 0.75x_5 & - & 2.75x_6 \\
x_2 & = & 0 & - & 0.5x_3 & + & 2x_4 & + & 0.25x_5 & - & 0.25x_6 \\
x_7 & = & 1 & + & 0.5x_3 & - & 4x_4 & - & 0.75x_5 & + & 2.75x_6
\end{array}$$

- $B = \{1, 2, 7\}$ and $N = \{3, 4, 5, 6\}$.
- The values of basic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_3 = 0$, $\hat{x}_4 = 0$, $\hat{x}_5 = 0$, and $\hat{x}_6 = 0$.
- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- By Row 0, x_3 is the entering variable.
- By Rows 1 and 2, each of x_1 and x_2 achieves the minimum ratio. We will break ties in favour of the basic variable with the smallest index, i.e., x_1 is the leaving variable.
- By Row 1, $x_3 = 0 - 2x_1 + 8x_4 + 1.5x_5 - 5.5x_6$. Substitute this in Rows 0, 2, and 3.

Dictionary 4

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & + & 29x_1 & - & 18x_4 & - & 15x_5 & + & 93x_6 \\
x_3 & = & 0 & - & 2x_1 & + & 8x_4 & + & 1.5x_5 & - & 5.5x_6 \\
x_2 & = & 0 & + & x_1 & - & 2x_4 & - & 0.5x_5 & + & 2.5x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}$$

- $B = \{3, 2, 7\}$ and $N = \{1, 4, 5, 6\}$.
- The values of basic variables are given by $\hat{x}_3 = 0$, $\hat{x}_2 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_4 = 0$, $\hat{x}_5 = 0$, and $\hat{x}_6 = 0$.
- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- By Row 0, x_4 is the entering variable due to the most negative reduced cost.
- By Row 2, x_2 is the leaving variable. We obtain $x_4 = 0 + 0.5x_1 - 0.5x_2 - 0.25x_5 + 1.25x_6$. Substitute this in Rows 0, 1, and 3.

Dictionary 5

$$\begin{array}{rcrcrcrcrcr} z & = & 0 & + & 20x_1 & + & 9x_2 & - & 10.5x_5 & + & 70.5x_6 \\ x_3 & = & 0 & + & 2x_1 & - & 4x_2 & - & 0.5x_5 & + & 4.5x_6 \\ x_4 & = & 0 & + & 0.5x_1 & - & 0.5x_2 & - & 0.25x_5 & + & 1.25x_6 \\ x_7 & = & 1 & - & x_1 & & & & & & \end{array}$$

- $B = \{3, 4, 7\}$ and $N = \{1, 2, 5, 6\}$.
- The values of basic variables are given by $\hat{x}_3 = 0$, $\hat{x}_4 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_5 = 0$, and $\hat{x}_6 = 0$.
- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- By Row 0, x_5 is the entering variable.
- By Rows 1 and 2, x_3 and x_4 both achieve the minimum ratio. We choose x_3 as the leaving variable.
- By Row 1, we obtain $x_5 = 0 + 4x_1 - 8x_2 - 2x_3 + 9x_6$. Substitute this in Rows 0, 2, and 3.

Dictionary 6

$$\begin{array}{rcrcrcrcrcr} z & = & 0 & - & 22x_1 & + & 93x_2 & + & 21x_3 & - & 24x_6 \\ x_5 & = & 0 & + & 4x_1 & - & 8x_2 & - & 2x_3 & + & 9x_6 \\ x_4 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_6 \\ x_7 & = & 1 & - & x_1 & & & & & & \end{array}$$

- $B = \{5, 4, 7\}$ and $N = \{1, 2, 3, 6\}$.
- The values of basic variables are given by $\hat{x}_5 = 0$, $\hat{x}_4 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_3 = 0$, and $\hat{x}_6 = 0$.

- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- By Row 0, x_6 is the entering variable due to the most negative reduced cost.
- By Row 2, x_4 is the leaving variable. We obtain $x_6 = 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4$. Substitute this in Rows 0, 1, and 3.

Dictionary 7

$$
\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 10x_1 & + & 57x_2 & + & 9x_3 & + & 24x_4 \\
x_5 & = & 0 & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
x_6 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}
$$

- $B = \{5, 6, 7\}$ and $N = \{1, 2, 3, 4\}$.
- The values of basic variables are given by $\hat{x}_5 = 0$, $\hat{x}_6 = 0$, and $\hat{x}_7 = 1$.
- The values of nonbasic variables are given by $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, $\hat{x}_3 = 0$, and $\hat{x}_4 = 0$.
- The objective function value is $\hat{z} = 0$.
- We are still at the same vertex $\hat{x} = [0, 0, 0, 0, 0, 0, 1]^T$ with new index sets B and N.
- Note that this is the same dictionary as in Dictionary 1!

20.3.1 Discussion

- In the presence of degeneracy, the simplex method may go through a number of dictionaries and come back to an earlier dictionary.
- This phenomenon is called *cycling*.
- In fact, the vertex remains the same and only the index sets B and N change at each iteration.
- Therefore, the simplex method may cycle indefinitely among a set of dictionaries without making any progress and may not terminate under the existence of degenerate vertices.
- Note that cycling cannot occur under nondegeneracy.
- Fortunately, there is a simple rule that prevents cycling.

20.4 Bland's Rule

Proposition 20.1. *At a degenerate vertex, consider the following rule, known as Bland's rule: If there is more than one nonbasic variable with a negative reduced cost, choose the variable with the smallest subscript as the entering variable. If there is more than one basic variable that achieves the minimum ratio, choose the variable with the smallest subscript as the leaving variable. Then, the simplex method never visits an earlier dictionary and therefore terminates after a finite number of iterations.*

Proof. The proof is based on a contradiction argument. Conceptually, it is not too difficult but the details are somewhat tedious and we therefore omit the proof. □

20.5 Example Revisited

We will now reconsider the same example as in Section 20.3. This time, we will choose the entering and leaving variables using Bland's rule at every degenerate vertex.

Dictionary 1

$$\begin{array}{rcrcrcrcrcr} z &=& 0 &-& 10x_1 &+& 57x_2 &+& 9x_3 &+& 24x_4 \\ x_5 &=& 0 &-& 0.5x_1 &+& 5.5x_2 &+& 2.5x_3 &-& 9x_4 \\ x_6 &=& 0 &-& 0.5x_1 &+& 1.5x_2 &+& 0.5x_3 &-& x_4 \\ x_7 &=& 1 &-& x_1 \end{array}$$

- $B = \{5,6,7\}$ and $N = \{1,2,3,4\}$.
- By Row 0, x_1 is the entering variable.
- By Rows 1 and 2, x_5 and x_6 are both candidates for the leaving variable. Choose x_5 as the leaving variable by Bland's rule.

Dictionary 2

$$\begin{array}{rcrcrcrcrcr} z &=& 0 &-& 53x_2 &-& 41x_3 &+& 204x_4 &+& 20x_5 \\ x_1 &=& 0 &+& 11x_2 &+& 5x_3 &-& 18x_4 &-& 2x_5 \\ x_6 &=& 0 &-& 4x_2 &-& 2x_3 &+& 8x_4 &+& x_5 \\ x_7 &=& 1 &-& 11x_2 &-& 5x_3 &+& 18x_4 &+& 2x_5 \end{array}$$

- $B = \{1,6,7\}$ and $N = \{2,3,4,5\}$.
- By Row 0, x_2 and x_3 are the candidates for entering variable. By Bland's rule, we choose x_2 as the entering variable.
- By Row 2, x_6 is the leaving variable.

Dictionary 3

$$\begin{array}{rcrcrcrcrcr} z &=& 0 &-& 14.5x_3 &+& 98x_4 &+& 6.75x_5 &+& 13.25x_6 \\ x_1 &=& 0 &-& 0.5x_3 &+& 4x_4 &+& 0.75x_5 &-& 2.75x_6 \\ x_2 &=& 0 &-& 0.5x_3 &+& 2x_4 &+& 0.25x_5 &-& 0.25x_6 \\ x_7 &=& 1 &+& 0.5x_3 &-& 4x_4 &-& 0.75x_5 &+& 2.75x_6 \end{array}$$

- $B = \{1, 2, 7\}$ and $N = \{3, 4, 5, 6\}$.
- By Row 0, x_3 is the entering variable.
- By Rows 1 and 2, x_1 and x_2 are both candidates for the leaving variable. Choose x_1 as the leaving variable by Bland's rule.

Dictionary 4

$$
\begin{array}{rcrcrcrcrcr}
z & = & 0 & + & 29x_1 & - & 18x_4 & - & 15x_5 & + & 93x_6 \\
x_3 & = & 0 & - & 2x_1 & + & 8x_4 & + & 1.5x_5 & - & 5.5x_6 \\
x_2 & = & 0 & + & x_1 & - & 2x_4 & - & 0.5x_5 & + & 2.5x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}
$$

- $B = \{3, 2, 7\}$ and $N = \{1, 4, 5, 6\}$.
- By Row 0, x_4 and x_5 are candidates for the entering variable. We choose x_4 as the entering variable by Bland's rule.
- By Row 2, x_2 is the leaving variable.

Dictionary 5

$$
\begin{array}{rcrcrcrcrcr}
z & = & 0 & + & 20x_1 & + & 9x_2 & - & 10.5x_5 & + & 70.5x_6 \\
x_3 & = & 0 & + & 2x_1 & - & 4x_2 & - & 0.5x_5 & + & 4.5x_6 \\
x_4 & = & 0 & + & 0.5x_1 & - & 0.5x_2 & - & 0.25x_5 & + & 1.25x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}
$$

- $B = \{3, 4, 7\}$ and $N = \{1, 2, 5, 6\}$.
- By Row 0, x_5 is the entering variable.
- By Rows 1 and 2, x_3 and x_4 are both candidates for the leaving variable. Choose x_3 as the leaving variable by Bland's rule.

Dictionary 6

$$
\begin{array}{rcrcrcrcrcr}
z & = & 0 & - & 22x_1 & + & 93x_2 & + & 21x_3 & - & 24x_6 \\
x_5 & = & 0 & + & 4x_1 & - & 8x_2 & - & 2x_3 & + & 9x_6 \\
x_4 & = & 0 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_6 \\
x_7 & = & 1 & - & x_1 & & & & & &
\end{array}
$$

- $B = \{5, 4, 7\}$ and $N = \{1, 2, 3, 6\}$.

- By Row 0, x_1 and x_6 the candidates for entering variable. By Bland's rule, we choose x_1 as the entering variable.
- By Row 2, x_4 is the leaving variable.
- In the previous case, we had used x_6 as the entering variable due to the most negative reduced cost. **Since the current vertex is degenerate, we ignore the most negative reduced cost and apply Bland's rule instead to choose the entering variable.**

Dictionary 7

$$\begin{array}{rcrcrcrcrcr}
z & = & 0 & + & 27x_2 & - & x_3 & + & 44x_4 & + & 20x_6 \\
x_5 & = & 0 & + & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_6 \\
x_1 & = & 0 & + & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_6 \\
x_7 & = & 1 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_6
\end{array}$$

- $B = \{5, 1, 7\}$ and $N = \{2, 3, 4, 6\}$.
- By Row 0, x_3 is the entering variable.
- By Row 3, x_7 is the leaving variable.

Dictionary 8

$$\begin{array}{rcrcrcrcrcr}
z & = & -1 & + & 30x_2 & + & 42x_4 & + & 18x_6 & + & x_7 \\
x_5 & = & 2 & - & 2x_2 & - & 4x_4 & + & 5x_6 & - & 2x_7 \\
x_1 & = & 1 & & & & & & & - & x_7 \\
x_3 & = & 1 & - & 3x_2 & + & 2x_4 & + & 2x_6 & - & x_7
\end{array}$$

- $B = \{5, 1, 3\}$ and $N = \{2, 4, 6, 7\}$.
- Note that $\hat{z} = -1 < 0$ and we therefore moved to a new vertex!
- By Row 0, the current vertex is optimal since all reduced costs are nonnegative.
- Therefore, an optimal solution is given by $x^* = [1, 0, 1, 0, 2, 0, 0]^T$ and the optimal value is $z^* = -1$.

20.6 Concluding Remarks

- Degeneracy may cause problems in the implementation of the simplex method.
- These issues can easily be circumvented by using Bland's rule.
- **Note that Bland's rule needs to be applied only when the current vertex is degenerate.**
- For a nondegenerate vertex, it is still reasonable to choose the nonbasic variable with the most negative reduced cost.

- Therefore, by using Bland's rule at each degenerate vertex, we conclude that the simplex method always terminates after a finite number of iterations.
- In the next lecture, we will address the issue of detecting infeasibility and computing an initial vertex if one is not easily identifiable.

Exercises

Question 20.1. *Explain what leads to cycling at a degenerate vertex and why the simplex method cannot cycle if every vertex is nondegenerate.*

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 21 — The Two-Phase Method

Lecturer: E. Alper Yıldırım — *Week: 7*

21.1 Outline

- The Two-Phase Method
- Driving Artificial Variables out of the Basis
- Review Problems

21.2 Overview

- The simplex method requires an initial basic feasible solution.
- In this lecture, we will discuss how to construct an initial basic feasible solution if one is not easily identifiable, and how to detect infeasibility.
- We will discuss a method that solves both problems simultaneously.

21.3 Easily Identifiable Basic Feasible Solutions

Suppose that a linear programming problem is in the following form:

$$\min\{c^T x : Ax \leq b, \quad x \geq \mathbf{0}\},$$

where $b \geq \mathbf{0}$.

- Then, we can define a vector of slack variables, one for each constraint and denoted by x^s, and convert the above problem into an equivalent linear programming problem in standard form:

$$\min\{c^T x : Ax + x^s = b, \quad x \geq \mathbf{0}, \quad x^s \geq \mathbf{0}\}.$$

- If we set $\hat{x} = \mathbf{0}$ and $\hat{x}^s = b \geq \mathbf{0}$, then we easily obtain an initial basic feasible solution.
- This is the starting solution we used in all of the previous examples.

21.4 General Case

As before, we will assume that the linear programming problem is in standard form, i.e.,

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}.$$

- Note that we may assume that $b \geq \mathbf{0}$ by multiplying the ith equality constraint by -1 if $b_i < 0$, where $i = 1, \ldots, m$.
- We also assume that A has full row rank.
- **Question 1:** Does there exist a feasible solution?
- **Question 2:** If the feasible region is nonempty, how can we find a vertex? (Recall that every nonempty polyhedron in standard form has at least one vertex.)

21.4.1 Basic Idea

- We define and add a nonnegative "artificial" variable a_i to the left-hand side of each equality constraint i, where $i = 1, \ldots, m$.
- Denote this vector of artificial variables by $a \in \mathbb{R}^m$.
- We obtain the following modified system:

$$Ax + a = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0}.$$

- Note that the set of all vectors $(x, a) \in \mathbb{R}^n \times \mathbb{R}^m$ that satisfy the above system is a polyhedron.
- Let us now define

$$\begin{aligned} \mathcal{P} &= \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\} \\ \tilde{\mathcal{P}} &= \{(x, a) \in \mathbb{R}^n \times \mathbb{R}^m : Ax + a = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0}\} \end{aligned}$$

Proposition 21.1. *Suppose that* $b \geq \mathbf{0}$.

(i) Then, $\tilde{\mathcal{P}}$ *is a nonempty polyhedron.*

(ii) Furthermore, $\mathcal{P}$ *is a nonempty polyhedron if and only if there exists a solution* $(\bar{x}, \bar{a}) \in \tilde{\mathcal{P}}$ *such that* $\bar{a} = \mathbf{0}$.

(iii) Finally, if A *has full row rank and* $(\hat{x}, \hat{a})$ *is a vertex of* $\tilde{\mathcal{P}}$ *such that* $\hat{a} = \mathbf{0}$*, then* $\hat{x}$ *is a vertex of* $\mathcal{P}$.

Proof. (i) Since $b \geq \mathbf{0}$, we can set $(\bar{x}, \bar{a}) = (\mathbf{0}, b)$. Then, $A\bar{x} + \bar{a} = b$ and $\bar{x} \geq \mathbf{0}$ and $\bar{a} \geq \mathbf{0}$, i.e., $(\bar{x}, \bar{a}) \in \tilde{\mathcal{P}}$, which implies that $\tilde{\mathcal{P}}$ is a nonempty polyhedron (regardless of whether $\mathcal{P}$ is nonempty or not).

(ii) Suppose that $\mathcal{P}$ is a nonempty polyhedron. Then, there exists $\bar{x} \in \mathbb{R}^n$ such that $A\bar{x} = b$ and $\bar{x} \geq \mathbf{0}$. Then, by defining $(\bar{x}, \bar{a}) = (\bar{x}, \mathbf{0})$, we obtain $(\bar{x}, \bar{a}) \in \tilde{\mathcal{P}}$.

Conversely, if there exists a solution $(\bar{x}, \bar{a}) \in \tilde{\mathcal{P}}$ such that $\bar{a} = \mathbf{0}$, then we clearly have $\bar{x} \in \mathcal{P}$.

(iii) Finally, suppose that A has full row rank and that $(\hat{x}, \hat{a})$ is a vertex of $\tilde{\mathcal{P}}$ such that $\hat{a} = \mathbf{0}$. Since $\hat{a} = \mathbf{0}$, let us define $\hat{B} = \{j \in \{1, \ldots, m\} : \hat{x}_j > 0\}$. Note that $|\hat{B}| \leq m$.

Case 1: If $|\hat{B}| = m$, we set $B = \hat{B}$ and $N = \{1, \ldots, n\} \backslash B$ and we obtain that $\hat{x}$ is a nondegenerate vertex of $\mathcal{P}$.

Case 2: If $|\hat{B}| < m$, since A has full row rank, we can choose $m - |\hat{B}|$ columns of A corresponding to $\hat{x}_j = 0$, add those indices to $\hat{B}$ to obtain an index set with $|B| = m$, and ensure that $A_B \in \mathbb{R}^{m \times m}$ is invertible. Therefore, $\hat{x}$ is a degenerate vertex of $\mathcal{P}$.

□

21.5 The Two-Phase Method

- By Proposition 21.1, we can try to solve the following *auxiliary* linear programming problem, called the Phase 1 Problem:

$$\text{(AUX)} \quad \min \left\{ \sum_{i=1}^{m} a_i : Ax + a = b, \quad x \geq \mathbf{0}, \quad a \geq \mathbf{0} \right\}.$$

- Note that (AUX) has a nonempty feasible region by Proposition 21.1 if $b \geq \mathbf{0}$.
- The optimal value of (AUX) is always nonnegative since $a \geq \mathbf{0}$ and the objective function is given by $\sum_{i=1}^{m} a_i$.
- Furthermore, if we set $\hat{x} = \mathbf{0}$ and $\hat{a} = b \geq \mathbf{0}$, we obtain an easily identifiable vertex, from which we can start the simplex method to solve (AUX).
- There are two possibilities:
  (i) **Case 1:** If the optimal value of (AUX) is positive, then $\mathcal{P} = \emptyset$ by Proposition 21.1 (i.e., the original linear programming problem is infeasible).
  (ii) **Case 2:** If the optimal value of (AUX) is zero, then (AUX) has an optimal vertex $(\hat{x}, \hat{a})$ such that $\hat{a} = \mathbf{0}$. Assuming that A has full row rank, then $\hat{x}$ is a vertex of $\mathcal{P}$ by Proposition 21.1 and can be used to start the simplex method to solve the original linear programming problem, called the Phase 2 Problem.

21.5.1 An Example

Consider the following linear programming problem:

$$\begin{array}{lrcrcrcrcrcl}
\min & 2x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & - & 2x_5 & & \\
\text{s.t.} & & & & & & & & & & & \\
 & x_1 & + & 3x_2 & & & + & 4x_4 & + & x_5 & = & 2 \\
 & x_1 & + & 2x_2 & & & - & 3x_4 & + & x_5 & = & 2 \\
 & -x_1 & - & 4x_2 & + & 3x_3 & & & & & = & 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}$$

- Since a vertex is not easily identifiable, we will use the two-phase method.
- We define and add nonnegative artificial variables a_1, a_2, and a_3 corresponding to the three equality constraints and try to minimize $a_1 + a_2 + a_3$ in Phase 1.

Phase 1 Problem

$$
\begin{array}{lrcrcrcrcrcrcrcrcl}
\min & & & & & & & & & & & a_1 & + & a_2 & + & a_3 & & \\
\text{s.t.} & & & & & & & & & & & & & & & & & \\
 & x_1 & + & 3x_2 & & & + & 4x_4 & + & x_5 & + & a_1 & & & & & = & 2 \\
 & x_1 & + & 2x_2 & & & - & 3x_4 & + & x_5 & & & + & a_2 & & & = & 2 \\
 & -x_1 & - & 4x_2 & + & 3x_3 & & & & & & & & & + & a_3 & = & 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & a_1 & , & a_2 & , & a_3 & \geq & 0
\end{array}
$$

- We will use $\hat{x}_1 = \hat{x}_2 = \hat{x}_3 = \hat{x}_4 = \hat{x}_5 = 0$ and $\hat{a}_1 = 2$, $\hat{a}_2 = 2$, and $\hat{a}_3 = 1$ as the initial vertex for the Phase 1 problem.
- Then, Rows 1, 2, and 3 of the starting dictionary are given by the following:

$$
\begin{array}{rcrcrcrcrcrcr}
a_1 & = & 2 & - & x_1 & - & 3x_2 & & & - & 4x_4 & - & x_5 \\
a_2 & = & 2 & - & x_1 & - & 2x_2 & & & + & 3x_4 & - & x_5 \\
a_3 & = & 1 & + & x_1 & + & 4x_2 & - & 3x_3 & & & &
\end{array}
$$

- The basic variables are a_1, a_2, and a_3.
- We have not yet added Row 0, given by $z = a_1 + a_2 + a_3$.
- **However, recall that we need to express the right-hand side of Row 0 in terms of nonbasic variables only** (i.e., x_1, x_2, x_3, x_4, and x_5).
- Therefore, we will substitute the expressions on the right-hand sides of Rows 1, 2, and 3 for a_1, a_2, and a_3, respectively, in Row 0.
- **We can apply the simplex method only after we satisfy this requirement (i.e., we need to ensure that our initial dictionary is in proper form).**

Phase 1 Dictionary 1

$$
\begin{array}{rcrcrcrcrcrcr}
z & = & 5 & - & x_1 & - & x_2 & - & 3x_3 & - & x_4 & - & 2x_5 \\
a_1 & = & 2 & - & x_1 & - & 3x_2 & & & - & 4x_4 & - & x_5 \\
a_2 & = & 2 & - & x_1 & - & 2x_2 & & & + & 3x_4 & - & x_5 \\
a_3 & = & 1 & + & x_1 & + & 4x_2 & - & 3x_3 & & & &
\end{array}
$$

- By Row 0, x_3 is the entering variable due to the most negative reduced cost.
- By Row 3, a_3 is the leaving variable.

Phase 1 Dictionary 2

$$\begin{array}{rcrcrcrcrcrcr}
z &=& 4 &-& 2x_1 &-& 5x_2 &-& x_4 &-& 2x_5 &+& a_3 \\
a_1 &=& 2 &-& x_1 &-& 3x_2 &-& 4x_4 &-& x_5 && \\
a_2 &=& 2 &-& x_1 &-& 2x_2 &+& 3x_4 &-& x_5 && \\
x_3 &=& \frac{1}{3} &+& \frac{1}{3}x_1 &+& \frac{4}{3}x_2 &&&&&-& \frac{1}{3}a_3
\end{array}$$

- By Row 0, x_2 is the entering variable due to the most negative reduced cost.
- By Row 1, a_1 is the leaving variable.

Phase 1 Dictionary 3

$$\begin{array}{rcrcrcrcrcrcr}
z &=& \frac{2}{3} &-& \frac{1}{3}x_1 &+& \frac{17}{3}x_4 &-& \frac{1}{3}x_5 &+& \frac{5}{3}a_1 &+& a_3 \\
x_2 &=& \frac{2}{3} &-& \frac{1}{3}x_1 &-& \frac{4}{3}x_4 &-& \frac{1}{3}x_5 &-& \frac{1}{3}a_1 && \\
a_2 &=& \frac{2}{3} &-& \frac{1}{3}x_1 &+& \frac{17}{3}x_4 &-& \frac{1}{3}x_5 &+& \frac{2}{3}a_1 && \\
x_3 &=& \frac{11}{9} &-& \frac{1}{9}x_1 &-& \frac{16}{9}x_4 &-& \frac{4}{9}x_5 &-& \frac{4}{9}a_1 &-& \frac{1}{3}a_3
\end{array}$$

- By Row 0, x_1 and x_5 have the same negative reduced cost. We arbitrarily pick x_1 as the entering variable.
- By Row 1 and Row 2, each of x_2 and a_2 achieves the same minimum ratio. We arbitrarily pick x_2 as the leaving variable.

