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Complex portfolio selection via convex mixed-integer quadratic programming: a survey

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Abstract

In this paper, we review convex mixed-integer quadratic programming approaches to deal with single-objective single-period mean-variance portfolio selection problems under real-world financial constraints. In the first part, after describing the original Markowitz's mean-variance model, we analyze its theoretical and empirical limitations, and summarize the possible improvements by considering robust and probabilistic models, and additional constraints. Moreover, we report some recent theoretical convexity results for the probabilistic portfolio selection problem. In the second part, we overview the exact algorithms proposed to solve the single-objective single-period portfolio selection problem with quadratic risk measure.

Keywords: portfolio selection; mixed-integer quadratic programming; convex MINLP; robust and probabilistic optimization; exact methods

1. Introduction

The first milestone in modern single-period portfolio selection theory is undoubtedly Harry Markowitz's (1952) seminal paper (for a historical perspective, see also Rubinstein, 2002 and surveys Wallingford, 1967; Constantinides and Malliaris, 1995; Elton and Gruber, 1997), in which the mean-variance portfolio optimization model was proposed for the first time. Although several ideas and results had already been introduced by de Finetti (1940) (see Barone, 2006, for the English translation of the first chapter "The problem in a single accounting period"), the contribution of the Italian mathematician was discovered only recently by the financial international community (see Rubinstein, 2006a, 2006b; Pressacco and Serafini, 2007) and was acknowledged by Harry Markowitz (2006) himself. The mean-variance approach is based on the fundamental observation that, according to what Markowitz (1952) states, the investors should try to increase their portfolio return and contemporaneously to decrease, as much as possible, its volatility or its risk (see also Markowitz, 1959; Markowitz and Todd, 2000). The portfolio variance is the most widely used measurement of

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the portfolio volatility; other possible risk measurements are reported, for instance, in Chang et al. (2009), Hoe et al. (2010), and Mansini et al. (2014). If the expected returns of the assets follow a Gaussian distribution (Hanoch and Levy, 1969) or the investor's utility function is quadratic (Castellani et al., 2005), then the mean-variance criterion is theoretically compatible with the expected utility hypothesis originally introduced by Bernoulli (1738) (see Sommer, 1954, for the English translation and see also von Neumann and Morgenstern, 1944; Fishburn, 1970; Huang and Litzenberger, 1988, for more modern approaches). As pointed out by Markowitz (2010, 2014), the previous assumptions are sufficient, but not necessary conditions. However, the assumption of Gaussian asset returns might be unrealistic: the probability distribution for the expected returns is generally leptokurtic (Mills, 1997).

The resulting mathematical program might not represent completely the problem solved nowadays by the practitioners, but it can be enriched with various constraints to model the different characteristics of the modern financial markets. Moreover, the mean-variance approach considers only the first- and second-order moments of the probability distribution of the returns: consequently, in specific situations, this approach might lead to counterintuitive or even paradoxical solutions (Copeland and Weston, 1988). However, this approach gives rise to different possible applications (see, for instance, the recent papers: Lejeune and Smatlı-Paç, 2013; Ji and Lejeune, 2015; Vinel and Krokhmal, 2017).

Kallberg and Ziemba (1983) compare the effects of different utility functions with respect to the optimal portfolios when the distribution of the expected returns is Gaussian, and show empirically that utility functions with similar absolute risk aversion indices—defined by Arrow (1970) and Pratt (1964), but originally introduced by de Finetti (1952) (see also Castellani et al., 2005; Montesano, 2009)—give rise to similar optimal portfolios.

Note that in this paper, we exclusively consider single-objective single-period mean-variance portfolio selection problems under real-world constraints. In particular, we focus on mathematical programming formulations and approaches based on mixed integer quadratic programming (MIQP). Consequently, we do not cover multiobjective optimization, multiple period models, or linear risk measurements.

The reminder of the paper is organized as follows. In the next section, we give the mathematical formulation of the portfolio problem according to Markowitz (1952) and we discuss its main drawbacks. Sections 3, 4, and 5 illustrate several ways to enrich the original formulation. Section 6 describes several equivalent mathematical reformulations for the probabilistic portfolio problem. In Section 7, we survey exact methods, proposed in the literature, to solve mean-variance portfolio problems. Conclusions are finally drawn in Section 8. A summary of the main notation used throughout the paper is reported in the Appendix. Sections 3.2 and 4.1–4.3 are sourced from the papers by Bonami and Lejeune (2009) and Lejeune (2011). Section 6.1 is based on Bonami and Lejeune (2009), while Sections 6.2–6.3 on Filomena and Lejeune (2012).

2. Portfolio optimization

We consider r possibly risky assets characterized by a mean return vector $\overline{\mu} \in \mathbb{R}^r$. Let $x \in \mathbb{R}^r_+$ be the vector whose generic entry represents the fraction of the portfolio value invested in asset j ($j = 1, \ldots, r$). For the moment, following Markowitz (1952), we assume that the entries of $\overline{\mu}$ and of the covariance return matrix $\overline{\Sigma} \in \mathbb{R}^{r \times r}$ are known precisely. In the mean-variance approach, we aim to minimize the portfolio variance $x^T \overline{\Sigma} x$ under the constraint that the portfolio return is at least equal to a given level $R > 0$:

$$\min \ x^T \overline{\Sigma} x \tag{1a}$$
$$\text{s.t.} \ \overline{\mu}^T x \geq R \tag{1b}$$
$$\mathbf{e}^T x = 1 \tag{1c}$$
$$x \geq 0, \tag{1d}$$

where $\mathbf{e} \in \mathbb{R}^r$ is the all-one vector. Then, by repeatedly solving problem (1) for different values of return R, we can compute the efficient frontier, that is, the set of the nondominated portfolios in the sense of Pareto optimality. In several cases (see, e.g., Bonami and Lejeune, 2009), an additional nonrisky asset with mean $\overline{\mu}_0$ and zero variance is also considered to algorithmically derive the efficient frontier (see Section 4.8). Several papers (see, e.g., Crama and Schyns, 2003) consider an equality version for the return constraint (1b), namely $\overline{\mu}^T x = R$.

Constraint (1c) ensures that the whole initial budget is invested in the portfolio and in several papers (see, e.g., Burdakov et al., 2016) it is substituted by

$$\mathbf{e}^T x \leq 1. \tag{2}$$

Constraint (1d) prevents short selling, that is, the possibility for the investor to sell financial assets not already in his/her portfolio. This financial operation is generally performed with speculative intents when the investor expects a bearish trend in the financial stock market. In case short selling is allowed, (1c) can be replaced by the constraint $\mathbf{e}^T x = 0$, which defines the so-called dollar neutral portfolio, by requiring the exposure on long part of the portfolio to be equal to the one on the short part. Several authors consider the case where the decision variables x represent the absolute amount invested per asset so that the inequality (1c) becomes $\mathbf{e}^T x \leq B$, where B is the investor's total initial budget.

