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RLT: A unified approach for discrete and continuous nonconvex optimization

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1 Introduction

Discrete and continuous nonconvex programming problems arise in a host of practical applications in the context of production, location-allocation, distribution, economics, game theory, process design, and engineering design situations. Several advances have been made in the development of branch-and-cut algorithms for discrete optimization problems and in polyhedral outer-approximation methods for continuous nonconvex programming problems. Often at the heart of these approaches is a sequence of linear programming problems that drive the solution process. The success of such algorithms is strongly linked to the strength or tightness of the linear programming representations employed.

This note focuses on my foray through the evolution of the *Reformulation-Linearization/Convexification Technique* (**RLT**), which affords a unifying framework for solving wide classes of both discrete and continuous nonconvex optimization problems to global optimality. The philosophy of this approach is predicated on constructing "*good*" models, that is, models that possess tight linear/convex programming relaxations, rather than simply "*mathematically correct*" models. As such, this methodology is an automatic reformulation technique for tightening model representations, and can be deployed to not only construct exact solution algorithms, but also to design powerful heuristic procedures.

To put our discussion in perspective, let us outline the development in Sherali and Adams (1990, 1994) that formally launched the RLT. Consider a mixed-integer zero-one linear programming problem whose feasible region X is defined in terms of some inequalities and equalities in binary variables $x = (x_1, \ldots, x_n)$ and a set of bounded continuous variables $y = (y_1, \ldots, y_m)$. Given a value of $d \in \{1, \ldots, n\}$, the RLT procedure constructs various polynomial factors of degree d comprised of the product of some d binary variables x_j or

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their complements $(1 - x_j)$. These factors are then used to multiply each of the constraints defining X (including the variable bounding restrictions), to create a (nonlinear) polynomial mixed-integer zero-one programming problem. Using the key relationship $x_j^2 = x_j$ for each binary variable $x_j, j = 1, \ldots, n$, substituting a variable w_J and v_{Jk}, respectively, in place of each nonlinear term of the type $\prod_{j \in J} x_j$, and $y_k \prod_{j \in J} x_j$, where $J \subseteq \{1, \ldots, n\}$, and relaxing integrality, the nonlinear polynomial program is re-linearized into a higher dimensional polyhedral set X_d defined in terms of the original variables (x, y) and the new variables (w, v). For X_d to be equivalent to X, it is only necessary to enforce x to be binary valued, with the remaining variables treated as continuous, since the binariness on the x-variables is shown to automatically enforce the required product relationships on the w- and v-variables. Denoting the projection of X_d onto the space of the original (x, y)-variables as X_{Pd}, we prove that as d varies from 1 to n, we get,

$$X_{P0} \supseteq X_{P1} \supseteq X_{P2} \supseteq \cdots \supseteq X_{Pn} \equiv \text{conv}(X),$$

where X_{P0} is the ordinary linear programming relaxation, and conv(X) represents the convex hull of X. The projection process also produces an algebraic representation of $X_{Pn} \equiv \text{conv}(X)$, which has a structure that can be exploited to derive facets in the original variable space for various special combinatorial optimization problems by viewing classes of extreme directions of the dual projection cone. Moreover, this hierarchy and convex hull relationship also holds true for polynomial 0-1 mixed-integer programs in which the problem is linear in the continuous variables for any fixed set of binary variables.

2 Stepping stones to the RLT

The invention of RLT as propounded above evolved through a gestation period of about a decade through a transition over many stepping stones. The idea originated in my dissertation (1977–1979) (under the direction of Mokhtar Bazaraa at Georgia Tech) when I studied the classical, notorious quadratic assignment problem (QAP), which can be stated as follows.

$$\textbf{QAP}: \text{Minimize} \quad \sum_{i=1}^{m-1} \sum_{j=1}^{m} \sum_{k=i+1}^{m} \sum_{\substack{\ell=1 \\ \ell \neq j}}^{m} c_{ijk\ell}\, x_{ij}\, x_{k\ell} \tag{1a}$$

$$\text{subject to} \quad \sum_{i=1}^{m} x_{ij} = 1, \quad \forall j = 1, \ldots, m \tag{1b}$$

$$\sum_{j=1}^{m} x_{ij} = 1, \quad \forall i = 1, \ldots, m \tag{1c}$$

$$x \text{ binary.} \tag{1d}$$

In lieu of this standard formulation, we examined the following linearized reformulation of the problem, which we showed to be equivalent to QAP (see Bazaraa and Sherali (1980)).