Phase 1 Dictionary 4

$$\begin{array}{rcrcrcrcrcrcr}
z &=& 0 &+& x_2 &+& 7x_4 &&&+& 2a_1 &+& a_3 \\
x_1 &=& 2 &-& 3x_2 &-& 4x_4 &-& x_5 &-& a_1 && \\
a_2 &=& 0 &+& x_2 &+& 7x_4 &&&+& a_1 && \\
x_3 &=& 1 &+& \frac{1}{3}x_2 &-& \frac{4}{3}x_4 &-& \frac{1}{3}x_5 &-& \frac{1}{3}a_1 &-& \frac{1}{3}a_3
\end{array}$$

- By Row 0, since all reduced costs are nonnegative, this solution is optimal.
- Since the optimal value of the Phase 1 Problem is $z^* = 0$, the original problem has a nonempty feasible region.
- However, we still have an artificial variable a_2 as a basic variable, with a value of zero.
- Note that this vertex is degenerate (see Case 2 of Proposition 21.1).
- We will drive the artificial variable a_2 out of the set of basic variables by interchanging it with one of the non-artificial nonbasic variables.

- By Row 2, each of the non-artificial nonbasic variables x_2 and x_4 has a nonzero coefficient.
- Let us arbitrarily pick x_2 and move it to the left-hand side of Row 2, while moving a_2 to the right-hand side.
- We will then substitute this expression for x_2 into Rows 1 and 3.

Phase 1 Dictionary 5

$$
\begin{array}{rcrcrcrcrcrcr}
z & = & 0 & & & & & + & a_1 & + & a_2 & + & a_3 \\
x_1 & = & 2 & + & 17x_4 & - & x_5 & + & 2a_1 & - & 3a_2 & & \\
x_2 & = & 0 & - & 7x_4 & & & - & a_1 & + & a_2 & & \\
x_3 & = & 1 & - & \frac{11}{3}x_4 & - & \frac{1}{3}x_5 & - & \frac{2}{3}a_1 & + & \frac{1}{3}a_2 & - & \frac{1}{3}a_3
\end{array}
$$

- Note that this dictionary is still optimal and there are no artificial basic variables.
- Since the optimal solution computed in the previous iteration is degenerate, we simply changed the labels of basic and nonbasic variables without actually changing the vertex.
- We will delete Row 0 and all artificial variables a_1, a_2, and a_3 as they are no longer needed.
- We are now ready to move to Phase 2. We will use the original objective function $z = 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5$.
- Since x_1, x_2, and x_3 are basic, we will substitute the right-hand sides of Rows 1, 2, and 3, respectively in Row 0.

Phase 2 Problem

Phase 2 Dictionary 1

$$
\begin{array}{rcrcrcr}
z & = & 7 & + & 3x_4 & - & 5x_5 \\
x_1 & = & 2 & + & 17x_4 & - & x_5 \\
x_2 & = & 0 & - & 7x_4 & & \\
x_3 & = & 1 & - & \frac{11}{3}x_4 & - & \frac{1}{3}x_5
\end{array}
$$

- This is the starting dictionary for Phase 2.
- $B = \{1, 2, 3\}$ and $N = \{4, 5\}$.
- Note that this basic feasible solution is degenerate.
- By Row 0, x_5 is the entering variable.
- By Row 1, x_1 is the leaving variable.

Phase 2 Dictionary 2

$$
\begin{array}{rcrcrcr}
z & = & -3 & + & 5x_1 & - & 82x_4 \\
x_5 & = & 2 & - & x_1 & + & 17x_4 \\
x_2 & = & 0 & & & - & 7x_4 \\
x_3 & = & \frac{1}{3} & + & \frac{1}{3}x_1 & - & \frac{28}{3}x_4
\end{array}
$$

- $B = \{5, 2, 3\}$ and $N = \{1, 4\}$.
- Note that this is also a degenerate basic feasible solution.
- By Row 0, x_4 is the entering variable.
- By Row 2, x_2 is the leaving variable.

Phase 2 Dictionary 3

$$
\begin{array}{rcrcrcr}
z & = & -3 & + & 5x_1 & + & \frac{82}{7}x_2 \\
x_5 & = & 2 & - & x_1 & - & \frac{17}{7}x_2 \\
x_4 & = & 0 & & & - & \frac{1}{7}x_2 \\
x_3 & = & \frac{1}{3} & + & \frac{1}{3}x_1 & + & \frac{4}{3}x_2
\end{array}
$$

- $B = \{5, 4, 3\}$ and $N = \{1, 2\}$.
- Note that this vertex is optimal since there are no negative reduced costs in Row 0.
- Note that this is the same solution as in the previous dictionary and only the index sets B and N have changed.
- Therefore, $x^* = [0, 0, 1/3, 0, 2]^T$ is an optimal solution and the optimal value of the original linear programming problem is $z^* = -3$.
- This is the end of Phase 2.

21.6 Outline of the Two-Phase Method

1. Define an artificial variable for each equality constraint and minimize the sum of the artificial variables in Phase 1.
2. Set all original variables to zero and all artificial variables to their corresponding right-hand side values and use this as an initial vertex in Phase 1.
3. **Make sure to put the initial dictionary in proper form by substituting each artificial variable in Row 0 by using their corresponding right-hand side expressions.**
4. If the optimal value of Phase 1 problem is positive, stop. The original problem is infeasible.

5. If the optimal value of Phase 1 problem is zero and there are no artificial basic variables, delete Row 0 and all artificial variables. Introduce the original objective function as the new Row 0 and **ensure that the initial dictionary in proper form (i.e., only nonbasic variables should appear on the right-hand side of Row 0 in the first dictionary of the Phase 2 problem**. Proceed to Phase 2.

6. If the optimal value of Phase 1 problem is zero and there is at least one artificial variable in the basis, drive all artificial variables out of the basis (i.e., replace each of them by a nonbasic and non-artificial variable) and perform Step 5.

Exercises

Question 21.1. *Suppose that you solve a linear programming problem in standard form with m equality constraints by using the two-phase method. Suppose that the optimal value at the end of Phase 1 is zero and that none of the artificial variables $a_1, \ldots, a_m$ is in the basis. Prove that Row 0 should read $z = 0 + \sum_{i=1}^{m} a_i$ in the final dictionary of Phase 1 (please see Phase 1 Dictionary 5 in the example above).*

22.1 Outline

- The Klee-Minty Example
- Exponential Complexity
- Review Problems

22.2 Overview

- In this lecture, we will discuss the efficiency of the simplex method, i.e., how fast the simplex method "solves" a linear programming problem.
- Note that the simplex method is an algorithm.
- Every iteration consists of simple arithmetic operations.
- Under appropriate assumptions and using Bland's rule in the presence of degeneracy, we know that the simplex method always terminates after a finite number of iterations.
- **Question:** Can we estimate the number of iterations on average? How about the worst case?

22.3 Number of Vertices

- Consider a linear programming problem in standard form, with m equality constraints and n variables:

$$\min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}.$$

- Since the simplex method only visits vertices, and the maximum number of vertices is given by $\binom{n}{m}$, we conclude that the number of iterations is at most $\binom{n}{m}$.
- However, this upper bound is probably loose because not every choice of index sets B and N necessarily yields a basic feasible solution.
- Furthermore, it is not clear if one can construct a polyhedron in standard form that has exactly $\binom{n}{m}$ vertices.
- Even if such a polyhedron exists, it is not clear if one can construct an objective function such that the simplex method visits each and every vertex.

22.4 An Example

Consider the following linear programming problem:

$$\begin{array}{lrcrcrcr}
\text{min} & -100x_1 & - & 10x_2 & - & x_3 & & \\
\text{s.t.} & & & & & & & \\
& x_1 & & & & & \leq & 1 \\
& 20x_1 & + & x_2 & & & \leq & 100 \\
& 200x_1 & + & 20x_2 & + & x_3 & \leq & 10000 \\
& x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

- Define slack variables x_4, x_5, and x_6 for each of the first three inequality constraints, respectively, and convert the problem into an equivalent problem in standard from.
- Note that the two-phase method is not needed since there is an easily identifiable starting vertex.
- Use $\hat{x} = [0, 0, 0, 1, 100, 10000]^T$ as the initial vertex to start the simplex method.

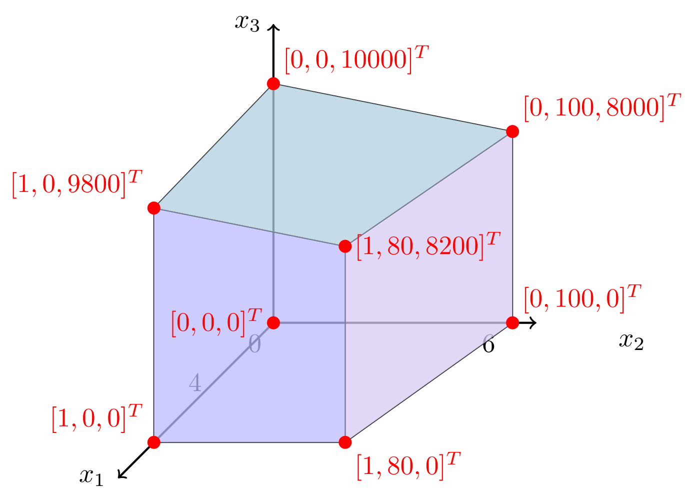


Figure 22.1: The feasible region of the problem in Section 22.4 (Courtesy of Prof John E. Mitchell)

22.4.1 Summary of Iterations

If you choose the nonbasic variable with the most negative reduced cost at each iteration, you will obtain the sequence of vertices outlined in Table 22.1.

The vertices visited by the simplex method on this example is illustrated in Figure 22.2.

Iteration	B	N	Vertex	$\hat{z}$
0	$\{\mathbf{4},5,6\}$	$\{\mathbf{1},2,3\}$	$\hat{x}=[0,0,0,1,100,10000]^T$	0
1	$\{1,\mathbf{5},6\}$	$\{\mathbf{2},3,4\}$	$\hat{x}=[1,0,0,0,80,9800]^T$	-100
2	$\{\mathbf{1},2,6\}$	$\{3,\mathbf{4},5\}$	$\hat{x}=[1,80,0,0,0,8200]^T$	-900
3	$\{4,2,\mathbf{6}\}$	$\{1,\mathbf{3},5\}$	$\hat{x}=[0,100,0,1,0,8000]^T$	-1000
4	$\{\mathbf{4},2,3\}$	$\{\mathbf{1},5,6\}$	$\hat{x}=[0,100,8000,1,0,0]^T$	-9000
5	$\{1,\mathbf{2},3\}$	$\{4,\mathbf{5},6\}$	$\hat{x}=[1,80,8200,0,0,0]^T$	-9100
6	$\{\mathbf{1},5,3\}$	$\{2,\mathbf{4},6\}$	$\hat{x}=[1,0,9800,0,80,0]^T$	-9900
7	$\{4,5,3\}$	$\{1,2,6\}$	$\hat{x}=[0,0,10000,1,100,0]^T$	-10000

Table 22.1: Summary of the simplex iterations on the example in Section 22.4

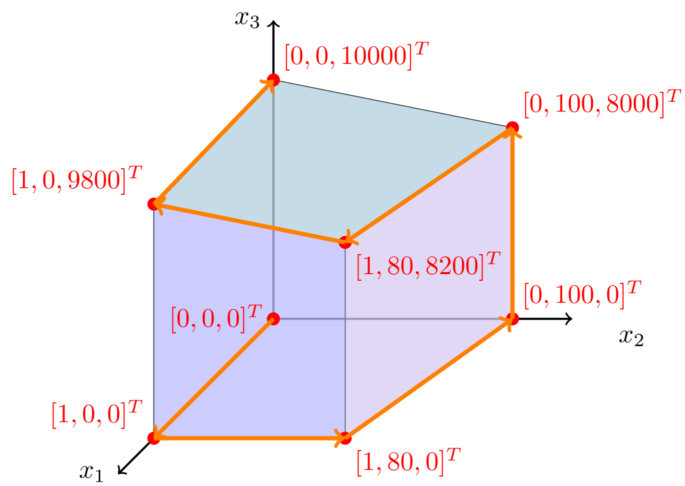


Figure 22.2: The vertices visited by the simplex method on the example in Section 22.4 (Courtesy of Prof John E. Mitchell)

22.4.2 Discussion

- In the original problem, $n = 3$ and the polyhedron has $2^n = 8$ vertices.
- Starting from the first vertex (i.e., the origin), the simplex method visits each vertex before finding the optimal vertex, i.e., it needs $2^n - 1$ iterations.
- Note that each vertex is nondegenerate and the objective function strictly improves at each iteration.
- This example can generalised to any value of n.

$$\begin{array}{llrcll}
\min & -\sum_{j=1}^{n} 10^{n-j} x_j & & & & \\
\text{s.t.} & & & & & \\
& 2\sum_{j=1}^{i-1} 10^{i-j} x_j \quad + & x_i & \le & 100^{i-1}, & i = 1,\dots,n \\
& & x_j & \ge & 0, & j = 1,\dots,n
\end{array}$$

22.5 Main Result

Proposition 22.1 (Klee and Minty, 1972)**.** *For each positive integer n, there is a linear programming problem with n variables such that the simplex method with the most reduced cost performs $2^n - 1$ iterations.*

Remark 22.1. *If $n = 50$ and you can perform 1,000,000 simplex iterations in a second, it would take you more than 35 years to solve the problem!*

22.6 Concluding Remarks

- This result shows that the worst-case performance of the simplex method can be quite poor.
- However, such examples seem to be rare.
- On average, the simplex method performs quite well, with the number of iterations increasing roughly linearly with $m + n$.

Remark 22.2. *It is still an open question whether there is a variant of the simplex method that does not require an exponential number of iterations in the worst case.*

Exercises

Question 22.1. *In the given example with 3 decision variables, the optimal vertex is in fact adjacent to the starting vertex. However, the simplex method with the most negative reduced cost rule fails to detect this and visits every other vertex before finding the optimal vertex.*

1. *What could have caused this behaviour of the simplex method?*
2. *Can you think of a remedy?*

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 23 Introduction to Duality Theory and Weak Duality Theorem

Lecturer: E. Alper Yıldırım *Week: 8*

23.1 Outline

- Relaxations
- Weak Duality Theorem
- Review Problems

23.2 Overview and Motivation

Consider a general optimization problem:

$$(\text{P}) \quad \min\{f(x) : x \in \mathcal{S}\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{S} \subseteq \mathbb{R}^n$.

Question: How can you find a lower bound on the optimal value of (P), denoted by z^*, without solving it?

We can replace (P) by a simpler but related problem:

1. Replace $\mathcal{S}$ by a set $\mathcal{S}_R \subseteq \mathbb{R}^n$ such that $\mathcal{S} \subseteq \mathcal{S}_R$ (e.g., by removing a subset of the constraints in (P))
2. Replace $f : \mathbb{R}^n \to \mathbb{R}$ by $f_R : \mathbb{R}^n \to \mathbb{R}$, where $f_R(x) \leq f(x)$ for each $x \in \mathcal{S}$.
3. Do both of the above.

23.3 Relaxation

Consider the following optimization problems:

$$\begin{array}{ll} (\text{P}) & \min\{f(x) : x \in \mathcal{S}\}, \\ (\text{R}) & \min\{f_R(x) : x \in \mathcal{S}_R\}. \end{array}$$

Definition 23.1. *For an optimization problem (P), the optimization problem (R) is called a* relaxation *of (P) if*

1. $\mathcal{S} \subseteq \mathcal{S}_R$, *and*
2. $f_R(x) \leq f(x)$ *for each* $x \in \mathcal{S}$.

23.3.1 Properties of Relaxations

Lemma 23.2. *Let (R) be a relaxation of (P). Then, $z_R^* \leq z^*$, where z_R^* and z^* denote the optimal values of (R) and (P), respectively.*

Proof. **Case 1:** If (P) is infeasible, then $z^* = +\infty$ by definition. We clearly have $z_R^* \leq z^*$, regardless of the value of z_R^*.

Case 2: Suppose that (P) has a nonempty feasible region. Let $\hat{x} \in \mathcal{S}$ be an arbitrary feasible solution of (P). Since $\mathcal{S} \subseteq \mathcal{S}_R$, we obtain $\hat{x} \in \mathcal{S}_R$. Since $\hat{x} \in \mathcal{S}$, we have $f_R(\hat{x}) \leq f(\hat{x})$. Therefore, for each feasible solution $\hat{x} \in \mathcal{S}$ of (P), we have $\hat{x} \in \mathcal{S}_R$ and $f_R(\hat{x}) \leq f(\hat{x})$. It follows that $z_R^* \leq z^*$. ☐

23.4 Back to Linear Programming

Consider a linear programming problem in standard form:

$$(\mathrm{P}) \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ are given, and $x \in \mathbb{R}^n$ denotes the decision variables.

- Note that $Ax = b$ if and only if $(a^i)^T x = b_i$ if and only if $b_i - (a^i)^T x = 0$ for each $i = 1, \ldots, m$, where $(a^i)^T$ is the ith row of A.

- For a given $x \in \mathbb{R}^n$, $b_i - (a^i)^T x$ measures the violation of the ith equality constraint, which can be negative, zero, or positive.

- We will allow the violation of equality constraints (i.e., remove them from (P)).

- However, we will introduce a "price" $y_i \in \mathbb{R}$ for each unit of violation of the ith equality constraint, where $i = 1, \ldots, m$.

- For a given vector of prices $y \in \mathbb{R}^m$, the total violation cost is given by

$$\sum_{i=1}^m y_i \left(b_i - (a^i)^T x\right) = \underbrace{\sum_{i=1}^m y_i b_i}_{y^T b} - \underbrace{\sum_{i=1}^m y_i \left((a^i)^T x\right)}_{y^T A x} = y^T b - y^T A x = y^T (b - Ax).$$

- We will (i) remove the equality constraints $Ax = b$ from (P) and (ii) add the total violation cost to the objective function:

$$(\mathrm{D}(y)) \quad \min\{c^T x + y^T (b - Ax) : x \geq \mathbf{0}\}, \quad y \in \mathbb{R}^m.$$

- For each fixed $y \in \mathbb{R}^m$, (D(y)) is a linear programming problem since

$$c^T x + y^T (b - Ax) = \underbrace{b^T y}_{constant} + (\underbrace{c - A^T y}_{fixed})^T x.$$

23.4.1 Relations Between Two Problems

Consider a linear programming problem in standard form:

$$\text{(P)} \quad \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\},$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ are given, and $x \in \mathbb{R}^n$ denotes the decision variables.

For a fixed $y \in \mathbb{R}^m$, consider the following linear programming problem:

$$\text{(D}(y)) \quad \min\{c^T x + y^T(b - Ax) : x \geq \mathbf{0}\}$$

Lemma 23.3. *For each $y \in \mathbb{R}^m$, (D(y)) is a relaxation of (P). Therefore, $z^*(y) \leq z^*$, where $z^*(y)$ and z^* denote the optimal values of (D(y)) and (P), respectively, i.e., the optimal value of (D(y)) is a lower bound on the optimal value of (P).*

Proof. Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, \quad x \geq \mathbf{0}\}$ and $\mathcal{P}_R = \{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$ denote the feasible regions of (P) and (D(y)), respectively. We clearly have $\mathcal{P} \subseteq \mathcal{P}_R$.

For a given $y \in \mathbb{R}^m$, let $\hat{x} \in \mathcal{P}$ be an arbitrary feasible solution of (P). Then, $A\hat{x} = b$ and $\hat{x} \geq \mathbf{0}$. The objective function value of (P) evaluated at $\hat{x}$ is given by $c^T\hat{x}$. Since $\mathcal{P} \subseteq \mathcal{P}_R$, we have $\hat{x} \in \mathcal{P}_R$. Consider the objective function value of (D(y)) evaluated at $\hat{x}$. We obtain

$$c^T\hat{x} + y^T \underbrace{(b - A\hat{x})}_{\mathbf{0}} = c^T\hat{x} \leq c^T\hat{x}.$$

Therefore, (D(y)) is a relaxation of (P). The last assertion follows from Lemma 23.2. □

- (D(y)) is called the *Lagrangian relaxation* of (P).
- By Lemma 23.3, for each price vector $y \in \mathbb{R}^m$, we have $z^*(y) \leq z^*$.
- For each $y \in \mathbb{R}^m$, recall that the objective function of (D(y)) is

$$c^T x + y^T(b - Ax) = \underbrace{b^T y}_{constant} + (c - A^T y)^T x.$$

- We therefore obtain

$$z^*(y) = \begin{cases} b^T y & \text{if } c - A^T y \geq \mathbf{0}, \\ -\infty & \text{otherwise.} \end{cases}$$

23.4.2 The Dual Problem

Consider the following linear programming problems:

$$\begin{aligned} &\text{(P)} && \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}, \\ &\text{(D}(y)) && \min\{c^T x + y^T(b - Ax) : x \geq \mathbf{0}\}, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$ are given, and $x \in \mathbb{R}^n$ denotes the decision variables.

- For any price vector $y \in \mathbb{R}^m$:
  (i) If $c - A^T y \geq \mathbf{0}$, then $z^*(y) = b^T y \leq z^*$.
  (ii) If $c - A^T y \not\geq \mathbf{0}$ (i.e., $c - A^T y$ has at least one negative component), then $z^*(y) = -\infty \leq z^*$ (i.e., we obtain a trivial lower bound on z^*).
- It is reasonable to ask for the best (i.e., the largest) lower bound on z^*:

$$\text{(D)} \quad \max\{b^T y : c - A^T y \geq \mathbf{0}\} = \max\{b^T y : A^T y \leq c\}$$

- (D) is a linear programming problem and is called the *dual problem*, whereas (P) is called the *primal problem*.
- $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are referred to as *primal* and *dual variables,* respectively.

23.5 Weak Duality Theorem

Consider the following pair of primal and dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

Proposition 23.1 (Weak Duality Theorem). *Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively. Then,*

(i) $b^T \bar{y} \leq c^T \bar{x}$;

(ii) if $b^T \bar{y} = c^T \bar{x}$, then $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ are optimal solutions of (P) and (D), respectively.

In addition,

(iii) if (P) is unbounded, then (D) is infeasible;

(iv) if (D) is unbounded, then (P) is infeasible;

Proof. (i) Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively. Since $c - A^T \bar{y} \geq \mathbf{0}$, we have $z^*(\bar{y}) = b^T \bar{y}$. It follows from Lemma 23.3 that $z^*(\bar{y}) = b^T \bar{y} \leq z^* \leq c^T \bar{x}$. The claim follows.

(ii) Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively, such that $b^T \bar{y} = c^T \bar{x}$. Then, by part (i), for any feasible solution $\hat{x} \in \mathbb{R}^n$ of (P), we have $b^T \bar{y} = c^T \bar{x} \leq c^T \hat{x}$. Therefore, $z^* = c^T \bar{x}$, where z^* denotes the optimal value of (P). A similar argument shows that $z_D^* = b^T \bar{y}$, where z_D^* denotes the optimal value of (D). It follows that $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ are optimal solutions of (P) and (D), respectively.

(iii) Suppose that (P) is unbounded. Suppose, for a contradiction, that (D) is not infeasible. Then, there exists $\bar{y} \in \mathbb{R}^m$ such that $c - A^T \bar{y} \geq \mathbf{0}$. By part (i), we obtain $z^*(\bar{y}) = b^T \bar{y} \leq z^*$, which contradicts that (P) is unbounded.

(iv) The proof of this assertion is very similar to that of part (iii) and is therefore omitted.

$\square$

23.6 Concluding Remarks and Outlook

- The linear programming problems (P) and (D) are closely related.
- Denoting the optimal values of (P) and (D) by z^* and z_D^*, respectively, we immediately obtain $z_D^* \leq z^*$.
- In the next lecture, we will show that $z_D^* = z^*$, provided that both (P) and (D) have nonempty feasible regions (known as the *strong duality theorem*).

Exercises

Consider the following pair of primal and dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

Question 23.1. *Suppose that we add a new equality constraint $a^T x = \alpha$ to the primal problem (P), where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$. Denote the modified problem by (P′) and its dual by (D′). Compare the optimal value of (D), denoted by z_D^*, and the optimal value of (D′), denoted by $z_{D'}^*$.*

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 24	Primal-Dual Symmetry and Strong Duality Theorem
Lecturer: E. Alper Yıldırım	*Week: 8*

24.1 Outline

- Primal-Dual Symmetry
- Strong Duality Theorem
- Review Problems

24.2 Quick Review

Consider the following pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- Denoting the optimal values of (P) and (D) by z^* and z_D^*, respectively, we have $z_D^* \leq z^*$.
- In this lecture, we will establish further properties between (P) and (D), including symmetry and strong duality.