Buchheim et al. (2015) introduce the budget constraint

$$v^T x \leq B, \tag{3}$$

where the decision variables x are the units of financial asset held in the investor's portfolio and $v \in \mathbb{R}^r_+$ is the vector of the costs per unit of corresponding asset.

Problem (1) is a convex continuous linearly constrained quadratic program, because, by definition, matrix $\overline{\Sigma}$ is symmetric and positive semidefinite; hence, we have a computationally tractable problem. However, the main drawback of this model consists in the sensitivity of the optimal solutions with respect to the input parameters (expected returns and covariance matrix), which are clearly unknown in real-world applications (see, e.g., Jobson and Korkie, 1981; Michaud, 1989; Jobson, 1991; Jorion, 1992; Broadie, 1993; Britten-Jones, 1999; Chen and Zhao, 2003; Ceria and Stubbs, 2006; Fastrich and Winker, 2012). Furthermore, when $\overline{\Sigma}$ is estimated starting from empirical measurements, it might happen that semidefiniteness is not directly satisfied and some ad hoc procedures are required (see Higham, 2002; Scherer, 2002; Cournéjols and Tütüncü, 2007).

Chopra (1993) empirically analyzes the effects of slight differences in the estimate. Best and Grauer (1991) conduct a theoretically rigorous analysis with computational results about the sensitivity of mean-variance efficient portfolios with respect to possible changes in asset means. Several papers study the instability and ill-conditioning of problem (1): for instance, Kallberg and Ziemba (1984) consider estimation errors in the investor's utility function and the mean vector and covariance matrix of the return distribution for normally distributed portfolio selection problems and observe that errors in the mean vector give rise to significant problems. Chopra and Ziemba (1993) show that the estimating errors with respect to the expected return means is generally one order of magnitude larger than the one corresponding to estimating errors in asset variances or covariances, assuming negative exponential utility function with joint normal distribution of returns.

3. Robust and probabilistic approaches

3.1. Robust approaches

The robust version of the mean-variance problem (1) has been considered in quite recent works (see the surveys of Fabozzi et al., 2010; Gabriel et al., 2014). It consists in assuming that the expected returns are uncertain and their expected values and variances belong to a given set. By supposing that the unknown input parameters belong to a given uncertainty set (see El Ghaoui and Lebret, 1997; Ben-Tal and Nemirovskii, 1998, 1999; El Ghaoui et al., 1998), it is possible to show some theoretical results.

Goldfarb and Iyengar (2003) establish that the robust portfolio selection problem can be formulated as a second-order cone programming (SOCP) (see Lobo et al., 1998; Ben-Tal and Nemirovskii, 2001; Alizadeh and Goldfarb, 2003; Boyd and Vandenberghe, 2004) for ellipsoidal uncertain sets.

Tütüncü and Koenig (2004) assume box uncertain sets both for the mean return vector and the covariance matrix. They show that the corresponding robust portfolio problem for the objective function $\mu^T x - \lambda x^T \Sigma x$, where λ is a given risk aversion coefficient, can be reformulated as a saddle-point problem.

Under the assumption that the return mean belongs to a convex polytope, whose vertices are known, Costa and Paiva (2002) prove that program (1) can be formulated as a linear matrix inequalities (LMIs) problem (see Boyd et al., 1994; Nesterov and Nemirovskii, 1994). El Ghaoui et al. (2003) show that, when the mean and the covariance are unknown, but bounded, the worst-case mean-variance portfolio selection problem can be reformulated as a semidefinite program (SDP) (see Nesterov and Nemirovskii, 1994; Vandenberghe and Boyd, 1996; Saigal et al., 2000; Ben-Tal and Nemirovskii, 2001).

Finally, Ye et al. (2012) introduce uncertain sets both for the mean vector and the second moment matrix of the returns, showing the connection between the fully robust portfolio selection problem with box uncertain set for the mean and ellipsoid uncertain set for the second moment of the returns and SOCP, SDP, and semiinfinite programming (see Žaković and Rumstem, 2002).

3.2. Probabilistic approach

This section, together with Sections 4.1–4.3, is entirely based on the papers by Bonami and Lejeune (2009) and Lejeune (2011).

Bonami and Lejeune (2009) take into account the uncertainty in the expected assets returns by formulating a stochastic programming problem including a probabilistic constraint, which imposes that the expected return of the optimal portfolio should be not less than a given return level R with a high probability $p > 0$.

Let ξ be the random vector representing the expected returns of the r risky assets. We assume that the random vector ξ admits a probability density function and the density function of $\xi^T x$ is strictly positive. Moreover, let $\mu \in \mathbb{R}^r$ with $\mu = \mathbb{E}[\xi]$ and $\Sigma = \mathbb{E}[(\xi - \mu)(\xi - \mu)^T]$ be the mean and the covariance matrix for the r-variate distribution of ξ, respectively. Formulation

$$\min \quad x^T \Sigma x \tag{4a}$$
$$\text{s.t.} \quad \mathbb{P}\left(\xi^T x \geq R\right) \geq p \tag{4b}$$
$$\mathbf{e}^T x = 1 \tag{4c}$$
$$x \geq 0 \tag{4d}$$

is usually referred to as the probabilistic Markowitz formulation and its deterministic equivalent defines a nonlinear optimization problem (NLP) (see Bonami and Lejeune, 2009; Filomena and Lejeune, 2012, 2013; Lejeune, 2014). Let $\psi = (\xi^T x - \mu^T x)/\sqrt{x^T \Sigma x}$ be the standardized random variable representing the normalized portfolio return. Equation (4b) can be equivalently rewritten, as follows:

$$\mathbb{P}\left(\xi^T x \geq R\right) = \mathbb{P}\left(\psi \geq \frac{R - \mu^T x}{\sqrt{x^T \Sigma x}}\right) = 1 - F_{(x)}\left(\frac{R - \mu^T x}{\sqrt{x^T \Sigma x}}\right), \tag{5}$$

where $F_{(x)}(\cdot)$ is the cumulative distribution of the standardized portfolio return. We assume that $F_{(x)}(\cdot)$ is a continue strictly increasing function. Moreover, we point out that the analytic form of the probability distribution F depends on the portfolio weights x. It follows that the probabilistic constraint (5) becomes

$$\begin{aligned} 1 - F_{(x)}\left(\frac{R - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \geq p \iff 1 - p \geq F_{(x)}\left(\frac{R - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \\ \iff \mu^T x + F_{(x)}^{-1}(1 - p)\sqrt{x^T \Sigma x} \geq R, \end{aligned} \tag{6}$$

where $F_{(x)}^{-1}(\cdot)$ is the inverse of the cumulative distribution $F_{(x)}(\cdot)$ and $F_{(x)}^{-1}(1 - p)$ is the $(1 - p)$-quantile of $F_{(x)}(\cdot)$. Therefore, the deterministic equivalent of optimization problem (4) corresponds to the following NLP (Kataoka, 1963):

$$\min \quad x^T \Sigma x \tag{7a}$$
$$\text{s.t.} \quad \mu^T x + F_{(x)}^{-1}(1 - p)\sqrt{x^T \Sigma x} \geq R \tag{7b}$$
$$\mathbf{e}^T x = 1 \tag{7c}$$
$$x \geq 0. \tag{7d}$$

In the following, we survey for which classes of probability distributions the problem can be reformulated as a SOCP (see, e.g., Ben-Tal and Nemirovskii, 2001; Alizadeh and Goldfarb, 2003; Boyd and Vandenberghe, 2004). We thus recall the definition of centrally distributed random variable.