$$\text{Minimize} \quad \sum_{i} \sum_{j} \sum_{k>i} \sum_{\ell \neq j} c_{ijk\ell}\, w_{ijk\ell} \tag{2a}$$

$$\text{subject to} \quad \sum_i x_{ij} = 1, \quad \forall j \tag{2b}$$

$$\sum_j x_{ij} = 1, \quad \forall i \tag{2c}$$

$$\sum_{i<k} \sum_{j \neq \ell} w_{ijk\ell} = (k-1)x_{k\ell}, \quad \forall k, \ell \tag{2d}$$

$$\sum_{k>i} \sum_{\ell \neq j} w_{ijk\ell} = (m-i)x_{ij}, \quad \forall i, j \tag{2e}$$

$$x \text{ binary}, \quad 0 \leq w \leq e, \tag{2f}$$

where e is an appropriate vector of ones.

Note that to generate the first-level RLT representation for QAP given by (1), we would have multiplied (1b) by $x_{k\ell}$, $\forall k, \ell \neq j$, and (1c) by $x_{k\ell}$, $\forall k \neq i, \ell$, then set $x_{k\ell}^2 = x_{k\ell}$, $\forall k, \ell$, which in effect prompts that $x_{kj}x_{k\ell} = 0$, $\forall j \neq \ell$, $x_{i\ell}x_{k\ell} = 0$, $\forall i \neq k$, and then we would have replaced the product $x_{ij}x_{k\ell}$ by $w_{ijk\ell}$, $\forall i < k, j \neq \ell$. Indeed, it is easy to verify that (2d) is produced by multiplying (1c) by $x_{k\ell}$, $\forall k > i$, linearizing via RLT, and then aggregating the resulting constraints by summing over all $i < k$, for each k, ℓ. Likewise, (2e) is produced by multiplying (1c) by $x_{k\ell}$, $\forall k < i$, linearizing via RLT, and then aggregating the resulting constraints by summing over all $i > k$, for each k, ℓ (and then switching indices (k, ℓ) with (i, j)). Hence, the problem given by (2) that we investigated was a relaxed form of the first-level RLT reformulation of QAP. This was just as well, given that I had to code a revised-simplex LP-based implementation of a Benders' decomposition approach to solve (2), and neither my software nor the existing hardware of the 1970s would have been able to handle a full-blown first-level RLT formulation! Nonetheless, we obtained better-than-previously-known results using this formulation on several standard test problems from the literature. It is gratifying to note that in recent years, using the first- and second-level RLT reformulations of the QAP, researchers have been able to optimally solve problems of size $m \approx 30$ (see the comprehensive survey of the QAP in Loiola et al. (2004)).

After obtaining my Ph.D. in 1979, I joined the faculty at Virginia Tech, where my research interests caught the fancy of two brilliant resident students—Suvrajeet Sen and Warren Adams. Suvrajeet, who relates that he had been living in the protective world of convex programs until then, was fascinated by the challenge of nonconvex optimization, and devoured my 1980 book (with Mike Shetty) on Disjunctive Programming, working with me for his dissertation on this topic as well as on extreme point mathematical programs. Warren, on the other hand, was intrigued by the reformulations I introduced him to related to a discrete location-allocation problem and the class of mixed-integer bilinear programs. Both these developments involved RLT-type liftings to generate tighter model representations (see Sherali and Adams (1984) and Adams and Sherali (1986, 1990, 1993)). I should relate here an anecdote regarding Warren, which I think defines his strong character and his intellectual passion. During my first year at Virginia Tech, Warren was offered a lucrative research assistantship by a senior colleague, and had to make a decision whether to accept that offer, or work with me while grading papers to etch a living. To my delight and honor, Warren preferred "struggle" to "intellectual compromise." He reminded me of a philosopher who once said that whenever he would get some income, he would buy a book; if he got more, then he would buy some food! Fortunately (for both of us!), I acquired a National Science Foundation grant by the end of my second year, and Warren put on some weight as a result! We continued our collaboration on generating the concepts of RLT after Warren's graduation in 1985, the name itself being coined in our 1990 and 1994 papers dealing respectively with pure 0-1 and mixed-integer 0-1 linear and polynomial programming problems. (Actually, the manuscript for Sherali and Adams (1994) was submitted and circulated in 1989.) This RLT for discrete optimization was further extended in Sherali, Adams, and Driscoll (1998) to generate a hierarchy of relaxations leading to the convex hull representation based on the use of more generalized constraint factors in the reformulation phase, in lieu of simply the bound-factors x_j and $(1 - x_j)$, for $j = 1, \ldots, n$. In addition, the proposed enhanced process embedded within its construction stronger logical implications than only $x_j^2 = x_j, \forall j = 1, \ldots, n$. As a result, not only did it subsume the previous development, but also, it provided the opportunity to exploit frequently-arising special structures such as generalized/variable upper bounds; covering, partitioning, and packing constraints; as well as sparsity. A further extension of RLT to handle general mixed-integer (as opposed to 0-1 mixed-integer) programs appears in Sherali and Adams (1999) (also, see our related paper in the accompanying set of volumes of this journal).