24.3 Primal-Dual Symmetry

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

Proposition 24.1. *The dual of (D) is equivalent to (P) (i.e., the dual of the dual is the primal). Therefore, the roles of the primal and the dual problem are symmetric.*

Proof. First, we need to transform (D) into standard form. Since $y \in \mathbb{R}^m$ is unrestricted, we replace it by the difference of $y^+ \in \mathbb{R}^m$ and $y^- \in \mathbb{R}^m$, where $y^+ \geq \mathbf{0}$ and $y^- \geq \mathbf{0}$. We also add nonnegative slack variables

$s \in \mathbb{R}^n$ to convert the inequality constraints into equality constraints. Finally, we multiply the objective function by -1 to convert it into minimization. We obtain

$$(\mathrm{D}') \quad \min\{-b^T y^+ + b^T y^- : A^T y^+ - A^T y^- + s = c, \quad y^+ \geq \mathbf{0},\ y^- \geq \mathbf{0},\ s \geq \mathbf{0}\}$$

Using $w \in \mathbb{R}^n$ as the dual variables, the dual of (D') is given by

$$(\mathrm{P}') \quad \max\{c^T w : Aw \leq -b, \quad -Aw \leq b, \quad w \leq \mathbf{0}\}$$

Replace w in (P') by $x = -w$. Note that $Aw = -b$, which implies that $Ax = b$. Since $w \leq \mathbf{0}$, we have $x \geq \mathbf{0}$. Finally, $-c^T w = c^T x$, which implies that maximizing $c^T w$ is equivalent to minimizing $c^T x$. We immediately obtain that (P') is equivalent to (P). □

24.4 Strong Duality Theorem

Consider the following primal-dual pair of linear programming problems:

$$\begin{array}{ll} (\mathrm{P}) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (\mathrm{D}) & \max\{b^T y : A^T y \leq c\} \end{array}$$

Proposition 24.2 (Strong Duality Theorem)**.** *Suppose that both the primal problem (P) and the dual problem (D) have nonempty feasible regions and that A has full row rank. Then, each of the two problems has a finite optimal value and $z_D^* = z^*$.*

Proof. Since (P) has a nonempty feasible region, it either has a finite optimal value or is unbounded. By weak duality, if (P) is unbounded, then (D) is infeasible, which contradicts the hypothesis. Therefore, (P) has a finite optimal value denoted by z^*. A similar argument shows that (D) has a finite optimal value denoted by z_D^*.

Since (P) is in standard form and (P) has a nonempty feasible region, (P) contains at least one basic feasible solution. Therefore, there is at least one optimal vertex $x^* \in \mathbb{R}^n$ of (P) computed by the simplex method. Let B and N denote the corresponding indices, where $|B| = m$, $|N| = n - m$, and $B \cap N = \emptyset$.

We have $x_B^* = (A_B)^{-1} b \geq \mathbf{0}$ and $x_N^* = \mathbf{0}$. Since x^* is an optimal vertex, the reduced costs of all nonbasic variables should be nonnegative, i.e.,

$$\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j \geq 0, \quad j \in N,$$

where $A^j \in \mathbb{R}^m$ denotes the $j$the column of A.

Consider the constraints of the dual problem given by $A^T y \leq c$, which can be rewritten as $(A^j)^T y \leq c_j$, or equivalently, as $c_j - (A^j)^T y \geq 0, j = 1, \ldots, n$.

Let us define $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$. Recall that

$$\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j = c_j - (A^j)^T \underbrace{\left((A_B)^{-1}\right)^T c_B}_{y^*} \geq 0, \quad j \in N,$$

which implies that $c_j - (A^j)^T y^* \geq 0$ for each $j \in N$. Recall that the reduced costs of basic variables are equal to zero (see Remark 4 in Section 18.4 in the lecture notes). Therefore, we have $c_j - (A^j)^T y^* = 0$ for each $j \in B$. It follows that y^* satisfies $A^T y^* \leq c$, i.e., y^* is a feasible solution of (D).

Finally, note that

$$b^T y^* = b^T \left((A_B)^{-1}\right)^T c_B = c_B^T \underbrace{(A_B)^{-1} b}_{x_B^*} = c_B^T x_B^* = c_B^T x_B^* + c_N^T \underbrace{x_N^*}_{\mathbf{0}} = c^T x^*.$$

Therefore, $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$ are feasible for (P) and (D), respectively, and $b^T y^* = c^T x^*$. By part (ii) of the Weak Duality Theorem, x^* and y^* are optimal solutions of (P) and (D), respectively. It follows that $z^* = c^T x^* = b^T y^* = z_D^*$, where z^* and z_D^* denote the optimal values of (P) and (D), respectively. The claim follows. □

24.4.1 Discussion

- By the proof of Proposition 24.2, the simplex method applied to solve the primal problem (P) computes an optimal solution of the dual problem (D) as a byproduct if (P) has a finite optimal value.
- If x^* is an optimal vertex of the primal problem (P) with index sets B and N, then $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is an optimal solution of the dual problem (D).
- Note that $b^T y^* = b^T \left((A_B)^{-1}\right)^T c_B = c_B^T (A_B)^{-1} b = c_B^T x_B^* = c^T x^* = z^*$ since $x_N^* = \mathbf{0}$.

24.5 Primal-Dual Relations

Consider the following primal-dual pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- By the Strong Duality Theorem, if (P) has a finite optimal value, then so does (D) and their optimal values are the same.
- By primal-dual symmetry, if (D) has a finite optimal value, then so does (P) and their optimal values are the same.
- By the Weak Duality Theorem, if (P) is unbounded, then (D) is infeasible.
- By primal-dual symmetry, if (D) is unbounded, then (P) is infeasible.
- **Question:** If (P) (or (D)) is infeasible, what can we infer about the status of (D) (or (P))?

24.5.1 An Example

Example 24.1. *Consider the following linear programming problem in standard form:*

$$\begin{array}{llrcrcrcrcr} (P) & \min & x_1 & - & 2x_2 & & & & & & \\ & s.t. & & & & & & & & & \\ & & x_1 & - & x_2 & - & x_3 & & & = & 2 \\ & & -x_1 & + & x_2 & & & - & x_4 & = & -1 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0. \end{array}$$

If you add the two equality constraints in (P), you get the following implied equality:

$$-x_3 - x_4 = 1 \Leftrightarrow x_3 + x_4 = -1.$$

Since $x \geq \mathbf{0}$, we immediately conclude that (P) is infeasible.

Consider now the dual problem:

$$\begin{array}{llrcrcr}
(D) & \max & 2y_1 & - & y_2 & & \\
& s.t. & & & & & \\
& & y_1 & - & y_2 & \leq & 1 \\
& & -y_1 & + & y_2 & \leq & -2 \\
& & -y_1 & & & \leq & 0 \\
& & & - & y_2 & \leq & 0 \\
& & y_1 & , & y_2 & \in & \mathbb{R}.
\end{array}$$

If you add the first two inequalities in (D), you obtain the following implied inequality:

$$0 \leq -1.$$

We immediately conclude that (D) is also infeasible.

As illustrated by Example 24.1, we may have examples in which both (P) and (D) are simultaneously infeasible.

24.5.2 Primal-Dual Relations

By using Weak Duality Theorem, Strong Duality Theorem, and Example 24.1, we arrive at the set of all possible relations between the primal problem (P) and the dual problem (D) outlined in Table 24.1.

		Dual Problem		
		Finite optimal value	Unbounded	Infeasible
Primal Problem	Finite optimal value	✓	✗	✗
	Unbounded	✗	✗	✓
	Infeasible	✗	✓	✓

Table 24.1: Possible primal-dual relations

24.6 Concluding Remarks

Consider the following primal-dual pair of linear programming problems:

$$\begin{array}{ll}
\text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\
\text{(D)} & \max\{b^T y : A^T y \leq c\}
\end{array}$$

- We established further properties between the primal problem (P) and the dual problem (D).
- We will continue to develop even further properties in the next lecture.

Exercises

Question 24.1. *Consider the following linear programming problems:*

$$\begin{array}{ll} (P1) & \min\{c^T x : Ax \leq b, \quad x \geq \mathbf{0}\} \\ (P2) & \min\{c^T x : Ax \geq b, \quad x \geq \mathbf{0}\} \end{array}$$

Convert each problem (P1) and (P2) into standard form. Take the dual of each one and simplify the dual problems.

Question 24.2. *Consider the linear programming problem:*

$$\begin{array}{llccccccccccc} (P) & \min & x_1 & + & x_2 & - & x_3 & & & & \\ & s.t. & & & & & & & & & \\ & & x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 2 \\ & & x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0. \end{array}$$

Show that (P) has a finite optimal value and find that optimal value without using the simplex method.

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 25 Complementary Slackness Property

Lecturer: E. Alper Yıldırım *Week: 9*

25.1 Outline

- Complementary Slackness Property
- Review Problems

25.2 Quick Review

Consider the following pair of linear programming problems:

$$\begin{array}{ll} (P) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- Denoting the optimal values of (P) and (D) by z^* and z_D^*, respectively, we have $z_D^* \leq z^*$.
- If both (P) and (D) have nonempty feasible regions, then $z_D^* = z^*$.
- In this lecture, we will establish the so-called complementary slackness property.

25.3 Complementary Slackness Property

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} (P) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^T y : A^T y \leq c\} \end{array}$$

- Recall that the matrix $A \in \mathbb{R}^{m \times n}$ can be expressed using its rows or columns:

$$A = \begin{bmatrix} (a^1)^T \\ \vdots \\ (a^m)^T \end{bmatrix}, \quad A = \begin{bmatrix} A^1 & \cdots & A^n \end{bmatrix},$$

where $a^i \in \mathbb{R}^n$ for each $i = 1, \ldots, m$, and $A^j \in \mathbb{R}^m$ for each $j = 1, \ldots, n$.

- A vector $x \in \mathbb{R}^n$ is feasible for (P) if and only if $b_i - (a^i)^T x = 0$ for each $i = 1, \ldots, m$, and $x_j \geq 0$ for each $j = 1, \ldots, n$.
- A vector $y \in \mathbb{R}^m$ is feasible for (D) if and only if $c_j - (A^j)^T y \geq 0$ for each $j = 1, \ldots, n$.

Proposition 25.1 (Complementary Slackness). *Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively. Then, $\bar{x}$ and $\bar{y}$ are optimal solutions of (P) and (D), respectively, if and only if*

$$\begin{aligned} \bar{x}_j \left(c_j - (A^j)^T \bar{y}\right) &= 0, \quad j = 1, \ldots, n, \\ \bar{y}_i \left(b_i - (a^i)^T \bar{x}\right) &= 0, \quad i = 1, \ldots, m. \end{aligned}$$

Proof. $\Rightarrow$: Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be optimal solutions of (P) and (D), respectively. By the Strong Duality Theorem, we have $c^T \bar{x} = b^T \bar{y}$. Since $A\bar{x} = b$, we obtain

$$c^T \bar{x} = (A\bar{x})^T \bar{y} = \bar{x}^T A^T \bar{y},$$

which implies that $\bar{x}^T (c - A^T \bar{y}) = 0$, or equivalently,

$$\sum_{j=1}^{n} \bar{x}_j \left(c_j - (A^j)^T \bar{y}\right) = 0.$$

Since $\bar{x}_j \geq 0$ and $c_j - (A^j)^T \bar{y} \geq 0$ for each $j = 1, \ldots, n$, it follows that

$$\bar{x}_j \left(c_j - (A^j)^T \bar{y}\right) = 0, \quad j = 1, \ldots, n.$$

Since $A\bar{x} = b$, we obtain $(a^i)^T \bar{x} = b_i$, or equivalently, $b_i - (a^i)^T \bar{x} = 0$ for each $i = 1, \ldots, m$. Therefore,

$$\bar{y}_i \left(b_i - (a^i)^T \bar{x}\right) = 0, \quad i = 1, \ldots, m.$$

$\Leftarrow$: Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively. Suppose that

$$\bar{x}_j \left(c_j - (A^j)^T \bar{y}\right) = 0, \quad j = 1, \ldots, n.$$

Then, adding up each of these expressions, we obtain

$$\sum_{j=1}^{n} \bar{x}_j \left(c_j - (A^j)^T \bar{y}\right) = 0.$$

Therefore, $c^T \bar{x} - \bar{x}^T A^T \bar{y} = 0$, i.e., $c^T \bar{x} = \bar{y}^T A \bar{x}$. By using $A\bar{x} = b$, we obtain $c^T \bar{x} = b^T \bar{y}$. By part (ii) of the Weak Duality Theorem, $\bar{x}$ and $\bar{y}$ are optimal solutions of (P) and (D), respectively. □

25.4 Discussion and Implications

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively. Then, by the Complementary Slackness Property, $\bar{x}$ and $\bar{y}$ are optimal solutions of (P) and (D), respectively, if and only if

$$\begin{aligned} \bar{x}_j \left(c_j - (A^j)^T \bar{y}\right) &= 0, \quad j = 1, \ldots, n, \\ \bar{y}_i \left(b_i - (a^i)^T \bar{x}\right) &= 0, \quad i = 1, \ldots, m. \end{aligned}$$

- Complementary slackness property establishes useful relations between the value of the jth primal variable and the "slackness" of the jth dual constraint, as well as the value of the ith dual variable and the "slackness" of ith primal constraint.
- Note that the second set of conditions is satisfied by any feasible solution $\bar{x} \in \mathbb{R}^n$ of (P).
- Recall that we had motivated the vector $y \in \mathbb{R}^m$ as unit prices for the violation of primal equality constraints.
- By the complementary slackness property, if one uses the "correct" prices $\bar{y} \in \mathbb{R}^m$ for constraint violations, then the total violation cost is equal to zero at any primal optimal solution $\bar{x}$!
- Suppose that $\bar{x} \in \mathbb{R}^n$ is an optimal solution of (P).
- Let $\bar{B} = \{j \in \{1, \ldots, n\} : \bar{x}_j > 0\}$ and $\bar{N} = \{j \in \{1, \ldots, n\} : \bar{x}_j = 0\}$.
- For each $j \in \bar{B}$, any dual optimal solution $\bar{y} \in \mathbb{R}^m$ should satisfy $c_j - (A^j)^T \bar{y} = 0$.
- If $\bar{x}$ is a nondegenerate optimal vertex, then $|\bar{B}| = m$ and $A_{\bar{B}} \in \mathbb{R}^{m \times m}$ is invertible.
- Then, $c_j - (A^j)^T \bar{y} = 0$ for each $j \in \bar{B}$ if and only if $c_{\bar{B}} - (A_{\bar{B}})^T \bar{y} = \mathbf{0}$ if and only if $\bar{y} = \left((A_{\bar{B}})^T\right)^{-1} c_{\bar{B}} = \left((A_{\bar{B}})^{-1}\right)^T c_{\bar{B}}$ (see the proof of the strong duality theorem).
- If $\bar{y} \in \mathbb{R}^m$ is an optimal solution of (D), then, for each $j = 1, \ldots, n$ such that $c_j - (A^j)^T \bar{y} > 0$, any primal optimal solution should satisfy $\bar{x}_j = 0$.

25.5 Concluding Remarks

- Complementary slackness property allows us to construct a primal (dual) optimal solution by starting from a dual (primal) optimal solution.
- In the next lecture, we will discuss the economic interpretation of the dual variables.

Exercises

Question 25.1. *Consider the linear programming problem:*

$$\begin{array}{llcccccccc} (P) & \min & x_1 & + & x_2 & - & x_3 & & & \\ & s.t. & & & & & & & & \\ & & x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 2 \\ & & x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0. \end{array}$$

Find an optimal solution of (P) without using the simplex method by relying on the information that $\bar{y} = [1/3, -1/3]^T$ *is an optimal solution of the dual problem (D).*

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 26 Economic Interpretation of the Dual Variables

Lecturer: E. Alper Yıldırım *Week: 9*

26.1 Outline

- Optimal Dual Variables as Shadow Prices
- Review Problems

26.2 Quick Review and Setup

Consider the following pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- So far, we established the following properties between (P) and (D):
  (i) Weak duality
  (ii) Strong duality
  (iii) Primal-dual symmetry
  (iv) Complementary slackness property
- In this lecture, we will give an economic interpretation of the dual optimal solution.
- We assume that both (P) and (D) have nonempty feasible regions and that $A \in \mathbb{R}^{m \times n}$ has full row rank.
- We assume that $x^* \in \mathbb{R}^n$ is a *nondegenerate* optimal vertex of (P) with corresponding index sets B and N.

26.3 Dual Variables and Optimal Value

Recall the following primal-dual pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- **Question:** Suppose that we wish to replace the right-hand side parameter b_i by $b_i + \delta$ in (P), where $i = 1, \ldots, m$ and $\delta \in \mathbb{R}$ is sufficiently small. How does the optimal value of (P) change?

- Since $x^* \in \mathbb{R}^n$ is a *nondegenerate* optimal vertex of (P) with corresponding index sets B and N, then $x_B^* = (A_B)^{-1}b > \mathbf{0}$, $x_N^* = \mathbf{0}$, and $\bar{c}_j = c_j - c_B^T(A_B)^{-1}A^j \geq 0$, where $j \in N$ and $A^j \in \mathbb{R}^m$ denotes the jth column of A.

- Note that the reduced costs do not depend on the right-hand side vector b.

- The values of basic variables $x_B^* = (A_B)^{-1}b$ do depend on b.

- Since b is replaced by $b + \delta e^i$, the new values of basic variables are given by $x_B^*(\delta) = (A_B)^{-1}(b + \delta e^i) = (A_B)^{-1}b + \delta(A_B)^{-1}e^i \geq \mathbf{0}$ for sufficiently small $\delta \in \mathbb{R}$ since $x_B^* = x_B^*(0) = (A_B)^{-1}b > \mathbf{0}$. Note that $x_N^*(\delta) = x_N^* = \mathbf{0}$.

- Therefore, the new vertex corresponding to the index sets B and N remains feasible and optimal for the modified problem if $\delta \in \mathbb{R}$ is sufficiently small.

- Note that the feasible region of (D) does not depend on the right-hand side vector b.

- Recall that $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is an optimal solution of (D) and y^* does not depend on b.

- Since reduced costs do not depend on b and remain nonnegative for the modified primal problem, it follows that y^* remains an optimal solution of the dual of the modified problem.

- Denoting the optimal value of the modified problem by $z^*(\delta)$, we obtain

$$z^*(\delta) = c_B^T(A_B)^{-1}(b + \delta e^i) = (y^*)^T(b + \delta e^i) = z^* + \delta y_i^*,$$

  where $z^* = z^*(0)$ denotes the optimal value of the original problem (P) and (D).

- Therefore, if the right-hand side vector b is replaced by $b + \delta e^i$ in (P), where $i = 1, \ldots, m$ and $\delta \in \mathbb{R}$ is sufficiently small, then $z^*(\delta) = z^* + \delta y_i^*$.

- y_i^* gives the rate of change of the optimal value with respect to changes in the right-hand side parameter b_i.

- We therefore refer to y_i^* as the *shadow price* associated with the $i$the primal constraint (or *marginal cost per unit increase in the right-hand side parameter* b_i).

- The dual optimal solution $y^* \in \mathbb{R}^m$ provides information about how the optimal value will change as the right-hand side vector $b \in \mathbb{R}^m$ changes slightly.

- More generally, if b is replaced by $b + \delta b'$, where $\delta \in \mathbb{R}$ is sufficiently small and $b' \in \mathbb{R}^m$, then $z^*(\delta) = z^* + \delta(y^*)^T b'$.

26.4 Another Interpretation of Dual Variables

In this section, we give another useful interpretation of dual variables using a real-life example.

26.4.1 Manufacturer's Problem: Product Mix

- A manufacturing facility produces n different products using m different resources.
- Each unit of product j requires a_{ij} units of resource i, $i = 1, \ldots, m$; $j = 1, \ldots, n$.
- Each unit of product j produced yields c_j units of profit, $j = 1, \ldots, n$, and b_i units of resource i are available, $i = 1, \ldots, m$.
- We wish to determine the best mix of products so as to maximize the total profit.
- Assume that each product can be produced in any fractional amount and that demand for each product is unlimited.

Decision Variables:

x_j: number of units of product j to be produced, $j = 1, \ldots, n$.

Optimization Model:

$$
\begin{array}{lllll}
\text{(MP)} & \max & \sum_{j=1}^{n} c_j x_j & & \\
& \text{s.t.} & & & \\
& & \sum_{j=1}^{n} a_{ij} x_j & \leq b_i, & i = 1, \ldots, m, \\
& & x_j & \geq 0, & j = 1, \ldots, n.
\end{array}
$$

26.4.2 Entrepreneur's Problem: Resource Pricing

- Suppose that an entrepreneur is interested in purchasing all available resources from the facility.
- **Question:** What are the "fair" unit prices for each of the m resources?
- By fair unit prices, we mean
  - (i) the entrepreneur should minimize her total purchasing cost;
  - (ii) the manufacturer should have an incentive to sell all of its resources at these prices.
- Let y_i denote the unit price of resource i, $i = 1, \ldots, m$.
- The total purchasing cost of the entrepreneur is given by $\sum_{i=1}^{m} b_i y_i$.
- By selling its resources, the manufacturer would give up c_j units of profit for each unit of product j not produced, $j = 1, \ldots, n$.
- Each unit of product j not produced would release a_{ij} units of resource i, $i = 1, \ldots, m$.
- Therefore, the unit prices y_i, $i = 1, \ldots, m$ should be such that

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, \ldots, n.$$

Decision Variables:

y_i: unit price of resource $i, \ i = 1, \ldots, m$.

Optimization Model:

$$
\begin{array}{llrcll}
\text{(EP)} & \min & \sum_{i=1}^{m} b_i y_i & & & \\
 & \text{s.t.} & & & & \\
 & & \sum_{i=1}^{m} a_{ij} y_i & \geq & c_j, & j = 1, \ldots, n, \\
 & & y_i & \geq & 0, & i = 1, \ldots, m.
\end{array}
$$

26.4.3 Discussion

- You can easily verify that the entrepreneur's problem (EP) is the dual of the manufacturer's problem (MP).
- By weak duality, the total purchase cost should be at least as large as the maximum total profit.
- By strong duality, the minimum total purchase cost equals the maximum total profit.
- Therefore, for any optimal solution $y^* \in \mathbb{R}^m$ of (EP), y_i^* is the fair unit price of resource $i, \ i = 1, \ldots, m$.
- y_i^* is the minimum unit price of resource $i, \ i = 1, \ldots, m$ the manufacturer should be willing to sell.
- Conversely, the manufacturer should be willing to pay at most y_i^* per each additional unit of resource $i, \ i = 1, \ldots, m$.
- Let $x^* \in \mathbb{R}^n$ denote an optimal solution of (MP). Recall that x_j^* denotes the optimal production quantity of product $j, \ j = 1, \ldots, n$.
- The manufacturer is indifferent between producing the optimal quantities $x^* \in \mathbb{R}^n$ or selling all of its m resources at fair prices $y^* \in \mathbb{R}^m$.
- If $x_j^* > 0$, then $c_j = \sum_{i=1}^{m} a_{ij} y_i^*$, since if $c_j < \sum_{i=1}^{m} a_{ij} y_i^*$, the manufacturer would prefer selling the resources to producing product $j, \ j = 1, \ldots, n$.
- Similarly, if $c_j < \sum_{i=1}^{m} a_{ij} y_i^*$, then $x_j^* = 0, \ j = 1, \ldots, n$.
- Therefore, $x_j^* \left(c_j - \sum_{i=1}^{m} a_{ij} y_i^* \right) = 0, \ j = 1, \ldots, n$.
- If $b_i > \sum_{j=1}^{n} a_{ij} x_j^*$, then we have $y_i^* = 0$ (i.e., there is no incentive for the manufacturer to buy additional units of resource $i, \ i = 1, \ldots, m$).
- If $y_i^* > 0$, then we have $b_i = \sum_{j=1}^{n} a_{ij} x_j^*, \ i = 1, \ldots, m$.
- Therefore, $y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j^* \right) = 0, \ i = 1, \ldots, m$.
- Observe that these are precisely the complementary slackness conditions.

