Attitude Heading Comparative Analysis of The Euler Kinematical Equations,The Poisson's Kinematical Equations(Strapdown Equation) and Quaternions Implemantation

Abstract

Attitude representation and propagation constitute fundamental components of flight dynamics, inertial navigation, and guidance systems. Different mathematical parameterizations of rigid body orientation exhibit distinct numerical properties, singularity behavior, and computational stability characteristics.

In this study, three classical attitude kinematic formulations The Euler angle kinematics, Poisson's kinematic equation and quaternions based propagation are implemented and compared using real gyroscope measurements. The angular velocity data are integrated within a Simulink framework, and the resulting Euler angles are analyzed to evaluate consistency, numerical behavior, and representation-induced artifacts.

The results demonstrate strong agreement in roll and pitch behavior across all methods, while yaw exhibits discontinuities due to angular wrapping. The study highlights the superior numerical robustness of quaternion propagation and discusses practical implications for aerospace applications.

1. Introduction

The orientation of a rigid body in three-dimensional space can be described using multiple mathematical representations. In aerospace engineering, attitude representation plays a central role in:

- Flight dynamics modeling
- Inertial navigation systems
- Guidance, navigation, and control (GNC) architectures
- Spacecraft and aircraft attitude determination

However, no single parameterization is universally optimal. Each representation involves trade-offs between physical interpretability, singularity behavior, computational efficiency, and numerical robustness.

This study investigates three widely used kinematic representations:

1. Euler angle kinematics
2. Direction Cosine Matrix (DCM) propagation via Poisson's equation
3. Quaternion-based propagation

All three formulations are implemented in parallel using identical gyroscope measurements. Their outputs are compared in Euler angle space to assess consistency and identify numerical artifacts.

2. Problem Formulation

The gyroscope provides body-frame angular velocity measurements:

$$\omega_b = [p \quad q \quad r]^T$$

where:

- p= roll rate
- q= pitch rate
- r= yaw rate

The objective is to propagate the rigid body orientation over time using these angular velocity measurements and compare the resulting attitude states obtained from different mathematical formulations.

3. Initial Conditions

The initial Euler angles are defined as:

$$\begin{aligned} \phi_0 &= 0 \\ \theta_0 &= 0.0059 \\ \psi_0 &= 0 \end{aligned}$$

From these initial angles:

- The initial Direction Cosine Matrix is constructed using a 3-2-1 rotation sequence.
- The initial quaternion is computed using MATLAB's angle2quat function.

This ensures all three methods start from identical physical initial conditions.

4. Attitude Representations

4.1 Euler Angle Kinematics

Euler angles describe orientation through three sequential rotations. For the 3-2-1 rotation sequence, the relationship between body rates and Euler angle rates is:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = T(\phi, \theta) \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

where $T(\phi, \theta)$is the transformation matrix that depends on the current Euler angles.

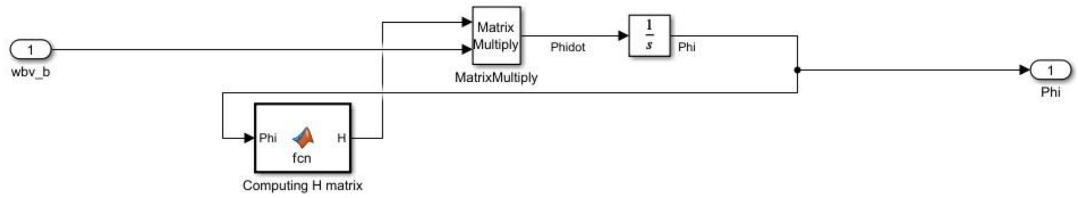

Figure 1 — Euler Kinematics Subsystem

Advantages

- Direct physical interpretability
- Intuitive representation of roll, pitch, and yaw

Limitations

- Singularities occur when $\theta = \pm 90°$(gimbal lock)
- Transformation matrix becomes ill-conditioned near singularity
- Not suitable for large-angle aggressive maneuvers

4.2 Poisson’s Equation

The orientation can also be represented by a rotation matrix C_{bv}, mapping vectors from the body frame to the inertial frame.

The evolution of the DCM follows Poisson’s equation:

$$\dot{C}_{bv} = C_{bv}\Omega$$

where $\Omega$is the skew-symmetric matrix formed from body rates:

$$\Omega = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

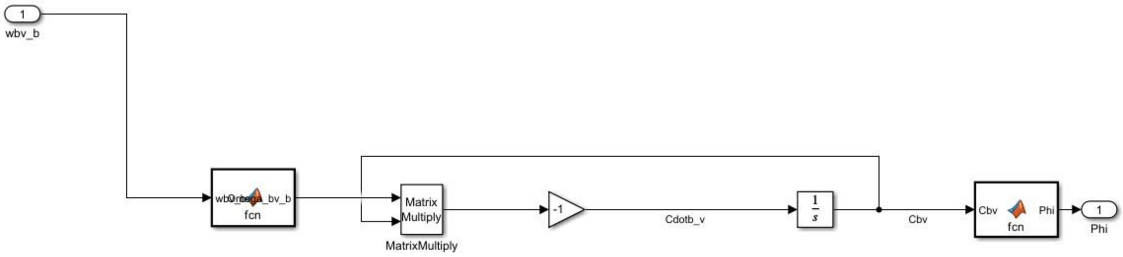

Figure 2 — Poisson's Subsystem

Advantages

- Globally nonsingular
- Direct geometric meaning
- Suitable for theoretical analysis