Definition 1 (Serfling, 2006). *Let $\xi \in \mathbb{R}^r$ be a random variable, whose probability density function is $f : \mathbb{R}^r \to \mathbb{R}$. If $f(\xi - \theta) = f(\theta - \xi)$, then ξ has a distribution that is centrally symmetric about $\theta \in \mathbb{R}^r$.*

Theorem 1 (Bonami and Lejeune, 2009). *Let $p \in [0.5, 1)$. If the probability distribution of $\xi^T x$ is centrally symmetric, then the deterministic constraint* (7b)*, equivalent to* (5)*, is a second-order cone constraint (SOCC).*

Therefore, optimization problem (4) is an SOCP because its objective function is convex quadratic and its feasible region is described by the intersection of a second-order cone and several linear constraints. Constraint $\mu^T x \geq R - F_{(x)}^{-1}(1-p)\sqrt{x^T \Sigma x}$ ensures that the expected portfolio return is greater than the given return plus a penalty term, which is function of the portfolio variance and is increasing with the confidence level p (Filomena and Lejeune, 2012).

We recall also the definition of the skewness of a multivariate distribution of a real-valued random variable ξ with mean μ and standard deviation σ (Avèrous and Meste, 1997):

$$\mathrm{skew}(\xi) = \frac{\mathbb{E}[\xi - \mu]^3}{\sigma^3}. \tag{8}$$

The skewness is basically an asymmetry index of the distribution: perfectly symmetric distributions have zero skewness.

Theorem 2 (Bonami and Lejeune, 2009). *Let $p \in [0.5, 1)$. If the skewness of the probability distribution of $\xi^T x$ is positive, then the deterministic constraint* (7b)*, equivalent to* (5)*, is an SOCC.*

The exact value of the $(1-p)$-quantile, $F_{(x)}(1-p)$, is known only for few probability distributions. If we assume, for example, that the distribution of the expected returns is Gaussian, which is a quite restrictive assumption (see, e.g., Fama, 1963, 1965; Mandelbrot, 1963; Rachev et al., 2005), but rather common in several theoretical frameworks (see Hanoch and Levy, 1969; Kallberg and Ziemba, 1983; Montesano, 2009), then the numerical values of quantiles $F_{(x)}^{-1}(1-p)$ of the normalized portfolio return ψ are computationally known.

4. Additional constraints

Beyond the ill-conditioning of problem (1), the other serious drawback of Markowitz's original proposal is represented by the fact that the problems faced by practitioners in real-world applications are generally characterized by several other restrictions besides the ones in problem (1) (see, e.g., Steinbach, 2001; Cournéjols and Tütüncü, 2007). Nevertheless, we can consider additional constraints to problems (1) or (4), which describe the most common characteristics observed in real-world financial markets (see Fabozzi et al., 2007; Lobo et al., 2007; Kolm et al., 2014; Lejeune, 2014; Mansini et al., 2014). However, this kind of constraints could make the efficient frontier discontinuous and more challenging to compute (Jobst et al., 2001).

4.1. Buy-in thresholds

Generally, investors avoid extremely small long positions in their portfolios, because, on the one side, they have a limited impact on the return value of the portfolio and, on the other side, they could be quite expensive with respect to finance fees and monitoring costs (Scherer and Martin, 2005). Long positions not belonging to a given range $[\underline{x}_j, \overline{x}_j] \subseteq [0, 1]$ of the total initial budget B can be prevented with the simple range constraint

$$\underline{x}_j \leq x_j \leq \overline{x}_j, \quad j = 1, \ldots, r. \tag{9}$$

Several authors (see, e.g., Chang et al., 2000; Jobst et al., 2001; Frangioni and Gentile, 2006; Cesarone et al., 2013) require x to be a semicontinuous variable (Sun et al., 2013), that is, they impose $x_j \in [\underline{x}_j, \overline{x}_j] \cup \{0\}$ for all $j = 1, \ldots, r$: they introduce extra binary variables $\delta \in \{0, 1\}^r$ such that, for all $j = 1, \ldots, r$, $\delta_j = 1$ if the investor holds the asset j, that is, if $x_j > 0$, and add the following constraints avoiding too small or huge holding positions:

$$\underline{x}_j \delta_j \leq x_j \leq \overline{x}_j \delta_j, \quad j = 1, \ldots, r. \tag{10}$$

Note that constraints (10) directly imply $0 \leq x_j \leq \delta_j$ for all $j = 1, \ldots, r$.

4.2. Round lot purchasing

Usually, investors consider financial operations regarding only lots of shares because in this way they can limit monitoring and purchasing/selling additional costs. Furthermore, for small private investors, splitting a large lot involves a premium, that has to be paid to the broker. The higher the cost of splitting large batch in single shares, the greater the impact of this kind of cost with respect to the optimal portfolios. Round lot purchasing constraints prescribe that investors hold, for the risky asset j ($j = 1, \ldots, r$), batches or lots of S_j stocks.

Let us define $\gamma \in \mathbb{Z}_+^r$ as a vector of general integer variables. We require that the number of the shares of asset j ($j = 1, \ldots, r$), namely $\eta_j \in \mathbb{Z}_+$, is an integer multiple of the lot-size S_j:

$$\eta_j = \gamma_j S_j, \quad j = 1, \ldots, r. \tag{11}$$

Let q_j be the market value of asset j ($j = 1, \ldots, r$) held in portfolio, then we have $\eta_j = x_j B / q_j$ and constraint (12) can be equivalently rewritten as follows:

$$x_j = \frac{q_j \gamma_j S_j}{B}, \quad j = 1, \ldots, r. \tag{12}$$

The reader is referred to Bonami and Lejeune (2009) for further discussion.

Mansini and Speranza (1999) show that finding a feasible solution of problem (1) with round lot constraints (11), upper bound on γ_j, that is, the number ($j = 1, \ldots, r$) of minimum lots, and bound constraints with respect to the total portfolio expenditure is NP-complete.