Exercises

Question 26.1. *Consider the following pair of primal-dual linear programming problems:*

$$\begin{array}{ll} (P) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^T y : A^T y \leq c\} \end{array}$$

Suppose that $x^* \in \mathbb{R}^n$ is a degenerate *optimal vertex of (P) with corresponding index sets B and N. Suppose that we wish to replace the right-hand side parameter b_i by $b_i + \delta$ in (P), where $i = 1, \ldots, m$ and $\delta \in \mathbb{R}$ is sufficiently small. Does the same analysis on the change of the optimal value as in the case of a nondegenerate optimal vertex remain valid? Why or why not?*

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 27	A Dual Perspective on the Simplex Method
Lecturer: E. Alper Yıldırım	*Week: 9*

27.1 Outline

- Simplex Method and Candidate Dual Solutions
- Review Problems

27.2 Quick Review and Setup

Consider the following pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- Recall that (P) can be solved using the simplex method.
- In this lecture, we will revisit the simplex method from the perspective of the dual problem (D).

27.2.1 Review of the Simplex Method

- Consider solving (P) using the simplex method in dictionary form. We assume that $A \in \mathbb{R}^{m \times n}$ has full row rank.
- Suppose that $\hat{x} \in \mathbb{R}^n$ is a vertex of the primal problem (P) with index sets B and N.
- We have $\hat{x}_N = \mathbf{0}$ and $\hat{x}_B = (A_B)^{-1} b \geq \mathbf{0}$. Therefore, $\hat{z} = c^T \hat{x} = c_B^T \hat{x}_B + c_N^T \hat{x}_N = c_B^T (A_B)^{-1} b$.
- The corresponding dictionary is given by

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

27.3 A Dual Perspective on the Simplex Method

In this section, we will develop a dual perspective on the simplex method. We will analyse an optimal vertex as well as an intermediate vertex encountered during the solution process of the primal problem (P).

27.3.1 Optimal Vertex

Recall the simplex method in dictionary form:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- If $\bar{c}_j \geq 0$ for each $j \in N$, then $\hat{x}$ is an optimal solution of (P) and we stop.
- Let $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$. The dual constraints are given by $(A^j)^T y \leq c_j,\ j = 1, \ldots, n$, or $c_j - (A^j)^T y \geq 0,\ j = 1, \ldots, n$.
- At $\hat{y}$, we have
  (i) $c_j - (A^j)^T\hat{y} = c_j - (A^j)^T\left((A_B)^{-1}\right)^T c_B = c_j - c_B^T(A_B)^{-1}A^j = \bar{c}_j \geq 0,\ j \in N$.
  (ii) $c_j - (A^j)^T\hat{y} = c_j - (A^j)^T\left((A_B)^{-1}\right)^T c_B = c_j - c_B^T(A_B)^{-1}A^j = \bar{c}_j = 0,\ j \in B$.
- Therefore, if $\hat{x}$ is an optimal solution of (P), then $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is a feasible solution of (D).
- Since $A\hat{x} = b$, we have $(a^i)^T\hat{x} = b_i$ or $b_i - (a^i)^T\hat{x} = 0$ for each $i = 1, \ldots, m$. Therefore, $\hat{y}_i\left(b_i - (a^i)^T\hat{x}\right) = 0$ for each $i = 1, \ldots, m$.
- Similarly, we have $\hat{x}_j\left(c_j - (A^j)^T\hat{y}\right) = 0$ for each $j = 1, \ldots, n$ since (i) $\hat{x}_j \geq 0$ and $\bar{c}_j = c_j - (A^j)^T\hat{y} = 0$ for each $j \in B$; and (ii) $\hat{x}_j = 0$ and $\bar{c}_j = c_j - (A^j)^T\hat{y} \geq 0$ for each $j \in N$.
- Therefore, $\hat{x}$ is feasible for (P), $\hat{y}$ is feasible for (D), and they satisfy the complementary slackness conditions.
- By the Complementary Slackness Conditions, we conclude that $\hat{x}$ is an optimal solution of (P), and $\hat{y}$ is an optimal solution of (D).

27.3.2 Intermediate Vertex

Again, recall the simplex method in dictionary form:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- Suppose now that $\hat{x}$ is an arbitrary intermediate vertex of (P) visited at one of the iterations of the simplex method, with index sets B and N.
- There is at least one index $j^* \in N$ such that $\bar{c}_{j^*} = c_{j^*} - c_B^T (A_B)^{-1} A^{j^*} < 0$.
- We can still define $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$.
- Note that $\hat{y}$ is not a feasible solution of (D) since $\bar{c}_{j^*} = c_{j^*} - c_B^T (A_B)^{-1} A^{j^*} = c_{j^*} - (A^{j^*})^T \hat{y} < 0$.
- We have $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ and $\hat{y}$ violates at least one of the constraints of (D).
- For each $j \in B$, we have $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j = c_j - (A^j)^T \hat{y} = 0$.
- Therefore, for each $j \in B$, the dual constraint $c_j - (A^j)^T y \geq 0$ is active at $\hat{y}$.
- Since $A_B \in \mathbb{R}^{m \times m}$ is invertible, we obtain $\text{span}\{A^j : j \in B\} = \mathbb{R}^m$.
- Since (D) has no equality constraints, it follows that $\hat{y}$ is a basic but infeasible solution of (D).
- We have $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ and $\hat{y}$ is a basic but infeasible solution of (D).
- For each $j \in B$, we have $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j = c_j - (A^j)^T \hat{y} = 0$. Therefore, we obtain $\hat{x}_j \left(c_j - (A^j)^T \hat{y}\right) = 0$ for each $j \in B$.
- Since $\hat{x}_j = 0$ for each $j \in N$, we obtain $\hat{x}_j \left(c_j - (A^j)^T \hat{y}\right) = 0$ for each $j \in N$.
- For each $i = 1, \ldots, m$, since $b_i - (a^i)^T \hat{x} = 0$, we obtain $\hat{y}_i \left(b_i - (a^i)^T \hat{x}\right) = 0$.
- Therefore, $\hat{x} \in \mathbb{R}^n$ and $\hat{y} \in \mathbb{R}^m$ satisfy the complementarity conditions, but not all complementary slackness conditions since $\hat{y}$ is not a feasible solution of (D).

27.4 Discussion and Concluding Remarks

- At each dictionary, the simplex method computes a basic feasible solution $\hat{x} \in \mathbb{R}^n$ of (P).
- A "candidate" dual solution is implicitly constructed by defining $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$.
- $\hat{y}$ is always a basic solution of (D). Furthermore, $\hat{x} \in \mathbb{R}^n$ and $\hat{y} \in \mathbb{R}^m$ satisfy the complementarity part of the complementary slackness conditions.
- The simplex method terminates at an optimal vertex $\hat{x}$ if and only if $\hat{y}$ is a basic feasible solution of (D).
- Otherwise, $\hat{y}$ is a basic but infeasible solution of (D) and the simplex method does not terminate at this dictionary.
- The simplex method maintains primal feasibility throughout each iteration, moving from one vertex of (P) to the next.
- At each dictionary, a candidate dual basic solution is constructed that satisfies the complementarity part of the complementary slackness conditions together with the current primal vertex.
- Therefore, the simplex method maintains primal feasibility and complementarity part of the complementary slackness conditions at each dictionary, and works towards dual feasibility.
- In the next lecture, we will study an alternative variant that maintains dual feasibility and complementarity part of the complementary slackness conditions at each dictionary, and works towards primal feasibility.

Exercises

Question 27.1. *Consider the following pair of primal-dual linear programming problems:*

$$
\begin{array}{ll}
(P) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\
(D) & \max\{b^T y : A^T y \leq c\}
\end{array}
$$

Assume that $A \in \mathbb{R}^{m \times n}$ has full row rank. Let $\hat{x} \in \mathbb{R}^n$ be a vertex of (P) with corresponding index sets B and N. Let $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ denote the corresponding candidate dual basic solution. Prove the following statement:

> *$\hat{y}$ is a degenerate basic solution of (D) if and only if there exists at least one index $j^* \in N$ such that $\bar{c}_{j^*} = c_{j^*} - c_B^T (A_B)^{-1} A^{j^*} = 0$.*

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 28	The Dual Simplex Method
Lecturer: E. Alper Yıldırım	*Week: 10*

28.1 Outline

- The Dual Simplex Method
- Review Problems

28.2 Quick Review and Setup

Consider the following pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- Recall that (P) can be solved using the simplex method.
- The simplex method for solving (P) maintains primal feasibility and the complementarity part of complementary slackness at each dictionary, and works towards dual feasibility.
- In this lecture, we will study an alternative variant, referred to as *the dual simplex method*, that maintains dual feasibility and the complementarity part of complementary slackness at each dictionary, and works towards primal feasibility.
- We assume that $A \in \mathbb{R}^{m \times n}$ has full row rank.
- Let $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ be two disjoint index sets such that $|B| = m$, $|N| = n - m$, and $A_B \in \mathbb{R}^{m \times m}$ is invertible.
- Let $\hat{x} \in \mathbb{R}^n$ be such that $\hat{x}_N = \mathbf{0}$ and $\hat{x}_B = (A_B)^{-1} b$. Note that $\hat{x}$ is a basic solution of (P).
- Suppose that $\hat{x}$ is infeasible for (P), i.e., there exists at least one $j \in B$ such that $\hat{x}_j < 0$.
- Suppose also that $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j \geq 0$ for each $j \in N$.

28.2.1 Corresponding Dictionary

Under the aforementioned assumptions, consider the corresponding dictionary given by

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- Note that the current dictionary does not correspond to a primal vertex since $\hat{x}_B = (A_B)^{-1}b \ngeq \mathbf{0}$.
- However, all reduced costs are nonnegative since $\bar{c}_j = c_j - c_B^T(A_B)^{-1}A^j \geq 0$ for each $j \in N$.
- Let $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$. Using the dual perspective, this dictionary is primal infeasible (i.e., $\hat{x}$ is a basic but infeasible solution of (P)), dual feasible (i.e., $\hat{y}$ is a basic feasible solution of (D)), and the complementarity part of complementary slackness conditions are satisfied by $\hat{x}$ and $\hat{y}$.
- **Question:** Is there a variant of the simplex method that maintains dual feasibility, the complementarity part of complementary slackness, and works towards primal feasibility?

28.3 Development of the Dual Simplex Method

Recall the the corresponding dictionary given by

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- To achieve primal feasibility, we need to make sure that the values of all basic variables are nonnegative.
- To maintain dual feasibility, we need to make sure that all Row 0 coefficients remain nonnegative at each dictionary.
- Recall that the complementarity part of the complementary slackness comes for free at each dictionary by the definition $\hat{y} = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$.

28.3.1 A Closer Look

Let us analyse the current dictionary in more detail:

$$\begin{aligned} z &= \hat{z} + \sum_{j\in N} \bar{c}_j x_j \\ x_i &= \hat{x}_i + \sum_{j\in N} \bar{a}_{ij} x_j, \quad i \in B. \end{aligned}$$

- Note that $\hat{z}$ denotes the objective function value and $\hat{x}_i,\ i \in B$ denotes the values of the basic variables at the current dictionary.
- We denote by $\bar{a}_{ij}$ the right-hand side coefficient of the nonbasic variable x_j corresponding to the row in which x_i is the basic variable.
- Suppose that $\hat{x}_p < 0$ for some $p \in B$.
- This suggests an incorrect choice of $p \in B$. We therefore would like to replace $p \in B$ with one of the indices $q \in N$.
- We have $x_p = \hat{x}_p + \sum_{j \in N} \bar{a}_{pj} x_j$ and $\hat{x}_p < 0$ for some $p \in B$. We wish to replace $p \in B$ with one of the indices $q \in N$:

$$x_q = -\frac{\hat{x}_p}{\bar{a}_{pq}} + \frac{1}{\bar{a}_{pq}} x_p - \sum_{j \in N \setminus \{q\}} \frac{\bar{a}_{pj}}{\bar{a}_{pq}} x_j$$

- Clearly, we need $\bar{a}_{pq} \neq 0$. Furthermore, if $\bar{a}_{pq} < 0$, then the value of x_q would be negative.
- Therefore, we need to pick $q \in N$ such that $\bar{a}_{pq} > 0$ since we would like to work towards primal feasibility.
- We have $\hat{x}_p < 0$ and need to pick $q \in N$ such that $\bar{a}_{pq} > 0$:

$$x_q = -\frac{\hat{x}_p}{\bar{a}_{pq}} + \frac{1}{\bar{a}_{pq}} x_p - \sum_{j \in N \setminus \{q\}} \frac{\bar{a}_{pj}}{\bar{a}_{pq}} x_j$$

- Substitute this expression for x_q in the right-hand side of the rows corresponding to the other basic variables $i \in B \setminus \{p\}$:

$$\begin{aligned} x_i &= \hat{x}_i + \bar{a}_{iq} x_q + \sum_{j \in N \setminus \{q\}} \bar{a}_{ij} x_j \\ &= \left(\hat{x}_i - \frac{\bar{a}_{iq} \hat{x}_p}{\bar{a}_{pq}} \right) + \frac{\bar{a}_{iq}}{\bar{a}_{pq}} x_p + \sum_{j \in N \setminus \{q\}} \left(\bar{a}_{ij} - \frac{\bar{a}_{iq} \bar{a}_{pj}}{\bar{a}_{pq}} \right) x_j. \end{aligned}$$

- We have $\hat{x}_p < 0$ and need to pick $q \in N$ such that $\bar{a}_{pq} > 0$.
- For each $i \in B \setminus \{p\}$, we obtain:

$$x_i = \left(\hat{x}_i - \frac{\bar{a}_{iq} \hat{x}_p}{\bar{a}_{pq}} \right) + \frac{\bar{a}_{iq}}{\bar{a}_{pq}} x_p + \sum_{j \in N \setminus \{q\}} \left(\bar{a}_{ij} - \frac{\bar{a}_{iq} \bar{a}_{pj}}{\bar{a}_{pq}} \right) x_j.$$

- Note that the value of x_i in the next dictionary may be larger, smaller, or remain the same depending on the sign of $\bar{a}_{iq}$.
- We have $\hat{x}_p < 0$ and need to pick $q \in N$ such that $\bar{a}_{pq} > 0$:

$$x_q = -\frac{\hat{x}_p}{\bar{a}_{pq}} + \frac{1}{\bar{a}_{pq}} x_p - \sum_{j \in N \setminus \{q\}} \frac{\bar{a}_{pj}}{\bar{a}_{pq}} x_j$$

- Now substitute this expression for x_q in the right-hand side of Row 0:

$$\begin{aligned} z &= \hat{z} + \bar{c}_q x_q + \sum_{j \in N \backslash \{q\}} \bar{c}_j x_j \\ &= \left(\hat{z} - \frac{\bar{c}_q \hat{x}_p}{\bar{a}_{pq}} \right) + \frac{\bar{c}_q}{\bar{a}_{pq}} x_p + \sum_{j \in N \backslash \{q\}} \left(\bar{c}_j - \frac{\bar{c}_q \bar{a}_{pj}}{\bar{a}_{pq}} \right) x_j. \end{aligned}$$

- We have $\hat{x}_p < 0$ and need to pick $q \in N$ such that $\bar{a}_{pq} > 0$. In Row 0, we obtain

$$z = \left(\hat{z} - \frac{\bar{c}_q \hat{x}_p}{\bar{a}_{pq}} \right) + \frac{\bar{c}_q}{\bar{a}_{pq}} x_p + \sum_{j \in N \backslash \{q\}} \left(\bar{c}_j - \frac{\bar{c}_q \bar{a}_{pj}}{\bar{a}_{pq}} \right) x_j$$

- In order to maintain dual feasibility, we need to ensure that the new Row 0 coefficients are all nonnegative.

- Since $\bar{c}_q \geq 0$ and $\bar{a}_{pq} > 0$, the Row 0 coefficient of the new nonbasic variable x_p is nonnegative.

- For each $j \in N \backslash \{q\}$, we need to ensure that

$$\bar{c}_j - \frac{\bar{c}_q \bar{a}_{pj}}{\bar{a}_{pq}} \geq 0 \Longleftrightarrow \bar{c}_j \bar{a}_{pq} \geq \bar{c}_q \bar{a}_{pj}, \quad j \in N \backslash \{q\}.$$

- Since $\bar{c}_j \geq 0$ and $\bar{a}_{pq} > 0$, we obtain $\bar{c}_j \bar{a}_{pq} \geq 0$.

- Since $\bar{c}_q \geq 0$, we only need to worry about $j \in N \backslash \{q\}$ such that $\bar{a}_{pj} > 0$.

- We have $\hat{x}_p < 0$ and need to pick $q \in N$ such that $\bar{a}_{pq} > 0$.

- We therefore need $\frac{\bar{c}_j}{\bar{a}_{pj}} \geq \frac{\bar{c}_q}{\bar{a}_{pq}}$ for each $j \in N \backslash \{q\}$ such that $\bar{a}_{pj} > 0$. Since $\bar{a}_{pq} > 0$, we obtain

$$\frac{\bar{c}_q}{\bar{a}_{pq}} = \min_{j \in N: \bar{a}_{pj} > 0} \frac{\bar{c}_j}{\bar{a}_{pj}}.$$

- Therefore, if we pick $q \in N$ accordingly, then we ensure that the next dictionary remains dual feasible.

- We have $\hat{x}_p < 0$ and need to pick $q \in N$ such that $\bar{a}_{pq} > 0$.

- In Row 0, we obtain

$$z = \left(\hat{z} - \frac{\bar{c}_q \hat{x}_p}{\bar{a}_{pq}} \right) + \frac{\bar{c}_q}{\bar{a}_{pq}} x_p + \sum_{j \in N \backslash \{q\}} \left(\bar{c}_j - \frac{\bar{c}_q \bar{a}_{pj}}{\bar{a}_{pq}} \right) x_j$$

- Since $\bar{c}_q \geq 0, \hat{x}_p < 0$ and $\bar{a}_{pq} > 0$, the new objective function value either remains the same or increases!

- This is expected since we would like to move to a better dual basic feasible solution and (D) is a **maximization** problem.

28.4 The Dual Simplex Method

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll}(\mathrm{P}) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\}\\ (\mathrm{D}) & \max\{b^T y : A^T y \leq c\}\end{array}$$

Initialisation

- We assume that $A \in \mathbb{R}^{m \times n}$ has full row rank.
- Let $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$ be two disjoint index sets such that $|B| = m$, $|N| = n - m$, and $A_B \in \mathbb{R}^{m \times m}$ is invertible.
- Let $\hat{x} \in \mathbb{R}^n$ be such that $\hat{x}_N = \mathbf{0}$ and $\hat{x}_B = (A_B)^{-1} b$. Note that $\hat{x}$ is a basic solution of (P).
- Suppose that $\hat{x}$ is infeasible for (P), i.e., there exists at least one $j \in B$ such that $\hat{x}_j < 0$.
- Suppose also that $\bar{c}_j = c_j - c_B^T (A_B)^{-1} A^j \geq 0$ for each $j \in N$.

Algorithm

1. **Leaving Variable:** Choose $p \in B$ such that $\hat{x}_p < 0$.
2. **Entering Variable:** Choose $q \in N$ such that
$$\frac{\bar{c}_q}{\bar{a}_{pq}} = \min_{j \in N : \bar{a}_{pj} > 0} \frac{\bar{c}_j}{\bar{a}_{pj}}.$$
3. Move x_q to the left-hand side and x_p to the right-hand side in the row in which x_p is the basic variable. Substitute this expression for x_q in the other rows including Row 0.
   (a) If the new values of basic variables are all nonnegative, then stop. We have an optimal dictionary.
   (b) Otherwise, $B \leftarrow (B \backslash \{p\}) \cup \{q\}$ and $N \leftarrow (N \backslash \{q\}) \cup \{p\}$. Go to Step 1.

28.5 Discussion and Concluding Remarks

- The dual simplex method can be used to solve (P) and (D) starting from a basic but infeasible solution of (P) with a corresponding feasible candidate dual solution.
- Dual feasibility and the complementarity part of the complementary slackness are maintained at each dictionary, and progress is made towards primal feasibility.
- The objective function value remains the same or increases since we are making progress in the dual problem (D).
- We first determine the leaving variable by looking at the values of basic variables in the current dictionary.

- We then determine the entering variable by performing a minimum ratio test using the ratios of reduced costs and coefficients of nonbasic variables in the right-hand side of the row corresponding to the leaving basic variable.

- The dual simplex method can be very useful for reoptimization of the primal problem (P) after replacing $b \in \mathbb{R}^m$ by $b' \in \mathbb{R}^m$ (see our discussion on sensitivity analysis and reoptimization in the subsequent lectures).

Exercises

Question 28.1. *Solve the following linear programming problem using the dual simplex method:*

$$\begin{array}{lrcrcrcrcr}
\min & x_1 & + & 2x_2 & + & x_3 & & & & \\
s.t. & & & & & & & & & \\
& 3x_1 & - & x_2 & - & x_3 & & & \leq & -3 \\
& x_1 & & & & & - & 4x_4 & \leq & -2 \\
& -3x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & \leq & 6 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 29 Sensitivity Analysis and Reoptimization – Part I

Lecturer: E. Alper Yıldırım *Week: 10*

29.1 Outline

- Sensitivity Analysis and Reoptimization: Changes in b
- Sensitivity Analysis and Reoptimization: Changes in c
- Review Problems

29.2 Motivation and Setup

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- In this lecture, we study how the optimal value and the optimal solutions of (P) and (D) change with respect to the changes in the problem parameters $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.
- This is an important problem due to
  - (i) uncertainty or incompleteness of the parameters
  - (ii) scenario analysis (what-if)
- Suppose that the pair of problems (P) and (D) are solved either using the usual (primal) simplex method or the dual simplex method.
- Suppose that $A \in \mathbb{R}^{m \times n}$ has full row rank and that $x^* \in \mathbb{R}^n$ is an optimal vertex of (P) with corresponding index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$, and $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is the corresponding dual optimal solution.
- There are two main questions of interest:
  - (i) **Sensitivity Analysis:** Under what conditions on the changes in the problem parameters b and c would the dictionary corresponding to the index sets B and N still remain optimal for the modified primal-dual pair of problems?
  - (ii) **Reoptimization:** If the dictionary corresponding to the index sets B and N is no longer optimal for the modified primal-dual pair of problems, how do we reoptimize effectively?

29.2.1 Observations

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- Changes in $b \in \mathbb{R}^m$ affect the feasible region of (P) but not the objective function of (P).
- Changes in $b \in \mathbb{R}^m$ affect the objective function of (D) but not the feasible region of (D).
- Changes in $c \in \mathbb{R}^n$ affect the objective function of (P) but not the feasible region of (P).
- Changes in $c \in \mathbb{R}^n$ affect the feasible region of (D) but not the objective function of (D).

29.2.2 Main Idea

- Suppose that the optimal dictionary is given by

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Suppose that some of the parameters given by $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are modified.
  - (i) Identify the parts of the optimal dictionary that would be affected by such a change.
  - (ii) Identify whether the primal and/or dual feasibility would be affected by such a change.
  - (iii) Derive conditions on the parameters such that primal and dual feasibility are maintained.
  - (iv) If the primal or dual feasibility is violated after the modification, determine which variant of the simplex method (primal or dual) would be appropriate to reoptimize the modified pair of primal and dual problems.

29.3 Changes in b

Recall the optimal dictionary for the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Let us first consider changes in $b \in \mathbb{R}^m$.

- Suppose that b_i is replaced by $b_i + \delta$, where $\delta \in \mathbb{R}$ and $i \in \{1, \ldots, m\}$.
- Therefore, $b \in \mathbb{R}^m$ is replaced by $b + \delta e^i$, where $e^i \in \mathbb{R}^m$ denotes the ith unit vector, $i \in \{1, \ldots, m\}$.
- Changes in $b \in \mathbb{R}^m$ only affect the values of the current basic variables and the objective function value at the current dictionary (i.e., only primal feasibility is affected).
- The current dictionary remains optimal for the modified primal-dual pair of problems if and only if

$$(A_B)^{-1}\left(b + \delta e^i\right) \geq \mathbf{0}.$$

- Note that

$$(A_B)^{-1}\left(b + \delta e^i\right) = x_B^* + \delta (A_B)^{-1} e^i \geq \mathbf{0} \Longleftrightarrow \delta (A_B)^{-1} e^i \geq -x_B^*.$$

- Therefore, we obtain m inequalities on $\delta \in \mathbb{R}$ such that the current dictionary remains optimal if and only if δ satisfies each of these inequalities simultaneously.
- As a function of $\delta \in \mathbb{R}$,
  (i) the values of basic variables are given by $x_B^*(\delta) = (A_B)^{-1}\left(b + \delta e^i\right) = x_B^* + \delta (A_B)^{-1} e^i$;
  (ii) the values of nonbasic variables are given by $x_N^*(\delta) = \mathbf{0}$;
  (iii) the corresponding dual solution is given by $y^*(\delta) = \left((A_B)^{-1}\right)^T c_B$;
  (iv) the objective function value is given by $z^*(\delta) = c_B^T (A_B)^{-1}\left(b + \delta e^i\right) = z^* + \delta c_B^T (A_B)^{-1} e^i$.
- Recall that

$$(A_B)^{-1}\left(b + \delta e^i\right) = x_B^* + \delta (A_B)^{-1} e^i \geq \mathbf{0} \Longleftrightarrow \delta (A_B)^{-1} e^i \geq -x_B^*.$$

- If we choose $\delta^* \in \mathbb{R}$ that does not satisfy each of these m inequalities simultaneously, then we can reoptimize using the dual simplex method since the updated current dictionary is primal infeasible but dual feasible.