3 Relationships with disjunctive programming

It is interesting to explore the connections between RLT and the exciting work on disjunctive programming originated by Balas (1974, 1985) (see also Sherali and Shetty (1980)). For simplicity in illustration consider the following 0-1 mixed-integer region defined by a single binary variable x_1 and some continuous variables $x_2 \in R^n$, where A_2 is $m \times n$, a_1 and b are m-vectors, and assume that Z is bounded.

$$Z = \{(x_1,\ x_2) : a_1x_1 + A_2x_2 \geq b, x_1 \text{ binary},\ x_2 \geq 0\}. \tag{3}$$

To construct conv(Z) (the convex hull of Z) by Balas' (1985) disjunctive programming technique, we could view Z via the disjunction that $x_1 = 1$ or $x_1 = 0$, leading to:

$$(A_2x_2 \geq b - a_1, x_2 \geq 0) \quad \text{or} \quad (A_2x_2 \geq b,\ x_2 \geq 0). \tag{4}$$

This produces conv(Z) in a higher dimensional space as

$$A_2x_2' \geq (b - a_1)x_1, \quad A_2x_2'' \geq b(1 - x_1), \quad 0 \leq x_1 \leq 1, \quad x_2' \geq 0, \quad x_2'' \geq 0, \tag{5a}$$

where

$$x_2 = x_2' + x_2''. \tag{5b}$$

Note that by the RLT process, we would multiply the defining inequalities (including $x_2 \geq 0$) in (3) by the bound-factors x_1 and $(1 - x_1)$, and then linearize the resulting polynomial constraints by substituting w_2 in place of the product term x_1x_2. This yields conv(Z) (see Sherali and Adams (1994)) as

$$A_2w_2 \geq (b - a_1)x_1, \quad A_2(x_2 - w_2) \geq b(1 - x_1), \quad 0 \leq x_1 \leq 1, \tag{6a}$$

$$w_2 \geq 0, \quad (x_2 - w_2) \geq 0. \tag{6b}$$

Observe that (6) is equivalent to (5) under the transformation $w_2 = x_2', (x_2 - w_2) = x_2''$. Notwithstanding this equivalence, there is a *key insight* that is resident in the derivation of (6), namely, that this convex hull representation is produced via a *reformulation step* of *multiplying* the original defining constraints of Z by the factors x_1 and $(1 - x_1)$ and then *linearizing* the resulting polynomial program by a *variable substitution process*.

This construct also permits us to derive convex hull representations for 0-1 polynomial mixed-integer programs as mentioned earlier. To illustrate this feature in the realm of disjunctive programming, suppose that $Z_1 = \{y: A_1 y \geq b_1\}$ and $Z_2 = \{y: A_2 y \geq b_2\}$ are two non-empty polytopes, and that we are interested in constructing $\text{conv}(Z_1 \cup Z_2)$. A 0-1 polynomial formulation of the disjunction that $y \in Z_1$ or $y \in Z_2$ can be constructed by using a single 0-1 variable x as follows:

$$A_1 y(1 - x) \geq b_1(1 - x) \tag{7a}$$

$$A_2 yx \geq b_2 x \tag{7b}$$

$$x \text{ binary.} \tag{7c}$$

By Sherali and Adams (1994), the convex hull representation of (7) can be constructed by multiplying the defining constraints of (7) by x and $(1 - x)$, and then substituting $w = yx$, noting that $x^2 = x$, i.e., $x(1 - x) = 0$. This yields the following representation of $\text{conv}(Z_1 \cup Z_2)$, which coincides with that produced by disjunctive programming methods.

$$A_1(y - w) \geq b_1(1 - x), \quad A_2 w \geq b_2 x, \quad 0 \leq x \leq 1. \tag{8}$$