4.3. Sector diversification

Generally, either there exist law limitations about the risk exposure (this is the case, for instance, of pension funds) or investors try to hold a representative portion of their portfolio in a prescribed number of asset categories or industrial sectors. However, in general, optimal portfolios for problem (1) are not well-diversified (Green and Hollifield, 1992). Usually, given are lower bound on the fraction of portfolio value held in specific sets of shares. For classical empirical analysis about financial benefits of a well-diversified portfolio, we refer the reader to Copeland and Weston (1988), Fama (1976), and Solnick (1975).

Let us assume that every asset can be allocated to a specific financial category and let C_k ($k = 1, \ldots, n$) be the index set of all risky assets connected with the category k. Moreover, we suppose that sets C_k define a partition of $\{1, \ldots, r\}$. We introduce a binary variable $\zeta_k \in \{0, 1\}$ for each financial category, such that $\zeta_k = 1$ if and only if the investment in financial category k ($k = 1, \ldots, n$) is above a prescribed minimum level $\underline{s}$:

$$\underline{s}\zeta_k \leq \sum_{j \in C_k} x_j \leq \underline{s} + (1 - \underline{s})\zeta_k. \tag{13}$$

Moreover, we have to consider an additional constraint to satisfy the diversification prescription (Bonami and Lejeune, 2009), which requires to hold portions of assets in at least $\underline{n} > 0$ categories:

$$\sum_{k=1}^{n} \zeta_k \geq \underline{n}. \tag{14}$$

4.4. Cardinality constraints

Beyond diversification requirements, asset managers (for instance in index tracking funds) wish to replicate as accurately as possible a market index with a limited number of financial agreements, namely $\overline{K} > 0$. This can be modeled through the following cardinality constraint:

$$\|x\|_0 = \sum_{j=1}^{r} \operatorname{sign}(|x_j|) \leq \overline{K}. \tag{15}$$

By introducing additional decision variables δ, already presented for constraints (10), we can straightforwardly reformulate the previous constraint in the following equivalent form Lejeune (2014):

$$\sum_{j=1}^{r} \delta_j \leq \overline{K}. \tag{16}$$

Bienstock (1996) (see also Shaw et al., 2008) shows that problem (1) with cardinality constraint (16) is NP-hard, even when $r = 3$. Several authors (see, e.g., Chang et al., 2000; Fernández and Gómez, 2007; Soleimani et al., 2009; Deng and Lin, 2010; Woodside-Oriakhi et al., 2011) consider an equality version for cardinality constraint (16) and propose mainly heuristic methods to solve the corresponding problem:

$$\sum_{j=1}^{r} \delta_j = \overline{K}. \tag{17}$$

Moreover, finding the $\overline{K}$ assets that should be included in the optimal portfolio is, in general, an NP-hard problem (Moral-Escudero et al., 2006).

Using the theoretical results in Scozzari and Tardella (2008) and Tardella (2004), and extending Cesarone et al. (2008, 2009, 2010), Cesarone et al. (2013) show that the problem (1) with cardinality constraints (16) has the same optimal solution of problem (1) with equality cardinality constraints (17) and reduce this kind of programs to Standard Quadratic Programming Problem (see Bomze, 1998; Bomze et al., 2008), avoiding to explicitly introduce binary variables and considering an exact tailored solving procedure, called increasing set algorithm. The Standard Quadratic Programming Problem consists in the minimization of a quadratic form over the standard simplex and is an NP-hard problem when the Hessian matrix of the objective function is indefinite, that is, if the Hessian matrix of the objective function is neither positive nor negative semidefinite (Bomze, 1998).

Di Gaspero et al. (2007) consider an "interval" version for the cardinality constraint (16): $\underline{K} \leq \sum_{j=1}^{r} \delta_j \leq \overline{K}$, where $\underline{K}$ and $\overline{K}$ are such that $1 \leq \underline{K} \leq \overline{K} \leq r$. Cardinality constraints are closely related to buy-in threshold constraints (Jobst et al., 2001). Finally, in several papers (see, e.g., Chang et al., 2000; Jobst et al., 2001; Cesarone et al., 2013), it is observed that the problem (1), with cardinality constraints (16) and with minimum and maximum buy-in thresholds (10), can be straightforwardly reformulated as a convex MIQP.

4.5. Sector capitalization

Sector capitalization constraints are introduced by Soleimani et al. (2009) to mathematically formulate the behavior of investors generally inclined to hold assets in financial sectors with higher capitalization value to reduce the total portfolio risk.

Let ℓ be the number of economic sectors and suppose, without loss of generality, that they are sorted in nonincreasing way according to their capitalization value. Define L_l as the set of assets for economic sector l ($l \in \{1, \ldots, \ell\}$). We introduce additional binary variables y_l such that

$$\frac{1}{M} \sum_{j \in L_l} \delta_j \leq y_l \leq M \sum_{j \in L_l} \delta_j \quad l \in \{1, \ldots, \ell\} \tag{18a}$$

$$\sum_{j \in L_l} \overline{\mu}_j + (1 - y_l) \geq \sum_{j \in L_{l+1}} \overline{\mu}_j \quad l \in \{1, \ldots, \ell - 1\}, \tag{18b}$$

where $M \in \mathbb{R}_+$ is a sufficiently large positive number. The "big-M" constraints (18) ensure that the assets belonging to the sectors with higher capitalization values have basically higher probability to be in the optimal portfolios than the ones belonging to sectors with corresponding lower capitalization values.

4.6. Turnover and trading

Frequently, investors already hold a portfolio $x^{(0)}$ and, because of mutations in the financial market or others, they want to change their portfolio, by considering the new financial environment and by limiting, however, the variations with respect to the portfolio already held (Perold, 1984).

Crama and Schyns (2003) propose to introduce restrictions on purchasing and selling variations. In particular, let $\overline{P}_j$ and $\overline{S}_j$ be, respectively, the maximum purchasing and selling levels for asset j ($j = 1, \ldots, r$), turnover constraints can be stated as follows:

$$\max\left\{x_j - x_j^{(0)}, 0\right\} \leq \overline{P}_j \quad j = 1, \ldots, r \tag{19a}$$

$$\max\left\{x_j^{(0)} - x_j, 0\right\} \leq \overline{S}_j \quad j = 1, \ldots, r. \tag{19b}$$

Because of fixed transaction costs (see Section 4.1), additional constraints are, generally, introduced to prevent small variations between portfolios. Let $\underline{P}_j$ and $\underline{S}_j$ be, respectively, the minimum purchasing and selling levels for asset j, trading disjunctive constraints can be stated as follows:

$$\left(x_j = x_j^{(0)}\right) \vee \left(x_j \leq x_j^{(0)} + \underline{P}_j\right) \vee \left(x_j \leq x_j^{(0)} - \underline{S}_j\right) \quad \text{for all } j = 1, \ldots, r.$$