29.3.1 More General Changes

- More generally, a similar analysis applies if b is replaced by $b + \delta b'$, where $\delta \in \mathbb{R}$ and $b' \in \mathbb{R}^m$, i.e., the current dictionary remains optimal if and only if

$$(A_B)^{-1}\left(b + \delta b'\right) = x_B^* + \delta (A_B)^{-1} b' \geq \mathbf{0} \Longleftrightarrow \delta (A_B)^{-1} b' \geq -x_B^*.$$

- If we choose $\delta^* \in \mathbb{R}$ that does not satisfy each of these m inequalities simultaneously, then we can reoptimize using the dual simplex method since the updated current dictionary is primal infeasible but dual feasible.

29.4 Changes in c

Recall the optimal dictionary for the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j \in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- We now focus on changes in $c \in \mathbb{R}^n$.
- Suppose that c_j is replaced by $c_j + \delta$, where $\delta \in \mathbb{R}$ and $j \in \{1, \ldots, n\}$.
- Changes in $c \in \mathbb{R}^n$ only affect the right-hand side of Row 0 (i.e., only dual feasibility is affected).
- We will identify two different cases:
  - (i) Changing the objective function coefficient of a nonbasic variable ($j \in N$)
  - (ii) Changing the objective function coefficient of a basic variable ($j \in B$)

29.4.1 Changes in the Objective Function Coefficient of a Nonbasic Variable

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j \in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- Suppose that c_j is replaced by $c_j + \delta$, where $\delta \in \mathbb{R}$ and $j \in N$.
- A change in $c_j,\ j \in N$ only affects the reduced cost $\bar{c}_j$.
- Therefore, the current dictionary remains optimal for the modified primal-dual pair of problems if and only if
$$c_j + \delta - c_B^T(A_B)^{-1}A^j = \bar{c}_j + \delta \geq 0 \iff \delta \geq -\bar{c}_j.$$
- We have $\bar{c}_j(\delta) = \bar{c}_j + \delta$ and everything else remains the same.
- If we choose $\delta^* \in \mathbb{R}$ that does not satisfy this inequality, then we can reoptimize using the primal simplex method since the updated current dictionary is primal feasible but dual infeasible.

29.4.2 Changes in the Objective Function Coefficient of a Basic Variable

$$
\begin{aligned}
z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\
x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j
\end{aligned}
$$

- Suppose now that c_j is replaced by $c_j + \delta$, where $\delta \in \mathbb{R}$ and $j \in B$.
- A change in c_B affects all the right-hand side entries in Row 0.
- Suppose that $j \in B$ is the ℓth basic variable, where $\ell \in \{1, \ldots, m\}$ (i.e., x_j is the basic variable in Row ℓ).
- Therefore, $c_B \in \mathbb{R}^m$ is replaced by $c_B(\delta) = c_B + \delta e^\ell$, where $e^\ell \in \mathbb{R}^m$.
- Since $c_B \in \mathbb{R}^m$ is replaced by $c_B(\delta) = c_B + \delta e^\ell$, where $e^\ell \in \mathbb{R}^m$, the current dictionary remains optimal if and only if

$$
c_k - \left(c_B + \delta e^\ell\right)^T (A_B)^{-1}A^k = \bar{c}_k - \delta (e^\ell)^T (A_B)^{-1}A^k \geq 0 \Longleftrightarrow \delta (e^\ell)^T (A_B)^{-1}A^k \leq \bar{c}_k, \quad k \in N.
$$

- Therefore, we obtain $\bar{c}_k(\delta) = \bar{c}_k - \delta(e^\ell)^T(A_B)^{-1}A^k,\ k \in N$. The primal solution is given by $x^*(\delta) = x^*$ and the corresponding dual solution by $y^*(\delta) = \left((A_B)^{-1}\right)^T (c_B + \delta e^\ell) = y^* + \delta \left((A_B)^{-1}\right)^T e^\ell$. The objective function value is given by $z^*(\delta) = \left(c_B + \delta e^\ell\right)^T (A_B)^{-1}b = z^* + \delta(e^\ell)^T(A_B)^{-1}b$.
- If we choose $\delta^* \in \mathbb{R}$ that does not satisfy all of these inequalities simultaneously, then we can reoptimize using the primal simplex method since the updated current dictionary is primal feasible but dual infeasible.

29.4.3 More General Changes

- More generally, if c is replaced by $c + \delta c'$, where $\delta \in \mathbb{R}$ and $c' \in \mathbb{R}^n$, we obtain

$$
\begin{aligned}
c_B(\delta) &= c_B + \delta c'_B \\
c_N(\delta) &= c_N + \delta c'_N
\end{aligned}
$$

- Row 0 coefficients can be updated accordingly and one can check conditions on δ for the nonnegativity of all reduced costs.
- Therefore, we obtain $\bar{c}_k(\delta) = \bar{c}_k + \delta c'_k - \delta(c'_B)^T(A_B)^{-1}A^k,\ k \in N$. The primal solution is given by $x^*(\delta) = x^*$ and the corresponding dual solution by $y^*(\delta) = \left((A_B)^{-1}\right)^T (c_B + \delta c'_B) = y^* + \delta\left((A_B)^{-1}\right)^T c'_B$. The objective function value is given by $z^*(\delta) = (c_B + \delta c'_B)^T (A_B)^{-1}b = z^* + \delta(c'_B)^T(A_B)^{-1}b$.
- We can use the primal simplex method to reoptimize if there is any negative reduced cost.

29.5 Concluding Remarks

- We discussed the effects of changes in $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ on the primal-dual optimal solutions and the optimal value.
- The same analysis can be extended to the case of simultaneously replacing b by $b+\delta b'$, and c by $c+\delta c'$, where $\delta \in \mathbb{R}$, $b' \in \mathbb{R}^m$, and $c' \in \mathbb{R}^n$.
- However, if δ^* does not satisfy all of the resulting inequalities for primal and dual feasibility simultaneously, we may lose both of them.
- In such a case, we need to solve the modified problem from scratch.
- In the next two lectures, we will discuss changes in $A \in \mathbb{R}^{m \times n}$, adding new variables to (P), and adding new constraints to (P).

Exercises

Question 29.1.

$$\begin{array}{ll} (P) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^T y : A^T y \leq c\} \end{array}$$

Consider the instance of (P) and (D) with the following parameters:

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 5 & 3 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 16 \end{bmatrix}, \quad c = \begin{bmatrix} -5 & -1 & 12 & 0 & -1 \end{bmatrix}^T.$$

Here is the optimal dictionary:

$$\begin{array}{rcrcrcrcr} z & = & -16 & + & 2x_2 & + & 12x_3 & + & x_4 \\ x_1 & = & 3 & - & 0.5x_2 & + & 0.5x_3 & - & 0.5x_4 \\ x_5 & = & 1 & - & 0.5x_2 & - & 2.5x_3 & + & 1.5x_4 \end{array}$$

For each change below, find the possible values of $\delta \in \mathbb{R}$ for which the dictionary remains optimal if (i) b_1 is replaced by $10 + \delta$, (ii) c_2 is replaced by $-1 + \delta$, and (iii) c_1 is replaced by $-5 + \delta$.

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 30	Sensitivity Analysis and Reoptimization – Part II
Lecturer: E. Alper Yıldırım	*Week: 10*

30.1 Outline

- Sensitivity Analysis and Reoptimization: Changes in A
- Sensitivity Analysis and Reoptimization: Adding a new variable
- Review Problems

30.2 Motivation and Setup

Consider the following pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- In the last lecture, we studied how the optimal value and the optimal solutions of (P) and (D) change with respect to the changes in the problem parameters $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, and how the modified problem can be reoptimized efficiently.
- In this lecture, we will study the same problem for the following additional cases:
  - (i) Changes in $A \in \mathbb{R}^{m \times n}$
  - (ii) Adding a new variable
- Suppose that the pair of problems (P) and (D) are solved either using the usual (primal) simplex method or the dual simplex method.
- Suppose that $A \in \mathbb{R}^{m \times n}$ has full row rank and that $x^* \in \mathbb{R}^n$ is an optimal vertex of (P) with corresponding index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$, and $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is the corresponding dual optimal solution.
- There are two main questions of interest:
  - (i) **Sensitivity Analysis:** Under what conditions on the changes in the problem parameters would the dictionary corresponding to the index sets B and N still remain optimal for the modified primal-dual pair of problems?
  - (ii) **Reoptimization:** If the dictionary corresponding to the index sets B and N is no longer optimal for the modified primal-dual pair of problems, how do we reoptimize effectively?

30.2.1 Main Idea Revisited

Suppose the optimal dictionary of the original pair of primal-dual linear programming problems is given by the following:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

(i) Identify the parts of the optimal dictionary that would be affected by such a change.

(ii) Identify whether the primal and/or dual feasibility would be affected by such a change.

(iii) Derive conditions on the parameters such that primal and dual feasibility are maintained.

(iv) If the primal or dual feasibility is violated after the modification, determine which variant of the simplex method (primal or dual) would be appropriate to reoptimize the modified pair of primal and dual problems.

30.3 Changes in A

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^Tx : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^Ty : A^Ty \leq c\} \end{array}$$

- First, we will focus on changes in the entries of the coefficient matrix $A \in \mathbb{R}^{m\times n}$.
- Changes in A affect the feasible region of (P) and the feasible region of (D) simultaneously.
- Therefore, we may lose both primal feasibility and dual feasibility.
- Let $A^j \in \mathbb{R}^m$ denote the $j$the column of $A, \ j = 1, \dots, n$. We will consider two cases:
  (i) Changes in $A^j, \ j \in N$
  (ii) Changes in $A^j, \ j \in B$

30.3.1 Changes in $A^j, \ j \in N$

Recall the optimal dictionary of the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \end{aligned}$$

- Let us first consider changes in $A^j, \ j \in N$.
- Suppose that A_{ij} is replaced by $A_{ij} + \delta$, where $\delta \in \mathbb{R}$, $i \in \{1, \ldots, m\}$, and $j \in N$.
- Therefore, $A^j \in \mathbb{R}^m$ is replaced by $A^j + \delta e^i$, where $e^i \in \mathbb{R}^m$ denotes the ith unit vector, $i \in \{1, \ldots, m\}$.
- Changes in $A^j, \ j \in N$ only affect the values of the reduced cost of the nonbasic variable x_j and the coefficients of x_j on the right-hand side of Rows 1 through m (i.e., only dual feasibility is affected).
- The current dictionary remains optimal for the modified primal-dual pair of problems if and only if

$$c_j - c_B^T (A_B)^{-1}(A^j + \delta e^i) \geq 0.$$

- Note that

$$c_j - c_B^T (A_B)^{-1}(A^j + \delta e^i) = \bar{c}_j - \delta c_B^T (A_B)^{-1} e^i \geq 0 \Longleftrightarrow \delta c_B^T (A_B)^{-1} e^i \leq \bar{c}_j.$$

- Therefore, the current dictionary remains optimal if and only if δ satisfies the inequality above. Note that primal-dual optimal solutions and the optimal value remain the same.
- If we choose $\delta^* \in \mathbb{R}$ that violates this inequality, we update the Row 0 coefficient of x_j using the above relation. We update the coefficients of x_j on the right-hand side of Rows 1 through m using $-(A_B)^{-1}(A^j + \delta e^i)$ and continue with the primal simplex method.
- More generally, a similar analysis applies if A^j is replaced by $A^j + \delta a'$, where $\delta \in \mathbb{R}$ and $a' \in \mathbb{R}^m$.

30.3.2 Changes in $A^j, \ j \in B$

Recall the optimal dictionary of the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Let us now consider changes in $A^j, \ j \in B$.
- Suppose that A_{ij} is replaced by $A_{ij} + \delta$, where $\delta \in \mathbb{R}$, $i \in \{1, \ldots, m\}$, and $j \in B$.
- Changes in $A^j, \ j \in B$ affect the basis matrix $A_B \in \mathbb{R}^{m \times m}$. Note that the entire dictionary is affected by such a change.
- The updated basis matrix may no longer be invertible.
- Even if it is invertible, we may lose both primal and dual feasibility.
- Solve the modified problem from scratch using the Two-Phase Method.

30.4 Adding a New Variable

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- Suppose now that we wish to add a new variable $x_{n+1} \in \mathbb{R}$ with a cost coefficient $c_{n+1} \in \mathbb{R}$ and the column $A^{n+1} \in \mathbb{R}^m$ in the coefficient matrix.
- The new coefficient matrix is given by $\tilde{A} = \begin{bmatrix} A^1 & \cdots & A^n & A^{n+1} \end{bmatrix} \in \mathbb{R}^{m \times (n+1)}$.
- We have $\text{rank}(\tilde{A}) \geq \text{rank}(A) = m$ and $\text{rank}(\tilde{A}) \leq m$. Therefore, $\tilde{A} \in \mathbb{R}^{m \times (n+1)}$ has full row rank.
- Defining $\tilde{x} \in \mathbb{R}^{n+1}$ by $\tilde{x}_j = x_j^*, \ j = 1, \ldots, n$, and $\tilde{x}_{n+1} = 0$, we obtain $\tilde{A}\,\tilde{x} = Ax^* + \mathbf{0} = b$ and $\tilde{x} \geq \mathbf{0}$.
- $\tilde{x} \in \mathbb{R}^{n+1}$ is a basic feasible solution of the new primal problem with index sets $\tilde{B} = B$ and $\tilde{N} = N \cup \{n+1\}$. Therefore, the new primal problem has a nonempty feasible region.
- On the dual side, we need to add a new constraint given by $(A^{n+1})^T y \leq c_{n+1}$. The new dual problem may or may not be infeasible.
- We will use $\tilde{x} \in \mathbb{R}^{n+1}$ as the starting primal vertex with corresponding index sets $\tilde{B} = B$ and $\tilde{N} = N \cup \{n+1\}$.
- We need to add a new column on the right-hand side of Rows $0, 1, 2, \ldots, m$ corresponding to the new nonbasic variable x_{n+1}.
- The reduced cost of x_{n+1} is $\bar{c}_{n+1} = c_{n+1} - c_B^T (A_B)^{-1} A^{n+1}$.
- The current dictionary remains optimal if and only if $\bar{c}_{n+1} \geq 0$.
- Otherwise, we compute the coefficients of x_{n+1} on the right-hand sides of Rows 1 through m using $-(A_B)^{-1} A^{n+1}$ and continue with the primal simplex method.

30.5 Concluding Remarks

- We discussed the effects of changes in $A \in \mathbb{R}^{m \times n}$ and adding a new variable on the primal-dual optimal solutions and the optimal value.
- Note that reoptimization of the modified problem can be performed efficiently if one of primal and dual feasibility is maintained.
- If the changes violate both primal and dual feasibility, we need to solve the modified problem from scratch.
- In the next lecture, we will discuss the effects of adding a new constraint on the primal-dual optimal solutions and the optimal value.

Exercises

Question 30.1.

$$\begin{array}{ll} (P) & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^T y : A^T y \leq c\} \end{array}$$

Consider the instance of (P) and (D) with the following parameters:

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 5 & 3 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 16 \end{bmatrix}, \quad c = \begin{bmatrix} -5 & -1 & 12 & 0 & -1 \end{bmatrix}^T.$$

Here is the optimal dictionary:

$$\begin{array}{rcrcrcrcr} z & = & -16 & + & 2x_2 & + & 12x_3 & + & x_4 \\ x_1 & = & 3 & - & 0.5x_2 & + & 0.5x_3 & - & 0.5x_4 \\ x_5 & = & 1 & - & 0.5x_2 & - & 2.5x_3 & + & 1.5x_4 \end{array}$$

(i) Find the possible values of $\delta \in \mathbb{R}$ for which the dictionary remains optimal if A_{22} is replaced by $3 + \delta$. (ii) What if A_{11} is replaced by 5? (iii) What if we add a new variable x_6 with $c_6 = -2$ and $A_6 = [1, -1]^T$?

MATH11111: Fundamentals of Optimization	Fall 2023
Lecture 31	Sensitivity Analysis and Reoptimization – Part III
Lecturer: E. Alper Yıldırım	*Week: 10*

31.1 Outline

- Sensitivity Analysis and Reoptimization: Adding a new equality constraint
- Sensitivity Analysis and Reoptimization: Adding a new inequality constraint
- Review Problems

31.2 Motivation and Setup

Consider the following pair of linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- (P) is the primal problem and (D) is the dual problem.
- In the last lecture, we studied how the optimal value and the optimal solutions of (P) and (D) change with respect to the changes in $A \in \mathbb{R}^{m \times n}$ and after adding a new variable, as well as how the modified problem can be reoptimized efficiently.
- In this lecture, we will study the same problem for the following additional cases:
  - (i) Adding a new equality constraint
  - (ii) Adding a new inequality constraint
- Suppose that the pair of problems (P) and (D) are solved either using the usual (primal) simplex method or the dual simplex method.
- Suppose that $A \in \mathbb{R}^{m \times n}$ has full row rank and that $x^* \in \mathbb{R}^n$ is an optimal vertex of (P) with corresponding index sets $B \subseteq \{1, \ldots, n\}$ and $N \subseteq \{1, \ldots, n\}$, and $y^* = \left((A_B)^{-1}\right)^T c_B \in \mathbb{R}^m$ is the corresponding dual optimal solution.
- There are two main questions of interest:
  - (i) **Sensitivity Analysis:** Under what conditions on the changes in the problem parameters would the dictionary corresponding to the index sets B and N still remain optimal for the modified primal-dual pair of problems?
  - (ii) **Reoptimization:** If the dictionary corresponding to the index sets B and N is no longer optimal for the modified primal-dual pair of problems, how do we reoptimize effectively?

31.2.1 Main Idea Revisited

Suppose that the optimal dictionary for the original primal-dual pair of linear programming problems is given by

$$
\begin{aligned}
z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\
x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j
\end{aligned}
$$

(i) Identify the parts of the optimal dictionary that would be affected by such a change.

(ii) Identify whether the primal and/or dual feasibility would be affected by such a change.

(iii) Derive conditions on the parameters such that primal and dual feasibility are maintained.

(iv) If the primal or dual feasibility is violated after the modification, determine which variant of the simplex method (primal or dual) would be appropriate to reoptimize the modified pair of primal and dual problems.

31.3 Adding a New Equality Constraint

Consider the following pair of primal-dual linear programming problems:

$$
\begin{array}{ll}
\text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\
\text{(D)} & \max\{b^T y : A^T y \leq c\}
\end{array}
$$

- Suppose now that we wish to add a new equality constraint $(a^{m+1})^T x = b_{m+1}$, where $a^{m+1} \in \mathbb{R}^n$ and $b_{m+1} \in \mathbb{R}$.
- If the optimal solution of the original primal problem $x^* \in \mathbb{R}^n$ satisfies $(a^{m+1})^T x^* = b_{m+1}$, then x^* is an optimal solution of the modified problem.
- Otherwise, x^* is not a feasible solution of the modified problem.
- The modified primal problem may not be feasible.
- Note that we need an additional basic variable corresponding to the new primal equality constraint.
- Let

$$
\tilde{A} = \begin{bmatrix} (a^1)^T \\ \vdots \\ (a^m)^T \\ (a^{m+1})^T \end{bmatrix} \in \mathbb{R}^{(m+1) \times n}
$$

- The new matrix $\tilde{A}$ may not have full row rank.

- Even if it has full row rank, it is not clear how to correctly identify the new basic variable.
- In the new dual problem, we need to define a new dual variable y_{m+1} corresponding to the new primal constraint.
- Let $\tilde{y} \in \mathbb{R}^{m+1}$ be given by $\tilde{y}_j = y^*_j$ for $j = 1, \ldots, m$, and $\tilde{y}_{m+1} = 0$, where $y^* \in \mathbb{R}^m$ is the optimal solution of the original dual problem.
- Since $\tilde{A}^T\tilde{y} = A^T y^* + \tilde{y}_{m+1}a^{m+1} = A^T y^* \leq c$, $\tilde{y} \in \mathbb{R}^{m+1}$ is a feasible solution of the modified dual problem.
- However, $\tilde{y} \in \mathbb{R}^{m+1}$ may not necessarily be a basic feasible solution.
- Solve the modified primal problem from scratch using the Two-Phase Method.

31.4 Adding a New Inequality Constraint

Consider the following pair of primal-dual linear programming problems:

$$\begin{array}{ll} \text{(P)} & \min\{c^T x : Ax = b, \quad x \geq \mathbf{0}\} \\ \text{(D)} & \max\{b^T y : A^T y \leq c\} \end{array}$$

- Suppose now that we wish to add a new inequality constraint $(a^{m+1})^T x \leq b_{m+1}$, where $a_{m+1} \in \mathbb{R}^n$ and $b^{m+1} \in \mathbb{R}$.
- If the optimal solution of the original primal problem $x^* \in \mathbb{R}^n$ satisfies $(a^{m+1})^T x^* \leq b_{m+1}$, then x^* is an optimal solution of the modified problem.
- Otherwise, x^* is not a feasible solution of the modified problem.
- The modified primal problem may be infeasible.
- Suppose that we add a new inequality constraint $(a^{m+1})^T x \leq b_{m+1}$ such that $(a^{m+1})^T x^* > b_{m+1}$.
- We need to convert the modified problem into standard form by defining a nonnegative slack variable x_{n+1} so that $(a^{m+1})^T x + x_{n+1} = b_{m+1}$.
- We also need a new basic variable corresponding to the new equality constraint.
- Let $\hat{x} \in \mathbb{R}^{n+1}$ be given by $\hat{x}_j = x^*_j$ for each $j = 1, \ldots, n$ and $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^*$, where $x^* \in \mathbb{R}^n$ is the optimal solution of the original primal problem.
- Then, $\hat{x} \in \mathbb{R}^{n+1}$ will satisfy all the equality constraints of the modified primal problem. However, it will be infeasible since $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^* < 0$.
- Let
$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ (a^{m+1})^T & 1 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}, \quad \tilde{b} = \begin{bmatrix} b \\ b_{m+1} \end{bmatrix} \in \mathbb{R}^{m+1}, \quad \tilde{c} = \begin{bmatrix} c \\ 0 \end{bmatrix} \in \mathbb{R}^{n+1}$$

Lemma 31.1. *Let $\tilde{A} \in \mathbb{R}^{(m+1)\times(n+1)}$ denote the new coefficient matrix.*

(i) $\tilde{A} \in \mathbb{R}^{(m+1)\times(n+1)}$ *has full row rank.*

(ii) If we define $\hat{x} \in \mathbb{R}^{n+1}$ such that $\hat{x}_j = x_j^*$ for each $j = 1, \dots, n$ and $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^*$, where $x^* \in \mathbb{R}^n$ is the optimal solution of the original primal problem, then $\hat{x} \in \mathbb{R}^{n+1}$ is a basic solution of the modified primal problem.

Proof. (i) Since $A \in \mathbb{R}^{m \times n}$ has full column rank, it follows that the first m rows of $\tilde{A} \in \mathbb{R}^{(m+1)\times(n+1)}$ are linearly independent. The last row of $\tilde{A}$ cannot be written as a linear combination of the first m rows of $\tilde{A}$ since the last component of the last row is 1 but the last components of all other rows are equal to 0. It follows that the rows of $\tilde{A}$ are linearly independent, i.e., $\tilde{A}$ has full row rank.

(ii) Let $\hat{x} \in \mathbb{R}^{n+1}$ be such that $\hat{x}_j = x_j^*$ for each $j = 1, \dots, n$ and $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^*$. Then, we have $\tilde{A}\hat{x} = \tilde{b}$, i.e., $\hat{x} \in \mathbb{R}^{n+1}$ will satisfy all of the equality constraints of the modified primal problem. Let us define the index sets $\tilde{B} = B \cup \{n+1\}$ and $\tilde{N} = N$. Note that $|\tilde{B}| = m+1$. By part (1) and Proposition 13.2, it suffices to show that $\tilde{A}_{\tilde{B}} \in \mathbb{R}^{(m+1)\times(m+1)}$ is invertible.

$$\tilde{A}_{\tilde{B}} = \begin{bmatrix} A_B & \mathbf{0} \\ (a_B^{m+1})^T & 1 \end{bmatrix}.$$

Using a similar argument as in part (1), since $A_B \in \mathbb{R}^{m \times m}$ is invertible, it follows that the first m rows of $\tilde{A}_{\tilde{B}}$ are linearly independent. The last row of $\tilde{A}_{\tilde{B}}$ cannot be written as a linear combination of the first m rows of $\tilde{A}_{\tilde{B}}$ since the last component of the last row is 1 but the last components of all other rows are equal to 0. It follows that the rows of $\tilde{A}_{\tilde{B}}$ are linearly independent, i.e., $\tilde{A}_{\tilde{B}}$ is invertible. By Proposition 13.2, $\hat{x} \in \mathbb{R}^{n+1}$ is a basic solution of the modified primal problem.