For the case in which (3) might include several 0-1 variables, based on the RLT construct (6), Balas, Ceria, and Cornuejols (1993) proposed a hierarchy of relaxations leading to the convex hull representation by applying (6) inductively, one variable at a time, interposed by a projection step onto the space of the original variables. They called this process a *lift-and-project* technique. However, the intermediate relaxations generated by Sherali and Adams (1990, 1994) are tighter. For example, the first-level RLT relaxation is itself tighter than the intersection of the convex hulls of the type (6) produced by each of the defining binary variables. We comment here that Lovasz and Schrijver (1991) have independently proposed another even tighter related hierarchy of relaxations leading to the convex hull representation, particularly by including in these relaxations certain semidefinite restrictions.

4 Expanding into the continuous domain

Given my co-interest in continuous nonlinear, nonconvex optimization, and noting the polynomial nature of constraints generated by the RLT process, even for originally linear problems, I began exploring in the late 1980s extensions of RLT to solve continuous nonlinear programs having general polynomial objective function and constraints to global optimality. Indeed, the binary restriction $x_j = 0$ or 1 can be posed in continuous space by the polynomial equation $x_j(1 - x_j) = 0$. Hence, it was evident that there was some latent unifying bridge between the worlds of discrete and continuous nonconvex programming problems, which heretofore resided in disjoint bodies of literature.

Around that time, a very bright student from Turkey, Cihan Tuncbilek, joined our Ph.D. program at Virginia Tech. I engaged Cihan in this research effort, and in 1992, we published the first RLT approach for solving such continuous nonconvex polynomial programming optimization problems. Here, in the reformulation phase, RLT constructs products of the bound factors $(x_j - \ell_j)$ and $(u_j - x_j)$, taken δ at a time, and restricts these polynomial terms to be nonnegative, where δ is the highest degree of any polynomial term appearing in the problem, and where each variable x_j in the problem is restricted by the bounds $\ell_j \le x_j \le u_j$, $\forall j = 1, \dots, n$. The resulting problem is then linearized by substituting a variable X_J for each product term $\prod_{j \in J} x_j$, where J might contain repeated indices, being a subset of the multi-set comprised of δ copies of $\{1, \dots, n\}$. Using this linear programming relaxation, and partitioning the problem based on splitting the bounding interval for a variable x_p that produces the highest discrepancy in the linear programming solution between $X_{J \cup p}$ and X_J over all J, a branch-and-bound algorithm was designed that was proven to converge to a global optimum to the original polynomial program. Special additional classes of polynomial valid inequalities were also proposed by Sherali and Tuncbilek (1997) to further tighten relaxations based on constraint-factors, grid factors, Lagrange interpolating polynomials, and simple semidefinite relations. While all these RLT constraints can quickly proliferate and encumber the solution of the problem, Sherali and Tuncbilek (1997) also proposed certain constraint filtering strategies that could be used to retain only those constraints that can potentially contribute to tighten the relaxation, and hence keep the size of the problem manageable.

While the RLT process leads to tight linear programming relaxations for the underlying discrete or continuous nonconvex problems being solved as discussed above, one has to contend with the repeated solutions of such large-scale linear programs. By the nature of the RLT process, these linear programs possess a special structure induced by the replicated products of the original problem constraints (or its subset) with certain designated variables. At the same time, this process injects a high level of degeneracy in the problem since blocks of constraints automatically become active whenever the factor expression that generated them turns out to be zero at any feasible solution, and the condition number of the bases can become quite large. As a result, simplex-based procedures and even interior-point methods experience difficulty in coping with such reformulated linear programs (see Adams and Sherali (1993) for some related computational experience). On the other hand, a Lagrangian duality-based scheme can not only exploit the inherent special structures, but can quickly provide near optimal primal and dual solutions for deriving tight lower and upper bounds. However, for a successful use of this technique, there are two critical issues. First, an appropriate formulation of the underlying Lagrangian dual must be constructed, and second, an appropriate nondifferentiable optimization technique must be employed to solve the Lagrangian dual problem. Sherali and Myers (1988) and Lim and Sherali (2006a, b) provide directions for developments along these lines.

These ideas have also evolved to solve more general classes of nonconvex factorable programming problems (Sherali and Wang, 2001). Furthermore, several engineering and process design problems lead to formidable optimization models that contain analytically complex objective and/or constraint functions that are either expensive to evaluate, or that might be available as only black-box functions. Such problems are also typically highly nonlinear and nonconvex in nature. Sherali and Ganesan (2003) have devised a new class of *pseudo-global optimization* methods to handle such problems based on using RLT in concert with successive polynomial approximations.