4.7. Benchmark constraints

Often, investors want to obtain a portfolio which is as close as possible to a benchmark (or target) portfolio x^B (Bertsimas et al., 1999). With respect to investments diversified across economic sectors, Bertsimas and Shioda (2006) introduce the following additional constraints to bound variances between the optimal and the target portfolios:

$$\left|\sum_{j \in S_l}(x_j - x_j^B)\right| \leq \varepsilon_l \qquad l = 1, \ldots, \ell. \tag{20}$$

4.8. Collateral constraints

Di Gaspero et al. (2011b) (see also Jacobs et al., 2005) discuss the following legal constraints for short selling portfolios imposed by U.S. Regulation T, a set of U.S. laws concerning the margin requirements for the collateral agreement. The complete text of the regulation is available at https://www.ecfr.gov/cgi-bin/text-idx?tpl=/ecfrbrowse/Title12/12cfr220_main_02.tpl. In particular, they introduce a free-risk asset with mean $\overline{\mu}_0$ and zero variance, the so-called collateral agreement, such that

$$x_0 \geq -a \sum_{j=1}^{r} \min\{0, x_j\} \tag{21a}$$

$$\sum_{j=0}^{r} |x_j| \leq 2, \tag{21b}$$

where $a \in \mathbb{N}_+$ is the security level for the collateral agreement.

In this case, the decision variables x are not constrained to be positive, since short selling is allowed, and variables δ_j defined in (10) are replaced by ternary variables $z \in \{-1, 0, 1\}^r$, such that, for each j ($j = 1, \dots, r$), $z_j = 1$ if the investor bought the asset j, that is, if $x_j > 0$, $z_j = -1$ if the investor sold the asset j, that is, if $x_j < 0$, and $z_j = 0$ if the investor does not hold asset j. Therefore, cardinality constraint (16) becomes $\sum_{j=1}^{r} |z_j| \leq \overline{K}$.

5. Objective functions and related constraints

Besides (1a), several different objective functions have been proposed in the literature to make problems (1) and (4) simpler with respect to computational tractability or to better model real behaviors of investors and money savers. We consider only objective functions involving quadratic risk measure, namely portfolio variance (for an exhaustive survey on approaches for portfolio selection problem with linear risk measures, we refer the interested reader to by Mansini et al. (2014, 2015).

5.1. Penalty functions

To define an unconstrained NLP, Bartholomew-Biggs and Kane (2009) introduce the following penalty function for problem (1) with minimum buy-in threshold constraints (9) with $\underline{x}_j := \underline{x}$ and $\overline{x}_j := 1$ for all $j \in \{1, \dots, r\}$,

$$\phi(x_j) = \frac{4\, x_j(x_j - \underline{x})}{\underline{x}^2}, \quad j = 1, \dots, r, \tag{22}$$

which is nonnegative when $x_j \leq 0$ or $x_j \geq \underline{x}$. Moreover, $-1 \leq \phi(x_j) < 0$ when $x_j \in (0, \underline{x})$, so that additional constraint (9) can be replaced by the following one:

$$\phi(x_j) \geq 0, \quad j = 1, \dots, r. \tag{23}$$

Therefore, an unconstrained NLP can be easily defined, by introducing additional continuous variables $s \in \mathbb{R}^r$, such that $x_j := s_j^2$ for all $j = 1, \dots, r$ and considering the resulting objective function, adjoint with three penalty terms, one replacing each set of constraints:

$$x^T \overline{\Sigma} x + \rho(1 - \mathbf{e}^T x)^2 + \rho\left(\frac{\overline{\mu}^T x}{R} - 1\right)^2 + \tau \sum_{j=1}^{r} \kappa_j(x_j)^2, \tag{24}$$

where ρ and τ are suitable positive parameters and $\kappa_j(x_j) := \min\{0, \phi(x_j)\}$ for all $j = 1, \dots, r$.

A similar approach is stated also for the round lot purchasing constraints (11) or (12), which can be replaced by the following constraints:

$$\kappa'_j(x_j) = \left(\frac{Bx_j}{q_j} - \left\lfloor \frac{Bx_j}{q_j} \right\rfloor\right)\left(1 - \left(\frac{Bx_j}{q_j} - \left\lfloor \frac{Bx_j}{q_j} \right\rfloor\right)\right) = 0, \quad j = 1, \ldots, r \tag{25}$$

where $\lfloor v \rfloor$ denotes the integer part of $v \in \mathbb{R}$.

However, round lot purchasing constraints (11) might make impossible satisfy at the same time request (1c): consequently, the following new quadratic risk measure (Mitchell and Braun, 2013) is considered:

$$\frac{x^T \overline{\Sigma} x}{(\mathbf{e}^T x)^2}, \tag{26}$$

leading to an alternative definition of (24):

$$\frac{x^T \overline{\Sigma} x}{(\mathbf{e}^T x)^2} + \rho\big(\min\{0, 1 - \mathbf{e}^T x\}\big)^2 + \rho\left(\frac{\overline{\mu}^T x}{R} - 1\right)^2 + \tau \sum_{j=1}^{r} \kappa'_j(x_j)^2. \tag{27}$$

Bartholomew-Biggs and Kane (2009) apply a DIRECT (DIviding RECTangles) type global algorithm (see Finkel, 2003; Gablonsky, 2001a, 2001b; Jones, 2001; Jones et al., 1993) to the previous unconstrained NLPs (24) and (27).

5.2. *Balanced objective functions*

Mean-variance portfolio selection problems (1) and (4) are naturally multiobjective optimization programs since usually investors want to gain the maximum profit at the minimum risk: these are, of course, conflicting targets that have to be considered at the same time.

Several authors (see, e.g., Chang et al., 2000) use standard (linear) scalarization techniques such as the Weighted Sum approach (see, e.g., Ehrgott, 2005). Namely, they consider the "balanced" objective function

$$\lambda(x^T \overline{\Sigma} x) - (1 - \lambda)(\overline{\mu}^T x), \tag{28}$$

where $\lambda \in [0, 1]$ is a parameter that represents investor's risk aversion. Let $\theta_1, \theta_2 \in \mathbb{R}_+$ be two parameters, a more general variant is proposed by Schaerf (2002):

$$\theta_1(x^T \overline{\Sigma} x) + \theta_2 \max\{0, \overline{\mu}^T x - R\}. \tag{29}$$

Schaerf (2002) discusses an iterative method to dynamically modify such parameters.