□

31.4.1 New Dictionary

Recall the optimal dictionary for the original pair of primal-dual linear programming problems:

$$\begin{aligned} z &= c_B^T (A_B)^{-1} b + \sum_{j \in N} \underbrace{\left(c_j - c_B^T (A_B)^{-1} A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1} b + \sum_{j \in N} \left(-(A_B)^{-1} A^j\right) x_j \end{aligned}$$

- Recall

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ (a^{m+1})^T & 1 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}, \quad \tilde{b} = \begin{bmatrix} b \\ b_{m+1} \end{bmatrix} \in \mathbb{R}^{m+1}, \quad \tilde{c} = \begin{bmatrix} c \\ 0 \end{bmatrix} \in \mathbb{R}^{n+1}$$

$$\tilde{B} = B \cup \{n+1\}, \quad \tilde{N} = N, \quad \tilde{A}_{\tilde{B}} = \begin{bmatrix} A_B & \mathbf{0} \\ (a_B^{m+1})^T & 1 \end{bmatrix}, \quad \tilde{c}_{\tilde{B}} = \begin{bmatrix} c_B \\ 0 \end{bmatrix},$$

$\hat{x} \in \mathbb{R}^{n+1}$, $\hat{x}_j = x_j^*$ for each $j = 1, \dots, n$ and $\hat{x}_{n+1} = b_{m+1} - (a^{m+1})^T x^*$.

- The new dictionary is given by

$$\begin{aligned} z &= \tilde{c}_{\tilde{B}}^T (\tilde{A}_{\tilde{B}})^{-1} \tilde{b} + \sum_{j \in \tilde{N}} (\tilde{c}_j - \tilde{c}_{\tilde{B}}^T (\tilde{A}_{\tilde{B}})^{-1} \tilde{A}^j) x_j \\ x_{\tilde{B}} &= (\tilde{A}_{\tilde{B}})^{-1} \tilde{b} + \sum_{j \in \tilde{N}} \left(-(\tilde{A}_{\tilde{B}})^{-1} \tilde{A}^j\right) x_j \end{aligned}$$

- By simple manipulations, we obtain

$$\left(\tilde{A}_{\tilde{B}}\right)^{-1} = \begin{bmatrix} A_B^{-1} & \mathbf{0} \\ -(a_B^{m+1})^T(A_B)^{-1} & 1 \end{bmatrix}.$$

- We substitute these expressions in the dictionary above:

$$\begin{aligned} z &= c_B^T(A_B)^{-1}b + \sum_{j\in N} \underbrace{\left(c_j - c_B^T(A_B)^{-1}A^j\right)}_{\bar{c}_j} x_j \\ x_B &= (A_B)^{-1}b + \sum_{j\in N} \left(-(A_B)^{-1}A^j\right) x_j \\ x_{n+1} &= \underbrace{b_{m+1} - (a^{m+1})^T x^*}_{<0} + \sum_{j\in N} \left((a_B^{m+1})^T(A_B)^{-1}A^j - a_j^{m+1}\right) x_j \end{aligned}$$

- The updated dictionary is dual feasible since $\bar{c}_j \geq 0$ for each $j \in N$.
- It is primal infeasible since the value of the new basic variable x_{n+1} is negative.
- We therefore use the dual simplex method to reoptimize the modified primal-dual pair of problems.

31.5 Concluding Remarks

- We discussed the effects of adding a new constraint on the primal-dual optimal solutions and the optimal value.
- Note that reoptimization of the modified problem can be performed efficiently if one of primal and dual feasibility is maintained.
- If the changes violate both primal and dual feasibility, we need to solve the modified problem from scratch.
- This lecture wraps up our discussion of linear programming. In the remaining lectures, we will discuss unconstrained nonlinear optimization.

Exercises

Question 31.1.

$$\begin{array}{ll} (P) & \min\{c^Tx : Ax = b, \quad x \geq \mathbf{0}\} \\ (D) & \max\{b^Ty : A^Ty \leq c\} \end{array}$$

Consider the instance of (P) and (D) with the following parameters:

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 \\ 5 & 3 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 16 \end{bmatrix}, \quad c = \begin{bmatrix} -5 & -1 & 12 & 0 & -1 \end{bmatrix}^T.$$

Here is the optimal dictionary:

$$\begin{array}{rcrcrcrcr} z & = & -16 & + & 2x_2 & + & 12x_3 & + & x_4 \\ x_1 & = & 3 & - & 0.5x_2 & + & 0.5x_3 & - & 0.5x_4 \\ x_5 & = & 1 & - & 0.5x_2 & - & 2.5x_3 & + & 1.5x_4 \end{array}$$

For each change below, determine if the optimal solution changes and how we should reoptimize.

(i) *Add a new constraint* $3x_1 - x_2 + x_3 - 2x_4 + x_5 = 10$.

(ii) *Add a new constraint* $3x_1 - x_2 + x_3 - 2x_4 + x_5 = 9$.

(iii) *Add a new constraint* $3x_1 - x_2 + x_3 - 2x_4 + x_5 \leq 12$.

(iv) *Add a new constraint* $3x_1 - x_2 + x_3 - 2x_4 + x_5 \leq 9$.

Lecture 32 Continuity and Differentiability – One-Dimensional Case

Lecturer: E. Alper Yıldırım *Week: –*

32.1 Outline

- Continuity
- Differentiability
- Review Problems

32.2 Motivation and Setup

- In the last part of this course, we will focus on unconstrained nonlinear optimization:

$$\text{(P)} \quad \min\{f(x) : x \in \mathbb{R}^n\},$$

  where $f : \mathbb{R}^n \to \mathbb{R}$ is a general (i.e., not necessarily linear) function.

- Continuity and differentiability play a very important role in optimization (as does linear algebra).

- In this lecture, we will review definitions of continuous functions and derivatives.

32.3 One-Dimensional Case: Continuity

- First, we will assume that $f : \mathbb{R} \to \mathbb{R}$, i.e., $n = 1$.

- For a sequence of real numbers $x^k, k = 1, 2, \ldots$, we have $\lim_{k \to \infty} x^k = \hat{x}$ if and only if $\lim_{k \to \infty} |x^k - \hat{x}| = 0$.

Definition 32.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\hat{x} \in \mathbb{R}$. We say that f is* continuous at $\hat{x}$ *if, for any sequence of real numbers $x^k, k = 1, 2, \ldots$ such that $\lim_{k \to \infty} x^k = \hat{x}$, we have*

$$\lim_{k \to \infty} f(x^k) = f(\hat{x}).$$

Otherwise, if the above equality is not satisfied for at least one sequence that converges to $\hat{x}$, or if there exists a sequence such that the limit on the left-hand side does not exist, then we say that f is discontinuous at $\hat{x}$. *If f is continuous at every $x \in \mathbb{R}$, then we say that f is a* continuous function *and write $f \in \mathbb{C}^0$. A function that is not continuous is said to be* discontinuous *and is denoted by $f \notin \mathbb{C}^0$.*

32.3.1 Continuity: Geometric Intuition

- A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $\hat{x} \in \mathbb{R}$ if for any sequence of real numbers $x^k, k = 1, 2, \ldots$ that converge to $\hat{x} \in \mathbb{R}$, the sequence of function values $f(x^k), k = 1, 2, \ldots$ should converge to $f(\hat{x})$.
- Geometrically, the function values should not have any "jumps" in a continuous function.
- **Role of Continuity:** Continuity of $f : \mathbb{R} \to \mathbb{R}$ at $\hat{x} \in \mathbb{R}$ allows us to ensure that $f(x)$ will be "close" to $f(\hat{x})$ whenever x is sufficiently close to $\hat{x}$.

Example 32.2. *Show that every linear function $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.*

Remark: There also exist many nonlinear continuous functions.

32.4 One-Dimensional Case: Differentiability

We will continue to assume that $f : \mathbb{R} \to \mathbb{R}$, i.e., $n = 1$.

Definition 32.3. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\hat{x} \in \mathbb{R}$. We say that f is* differentiable at $\hat{x}$ *if, for any sequence of real numbers $x^k \in \mathbb{R}\backslash\{\hat{x}\}, k = 1, 2, \ldots$ such that $\lim\limits_{k\to\infty} x^k = \hat{x}$, there exists a real number $\alpha \in \mathbb{R}$ such that*

$$\lim_{k\to\infty} \frac{f(x^k) - f(\hat{x})}{x^k - \hat{x}} = \alpha.$$

In this case, we say that f is differentiable at $\hat{x}$ *and we denote it by $f'(\hat{x}) = \alpha$. Otherwise, if the above limit does not exist for at least one sequence that converges to $\hat{x}$, or if there exist two different sequences that converge to two different real numbers α_1 and α_2, then we say that f is not* differentiable at $\hat{x}$. *If f is differentiable at every $x \in \mathbb{R}$, then we say that f is a* differentiable function. *A function that is not differentiable is said to be* nondifferentiable.

32.4.1 Differentiability: Geometric Intuition

- If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $\hat{x} \in \mathbb{R}$, then, for any sequence of real numbers $x^k \in \mathbb{R}\backslash\{\hat{x}\}, k = 1, 2, \ldots$ that converge to $\hat{x} \in \mathbb{R}$, we have
$$\lim_{k\to\infty} \frac{f(x^k) - f(\hat{x})}{x^k - \hat{x}} = \alpha = f'(\hat{x}).$$
- Geometrically, the left-hand side of the limit measures the ratio of the difference of the function values to the difference of the values of the function arguments. As such, the derivative measures the "rate of a change" of f at $\hat{x} \in \mathbb{R}$, i.e., $f'(\hat{x})$ is the slope of the tangent line to the graph of the function at $\hat{x} \in \mathbb{R}$.
- **Role of Differentiability:** Differentiability of $f : \mathbb{R} \to \mathbb{R}$ at $\hat{x} \in \mathbb{R}$ allows us to estimate how fast $f(x)$ will change whenever x is sufficiently close to $\hat{x}$.
- If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $\hat{x} \in \mathbb{R}$, then, $f : \mathbb{R} \to \mathbb{R}$ is continuous at $\hat{x} \in \mathbb{R}$. However, the converse relation does not hold in general.

Example 32.4. *Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is a continuous function and is differentiable at each $\hat{x} \in \mathbb{R}\backslash\{0\}$.*

32.4.2 Derivative as a Function

- Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
- Then, $f' : \mathbb{R} \to \mathbb{R}$ yields another function.
- If $f' : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f : \mathbb{R} \to \mathbb{R}$ is said to be continuously differentiable and is denoted by $f \in \mathbb{C}^1$.
- If $f' : \mathbb{R} \to \mathbb{R}$ is differentiable at $\hat{x} \in \mathbb{R}$, we denote it by $f''(\hat{x})$.
- If $f'' : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f : \mathbb{R} \to \mathbb{R}$ is said to be twice continuously differentiable and is denoted by $f \in \mathbb{C}^2$.

32.4.3 Interpretation of Derivatives

Lemma 32.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f \in \mathbb{C}^1$ and let $\hat{x} \in \mathbb{R}$. If $f'(\hat{x}) < 0$, then there exists a real number $\lambda^* > 0$ such that $f(\hat{x} + \lambda) < f(\hat{x})$ for all $\lambda \in (0, \lambda^*)$. Similarly, if $f'(\hat{x}) > 0$, then there exists a real number $\lambda^* > 0$ such that $f(\hat{x} - \lambda) < f(\hat{x})$ for all $\lambda \in (0, \lambda^*)$ (i.e., if $f'(\hat{x}) < 0$, then f is a decreasing function at $\hat{x}$ and if $f'(\hat{x}) > 0$, then f is an increasing function at $\hat{x}$).*

Proof. Suppose that $f'(\hat{x}) < 0$. Since $f \in \mathbb{C}^1$, there exists a real number $\lambda^* > 0$ such that $f'(\hat{x} + \lambda) < 0$ for all $\lambda \in (0, \lambda^*)$. By the fundamental theorem of calculus, for any $\lambda \in (0, \lambda^*)$, we have

$$f(\hat{x} + \lambda) - f(\hat{x}) = \int_{\hat{x}}^{\hat{x}+\lambda} \underbrace{f'(x)}_{<0}\, dx < 0.$$

Therefore, we obtain $f(\hat{x} + \lambda) < f(\hat{x})$ for all $\lambda \in (0, \lambda^*)$.

The proof of the other assertion is very similar. □

32.4.4 Chain Rule

Lemma 32.6 (Chain Rule)**.** *Let $\hat{x} \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function at $\hat{x}$ such that $g(\hat{x}) = \hat{y}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at $\hat{y}$. Let $(f \circ g)(x) = f(g(x))$. Then,*

$$(f \circ g)'(\hat{x}) = f'(\hat{y})\, g'(\hat{x}) = f'(g(\hat{x}))\, g'(\hat{x}).$$

Proof. Let $\hat{x} \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function at $\hat{x}$ such that $g(\hat{x}) = \hat{y}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at $\hat{y}$. Let $(f \circ g)(x) = f(g(x))$ and let $x^k \in \mathbb{R}\backslash\{\hat{x}\}, k = 1, 2, \ldots$ be a sequence such that $\lim_{k\to\infty} x^k = \hat{x}$. Then,

$$(f \circ g)'(\hat{x}) = \lim_{k\to\infty} \frac{f(g(x^k)) - f(g(\hat{x}))}{x^k - \hat{x}}.$$

Case 1: Suppose that for each real number $\epsilon > 0$, there exists a real number $x_\epsilon \in (\hat{x} - \epsilon, \hat{x} + \epsilon)\backslash\{\hat{x}\}$ such that $g(x_\epsilon) = g(\hat{x})$. Then, choosing $x^k = x_{1/k},\ k = 1, 2, \ldots$, we obtain $\lim_{k\to\infty} x^k = \hat{x}$. Then,

$$(f \circ g)'(\hat{x}) = \lim_{k\to\infty} \frac{f(g(x^k)) - f(g(\hat{x}))}{x^k - \hat{x}} = \lim_{k\to\infty} \frac{f(g(\hat{x})) - f(g(\hat{x}))}{x^k - \hat{x}} = 0.$$

Since $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $\hat{x}$, we obtain

$$g'(\hat{x}) = \lim_{k\to\infty} \frac{g(x^k) - g(\hat{x})}{x^k - \hat{x}} = \lim_{k\to\infty} \frac{g(\hat{x}) - g(\hat{x})}{x^k - \hat{x}} = 0.$$

Therefore, $0 = (f \circ g)'(\hat{x}) = f'(g(\hat{x}))\, g'(\hat{x}) = 0$.

Case 2: Suppose that there exists a real number $\epsilon^* > 0$ such that we have $g(x) \neq g(\hat{x})$ for each $x \in (\hat{x} - \epsilon^*, \hat{x} + \epsilon^*)\backslash\{\hat{x}\}$. Let $x^k \in (\hat{x} - \epsilon^*, \hat{x} + \epsilon^*)\backslash\{\hat{x}\}, k = 1, 2, \ldots$ be a sequence that converges to $\hat{x} \in \mathbb{R}$. Therefore,

$$(f \circ g)'(\hat{x}) = \lim_{k\to\infty} \frac{f(g(x^k)) - f(g(\hat{x}))}{x^k - \hat{x}} = \lim_{k\to\infty} \frac{f(g(x^k)) - f(g(\hat{x}))}{g(x^k) - g(\hat{x})} \cdot \frac{g(x^k) - g(\hat{x})}{x^k - \hat{x}}.$$

Since $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $\hat{x}$, $g : \mathbb{R} \to \mathbb{R}$ is continuous at $\hat{x}$. Therefore, $\lim\limits_{k\to\infty} g(x^k) = g(\hat{x})$. The assertion now follows from the definition of the derivative and the fact that the limit of the product of two sequences is equal to the product of the limits of each sequence. □

32.5 Discussion and Concluding Remarks

- Continuity and differentiability play an important role in terms of predicting the behaviour of a function $f : \mathbb{R} \to \mathbb{R}$ in the vicinity of a real number $\hat{x} \in \mathbb{R}$.
- If $f'(\hat{x}) < 0$, then f is a decreasing function at $\hat{x}$ and if $f'(\hat{x}) > 0$, then f is an increasing function at $\hat{x}$.
- If we are interested in minimizing a function $f \in \mathbb{C}^1$ over $x \in \mathbb{R}$ and we have a point $\hat{x} \in \mathbb{R}$ such that $f'(\hat{x}) \neq 0$, then we may reduce the function value by moving from $\hat{x}$ in the direction $-f'(\hat{x})$ at least for a while.
- In the next lecture, we will consider the multivariate case.

Exercises

Question 32.1. *Let $\hat{x} \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two differentiable functions at $\hat{x}$. Let $h(x) = f(x)g(x)$. Show that $h : \mathbb{R} \to \mathbb{R}$ is differentiable at $\hat{x}$ and derive an expression for $h'(\hat{x})$.*

MATH11111: Fundamentals of Optimization	**Fall 2023**
Lecture 33	Continuity and Differentiability – Multivariate Case
Lecturer: E. Alper Yıldırım	*Week: –*

33.1 Outline

- Continuity
- Differentiability
- Positive Semidefinite and Positive Definite Matrices
- Review Problems

33.2 Motivation and Setup

- Recall that our focus is on unconstrained nonlinear optimization:

$$\text{(P)} \quad \min\{f(x) : x \in \mathbb{R}^n\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a general (i.e., not necessarily linear) function.

- In the last lecture, we discussed continuity and differentiability in the one-dimensional case.
- In this lecture, we will extend these notions to the multivariate functions.

33.3 Multivariate Case: Convergent Sequences

- Let $f : \mathbb{R}^n \to \mathbb{R}$.
- For a sequence of vectors $x^k \in \mathbb{R}^n, k = 1, 2, \ldots$, we have $\lim_{k \to \infty} x^k = \hat{x}$, where $\hat{x} \in \mathbb{R}^n$, if and only if $\lim_{k \to \infty} \|x^k - \hat{x}\| = \lim_{k \to \infty} \left((x^k - \hat{x})^T (x^k - \hat{x})\right)^{1/2} = 0$.
- Note that $\lim_{k \to \infty} x^k = \hat{x}$ if and only if $\lim_{k \to \infty} x_j^k = \hat{x}_j$ for each $j = 1, \ldots, n$.

33.4 Multivariate Case: Continuity

Definition 33.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\hat{x} \in \mathbb{R}^n$. We say that f is* continuous at $\hat{x}$ *if, for any sequence of vectors $x^k \in \mathbb{R}^n, k = 1, 2, \ldots$ such that $\lim_{k \to \infty} x^k = \hat{x}$, we have*

$$\lim_{k \to \infty} f(x^k) = f(\hat{x}).$$

Otherwise, if the above equality is not satisfied for at least one sequence that converges to $\hat{x}$*, or if there exists a sequence such that the limit on the left-hand side does not exist, then we say that* f *is* discontinuous at $\hat{x}$*. If* f *is continuous at every* $x \in \mathbb{R}^n$*, then we say that* f *is a* continuous function *and write* $f \in \mathbb{C}^0$*. A function that is not continuous is said to be* discontinuous *and is denoted by* $f \notin \mathbb{C}^0$*.*

Role of Continuity: Continuity of $f : \mathbb{R}^n \to \mathbb{R}$ at $\hat{x} \in \mathbb{R}^n$ ensures that $f(x)$ will be "close" to $f(\hat{x})$ whenever $x \in \mathbb{R}^n$ is sufficiently close to $\hat{x}$.

Example 33.2. *Show that every linear function* $f : \mathbb{R}^n \to \mathbb{R}$ *is a continuous function.*

Remark: There also exist many nonlinear continuous functions.

33.5 Multivariate Case: Differentiability

Definition 33.3. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $\hat{x} \in \mathbb{R}^n$*. We say that* f *is* differentiable at $\hat{x}$ *if, for any sequence of vectors* $x^k \in \mathbb{R}^n \backslash \{\hat{x}\}, k = 1, 2, \ldots$ *such that* $\lim_{k \to \infty} x^k = \hat{x}$*, there exists a vector* $g \in \mathbb{R}^n$ *such that*

$$\lim_{k \to \infty} \frac{f(x^k) - f(\hat{x}) - g^T(x^k - \hat{x})}{\|x^k - \hat{x}\|} = 0.$$

In this case, we say that f *is* differentiable at $\hat{x}$ *and we denote it by* $f'(\hat{x}) = g^T$*. Otherwise, if the above limit does not exist or is not equal to zero for at least one sequence that converges to* $\hat{x}$*, then we say that* f *is not* differentiable at $\hat{x}$*. If* f *is differentiable at every* $x \in \mathbb{R}^n$*, then we say that* f *is a* differentiable function*. A function that is not differentiable is said to be* nondifferentiable.

33.5.1 Differentiability: Geometric Intuition

- Recall that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $\hat{x} \in \mathbb{R}$ if, for any sequence of real numbers $x^k \in \mathbb{R} \backslash \{\hat{x}\}, k = 1, 2, \ldots$ that converge to $\hat{x} \in \mathbb{R}$, we have

$$\lim_{k \to \infty} \frac{f(x^k) - f(\hat{x})}{x^k - \hat{x}} = f'(\hat{x}) \iff \lim_{k \to \infty} \frac{f(x^k) - f(\hat{x}) - f'(\hat{x})(x^k - \hat{x})}{x^k - \hat{x}} = 0.$$

- Let $\ell : \mathbb{R} \to \mathbb{R}$ be given by $\ell(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$. Then,

$$\lim_{k \to \infty} \frac{f(x^k) - \ell(x^k)}{x^k - \hat{x}} = 0.$$

- Therefore, the function $\ell(x)$ provides an increasingly better approximation of the function values as x gets closer to $\hat{x}$.
- However, $\ell(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$ is precisely the tangent line to the graph of the function at $\hat{x} \in \mathbb{R}$.
- For $f : \mathbb{R}^n \to \mathbb{R}$, f is *differentiable at* $\hat{x} \in \mathbb{R}^n$ if, for any sequence of vectors $x^k \in \mathbb{R}^n \backslash \{\hat{x}\}, k = 1, 2, \ldots$ such that $\lim_{k \to \infty} x^k = \hat{x}$, there exists a vector $g \in \mathbb{R}^n$ such that

$$\lim_{k \to \infty} \frac{f(x^k) - f(\hat{x}) - g^T(x^k - \hat{x})}{\|x^k - \hat{x}\|} = 0.$$

- Defining $\ell : \mathbb{R}^n \to \mathbb{R}$ by $\ell(x) = f(\hat{x}) + g^T(x - \hat{x}) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$, we similarly obtain that $\ell(x)$ provides an increasingly better approximation of the function values as x gets closer to $\hat{x}$.
- The graph of the function ℓ is the tangent hyperplane to the function f at $\hat{x}$.
- Similarly, if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\hat{x} \in \mathbb{R}^n$, then, $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\hat{x} \in \mathbb{R}^n$. However, the converse relation does not hold in general.

33.5.2 Understanding the Derivative

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\hat{x} \in \mathbb{R}^n$.
- Define one-dimensional functions $\phi_j(\lambda) = f(\hat{x} + \lambda e^j)$, where $j = 1, \ldots, n$ and $e^j \in \mathbb{R}^n$ denotes the jth unit vector.
- By the chain rule, we have $\phi_j'(\lambda) = f'(\hat{x} + \lambda e^j)e^j$. Therefore, $\phi_j'(0) = f'(\hat{x})e^j, \; j = 1, \ldots, n$.
- We therefore obtain $f'(\hat{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\hat{x}) & \frac{\partial f}{\partial x_2}(\hat{x}) & \cdots & \frac{\partial f}{\partial x_n}(\hat{x}) \end{bmatrix} \in \mathbb{R}^{1 \times n}$, where $\frac{\partial f}{\partial x_j}(\hat{x})$ denotes the partial derivative of f with respect to $x_j, \; j = 1, \ldots, n$.