Limitations

- Requires maintaining orthonormality:

$$C_{bv}^T C_{bv} = I$$

- Numerical drift may violate orthogonality over long integrations
- Computationally heavier (9 states)

4.3 Quaternion Kinematics

Quaternions provide a compact four-parameter representation of orientation:

$$q = [q_0 \quad q_1 \quad q_2 \quad q_3]^T$$

The propagation equation is:

$$\dot{q} = \frac{1}{2}\Omega_q q$$

where Ω_q is constructed from angular velocity components.

The unit-norm constraint must be preserved:

$$\| q \| = 1$$

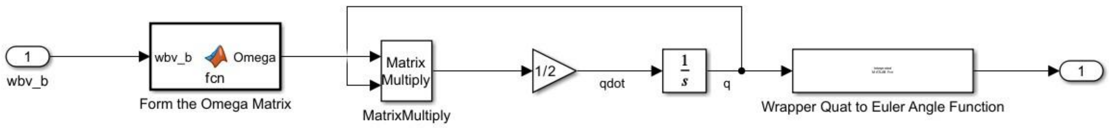

Figure 3 — Quaternion Subsystem

Advantages

- No singularities
- Compact representation (4 states)
- Numerically stable
- Widely used in spacecraft and aircraft systems

Limitation

- Less intuitive physical interpretation
- Requires normalization

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5. Numerical Implementation

The three formulations are implemented in Simulink using:

- Identical angular velocity inputs
- Identical initial conditions
- Parallel integration structures

After propagation, Euler angles are extracted from the DCM and quaternion solutions for direct comparison.

The simulation is run over the duration of the gyroscope dataset.

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6. Results and Discussion

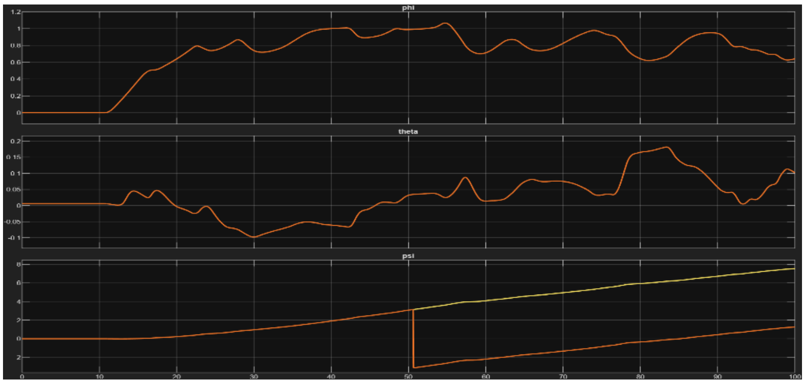

Figure 4- Time histories of Euler angles (ϕ, θ, ψ) obtained from three independent kinematic propagation methods. While roll and pitch responses remain consistent across all formulations, yaw exhibits a discontinuity due to angular wrapping in the Euler angle representation.

6.1 Roll and Pitch Behavior

The roll (ϕ) and pitch (θ) responses show strong agreement across all three formulations. This confirms:

- Consistency of the kinematic equations
- Correct implementation of integration structures
- Absence of singularity within the tested range

No noticeable divergence was observed in roll and pitch behavior.

6.2 Yaw Behavior and Discontinuity

Yaw (ψ) exhibits a visible discontinuity around the mid-simulation period. This is not a physical instability but rather a representation artifact.

Euler angles are typically constrained to:

$$\psi \in (-\pi, \pi]$$

When the yaw angle crosses $\pm\pi$, it wraps around, producing an apparent jump. This discontinuity:

- Does not reflect actual physical rotation
- Does not indicate numerical instability

- Is purely due to angular wrapping

Quaternion propagation itself remains continuous in 4D space. The discontinuity arises during conversion back to Euler angles.

6.3 Numerical Robustness Comparison

Method	Singularities	Drift Risk	Computational Cost	Robustness
Euler	Yes	Low	Low	Moderate
DCM	No	Yes	High	High
Quaternion	No	Very Low	Moderate	Very High

Quaternion formulation demonstrates superior stability and is the preferred choice for practical aerospace applications.

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7. Conclusions

This study compared three classical rigid body attitude propagation methods using real gyroscope data.

Key conclusions:

1. All three formulations produce consistent roll and pitch responses.
2. Euler angles suffer from representation discontinuities due to angular wrapping.
3. The DCM method is geometrically robust but requires orthonormality control.
4. Quaternion propagation provides the most numerically stable and singularity-free formulation.

For aerospace guidance, navigation, and control systems, quaternion-based propagation is strongly recommended as the primary state representation.