Bertsimas and Shioda (2006) (see also Bertsimas et al., 1999) introduce an extended "balanced" objective function, considering also trading and turnover requirements with respect to a given initial portfolio $x^{(0)}$:

$$\frac{1}{2} x^T \overline{\Sigma} x - \overline{\mu}^T x + \sum_{j=1}^{r} \iota_j \big(x_j - x_j^{(0)}\big)^2, \tag{30}$$

where $\iota_j > 0$ is a coefficient for asset j that models the symmetric purchasing/selling impact with respect to the stock price. Finally, Tadonki and Vial (2003) and Shaw et al. (2008) consider constant and linear transaction costs embedded in a quadratic "balanced" objective function, respectively, namely:

$$\lambda_1(x^T\overline{\Sigma}x) - \lambda_2(\overline{\mu}^T x) + c^T\delta \tag{31}$$

$$\lambda_1(x^T\overline{\Sigma}x) - \lambda_2(\overline{\mu}^T x) + c^T x, \tag{32}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}_+$ are two positive scalars, $c \in \mathbb{R}^r_+$ is a vector, whose entries represent the transaction costs for the portfolio assets and $\delta \in \mathbb{R}^r$ is the binary vector defined in constraints (10).

6. Compact reformulations

In this section, we present several different possible reformulations and approximations for the mean-variance portfolio optimization problem. Section 6.1 is based on Bonami and Lejeune (2009), while Sections 6.2–6.3 on Filomena and Lejeune (2012).

6.1. SOCC inner approximations

As observed in Section 3.2, given a probability distribution on the portfolio returns, it is not always possible to write the problem (7) in a closed form: the exact value for the quantile $F^{-1}_{(x)}(1-p)$ is known only for special distributions (e.g., normal distribution, Student distribution, uniform distribution on an ellipsoid). However, if the probability distribution of the expected returns is only partially known, the value of its quantiles can be approximately computed using several well-known probability inequalities (Lejeune, 2014), such as, for example, Cantelli (Bonami and Lejeune, 2009), Chebyshev (Bonami and Lejeune, 2009), and Camp-Meidell (Lejeune, 2011) inequalities (see also Hardy et al., 1934; Ion, 2001; Lin and Bai, 2011; Marshall et al., 2011).

Theorem 3 (Bonami and Lejeune, 2009). *Assume the first and the second moments of the probability distribution of the portfolio return are finite. The SOCC*

$$\mu^T x - \sqrt{\frac{p}{1-p}}\sqrt{x^T\Sigma x} \geq R \tag{33}$$

is an approximation of the chance constraint (4b).

Theorem 4 (Bonami and Lejeune, 2009). *Assume the first and the second moment of the probability distribution of the portfolio return are finite and the distribution is symmetric. The SOCC*

$$\mu^T x - \sqrt{\frac{1}{2\,(1-p)}}\sqrt{x^T\Sigma x} \geq R \tag{34}$$

is an approximation of the chance constraint (4b).

Theorem 5 (Lejeune, 2011). *Assume the first and the second moment of the probability distribution of the portfolio return are finite and the distribution is symmetric and unimodal. The SOCC*

$$\mu^T x - \sqrt{\frac{1}{9\,(1-p)}}\,\sqrt{x^T \Sigma x} \geq R \tag{35}$$

is an approximation of the chance constraint (4b).

The approximation given by Theorem 4 for a symmetric probability distribution is tighter than the one given by Theorem 3 and the approximation given by Theorem 5 for a symmetric unimodal probability distribution is tighter than the one given by Theorem 4 (see Bonami and Lejeune, 2009; Lejeune, 2011).

6.2. Variance reformulation

Given the symmetric positive definite matrix Σ, we consider its Cholesky decomposition $\Sigma = CC^T$, where $C \in \mathbb{R}^{r\times r}$ is a lower triangular matrix. From a computational viewpoint, the Cholesky decomposition is twice faster and more stable than *LU* factorization or Gauss elimination method (see Meinguet, 1983; Kielbasinski, 1987; Sun, 1992) and it is implemented in high-performance computing numerical software libraries (see Dongarra et al., 1979; Blackford et al., 1997; Anderson et al., 1999).

Note that the Cholesky decomposition exists and is unique if matrix Σ is positive definite (see Householder, 1964; Golub and Van Loan, 1996) and this property is verified by the covariance matrix, if we exclude the case of exact collinearity of the random variables, that is, we assume that none of the risky asset can be exactly replicated by a linear combination of the other ones. The hypotheses to apply Cholesky decomposition to positive semidefinitive matrices are identified in Higham (1990), Householder (1964), and Moler and Stewart (1978) and error analysis is instead formally stated in Moler and Stewart (1978) for idempotent matrices and in Higham (1990) for the general case.

By assuming positive definiteness for covariance matrix Σ and introducing nonnegative decision variable $h \in \mathbb{R}_+$, we obtain the following problem, equivalent to (7):

$$\min_{x,h} \quad \|C^T x\|_2^2 \tag{36a}$$

$$\text{s.t.} \quad \mu^T x - R \geq h \tag{36b}$$

$$F_{(x)}^{-1}\,(1-p)\,\|C^T x\|_2 \geq -h \tag{36c}$$

$$\mathbf{e}^T x = 1 \tag{36d}$$

$$x \geq 0, \quad h \geq 0. \tag{36e}$$

Theorem 6 (Filomena and Lejeune, 2012). *Program (36) is equivalent to the following NLP:*

$$\min_{x,h} \; h$$

$$\text{s.t. (36b), (36c), (36d), (36e).} \tag{37}$$

Filomena and Lejeune (2012) observe that to mathematically compute the variance in problems (36) and (37), the estimate of only $r(r+1)/2$ covariance terms is needed; however, the estimation procedure might result into a covariance matrix, which is not positive semidefinitive.

6.3. Period-separable reformulation

As pointed out by Filomena and Lejeune (2012), the variance of the portfolio can be reformulated as the Euclidean norm of a vector, whose number of components T corresponds to the number of data points by using the following preliminary result.

Theorem 7 (Konno and Suzuki, 1992). *Let ν_{jt} be the (observed) return of asset j at time t and introduce the extra variables $b_t = \sum_{j=1}^{r}(\nu_{jt} - \mu_j)x_j$ $(t = 1, \dots, T)$. The variance of the portfolio return can be rewritten as*

$$x^T \Sigma x = \frac{1}{T}\|b\|_2^2.$$

The probabilistic Markowitz portfolio model (4) can be reformulated as the following convex NLP:

$$\min_{x,h,b} \quad \frac{1}{T}\|b\|_2^2 \tag{38a}$$

$$\text{s.t.} \quad \mu^T x - R \geq h \tag{38b}$$

$$\frac{F_{(x)}^{-1}(1-p)}{\sqrt{T}}\|b\|_2 \geq -h \tag{38c}$$

$$b_t - \sum_{j=1}^{r}(\nu_{jt} - \mu_j)\, x_j = 0, \quad t = 1, \dots, T \tag{38d}$$

$$\mathbf{e}^T x = 1 \tag{38e}$$

$$x \geq 0, \quad h \geq 0. \tag{38f}$$

Finally, we can consider the corresponding equivalent epigraph formulation of problem (38):

$$\min_{x,h,b} \quad h$$

$$\text{s.t.} \quad \text{(38b), (38c), (38d), (38e), (38f).}$$

7. Exact algorithms

Mean-variance portfolio selection problem with the constraints introduced in Section 4 gives rise to a convex MIQP, which is at least as difficult as NP-hard, because it includes the mixed integer linear problem (MILP) as special case (Kannan and Monma, 1978; Garey and Johnson, 1979). Nowadays, MIQPs can be solved via commercial and open-source solvers (see, e.g.,Vigerske and Bussieck, 2010; Bonami et al., 2012; D'Ambrosio and Lodi, 2013). In this section, we overview specialized and more efficient computational procedures recently proposed in the literature.