33.5.3 Gradient as a Function

- Let us define

$$\nabla f(x) = (f'(x))^T = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix} \in \mathbb{R}^n.$$

- ∇f is called the gradient of f.
- Therefore, $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$.
- ∇f is continuous (differentiable) at $\hat{x} \in \mathbb{R}^n$ if each of the functions $\frac{\partial f}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}, \; j = 1, \ldots, n$ is continuous (differentiable) at $\hat{x} \in \mathbb{R}^n$.
- If $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, then $f : \mathbb{R}^n \to \mathbb{R}$ is said to be continuously differentiable and is denoted by $f \in \mathbb{C}^1$.

33.5.4 Hessian Matrix

- The derivative of $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\nabla f'(x) = \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

- $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ is called the Hessian matrix.
- If the second partial derivatives of f are all continuous functions, then $f : \mathbb{R}^n \to \mathbb{R}$ is said to be twice continuously differentiable and is denoted by $f \in \mathbb{C}^2$. In this case, $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ is a symmetric matrix, i.e., the order of partial derivatives does not matter:

$$\left(\nabla^2 f(x)\right)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \left(\nabla^2 f(x)\right)_{ji}, \quad 1 \le i < j \le n.$$

33.6 Positive Semidefinite and Positive Definite Matrices

- Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric matrix.
- Q is a positive semidefinite matrix if all of its eigenvalues are nonnegative.
- Equivalently, $Q \in \mathbb{R}^{n\times n}$ is positive semidefinite if

$$x^T Q x \ge 0, \quad \forall x \in \mathbb{R}^n.$$

- Q is positive definite if all of its eigenvalues are positive.
- Equivalently, $Q \in \mathbb{R}^{n\times n}$ is positive definite if

$$x^T Q x > 0, \quad \forall x \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Lemma 33.4 (2×2 Positive Semidefinite (Definite) Matrices)**.** *Let $Q \in \mathbb{R}^{2\times 2}$ be a symmetric matrix.*

(i) Q is positive semidefinite if and only if $Q_{11} \ge 0$, $Q_{22} \ge 0$, and $\det(Q) \ge 0$.

(ii) Q is positive definite if and only if $Q_{11} > 0$, $Q_{22} > 0$, and $\det(Q) > 0$.

Proof. We will only prove (i), since the proof of (ii) is very similar.

$\Rightarrow$: Suppose that Q is positive semidefinite. Then, for every $x \in \mathbb{R}^2$, we have $x^T Q x \ge 0$. In particular, if we pick $x = e^1 \in \mathbb{R}^2$, then $(e^1)^T Q e^1 = Q_{11} \ge 0$. Similarly, if we pick $x = e^2 \in \mathbb{R}^2$, then $(e^2)^T Q e^2 = Q_{22} \ge 0$. Finally, since the determinant is equal to the product of the eigenvalues and since each eigenvalue is nonnegative, we obtain $\det(Q) \ge 0$.

$\Leftarrow$: Suppose that $Q \in \mathbb{R}^{2\times 2}$ is a symmetric matrix such that $Q_{11} \ge 0$, $Q_{22} \ge 0$, and $\det(Q) \ge 0$. Then, for any vector $x = [x_1, x_2]^T \in \mathbb{R}^2$, we have

$$\begin{aligned} x^T Q x &= Q_{11} x_1^2 + 2Q_{12} x_1 x_2 + Q_{22} x_2^2 \\ &= \left(\sqrt{Q_{11}} x_1 - \sqrt{Q_{22}} x_2\right)^2 + 2\sqrt{Q_{11}}\sqrt{Q_{22}} x_1 x_2 + 2Q_{12} x_1 x_2 \\ &= \left(\sqrt{Q_{11}} x_1 + \sqrt{Q_{22}} x_2\right)^2 - 2\sqrt{Q_{11}}\sqrt{Q_{22}} x_1 x_2 + 2Q_{12} x_1 x_2 \end{aligned}$$

Since $\det(Q) = Q_{11}Q_{22} - Q_{12}^2 \ge 0$, $Q_{11} \ge 0$, and $Q_{22} \ge 0$, we obtain $-\sqrt{Q_{11}}\sqrt{Q_{22}} \le Q_{12} \le \sqrt{Q_{11}}\sqrt{Q_{22}}$. Therefore, if the vector $x = [x_1, x_2]^T \in \mathbb{R}^2$ satisfies $x_1 x_2 \ge 0$, then $x^T Q x \ge 0$ by the second equation above. If, on the other hand, $x_1 x_2 < 0$, then $x^T Q x \ge 0$ by the third equation above. We conclude that Q is positive semidefinite. $\square$

33.7 Discussion and Concluding Remarks

- We extended the definitions of continuity and differentiability to the multivariate case.
- In the next lectures, we will discuss the role of the first and second derivatives in optimization.

Exercises

Question 33.1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a linear function. Compute the gradient* $\nabla f(x)$ *by using partial derivatives. Verify that the definition of the derivative is satisfied. Compute the Hessian* $\nabla^2 f(x)$*. Is* $\nabla^2 f(x)$ *positive semidefinite? Is* $\nabla^2 f(x)$ *positive definite?*

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 34 Optimality Conditions – One-Dimensional Case

Lecturer: E. Alper Yıldırım *Week: –*

34.1 Outline

- First-Order Necessary Conditions
- Second-Order Necessary Conditions
- Second-Order Sufficient Conditions
- Review Problems

34.2 Motivation and Setup

- Recall that our focus is on unconstrained nonlinear optimization:

$$\text{(P)} \quad \min\{f(x) : x \in \mathbb{R}^n\},$$

  where $f : \mathbb{R}^n \to \mathbb{R}$ is a general (i.e., not necessarily linear) function.

- In the last two lectures, we discussed continuity and differentiability in the one-dimensional case and in the multivariate case.

- In this lecture, we will focus on the roles of continuity and differentiability in unconstrained optimization.

34.3 Local and Global Minimizers - One-Dimensional Case

- Let $f : \mathbb{R} \to \mathbb{R}$, i.e., $n = 1$.

- We are interested in the unconstrained optimization problem given by

$$\text{(P)} \quad \min\{f(x) : x \in \mathbb{R}\}.$$

Definition 34.1. *Let $f : \mathbb{R} \to \mathbb{R}$. We say that $x^* \in \mathbb{R}$ is a* global minimizer *of (P) if*

$$f(x^*) \leq f(x), \quad \forall x \in \mathbb{R}.$$

If $f(x^*) < f(x)$ for each $x \in \mathbb{R} \backslash \{x^*\}$, then $x^* \in \mathbb{R}$ is a strict global minimizer *of (P).*

We say that $x^* \in \mathbb{R}$ is a local minimizer *of (P) if there exists a real number $\delta > 0$ such that*

$$f(x^*) \leq f(x), \quad \forall x \in (x^* - \delta, x^* + \delta).$$

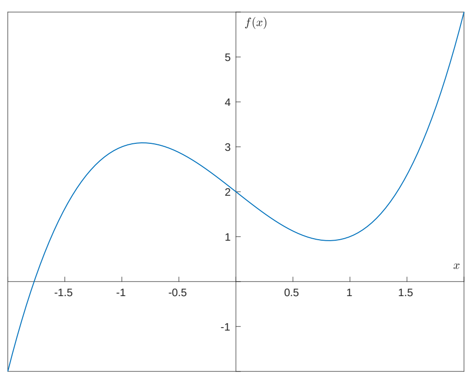
If $f(x^*) < f(x)$ for each $x \in (x^* - \delta, x^* + \delta) \backslash \{x^*\}$, then $x^* \in \mathbb{R}$ is a strict local minimizer *of (P).*

Remarks

- Replacing each $\leq$ by $\geq$ (and each $<$ by $>$) each in the above definition, we obtain the definitions of (strict) global and (strict) local maximizer, respectively.
- In general, an unconstrained optimization problem (P) may not have any local or global minimizers (e.g., $f(x) = x^3$).
- If there exists a sequence of real numbers $x^k, k = 1, 2, \dots$ such that $\lim_{k\to\infty} f(x^k) = -\infty$, then we define the optimal value to be $-\infty$ and there is no global minimizer.

34.3.1 An Example

$$f(x) = x^3 - 2x + 2$$



In this example, we have one strict local minimizer and one strict local maximizer but no global minimizers or global maximizers.

34.4 First-Order Necessary Optimality Conditions

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the unconstrained optimization problem given by

$$\text{(P)} \quad \min_{x\in\mathbb{R}} f(x)$$

Proposition 34.1 (First-Order Necessary Optimality Conditions)**.** *Let $f \in \mathbb{C}^1$. If $x^* \in \mathbb{R}$ is a local or global minimizer of (P), then $f'(x^*) = 0$.*

Proof. Let $f \in \mathbb{C}^1$ and let $x^* \in \mathbb{R}$ be a local minimizer of (P). Then, there exists a $\delta > 0$ such that $f(x^*) \leq f(x)$ for all $x \in (x^* - \delta, x^* + \delta)$.

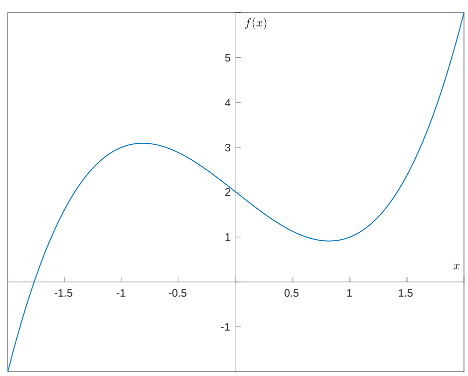
Suppose, for a contradiction, that $f'(x^*) \neq 0$. By Lemma 32.5, if $f'(x^*) < 0$, then there exists a real number $\lambda^* > 0$ such that $f(x^* + \lambda) < f(x^*)$ for all $\lambda \in (0, \lambda^*)$, contradicting that x^* is a local minimizer of (P). Similarly, if $f'(x^*) > 0$, then we again obtain a contradiction by Lemma 32.5. It follows that $f'(x^*) = 0$. Since a global minimizer is also a local minimizer, the assertion follows. □

Remarks

- Note that Proposition 34.1 provides a necessary condition for local and global optimality.
- Let $g : \mathbb{R} \to \mathbb{R}$. Since maximizing g is equivalent to minimizing $-g$, it follows that the same necessary condition also applies to a local and global maximizer.
- However, this condition is not sufficient. If $f(x) = x^3$, then $f'(x) = 3x^2$. For $\hat{x} = 0$, we have $f'(\hat{x}) = 0$, but $\hat{x}$ is neither a local minimizer nor a local maximizer.
- Therefore, finding a real number $\hat{x} \in \mathbb{R}$ such that $f'(\hat{x}) = 0$ does not necessarily imply that $\hat{x}$ is a local minimizer or a local maximizer.

34.4.1 Example Revisited

- If $f(x) = x^3 - 2x + 2$, then $f'(x) = 3x^2 - 2$, and $f'(\hat{x}) = 0$ if and only if $\hat{x} = \pm\sqrt{2/3}$.
- $-\sqrt{2/3}$ is a (strict) local maximizer whereas $\sqrt{2/3}$ is a (strict) local minimizer.



34.5 Second-Order Necessary Optimality Conditions

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the unconstrained optimization problem given by

$$\text{(P)} \quad \min_{x \in \mathbb{R}} f(x)$$

Proposition 34.2 (Second-Order Necessary Optimality Conditions). *Let $f \in \mathbb{C}^2$. If $x^* \in \mathbb{R}$ is a local or global minimizer of (P), then $f'(x^*) = 0$ and $f''(x^*) \geq 0$.*

Proof. Let $f \in \mathbb{C}^2$ and let $x^* \in \mathbb{R}$ be a local or global minimizer of (P). Then, there exists a real number $\delta > 0$ such that $f(x^*) \leq f(x)$ for all $x \in (x^* - \delta, x^* + \delta)$. By Proposition 34.1, we have $f'(x^*) = 0$.

Suppose, for a contradiction, that $f''(x^*) < 0$. Since $f \in \mathbb{C}^2$, there exists a real number $\lambda^* > 0$ such that $f''(x^* + \lambda) < 0$ for all $\lambda \in (0, \lambda^*)$. By the fundamental theorem of calculus, for any $\lambda \in (0, \lambda^*)$, we have

$$f'(x^* + \lambda) - \underbrace{f'(x^*)}_{0} = \int_{x^*}^{x^*+\lambda} \underbrace{f''(x)}_{<0}\, dx < 0.$$

Therefore, $f'(x^* + \lambda) < 0$ for each $\lambda \in (0, \lambda^*)$. Applying the fundamental theorem of calculus once again, for any $\lambda \in (0, \lambda^*)$, we have

$$f(x^* + \lambda) - f(x^*) = \int_{x^*}^{x^*+\lambda} \underbrace{f'(x)}_{<0}\, dx < 0,$$

which contradicts our hypothesis that x^* is a local minimizer of (P). It follows that $f''(x^*) \geq 0$. Since a global minimizer is also a local minimizer, the assertion follows.

Remark: Note how continuity is used in the proof. □

Remarks

- Note that Proposition 34.2 provides a necessary second-order condition for local and global optimality.
- Let $g : \mathbb{R} \to \mathbb{R}$. Since maximizing g is equivalent to minimizing $-g$, it follows that the necessary condition is replaced by $g''(x^*) \leq 0$ for a local and global maximizer.
- However, this condition is still not sufficient. If $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$. For $\hat{x} = 0$, we have $f'(\hat{x}) = 0$ and $f''(\hat{x}) = 0$, but $\hat{x}$ is neither a local minimizer nor a local maximizer.
- Therefore, finding a real number $\hat{x} \in \mathbb{R}$ such that $f'(\hat{x}) = 0$ and $f''(\hat{x}) \geq 0$ does not necessarily imply that $\hat{x}$ is a local minimizer (a similar remark applies for a local maximizer).

34.6 Second-Order Sufficient Optimality Conditions

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the unconstrained optimization problem given by

$$\text{(P)} \quad \min_{x \in \mathbb{R}} f(x)$$

Proposition 34.3 (Second-Order Sufficient Optimality Conditions)**.** *Let $f \in \mathbb{C}^2$ and let $x^* \in \mathbb{R}$. If $f'(x^*) = 0$ and $f''(x^*) > 0$, then x^* is a strict local minimizer of (P).*

Proof. Let $f \in \mathbb{C}^2$ and let $x^* \in \mathbb{R}$. Suppose that $f'(x^*) = 0$ and $f''(x^*) > 0$. Since $f \in \mathbb{C}^2$, there exists a real number $\lambda^* > 0$ such that $f''(x^* + \lambda) > 0$ for all $\lambda \in (-\lambda^*, \lambda^*)$.

By the fundamental theorem of calculus, for any $\lambda \in (0, \lambda^*)$, we have

$$f'(x^* + \lambda) - \underbrace{f'(x^*)}_{0} = \int_{x^*}^{x^*+\lambda} \underbrace{f''(x)}_{>0}\, dx > 0.$$

A similar argument shows that $f'(x^* + \lambda) < 0$ for any $\lambda \in (-\lambda^*, 0)$. Applying the fundamental theorem of calculus once again, for any $\lambda \in (0, \lambda^*)$, we have

$$f(x^* + \lambda) - f(x^*) = \int_{x^*}^{x^*+\lambda} \underbrace{f'(x)}_{>0} \, dx > 0.$$

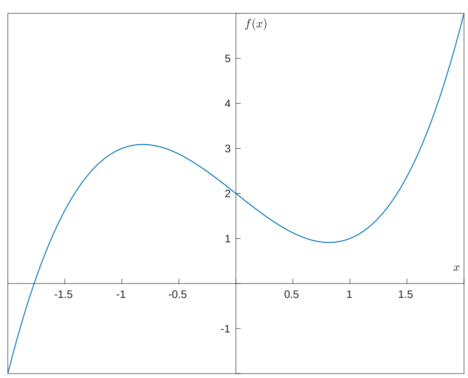
A similar argument shows that $f(x^* + \lambda) > f(x^*)$ for any $\lambda \in (-\lambda^*, 0)$. It follows that $f(x) > f(x^*)$ for each $x \in (x^* - \lambda, x^* + \lambda) \backslash \{x^*\}$. Therefore, x^* is a strict local minimizer of (P). □

Remarks

- In comparison with the second-order necessary conditions, the second-order sufficient conditions require stronger assumptions on $x^* \in \mathbb{R}$.
- However, the conclusion is also stronger.
- Note also that the sufficient conditions are not necessary to be a strict local minimizer (e.g., if $f(x) = x^4$, then $x^* = 0$ is a strict local (in fact, global) minimizer but does not satisfy the second-order sufficient conditions).

34.6.1 Example Revisited

- If $f(x) = x^3 - 2x + 2$, then $f'(x) = 3x^2 - 2$, and $f''(x) = 6x$.
- By Proposition 34.3, $-\sqrt{2/3}$ is a strict local maximizer whereas $\sqrt{2/3}$ is a strict local minimizer.



34.7 Concluding Remarks

- We discussed how continuity and differentiability play a key role in optimality conditions for unconstrained optimization in the one-dimensional case.
- We will discuss the optimality conditions in the multivariate case in the next lecture.

Exercises

Question 34.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be two functions such that* $f \in \mathbb{C}^2$ *and* $g \in \mathbb{C}^2$*. Let* $x^* \in \mathbb{R}$ *be such that* x^* *satisfies the second-order necessary conditions to be a local minimizer of* f *and second-order necessary conditions to be a local maximizer of* g*. Show that* x^* *satisfies the second-order necessary conditions to be either a local minimizer or a local maximizer of the function* $f + g : \mathbb{R} \to \mathbb{R}$*.*

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 35 — Optimality Conditions – Multivariate Case

Lecturer: E. Alper Yıldırım *Week: –*

35.1 Outline

- First-Order Necessary Conditions
- Second-Order Necessary Conditions
- Second-Order Sufficient Conditions
- Review Problems

35.2 Motivation and Setup

- Recall that our focus is on unconstrained nonlinear optimization:

$$\text{(P)} \quad \min\{f(x) : x \in \mathbb{R}^n\},$$

  where $f : \mathbb{R}^n \to \mathbb{R}$ is a general (i.e., not necessarily linear) function.

- In the last lecture, we focused on the roles of continuity and differentiability in unconstrained optimization in the one-dimensional case.
- In this lecture, we will focus on the roles of continuity and differentiability in unconstrained optimization in the multivariate case.
- In particular, we will heavily rely on optimality conditions in the one-dimensional case.

35.3 Local and Global Minimizers - Multivariate Case

- Let $f : \mathbb{R}^n \to \mathbb{R}$.
- We are interested in the unconstrained optimization problem given by

$$\text{(P)} \quad \min_{x \in \mathbb{R}^n} f(x)$$

Definition 35.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that $x^* \in \mathbb{R}^n$ is a* global minimizer *of (P) if*

$$f(x^*) \leq f(x), \quad \forall x \in \mathbb{R}^n.$$

If $f(x^*) < f(x)$ for each $x \in \mathbb{R}^n \backslash \{x^*\}$, then $x^* \in \mathbb{R}^n$ is a strict global minimizer *of (P).*

We say that $x^* \in \mathbb{R}^n$ is a local minimizer *of (P) if there exists a real number $\delta > 0$ such that*

$$f(x^*) \leq f(x), \quad \forall x \in \mathcal{B}(x^*, \delta),$$

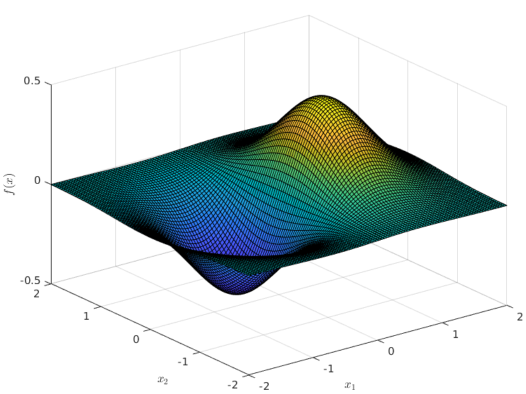
where $\mathcal{B}(x^*, \delta) = \{x \in \mathbb{R}^n : \|x^* - x\| < \delta\}$. If $f(x^*) < f(x)$ for each $x \in \mathcal{B}(x^*, \delta) \backslash \{x^*\}$, then $x^* \in \mathbb{R}^n$ is a strict local minimizer *of (P).*

Remarks

- Replacing each $\leq$ by $\geq$ (and each $<$ by $>$) in the above definition, we obtain the definitions of (strict) global and (strict) local maximizer, respectively.
- As in the one-dimensional case, an unconstrained optimization problem (P) may not have any local or global minimizers (e.g., $f(x) = -x_1^3 + x_2^3$).
- If there exists a sequence of vectors $x^k \in \mathbb{R}^n, \ k = 1, 2, \ldots$ such that $\lim\limits_{k\to\infty} f(x^k) = -\infty$, then we define the optimal value to be $-\infty$ and there is no global minimizer.

35.3.1 An Example

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x) = x_1 e^{-(x_1^2+x_2^2)}$$



In this example, we have one strict local (and global) minimizer and one strict local (and global) maximizer.

35.4 First-Order Necessary Optimality Conditions

Let $f : \mathbb{R}^n \to \mathbb{R}$ and consider the unconstrained optimization problem given by

$$\text{(P)} \quad \min_{x\in\mathbb{R}^n} f(x)$$

Proposition 35.1 (First-Order Necessary Optimality Conditions)**.** *Let $f \in \mathbb{C}^1$. If $x^* \in \mathbb{R}^n$ is a local or global minimizer of (P), then $\nabla f(x^*) = \mathbf{0} \in \mathbb{R}^n$.*

Proof. Let $f \in \mathbb{C}^1$ and let $x^* \in \mathbb{R}^n$ be a local minimizer of (P). Then, there exists a $\delta > 0$ such that $f(x^*) \leq f(x)$ for all $x \in \mathcal{B}(x^*, \delta)$.

Fix any direction $d \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and consider the one-dimensional function $\phi_d : \mathbb{R} \to \mathbb{R}$ given by $\phi_d(\lambda) = f(x^* + \lambda d)$. Note that ϕ_d is the restriction of f along the line $x^* + \lambda d$, where $\lambda \in \mathbb{R}$. Therefore, $\phi_d \in \mathbb{C}^1$. Furthermore, for any $\lambda \in (-\delta/\|d\|, \delta/\|d\|)$, we obtain $x^* + \lambda d \in \mathcal{B}(x^*, \delta)$. Since $f(x^*) \leq f(x)$ for all $x \in \mathcal{B}(x^*, \delta)$, it follows that $\phi_d(0) = f(x^*) \leq \phi_d(\lambda) = f(x^* + \lambda d)$ for all $\lambda \in (-\delta/\|d\|, \delta/\|d\|)$. It follows that $\lambda^* = 0$ is a local minimizer of $\phi_d : \mathbb{R} \to \mathbb{R}$. By the chain rule, $\phi_d'(\lambda) = f'(x^* + \lambda d)d = \nabla f(x^* + \lambda d)^T d$. By Proposition 34.1, we obtain $\phi_d'(0) = \nabla f(x^*)^T d = 0$. Since this equality holds for each $d \in \mathbb{R}^n$, it follows that $\nabla f(x^*) = \mathbf{0} \in \mathbb{R}^n$. Since a global minimizer is also a local minimizer, the assertion follows. $\square$

Remarks

- Note that Proposition 35.1 provides a necessary condition for local and global optimality.
- Let $g : \mathbb{R}^n \to \mathbb{R}$. Since maximizing g is equivalent to minimizing $-g$, it follows that the same necessary condition also applies to a local and global maximizer.
- However, this condition is not sufficient. If $f(x) = x_1^3 - x_2^3$, then $\nabla f(x) = [3x_1^2, -3x_2^2]^T$. For $\hat{x} = \mathbf{0} \in \mathbb{R}^2$, we have $\nabla f(\hat{x}) = \mathbf{0} \in \mathbb{R}^2$, but $\hat{x}$ is neither a local minimizer nor a local maximizer.
- Therefore, finding a vector $\hat{x} \in \mathbb{R}^n$ such that $\nabla f(\hat{x}) = \mathbf{0}$ does not necessarily imply that $\hat{x}$ is a local minimizer or a local maximizer.
- By the proof of Proposition 35.1, if $\nabla f(\hat{x}) \neq \mathbf{0}$ for some $\hat{x} \in \mathbb{R}^n$, then we can decrease f by moving from $\hat{x}$ in any direction $d \in \mathbb{R}^n$ such that $\nabla f(\hat{x})^T d < 0$ at least for a while (such directions are called *descent directions*).