It is worth mentioning that as RLT and other related methods for continuous nonconvex programs have matured, an osmosis of different concepts from the domain of discrete optimization has occurred. This includes preprocessing techniques for range reductions (see Sahinidis (1996) and Sherali and Tuncbilek (1995, 1997)) and branch-and-cut ideas (see Audet et al. (2000) and Vandenbussche and Nemhauser (2005)).

A historical footnote in closing this section might be in order. Many colleagues have inquired as to whatever happened to Cihan Tuncbilek, who co-authored many seminal papers in this field with me, but then mysteriously dropped off into oblivion. After completing his Ph.D. in 1994, Cihan went for military service. He found this experience excruciatingly boring, except for the firing of a single bullet that each cadet was handed with an air of precious fanfare! Many times, during enforced recreational events during the night, Cihan was caught reading a book by a flashlight, for which he was duly punished. But during this visit home, Cihan also found his life-partner, whose sole constraint on him was that he should settle in his home country. Realizing that academicians there need to work at a second job in order to make ends meet, this constraint in concert with Cihan's objective function caused him to abandon OR. Nonetheless, Cihan is now a successful banker who, from time-to-time, dabbles in portfolio optimization.

5 Enhancing RLT via semidefinite cuts

To illustrate the underlying concept here, consider the quadratic polynomial programming problem. Note that the new RLT variables in this context are represented by the $n \times n$ matrix $X \equiv [xx^T]_L$, where $[\cdot]_L$ represents the linearization of the expression $[\cdot]$ under the RLT variable substitution process. Observe that since xx^T is symmetric and positive semidefinite (denoted $\succeq 0$), we could require that $X \succeq 0$, as opposed to simply enforcing nonnegativity on this matrix. In fact, a stronger implication in this same vein is obtained by considering

$$x_{(1)} = \begin{bmatrix} 1 \\ x \end{bmatrix}, \text{ and defining the matrix } M_1 \equiv \left[x_{(1)} x_{(1)}^T\right]_L = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}, \tag{9}$$

and requiring that $M_1 \succeq 0$.

In lieu of solving the resulting semidefinite programming relaxations, which might detract from the ease and convenience of relying on LP relaxations, Sherali and Fraticelli (2002) have proposed the use of a class of RLT constraints known as *semidefinite cuts* that are predicated on the fact that

$$M_1 \succeq 0 \Leftrightarrow \alpha^T M_1 \alpha = \left[(\alpha^T x_{(1)})^2\right]_L \geq 0, \quad \forall \alpha \in R^{n+1},\ \|\alpha\| = 1. \tag{10}$$

Accordingly, given a certain solution $(\bar{x},\ \bar{X})$ to the RLT-based LP relaxation for the underlying quadratic polynomial program for which $\bar{X} \neq \bar{x}\,\bar{x}^T$, Sherali and Fraticelli (2002) invoke (10) to check in polynomial time having a worst-case complexity $O(n^3)$ whether or not $\bar{M}_1 \succeq 0$, where $\bar{M}_1$ evaluates M_1 at the solution $(\bar{x},\ \bar{X})$. In case that $\bar{M}_1$ is not positive semidefinite, they show that this process also automatically generates an $\bar{\alpha} \in R^{n+1}$ such that $\bar{\alpha}^T\, \bar{M}_1\, \bar{\alpha} < 0$, which in turn yields the semidefinite cut

$$\bar{\alpha}^T M_1\, \bar{\alpha} = \left[\left(\bar{\alpha}^T x_{(1)}\right)^2\right]_L \geq 0. \tag{11}$$

Several alternative schemes of implementing this concept by replacing the vector $x_{(1)}$ in (9) by higher-order polynomial terms have been proposed by Sherali and Desai (2005), and ongoing investigations exhibit a promise of enhancing the solvability of many challenging problems using this idea.

6 Applications

RLT has enabled the solution of several difficult practical operational and design problems, many of which had heretofore defied solution. These include different classes of location-allocation problems, facility layout problems, engineering design problems such as communication and water distribution network design, ship design, automobile engine design, national airspace air-traffic management, risk management and resource allocation problems, hard and fuzzy clustering problems, airline operational problems, and stochastic mixed-integer optimization problems and related applications. My home page (*http://www.ise.vt.edu/sherali*) contains references to these works, developed in conjunction with several graduate students.

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