Bienstock (1996) proposes a tailored branch-and-cut (BC) procedure to solve the cardinality constrained portfolio problem, where (16) is replaced with the "surrogate" constraint:

$$\sum_{j=1}^{r} \frac{x_j}{\overline{x}_j} \leq \overline{K}. \tag{39}$$

Several types of cutting planes, namely mixed-integer rounding inequalities, knapsack cuts, intersection cuts, and disjunctive cuts are also considered in the same paper.

Bertsimas and Shioda (2006) develop a BC algorithm where at each node of branch-and-bound (BB) tree, the convex continuous relaxation of problem (1) with cardinality (16) and buy-in (10) constraints is solved by means of Lemke's method (Cottle et al., 2009). The portfolio problem with objective function (31) and cardinality (16) and buy-in (10) constraints was solved by Tadonki and Vial Tadonki and Vial (2003) with BB techniques together with a Bender decomposition approach.

Lee and Mitchell (2000) describe a parallel BB framework for the cardinality constrained portfolio selection problem, in which each node is approximated by means of sequential quadratic programming (SQP) and each quadratic subproblem is solved via an interior-point method (see, e.g., Nemirovskii and Todd, 2008; Nesterov and Nemirovskii, 1994).

Frangioni and Gentile (2006) solve problem (1) with minimum and maximum buy-in thresholds additional constraints (10) with a BC method improved by using perspective cuts (see also Frangioni et al., 2016), a family of valid inequalities, related to the perspective function (see Hiriart-Urruty and Lemaréchal, 1999a, 1999b) and to the convex envelope of the objective function (see Günlük and Linderoth, 2012). Zheng et al. (2014) propose a difference of convex functions approach to the cardinality constrained quadratic program, by replacing the cardinality constraint (15) with the following piecewise linear approximation:

$$\frac{1}{\omega}\left(\|x\|_1 - \sum_{j=1}^{r} \max\{x_j - \omega, 0\} + \max\{-x_j - \omega, 0\}\right) \leq 0, \tag{40}$$

where $\omega > 0$ is a given parameter. Nonsmooth approximation (1) with constraint (40) is solved by means of successive convex approximation method. This algorithm determines a Karush–Kuhn–Tucker (KKT) point or defines a sequence of points converging to a KKT point for the ω-parametrized approximation. Moreover, the authors show that, letting $\omega \to 0^+$, the optimal value of the approximate problem approaches the optimal value of the original problem.

Shaw et al. (2008) solve cardinality-constrained portfolio problem under the assumption that vector $\overline{\mu}$ of assets returns can be decomposed according to a multiple factor model (Castellani et al., 2005), that is, $\overline{\mu} = \Xi f + u$, where r' represents the number of different factors, $\Xi \in \mathbb{R}^{r \times r'}$ is the sensitivity-factor matrix, $f \in \mathbb{R}^{r'}$ is the factor-return vector, and $u \in \mathbb{R}^r$ is the asset-specific (nonfactor) returns vector. A Lagrangian relaxation of the problem is then solved by means of subgradient procedure (Rockafellar, 1981) and embedded in a BB framework.

Bonami and Lejeune (2009) deal with the deterministic equivalent (7) of the probabilistic portfolio selection problem with buy-in threshold (10), round lot purchasing (12), and diversification (13) and (14) constraints, by proposing a nonlinear BB algorithm (Belotti et al., 2013) with tailored branching rules. Buchheim et al. (2015) consider portfolio selection problem with objective function (28), constraints (1d) and (3), and integrality requirement on the decision variables,
that is,

$$x \in \mathbb{Z}^r, \tag{41}$$

which represents the units of assets held in the investor's portfolio. They introduce a new BB algorithm, where the continuous relaxation is solved through an efficient Frank–Wolfe type method with nonmonotone Armijo line-search.

Burdakov et al. (2016) deal with the cardinality constrained portfolio problem, by introducing an NLP reformulation, whose global minima are the same of the ones of the original problem. The NLP is solved via a sequence of regularized programs (see Kanzow and Schwartz, 2013).

We end this section with Table 1, which summarizes the main characteristics of the papers described above. In particular, the columns report the authors, the year of publication of the paper, the objective function and constraints of the tackled problem, the proposed algorithm, the competitors employed as benchmarks, and the instances that were used for the computational experiments.

8. Conclusions

In this survey, we have introduced the original basic framework and the possible improvements for the Markowitz portfolio selection problem in terms of mathematical modeling and robustness. In fact, in practical application the return and the variance of the assets are unknown and are estimated from observed data: the optimal solutions of the original Markowitz model is not robust with respect to variations in the estimation of the input parameters. Moreover, the original Markowitz work does not include the real-world specifications observed in the financial markets. The first limitation is faced via robust and probabilistic approaches. In the first case, we assume the assets returns (and eventually their variances) are unknown, but belonging to a given uncertain set, while in the second one we substitute the return constraint with a probabilistic version, requiring the return of the portfolio to be above a given level with high probability. The second limitation is handled with additional constraints, as, for example, buy-in threshold, round lot purchasing, diversification, and cardinality constraints.

Furthermore, several variants of the original objective function have been proposed in the literature for computational and algorithmic reasons (for instance, by introducing several penalty terms in the objective to obtain an unconstrained optimization problem) or for better modeling the real-word financial markets (e.g., by considering constant or linear transaction costs). Then, we discussed possible exact and (inner) approximate reformulations for the probabilistic version of the mean-variance portfolio problem. Finally, in the last part of the paper, we have reviewed exact tailored algorithms designed to solve the deterministic and the probabilistic mean-variance portfolio selection problem to optimality.