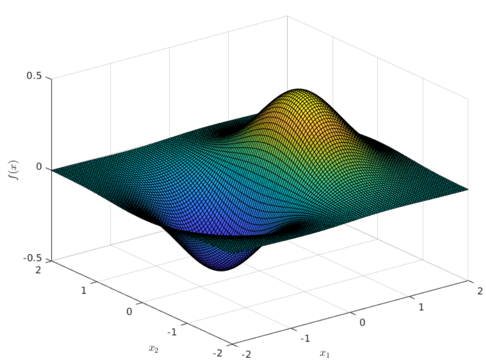
35.4.1 Example Revisited

- If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x) = x_1 e^{-(x_1^2 + x_2^2)}$, then

$$\nabla f(x) = \begin{bmatrix} e^{-(x_1^2+x_2^2)} - 2x_1^2 e^{-(x_1^2+x_2^2)} \\ -2x_1 x_2 e^{-(x_1^2+x_2^2)} \end{bmatrix},$$

  and $\nabla f(\hat{x}) = \mathbf{0}$ if and only if $\hat{x}_1 = \pm 1/\sqrt{2}$ and $\hat{x}_2 = 0$.

- $[-1/\sqrt{2}, 0]^T$ is a (strict) local and global minimizer whereas $[1/\sqrt{2}, 0]^T$ is a (strict) local and global maximizer.



35.5 Second-Order Necessary Optimality Conditions

Let $f : \mathbb{R}^n \to \mathbb{R}$ and consider the unconstrained optimization problem given by

$$\text{(P)} \quad \min_{x \in \mathbb{R}^n} f(x)$$

Proposition 35.2 (Second-Order Necessary Optimality Conditions)**.** *Let $f \in \mathbb{C}^2$. If $x^* \in \mathbb{R}^n$ is a local or global minimizer of (P), then $\nabla f(x^*) = \mathbf{0} \in \mathbb{R}^n$ and $\nabla^2 f(x^*) \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite.*

Proof. Let $f \in \mathbb{C}^2$ and let $x^* \in \mathbb{R}^n$ be a local minimizer of (P). Then, there exists a real number $\delta > 0$ such that $f(x^*) \leq f(x)$ for all $x \in \mathcal{B}(x^*, \delta)$. By Proposition 35.1, we have $\nabla f(x^*) = \mathbf{0}$.

Arguing similarly as in the proof of Proposition 35.1, for any direction $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, we define one-dimensional functions $\phi_d : \mathbb{R} \to \mathbb{R}$ given by $\phi_d(\lambda) = f(x^* + \lambda d)$. We obtain $\phi_d \in \mathbb{C}^2$. It follows that $\lambda^* = 0$ is a local minimizer of $\phi_d : \mathbb{R} \to \mathbb{R}$ for each direction $d \in \mathbb{R}^n$. By Proposition 34.2, we obtain $\phi_d''(0) \geq 0$. We have $\phi_d'(\lambda) = f'(x^* + \lambda d)d = \nabla f(x^* + \lambda d)^T d$. Note that

$$\phi_d'(\lambda) = \sum_{j=1}^{n} \underbrace{\frac{\partial f}{\partial x_j}}_{g_j}(x^* + \lambda d) d_j = \sum_{j=1}^{n} g_j(x^* + \lambda d) d_j.$$

Using the chain rule once again, we obtain

$$\phi_d''(\lambda) = \sum_{j=1}^{n} \left(\nabla g_j(x^* + \lambda d)^T d \right) d_j.$$

Using the definition of the gradient and partial derivatives, you can easily verify that $\phi_d''(\lambda) = d^T \nabla^2 f(x^* + \lambda d)^T d$. Therefore, $\phi_d''(0) = d^T \nabla^2 f(x^*)^T d \geq 0$. Since this inequality holds for any direction $d \in \mathbb{R}^n$, we conclude that $\nabla^2 f(x^*) \in \mathbb{R}^{n \times n}$ is positive semidefinite. It is symmetric since $f \in \mathbb{C}^2$. Since a global minimizer is also a local minimizer, the assertion follows. □

Remarks

- Note that Proposition 35.2 provides a necessary second-order condition for local and global optimality.
- Let $g : \mathbb{R}^n \to \mathbb{R}$. Since maximizing g is equivalent to minimizing $-g$, it follows that the necessary condition is replaced by $-\nabla^2 g(x^*) \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite for a local and global maximizer.
- However, this condition is still not sufficient. If $f(x) = x_1^3 - x_2^3$, then $\nabla f(x) = [3x_1^2, -3x_2^2]^T$ and

  $$\nabla^2 f(x) = \begin{bmatrix} 6x_1 & 0 \\ 0 & -6x_2 \end{bmatrix}.$$

  For $\hat{x} = \mathbf{0} \in \mathbb{R}^2$, we have $\nabla f(\hat{x}) = \mathbf{0} \in \mathbb{R}^2$ and $\nabla^2 f(\hat{x}) = \mathbf{0} \in \mathbb{R}^{2 \times 2}$. Both $\nabla^2 f(\hat{x})$ and $-\nabla^2 f(\hat{x})$ are symmetric and positive semidefinite, but $\hat{x}$ is neither a local minimizer nor a local maximizer.
- Therefore, finding a vector $\hat{x} \in \mathbb{R}^n$ such that $\nabla f(\hat{x}) = \mathbf{0} \in \mathbb{R}^n$ and $\nabla^2 f(\hat{x}) \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite does not necessarily imply that $\hat{x}$ is a local minimizer (a similar remark applies for a local maximizer).

35.6 Second-Order Sufficient Optimality Conditions

Let $f : \mathbb{R}^n \to \mathbb{R}$ and consider the unconstrained optimization problem given by

$$\text{(P)} \quad \min_{x \in \mathbb{R}^n} f(x)$$

Proposition 35.3 (Second-Order Sufficient Optimality Conditions)**.** *Let $f \in \mathbb{C}^2$ and let $x^* \in \mathbb{R}^n$. If $\nabla f(x^*) = \mathbf{0} \in \mathbb{R}^n$ and $\nabla^2 f(x^*) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, then x^* is a strict local minimizer of (P).*

Proof. Let $f \in \mathbb{C}^2$ and let $x^* \in \mathbb{R}^n$ be such that $\nabla f(x^*) = \mathbf{0} \in \mathbb{R}^n$ and $\nabla^2 f(x^*) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Note that all eigenvalues of $\nabla^2 f(x^*) \in \mathbb{R}^{n\times n}$ are strictly positive.

Since $f \in \mathbb{C}^2$ and all eigenvalues of a symmetric matrix change continuously, it follows that there exists a real number $\delta > 0$ such that $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite for all $x \in \mathcal{B}(x^*, \delta)$.

Arguing similarly as in the proof of Proposition 35.2, for any direction $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, we define one-dimensional functions $\phi_d : \mathbb{R} \to \mathbb{R}$ given by $\phi_d(\lambda) = f(x^* + \lambda d)$. We obtain $\phi_d \in \mathbb{C}^2$. Since $\phi_d'(0) = \nabla f(x^*)^T d = 0$ and $\phi_d''(\lambda) = d^T \nabla^2 f(x^* + \lambda d)^T d > 0$ for all $\lambda \in (-\delta/\|d\|, \delta/\|d\|)$, it follows from a similar argument as in the proof of Proposition 34.3 that we obtain $\phi_d(\lambda) = f(x^* + \lambda d) > \phi_d(0) = f(x^*)$ for all $\lambda \in (-\delta/\|d\|, \delta/\|d\|)\backslash\{0\}$. It follows that $f(x) > f(x^*)$ for each $x \in \mathcal{B}(x^*, \delta)\backslash\{x^*\}$. We conclude that x^* is a strict local minimizer of (P). $\square$

Remarks

- In comparison with the second-order necessary conditions, the second-order sufficient conditions require stronger assumptions on $x^* \in \mathbb{R}^n$.
- However, the conclusion is also stronger.
- Note also that the sufficient conditions are not necessary to be a strict local minimizer (e.g., if $f(x) = x_1^4 + x_2^4$, then $x^* = \mathbf{0} \in \mathbb{R}^2$ is a strict local (in fact, global) minimizer but does not satisfy the second-order sufficient conditions).

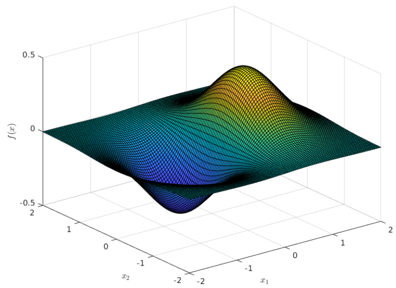
35.6.1 Example Revisited

- If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x) = x_1 e^{-(x_1^2+x_2^2)}$, then

$$\nabla^2 f(x) = \begin{bmatrix} e^{-(x_1^2+x_2^2)}(4x_1^3 - 6x_1) & e^{-(x_1^2+x_2^2)}(4x_1^2 x_2 - 2x_2) \\ e^{-(x_1^2+x_2^2)}(4x_1^2 x_2 - 2x_2) & e^{-(x_1^2+x_2^2)}(4x_1 x_2^2 - 2x_1) \end{bmatrix}.$$

  and $\nabla f(\hat{x}) = \mathbf{0}$ if and only if $\hat{x}_1 = \pm 1/\sqrt{2}$ and $\hat{x}_2 = 0$.

- $[-1/\sqrt{2}, 0]^T$ is a (strict) local and global minimizer since $\nabla^2 f(\hat{x})$ is positive definite whereas $[1/\sqrt{2}, 0]^T$ is a (strict) local and global maximizer since $-\nabla^2 f(\hat{x})$ is positive definite.



35.7 Concluding Remarks

- We discussed the key role of continuity and differentiability in optimality conditions for unconstrained optimization in the multivariate case.
- We will discuss the role of convexity in unconstrained optimization in the next lecture.

Exercises

Question 35.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two functions such that $f \in \mathbb{C}^2$ and $g \in \mathbb{C}^2$. Let $x^* \in \mathbb{R}^n$ be such that x^* satisfies the second-order necessary conditions to be a local minimizer of f and second-order necessary conditions to be a local maximizer of g. Is it necessarily true that $x^* \in \mathbb{R}^n$ satisfies the second-order necessary conditions to be either a local minimizer or a local maximizer of the function $f + g : \mathbb{R}^n \to \mathbb{R}$? Either prove the statement or give a counterexample.*

MATH11111: Fundamentals of Optimization **Fall 2023**

Lecture 36 Convexity and Unconstrained Optimization

Lecturer: E. Alper Yıldırım *Week: –*

36.1 Outline

- First-Order Characterisation of Convexity
- First-Order Necessary and Sufficient Optimality Conditions
- Second-Order Characterisation of Convexity
- Review Problems

36.2 Motivation and Setup

- Recall that our focus is on unconstrained nonlinear optimization:

$$\text{(P)} \quad \min\{f(x) : x \in \mathbb{R}^n\},$$

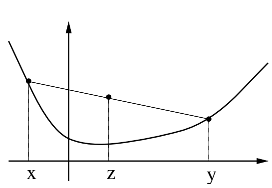
  where $f : \mathbb{R}^n \to \mathbb{R}$ is a general (i.e., not necessarily linear) function.

- In this last lecture, we focus on the case in which $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.
- We aim to establish some useful implications of convexity in unconstrained optimization.

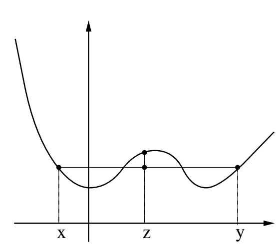
36.3 Convex Functions Revisited

- Let $f : \mathbb{R}^n \to \mathbb{R}$.
- Recall that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function if, for all $x \in \mathbb{R}^n$, for all $y \in \mathbb{R}^n$, and for all real numbers $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$



Convex function



Nonconvex function

36.4 One-Dimensional Case

- First, we will focus on the one-dimensional case, i.e., we will consider $f : \mathbb{R} \to \mathbb{R}$.
- Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $\hat{x} \in \mathbb{R}$. Then, for any sequence of real numbers $x^k \in \mathbb{R}\backslash\{\hat{x}\},\ k = 1, 2, \ldots$ such that $\lim_{k\to\infty} x^k = \hat{x}$, we have

$$\lim_{k\to\infty} \frac{f(x^k) - f(\hat{x}) - f'(x)(x^k - \hat{x})}{x^k - \hat{x}} = \lim_{k\to\infty} \frac{f(x^k) - \ell(x^k)}{x^k - \hat{x}} = 0,$$

  where $\ell : \mathbb{R} \to \mathbb{R}$ is given by $\ell(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$.
- Recall that $\ell(x)$ provides an increasingly better approximation of the function values $f(x)$ as x gets closer to $\hat{x}$ and its graph is precisely the tangent line to the graph of the function at $\hat{x} \in \mathbb{R}$.

36.4.1 First-Order Characterisation of Convexity

Proposition 36.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then, $f : \mathbb{R} \to \mathbb{R}$ is a convex function if and only if, for any $\hat{x} \in \mathbb{R}$,*

$$f(x) \geq \ell(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x}), \quad \textit{for all } x \in \mathbb{R}.$$

Proof. $\Rightarrow$: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable convex function. Let $\hat{x} \in \mathbb{R}$ be an arbitrary real number. Clearly, $f(\hat{x}) = \ell(\hat{x})$. Therefore, for any real number $x \in \mathbb{R}\backslash\{\hat{x}\}$ and any real number $\lambda \in [0, 1]$, $f(\lambda\hat{x} + (1 - \lambda)x) \leq \lambda f(\hat{x}) + (1 - \lambda)f(x)$, which can be rewritten as

$$f(\hat{x} + (1 - \lambda)(x - \hat{x})) \leq (1 - \lambda)(f(x) - f(\hat{x})) + f(\hat{x}).$$

For any $\lambda \in [0, 1)$, we obtain

$$\frac{f(\hat{x} + (1 - \lambda)(x - \hat{x})) - f(\hat{x})}{1 - \lambda} \leq f(x) - f(\hat{x}).$$

Consider the following sequence of real numbers given by $x^k = \hat{x} + (1/k)(x - \hat{x}),\ k = 1, 2, \ldots$. Note that $x^k \neq \hat{x},\ k = 1, 2, \ldots$ and $\lim_{k\to\infty} x^k = \hat{x}$.

Using $x^k = \hat{x} + (1/k)(x - \hat{x}),\ k = 1, 2, \ldots$, we obtain

$$\begin{aligned} f'(\hat{x}) &= \lim_{k\to\infty} \frac{f(x^k) - f(\hat{x})}{x^k - \hat{x}} \\ &= \lim_{k\to\infty} \frac{f(\hat{x} + (1 - \overbrace{(1 - 1/k)}^{\lambda^k})(x - \hat{x})) - f(\hat{x})}{(1 - \underbrace{(1 - 1/k)}_{\lambda^k})(x - \hat{x})} \\ &\leq \frac{f(x) - f(\hat{x})}{x - \hat{x}}, \end{aligned}$$

which implies that $f(x) \geq \ell(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$.

$\Leftarrow$: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and, for any $\hat{x} \in \mathbb{R}$, $f(x) \geq \ell(x) = f(\hat{x}) + f'(\hat{x})(x - \hat{x})$ for all $x \in \mathbb{R}$. Let $x^1 \in \mathbb{R}$ and $x^2 \in \mathbb{R}$ be two arbitrary real numbers and let $\lambda \in [0, 1]$. Let us define $\hat{x} = \lambda x^1 + (1 - \lambda)x^2 \in \mathbb{R}$. Then,

$$\begin{aligned} f(x^1) &\geq f(\hat{x}) + f'(\hat{x})(x^1 - \hat{x}) \\ f(x^2) &\geq f(\hat{x}) + f'(\hat{x})(x^2 - \hat{x}) \end{aligned}$$

Multiplying the first inequality by $\lambda \geq 0$ and the second one by $1 - \lambda \geq 0$, we obtain $\lambda f(x^1) + (1 - \lambda)f(x^2) \geq f(\hat{x}) = f(\lambda x^1 + (1 - \lambda)x^2)$. Therefore, f is a convex function.

□

Remark 36.1. *A differentiable function $f : \mathbb{R} \to \mathbb{R}$ is a convex function if and only if the tangent line to the graph of the function at $\hat{x} \in \mathbb{R}$ lies below the graph of f at any real number $\hat{x} \in \mathbb{R}$.*

36.4.2 First-Order Necessary and Sufficient Optimality Conditions

Consider the following unconstrained optimization problem:

$$(\text{P}) \quad \min_{x \in \mathbb{R}^n} f(x)$$

Corollary 36.1 (First-Order Necessary and Sufficient Optimality Conditions)**.** *Let $f \in \mathbb{R} \to \mathbb{R}$ be a $\mathbb{C}^1$ convex function. Then, $x^* \in \mathbb{R}$ is a global minimizer of (P) if and only if $f'(x^*) = 0$.*

Proof. $\Rightarrow$: Follows from Proposition 34.1.

$\Leftarrow$: If $f'(x^*) = 0$, then, by Proposition 36.1,

$$f(x) \geq f(x^*) + f'(x^*)(x - x^*) = f(x^*), \quad \text{for all } x \in \mathbb{R}.$$

Therefore, $x^* \in \mathbb{R}$ is a global minimizer of (P). □

36.4.3 Second-Order Characterisation of Convexity

Proposition 36.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then, $f : \mathbb{R} \to \mathbb{R}$ is a convex function if and only if $f''(x) \geq 0$ for all $x \in \mathbb{R}$.*

Proof. $\Rightarrow$: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable convex function. Let $\hat{x} \in \mathbb{R}$ be an arbitrary real number and let $x^k \in \mathbb{R}\backslash\{\hat{x}\},\ k = 1, 2, \ldots$ be a sequence such that $\lim\limits_{k\to\infty} x^k = \hat{x}$. By Proposition 36.1, for each $k = 1, 2, \ldots$, we have

$$\begin{aligned} f(x^k) &\geq f(\hat{x}) + f'(\hat{x})(x^k - \hat{x}) \\ f(\hat{x}) &\geq f(x^k) + f'(x^k)(\hat{x} - x^k) \end{aligned}$$

Rearranging the inequalities, we obtain

$$f'(\hat{x})(x^k - \hat{x}) \leq f(x^k) - f(\hat{x}) \leq f'(x^k)(x^k - \hat{x}).$$

Therefore, $(f'(x^k) - f'(\hat{x}))(x^k - \hat{x}) \geq 0$. Dividing both sides by $(x^k - \hat{x})^2 > 0$, it follows that

$$f''(\hat{x}) = \lim_{k\to\infty} \frac{f'(x^k) - f'(\hat{x})}{x^k - \hat{x}} \geq 0.$$

$\Leftarrow$: Suppose that $f''(x) \geq 0$ for all $x \in \mathbb{R}$. By the fundamental theorem of calculus, for any $x \in \mathbb{R}$ and $\hat{x} \in \mathbb{R}$, we have

$$f(x) - f(\hat{x}) = \int_{\hat{x}}^{x} f'(t)dt = \int_{\hat{x}}^{x} (f'(t) - f'(\hat{x}) + f'(\hat{x}))dt.$$

Applying the fundamental theorem of calculus once again, we obtain

$$f'(t) - f'(\hat{x}) = \int_{\hat{x}}^{t} f''(u)du.$$

Therefore,

$$f(x) - f(\hat{x}) = \int_{\hat{x}}^{x} \left(\int_{\hat{x}}^{t} \underbrace{f''(u)}_{\geq 0} du \right) dt + \int_{\hat{x}}^{x} f'(\hat{x})dt \geq f'(\hat{x})(x - \hat{x}).$$

By Proposition 36.1, f is a convex function.

□

36.5 Multivariate Case

- Suppose now that $f : \mathbb{R}^n \to \mathbb{R}$.

Proposition 36.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then, f is a convex function if and only if $\phi_d : \mathbb{R} \to \mathbb{R}$ given by $\phi_d(\alpha) = f(\hat{x} + \alpha d)$ is a convex function for each $\hat{x} \in \mathbb{R}^n$ and each $d \in \mathbb{R}^n$.*

Proof. $\Rightarrow$: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\hat{x} \in \mathbb{R}^n$ be an arbitrary vector. Pick any arbitrary direction $d \in \mathbb{R}^n$ and consider the function $\phi_d : \mathbb{R} \to \mathbb{R}$ given by $\phi_d(\alpha) = f(\hat{x} + \alpha d)$. For any real numbers $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$, and $\lambda \in [0, 1]$, we have

$$\begin{aligned}
\phi_d(\lambda\alpha_1 + (1-\lambda)\alpha_2) &= f(\hat{x} + (\lambda\alpha_1 + (1-\lambda)\alpha_2)d) \\
&= f(\lambda(\hat{x} + \alpha_1 d) + (1-\lambda)(\hat{x} + \alpha_2 d)) \\
&\leq \lambda f(\hat{x} + \alpha_1 d) + (1-\lambda) f(\hat{x} + \alpha_2 d) \\
&= \lambda\phi_d(\alpha_1) + (1-\lambda)\phi_d(\alpha_2).
\end{aligned}$$

Therefore, $\phi_d : \mathbb{R} \to \mathbb{R}$ is a convex function.

$\Leftarrow$: Suppose that $\phi_d : \mathbb{R} \to \mathbb{R}$ given by $\phi_d(\alpha) = f(\hat{x} + \alpha d)$ is a convex function for each $\hat{x} \in \mathbb{R}^n$ and each $d \in \mathbb{R}^n$. Let $\hat{x} \in \mathbb{R}^n$ be an arbitrary vector and let $x \in \mathbb{R}^n$. Define $d = x - \hat{x} \in \mathbb{R}^n$. Then,

$$\begin{aligned}
f(\lambda\hat{x} + (1-\lambda)x) &= f(\hat{x} + (1-\lambda)d) \\
&= \phi_d(1-\lambda) \\
&= \phi_d(\lambda(0) + (1-\lambda)(1)) \\
&\leq \lambda\phi_d(0) + (1-\lambda)\phi_d(1) \\
&= \lambda f(\hat{x}) + (1-\lambda) f(x).
\end{aligned}$$

Therefore, f is a convex function. □

36.5.1 Implications on First-Order Conditions

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Then, $\phi_d : \mathbb{R} \to \mathbb{R}$ given by $\phi_d(\alpha) = f(\hat{x} + \alpha d)$ is a differentiable convex function for each $\hat{x} \in \mathbb{R}^n$ and each $d \in \mathbb{R}^n$, and its derivative is given by $\phi_d'(\alpha) = \nabla f(\hat{x} + \alpha d)^T d$.
- For any $x \in \mathbb{R}^n$ and any $\hat{x} \in \mathbb{R}^n$, let $d = x - \hat{x} \in \mathbb{R}^n$. Therefore,

$$f(x) - f(\hat{x}) = \phi_d(1) - \phi_d(0) \geq \phi_d'(0)(1 - 0) = \nabla f(\hat{x})^T d = \nabla f(\hat{x})^T (x - \hat{x}).$$

- By Propositions 36.1 and 36.3, if the above inequality holds for any $x \in \mathbb{R}^n$ and any $\hat{x} \in \mathbb{R}^n$, then f is a convex function.
- Therefore, for a convex $\mathbb{C}^1$ function $f : \mathbb{R}^n \to \mathbb{R}$, $x^* \in \mathbb{R}^n$ is a global minimizer of f if and only if $\nabla f(x^*) = \mathbf{0}$.

36.5.2 Implications on Second-Order Conditions

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable convex function. Then, $\phi_d : \mathbb{R} \to \mathbb{R}$ given by $\phi_d(\alpha) = f(\hat{x}+\alpha d)$ is a twice differentiable convex function for each $\hat{x} \in \mathbb{R}^n$ and each $d \in \mathbb{R}^n$, and its second derivative is given by $\phi_d''(\alpha) = d^T \nabla^2 f(\hat{x} + \alpha d)^T d$.
- Therefore, by Proposition 36.2,

$$\phi_d''(0) = d^T \nabla^2 f(\hat{x}) d \geq 0, \quad \text{for all } \hat{x} \in \mathbb{R}^n, \quad \text{for all } d \in \mathbb{R}^n.$$

- By Proposition 36.2, f is a convex function if and only if $\nabla^2 f(x) \in \mathbb{R}^{n \times n}$ is positive semidefinite for each $x \in \mathbb{R}^n$.

36.6 Concluding Remarks

- For a convex function $f : \mathbb{R}^n \to \mathbb{R}$, any local minimizer is in fact a global minimizer.
- In comparison with the general case, we obtain simpler and a complete set of optimality conditions.
- Differentiable and twice differentiable convex functions admit alternative characterisations that may be easier to check than directly verifying the definition of a convex function.

Exercises

Question 36.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function and let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex and nondecreasing function. Prove that $f \circ g : \mathbb{R} \to \mathbb{R}$ given by $(f \circ g)(x) = f(g(x))$ is a convex function by using the second-order characterisation of convexity.*