Table 1
Exact approaches to mean-variance portfolio selection problems (see also Mansini et al., 2014 and references therein)

Author(s)	Year	Objective	Constraints	Algorithm	Benchmark(s)	Instances (number and type)
Bienstock (1996)	1996	(1a)	(1c), (1b), (1d), (9), (16)	Branch-and-Cut	–	13 problems (real-life data)
Lee and Mitchell (2000)	2000	(1a)	(1c), (1b), (1d), (16)	Branch-and-Bound	–	5 portfolio instances and 16 instances from MIPLIB
Tadonki and Vial (2003)	2003	(31)	(1c), (1b), (1d), (10), (16)	Branch-and-Bound	–	OR-Library Beasley (1990, 1996)
Frangioni and Gentile (2006)	2006	(1a)	(1c), (1b), (1d), (10)	Branch-and-Cut	CPLEX 8.0	20 self-generated (see Pardalos and Rodgers, 1990)
Shaw et al. (2008)	2008	(32)	(9), (16)	Branch-and-Bound	CPLEX 8.1	8 real representative instances
Bartholomew-Biggs and Kane (2009)	2009	(24)	–	DIRECT	–	2 illustrative examples
Bartholomew-Biggs and Kane (2009)	2009	(27)	–	DIRECT	–	2 illustrative examples
Bertsimas and Shioda (2006)	2009	(30)	(1c), (1d), (9), (16), (20)	Branch-and-Bound	CPLEX 8.1	50 randomly generated
Bonami and Lejeune (2009)	2009	(1a)	(1c), (1d), (7b), (10)	Branch-and-Bound	Bonami et al. (2008), CPLEX 10.1	36 self-generated instances
			(12), (13), (14), (41)		MINLP_BB Leyffer (2003)	
Zheng et al. (2014)	2012	(1a)	(1c), (1b), (1d), (9), (15)	SQA Algorithm	Lu and Zhang (2013)	Frangioni and Gentile (2006)
Cesarone et al. (2013)	2013	(1a)	(1c), (1b), (1d), (10), (16)	Increasing Set	Di Gaspero et al. (2011a), Moral-Escudero et al. (2006),	Chang et al. (2000)
					Ruiz-Torrubiano and Suarez (2010), Schaerf (2002)	5 additional data sets
Zheng et al. (2014)	2013	(1a)	(1c), (1b), (1d), (10), (16)	Branch-and-Cut	–	Frangioni and Gentile (2006) OR-Library Beasley (1990, 1996)
Buchheim et al. (2015)	2015	(28)	(1d), (3), (41)	Branch-and-Bound with Frank-Wolfe	CPLEX 12.6	Cesarone et al. (2013)
Burdakov et al. (2016)	2016	(1a)	(1c), (1d), (2), (9), (16)	Regularization Method	GUROBI 5.6.2	Frangioni and Gentile (2006)

References are sorted in chronological order; papers published in the same year are sorted according to alphabetic order of the last name of the corresponding first author.

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Appendix

Notation

$r \in \mathbb{N}_+$	number of possibly risky assets
$r' \in \mathbb{N}_+$	number of number of different factors
$n \in \mathbb{N}_+$	number of financial categories
$\ell \in \mathbb{N}_+$	number of economic sectors
$R \in \mathbb{N}_+$	minimum return level for the portfolio
$p \in \mathbb{N}_+$	confidence level for the probabilistic return constraint
$B \in \mathbb{N}_+$	total investor initial budget
$\bar{s} \in \mathbb{N}_+$	prescribed minimum level per financial category
$\underline{n} \in \mathbb{N}_+$	minimum number of categories with positive positions
$\underline{K}, \overline{K} \in \mathbb{N}_+$	minimum and maximum number of assets in the portfolio
$\overline{P} \in \mathbb{R}_+$	maximum purchasing level per asset
$\overline{S} \in \mathbb{R}_+$	maximum selling level per asset
$T \in \mathbb{N}_+$	number of time period for observing the returns of quoted assets
$m \in \mathbb{N}_+$	number of factors in the multiple factor model
$\overline{\mu} \in \mathbb{R}^r$	mean return vector of the assets
$\xi \in \mathbb{R}^r$	random vector of expected returns
$\mu \in \mathbb{R}^r$	mean of the r-variate distribution of ξ
$\psi \in \mathbb{R}^r$	normalized portfolio return
$\underline{x}, \overline{x} \in \mathbb{R}^r$	lower and upper bound for the fraction of the portfolio value
$S \in \mathbb{R}^r$	size of the batches of the assets
$q \in \mathbb{R}^r_+$	market value of the quoted assets
$x^{(0)} \in \mathbb{R}^r$	fraction of the portfolio value already invested
$x^B \in \mathbb{R}^r$	benchmark (or target) portfolio
$c \in \mathbb{R}^r$	transaction costs per asset
$v \in \mathbb{R}^r$	costs per unit of asset
$f \in \mathbb{R}^m$	factor-return vector
$u \in \mathbb{R}^r$	asset-specific (non-factor) returns vector
$\Xi \in \mathbb{R}^{r\times r'}$	sensitivity-factor matrix
$\Sigma \in \mathbb{R}^{r\times r}$	covariance matrix of the r-variate distribution of ξ
$\overline{\Sigma} \in \mathbb{R}^{r\times r}$	covariance return matrix of the assets
$L_l \subseteq \{1, \dots, r\}$	set of assets for economic sector l $(l = 1, \dots, \ell)$
$C_k \subseteq \{1, \dots, r\}$	set of indexes of all risky assets connected with the category k $(k = 1, \dots, n)$

Decision variables:

$x \in \mathbb{R}^r$	(fraction of the) portfolio value invested per asset
$\eta \in \mathbb{Z}^r_+$	integer multiple of the lot-size S
$\delta \in \{0,1\}^r$	additional binary variables such that $\delta_j = 1$ $(j \in \{1, \dots, r\})$ iff $x_j > 0$ $(j \in \{1, \dots, r\})$
$\gamma \in \mathbb{Z}^r_+$	additional vector of general integer variables
$\zeta \in \{0,1\}^n$	additional binary variables such that $\zeta_k = 1$ $(k \in \{1, \dots, n\})$ iff $\sum_{j\in C_k} x_j \le \bar{s}$
$y \in \{0,1\}^\ell$	additional binary variables such that $y_l = 1$ $(l \in \{1, \dots, \ell\})$ iff $\sum_{j\in L_l} \delta_j = 1$
$z \in \{-1,0,1\}^r$	additional ternary variables such that $z_j = 1$ $(j \in \{1, \dots, r\})$ iff $x_j > 0$ $(j \in \{1, \dots, r\})$ and $z_j = -1$ $(j \in \{1, \dots, r\})$ iff $x_j < 0$ $(j \in \{1, \dots, r\})$
$s \in \mathbb{R}^r$	additional continuous variables such that $s_j = \sqrt{x_j}$ $(j \in \{1, \dots, r\})$
$b \in \mathbb{R}^T$	additional continuous variables such that $b_t = \sum_{j=1}^r (\nu_{jt} - \overline{\mu}_j)x_j$ $(t \in \{1, \dots, T\})$

Functions:

$F_{(x)} : \mathbb{R} \to \mathbb{R}$	cumulative distribution of the normalized portfolio return
$\phi : \mathbb{R} \to \mathbb{R}$	penalty function for the minimum buy-in threshold constraint (9)
$\theta : \mathbb{R} \to \mathbb{R}$	penalty function for the round lot purchasing constraints (11)